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ISOMETRY ACTIONS AND GEODESICS ORTHOGONAL TO SUBMANIFOLDS

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Abstract. We obtain a condition, involving geodesics orthogonal to tangent vectors, which implies that a submanifold must be contained in a level set of a Lipschitz function. One application is the following theorem. Let $f : \Sigma \to M$ be a differentiable immersion of a connected manifold Σ in a complete noncompact manifold with nonnegative sectional curvature. Fix a ray σ in M and assume that for all point $p \in \Sigma$ and $v \in T_p\Sigma$ there exists a vector η orthogonal to $df_p v$ such that the geodesic γ_η tangent to η at p is a ray asymptotic to σ. Then $f(\Sigma)$ is contained in a horosphere of M associated with σ. Another theorem study those ideas in the context of space forms, establishing a set of equivalent conditions on a submanifold so that it is locally contained in a hypersurface invariant under the action of isometries which fix points in a given totally geodesic complete submanifold.

1. Introduction

A simple well-known fact says that if $f : \Sigma \to \mathbb{R}^n$ is an immersion satisfying that at each point of $f(\Sigma)$ there exists a normal line intersecting a fixed point $p \in \mathbb{R}^n$ then $f(\Sigma)$ is contained in a round sphere centered at p. In this paper we will provide two generalizations of this fact, obtaining also an application to horospheres in complete noncompact manifolds of nonnegative sectional curvature, or in Hadamard manifolds. Some results in this paper will just require the weak notion of a differentiable map $f : \Sigma \to \Omega$ in the sense that for any $x \in \Sigma$ there exists a derivative $df_x : T_p\Sigma \to T_{f(x)}\Omega$ which is a first order approximation of f when it is written in local coordinates. In other results we will need the hypothesis that f is of class C^1.

Let us fix some notations. For an arbitrary subset C of a Riemannian manifold M and $r \geq 0$ we set:

$$\mathcal{S}(C,r) = \left\{x \in M \;\middle|\; d(x,C) = r\right\},$$

$$B(C,r) = \left\{x \in M \;\middle|\; d(x,C) < r\right\}; \quad \bar{B}(C,r) = \left\{x \in M \;\middle|\; d(x,C) \leq r\right\},$$

where d is the distance function. Given a tangent vector v in some point in a Riemannian manifold, we will denote by γ_v a geodesic satisfying $\gamma_v'(0) = v$. The domain of γ_v will be specified in each case.

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Busemann functions are very important in the study of complete and noncompact manifolds, specially under curvature conditions. In the case of nonnegative curvature we obtained the following result (compare with condition (C) in Theorem 3 below).

Theorem 1. *Let $f : \Sigma \to M$ be a differentiable immersion of a connected manifold Σ in a complete noncompact manifold with nonnegative sectional curvature. Let σ be a ray in M. Assume that for all point $p \in \Sigma$ and $v \in T_p\Sigma$ there exists a nontrivial vector $\eta \in T_{f(p)}M$ orthogonal to df_pv such that the geodesic $\gamma_\eta : [0, +\infty) \to M$ is a ray asymptotic to σ. Then $f(\Sigma)$ is contained in a horosphere of M associated with σ.*

The above result can be proved by using the following general result.

Theorem 2. *Let $f : \Sigma \to M$ be a differentiable immersion of a connected manifold Σ in a Riemannian manifold M. Let $G : M \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $C > 0$. Assume that for all $p \in \Sigma$ and any $v \in T_p\Sigma$ there exists a nontrivial vector $\eta \in T_{f(p)}M$ orthogonal to df_pv such that the geodesic $\gamma_\eta : [0, 1] \to M$ satisfies that*

$$\big|G(f(p)) - G(\gamma_\eta(1))\big| = C\,L(\gamma_\eta). \tag{1}$$

Then $f(\Sigma)$ is contained in a level set $G^{-1}(\{d\})$ and η is orthogonal to any $\beta'(0)$ such that $\beta : (-\epsilon, \epsilon) \to G^{-1}(\{d\})$ is a curve differentiable at $s = 0$. In particular η is orthogonal to $df_p(T_p\Sigma)$.

Let M be a Hadamard manifold. It is well known that M admits a natural compactification $\bar{M} = M \cup M(\infty)$, where the ideal boundary $M(\infty)$ consists of the asymptotic classes $\gamma(\infty)$ of rays γ in M (see [EO'N] or Chapter 3 of [BGS]). Theorem 2 also implies the following result.

Corollary 1. *Let $f : \Sigma \to M$ be a differentiable immersion of a connected manifold Σ in a Hadamard manifold M. Fix $x_0 \in M(\infty)$ and assume that for all point $p \in \Sigma$ and $v \in T_p\Sigma$ there exists a nontrivial vector $\eta \in T_{f(p)}M$ orthogonal to df_pv such that the geodesic ray $\gamma_\eta : [0, +\infty) \to M$ satisfies that $\gamma_\eta(\infty) = x_0$. Then $f(\Sigma)$ is contained in a horosphere of M associated with x_0.*

Remark 1. It should be observed that Corollary 1 could be proved without using Theorem 2, by taking account that the gradient of a Busemann function in a Hadamard manifold is of class C^1 and defined everywhere.

Remark 2. The reciprocal for Theorem 2 is not true. Example 5.1 below presents a Lipschitz function G on a manifold M, whose level sets are smooth submanifolds. In this example there exist a point p in some level set Σ and a vector $v \in T_p\Sigma$ such that for any $\eta \in T_pM - \{0\}$ orthogonal to v the geodesic γ_η does not satisfy (1).

Given an arbitrary subset $\mathcal{A}$ of a manifold M the distance function from $\mathcal{A}$ is Lipschitz with Lipschitz constant 1 and it vanishes on $\mathcal{A}$. Thus we may apply Theorem 2 to obtain the following corollary, in which a reciprocal is true if $\mathcal{A}$ is a closed set.

Corollary 2. *Let $f : \Sigma \to M$ be a differentiable immersion of a connected manifold Σ in a Riemannian manifold M. Let $\mathcal{A} \subset M$ be an arbitrary subset. Assume that for all $p \in \Sigma$ and $v \in T_p\Sigma$ there exists a vector $\eta \in T_{f(p)}M$ orthogonal to df_pv such that the geodesic $\gamma_\eta : [0,1] \to M$ satisfies that $\gamma_\eta(1) \in \mathcal{A}$ and*

$$L(\gamma_\eta) = d\big(f(p), \mathcal{A}\big). \tag{2}$$

Then $f(\Sigma)$ is contained in $\mathcal{S}(\mathcal{A}, r_0)$ for some constant $r_0 \geq 0$. Reciprocally, if $f : \Sigma \to M$ is a differentiable immersion of a connected manifold Σ in a Riemannian manifold M such that $f(\Sigma) \subset \mathcal{S}(\mathcal{A}, r_0)$ for some constant $r_0 \geq 0$, where $\mathcal{A}$ is a closed set, then for any $p \in M$ there exists a vector η orthogonal to $df_p(T_p\Sigma)$ such that $\gamma_\eta : [0,1] \to M$ satisfies (2).

In the above results two kind of hypotheses about a differentiable immersion $f : \Sigma \to M$ emerged. The weakest one says that for any point $p \in \Sigma$ and any $v \in T_p\Sigma$ there exists a vector η orthogonal to df_pv such that the geodesic γ_η satisfies certain condition. The other one says that for any point $p \in \Sigma$ there exists some vector η orthogonal to $df_p(T_p\Sigma)$ such that γ_η satisfies the same condition. These theorems suggest that we could prove another results by changing the condition on γ_η, for example asking that this geodesic intersects some totally geodesic submanifold. The next theorem will apply these ideas in the context of space forms.

We will denote by $\mathbb{Q}^n_c$ the complete simply-connected n-dimensional manifold of constant curvature c. Let $W = W^j$ denote a complete connected j-dimensional totally geodesic submanifold of $\mathbb{Q}^n_c$. If $c \leq 0$ there exists a natural projection $\pi_W : \mathbb{Q}^n_c \to W$ satisfying $\pi_W(q) = \gamma(1)$, where $\gamma : [0,1] \to \mathbb{Q}^n_c$ is the unique geodesic with $\gamma(0) = q$, $\gamma(1) \in W$ and the length $L(\gamma) = d(q, W)$. To obtain an equivalent definition for π_W, we may consider the normal bundle

$$\nu(W) = \Big\{(x, v) \;\Big|\; x \in W,\, v \in (T_xW)^\perp\Big\},$$

where $(T_xW)^\perp$ denotes the orthogonal complement of T_xW relatively to $T_x(\mathbb{Q}^n_c)$. It is well known that the normal exponential map $\exp^\perp : \nu(W) \to \mathbb{Q}^n_c$ is a diffeomorphism if $c \leq 0$. Then we may define $\pi_W : \mathbb{Q}^n_c \to W$ by $\pi_W(\exp^\perp(p, v)) = p$, for $(p, v) \in \nu(W)$. It is very easy to see that both definitions coincide.

Now we recall how this projection may be defined in the case $c > 0$, where the domain of π_W is the complement of a measure-zero set. We first set

$$V_W = \mathcal{S}\left(W, \frac{\pi}{2\sqrt{c}}\right).$$

It is well-known (see Lemma 3.3 below) that V_W is a totally geodesic sphere of dimension $n - j - 1$ if $j \leq n - 2$, and V_W consists of two antipodal points if $j = n-1$. Set

$$B_W = \left\{ (x,v) \in \nu(W) \;\Big|\; |v| < \frac{\pi}{2\sqrt{c}} \right\}.$$

If $j \geq 1$, it is well known that $\mathbb{Q}_c^n = \bar{B}\left(W, \frac{\pi}{2\sqrt{c}}\right)$, that the map $\exp^\perp|_{B_W} : B_W \to (\mathbb{Q}_c^n - V_W)$ is a diffeomorphism and that $\exp^\perp(\partial B_W) = V_W$, where ∂B_W denotes the boundary of the closure $\bar{B}_W$ (see Lemma 3.3 below). Thus we may define the projection $\pi_W : (\mathbb{Q}_c^n - V_W) \to W$ by $\pi_W(\exp^\perp(p,v)) = p$. In other words, for $q \in \mathbb{Q}_c^n - V_W$ it holds that $\pi_W(q) = \gamma(1)$, where $\gamma : [0,1] \to \mathbb{Q}_c^n$ is the unique geodesic with $\gamma(0) = q$, $\gamma(1) \in W$ and $L(\gamma) = d(q,W)$. If $j = 0$ and W is a point, the map $\pi_W : \mathbb{Q}_c^n \to W$ may be defined as the constant map.

We denote by G_W the group of isometries of $\mathbb{Q}_c^n$ that fix each point in W. Let $\Sigma \subset \mathbb{Q}_c^n$ be a connected embedded differentiable submanifold of the space form $\mathbb{Q}_c^n$. Let $W = W^j$ be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$. We will consider the following properties:

(A) For each point $q \in \Sigma$ there exists a neighborhood U of q in Σ such that U is contained in an embedded hypersurface M of class C^k of $\mathbb{Q}_c^n$, with $k \geq 1$, which is invariant under the action of G_W.

(B) For any point $q \in \Sigma$, there exists a vector $\eta \in T_q(\mathbb{Q}_c^n)$ orthogonal to Σ such that the geodesic γ_η intersects W.

(C) For any point $q \in \Sigma$ and any vector $v \in T_q\Sigma$ with $(d\pi_W)_q v = 0$, there exists a vector $\eta \in T_q(\mathbb{Q}_c^n)$ orthogonal to v such that the geodesic γ_η intersects W.

(D) For each point $q \in \Sigma$ there exists a neighborhood U of q in Σ such that U is contained in an embedded differentiable hypersurface M of $\mathbb{Q}_c^n$ which is invariant under the action of G_W.

In the case $j = 0$, we may use Corollary 2 to obtain the following

Corollary 3. *Let Σ be a differentiable embedded submanifold of $\mathbb{Q}_c^n$ and W a point not contained in Σ. Then Properties* (A), (B), (C) *and* (D) *are equivalent. If one of these properties occurs, then M is in fact a sphere centered at W and $\Sigma \subset M$.*

The next theorem deals with the case $j \geq 1$.

Theorem 3. *Let Σ be a connected embedded differentiable submanifold of $\mathbb{Q}_c^n$. Let $W = W^j$ be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$, with $j \geq 1$. Assume that $\Sigma \cap W = \emptyset$ and that the map $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion. In the case $c > 0$ assume further that $\Sigma \cap V_W = \emptyset$. Then we have that* (D) *implies* (B) *if $c \geq 0$, and* (D) *implies* (C) *for any c. If further Σ is of class C^k, for some $k \geq 1$, then it holds that:*

(i) If $c = 0$ then (A), (B) *and* (C) *are equivalent;*

(ii) If $c > 0$ then (A) *and* (B) *are equivalent;*
(iii) If $c < 0$ then (A) *and* (C) *are equivalent.*

Remark 3. Note that (B) implies (C) trivially. Thus if Σ is of class C^k for some $k \geq 1$ we obtain from Theorem 3 the following sequence of implications

$$\text{(B)} \implies \text{(A)} \implies \text{(D)} \implies \text{(C)}$$

for all values of c.

Remark 4. It is simple to show that (C) is always true if $c > 0$ (see Proposition 4.1).

Remark 5. If Σ is of class C^k for some $k \geq 1$, we will see that several implications that do not appear in Theorem 3 or in Remark 3 fail (see Section 5). We will also see in Section 5 that the assumption that $(\pi_W)|_\Sigma$ is a submersion may not be dropped. In Proposition 4.2 we see that if $c \leq 0$ and Σ is a hypersurface of $\mathbb{Q}_c^n$, then property (C) implies that $(\pi_W)|_\Sigma$ is a submersion.

Example 5.6 presents a nontrivial situation in $\mathbb{R}^4$ in which Theorem 3 holds. In this example, if $p \in W - \{(0,0,0,0),(0,0,1,0)\}$ then the complete totally geodesic submanifold of maximal dimension which is orthogonal to W at p intersects Σ in infinitely many isolated points.

Question 1.1. *Let Σ be a connected embedded differentiable submanifold of $\mathbb{Q}_c^n$. Let $W = W^j$ be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$, with $j \geq 1$. Assume that $\Sigma \cap W = \emptyset$ and that the map $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion. In the case $c > 0$ assume further that $\Sigma \cap V_W = \emptyset$. Is it true that the following assertions hold?*

. *If $c = 0$ then* (D), (B) *and* (C) *are equivalent;*
. *If $c > 0$ then* (D) *and* (B) *are equivalent;*
. *If $c < 0$ then* (D) *and* (C) *are equivalent.*

The following theorem studies the situation when $(\pi_W)|_\Sigma$ is not a submersion but it still has constant rank. It is very surprising to realize that (B) and (C) are not necessary in this case to obtain (A).

Theorem 4. *Let Σ be a connected embedded submanifold of class C^k of $\mathbb{Q}_c^n$, with $k \geq 1$. Let $W = W^j$ be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$, with $j \geq 1$. Assume that $\Sigma \cap W = \emptyset$ and that the map $(\pi_W)|_\Sigma : \Sigma \to W$ has constant rank $i < j$. In the case $c > 0$ assume further that $\Sigma \cap V_W = \emptyset$. Then Property* (A) *holds.*

Given a map $g : \Sigma \to \Omega$ of class C^1, it is well known that there exists an open dense subset $\Omega = \cup_\lambda U_\lambda$ of Σ, where each U_λ is an open subset such that $f|_{U_\lambda}$ has constant rank (see for example [L]). Thus Theorems 3 and 4 imply together the following result, where we do not need to assume that π_W is a submersion.

Corollary 4. *Let Σ be an embedded connected submanifold of class C^k of $\mathbb{Q}_c^n$, with $k \geq 1$. Let $W = W^j$ be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$, with $j \geq 1$. Assume that $\Sigma \cap W = \emptyset$. In the case $c > 0$ assume further that $\Sigma \cap V_W = \emptyset$. Then there exists an open dense subset of Σ such that* (B) *implies* (A) *if $c \geq 0$, and* (C) *implies* (A) *if $c < 0$.*

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2. Distance function from subsets

Proof of Theorem 2. By the connectedness of Σ, it suffices to show that $G \circ f$ is locally constant. Let V be a neighborhood of a point x_0 in Σ such that the restriction $f|_V : V \to M$ is an embedding and denote by $\Sigma' = f(V)$. Fix distinct points $p, q \in \Sigma'$ and consider a differentiable curve $\alpha : [a, b] \to \Sigma'$ with $\alpha(a) = p$ and $\alpha(b) = q$ parameterized by arc length. Let $\rho : [a, b] \to \mathbb{R}$ be given by $\rho(s) = G(\alpha(s))$. By using that G is a Lipschitz function we have that

$$|\rho(s) - \rho(t)| = |G(\alpha(s)) - G(\alpha(t))| \leq C d(\alpha(s), \alpha(t)) \leq C L\left(\alpha|_{[s,t]}\right) = C|s - t|.$$

Thus, since ρ is a Lipschitz function, it must be differentiable almost everywhere and satisfy the equality $\rho(b) = \rho(a) + \int_a^b \rho'(s)ds$. We fix $s_0 \in (a, b)$ such that $\rho'(s_0)$ exists.

Claim 2.1. $\rho'(s_0) = 0$.

In fact, by hypothesis, there exists a nontrivial geodesic $\gamma : [0, 1] \to M$ satisfying

(i) $\gamma(0) = \alpha(s_0)$;
(ii) $\gamma'(0)$ is orthogonal to $\alpha'(s_0)$;
(iii) $CL(\gamma) = |G(\alpha(s_0)) - G(\gamma(1))| = |\rho(s_0) - G(\gamma(1))|$.

Since $L(\gamma) > 0$ it follows that $G(\alpha(s_0)) - G(\gamma(1)) \neq 0$. By replacing G by $-G$ if necessary, we may assume that $G(\alpha(s_0)) - G(\gamma(1)) > 0$. Now we choose $0 < t_0 < 1$ sufficiently small so that $\alpha(s_0)$ is contained in a strongly convex open ball $B \subset M$ centered at $\gamma(t_0)$. Choose $0 < \delta < \epsilon$ sufficiently small so that $I = (s_0 - \delta, s_0 + \delta) \subset (a, b)$, $\alpha([s_0 - \delta, s_0 + \delta]) \subset B$ and $G(\alpha(s)) - G(\gamma(1)) > 0$ for all $s \in I$. Consider the smooth map $r : B \to \mathbb{R}$ given by $r(x) = d(\gamma(t_0), x)$ and the map $h : I \times [0, 1] \to M$ given by

(a) $h(s, t) = \exp_{\alpha(s)}\left(\frac{t}{t_0}\left(\exp_{\alpha(s)}^{-1} \gamma(t_0)\right)\right)$, for $s \in I$ and $t \in [0, t_0]$;
(b) $h(s, t) = \gamma(t)$, for $s \in I$ and $t \in [t_0, 1]$.

Consider the curve $h_s : t \in [0, 1] \mapsto h(s, t)$. Note that $L(h_s) = L(\gamma|_{[t_0,1]}) + r(\alpha(s))$. Since α is differentiable we obtain that

$$\begin{aligned} \frac{d}{ds} L(h_s)\Big|_{s=s_0} &= (r \circ \alpha)'(s_0) = \left\langle \nabla r(\alpha(s_0)), \alpha'(s_0)\right\rangle \\ &= \left\langle -\gamma'(0), \alpha'(s_0)\right\rangle = 0. \end{aligned} \tag{3}$$

Since $h_s(0) = \alpha(s)$ and $h_s(1) = \gamma(1)$ we have that

$$\begin{aligned} C\,L(h_s) &\geq C d(\alpha(s), \gamma(1)) \geq |G(\alpha(s)) - G(\gamma(1))| = G(\alpha(s)) - G(\gamma(1)) \\ &= \rho(s) - G(\gamma(1)), \end{aligned} \tag{4}$$

for all $s \in I$. Thus, using (iii), (3) and (4), we obtain that

$$\begin{aligned} \rho'(s_0) &= \lim_{\substack{s \to s_0 \\ s > s_0}} \frac{\rho(s) - \rho(s_0)}{s - s_0} \leq \lim_{\substack{s \to s_0 \\ s > s_0}} \frac{C\,L(h_s) + G(\gamma(1)) - \rho(s_0))}{s - s_0} \\ &= \lim_{\substack{s \to s_0 \\ s > s_0}} \frac{C\,L(h_s) - C\,L(h_{s_0})}{s - s_0} = C \left.\frac{d}{ds}\right|_{s=s_0} L(h_s) = 0 \end{aligned}$$

and

$$\begin{aligned} \rho'(s_0) &= \lim_{\substack{s \to s_0 \\ s < s_0}} \frac{\rho(s) - \rho(s_0)}{s - s_0} \geq \lim_{\substack{s \to s_0 \\ s < s_0}} \frac{C\,L(h_s) + G(\gamma(1)) - \rho(s_0))}{s - s_0} \\ &= \lim_{\substack{s \to s_0 \\ s < s_0}} \frac{C\,L(h_s) - C\,L(h_{s_0})}{s - s_0} = C \left.\frac{d}{ds}\right|_{s=s_0} L(h_s) = 0. \end{aligned}$$

Thus it holds that $\rho'(s_0) = 0$ and Claim 2.1 is proved.

Thus, using that $\rho(b) = \rho(a) + \int_a^b \rho'(s)ds = \rho(a)$, we obtain that $G|_{\Sigma'}$ is constant. Since Σ is connected we have that the function $G \circ f$ is constant. This implies that $f(\Sigma)$ is contained in a level set $G^{-1}(\{d\})$ for some $d \in \mathbb{R}$.

We have that $|G(\gamma(1)) - G(\gamma(0))| \leq C\,d(\gamma(1), \gamma(0)) \leq C\,L(\gamma)$. Thus (1) implies that $d(\gamma(1), \gamma(0)) = L(\gamma)$, hence γ is a minimal geodesic.

Now we consider some curve $\beta : (-\epsilon, \epsilon) \to G^{-1}(\{d\})$, for some positive $\epsilon > 0$, such that β is differentiable at $s = 0$ and $\beta(0) = \gamma(0)$. We claim that $\gamma'(0)$ is orthogonal to $\beta'(0)$. If this is not true, by changing the orientation of β, if necessary, we may assume that $\langle \gamma'(0), \beta'(0) \rangle > 0$. We define the smooth function r and the map $h_s(t) = h(s,t)$ as above, with β replacing α. By using the smoothness of r and the differentiability of β at $s = 0$ as in (3) we obtain that $\frac{d}{ds} L(h_s)|_{s=0} = \langle -\gamma'(0), \beta'(0) \rangle < 0$. Then we have for small $0 < s < \epsilon$ that $d(\gamma(1), \beta(s)) \leq L(h_s) < L(h_0) = d(\gamma(1), \gamma(0)) = L(\gamma)$. Hence we arrive to

$$|G(\gamma(1)) - G(\gamma(0))| = |G(\gamma(1)) - G(\beta(s))| \leq C\,d(\gamma(1), \beta(s)) < C\,L(\gamma),$$

and this contradicts (1). Theorem 2 is proved. □

Proof of Corollary 2. Let $\mathcal{A} \subset M$ be an arbitrary subset and consider a differentiable immersion $f : \Sigma \to M$ of a connected manifold Σ satisfying that for all $p \in \Sigma$ and $v \in T_p\Sigma$ there exists a vector $\eta \in T_{f(p)}M$ orthogonal to $df_p v$ such that the geodesic $\gamma_\eta : [0,1] \to M$ satisfies that $\gamma_\eta(1) \in \mathcal{A}$ and γ_η satisfies (2). Consider the Lipschitz function $G : M \to \mathbb{R}$ given by $G(x) = d(x, \mathcal{A})$ with Lipschitz constant $C = 1$. Set $A = (G \circ f)^{-1}((0, +\infty))$ and $F = (G \circ f)^{-1}(\{0\})$. The function $G \circ f$ is constant on F and, by using Theorem 2, it is constant on each connected component of the open subset A. We conclude that the image of $G \circ f$ is countable. On the other hand, the connectedness of Σ implies that the image of $G \circ f$ is an interval, hence it must be a point. As a consequence $f(\Sigma)$ is contained in $\mathcal{S}(\mathcal{A}, r_0)$ for some $r_0 \geq 0$.

Now consider a differentiable immersion $f : \Sigma \to M$ of a connected manifold Σ such that $f(\Sigma)$ is contained in $\mathcal{S}(\mathcal{A}, r_0)$ for some $r_0 \geq 0$, where $\mathcal{A}$ is a closed subset of M. If $r_0 = 0$, we just take $\eta = 0$ for any $p \in \Sigma$ and any $v \in T_p\Sigma$, and we will have trivially that (2) holds. Thus we will assume that $r_0 > 0$. Fix $p \in \Sigma$. Since $\mathcal{A}$ is closed, there exists a point $q \in \mathcal{A}$ such that $d(f(p), q) = d(f(p), \mathcal{A}) = r$. Then there exists a nontrivial minimal geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = p, \gamma(1) = q$ and $L(\gamma) = d(p, \mathcal{A}) = r_0$. We claim that $\eta = \gamma'(0)$ is orthogonal to $df_p(T_p\Sigma)$. To see this we take a vector $v \in T_p\Sigma$ and consider a differentiable curve $\beta : (-\epsilon, \epsilon) \to f(\Sigma)$ such that $\beta(0) = f(p)$ and $\beta'(0) = df_p v$. Since γ_η satisfies (1), we have by Theorem 2 that $\langle \eta, \beta'(0) \rangle = 0$, hence η is orthogonal to $df_p(T_p\Sigma)$. Corollary 2 is proved. $\square$

Proof of Corollary 3. To prove Corollary 3, we first prove the following

Claim 2.2. (C) *implies* (A).

Assume that Property (C) holds. Thus for any $q \in \Sigma$ and any $v \in T_q\Sigma$, there exists a nontrivial vector η orthogonal to v such that the image of the geodesic γ_η contains W. By replacing η by $-\eta$, if necessary in the case $c > 0$, we may assume that there exists $d > 0$ such that $\gamma_\eta|_{[0,d]}$ is minimizing and $\gamma_\eta(d) = W$. Thus we may apply Corollary 2 obtaining that Σ is contained in a sphere M centered at W. This sphere is a smooth hypersurface which is invariant under G_W, hence Property (A) holds.

Claim 2.3. (D) *implies* (B).

Assume that Property (D) holds. Fix $q \in \Sigma$. Then there exists a neighborhood U of q and a differentiable hypersurface M of $\mathbb{Q}_c^n$ containing U such that M is invariant under the action of G_W. Set $r_0 = d(q, W)$. We claim that $\mathcal{S}(W, r_0) \subset M$. Indeed, fix $z \in \mathcal{S}(W, r_0)$. Since G_W is transitive on $\mathcal{S}(W, r_0)$, there exists $\phi \in G_W$ such that $\phi(q) = z$. Since M is invariant under the action of G_W, we have that $z \in M$, hence $\mathcal{S}(W, r_0) \subset M$. Since $\mathcal{S}(W, r_0)$ is a closed manifold contained in the manifold M of the same dimension, we conclude that $\mathcal{S}(W, r_0) = M$. Fix a minimal geodesic $\gamma : [0, 1] \to \mathbb{Q}_c^n$ from q to W. It holds that $\gamma'(0)$ is orthogonal to $T_q\big(\mathcal{S}(W, r_0)\big) = T_qM$, hence $\gamma'(0)$ is orthogonal to $T_q\Sigma$. Thus Property (B) holds.

Since trivially we have that (A) implies (D), and (B) implies (C), we conclude from the above claims that (A), (B), (C) and (D) are equivalent if $j = 0$. The proof of Claim 2.2 implies that if one of the four equivalent properties occurs then M is a sphere centered at W which contains Σ. $\square$

Proof of Theorem 1 and Corollary 1. Let M be a complete and noncompact manifold. For any $z \in M$ it is well known that there exists a ray $\alpha : [0, +\infty) \to M$ starting at z, which by definition satisfies that

$L\left(\alpha|_{[0,t]}\right) = d\big(z, \alpha(t)\big)$ for any $t \geq 0$. Fix a point $q \in M$. A unit speed ray $\gamma : [0, +\infty) \to M$ starting at q is said to be asymptotic to α if there exists a sequence $t_k \to +\infty$ and a sequence of unit speed minimal geodesics $\gamma_k : [0, d_k] \to M$ from q to $\alpha(t_k)$ such that $\gamma_k'(0) \to \gamma'(0)$. It is well known and easy to see that for any $q \in M$ there exists a ray γ starting at q which is asymptotic to α.

We recall that a horosphere of M associated with α is a level set of the Busemann function $h_\alpha : M \to \mathbb{R}$ given by

$$h_\alpha(x) = \lim_{t \to +\infty} d\big(x, \alpha(t)\big) - t.$$

It is well known that h_α is a Lipschitz function with Lipschitz constant 1. If γ is asymptotic to α it is also well known (see for example Proposition 41 in [P]) that

$$h_\alpha\big(\gamma(0)\big) - h_\alpha\big(\gamma(t)\big) = t, \text{ for any } \ t \geq 0. \tag{5}$$

From now on we will assume that one of two possibilities holds: or M is a Hadamard manifold, or M has sectional curvature $K \geq 0$. In both cases there exists a standard compactification $\tilde{M} = M \cup M(\infty)$. As usual we denote by $\gamma(\infty)$ the point in $M(\infty)$ associated to a ray γ. If M is a Hadamard manifold we have that $\gamma(\infty) = \alpha(\infty)$ if and only if γ is asymptotic to α (see, for example, [BGS]). In the case $K \geq 0$ the fact that the ray γ is asymptotic to α implies that $\gamma(\infty) = \alpha(\infty)$ (see for example Lemma 2.2 in [M]). We observe that the reciprocal is not true in this case. Indeed, there exist examples of complete noncompact manifolds with $K \geq 0$ with some rays γ, α satisfying $\gamma(\infty) = \alpha(\infty)$ and γ not being asymptotic to α.

Let $f : \Sigma \to M$ be a differentiable immersion and assume that there exists a unit speed ray σ in M such that for any point $p \in \Sigma$ and $v \in T_p\Sigma$ there exists a unit vector $\eta \in T_{f(p)}M$ orthogonal to $df_p v$ such that $\gamma_\eta : [0, +\infty) \to M$ is a ray asymptotic to σ.

Now fix $p \in \Sigma$ and $v \in T_p\Sigma$. By hypothesis there exists a unit vector $\eta \in T_{f(p)}M$ orthogonal to $df_p v$ satisfying that γ_η is a ray asymptotic to σ. Thus we have by (5) that

$$h_\sigma\big(\gamma_\eta(0)\big) - h_\sigma\big(\gamma_\eta(1)\big) = 1 = L\left(\gamma_\eta|_{[0,1]}\right).$$

Thus we can apply Theorem 2 with $G = h_\sigma$ to obtain that $G \circ f$ is constant. Theorem 1 and Corollary 1 are proved. □

3. Some basic facts about space forms

The purpose of this section is to recall some basic facts about the geometry of $\mathbb{Q}_c^n$, which will be needed in the proof of Theorem 3.

Lemma 3.1. Let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in $\mathbb{Q}_c^n$ where each $\gamma_i : [0, a_i] \to \mathbb{Q}_c^n$ is a minimal unit speed geodesic with $\gamma_i(a_i) = \gamma_{i+1}(0)$ for $i = 1, 2$ and $\gamma_3(a_3) = \gamma_1(0)$. Then there exists a totally geodesic surface $N^2 \subset \mathbb{Q}_c^n$ which is isometric to $\mathbb{Q}_c^2$ and contains the images of γ_1, γ_2 and γ_3.

Proof. There exists a plane $P_{12} \subset T_{\gamma_1(a_1)}(\mathbb{Q}_c^n)$ containing $\gamma_1'(a_1)$ and $\gamma_2'(0)$. Since $\mathbb{Q}_c^n$ has constant sectional curvature it holds that $N_{12} = \exp_{\gamma_1(a_1)}\big(P_{12}\big)$ is a totally geodesic surface of $\mathbb{Q}_c^n$ which clearly contains the images of γ_1 and γ_2. Similarly there exists a totally geodesic complete surface N_{13} containing the images of γ_1 and γ_3. If either $c > 0$ and $L(\gamma_3) < \frac{\pi}{\sqrt{c}}$, or if $c \leq 0$, there exists a unique unit speed minimal geodesic $\tau : [0, a_3] \to \mathbb{Q}_c^n$ from $\gamma_3(0)$ to $\gamma_3(a_3)$. The fact that τ and γ_3 are unit speed minimal geodesics implies that $\tau = \gamma_3$ and its image is contained in N_{12}. It remains to consider the case that $c > 0$ and $L(\gamma_3) = \frac{\pi}{\sqrt{c}}$. If this occurs we have that $\gamma_1(0)$ and $\gamma_2(a_2)$ are antipodal points, hence we have by triangle inequality that the images of γ_1 and γ_2 are contained in the same geodesic circle, hence the images of γ_1, γ_2 and γ_3 are contained in N_{13}. Lemma 3.1 is proved. □

Lemma 3.2. For $c > 0$, let $\gamma : [0, a] \to \mathbb{Q}_c^n$ be a unit speed geodesic, with $0 < a \leq \frac{\pi}{\sqrt{c}}$. Let $\alpha : [0, t_0] \to \mathbb{Q}_c^n$ and $\beta : [0, s_0] \to \mathbb{Q}_c^n$ be unit speed geodesics satisfying that:

(i) $\alpha(0) = \gamma(0)$ and $\beta(0) = \gamma(a)$;
(ii) $\langle \alpha'(0), \gamma'(0)\rangle = \langle \beta'(0), \gamma'(a)\rangle = 0$;
(iii) $\alpha(t_0) = \beta(s_0)$ and $0 \leq t_0 \leq s_0 \leq \frac{\pi}{\sqrt{c}}$.

Then $s_0 \geq \frac{\pi}{2\sqrt{c}}$ and $\beta'(0)$ is the parallel transport of $\alpha'(0)$ along γ.

Proof. Since $L(\gamma), L(\alpha), L(\beta) \leq \frac{\pi}{\sqrt{c}}$, we have that γ, α and β are minimal geodesics. Thus by Lemma 3.1 there exists a totally geodesic surface $N^2 \subset \mathbb{Q}_c^n$ which is isometric to $\mathbb{Q}_c^2$ containing the images of γ, α and β. By using an isometry we may assume that N is a sphere centered at the origin in $\mathbb{R}^3$. Consider the map $\varphi : \mathbb{R} \times \mathbb{R} \to N$ given by:

$$\varphi(u, v) = \frac{1}{\sqrt{c}}\Big(\cos\big(u\sqrt{c}\big)\cos\big(v\sqrt{c}\big),\, \sin\big(u\sqrt{c}\big)\cos\big(v\sqrt{c}\big),\, \sin\big(v\sqrt{c}\big)\Big). \tag{6}$$

By using again a convenient isometry we may assume that $\gamma(u) = \varphi(u, 0)$ for $0 \leq u \leq a$ and $\alpha(t) = \varphi(0, t)$, for $0 \leq t \leq t_0$. Consider $\tilde{\beta}(s) = \varphi\big(a, s\big)$, for $0 \leq s \leq \frac{\pi}{\sqrt{c}}$.

Claim 3.1. $\tilde{\beta}|_{[0,s_0]} = \beta$.

Indeed, if Claim 3.1 is not true, we obtain from assumption (ii) in Lemma 3.2 that $\beta(s) = \tilde{\beta}(-s)$. But this, together with (6), implies that the third coordinate of $\beta(s)$ is negative if $0 < s < \frac{\pi}{\sqrt{c}}$. Since the third coordinate of $\alpha(t)$ is positive if $0 < t < \frac{\pi}{\sqrt{c}}$, we obtain from assumption (iii) that $t_0, s_0 \in \Big\{0, \frac{\pi}{\sqrt{c}}\Big\}$. Since $\alpha(0) = \gamma(0) \neq \gamma(a) = \beta(0)$, we obtain that $t_0 = s_0 = \frac{\pi}{\sqrt{c}}$. Thus we have that $-\alpha(0) = \alpha\left(\frac{\pi}{\sqrt{c}}\right) = \beta\left(\frac{\pi}{\sqrt{c}}\right) = -\beta(0)$, hence $\gamma(0) = \alpha(0) = \beta(0) = \gamma(a)$, which contradicts the fact that $0 < a \leq \frac{\pi}{\sqrt{c}}$.

From Claim 3.1 we obtain that $\beta'(0)$ is the parallel transport of $\alpha'(0)$ along γ.

By using Claim 3.1 and (6), it is straightforward to obtain that either $a = \frac{\pi}{\sqrt{c}} = s_0 + t_0$, or $0 < a < \frac{\pi}{\sqrt{c}}$ and $s_0 = t_0 = \frac{\pi}{2\sqrt{c}}$. In both cases we have that $s_0 \geq \frac{\pi}{2\sqrt{c}}$. Lemma 3.2 is proved. $\square$

Lemma 3.3. For $c > 0$, let $W = W^j \subset \mathbb{Q}^n_c$, with $j \geq 1$, be a compact connected totally geodesic submanifold and $B_W = \left\{(x, v) \in \nu(W) \mid |v| < \frac{\pi}{2\sqrt{c}}\right\}$. Then it holds that:

(i) V_W is a totally geodesic $(n-j-1)$-sphere if $1 \leq j \leq n-2$ and consists of two antipodal points if $j = n-1$;
(ii) $\mathbb{Q}^n_c = \bar{B}\left(W, \frac{\pi}{2\sqrt{c}}\right)$;
(iii) $\exp^\perp|_{B_W} : B_W \to (\mathbb{Q}^n_c - V_W)$ is a diffeomorphism;
(iv) $\exp^\perp(\partial B_W) = V_W$.

Proof. We may view $\mathbb{Q}^n_c$ as a sphere centered at the origin in $\mathbb{R}^{n+1}$. The totally geodesic sphere W spans a $(j+1)$-dimensional linear subspace $E = \operatorname{span}(W) \subset \mathbb{R}^{n+1}$. If $E^\perp$ denotes the orthogonal complement of E in $\mathbb{R}^{n+1}$, then $V_W = E^\perp \cap \mathbb{Q}^n_c$, which easily implies Item (i) in Lemma 3.3.

To prove Item (ii), we consider a point $q \in \mathbb{Q}^n_c - W$. Take a unit speed minimal geodesic $\gamma : [0, d] \to \mathbb{Q}^n_c$ satisfying that $\gamma(0) = q, \gamma(d) \in W$ and $d = L(\gamma) = d(q, W)$. Consider the geodesic $\sigma : [0, d_0] \to \mathbb{Q}^n_c$ such that $\sigma(0) = q$, $\sigma'(0) = -\gamma'(0)$ and $d + d_0 = \frac{\pi}{\sqrt{c}}$. Since $\sigma(d_0)$ is the antipodal point of $\gamma(d)$, we have that $\sigma(d_0) \in W$, hence $d_0 = L(\sigma) \geq d$, which implies that $d \leq \frac{\pi}{2\sqrt{c}}$ and proves Item (ii).

Claim 3.2. $\exp^\perp|_{B_W}$ *is injective.*

We fix $(p_1, v_1), (p_2, v_2) \in B_W$ and consider the unit speed geodesics $\sigma_i : [0, |v_i|] \to \mathbb{Q}^n_c$ given by $\sigma_i(t) = \exp^\perp\left(p_i, \frac{t}{|v_i|} v_i\right)$, for $i = 1, 2$. Since $L(\sigma_i) = |v_i| \leq \frac{\pi}{\sqrt{c}}$ for $i = 1, 2$, each σ_i is a minimal geodesic. There exists a unit speed minimal geodesic $\gamma : [0, a] \to W$ from p_1 to p_2. Note that v_1 and v_2 are orthogonal to γ. Since $|v_i| < \frac{\pi}{2\sqrt{c}}$ for $i = 1, 2$, Lemma 3.2 implies that $\sigma_1(|v_1|) \neq \sigma_2(|v_2|)$, hence $\exp^\perp(p_1, v_1) \neq \exp^\perp(p_2, v_2)$. Thus $\exp^\perp$ is injective on B_W.

Claim 3.3. $\exp^\perp|_{B_W}$ *is a diffeomorphism onto its open image.*

Indeed, fix $(p, v) \in \nu(W)$ with $v \neq 0$ at which $d(\exp^\perp)_{(p,v)}$ is singular. Consider the unit speed geodesic $\sigma : [0, |v|] \to \mathbb{Q}^n_c$ given by $\sigma(t) = \exp^\perp\left(p, \frac{t}{|v|} v\right)$. We know that there exists a nontrivial Jacobi field J along σ such that $J(0) \in T_pW$, $J'(0)$ is orthogonal to T_pW and $J(|v|) = 0$ (see for example [dC]). These conditions imply that $J(t)$ is orthogonal to $\sigma'(t)$ for all $0 \leq t \leq |v|$. Let P be the parallel transport of $J(0)$ along σ. Thus we have that $J(t) = \cos(t\sqrt{c})P(t)$, hence $|v| \geq \frac{\pi}{2\sqrt{c}}$. As a consequence the map $\exp^\perp$ has no singular points in B_W, hence $\exp^\perp|_{B_W}$ is a diffeomorphism onto its open image.

Claim 3.4. $\exp^\perp|_{B_W}$ *is a diffeomorphism onto* $B\left(W, \frac{\pi}{2\sqrt{c}}\right)$.

In fact, fix $q \in B\left(W, \frac{\pi}{2\sqrt{c}}\right)$. Then there exists a unit speed minimal geodesic $\gamma : [0,d] \to \mathbb{Q}^n_c$ from q to W with $d = L(\gamma) = d(q,W)$. By the first variation formula we have that $\gamma'(d)$ is orthogonal to $T_{\gamma(d)}W$, hence $q = \exp^\perp(\gamma(d), -d\gamma'(d))$ with $|-d\gamma'(d)| = d < \frac{\pi}{2\sqrt{c}}$. Thus $B\left(W, \frac{\pi}{2\sqrt{c}}\right) \subset \exp^\perp(B_W)$. Reciprocally, let us consider a point $(p,v) \in B_W$. The geodesic $\sigma(t) = \exp^\perp\left(p, \frac{t}{|v|}v\right)$ for $0 \le t \le |v|$ has length $L(\sigma) = |v| < \frac{\pi}{2\sqrt{c}}$, hence $d(\sigma(|v|), W) < \frac{\pi}{2\sqrt{c}}$. As a consequence we obtain that $\exp^\perp(B_W) = B\left(W, \frac{\pi}{2\sqrt{c}}\right)$, hence $\exp^\perp : B_W \to B\left(W, \frac{\pi}{2\sqrt{c}}\right)$ is a diffeomorphism.

From Item (ii) in Lemma 3.3 we have that

$$\mathbb{Q}^n_c - V_W = B\left(W, \frac{\pi}{2\sqrt{c}}\right), \tag{7}$$

and thus Item (iii) follows from Claim 3.4.

To prove Item (iv), we first take $(p,v) \in \partial B_W$. Then $|v| = \frac{\pi}{2\sqrt{c}}$. Consider a sequence $0 < t_n < 1$ with $t_n \to 1$. We have that $d\left(\exp^\perp(p, t_n v), W\right) = |t_n v| \to |v| = \frac{\pi}{2\sqrt{c}}$. Thus $d\left(\exp^\perp(p,v), W\right) = \frac{\pi}{2\sqrt{c}}$, hence $\exp^\perp(p,v) \in V_W$. Reciprocally, take $q \in V_W$. Then there exists a unit speed shortest geodesic $\gamma : [0,d] \to \mathbb{Q}^n_c$ with $L(\gamma) = d = \frac{\pi}{2\sqrt{c}}$. Then $q = \exp^\perp\left(\gamma(d), -\frac{\pi}{2\sqrt{c}}\gamma'(d)\right) \in \exp^\perp(\partial B_W)$. Lemma 3.3 is proved. $\square$

Given $p \in \mathbb{Q}^n_c$ and a complete connected totally geodesic submanifold $W = W^j$, $j \ge 1$, set $S_{pW} = \exp_p\left((T_pW)^\perp\right)$. It is well known that S_{pW} is a complete totally geodesic submanifold. If $c > 0$ we have that S_{pW} is a round sphere with dimension $n - j$.

Lemma 3.4. Consider a complete connected totally geodesic submanifold $W = W^j$, with $j \ge 1$. Then the map $\pi_{_W}$ is a submersion on its domain. Furthermore for $p \in W$ it holds that $\pi_{_W}^{-1}(\{p\}) = S_{pW}$ in the case $c \le 0$, and $\pi_{_W}^{-1}(\{p\}) = S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right)$ in the case $c > 0$. In particular for $q \in \pi_{_W}^{-1}(\{p\})$ the kernel $\operatorname{Ker}\big((d\pi_{_W})_q\big) = T_q(S_{pW})$.

Proof. We have that $\exp^\perp : \nu(W) \to \mathbb{Q}^n_c$ is a diffeomorphism if $c \le 0$, and Lemma 3.3 implies that $\exp^\perp : B_W \to \mathbb{Q}^n_c - V_W$ is a diffeomorphism if $c > 0$. If $D(\pi_{_W})$ denotes the domain of $\pi_{_W}$, we have that $\pi_{_W}\left(\exp^\perp(p,v)\right) = p$ if $\exp^\perp(p,v) \in D(\pi_{_W})$. Thus we conclude that $\pi_{_W}$ is a submersion on $D(\pi_{_W})$.

Fix $p \in W$ and $q \in S_{pW}$. There exist unit speed minimal geodesics $\gamma : [0,d] \to \mathbb{Q}^n_c$ and $\sigma : [0,e] \to \mathbb{Q}^n_c$ satisfying: $\gamma(0) = q$, $\gamma(d) = p$, $d = L(\gamma) = d(q,p)$, $\sigma(0) = q$, $\sigma(e) \in W$ and $e = L(\sigma) = d(q,W)$. Clearly we have that $e \le d$.

Claim 3.5. *If* $c \le 0$ *then* $S_{pW} \subset \pi_{_W}^{-1}(\{p\})$.

Since $c \leq 0$, the set S_{pW} is convex, hence the image of γ is contained in S_{pW}. As a consequence we have that $\gamma'(d)$ is orthogonal to T_pW. We also know that $\sigma'(e)$ is orthogonal to W, by the first variation formula. Thus the fact that $\exp^\perp : \nu(W) \to \mathbb{Q}^n_c$ is a diffeomorphism implies that $d = e$ and $\gamma = \sigma$. In particular we have that $\pi_W(q) = \sigma(e) = \gamma(d) = p$, hence $S_{pW} \subset \pi_W^{-1}(\{p\})$.

Claim 3.6. *If $c > 0$ then $S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right) \subset \pi_W^{-1}(\{p\})$.*

Assume that $q \in S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right)$. Since $S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right)$ is strongly convex, the image of γ is contained in S_{pW}, hence $\gamma'(d)$ is orthogonal to T_pW. Since $e \leq d < \frac{\pi}{2\sqrt{c}}$, we obtain from (7) that $q \in B\left(W, \frac{\pi}{2\sqrt{c}}\right) = \mathbb{Q}^n_c - V_W$. Since $\gamma'(d)$ is orthogonal to T_pW, we have that

$$q = \exp^\perp\big(p, -d\gamma'(d)\big) = \exp^\perp\big(\sigma(e), -e\sigma'(e)\big). \tag{8}$$

Since $\exp^\perp|_{B_W} : B_W \to \mathbb{Q}^n_c - V_W$ is a diffeomorphism we conclude from (8) that $\pi_W(q) = \sigma(e) = p$, hence $S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right) \subset \pi_W^{-1}(\{p\})$.

Claim 3.7. *If $c \leq 0$ then $\pi_W^{-1}(\{p\}) \subset S_{pW}$. If $c > 0$ then $\pi_W^{-1}(\{p\}) \subset S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right)$.*

For $c \in \mathbb{R}$, assume that $z \in \pi_W^{-1}(\{p\})$. Then there exists a unit speed minimal geodesic $\tau : [0, u] \to \mathbb{Q}^n_c$ such that $\tau(0) = z$, $\tau(u) = p$ and $u = L(\tau) = d(z, W) = d(z, p)$. Since $\tau'(u)$ is orthogonal to T_pW then $\tau'(d) \in T_p(S_{pW})$. Thus the fact that S_{pW} is totally geodesic implies that the image of τ must be contained in S_{pW}, hence $z \in S_{pW}$. We conclude that $\pi_W^{-1}(\{p\}) \subset S_{pW}$. Now assume that $c > 0$. Since $z \in \pi_W^{-1}(\{p\}) \subset D(\pi_W)$ and $c > 0$ we have by (7) that $d(z, p) = d(z, W) < \frac{\pi}{2\sqrt{c}}$. Thus we obtain that $\pi_W^{-1}(\{p\}) \subset S_{pW} \cap B\left(p, \frac{\pi}{2\sqrt{c}}\right)$.

Finally, since π_W is a submersion, we have that for any point $z \in D(\pi_W)$ it holds that $\mathrm{Ker}(d(\pi_W)_z) = T_z\left(\pi_W^{-1}(\pi_W(z))\right)$. Thus we have that, for $q \in \pi_W^{-1}(\{p\})$, it holds that $\mathrm{Ker}\big((d\pi_W)_q\big) = T_q(S_{pW})$. Lemma 3.4 is proved. □

Before stating the next lemma, we recall that, if $c > 0$, the group $O(n+1)$ of linear isometries of $\mathbb{R}^{n+1}$ is isomorphic to the group of isometries of $\mathbb{Q}^n_c$ by the restriction map, where we identify $\mathbb{Q}^n_c$ with a sphere centered at the origin. It is well known and easy to show by using orthonormal bases that $O(n+1)$ acts transitively on $\mathbb{Q}^n_c$.

Lemma 3.5. Consider a complete totally geodesic connected submanifold $W = W^j$ of $\mathbb{Q}^n_c$, with $j \geq 1$ and fix $p \in W$ and $r > 0$. Then G_W acts transitively on $S' = \mathcal{S}(p, r) \cap S_{pW}$.

Proof. Consider first the case $c = 0$. By using a convenient isometry we may assume that $W = \mathbb{R}^j \times \{O'\}$, where O' is the zero vector in $\mathbb{R}^{n-k}$. Given an isometry ϕ of $Q_0^n = \mathbb{R}^n$, we have that $\phi \in G_W$ if and only if $\phi|_W$ is the identity map on W and $\phi|_{S_{pW}}$ is an orthogonal transformation on the Euclidean space S_{pW}. In particular G_W acts transitively on S'.

Now assume that $c > 0$. Identify $\mathbb{Q}_c^n$ as the Euclidean sphere centered at the origin $O \in \mathbb{R}^{n+1}$ and radius $\frac{1}{\sqrt{c}}$. Set $E = \mathrm{span}(W) \subset \mathbb{R}^{n+1}$. Given $u \in \mathbb{Q}_c^n$ we will denote by Ou the vector in $\mathbb{R}^{n+1}$ from O to u. Fix $q, z \in S'$. By considering angles relatively to the origin, we have that $\measuredangle(Op, Oq) = \measuredangle(Op, Oz) = r\sqrt{c}$. Let $E^\perp$ be the orthogonal complement of E in $\mathbb{R}^{n+1}$. By using the orthogonal decomposition $\mathrm{span}(S_{pW}) = \mathrm{span}(\{Op\}) \oplus E^\perp$, we have that

$$Oq = \cos\left(r\sqrt{c}\right)\, Op + \sin\left(r\sqrt{c}\right)\, v, \quad Oz = \cos\left(r\sqrt{c}\right)\, Op + \sin\left(r\sqrt{c}\right)\, w, \tag{9}$$

for some $v, w \in E^\perp$ with $|v| = |w| = \frac{1}{\sqrt{c}}$. We obtain for $E^\perp$ two orthogonal bases $f_1 = v,\ f_2, \cdots, f_{n-j}$ and $g_1 = w,\ g_2, \cdots, g_{n-j}$, with $|f_i| = |g_i| = \frac{1}{\sqrt{c}}$, for $i = 1, \cdots, n-j$. Now consider the orthogonal transformation ϕ which is the identity map on E and such that $\phi(f_i) = g_i$, for $i = 1, \cdots, n-j$. From (9) we have that $\phi(Oq) = Oz$, hence G_W acts transitively on S'.

Finally we consider the case that $c < 0$. We will use the model $\mathbb{Q}_c^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. By using a convenient isometry on $\mathbb{Q}_c^n$ we may assume that W is a j-dimensional vertical Euclidean half-space containing the origin $O \in \mathbb{R}^n$ and that $p = (0, \cdots, 0, s_0) \in W$. If $n = 2$ then $j = 1$ and $S(p, r)$ consists of two points $z = (x, y)$ and $w = (-x, y)$, with $x \neq 0$ and $y > 0$. Then the reflection which fix the line W maps z to w. Thus, from now on we will assume that $n \geq 3$. Let S_0 be the Euclidean sphere centered at O and radius s_0. We will denote by L' the spherical length on S_0 and by $S'(x, s)$ the sphere centered at x of radius s, with respect to the round metric of S_0.

Claim 3.8. *$\mathcal{S}(p, r) \cap S_0 = S'(p, r')$ for some radius $0 < r' < \frac{\pi s_0}{2}$.*

We fix a point $z \in \mathcal{S}(p, r) \cap S_0$. Since $S_0 \cap \mathbb{Q}_c^n$ is a convex hypersurface, there exists a unique unit speed minimal geodesic $\gamma : [0, r] \to \mathbb{Q}_c^n$ from p to z and the image of γ is a geodesic arc in S_0. Set $r' = L'(\gamma)$. We will first prove that $\mathcal{S}(p, r) \cap S_0$ is contained in $S'(p, r')$. Indeed, fix a point $w \in \mathcal{S}(p, r) \cap S_0$. Consider a unit speed minimal geodesic $\sigma : [0, r] \to \mathbb{Q}_c^n$ from p to w and we know by convexity of S_0 that the image of σ is a geodesic arc in S_0. Since $\gamma'(0)$ and $\sigma'(0)$ are horizontal vectors at p, it is easy to obtain, by using orthogonal bases, that there exists an orthogonal transformation ϕ on $\mathbb{R}^n$ satisfying that $\phi\big(\mathbb{R}^{n-1} \times \{0\}\big) = \mathbb{R}^{n-1} \times \{0\}$ that maps the image of γ to a geodesic arc in S_0 that is tangent to $\sigma'(0)$ at p. Since $\phi|_{\mathbb{Q}_c^n}$ is an isometry of $\mathbb{Q}_c^n$, we conclude that $r = L(\gamma) = L(\phi \circ \gamma)$, hence $\phi \circ \gamma = \sigma$. Since ϕ is an orthogonal transformation, we have also that $L'(\gamma) = L'(\phi \circ \gamma) = L'(\sigma) = r'$. Since γ and σ start at p and do not intersect

$\mathbb{R}^{n-1} \times \{0\}$, we see that $0 < r' < \frac{\pi s_0}{2}$. In particular the images of γ and σ are minimal geodesic arcs in S_0, hence w belong to $S'(p,r')$. We proved that $\mathcal{S}(p,r) \subset S'(p,r')$. Since $n \geq 3$, we may use the fact that $\mathcal{S}(p,r) \cap S_0$ and $S'(p,r')$ are topological spheres of the same dimension $n-2$, hence this inclusion must be an equality. Claim 3.8 is proved.

Let ψ be either a horizontal translation, or a homothety or inversion with center in $\mathbb{R}^{n-1} \times \{0\}$. It is easy to see that $\psi|_{\mathbb{Q}^n_c}$ belongs to G_W if, and only if, ψ is the identity map. Thus a map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $\phi|_{\mathbb{Q}^n_c}$ belongs to G_W if, and only if, it is an orthogonal transformation which is the identity map on W (indeed, note that this condition implies that the line containing O and p has its points fixed by ϕ and then horizontal vectors are mapped to horizontal ones). Set $W_S = S_0 \cap \big(\mathrm{span}(W)\big)$. By using orthogonal bases we see that an orthogonal transformation ϕ satisfies that $\phi|_W$ is the identity map on W if, and only if, $\phi|_{W_S}$ is the identity map on W_S. Thus the restriction map induces a natural isomorphism from G_W to the subgroup $G_{(W_S)}$ of isometries on S_0 that fix any point in W_S. Note that $S_{pW} = S_{p(W_S)} \cap \mathbb{Q}^n_c$. Since $S'(p,r') \subset \mathbb{Q}^n_c$, we obtain from Claim 3.8 that

$$\begin{aligned} S' = \mathcal{S}(p,r) \cap S_{pW} &= \mathcal{S}(p,r) \cap S_0 \cap S_{pW} \\ &= S'(p,r') \cap S_{pW} = S'(p,r') \cap S_{p(W_S)} \cap \mathbb{Q}^n_c \\ &= S'(p,r') \cap S_{p(W_S)}. \end{aligned}$$

By the case $c' > 0$ applied to $S_0 = \mathbb{Q}^{n-1}_{c'}$ with $c' = \frac{1}{s_0^2}$, we conclude that G_{W_S} acts transitively on S', hence G_W acts transitively on S'. Lemma 3.5 is proved. $\square$

We observe that the convexity of S_{pW} implies that $S'(p,r_0)$ is the sphere centered at p and radius r_0 with respect to the Riemannian metric of S_{pW}, but we will not use this fact.

4. **Isometry actions and submanifolds**

The purpose of this section is to prove Theorems 3, 4 and Corollary 4. We first prove the following

Lemma 4.1. Let $\Sigma \subset \mathbb{Q}^n_c$ be a differentiable embedded connected submanifold with $c > 0$. Let $W = W^j$, with $j \geq 1$, be a closed connected totally geodesic submanifold of $\mathbb{Q}^n_c$ and fix a point $q \in \Sigma \cap (\mathbb{Q}^n_c - \{W \cup V_W\})$. Assume that the map $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion at q and that there exists a vector $\eta \in T_q(\mathbb{Q}^n_c)$ orthogonal to Σ such that the geodesic γ_η intersects W. Consider a shortest unit speed geodesic $\gamma : [0, r_0] \to \mathbb{Q}^n_c$ from q to W, namely, assume that $\gamma(0) = q$, $\gamma(r_0) = p = \pi_W(q) \in W$ and $L(\gamma) = r_0 = d(q,W)$. Then it holds that $\langle \eta, \gamma'(0) \rangle \neq 0$.

Proof. Consider the totally geodesic sphere $S = S_{pW}$ as in Lemma 3.4. Since $\gamma'(r_0) \in (T_pW)^\perp = T_pS$ it follows that the image of γ is contained in S, hence $q \in S$.

Since γ_η intersects W and $q = \gamma_\eta(0) \notin W$ we have easily that $\eta \neq 0$. Without loss of generality we will assume that $|\eta| = 1$.

Claim 4.1. *There exists* $0 < s_0 < \frac{\pi}{\sqrt{c}}$ *such that* $u = \gamma_\eta(s_0) \in W$.

Indeed, the intersection between the image of γ_η and W occurs in two antipodal points. Since $\gamma_\eta(0) \notin W$ then $\gamma_\eta\left(\frac{\pi}{\sqrt{c}}\right) \notin W$. We know that the image of γ_η is contained in a closed geodesic of length $\frac{2\pi}{\sqrt{c}}$. Then there exists two values $0 < s_0 < \frac{\pi}{\sqrt{c}}$ and $\frac{\pi}{\sqrt{c}} < s_1 = s_0 + \frac{\pi}{\sqrt{c}} < \frac{2\pi}{\sqrt{c}}$ such that $\gamma_\eta(s_0), \gamma_\eta(s_1) \in W$.

Now we assume by contradiction that $\left\langle \gamma_\eta'(0), \gamma'(0) \right\rangle = \langle \eta, \gamma'(0) \rangle = 0$. This fact and the inequalities $0 < r_0 < \frac{\pi}{2\sqrt{c}}$ and $0 < s_0 < \frac{\pi}{\sqrt{c}}$ imply together that $p \neq u$. Thus there exists a minimal unit speed geodesic $\mu : [0, t_0] \to W$ satisfying that $\mu(0) = p$ and $\mu(t_0) = u$. Since $\mu'(0) \in T_pW$ we have that $\mu'(0)$ is orthogonal to $\gamma'(r_0)$. Since $u = \gamma_\eta(s_0) = \mu(t_0)$ and μ and γ_η are minimal unit speed geodesics orthogonal to γ, we may apply Lemma 3.2 to conclude that $\mu'(0)$ is the parallel transport of η along γ.

We may write a direct sum $T_q\Sigma = (T_qS \cap T_q\Sigma) \oplus V$ and set $j = \dim(W)$ the dimension of W. Since $V \subset T_q\Sigma$ we have that $V \cap T_qS = V \cap (T_qS \cap T_q\Sigma) = \{0\}$. Since $(\pi_W)|_\Sigma$ is a submersion, we have that $T_pW = (d\pi_W)_q(T_q\Sigma) = (d\pi_W)_q(V)$ by Lemma 3.4, and hence we have that $\dim(V) \geq j$.

Let $P : T_q(\mathbb{Q}_c^n) \to T_p(\mathbb{Q}_c^n)$ be the parallel transport along γ. Since $V \subset T_q\Sigma$ we obtain that η is orthogonal to the linear space V. Since $P(\eta) = \mu'(0)$ we obtain that $\mu'(0)$ is orthogonal to the image $P(V)$. Since $\mu'(0) \in T_pW$ it must be orthogonal to T_pS. Thus we have that $\mu'(0)$ is orthogonal to $(P(V) + T_pS)$. Furthermore it holds that

$$P(V) \cap T_pS = P(V) \cap P(T_qS) = P(V \cap T_qS) = \{0\}.$$

We conclude that

$$\dim(P(V) + T_pS) = \dim(P(V)) + \dim(T_pS) \geq j + (n - j) = n,$$

hence $P(V) + T_pS = T_p(\mathbb{Q}_c^n)$ and $\mu'(0) = 0$. This contradicts the fact that $|\mu'(0)| = 1$. Lemma 4.1 is proved. □

Proof of Theorem 3. Let $W = W^j$, with $j \geq 1$, be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$ and Σ be an embedded connected differentiable submanifold of $\mathbb{Q}_c^n$ such that $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion and $\Sigma \cap W = \emptyset$. If $c > 0$ assume further that $\Sigma \cap V_W = \emptyset$.

First we will prove that (D) implies (B) in the case $c \geq 0$, and that (D) implies (C) for any c. For this we fix $q \in \Sigma$ and a small neighborhood $\mathcal{U}$ of q in Σ which is contained in an embedded differentiable hypersurface M in $\mathbb{Q}_c^n$ that is invariant under the action of G_W.

Let $\gamma : [0, r_0] \to \mathbb{Q}_c^n$ be the unit speed shortest geodesic from q to W, namely, assume that $\gamma(0) = q$, $\gamma(r_0) = p \in W$ and $L(\gamma) = r_0 = d(q, W)$, hence $p = \pi_W(q) = \gamma(r_0)$. Set $S = S_{pW}$ and $S' = S'(p, r_0) = \mathcal{S}(p, r_0) \cap S$.

Claim 4.2. $S' \subset M$

In fact, we know by Lemma 3.5 that G_W acts transitively on S'. Fix $\bar{q} \in S'$. Then there exists an isometry $\phi \in G_W$ such that $\phi(q) = \bar{q}$. Since $q \in M$ and M is invariant under the action of G_W, we obtain that $\bar{q} = \phi(q) \in M$, hence $S' \subset M$.

Claim 4.3. $\gamma'(0) \notin T_qM$.

In fact, since $(\pi_W)|_\Sigma$ is a submersion we obtain that

$$T_pW = d(\pi_W)_q(T_q(\Sigma)) \subset d(\pi_W)_q(T_qM),$$

hence $d(\pi_W)_q(T_qM) = T_pW$. Furthermore we have that

$$T_q(S') \subset (T_qS \cap T_qM) = \mathrm{Ker}(d\pi_W)_q \cap T_qM = \mathrm{Ker}(d(\pi_W|_M)_q). \tag{10}$$

We obtain that $\mathrm{dim(Ker(d(\pi_W|_M)_q)) = dim(M) - dim(W) = n - 1 - j}$, where $j = \dim(W)$. Since $\dim(S') = \dim(S) - 1 = n - j - 1$, we obtain from (10) that

$$\mathrm{Ker(d(\pi_W|_M)_q) = T_q(S')}. \tag{11}$$

Now assume by contradiction that $\gamma'(0) \in T_qM$. Since $\gamma'(0) \in \mathrm{Ker(d\pi_W)_q}$ and is orthogonal to T_qS' we have that $\mathrm{dim(Ker(d(\pi_W|_M)_q)) \geq 1 + dim(S')}$, which is a contradiction. Claim 4.3 is proved.

Claim 4.4. (D) *implies* (C).

In fact, take $v \in T_q\Sigma$ with $(d\pi_W)_q(v) = 0$. In particular we have that $v \in \mathrm{Ker(d(\pi_W|_M)_q)}$. By (11) we have that $v \in T_qS'$, hence v is orthogonal to $\gamma'(0)$. Since the geodesic γ intersects W at p, we conclude that (C) holds (by taking $\eta = \gamma'(0)$). Claim 4.4 is proved.

Let $P : T_q(\mathbb{Q}^n_c) \to T_p(\mathbb{Q}^n_c)$ be the parallel transport along γ. Take $V \subset T_q(\mathbb{Q}^n_c)$ such that $T_pW = P(V)$.

Claim 4.5. *$T_q(\mathbb{Q}^n_c)$ is an orthogonal direct sum $\mathbb{R}\gamma'(0) \oplus T_qS' \oplus V$.*

In fact, since S is totally geodesic and T_pW is orthogonal to T_pS it follows that $V = P^{-1}(T_pW)$ is orthogonal to $T_qS = \mathbb{R}\gamma'(0) + T_qS'$. And clearly we have that $\gamma'(0)$ is orthogonal to T_qS'.

Claim 4.6. (D) *implies* (B) *if $c \geq 0$.*

In fact, take a unit vector $\eta \in (T_qM)^\perp$. From Claim 4.5 we may write $\eta = a\gamma'(0) + u + \xi$, with $a \in \mathbb{R}$, $u \in T_qS'$ and $\xi \in V$. Since η is orthogonal to M and $T_qS' \subset T_qM$ we obtain that $u = 0$. If $a = 0$ then $\langle \eta, \gamma'(0) \rangle = 0$, hence $\gamma'(0) \in T_qM$ which contradicts Claim 4.3. Thus we obtain that $a \neq 0$. If η and $\gamma'(0)$ are linearly dependent, then (B) holds, since γ intersects W, hence we are done in this case. Thus from now on we may assume that $\xi \neq 0$.

We consider the unique totally geodesic surface N^2 of constant curvature c such that $T_q(N^2)$ agrees with the plane generated by $\gamma'(0)$ and ξ. In particular the images of γ and γ_η are contained in N^2. By construction we have that $w = P\xi \in T_pW$. Since N^2 is totally geodesic and $\xi \in T_q(N^2)$ it holds that $w = P\xi \in T_p(N^2)$, hence the image of the geodesic γ_w is contained in N^2. If $c > 0$, the images of γ_η and γ_w must intersect, since they are nontrivial geodesics of the 2-dimensional sphere N^2, which implies that (B) holds. If $c = 0$ and γ_η does not intersect γ_w then they are parallel to each other. Since γ_w is orthogonal to $\gamma'(r_0)$ we will have that η is orthogonal to $\gamma'(0)$ which contradicts the fact that $a \neq 0$. This contradiction concludes the proof of Claim 4.6.

From now on we will assume that Σ has the additional property that it is of class C^k, for some $k \geq 1$. Since (A) implies (D), and (B) implies (C), we have from Claims 4.4 and 4.6 that (A) implies (B) if $c \geq 0$, and (A) implies (C) for all values of c. Thus, to conclude the proof of Theorem 3 we just need to prove that Property (A) holds if one of the following conditions hold:

(I) $c > 0$ and Property (B) holds;
(II) $c \leq 0$ and Property (C) holds.

Thus we will assume that (I) or (II) holds and we will prove that each sufficiently small open subset of Σ is contained in a hypersurface invariant under the action of G_W.

Fix $q \in \Sigma$. Consider a unit speed shortest geodesic $\gamma : [0, r_0] \to \mathbb{Q}^n_c$ from q to W, namely, assume that $\gamma(0) = q$, $\gamma(r_0) = p = \pi_W(q) \in W$ and $L(\gamma) = d(q, W) = r_0$. Set $S = S_{pW}$. Since $\gamma'(r_0) \in T_pS$ it follows that the image of γ is contained in S, hence $q \in S$.

Fix $v \in T_q\Sigma$ with $(d\pi_W)_q v = 0$. By Lemma 3.4 we have that $v \in T_qS$.

Claim 4.7. $\langle v, \gamma'(0)\rangle = 0$.

In fact, by using (I) or (II), we may choose a vector $\eta \in T_q(\mathbb{Q}^n_c)$ such that the geodesic γ_η intersects W and one of the following properties holds:

(a) η is orthogonal to $T_q\Sigma$ and $c > 0$;
(b) η is orthogonal to v and $c \leq 0$.

Recall that $\eta \neq 0$ since $q \notin W$ and $\gamma_\eta(\mathbb{R})$ intersects W. Without loss of generality we will assume that $|\eta| = 1$. If η and $\gamma'(0)$ are linearly dependent Claim 4.7 follows trivially. Thus we may assume that η and $\gamma'(0)$ are linearly independent.

In the case $c \leq 0$ the intersection between γ_η and W occurs at a unique point $u = \gamma_\eta(s_0) \in W$. If $c > 0$ there exists $0 < s_0 < \frac{\pi}{\sqrt{c}}$ such that $u = \gamma_\eta(s_0) \in W$ (see Claim 4.1 in the proof of Lemma 4.1). In both cases the geodesic $\gamma_\eta : [0, s_0] \to \mathbb{Q}^n_c$ is the unique minimal unit speed geodesic joining q and u. We have that $p \neq u$ because of the two following facts: (i) γ and γ_η are the unique minimal unit speed geodesics from q to p and q to u, respectively; (ii) η and $\gamma'(0)$ are linearly independent. Thus we obtain that there exists a minimal unit speed geodesic $\mu : [0, t_0] \to W$ with $t_0 > 0$, satisfying that $\mu(0) = p$, $\mu(t_0) = u$.

Now we assert that

$$\big\langle \eta, \gamma'(0) \big\rangle \neq 0. \tag{12}$$

In fact, if $c \leq 0$ and (12) is false, the lines γ_η and μ are orthogonal to γ which implies that they cannot intersect in the point u, which is a contradiction. In the case $c > 0$, the assertion (12) follows from Lemma 4.1.

Let $P : T_q(\mathbb{Q}^n_c) \to T_p(\mathbb{Q}^n_c)$ be the parallel transport along γ. We claim that

$$P(\eta) \text{ and } \mu'(0) \text{ are linearly independent.} \tag{13}$$

In fact, if (13) is not true we have that $P(\eta) = \pm\mu'(0)$. Since $\mu'(0) \in T_pW$ and $\gamma'(r_0) \in T_pS$ it holds that $\langle \mu'(0), \gamma'(r_0) \rangle = 0$, hence we have that

$$\big\langle \eta, \gamma'(0) \big\rangle = \big\langle P(\eta), P(\gamma'(0)) \big\rangle = \big\langle P(\eta), \gamma'(r_0) \big\rangle = \pm \big\langle \mu'(0), \gamma'(r_0) \big\rangle = 0,$$

which contradicts (12).

Now we assert that

$$\big\langle P(v), \mu'(0) \big\rangle = \langle P(v), P(\eta) \rangle = 0. \tag{14}$$

Indeed, the equality $\langle P(v), P(\eta) \rangle = 0$ follows directly from the equality $\langle v, \eta \rangle = 0$, which follows from (a) or (b). Since $v \in T_qS$ and S is totally geodesic we obtain that $P(v) \in T_pS = (T_pW)^\perp$. This implies that $\langle P(v), \mu'(0) \rangle = 0$.

By Lemma 3.1 there exists a complete totally geodesic surface N^2 containing the images of γ, γ_η and μ. Since $\eta \in T_q(N^2)$ and N^2 is totally geodesic it follows that $P(\eta) \in T_p(N^2)$. Thus (13) implies that $P(\eta)$ and $\mu'(0)$ form a basis for $T_p(N^2)$. From (14) we obtain that $P(v)$ is orthogonal to $T_p(N^2)$, which implies that

$$\big\langle v, \gamma'(0) \big\rangle = \big\langle P(v), P(\gamma'(0)) \big\rangle = \big\langle P(v), \gamma'(r_0) \big\rangle = 0,$$

since $\gamma'(r_0) \in T_p(N^2)$. Claim 4.7 is proved.

Now we are in position to prove that (A) holds under condition (I) or (II) above. To do this we fix $q \in \Sigma$. Since $(\pi_W)|_\Sigma$ is a submersion and Σ is a manifold of class C^k, with $k \geq 1$, there exists a C^k diffeomorphism $h : D \times \mathcal{V} \to \mathcal{U}$ satisfying that $\pi_W(h(x, y)) = y$ for any $(x, y) \in D \times \mathcal{V}$, where $\mathcal{U} \subset \Sigma$ is a small open neighborhood of q, $\mathcal{V} = \pi_W(\mathcal{U})$ is an open subset of W, and D is an open disk in $\mathbb{R}^{m-j}$ with $m = \dim(\Sigma)$ and $j = \dim(W)$. Since $\mathcal{U} \subset \Sigma$ we have that $\mathcal{U} \cap (W \cup V_W) = \emptyset$.

Write $q = h(x_q, p)$ for some $(x_q, p) \in D \times \mathcal{V}$ and note that $\pi_W(q) = p$. Define the C^k map $\xi : \mathcal{V} \to \mathcal{U}$ given by

$$\xi(y) = h(x_q, y).$$

Claim 4.8. *For any $y \in \mathcal{V}$ and $z, \tilde{z} \in h(D \times \{y\})$, it holds that $d(z, W) = d(\tilde{z}, W)$.*

In fact, for any $x \in D$, we have that $\pi_W(h(x, y)) = y$, hence $\pi_W(u) = y$ for any $u \in h(D \times \{y\})$. Thus any vector v tangent to $h(D \times \{y\})$ in u

must satisfy that $(d\pi_W)_u v = 0$. By Claim 4.7 it holds that $\langle v, \gamma'(0)\rangle = 0$ where $\gamma : [0, r_0] \to \mathbb{Q}^n_c$ is the unit speed shortest geodesic from u to W, namely, it satisfies that $\gamma(0) = u, \gamma(r_0) = y$ and $L(\gamma) = r_0 = d(u, W)$. Thus we may apply Corollary 2 to conclude that $d(z, W) = d(\tilde{z}, W)$ for all $z, \tilde{z} \in h(D \times \{y\})$. Claim 4.8 is proved.

Given $z \in \mathcal{U}$, it holds that $z = h(x, \pi_W(z))$ for some $x \in D$. We also have that $\xi(\pi_W(z)) = h(x_q, \pi_W(z))$. Hence z and $\xi(\pi_W(z))$ belong to $h(D \times \{\pi_W(z)\})$. We conclude from Claim 4.8 that

$$d(z, W) = d(\xi(\pi_W(z)), W). \tag{15}$$

We define the C^k function $r : \mathcal{V} \to (0, \infty)$ given by $r(y) = d(\xi(y), W)$. Consider the following set

$$M = \bigcup_{y \in \mathcal{V}} S'(y, r(y)),$$

where we denote $S'(y, s) = \mathcal{S}(y, s) \cap S_{yW}$.

Claim 4.9. *The set M is invariant under the action of the group G_W.*

In fact, fix an isometry $\phi \in G_W$ and $y \in \mathcal{V}$. For $w \in T_yW$ and $v \in T_y(S_{yW}) = (T_yW)^\perp$ we have that

$$\langle d\phi_y v, w\rangle = \langle d\phi_y v, d\phi_y w\rangle = \langle v, w\rangle = 0,$$

hence $d\phi_y(T_y(S_{yW})) \subset (T_yW)^\perp = T_y(S_{yW})$. By the injectivity of $d\phi_y$ and an argument on dimension we conclude that $d\phi_y(T_y(S_{yW})) = T_y(S_{yW})$. From this and the fact that S_{yW} and $\phi(S_{yW})$ are totally geodesic it follows that

$$\phi(S_{yW}) = S_{yW}. \tag{16}$$

Since $\phi(y) = y$, for any $z \in S'(y, r(y))$ we have that

$$r(y) = d(y, z) = d(\phi(y), \phi(z)) = d(y, \phi(z)),$$

hence $\phi\big(S'(y, r(y)\big) \subset \mathcal{S}(y, r(y))$. Thus we have from (16) that

$$\phi\big(S'(y, r(y))\big) \subset \mathcal{S}(y, r(y)) \cap S_{yW} = S'(y, r(y)).$$

Claim 4.9 is proved.

Claim 4.10. *The set M contains $\mathcal{U}$.*

In fact, take $z \in \mathcal{U}$. Set $y = \pi_W(z) \in \mathcal{V}$. To prove Claim 4.10 it suffices to prove that $z \in S'(y, r(y))$. By Lemma 3.4 we have that $z \in S_{yW}$. By (15) we obtain that

$$d(z, y) = d(z, \pi_W(z)) = d(z, W) = d(\xi(\pi_W(z)), W) = d(\xi(y), W) = r(y).$$

Thus $z \in \mathcal{S}(y, r(y)) \cap S_{yW} = S'(y, r(y))$. Claim 4.10 is proved.

Claim 4.11. *The set M is an embedded hypersurface of class C^k.*

In fact, let $\nu_1(\mathcal{V}) = \{(y,v) \mid y \in \mathcal{V},\ v \in (T_y\mathcal{V})^\perp \text{ with } |v| = 1\}$ denote the unit normal fiber bundle over $\mathcal{V}$. We define the C^k map $\psi : \nu_1(\mathcal{V}) \to \mathbb{Q}_c^n$ given by

$$\psi(y,v) = \exp^\perp(y, r(y)v)$$

and $\varphi : M \to \nu_1(\mathcal{V})$ given by

$$\varphi(z) = \left(\pi_1\left(\left(\exp^\perp\right)^{-1}(z)\right), \frac{\pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)}{\left|\pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)\right|}\right),$$

where π_1 and π_2 are the natural projections given by $\pi_1(y,v) = y$ and $\pi_2(y,v) = v$. It is clear that $\psi((y,v)) \in S_{yW}$ and $d\big(\psi((y,v)), y\big) = r(y)$. Hence we have that $\psi((y,v)) \in S'(y, r(y))$. Thus we obtain that $\psi(\nu_1(\mathcal{V})) \subset M$. Furthermore we have that φ is the restriction of a C^∞ map defined in $\mathbb{Q}_c^n - W$ in the case $c \leq 0$ and defined in $\mathbb{Q}_c^n - (W \cup V_W)$ in the case $c > 0$.

It is straightforward to show that $\varphi(\psi(y,v)) = (y,v)$. We will show that $\psi(\varphi(z)) = z$ for all $z \in M$. Set

$$y = \pi_1\left(\left(\exp^\perp\right)^{-1}(z)\right) \quad \text{and} \quad v = \frac{\pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)}{\left|\pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)\right|}.$$

With this notation we have that $\varphi(z) = (y,v)$. Note that

$$\pi_W(z) = \pi_1\left(\left(\exp^\perp\right)^{-1}(z)\right) = y.$$

By (15) we have that

$$\left|\pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)\right| = d(z,W) = d(\xi(\pi_W(z)), W) = r(\pi_W(z)) = r(y),$$

which implies that $r(y)v = \pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)$. Thus we have that

$$\begin{aligned}
\psi(\varphi(z)) &= \psi(y,v) = \exp^\perp(y, r(y)v) = \exp^\perp\left(\pi_1\left(\left(\exp^\perp\right)^{-1}(z)\right), r(y)v\right) \\
&= \exp^\perp\left(\pi_1\left(\left(\exp^\perp\right)^{-1}(z)\right), \pi_2\left(\left(\exp^\perp\right)^{-1}(z)\right)\right) \\
&= z.
\end{aligned}$$

We conclude that $M = \psi(\nu_1(\mathcal{V}))$ and ψ is a C^k diffeomorphism. Hence M is an embedded C^k hypersurface of $\mathbb{Q}_c^n$. Claim 4.11 is proved.

It follows from Claims 4.9, 4.10 and 4.11 that Property (A) holds. Theorem 3 is proved. □

The following proposition was mentioned in Remark 4.

Proposition 4.1. *Let Σ be a differentiable connected submanifold of $\mathbb{Q}_c^n$ with $c > 0$ and $W \subset \mathbb{Q}_c^n$ be a complete connected totally geodesic submanifold with $\Sigma \cap \{W \cup V_W\} = \emptyset$. Then Property* (C) *is true.*

Proof. Fix $q \in \Sigma$ and $v \in T_q\Sigma$ with $d(\pi_W)_q v = 0$. Thus it holds that $v \in T_q(S_{pW})$, where $p = \pi_W(q)$ (see Lemma 3.4). Let $\gamma : [0, r_0] \to S_{pW}$ be a unit speed minimal geodesic from q to p satisfying $L(\gamma) = r_0 = d(q, W)$. Fix a unit vector $w \in T_pW$. Let $\eta \in T_q(\mathbb{Q}^n_c)$ be given by the parallel transport of w along γ. Since S_{pW} is totally geodesic and w is orthogonal to $T_p(S_{pW})$ we have that η is orthogonal to $T_q(S_{pW})$, hence it is orthogonal to v. By using the unique totally geodesic surface N^2 such that $T_q(N^2) = \text{span}\{\gamma'(0), \eta\}$ we obtain that γ_η intersects γ_w, hence it intersects W. Proposition 4.1 is proved. □

The next proposition was mentioned in Remark 5.

Proposition 4.2. *Let Σ be a differentiable hypersurface of $\mathbb{Q}^n_c$ with $c \leq 0$ and $W \subset \mathbb{Q}^n_c$ be a complete connected totally geodesic submanifold with $\Sigma \cap W = \emptyset$. Then Property* (C) *implies that $(\pi_W)|_\Sigma$ is a submersion.*

Proof. Assume by contradiction that (C) holds and that $(\pi_W)|_\Sigma$ is not a submersion. Then there exists $q \in \Sigma$ such that $d((\pi_W)|_\Sigma)_q : T_q\Sigma \to T_pW$ is not surjective. Consider as above a shortest unit speed geodesic $\gamma : [0, r_0] \to \mathbb{Q}^n_c$ from q to W, namely, assume that $\gamma(0) = q$, $\gamma(r_0) = p = \pi_W(q) \in W$ and $L(\gamma) = r_0 = d(q, W)$. We consider again the totally geodesic submanifold $S_{pW} = \pi_W^{-1}(\{p\})$ (see Lemma 3.4).

Since $d((\pi_W)|_\Sigma)_q$ is not surjective, it holds that the intersection between Σ and S_{pW} is not transversal at q. In fact, if $T_q\Sigma + T_q(S_{pW}) = T_q(\mathbb{Q}^n_c)$ then we have by Lemma 3.4 that $(d\pi_W)_q(T_q\Sigma) = (d\pi_W)_q(T_q(\mathbb{Q}^n_c)) = T_pW$, which contradicts the hypothesis that $d((\pi_W)|_\Sigma)_q$ is not surjective.

Since Σ is a hypersurface and it does not intersect S_{pW} transversally at q, we conclude that $T_q(S_{pW}) \subset T_q\Sigma$, hence $\gamma'(0) \in T_q\Sigma$. Since $d(\pi_W)_q(\gamma'(0)) = 0$, Property (C) implies that there exists a unit vector η orthogonal to $\gamma'(0)$ such that the geodesic γ_η intersects W. However the facts that $c \leq 0$, W is totally geodesic and η is orthogonal to $\gamma'(0)$ imply together that γ_η may not intersect W, which give us a contradiction. Proposition 4.2 is proved. □

Proof of Theorem 4. Under the hypotheses of Theorem 4, if $c \leq 0$ we denote by $\exp^\perp_W$ the diffeomorphism given by the exponential map $\exp^\perp_W : \nu(W) \to \mathbb{Q}^n_c$. In the case $c > 0$ $\exp^\perp_W$ will denote the exponential map $\exp^\perp_W : B_W \to \mathbb{Q}^n_c - V_W$. We recall that by definition $B_W \subset \nu(W)$.

Fix $q \in \Sigma$. Since $\pi_W : \Sigma \to W$ has constant rank $i < j$, the constant rank theorem implies that there exists a neighborhood U of q in Σ such that $\pi_W(U)$ is an embedded C^k submanifold of W of dimension $i < j = \dim(W)$. Then there exists a C^k hypersurface $S_1 = S_1^{j-1}$ of W containing $\pi_W(U)$. Consider the set $M = \exp^\perp_W\big(\nu(W) \cap \nu(S_1)\big)$, in the case $c \leq 0$, and $M = \exp^\perp_W\big(B_W \cap \nu(S_1)\big)$, in the case $c > 0$. Note that M is a union of images of geodesics orthogonal to W which start at points in S_1. The domain of $\exp^\perp_W$ is a fiber bundle over S_1 with fiber of dimension $n - j$. Thus its dimension is $(j-1) + (n-j) = n-1$. Since $\exp^\perp_W$ is a diffeomorphism on its domain (see

Lemma 3.3 in the case $c > 0$), we have that M is an embedded hypersurface of $\mathbb{Q}_c^n$.

Claim 4.12. *$U \subset M$.*

Indeed take $z \in U$ and $p = \pi_W(z) \in \pi_W(U)$. There exists a unique unit speed minimal geodesic $\gamma : [0, d] \to \mathbb{Q}_c^n$ from z to p such that $L(\gamma) = d = d(z, W)$. Thus we have that $z = \exp^{\perp}\big(p, -d\gamma'(d)\big)$. Note that $-d\gamma'(d) \in (T_pW)^{\perp} \subset \big(T_p(S_1)\big)^{\perp}$. Since we also have that $p \in \pi_W(U) \subset S_1$ we conclude that $z \in \exp^{\perp}\big(\nu(W) \cap \nu(S_1)\big)$. Thus, if $c \leq 0$ we have that $z \in M$. In the case that $c > 0$ we have by hypothesis that $z \in \mathbb{Q}_c^n - (W \cup V_W)$, hence by Lemma 3.3 we have that $|-d\gamma'(d)| = d < \frac{\pi}{2\sqrt{c}}$, which implies that $\big(p, -d\gamma'(d)\big) \in B_W$. We conclude that $z \in \exp^{\perp}\big(B_W \cap \nu(S_1)\big) = M$. Claim 4.12 is proved.

Claim 4.13. *M is invariant under the action of G_W.*

Take $\phi \in G_W$ and $z \in M$. Then $z = \exp_W^{\perp}(p, v)$, for some $(p, v) \in \nu(W) \cap \nu(S_1)$, hence $p \in S_1 \subset W$. Since $(p, v) \in \nu(W)$ it holds that v is orthogonal to W at p, hence $v \in T_p(S_{pW})$. If $v = 0$ then $z = p \in W$, hence $\phi(z) = z \in M$. For $v \neq 0$, consider the unit speed geodesic $\gamma : \big[0, |v|\big] \to \mathbb{Q}_c^n$ given by $\gamma(t) = \exp_W^{\perp}\left(p, \frac{t}{|v|}v\right)$ from p to z. Since S_{pW} is totally geodesic and $v \in T_p(S_{pW})$, we have that $\gamma\big([0, |v|]\big) \subset S_{pW}$. Set $\sigma = \phi \circ \gamma$. By (16) we have that $\phi(S_{pW}) = S_{pW}$, hence $\sigma\big([0, |v|]\big) \subset S_{pW}$. Since $p \in W$ and $\phi \in G_W$, it holds that $\phi(p) = p = \sigma(0)$ and that $\sigma'(0) \in T_p(S_{pW}) = (T_pW)^{\perp} \subset (T_p(S_1))^{\perp}$. Thus we have that

$$\phi(z) = \phi\big(\gamma(|v|)\big) = \sigma(|v|) = \exp_W^{\perp}\big(p, |v|\sigma'(0)\big) \in \exp_W^{\perp}\big(\nu(W) \cap \nu(S_1)\big).$$

If $c \leq 0$ we conclude that $\phi(z) \in M$. If $c > 0$, the fact that $(p, v) \in B_W$ implies that $|v| < \frac{\pi}{2\sqrt{c}}$, hence $\big||v|\sigma'(0)\big| = |v| < \frac{\pi}{2\sqrt{c}}$, hence $\phi(z) \in \exp^{\perp}\big(B_W \cap \nu(S_1)\big) = M$. Claim 4.13 is proved.

By Claims 4.12 and 4.13 we conclude that Σ satisfies (A). Theorem 4 is proved. $\square$

Proof of Corollary 4. We know that there exists an open set $\Omega \subset \Sigma$ such $\Omega = \cup_\lambda U_\lambda$, where each U_λ is an open set where π_W has constant rank. Fix $q \in \Omega$. If π_W is a submersion at p, we apply Theorem 3 and obtain that (B) implies (A) if $c \geq 0$ and (C) implies (A) if $c < 0$. If the rank of π_W is some $i < j$ at p, there exists some open set U_λ containing p such that π_W has rank i on U_λ. By Theorem 4 we conclude that (A) holds. Corollary 4 is proved. $\square$

5. Examples

The following example was cited in Remark 2 above.

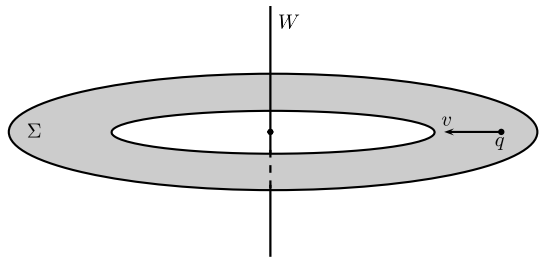

Figure 1. Referred to Example 5.2.

Example 5.1. Consider the function $G : \mathbb{R}^n \to \mathbb{R}$, given by $G(x_1, \cdots, x_n) = \arctan(x_1)$, which has Lipschitz constant $C \geq 1$. Set $\Sigma = G^{-1}(\{0\}) = \{0\} \times \mathbb{R}^{n-1}$, and let $f : \Sigma \to \mathbb{R}^n$ be the inclusion map. Fix $p = (0, \cdots, 0)$ and $v = e_2 = (0, 1, 0, \cdots, 0) \in T_p\Sigma$. Now consider any nontrivial vector $\eta = (a_1, 0, a_3, \cdots, a_n)$ orthogonal to v. We claim that $\gamma_\eta : [0, 1] \to \mathbb{R}^n$, given by $\gamma_\eta(t) = t(a_1, 0, a_3, \cdots, a_n)$, does not satisfy Equation (1) above. Indeed, if γ_η satisfies (1), we have $|G(\gamma_\eta(1)) - G(\gamma_\eta(0))| = C\, L(\gamma_\eta)$, hence

$$|\arctan(a_1)| \;=\; C\sqrt{a_1^2 + a_3^2 + \cdots + a_n^2} \;\geq\; |a_1|,$$

which implies that $a_1 = 0$. Thus we have that $\sqrt{a_1^2 + a_3^2 + \cdots + a_n^2} = 0$, hence $\eta = 0$, which is a contradiction.

The following example (see Figure 1) shows that in the case $c \leq 0$ the assumption that $(\pi_W)|_\Sigma$ is a submersion is essential to obtain that (A) implies (C) (compare with Proposition 4.1).

Example 5.2. Let W be a complete connected totally geodesic submanifold of $\mathbb{Q}_c^n$, with $c \leq 0$. Take $S_{pW} = \pi_W^{-1}(\{p\})$, for some $p \in W$. Fix $0 \leq a < b \leq \infty$ and set

$$\Sigma = \{z \in S_{pW} \mid a < d(z, W) < b\}.$$

It is easy to see that Σ is invariant under the action of G_W, hence it satisfies (A). Fix $q \in \Sigma$ and $v = \gamma'(0)$, where γ is the unit speed geodesic from q to p. For any vector $\eta \in T_q(\mathbb{Q}_c^n)$ orthogonal to v, the geodesic γ_η does not intersect W, hence (C) does not hold.

According to Theorem 3 we have that (B) implies (A) for any space form of constant curvature $c \in \mathbb{R}$. However, the next example shows that without the condition that $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion this implication may fail.

Example 5.3. Consider the cone and cylinder given, respectively, by

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z < 1, (z-1)^2 = x^2 + y^2\}$$

and

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 0, x^2 + y^2 = 1\},$$

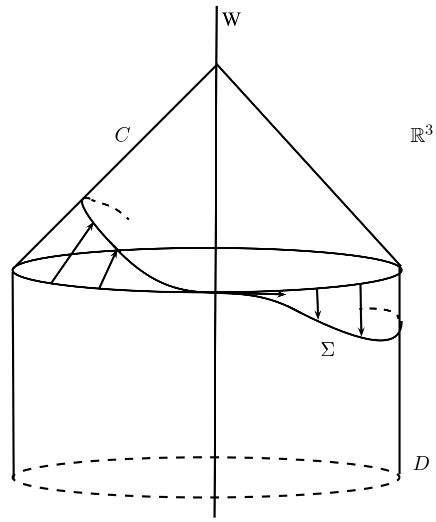

FIGURE 2. Referred to Example 5.3

and let $\Sigma, W \subset \mathbb{R}^3$ be as in Figure 2. More precisely, consider smooth functions $\mu, \nu : (-\epsilon, \epsilon) \to [0, +\infty)$ for some small $\epsilon > 0$, satisfying that

$$\begin{cases} \mu(t) = 0 \text{ for all } t \leq 0; & \mu(t) > 0 \text{ for all } t > 0; \\ \nu(t) > 0 \text{ for all } t \leq 0; & \nu(t) = 0 \text{ for all } t > 0. \end{cases}$$

Consider the curve $\alpha(t) = (\cos(t), \sin(t), 0)$, with $t \in \mathbb{R}$. Let $\beta : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be the smooth curve given by

$$\beta(t) = \alpha(t) + \nu(t)(-\cos(t), -\sin(t), 1) + \mu(t)(0, 0, -1).$$

Let Σ be the image of β and W the z-axis. It is easy to see that Σ is a smooth embedded submanifold if ϵ is sufficiently small. We have that $\beta(t)$ belongs to the cone C if $t < 0$ and to the cylinder D if $t \geq 0$. Thus it is not difficult to see that Σ satisfies (B) in Theorem 3. Note that any submanifold containing a small neighborhood of $\alpha(0)$ in Σ and invariant under the G_W action should contain an open neighborhood of $\alpha(0)$ in the non-smooth continuous hypersurface $C \cup D$, which implies that Σ does not satisfy (A). Note that $(\pi_W)|_\Sigma$ is not a submersion at the point $\beta(0)$, since

$$d(\pi_W)_{\beta(0)}\beta'(0) = \pi_W(\beta'(0)) = \pi_W(0, 1, 0) = 0.$$

The following example shows that Theorem 3 is sharp in the sense that (A) does not imply (B) in the case $c < 0$ (see Remark 5 in the Introduction).

Example 5.4. Consider the hyperbolic space $\mathbb{H}^3$ in the half space model $\mathbb{R}^3_+ = \{(x, y, z) \mid z > 0\}$. Let $W = \{(0, 0, z) \mid z > 0\}$ be a vertical (totally geodesic) line in $\mathbb{H}^3$. Let $\Sigma = \{(x, y, z) \mid x^2 + y^2 = 1, z > 0\} \subset \mathbb{H}^3$ be the cylinder of axis W and Euclidean radius 1. We first verify that $(\pi_W)|_\Sigma$ is a submersion. For this we take $q = (x, y, z) \in \Sigma$ and the curve $\alpha : (0, +\infty) \to \Sigma$ given by $\alpha(t) = (x, y, t)$. We have that $\alpha(z) = q$ and $\alpha'(z) = (0, 0, 1)$.

Set $\beta : (0, +\infty) \to W$ given by $\beta(t) = (0, 0, \sqrt{1+t^2})$. It is easy to see that $\beta(t) = \pi_W(\alpha(t))$, hence we obtain that

$$(d(\pi_W)|_\Sigma)_q(\alpha'(z)) = \beta'(z) = \left(0, 0, \frac{z}{\sqrt{1+z^2}}\right) \neq 0,$$

hence we have that $(d(\pi_W)|_\Sigma)_q : T_q\Sigma \to T_{\pi_W(q)}W$ is surjective and $(\pi_W)|_\Sigma$ is a submersion. Since Σ is a hypersurface invariant under rotations around W we see that Σ satisfies (A). Now we will verify that Σ does not satisfy (B). We choose $q = (x, y, z) \in \Sigma$ with $0 < z \leq 1$. For any unit vector η orthogonal to Σ the geodesic γ_η will not intersect W since it is contained in the Euclidean sphere of center $(x, y, 0)$ and radius z (see Figure 3). This shows that (A) does not imply (B) in the case $c < 0$.

The next example shows that Theorem 3 may not be improved to obtain that (C) implies (A) in the case $c > 0$ (see Remarks 4, 5).

Example 5.5. Consider the standard unit sphere S^3 and the natural totally geodesic inclusion $S^2 \subset S^3$. Consider on S^2 the image W of a closed geodesic on S^2 (see Figure 4). Let Σ be an open subset of S^2 satisfying that $\Sigma \cap \{W \cup V_W\} = \emptyset$. Clearly we have that $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion. First we will see that Σ satisfies (C). In fact, fix a point q on Σ and any unit vector $v \in T_p\Sigma$. Choose a unit vector $\eta \in T_qS^2$ orthogonal to v. The geodesic γ_η must remain contained in S^2, hence it will intersect W and (C) holds. Now we will see that (A) does not hold. We observe that, since Σ is an open subset of S^2, the union of orbits $\mathcal{V} = \bigcup_{x\in\Sigma} G_W(x)$ is an open subset of S^3. Thus any submanifold M containing Σ and invariant under the action of G_W must contain $\mathcal{V}$, hence M may not be a hypersurface. We conclude that Σ does not satisfy (A).

The example below presents a nontrivial situation where Theorem 3 applies.

Example 5.6. Consider the map $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^4$ given by $f(x, y) = (x, y, e^x \cos y, e^x \sin y)$ and let Σ be the image of f. Set $W = \{(0,0)\} \times \mathbb{R}^2$

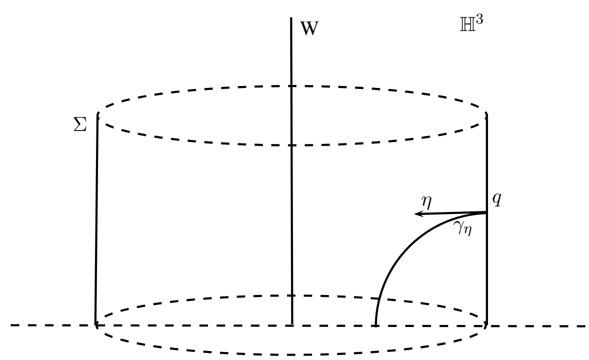

Figure 3. Reffered to Example 5.4

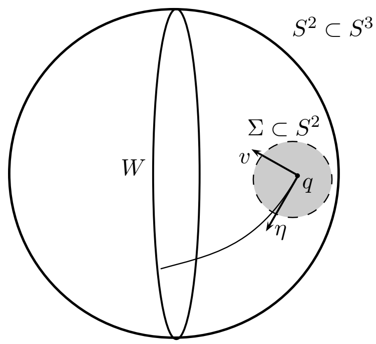

FIGURE 4. Reffered to Example 5.5

and consider the natural projection $\pi_W : \mathbb{R}^4 \to W$. We claim that Σ and W satisfy the hypotheses of Theorem 3, and that any plane orthogonal to W at a point $p \in W - \{(0,0,1,0),(0,0,0,0)\}$ intersects Σ in infinitely many isolated points (see Remark 5). In fact, we first note that $\Sigma \cap W = \emptyset$. We have that

$$\frac{\partial f}{\partial x} = (1, 0,\, e^x \cos y,\, e^x \sin y) \quad \text{and} \quad \frac{\partial f}{\partial y} = (0, 1,\, -e^x \sin y,\, e^x \cos y), \tag{17}$$

hence f is an immersion. Since Σ is a smooth graph we conclude that Σ is a smooth embedded submanifold. The vectors

$$\pi_W\left(\frac{\partial f}{\partial x}\right) = (0,0,\, e^x \cos y,\, e^x \sin y),\ \pi_W\left(\frac{\partial f}{\partial y}\right) = (0,0,\, -e^x \sin y,\, e^x \cos y)$$

are linearly independent, hence $(\pi_W)|_\Sigma : \Sigma \to W$ is a submersion. Now we will see that Item (B) in Theorem 3 is satisfied. To obtain this it suffices to prove that $\left(q + (T_q\Sigma)^\perp\right) \cap W \neq \emptyset$, for any $q = f(x,y) \in \Sigma$. By a simple computation using (17) we obtain that

$$(T_q\Sigma)^\perp = \{(-c\, e^x \cos y - d\, e^x \sin y, c\, e^x \sin y - d\, e^x \cos y,\, c,\, d) \bigm| c, d \in \mathbb{R}\}.$$

Thus we have $\left(q + (T_q\Sigma)^\perp\right) \cap W \neq \emptyset$ if and only if the linear system

$$\begin{cases} x = c\, e^x \cos y + d\, e^x \sin y, \\ y = -c\, e^x \sin y + d\, e^x \cos y \end{cases}$$

has a solution, and this is the case. Now, take

$$p = (0,0,\alpha,\beta) \in W - \{(0,0,1,0),(0,0,0,0)\}.$$

We will see that the plane $S_{pW} = p + W^\perp$ intersects Σ at infinitely many isolated points. To see this, note that $S_{pW} = \{(u,v,\alpha,\beta) \bigm| u,v \in \mathbb{R}\}$. Thus an easy computation shows that

$$S_{pW} \cap \Sigma = \left\{\left(\log\left(\sqrt{\alpha^2+\beta^2}\right), \theta + 2k\pi, \alpha, \beta\right) \bigm| k \in \mathbb{Z}\right\},$$

where θ is any angle satisfying $\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$ and $\sin\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$. Our claim is proved.

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