Politecnico of Turin - University of Turin
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On semiseparability, semifunctors and conditions up to retracts

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Summary of the thesis

In this thesis, we present the results investigated mainly in [1], [2], [5].

We present the notion of *semiseparable* functor, introduced in [1], and its main properties that extend known results for separable functors. Separability plays a central role in several topics in Algebra, Number Theory and Algebraic Geometry. Separable functors were introduced in [8] in order to reinterpret categorically the theory of separable field extensions, and of separable algebras. Explicitly, a functor $F : \mathcal{C} \to \mathcal{D}$ is *separable* if its associated natural transformation $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-)$, $f \mapsto Ff$, has a left inverse. On the other hand, requiring that $\mathcal{F}$ has a right inverse yields the notion of *naturally full* functor [3]. We have defined a functor $F : \mathcal{C} \to \mathcal{D}$ to be *semiseparable* if $\mathcal{F}$ is a regular natural transformation, i.e., if there is a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. It is known that a separable functor is faithful whereas a naturally full functor is full: semiseparability allows to reverse these implications and to treat separable and naturally full functors in a unified way. A suitable idempotent natural transformation and a canonical factorization (as a naturally full functor followed by a separable functor) can be attached to any semiseparable functor. We provide characterizations of semiseparibility for functors that are part of an adjunction, focusing mainly on functors attached to morphisms of rings and coalgebras, comodule categories over corings, and bimodules.

We deal with semiseparable functors in the context of Eilenberg-Moore categories and idempotent complete categories, as investigated in [2]. We present the notions of *coreflections up to retracts* (*reflections up to retracts*, respectively), i.e., functors whose idempotent completion admits a fully faithful left (resp., right) adjoint, and *bireflections up to retracts*, a stronger notion involving both a left and right adjoint. We discuss semiseparability with respect to these functors. One of the main results we have proved in this setting is that a right (resp., left) adjoint functor is semiseparable if, and only if, the associated (co)monad is (co)separable and the (co)comparison functor is a bireflection up to retracts, recovering known results in the separable case [6]. As a consequence, the semiseparability of a right adjoint functor entails an equivalence after idempotent completion between the Kleisli category of free modules over the associated monad and the Eilenberg-Moore category of modules over that monad. We have then considered the context of pre-triangulated categories, providing conditions for the Eilenberg-Moore category of modules to inherit the pre-triangulation from the base category by means of semiseparability, extending a result proved in [4] for the separable case. We prove similar results for the Eilenberg-Moore category of comodules and for the Kleisli category.

Another aim of this thesis is to show how several properties of functors, such as faithfulness, (natural) fullness, (semi)separability, can be formulated for semifunctors, as studied in [5]. The notion of semifunctor appeared in [7] and it is defined as a functor that does not necessarily preserve identities. We present the notion of *semifullness* (and then *semifull faithfulness* and *natural semifullness*) for semifunctors. We describe "semisplitting properties" for seminatural transformations and the corresponding ones for morphisms whose source or target is the image of a semifunctor. We characterize (natural) semifullness and (semi)separability for semifunctors that are part of a semiadjunction, in terms of semisplitting conditions for the unit and counit. We show examples of semifunctors on which we have tested these properties.

In the thesis we provide some other original results and we start exploring some variations of semiseparability.

References

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