Robust energy-maximising control of wave energy systems under input uncertainty

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***Abstract*— Motivated by the ubiquitous presence of input uncertainty in the wave energy control problem, we propose, in this paper, a robust energy-maximising framework which explicitly considers potential wave excitation force deviations in the computation of the optimal control law, while systematically respecting state and input constraints. In particular, this is achieved by a suitable *moment-based* characterisation for the input uncertainty, taking into consideration an appropriate convex uncertainty set. The concept of moments is combined with well-known robust optimisation principles, by proposing a worst-case performance approach. We show that this novel moment-based robust optimal control framework always admits a unique global energy-maximising solution, hence leading to a computationally efficient robust solution. The performance of the proposed controller is illustrated by means of a case study, considering a heaving point absorber WEC.**

I. Introduction

Energy-maximising control of wave energy converters (WECs) has proven to be a fundamental stepping stone towards effective commercialisation of wave energy technology [1]. The control problem for WECs naturally lends itself towards optimal control theory: the energy-maximising control design procedure for WECs can be cast as an optimal control problem (OCP), where the objective is to maximise the absorbed energy from incoming ocean waves, while respecting the physical limitations associated with both device, and power take-off (PTO) actuator, characteristics.

Though system uncertainty is effectively ubiquitous within hydrodynamic WEC modelling (see [2]), is not the only source of error inherently present in the WEC energy-maximising optimal control problem: Given that the wave excitation force, which is a key variable in the OCP, is, in general, not measurable, approximations are virtually always required, computed by means of input estimation (instantaneous values) and forecasting (future values) techniques (see [3]). In other words, *input uncertainty* is also ubiquitous. Moreover, to the best of the authors' knowledge, robustness with respect to errors in the estimation and forecasting of the excitation effect, *i.e.* input uncertainty, has not been yet explicilty addressed in the WEC control literature.

Motivated by the discussion provided above, we propose, in this paper, an energy-maximising moment-based framework which explicitly considers *input* uncertainty in the computation of the optimal control law, while systematically respecting motion and PTO (state and input) constraints. In particular, and inspired by the robust approach for WECs under parametric uncertainty presented in [4], this is achieved by a suitable moment-based characterisation for the input uncertainty, taking into consideration an appropriate uncertainty set, written in terms of a convex polytope defined over a real vector space. To this end, the concept of moments is combined with the robust optimisation principles considered in [5], [6], by proposing a worst-case performance (WCP) approach. We show that this novel moment-based robust optimal control framework always admits a unique global energy-maximising solution, preserving all the appealing characteristics of the (nominal) strategy developed in [2], hence leading to a computationally efficient robust control solution for WECs. The performance of the proposed controller is illustrated and analysed by means of a case study, considering a heaving point absorber WEC.

A. Notation

$\mathbb{N}_q$ indicates the set of all positive natural numbers up to q, *i.e.* $\mathbb{N}_q = \{1, 2, \dots, q\}$. $\mathbf{1}_{n\times m}$ is used to denote a $n \times m$ Hadamard identity matrix. The spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\lambda(A)$. The symbol $\bigoplus$ denotes the direct sum of n matrices, *i.e.* $\bigoplus_{i=1}^{n} A_i = \text{diag}(A_1, A_2, \dots, A_n)$. The *Kronecker product* between two matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. The *Kronecker sum* of two matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{k\times k}$ is denoted as $P_1 \hat{\oplus} P_2$. The convolution between two functions f and g is denoted as $f * g$. The symbol $e_{ij}^{q} \in \mathbb{R}^{q\times q}$ denotes a matrix with 1 in the ij-entry and 0 elsewhere, while $\varepsilon_n \in \mathbb{R}^n$ denotes a vector with odd entries equal to 1 and even entries equal to 0.

II. Preliminaries on Moment-based theory

We provide, in this section, a brief summary of moment-based theory for linear systems [7]. Let us consider a single-input single-output system, defined, for $t \in \mathbb{R}^+$, by the following set of differential equations[1]

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and the triple of (constant) matrices (A, B, C) is dimensioned accordingly. Assume that system (1) is minimal. Let the external input u, be written as the output of the so-called *signal generator*:

$$\dot{\xi} = S\xi, \quad u = L\xi, \tag{2}$$

with $\xi(t) \in \mathbb{R}^\nu$, $S \in \mathbb{R}^{\nu\times\nu}$ and $L^\intercal \in \mathbb{R}^\nu$.

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[1]The dependence on t is dropped when clear from the context.

Lemma 1. [7] *Consider system* (1) *and the autonomous signal generator* (2)*. Assume that the triple* $(L, S, \xi(0))$ *is minimal,* $\lambda(A) \subset \mathbb{C}_{<0}$*,* $\lambda(S) \subset \mathbb{C}^0$ *and the eigenvalues of* S *are simple. Then, there is a unique matrix* $\Pi \in \mathbb{R}^{n\times\nu}$ *which solves the Sylvester equation* $A\Pi + BL = \Pi S$*, and the steady-state response of the output of the interconnected system* (1)-(2) *is* $y_{ss}(t) = C\Pi\xi(t)$.

Definition 1. The matrix $\underline{Y} = C\Pi$ is the *moment* of system (1) at the signal generator (2).

Remark 1. From now on, we refer to the matrix $\underline{Y}$ as the *moment-domain equivalent* of y.

III. Optimal control of WECs: Fundamentals

This section briefly recalls linear control-oriented modelling for a 1-degree-of-freedom (DoF) device[2] (see, for instance, [9]). The equation of motion for such a WEC can be expressed in terms of the following system Σ:

$$\Sigma : \left\{ \ddot{z} = \mathcal{M}\left(-k_{\mathrm{r}} * \dot{z} - s_{\mathrm{h}}z + f_{\mathrm{e}} - u\right),\ y = \dot{z}. \right. \tag{3}$$

where $z : \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto z(t)$, is the device excursion (displacement), $f_{\mathrm{e}} : \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto f_{\mathrm{e}}(t)$, the wave excitation force (external uncontrollable input due to the incoming wave field), $k_r : \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto k_r(t)$, is the (causal) radiation impulse response function containing the memory effect of the fluid response, $s_{\mathrm{h}} \in \mathbb{R}$ denotes the hydrostatic stiffness, which depends upon the device geometry, and $\mathcal{M} \in \mathbb{R}_{>0}$ is the inverse of the generalised mass of the device (see [9]). The control input $u : \mathbb{R}^+ \to \mathbb{R}$, supplied by means of the so-called power take-off (PTO) system, plays a key role in the optimisation of the operation of wave energy devices, as further discussed in Section III-B.

A. Nominal moment-based representation of a WEC

Aiming to consider the results recalled in Section II for this 1-DoF WEC case, the nominal equation of motion characterising the WEC system Σ is written using the following equivalent representation:

$$\Sigma^0 : \left\{ \dot{w} = Aw + B(f_e - k_r * Cw - u),\ \ y^0 = Cw, \right. \tag{4}$$

for $t \in \mathbb{R}^+$, where $w(t) = \begin{bmatrix} z(t) & \dot{z}(t) \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^2$ contains displacement and velocity for the (single) DoF involved in the equation of motion, and the (constant) matrices $A \in \mathbb{R}^{2\times2}$, $B \in \mathbb{R}^2$ and $C^{\mathsf{T}} \in \mathbb{R}^2$ are defined as

$$A = \begin{bmatrix} 0 & 1 \\ -\mathcal{M}s_h & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \mathcal{M} \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathsf{T}}. \tag{5}$$

The mappings corresponding to both external inputs, *i.e.* f_e and u, are written in terms of a signal generator (analogously to the case of equation (2)), *i.e.*

$$\mathcal{G} : \left\{ \dot{\xi} = S\xi,\ f_e = L_e\xi,\ u = L_u\xi, \right. \tag{6}$$

for $t \in \mathbb{R}^+$, with $\xi(t) \in \mathbb{R}^\nu$, $S \in \mathbb{R}^{\nu\times\nu}$ and $L^{\mathsf{T}} \in \mathbb{R}^\nu$. Recalling that ocean waves are commonly represented as a finite sum of harmonics of a (sufficiently small) fundamental frequency ω_0 [10], we define the finite-set $\mathscr{F} = \{p\omega_0\}_{p=1}^{f} \subset \mathbb{R}^+$, with $f \in \mathbb{N}_{\geq 1}$, and let S in (6) be

$$S = \bigoplus_{p=1}^{f} \begin{bmatrix} 0 & p\omega_0 \\ -p\omega_0 & 0 \end{bmatrix}, \tag{7}$$

where $\nu = 2f$, and hence $\lambda(S) = (j\mathscr{F}) \cup (-j\mathscr{F}) \subset \mathbb{C}^0$. Without any loss of generality, the initial condition on the signal generator is chosen as $\xi(0) = \varepsilon_\nu \in \mathbb{R}^\nu$. Finally, we introduce the following two assumptions.

Assumption 1. The pair $(S, L_e - L_u)$ is observable.

Assumption 2. The zero equilibrium of $\dot{w} = Aw - k_r * Cw$ is asymptotically stable.

Assumption 2 is, in practice, without loss of generality (see [11]). We now recall the following result from [2].

Lemma 2. [2] *Suppose Assumptions 1 and 2 hold. Then, the moment-domain equivalent of the output* y^0 *of system* (4) *can be uniquely determined as*

$$\underline{Y}^0 = (L_{\mathrm{e}} - L_{\mathrm{u}})\Phi_{\mathscr{R}}^{\mathsf{T}}, \tag{8}$$

where the matrix $\Phi_{\mathscr{R}} \in \mathbb{R}^{\nu\times\nu}$ *is defined as*

$$\Phi_{\mathscr{R}} = (\mathbb{I}_\nu \otimes C)(S \hat{\oplus} A + \mathscr{R}^{\mathsf{T}} \otimes -BC)^{-1}(\mathbb{I}_\nu \otimes -B), \tag{9}$$

and the block-diagonal operator $\mathscr{R} \in \mathbb{R}^{\nu\times\nu}$*, characterising the radiation effects in the moment-domain, is given by*

$$\mathscr{R} = \bigoplus_{p=1}^{f} \begin{bmatrix} \Re\{K_{\mathrm{r}}(p\omega_0)\} & \Im\{K_{\mathrm{r}}(p\omega_0)\} \\ -\Im\{K_{\mathrm{r}}(p\omega_0)\} & \Re\{K_{\mathrm{r}}(p\omega_0)\} \end{bmatrix}, \tag{10}$$

with $K_{\mathrm{r}} : \mathbb{R} \to \mathbb{C}$*,* $\omega \mapsto K_{\mathrm{r}}(\omega)$*, the (well-defined) Fourier transform of the impulse response function* k_{r}.

B. Optimal control problem for WECs

The WEC control design entails an *energy-maximisation* criterion, where the objective is to maximise the absorbed energy from incoming ocean waves over a finite time interval $\mathcal{T} = [0, T] \subset \mathbb{R}^+$. This procedure can be cast as an OCP, with *objective function* $\mathcal{J} : \mathbb{R} \to \mathbb{R}$, $u \mapsto J(u)$, defined as

$$\mathcal{J}(u) = \frac{1}{T}\int_{\mathcal{T}} u(\tau)\dot{z}(\tau)d\tau. \tag{11}$$

Given the control objective function defined in (11) and the governing dynamics of the WEC in (3), the energy-maximising OCP can be formally posed as follows[3].

Problem 1 (Energy-maximising OCP)**.** *Find an optimal control input* $u^{opt} : \mathcal{T} \to \mathbb{R}$ *such that*

$$u^{opt} = \arg\max_{u}\ \mathcal{J}(u),\ \textit{s.t.} : \textit{WEC dynamics } (3), \tag{12}$$

[2] We consider a 1-DoF device since an analogous analysis can be carried out for multi-DoF devices, by following *e.g.* [8].

[3] State and input constraints are included to the OCP in Section IV.

IV. Moment-based control under input uncertainty

This section details a moment-based energy-maximising control framework which is robust with respect to *input* uncertainty. From now on, the *approximated* wave excitation force signal (also referred as either *wave excitation force estimate* or simply *wave excitation force*, when clear from the context), arising from both estimation and forecasting processes, is denoted as $\tilde{f}_e$. To be precise, suppose now that the input mapping $\tilde{f}_e$ is uncerain, and let the so-called *nominal* signal generator $\mathcal{G}^0$ be defined as,

$$\mathcal{G}^0 : \{\dot{\xi} = S\xi,\ \tilde{f}_e^0 = L_e^0\xi,\ u = L_u\xi, \tag{13}$$

where S and L_u are defined as in (6), and $L_e^0 \in \mathbb{R}^{1\times\nu}$ characterises the *nominal wave excitation force* $\tilde{f}_e^0$.

Remark 2. From now on, the matrix L_e^0 is referred to as the *nominal output vector* for the wave excitation force, associated with the nominal signal generator (13).

Suppose now that the excitation force affected by uncertainty, denoted as $\tilde{f}_e^\Delta$, is defined in terms of ξ as $\tilde{f}_e^\Delta = L_e^\Delta\xi$, with $L_e^\Delta \in \mathbb{R}^{1\times\nu}$. To be precise with respect to the 'type' of (input) uncertainty considered, the definition of the so-called *uncertain signal generator* $\mathcal{G}^\Delta$ is now provided.

Definition 2 (Uncertain signal generator)**.** Let S and L_u be as in (6). The exogenous system $\mathcal{G}^\Delta$ is termed an *uncertain signal generator* if it is described by the set of equations,

$$\mathcal{G}^\Delta : \{\dot{\xi} = S\xi,\ \tilde{f}_e^\Delta = (L_e^0 + L_e^0\Delta)\xi,\ u = L_u\xi, \tag{14}$$

where L_e^0 is as in equation (13), and $\Delta \in \mathbb{R}^{\nu\times\nu}$ is such that,

$$L_e^0\Delta = L_e^\Delta - L_e^0. \tag{15}$$

Definition 2 can be interpreted from a 'traditional' robust control theory viewpoint. In particular, it stems directly from considering that the nominal signal generator $\mathcal{G}^0$ is affected by a *multiplicative output uncertainty* [12] in the output associated with the wave excitation force. This output uncertainty is characterised by a stable and minimal linear time-invariant (LTI) system H^Δ, with input $\tilde{f}_e^0$ (*i.e.* the nominal excitation force), and where the moment-domain equivalent of its output d^Δ can always be uniquely[4] computed as $\underline{D^\Delta} = L_e^0\Delta$, with $\Delta \in \mathbb{R}^{\nu\times\nu}$.

Remark 3. From now on, the matrix L_e^Δ is referred to as the *uncertain output vector* for the wave excitation force, associated with the uncertain signal generator (14).

Remark 4. The computation of the matrix Δ follows immediately from Lemma 1. Furthermore, given the specific structure of the matrices involved in (13), it is straightforward to show that the matrix Δ can always be written as,

$$\Delta = \bigoplus_{p=1}^{f} \begin{bmatrix} {}^p\delta^+ & {}^p\delta^- \\ -{}^p\delta^- & {}^p\delta^+ \end{bmatrix}, \tag{16}$$

where $\{{}^p\delta^+, {}^p\delta^-\}_{p=1}^f \subset \mathbb{R}$.

[4]Uniqueness of $\underline{D^\Delta}$ is a direct consequence of the linearity and internal stability of H^Δ (see Section II).

Not only is Δ always structured as (16), but it can also be fully characterised in terms of a vector $\delta \in \mathbb{R}^\nu$, which is especially useful for the upcoming results. The nature of this vector is formalised in the following. Let $\delta \in \mathbb{R}^\nu$ be the *uncertainty vector*, defined as $\delta = \sum_{p=1}^f e_p^f \otimes \begin{bmatrix} {}^p\delta^+ & {}^p\delta^- \end{bmatrix}^\intercal$. Following Remark 4, it is clear that the matrix Δ can always be written in terms of the uncertainty vector δ, provided a suitable mapping $\delta \mapsto \Delta$ is defined. In particular, the following mapping $\Gamma : \mathbb{R}^\nu \to \mathbb{R}^{\nu\times\nu}$, $\delta \mapsto \Gamma(\delta)$, is proposed:

$$\Gamma(\delta) = \bigoplus_{p=1}^{f} \Big(\delta^\intercal e_{2p-1}^{2f}\Big)\mathbb{I}_2 + \Big(\delta^\intercal e_{2p}^{2f}\Big)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Delta. \tag{17}$$

Remark 5. The mapping Γ is *linear* in the argument δ. As discussed in Section IV-B, the linearity of the mapping Γ plays a fundamental role in the tractability of the proposed robust moment-based energy-maximising procedure.

To pose a robust moment-based energy-maximising formulation for WECs under input uncertainty, we now provide an expression for the moment-domain equivalent of the device velocity $\underline{Y}$, considering the uncertain wave excitation force $\tilde{f}_e^\Delta$. This is, indeed, straightforward to compute from the result of Lemma 2, by introducing the following standing assumption.

Assumption 3. The pair $(S, L_e^0 + L_e^0\Delta - L_u)$ is observable.

Suppose Assumptions 2 and 3 hold. Then, the moment-domain equivalent of the WEC velocity under the effect of input uncertainty can be directly computed via Lemma 2, *i.e.*

$$\underline{Y} = (L_e^0 + L_e^0\Delta - L_u)\Phi_{\mathscr{R}}^\intercal, \tag{18}$$

where the matrix $\Phi_{\mathscr{R}}^\intercal$ is defined as in (9).

A. Formulation of the OCP using moments

Given a fixed uncertainty vector δ, the results presented in Section IV can be used to approximate the energy-maximising OCP of Problem 1 for the case of an uncertain wave excitation force $\tilde{f}_e^\Delta$, by making explicit use of the connection between moments and the steady-state behaviour of the corresponding perturbed system. In the following, we show that the OCP (12) can be approximated in terms of a finite-dimensional *strictly concave* QP problem, using the associated moment-based representations.

Assumption 4. Let $\psi_p = C(jp\omega_0\mathbb{I}_2 - T_p)^{-1}B$, with $T_p = A - BK_r(p\omega_0)C$. Then, $\Re\{\psi_p\} > 0$, for all $p \in \mathbb{N}_f$.

Proposition 1. *Consider the OCP* (12) *and suppose Assumptions 2 and 3 hold. Then, given a fixed uncertainty vector δ, the optimal control law u^{opt}, which maximises the objective function $\mathcal{J}$ for the uncertain signal generator $\mathcal{G}^\Delta$, can be approximated in the moment-domain as $u^{opt} = L_{\mathrm{u}}^{opt}\xi$, where L_{u}^{opt} is the solution of the QP problem*

$$L_{\mathrm{u}}^{opt} = \arg\max_{L_{\mathrm{u}}^\intercal \in \mathbb{R}^\nu} -\frac{1}{2}L_{\mathrm{u}}\Phi_{\mathscr{R}}^\intercal L_{\mathrm{u}}^\intercal + \frac{1}{2}L_{\mathrm{e}}^0(\mathbb{I}_\nu + \Gamma(\delta))\Phi_{\mathscr{R}}^\intercal L_{\mathrm{u}}^\intercal \tag{19}$$

Proposition 2. *Suppose Assumption 4 holds. Then, the QP problem defined in* (19) *has a unique global energy-maximising solution.*

Remark 6. The condition expressed in Assumption 4 coincides with the condition for global optimality for the nominal WEC system presented in [2]. In particular, $\text{sign}\{\Re\{\psi_p\}\} = \text{sign}\{\Re\{K(p\omega_0)\}\} > 0$ holds for all $p \in \mathbb{N}_f$ as a consequence of the passivity property of radiation forces [9], so that Assumption 4 is, in practice, without loss of generality.

Remark 7. Unlike the case of [4], the QP formulation presented in (19) is *always strictly concave independently on the definition of* the uncertainty δ: Given *any* fixed uncertainty vector δ, there exists a *unique global energy-maximising solution* for the moment-based WEC optimal control problem under input uncertainty as formulated herein.

B. Robust formulation under input uncertainty

In this section, the moment-domain optimisation problem (19) is re-formulated based on the underlying principles of robust optimisation theory, developed in key studies such as [6]. The underpinning concept behind this approach originates in the field of decision theory, and is known as *Wald's Minimax (Maximin) paradigm* [13], also commonly referred to as the *Worst-Case Performance* (WCP) method. We now introduce the following standing assumption.

Assumption 5. The uncertainty vector δ is such that $\delta \in \mathcal{P}$, where $\mathcal{P} \subset \mathbb{R}^\nu$ is a convex $\mathcal{V}$-polytope[5], defined as the convex hull of a finite set of N_V points (vertices) in space $V_\delta = \{\delta_1^V, \dots \delta_{N_V}^V\}$, *i.e.* $\mathcal{P} = \text{conv}\{V_\delta\}$.

Problem 2 (Robust energy-maximising OCP under input uncertainty)**.** *Suppose Assumptions 2-5 hold. Find the optimal WCP control input $u^{RC} = L_\text{u}^{RC}\xi$, with L_u^{RC} the solution of the maximin problem*

$$L_\text{u}^{RC} = \arg \max_{L_\text{u}^\intercal \in \mathbb{R}^\nu} \arg \min_{\delta \in \mathcal{P}} -\frac{1}{2} L_\text{u} \Phi_{\mathscr{R}}^\intercal L_\text{u}^\intercal + \frac{1}{2} L_\text{e}^0 (\mathbb{I}_\nu + \Gamma(\delta)) \Phi_{\mathscr{R}}^\intercal L_\text{u}^\intercal. \tag{20}$$

Remark 8. Assumption 5 plays a fundamental role in the definition of Problem 2: as a direct consequence of the linearity of the mapping Γ, (20) is *affine* in δ. Taking into account that the QP problem in L_u is always strictly concave for any δ under Assumption 4 (see also Remark 7), the solution of the moment-based robust formulation under input uncertainty (20) is reached on the convex hull of the uncertainty set $\mathcal{P}$. Since $\mathcal{P}$ is convex by Assumption 5, the solution of (20) lies precisely at one of the vertices of $\mathcal{P}$, and it is sufficient to solve (20) only for the finite-set of N_V vertices V_δ (see [14], [15]).

Two main conclusions can be directly extracted from Remark 8. Firstly, the robust energy-maximising framework, defined in Problem 2, *always admits a unique globally optimal* solution. Secondly, the optimisation problem to be solved is tractable, *i.e.* it is sufficient to solve (20) for a small number of 'points', characterising the vertices of the uncertainty polytope. This directly implies that (20) can be solved using efficient minimax optimisation routines.

[5]The reader is referred to [12] for further detail in convex polytopes and their use in robust control applications.

C. Handling of state and input constraints

We now consider constraints on both the displacement and velocity of the WEC, and the exerted control force, and we map these using their respective moment-domain equivalents under input uncertainty[6], as

$$\mathscr{C} : \begin{cases} |z(t)| \leq Z_\text{max}, \\ |\dot{z}(t)| \leq Y_\text{max}, \\ |u(t)| \leq U_\text{max}, \end{cases} \mapsto \begin{cases} |\underline{Y} S^{-1} \xi(t)| \leq Z_\text{max}, \\ |\underline{Y} \xi(t)| \leq Y_\text{max}, \\ |L_u \xi(t)| \leq U_\text{max}. \end{cases} \tag{21}$$

with $t \in \mathcal{T}$, and where $\{Z_\text{max}, Y_\text{max}, U_\text{max}\} \subset \mathbb{R}^+$. Let $\mathcal{T}_c = \{t_i\}_{i=1}^{N_c} \subset \mathcal{T}$, be a finite set of (specified) uniformly-spaced time instants, with $N_c \in \mathbb{N}_{\geq 1}$. The constraints defined in (21) can be enforced at the set $\mathcal{T}_c$, *i.e.* using a collocation approach, via the definition of the following matrices: Let $\Lambda \in \mathbb{R}^{\nu \times N_c}$ and $\Upsilon \in \mathbb{R}^{\nu \times 2N_c}$ be

$$\Lambda = \begin{bmatrix} \xi(t_1) & \dots & \xi(t_{N_c}) \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} \Lambda & -\Lambda \end{bmatrix}. \tag{22}$$

Remark 9. Given that it is sufficient to solve the robust formulation defined in Problem 2 *only* at the finite-set of vertices of the convex polytope $\mathcal{P}$ (see Remark 8), we incorporate state constraints into Problem 2 such that these are satisfied at every element of the set V_δ. This effectively ensures that the state constraints in (21) are consistently fulfilled *for every* $\delta \in \mathcal{P}$ [14], [15].

Proposition 3. *Consider Problem 2 and the set of constraints $\mathscr{C}$ in* (21)*. The state and (control) input constrained robust moment-based energy-maximising control law $u^{RC} = L_\text{u}^{RC}\xi$, can be computed in terms of the maximin problem,*

$$L_\text{u}^{RC} = \arg \max_{L_\text{u}^\intercal \in \mathbb{R}^\nu} \arg \min_{\delta \in V_\delta} -\frac{1}{2} L_\text{u} \Phi_{\mathscr{R}}^\intercal L_\text{u}^\intercal + \frac{1}{2} L_\text{e}^0 (\mathbb{I}_\nu + \Gamma(\delta)) \Phi_{\mathscr{R}}^\intercal L_\text{u}^\intercal, \tag{23}$$

$$s.t. : \left\{ L_\text{u} \mathcal{A}_z^{RC} \leq \mathcal{B}_z^{RC},\ L_\text{u} \mathcal{A}_{\dot{z}}^{RC} \leq \mathcal{B}_{\dot{z}}^{RC},\ L_\text{u} \mathcal{A}_u \leq \mathcal{B}_u, \right.$$

where the pair of matrices $(\mathcal{A}_u, \mathcal{B}_u)$ is as in (25)*,*

$$\begin{aligned} \mathcal{A}_z^{RC} &= \mathcal{A}_z \otimes \mathbf{1}_{1 \times N_V}, \quad \mathcal{B}_z^{RC} = \begin{bmatrix} \mathcal{B}_z^{\delta_1^V} & \dots & \mathcal{B}_z^{\delta_N^V} \end{bmatrix}, \\ \mathcal{A}_{\dot{z}}^{RC} &= \mathcal{A}_{\dot{z}} \otimes \mathbf{1}_{1 \times N_V}, \quad \mathcal{B}_{\dot{z}}^{RC} = \begin{bmatrix} \mathcal{B}_{\dot{z}}^{\delta_1^V} & \dots & \mathcal{B}_{\dot{z}}^{\delta_N^V} \end{bmatrix}, \end{aligned} \tag{24}$$

with $\mathcal{A}_z = -\Phi_{\mathscr{R}}^\intercal S^{-1} \Upsilon$, $\mathcal{A}_{\dot{z}} = -\Phi_{\mathscr{R}}^\intercal \Upsilon$, and

$$\begin{aligned} \mathcal{B}_z^{\delta_i^V} &= Z_{max} \mathbf{1}_{1 \times 2N_c} + L_\text{e}^0 (\mathbb{I}_\nu + \Gamma(\delta_i^V)) \mathcal{A}_z, \\ \mathcal{B}_{\dot{z}}^{\delta_i^V} &= Y_{max} \mathbf{1}_{1 \times 2N_c} + L_\text{e}^0 (\mathbb{I}_\nu + \Gamma(\delta_i^V)) \mathcal{A}_{\dot{z}}, \end{aligned} \tag{25}$$

for all $\delta_i^V \in V_\delta$.

V. Case study

This section analyses and illustrates the performance of the robust moment-based energy-maximising control framework, presented in this paper, for the case of WECs under input uncertainty. To be precise, a spherical heaving point absorber WEC is considered, with a radius of 2.5 [m] (see [2] for a detailed description of the device). Motivated by the analysis

[6]Note that the moment-domain equivalent of the displacement z can be expressed as $\underline{Y} S^{-1}$, following the result of [16, Proposition 1].

performed in [17], the existence of errors in the instantaneous phase (*i.e.* time-delays) of the wave excitation force is first considered, in Sections V-A and V-B, to characterise the input uncertainty. This error source can cause significant losses in terms of energy-maximising performance for the case of the nominal controller, if the time delay is sufficiently large [17]. Finally, Section V-C considers constant deviations in instantaneous amplitude, showing that the presented robust strategy is able to consistently fulfill constraint specifications for the defined uncertainty set.

A. On the definition of the uncertainty polytope $\mathcal{P}$

We outline a methodology to compute the polytope characterising input uncertainty, arising from the presence of errors in the instantaneous phase of the wave excitation force. Note that an analogous procedure can be followed for any other source of uncertainty, as long as it can be represented in terms of an LTI system H^{Δ} (as per discussed in Section IV).

Suppose the dynamic matrix $S \in \mathbb{R}^{\nu\times\nu}$ is given, and let $\tilde{f}_e^0$ be the nominal excitation force, generated in terms of the nominal signal generator $\mathcal{G}^0$ (see equation (13)). Let the 'perturbed' excitation be defined as $\tilde{f}_e^{\Delta} = \tilde{f}_e^0(t-\tau)$, *i.e.* as a 'delayed' version of the nominal wave excitation $\tilde{f}_e^0$. Suppose $\tau \in \mathcal{F} \subset \mathbb{R}$, where, for this case study, $\mathcal{F} = [-1.25, 0.75]$ [s] (which corresponds with a shift in time τ between -1.25 and 0.75 seconds). To consider the robust moment-based control framework the input uncertainty generated by the set $\mathcal{F}$ needs to be written in terms of an uncertain polytope $\mathcal{P}$. Before proposing a method to compute such a set $\mathcal{P}$, note that the following relation,

$$\tilde{f}_e^0(t-\tau) = L_e^0 e^{S(t-\tau)}\xi(0) = L_e^0 e^{-S\tau}\xi(t), \qquad (26)$$

holds, so that, for a given $\tau \in \mathcal{F}$, the matrix $\Delta \in \mathbb{R}^{\nu\times\nu}$, characterising the so-called uncertain signal generator $\mathcal{G}^{\Delta}$ (see Definition 2 and equation (15)), can be directly obtained as $\Delta = e^{-S\tau}$. Based on the discussion provided above, we propose the following procedure.

1) Discretise the set $\mathcal{F}$, *i.e.* construct the finite-set $\mathcal{F}^{\Delta} = \{\tau_i\}_{i=1}^{N^{\Delta}} \subset \mathcal{F}$, containing $N^{\Delta} \in \mathbb{N}_{\geq 1}$ possible values for the wave excitation input delay.
2) Compute the set of matrices $\{\Delta_i\}_{i=1}^{N^{\Delta}} \subset \mathbb{R}^{\nu\times\nu}$, corresponding with each delay τ_i in the set $\mathcal{F}^{\Delta}$, using the relation posed in equation (26).
3) Construct the set of *uncertainty vectors* $\{\delta_i\}_{i=1}^{N^{\Delta}} \subset \mathbb{R}^{\nu}$, corresponding with each matrix in the set $\{\Delta_i\}_{i=1}^{N^{\Delta}}$, which can be done straightforwardly following the structure of Δ in (16).
4) Finally, compute the polytope $\mathcal{P}$ as the *convex hull* of the set $\{\delta_i\}_{i=1}^{N^{\Delta}}$, and extract the corresponding set of vertices, *i.e.* construct the set V_{δ}.

B. Performance: Energy-absorption in regular seas

This section evaluates the performance of the robust moment-based framework proposed throughout Section IV, driven by a *regular* uncertain wave excitation. From now on, the following convention is adopted, to define two different assessment (performance) scenarios:

- *Nominal performance* (nominal control - uncertain input): the optimal control input is computed using the *nominal* wave excitation force input, and the WEC system is driven by the uncertain signal generator $\mathcal{G}^{\Delta}$, characterised by the uncertainty vector δ.
- *Robust performance* (robust control - uncertain input): the optimal control input is computed using the *robust* approach proposed in this paper, *i.e.* the control law explicitly considers the knowledge of the (input) uncertainty polytope $\mathcal{P}$, and the WEC system is driven by the uncertain signal generator $\mathcal{G}^{\Delta}$, characterised by δ.

Figure 1 shows (state and input unconstrained[7]) *nominal* and *robust* performance results, in terms of energy absorption for different 'levels' of uncertainty in the delay parameter τ (in seconds), where the input waves are regular, with height $H_w = 2$ [m], and different wave periods $T_w \in [6, 10]$ [s]. The case of nominal performance is denoted with black circles. The robust performance (denoted with diamonds), corresponds with the robust moment-based controller computed with the uncertainty polytope $\mathcal{P}$, considering a signal generator according to each T_w analysed. Note that the WCP occurs when the delay takes the boundary value -1.25 [s].

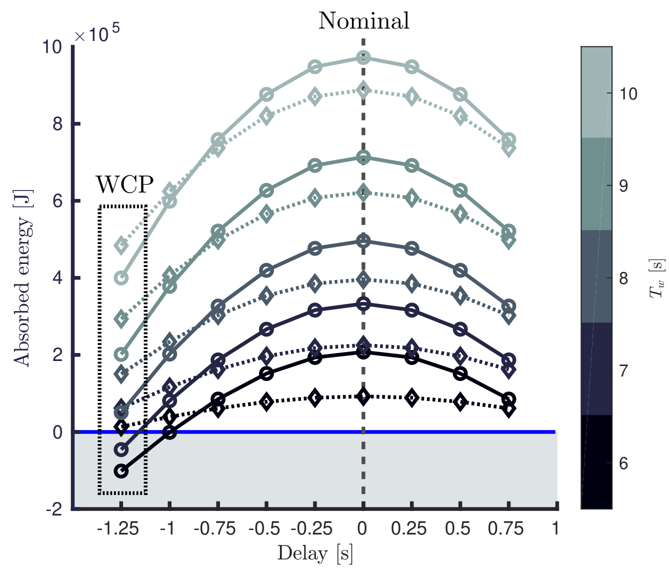

Fig. 1. Nominal performance (circles), and robust performance (diamonds), as a function of the input delay τ (in seconds), for input waves with $T_w \in [6, 10]$ [s]. The performance for the nominal wave input (*i.e.* with zero delay) is indicated with a vertical dashed line. A value below zero (solid-blue line) indicates negative energy absorption.

Remark 10. For $\tau < -1$, the nominal performance drops below zero, *i.e.* the controlled device 'drains' energy from the grid, which is effectively consistent with the sensitivity results presented in [17]. In contrast, the robust controller is *always* able to deliver positive energy absorption even in a worst-case scenario, for each of the wave periods analysed. This is effectively a fundamental feature of the proposed framework, which always guarantees positive results in terms of absorbed energy, as a consequence of WCP approach.

[7] Almost identical conclusions can be drawn for the constrained case.

Remark 11. Following the discussion provided immediately above, note that the robust performance case is conservative *by definition*, given that it optimises for the worst-case scenario, in terms of the (defined) uncertainty.

C. Performance: Constraint satisfaction in irregular seas

We now analyse the performance of the robust moment-based strategy in terms of constraint satisfaction, under irregular wave excitation. In particular, we analyse the performance of the strategy subject to *state* constraints. The uncertainty is now considered to be defined in terms of the *instantaneous amplitude* of the excitation force, *i.e.* $\tilde{f}_e^{\Delta} = \tau \tilde{f}_e^0(t)$, with $\tau \in \mathcal{F} \subset \mathbb{R}$, and where $\mathcal{F} = [0.75, 1.25]$.

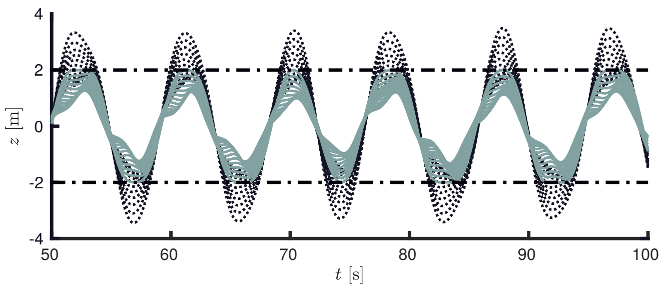

Fig. 2. Displacement results for nominal (dotted) and robust (solid) moment-based control techniques, for different uncertain signal generators, all of which have been computed by a random generation of input uncertainty. The state constraints are indicated with dash-dotted horizontal lines.

The results presented in this section are computed considering irregular input waves, generated from a JONSWAP SDF S_w [18] with significant wave height $\bar{H}_w = 2$ [m], peak period $\bar{T}_w = 9$ [s], and peak enhancement factor $\gamma = 3.3$. Figure 2 shows time-snippets of displacement, for both nominal (dotted), and robust (solid) moment-based control strategies, for different uncertain signal generators, all of which have been computed by a random generation of uncertainty vectors δ in the defined polytope $\mathcal{P}$. To be precise, each of these systems corresponds with a different value of instantaneous amplitude, lying in the set $\mathcal{F}$. Unlike the motion results arising from the nominal moment-based controller, the presented robust framework is able to consistently respect the defined state constraint $Z_{\text{max}} = 2$ [m].

VI. Conclusions

We present a robust moment-based approach which effectively incorporates input uncertainty in the WEC OCP, by a suitable definition of an uncertainty polytope in the moment-domain, and exploiting the underpinning concept of the WCP method. The proposed control law is computed in terms of an optimisation procedure, formulated as a minimax problem, which has to be solved *only* at the set of vertices of the polytope, as a result of the nature of the objective function, arising from mapping the state variables, and both the external and control inputs, into their respective moment-based representations. As a result, the presented framework provides a computationally efficient robust optimal control method, which is able to maximise energy absorption while consistently respect state and input constraint limitations under the presence of input uncertainty.

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