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RESEARCH

An Approach to Growth Mechanics Based on the Analytical Mechanics of Nonholonomic Systems

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Abstract
Motivated by the convenience, in some biomechanical problems, of interpreting the mass balance law of a growing medium as a nonholonomic constraint on the time rate of a structural descriptor known as growth tensor, we employ some results of analytical mechanics to show that such constraint can be studied variationally. Our purpose is to move a step forward in the formulation of a field theory of the mechanics of volumetric growth by defining a Lagrangian function that incorporates the nonholonomic constraint of the mass balance. The knowledge of such Lagrangian function permits, on the one hand, to deduce the dynamic equations of a growing medium as the result of a variational procedure known as Hamilton–Suslov Principle (clearly, up to non-potential generalized forces that are accounted for by extending this procedure), and, on the other hand, to study the symmetries and conservation laws that pertain to a given growth problem. While this second issue is not investigated in this work, we focus on the first one, and we show that the Euler–Lagrange equations of the considered growing medium, which describe both its motion and the evolution of the growth tensor, can be obtained by reformulating a variational method developed by other authors. We discuss the main features of this method in the context of growth mechanics, and we show how our procedure is able to improve them.

Keywords Growth mechanics · Nonholonomic constraints · Quasi-velocities · Hamilton–Suslov variational principle

Mathematics Subject Classification 74Axx · 74Cxx · 37J60 · 70Hxx · 74L15

1 Introduction

The primary purpose of this work is to construct a *quasi-variational theory* of the mechanics of volumetric growth. By "quasi-variational" we mean that, even though we formulate a field

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theory of growth, with its own Lagrangian density function and action functional, we also need to consider peculiar interactions that do not admit a potential, not even in generalized sense. These interactions can rather be described by force-like entities [29] (see also [15], and [54, 55] for the rationale behind this approach), which have to be deduced constitutively when they are regarded as internal, or assigned phenomenologically when they are rated as external. In both cases, however, they can be so complicated and problem specific, that they cannot be determined from a single scalar function. For this reason, such forces are referred to as non-potential, or *"polygenic"* in the terminology of [67], while those that can be obtained by differentiation of a generalized potential are said to be *"monogenic"*, or "*single-generated*" [67].

While non-potential forces must be allotted in a model of growth to study biologically relevant situations, the Lagrangian density function of a growing medium is *"apodeictic"*,[1] i.e., it represents a model that is "true" by itself, since it is assigned on the basis of all the hypotheses done on the body (see also [24]). In other words, the Lagrangian density function is constructed so as to account for all the items of information that can be "condensed" in one (pseudo-)scalar quantity. Therefore, if the body is assumed to be hyperelastic, and if one can find, or design, interactions that admit generalized potentials, such as inertial forces (although they are often negligible), gravity, and forces acting on the body's internal structure, then the Lagrangian function will consist of the body's kinetic energy, strain energy, and all the other potential terms that participate in the body's dynamics.

1.1 Why a Lagrangian Theory of Growth: Advantages and Problems

Aside from mathematical elegance, a Lagrangian theory of growth has some advantages. To mention a few: it is self-contained; up to non-potential forces, it encloses both the *"direct"* and the *"configurational"* dynamics of a body [32, 54, 96], and it unfolds each of them depending on the variations that are performed on the arguments of the body's Lagrangian function (here, the adjectives *"direct"* and *"configurational"* are intended as in [32], and [54], respectively); through Noether's Theorem (see, e.g., [34, 53, 57, 85, 96]), it provides the natural device for investigating the symmetries of a body, the related conservation laws, and the symmetry breaking brought about by growth along with its consequences on the body's overall dynamics [53]. In addition, a field-theoretical approach to growth can be inherently geometrized [45, 108, 110], so as to account for the phenomenological aspects connected with the incompatibility of the distortions induced by growth [101], and it supplies the basis for including other theories, like that of micromorphic media [59].

Yet, to benefit from all these advantages, one has to answer the question as to whether the Lagrangian density function of a growing body is able to account also for the core feature of the mechanics of volumetric growth, which is the presence of mass sources that, acting in the body's interior, trigger its variation of mass [104]. These sources are positive when mass increases, as is the case for cell proliferation in tumors, and negative when mass is depleted due to removal processes such as necrosis or apoptosis [6, 83]. In both situations, they appear in the body's mass balance law, and must be characterized very accurately in order to capture the combination of the biochemical, biophysical, and mechanical stimuli from which they originate (see, e.g., [28, 29]).

[1] We are thankful to Prof. Gaetano Giaquinta (1945–2016) for teaching us the meaning of this word in relation to Lagrangian functions.

1.2 Mass Sources: A Posteriori and a Priori Approaches

To our knowledge, two distinct paths can be followed for characterizing mass sources in accordance with experimental evidences. As reported in [48], the functional form expressing a mass source is called *"growth law"*, and it can be determined by employing *"a posteriori approaches"* or *"a priori approaches"*. Both rely on the Bilby–Kröner–Lee (BKL) decomposition of the deformation gradient tensor of the growing medium under study. The anelastic factor of such decomposition, termed *growth tensor*, is identified with the descriptor of the medium's structural transformations accompanying its growth. A crucial aspect is that the mass balance law of the medium can be recast in the form of a relationship between the mass source active in it and the trace of the time rate of the growth tensor (see, e.g., [3, 5, 32, 73, 104]).

In the a posteriori approaches, an initial and boundary value problem is formulated for determining both the motion and the growth tensor of the body, and, once the growth tensor is known, the source of mass is obtained *"a posteriori"* [48, 49] by setting it equal to the trace of the growth tensor time rate [3, 32, 53]. It should be noticed that, even describing the evolution of the growth tensor very accurately, the growth laws obtained with the a posteriori approaches, being calculated quantities, may exhibit discrepancies with the ones observed experimentally.

In the a priori approaches, the growth law is prescribed by the modeler to reproduce experiments [7, 9, 44], to comply with phenomenology [1, 2, 4–6, 20, 73], or to test the response of a medium to a mass source designed to match some target biomechanical behavior. This may occur, for instance, in control problems, or when a specific medicament is analyzed. In all these cases, using the knowledge of the given growth law in the mass balance of the medium under study amounts to imposing a condition on the time rate of its growth tensor. If the latter is viewed as a generalized kinematic variable [28, 29], this condition acquires the meaning of a *constraint* [47, 50]. In particular, unless very specific growth laws are considered, this constraint can only be expressed as a differential relationship, and is thus classified as *nonholonomic* [40, 67, 90, 94], i.e., it cannot be obtained by time differentiation of a scalar function of the sole growth tensor, material points and time. We recall that, on the contrary, a constraint is said to be holonomic when the converse is true [67].

As discussed in [47, 50], interpreting the mass balance of a growing medium as a nonholonomic constraint on the growth tensor ensures that the evolution of this quantity complies with the growth law taken as target, while granting the freedom of modeling other interactions as necessary. Hence, no further restrictions are placed on the growth tensor, if unneeded (see, e.g., [7, 9], and [47] for some remarks on this issue), and its dynamics is dictated by the constraint and the balance of the configurational forces obtained through the quasi-variational procedure outlined in the forthcoming sections.

1.3 Nonholonomic Constraints: Kozlov's Method and Its Modifications

The considerations reported so far lead to the fundamental question as to whether a nonholonomic constraint can be handled *variationally*. While the answer is affirmative for holonomic constraints, which can be appended to the Lagrangian function of a given mechanical system through the Lagrange multiplier technique (see, e.g., [67]), the extension of such procedure to nonholonomic constraints is not trivial, and has been the subject of a whole branch of literature. In particular, this was the main point of the works by Kozlov [61–65], who developed a formulation of analytical mechanics in which it was claimed that Hamilton's Principle of stationary action can be employed also to Lagrangian functions augmented with the Lagrange multiplier method applied to nonholonomic constraints. In the literature, Kozlov's approach is termed "Vakonomic Method" (VM).

If, on the one hand, exploiting Kozlov's idea would allow to cast the nonholonomic constraint on the growth tensor in the variational picture which we are looking for —and that, as previously discussed, is the primary scope of our work—, on the other hand, a lot of caution is necessary to "import" Kozlov's method *as is*. Indeed, many critiques have been raised towards it (see, e.g., [60, 69, 70]), because, in several cases, it has been proven to be inconsistent with the well consolidated results of the analytical mechanics of nonholonomic systems, based on *Extended Hamilton's Method* or, equivalently, on the *d'Alembert–Lagrange Principle* [38, 67]. In the sequel, we shall refer to both approaches as the *"Traditional Non-Holonomic Method"* (TNHM) [95].

In spite of the problems related to the VM, a work by Llibre et al. [71] proposes a *"Modified Vakonomic Method"* (MVM) [71], which, for the class of constraints analyzed by the authors, is able to save the idea of handling nonholonomic constraints variationally. This is achieved by raising a technical issue: namely, the variations performed on the generalized velocities restricted by the considered nonholonomic constraints should not be taken equal to the time derivatives of the variations of the associated Lagrangian parameters. This non-commutativity between time differentiation and variation of a given kinematic descriptor is known as *"transpositional relation"* [58, 71, 90], and constitutes a fundamental concept of the mechanics of nonholonomic systems.

Llibre et al. [71] employ transpositional relations in conjunction with a variational procedure referred to as *Hamilton–Suslov Variational Principle* [103, 106], which they apply to a Lagrangian function augmented with the considered nonholonomic constraints, just as Kozlov would do, but taking the variations on the system's generalized velocities consistently with the transpositional relations. Moreover, Llibre et al. [71] develop their MVM in two ways, which they formalize in two corresponding theorems (see Theorem 1 and Theorem 3 of [71]). In the present work, we are interested in comparing our approach with the formulation of the MVM provided in their Theorem 1, and in studying how it applies to our growth problem. Hence, from here on, Llibre et al.'s MVM [71] refers to their Theorem 1.

1.4 Main Novelties of Our Work

While the MVM has been recently reviewed in [95], in the present work we investigate whether the MVM can be used for handling variationally the nonholonomic constraint placed on the growth tensor. Although for this purpose we take much inspiration from [71], we find that we need to reformulate it remarkably in order to reach our goal. Indeed, rather than adhering to the theory presented in [71], we follow a different path, which, to a certain extent, could be regarded as the "inverse" of the one developed in [71]. However, also other noticeable differences arise, and the main novelties of our work are:

- N1. Upon considering the mass balance law of the growing medium under study as a nonholonomic constraint on the growth tensor [47, 50], we determine the transpositional relations associated with it by having recourse to the concept of *quasi-velocities* [13, 90], which we suitably adapt to the problem at hand. We remark that quasi-velocities constitute a pillar of the analytical mechanics of nonholonomic systems, but, to the best of our knowledge, they have not been employed to describe growth, yet. For our purposes, we use the algebra of fourth-order tensors, as shown in Sects. 3.1 and 3.2.
- N2. We show that, by means of our reformulation of the MVM by Llibre et al. [71], it is possible to obtain the full equivalence between our approach and the TNHM (see Sect. 4). This is the *core result* of our work because, starting from the dynamic equations of a

growing body, written in the system of the quasi-velocities (Sect. 4.1), it allows us to conclude that one can determine a Lagrangian density function even in the presence of the nonholonomic constraint on the growth tensor (Sect. 4.2).

N3. We analyze in detail the quasi-static case, since it is the most relevant one in the biomechanical problems of interest, and we show that our method is able to recover other formulations [3, 29, 84, 92] (Sect. 4.3).

N4. We highlight the main differences between our results and those of Llibre et al. [71], and we provide a theorem and a corollary to state the conditions under which the latter ones can be used for modeling growth.

To present our results, we review some well-established formulations of growth mechanics based on the Principle of Virtual Work [3, 18, 19, 28, 29, 48–50, 92] (Sect. 2.3.1) and the Extended Hamilton Method [11, 53, 67] (Sect. 2.3.2). Although both of them are rather consolidated, it is important for us to recapitulate their most essential logical steps to compare the resulting dynamic equations with those obtained in the present work (Sect. 4).

2 Growth and Nonholonomic Continuum Systems

To express the ideas presented in the sequel, it is convenient to start with the presentation of the main notation used throughout this work.

2.1 Notation

Let us denote by $\mathscr{B}$ the reference placement of the medium under investigation (an open subset of the three-dimensional Euclidean space $\mathscr{S}$), by $\partial\mathscr{B}$ its boundary, and by $\mathscr{I}$ the *time line* [81].

In our setting, $\mathscr{B}$ is assumed to be a smooth differentiable manifold, endowed, for all $X \in \mathscr{B}$, with the metric tensor $\boldsymbol{G}(X) : T_X\mathscr{B} \to T_X^*\mathscr{B}$, where $T_X\mathscr{B}$ and $T_X^*\mathscr{B}$ are the tangent space and cotangent space of $\mathscr{B}$ at $X \in \mathscr{B}$, respectively. For future use, we also introduce the tangent bundle $T\mathscr{B} := \cup_{X\in\mathscr{B}}\big(\{X\} \times T_X\mathscr{B}\big)$ and the cotangent bundle $T^*\mathscr{B} := \cup_{X\in\mathscr{B}}\big(\{X\} \times T_X^*\mathscr{B}\big)$.

By letting $(Z^I)_{I=1}^3$ and $(X^A)_{A=1}^3$ be a system of Cartesian and curvilinear coordinates, and $(\Phi^I)_{I=1}^3$ the collection of real-valued C^∞-functions such that $Z^I = \Phi^I(X^1, X^2, X^3)$, for $I = 1, 2, 3$, with non-singular Jacobian $[\partial_K \Phi^I]_{I,K=1}^3$ [81], the components of $\boldsymbol{G}$ are given by $G_{AB} = \delta_{IK}\partial_A\Phi^I\partial_B\Phi^K$, where δ_{IK} is the Kronecker Delta [81]. Together with $\boldsymbol{G}$, $\mathscr{B}$ is endowed with an affine connection, which, for our purposes, can be taken equal to the one induced by the chosen curvilinear coordinates.

Let us consider the list of the kinematic and space-time variables that are necessary for our minimal description of the medium's volumetric growth:

$$\natural := (\chi, D\chi, \boldsymbol{F}, \boldsymbol{K}; \dot{\chi}, \dot{\overline{D\chi}} \equiv \mathrm{Grad}\dot{\chi}, \dot{\boldsymbol{F}}, \dot{\boldsymbol{K}}; \mathcal{X}, \mathcal{T}). \tag{1}$$

Each entry of $\natural$ is a function defined over the Cartesian product $\mathscr{B} \times \mathscr{I}$, and valued in an appropriate set of points, or in a vector or tensor space. To be specific, the following identifications apply:

1. $\chi : \mathscr{B} \times \mathscr{I} \to \mathscr{S}$ defines, for varying time $t \in \mathscr{I}$, the one-parameter family of embeddings $\chi(\cdot, t) : \mathscr{B} \to \mathscr{S}$ mapping the points $X \in \mathscr{B}$ in the three-dimensional Euclidean space $\mathscr{S}$ at time $t \in \mathscr{I}$.

2. For each pair $(X,t) \in \mathscr{B} \times \mathscr{I}$, $D\chi(X,t) : T_X\mathscr{B} \to T_{\chi(X,t)}\mathscr{S}$ is the two-point tensor defining the Jacobian tensor of $\chi(\cdot,t)$ at $X \in \mathscr{B}$. Here, $T_{\chi(X,t)}\mathscr{S}$ is the tangent space of $\mathscr{S}$ at $\chi(X,t) \in \mathscr{S}$. The tensor $D\chi(X,t)$ represents the deformation gradient tensor of the body, and with respect to two local coordinate systems, covering a neighborhood of $X \in \mathscr{B}$ and a neighborhood of $x = \chi(X,t) \in \mathscr{S}$, respectively, the components of $D\chi$ are given by the partial derivatives $[D\chi]^a{}_A \equiv \partial_A \chi^a \equiv \partial\chi^a/\partial X^A$, with $a, A = 1, 2, 3$.
3. $\boldsymbol{F}$ is an "auxiliary" deformation gradient tensor field, which, for our purposes, is regarded as a generalized kinematic variable on its own. Later, it will be identified with $D\chi$. This is done in order to unfold a variational procedure similar to the Hu–Washizu variational principle [14].
4. $\boldsymbol{K}$ is referred to as *growth tensor* (see, e.g., [5]), and represents the time-dependent anelastic tensor field describing the structural distortions associated with growth [5, 45]. The tensor $\boldsymbol{K}(X,t)$ maps the vectors of $T_X\mathscr{B}$ into the linear vector space, denoted by $\mathscr{N}_X(t)$ [26, 52], that defines the *natural state* of $T_X\mathscr{B}$ at time $t \in \mathscr{I}$ (see, e.g., [18, 29, 45, 102]). Hence, we can write $\boldsymbol{K}(X,t) : T_X\mathscr{B} \to \mathscr{N}_X(t)$. We recall that $T_X\mathscr{B}$ is referred to as *"body element"* in [29]. For a given $t \in \mathscr{I}$, $\mathscr{N}(t) := \cup_{X\in\mathscr{B}}\big(\{X\} \times \mathscr{N}_X(t)\big)$ denotes the bundle of linear spaces representing the natural state of the body at time t. Once $\mathscr{N}(t)$ is introduced, we indicate with $\boldsymbol{K}(\cdot,t)$ the tensor field $\boldsymbol{K}(\cdot,t) : \mathscr{B} \to \mathscr{N}(t) \otimes T^*\mathscr{B}$. Moreover, we also define the collection of natural states $\mathscr{N} := \cup_{t\in\mathscr{I}}\big(\cup_{X\in\mathscr{B}}\big(\{X\} \times \mathscr{N}_X(t)\big)\big)$.
5. $\dot{\chi} : \mathscr{B} \times \mathscr{I} \to T\mathscr{S}$ is the (Lagrangian) velocity field associated with χ, so that $\dot{\chi}(X,t) \in T_{\chi(X,t)}\mathscr{S}$. The superimposed dot means $\dot{\chi}(X,t) \equiv \partial_t\chi(X,t)$. Analogously, $\dot{\boldsymbol{F}} \equiv \partial_t\boldsymbol{F}$ and $\dot{\boldsymbol{K}} \equiv \partial_t\boldsymbol{K}$.
6. $\mathcal{X} : \mathscr{B} \times \mathscr{I} \to \mathscr{B}$ and $\mathcal{T} : \mathscr{B} \times \mathscr{I} \to \mathscr{I}$ denote the projections $\mathcal{X}(X,t) = X$ and $\mathcal{T}(X,t) = t$. For any physical quantity f defined as a function $\hat{f}$ of $\boldsymbol{F}$ and $\boldsymbol{K}$, and exhibiting explicit dependence on points and time, we write $f = \hat{f} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$ and $f(X,t) = \hat{f}(\boldsymbol{F}(X,t), \boldsymbol{K}(X,t), X, t)$ [36].

For any second-order tensor $\boldsymbol{T}$, we use the wordings "$\boldsymbol{A}$-deviatoric part" and "$\boldsymbol{A}$-spherical part" of $\boldsymbol{T}$, with $\boldsymbol{A}$ being an appropriate non-singular second-order tensor, to indicate $\boldsymbol{T} - \frac{1}{3}\mathrm{tr}(\boldsymbol{A}^{-1}\boldsymbol{T})\boldsymbol{A}$ and $\frac{1}{3}\mathrm{tr}(\boldsymbol{A}^{-1}\boldsymbol{T})\boldsymbol{A}$, respectively.

To perform operations involving vectors, tensors, and their dual entities, we employ the notation of duality pairs between a generic vectorial or tensorial quantity $\boldsymbol{V}$ and its dual entity $\boldsymbol{\Omega}$. Hence, we denote by $\langle\boldsymbol{\Omega}|\boldsymbol{V}\rangle$ the real-valued application of the linear map $\boldsymbol{\Omega}$ to $\boldsymbol{V}$. For example, if $\boldsymbol{V}$ is a vector and $\boldsymbol{\Omega}$ is a co-vector, we obtain $\langle\boldsymbol{\Omega}|\boldsymbol{V}\rangle = \Omega_A V^A$. Similarly, if $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ are mixed second-order tensors with components $V^L{}_M$ and $\Omega_P{}^Q$ in some coordinate system, then we find $\langle\boldsymbol{\Omega}|\boldsymbol{V}\rangle = \Omega_A{}^B V^A{}_B = \mathrm{tr}(\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{V})$.

Given two fourth-order tensors $\mathbb{L}$ and $\mathbb{K}$ having components, e.g., $\mathbb{L}^A{}_B{}^C{}_D$ and $\mathbb{K}_P{}^Q{}_{RS}$, we define the operation $\mathbb{L} \diamond \mathbb{K}$ as the contraction of the second pair of indices of the first tensor with the first pair of indices of the second tensor, i.e., in components, $[\mathbb{L} \diamond \mathbb{K}]^A{}_{BRS} := \mathbb{L}^A{}_B{}^M{}_N \mathbb{K}_M{}^N{}_{RS}$.

By viewing a fourth-order tensor $\mathbb{L}$ as a linear map $\mathbb{L} : \mathscr{U} \to \mathscr{V}$ between the spaces of second-order tensors $\mathscr{U}$ and $\mathscr{V}$, possibly of different kind, we write $\mathbb{L}[\boldsymbol{U}] = \boldsymbol{V}$ to denote the application of $\mathbb{L}$ to $\boldsymbol{U} \in \mathscr{U}$ returning $\boldsymbol{V} \in \mathscr{V}$. For example, if $\mathbb{L}$ and $\boldsymbol{U}$ have components $\mathbb{L}^A{}_B{}^C{}_D$ and $U_P{}^Q$, then $\boldsymbol{V}$ has components given by $V^A{}_B = \mathbb{L}^A{}_B{}^M{}_N U_M{}^N$. The transpose of $\mathbb{L}$ is defined through $\langle\boldsymbol{\Omega}|\mathbb{L}[\boldsymbol{U}]\rangle = \langle\mathbb{L}^{\mathrm{T}}[\boldsymbol{\Omega}]|\boldsymbol{U}\rangle$, and $\mathbb{L}^{\mathrm{T}}$ has components $[\mathbb{L}^{\mathrm{T}}]^C{}_D{}^A{}_B$.

A glossary of the most important terminology and a table collecting the most recurrent symbols are supplied in the Appendix.

2.2 An Overview on Growth: From *Geometric* to *Configurational* Mechanics

The process of volumetric growth manifests itself through a source of mass, hereafter denoted by R, and yields a variation in time of the mass density of the body under study. This is captured by the mass balance law, which, in the body's reference placement $\mathscr{B}$, is given by $\dot{\varrho}_{\mathrm{R}} = \varrho_{\mathrm{R}} R$, with ϱ_{R} being the body's mass density per unit volume of $\mathscr{B}$ (see, e.g., [32]).

In general, the variation of ϱ_{R} represents a reorganization of the internal structure of the body that is virtually independent of deformation. Moreover, it introduces inhomogeneities [30, 85, 91] that do not amount only to nonuniform redistributions of mass within the body, and that, similarly to dislocations [12, 66, 107], cannot be eliminated by deformation alone.

Growth can be accompanied also by other structural reorganizations, which, although possibly related to R, do not directly induce changes of ϱ_{R}. All these structural processes lead to *incompatible rearrangements* of the body elements, often termed *anelastic*. When such rearrangements occur, the body elements tend to find themselves in a state in which they do not *"fit together"* [45]. This produces *residual stresses* [86, 101], which are the mechanical manifestation of incompatibility [86, 100].

Incompatibility is a geometric concept expressing that, in general, the above mentioned rearrangements cannot be reduced to maps transforming $\mathscr{B}$ into other body material manifolds in the Euclidean space. For these reasons, a second-order tensor field —in fact, $\boldsymbol{K}$— that is not defined as the tangent map [81] of a deformation is a suitable descriptor for growth and for the other structural reorganization processes related to it.

From the mechanical point of view, the residual stresses accumulated in the body elements in response to growth can be relaxed by virtually isolating each body element from the other ones, and letting it grow alone. By doing this, the body element undergoing growth will be in a stress-free state at each time $t \in \mathscr{I}$. This state is, in fact, the linear space $\mathscr{N}_X(t)$ introduced in Sect. 2.1, and the ideal operation of relaxation is $\boldsymbol{K}(X,t) : T_X\mathscr{B} \to \mathscr{N}_X(t)$.

The incompatibility of $\boldsymbol{K}$, i.e., its intrinsic non-integrability, leads to frame the mechanics of growth within non-Euclidean geometry. Indeed, it allows to introduce a non-Euclidean metric and affine connections by means of which several geometric settings can be studied, such as the Riemannian, Riemann–Cartan, Weitzenböck, or Weyl manifolds (see, e.g., [45, 74–80, 108–110]).

By referring to *configuration* of a body as the manifold described by its deformation *and* growth tensor $\boldsymbol{K}$, one can augment its kinematics. This way, it is possible to resolve, aside deformation, the structural changes due both to the mass variation and to the other reorganization processes associated with it. This fact suggests that *configurational mechanics* [54] is a natural framework to study growth.

In our work, we consider the minimal context of a theory of grade zero in $\boldsymbol{K}$ [29] (see also [15] for plasticity). This choice is dictated by simplicity, but it allows to study the geometric aspects of growth as "byproducts" of our theory, and is sufficient to handle growth as a problem of configurational mechanics.

2.3 Growth Mechanics as a Constrained Field Theory

In this section, we review the peculiar points of some previous works [47–50].

The continuum theories of volumetric growth in monophasic media often adopt the Bilby–Kröner–Lee (BKL) decomposition of $\boldsymbol{F}$ (see, e.g., [4, 5, 9, 26, 32, 41, 45, 53, 72, 73]), and recast the mass balance law in the form of a differential relationship between $\boldsymbol{K}$ and the (rescaled) source of mass R, i.e.,

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle \equiv \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = R. \tag{2}$$

In the BKL decomposition $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}}\boldsymbol{K}$, $\boldsymbol{F}_{\mathrm{e}}$ denotes the tensor field of the elastic distortions that accommodate for the growth distortions, described by $\boldsymbol{K}$. It follows that $J := \det \boldsymbol{F} > 0$ is given by $J = J_{\mathrm{e}} J_{\boldsymbol{K}}$, with $J_{\mathrm{e}} := \det \boldsymbol{F}_{\mathrm{e}} > 0$ and $J_{\boldsymbol{K}} := \det \boldsymbol{K} > 0$. For details, see, e.g., [45] and the references therein.

Starting from the mass balance law $\dot{\varrho}_{\mathrm{R}} = \varrho_{\mathrm{R}} R$, and exploiting the BKL decomposition, Equation (2) is obtained under the assumption that, for each $X \in \mathscr{B}$ and time $t \in \mathscr{I}$, $\dot{\varrho}_{\mathrm{R}}(X, t)$ is "absorbed" by the volumetric part of the ideal relaxation process $\boldsymbol{K}(X, t) : T_X\mathscr{B} \to \mathscr{N}_X(t)$. To this end, we decompose $\boldsymbol{K}$ as $\boldsymbol{K} = J_{\boldsymbol{K}}^{1/3}\boldsymbol{K}_{\mathrm{u}}$, where $J_{\boldsymbol{K}}$ accounts for the volume change of the body elements from the reference placement to the body's natural state, and $\boldsymbol{K}_{\mathrm{u}}$ describes the isochoric (volume-preserving) part of $\boldsymbol{K}$. Then, if ϱ is the mass density in the current placement of the body, so that $\varrho_{\mathrm{R}} := J\varrho$, we write ϱ_{R} as $\varrho_{\mathrm{R}} = J_{\boldsymbol{K}}\varrho_{\nu}$, where $\varrho_{\nu} := J_{\mathrm{e}}\varrho$ is such that $\varrho_{\nu}(X, t)$ defines the mass density of the body element attached at X in its relaxed state at time t. However, due to the assumption that has been made, the function $\varrho_{\nu}(X, \cdot)$ can be taken constant in time, and, thus, upon dropping the dependence on X, it holds that $\dot{\varrho}_{\mathrm{R}} = \dot{J}_{\boldsymbol{K}}\varrho_{\nu}$. Finally, because of the chain of identities $\dot{J}_{\boldsymbol{K}} = J_{\boldsymbol{K}}\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = J_{\boldsymbol{K}}\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}}\rangle$, we obtain $\dot{\varrho}_{\mathrm{R}} = J_{\boldsymbol{K}}\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}}\rangle\varrho_{\nu} = \varrho_{\mathrm{R}}\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}}\rangle$, which yields Equation (2).

Equation (2) places a condition both for $\boldsymbol{K}$ and for R, and it can be viewed either as a way for defining R, once $\boldsymbol{K}$ is determined (see, e.g., [30, 32, 52, 53]), or as a *constraint* on $\boldsymbol{K}$ [47, 50], if R is assumed to be given from the outset, for example, phenomenologically [1, 4–6, 83, 84].

In the present work, we concentrate on the second point of view, which we refer to as *"a priori approach"* [48]. Moreover, following the phenomenological framework developed for tumor growth in [83, 84], we hypothesize that R can be expressed as a function of $\boldsymbol{F}$ and $\boldsymbol{K}$ through an appropriate function of mechanical stress. In addition, R must be related to chemical factors as well as to any other factor that enhances or hinders growth. Hence, we set

$$R := \hat{R} \circ \natural_{\gamma}, \qquad \natural_{\gamma} := (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T}). \tag{3}$$

We view Equation (2) as a constraint. This way, one is sure of describing growth as necessary, with the possibility of developing a dynamic model for the full tensor field $\boldsymbol{K}$ [47, 50]. This procedure permits to consider the remodeling that accompanies growth for any type of material, without the necessity of "guessing" the form of $\boldsymbol{K}$ on the basis of material symmetries (see, e.g., [8, 9, 73]). Thus, we write:

$$\mathcal{C} \equiv \hat{\mathcal{C}} \circ \natural_{\mathrm{c}} := \langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}}\rangle - \hat{R} \circ \natural_{\gamma} = 0, \qquad \natural_{\mathrm{c}} := (\boldsymbol{F}, \boldsymbol{K}; \dot{\boldsymbol{K}}; \mathcal{X}, \mathcal{T}). \tag{4}$$

Remark 1 (Nonholonomic nature of the constraint) If R is zero, no variation of mass occurs, and the constraint (4) becomes $\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}}\rangle = 0$, which is holonomic. Indeed, one can take $h := \log J_{\boldsymbol{K}}$, to retrieve $\dot{h} = \langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}}\rangle = 0$. In such a situation, $\boldsymbol{K}$ is constrained to be isochoric, as is often assumed in the biomechanics of remodeling (see e.g. [6, 97, 98]). There can also be other functional forms of $\hat{R}$ that make the constraint (4) holonomic (see, e.g., [47]), but they are rather special. Hence, with the purpose of virtually including any biologically plausible form of $\hat{R}$, we regard the constraint (4) as *nonholonomic* with respect to $\boldsymbol{K}$. This means that no scalar function $h := \hat{h} \circ \natural_{\gamma}$ exists, such that $\dot{h} = \hat{\mathcal{C}} \circ \natural_{\mathrm{c}} = 0$. In particular, $\hat{\mathcal{C}} \circ \natural_{\mathrm{c}}$ is affine in $\dot{\boldsymbol{K}}$.

Remark 2 (Biological scenario and differentiability of R) In some studies on tumor growth (see, e.g., [83]), $\hat{R}$ depends on $\boldsymbol{F}$ and $\boldsymbol{K}$ through mechanical stress. More specifically, one is interested in capturing the inhibitory effect that compressive stresses exert on the cell proliferation processes, like mitosis [16], which precedes vascularization. In this stage, the mass variation of the tumor is mainly due to the accessibility of the tumor cells to nourishment, which is supplied in the form of *nutrient* chemical substances, such as glucose and oxygen [1, 5, 6, 16, 83, 84]. The concentration of nutrients at each point $X \in \mathscr{B}$ and time $t \in \mathscr{I}$, hereafter denoted by $\mathfrak{c}(X, t)$, evolves by following a diffusion-reaction equation coupled with the other mechanical variables of the problem (see e.g. [5, 16, 49, 84]). To model the influence of stress on growth, $\hat{R}$ can be related to the positive part of the mechanical pressure $\wp := -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor of the medium, expressed as a function of $\boldsymbol{F}$ and $\boldsymbol{K}$, as is the case when the mechanical response of the tumor is hypothesized to be elastic. Hence, upon setting $\langle\wp\rangle_+ = \frac{1}{2}(\wp + |\wp|)$ for the positive part of $\wp$, the effect of mechanical stress is switched off for $\wp \leq 0$, and switched on for $\wp > 0$ [83]. In particular, $\hat{R}$ decreases with increasing $\wp$. Finally, by expressing $\wp$ constitutively as $\wp = \hat{\wp} \circ \natural_\gamma$, $\hat{R}$ is made dependent on $\boldsymbol{F}$ and $\boldsymbol{K}$. To account for these facts, and imitating an expression of $\hat{R}$ prescribed in [83, 84], Grillo&Di Stefano [48, 49] suggested the functional form

$$R \equiv \hat{R} \circ \natural_\gamma := \zeta_{\mathrm{a}} \left\langle \frac{\mathfrak{c} - \mathfrak{c}_{\mathrm{cr}}}{\mathfrak{c}_{\mathrm{env}} - \mathfrak{c}_{\mathrm{cr}}} \right\rangle_+ \left[1 - \frac{\alpha \langle \hat{\wp} \circ \natural_\gamma \rangle_+}{\sigma_{\mathrm{c}} + \langle \hat{\wp} \circ \natural_\gamma \rangle_+} \right] - \zeta_{\mathrm{r}} \left\langle 1 - \frac{\mathfrak{c}}{\mathfrak{c}_{\mathrm{cr}}} \right\rangle_+ . \tag{5}$$

Here, ζ_{a} and ζ_{r} are non-negative, constant material coefficients associated with mass "accretion" and "resorption", respectively, having units of the reciprocal of time [48]; $\mathfrak{c}_{\mathrm{cr}}$ is a constant threshold value of the nutrients' concentration; $\mathfrak{c}_{\mathrm{env}} > \mathfrak{c}_{\mathrm{cr}}$ is a constant value of the nutrients' concentration in the tumor's environment [48]; $\alpha \geq 0$ is a non-dimensional material constant; σ_{c} is a constant characteristic reference value of stress. Growth laws of the kind given in Equation (5) render $\hat{R}$ continuous but not everywhere differentiable, because $\langle\wp\rangle_+$ is not differentiable in $\wp = 0$. Thus, when the differentiability of $\hat{R}$ is needed, $\langle\wp\rangle_+$ is mollified, and $\hat{R}$ is taken to be at least C^1.

2.3.1 Principle of Virtual Work and Nonholonomic Constraint on $\boldsymbol{K}$

Following the methodology outlined in [47, 50], which, in turn, is based on the approaches developed in [15, 29], the constraint (4) has to be appended to the Principle of Virtual Work (PVW), formulated for the growing body under study. To this end, it is necessary to introduce the virtual variations of the basic kinematic descriptors χ, $\boldsymbol{F}$, and $\boldsymbol{K}$. Thus, since the present context is of grade one in χ, and of grade zero in $\boldsymbol{F}$ and $\boldsymbol{K}$ [28, 29], we write

$$(\chi, D\chi, \boldsymbol{F}, \boldsymbol{K}; \delta\chi, \delta D\chi \equiv \mathrm{Grad}\delta\chi, \delta\boldsymbol{F}, \delta\boldsymbol{K}; \mathcal{X}, \mathcal{T}). \tag{6}$$

Within the *"canonical doctrine"* [50], the *varied form* of the constraint to be appended to the PVW is supplied by the so-called *Chetaev condition* [35, 38, 40, 71, 94], which holds true for *"ideal"* constraints [71], and reads [50]

$$\begin{aligned} &\mathrm{Ch}_{\natural_{\mathrm{c}}}(\delta\boldsymbol{K}) := \langle \partial_{\dot{\boldsymbol{K}}} \hat{\mathcal{C}} \circ \natural_{\mathrm{c}} | \delta\boldsymbol{K} \rangle = 0 \\ \Rightarrow \quad &\mathrm{Ch}_{\natural_{\mathrm{c}}}(\delta\boldsymbol{K}) = \langle \boldsymbol{K}^{-\mathrm{T}} | \delta\boldsymbol{K} \rangle \equiv \langle \boldsymbol{I}^{\mathrm{T}} | \boldsymbol{K}^{-1}\delta\boldsymbol{K} \rangle \equiv \mathrm{tr}(\boldsymbol{K}^{-1}\delta\boldsymbol{K}) = 0, \end{aligned} \tag{7}$$

where $\boldsymbol{I}^{\mathrm{T}} : T^*\mathscr{B} \to T^*\mathscr{B}$ is the transpose of the identity tensor [81].

In conjunction with Equation (4), and in order to postulate the PVW in a form *à la* Hu–Washizu [14], we enforce the condition that $\boldsymbol{F}$ must be equal to $D\chi$ at all times and at all points, thereby introducing the auxiliary constraint $\boldsymbol{\mathcal{C}}_{\mathrm{a}} := D\chi - \boldsymbol{F} = \boldsymbol{O}$, where $\boldsymbol{O}$ is the null second-order tensor field. By introducing the tensorial Lagrange multiplier $\boldsymbol{T}$, this condition can be associated with the weak forms

$$\langle \boldsymbol{T} \mid D\chi - \boldsymbol{F}\rangle = 0, \qquad \langle \delta\boldsymbol{T} \mid D\chi - \boldsymbol{F}\rangle + \langle \boldsymbol{T} \mid \mathrm{Grad}\delta\chi - \delta\boldsymbol{F}\rangle = 0. \tag{8}$$

By assuming that, in addition to the prescribed constraints, growth occurs under the action of the generalized forces $\boldsymbol{Y}_{\mathrm{u}}$ and $\boldsymbol{Z}$ [15, 29], introduced by duality with the generalized virtual displacement $\boldsymbol{K}^{-1}\delta\boldsymbol{K}$, and interpreted as internal and external, respectively, the PVW can be cast in the form [47, 50]

$$\mathcal{W}_{\mathrm{int}} + \mathcal{W}_{\mathrm{c}} = \mathcal{W}_{\mathrm{ext}}, \tag{9}$$

where $\mathcal{W}_{\mathrm{int}}$, $\mathcal{W}_{\mathrm{c}}$, and $\mathcal{W}_{\mathrm{ext}}$ are the "internal", "constrained", and "external" virtual work, respectively, and are defined by

$$\mathcal{W}_{\mathrm{int}} := \int_{\mathscr{B}} \big\{\langle \boldsymbol{P}|\delta\boldsymbol{F}\rangle + \langle \boldsymbol{Y}_{\mathrm{u}}|\boldsymbol{K}^{-1}\delta\boldsymbol{K}\rangle\big\}, \tag{10a}$$

$$\mathcal{W}_{\mathrm{c}} := \int_{\mathscr{B}} \big\{\langle \mu\, \boldsymbol{I}^{\mathrm{T}}|\boldsymbol{K}^{-1}\delta\boldsymbol{K}\rangle + \langle \boldsymbol{T}|\mathrm{Grad}\delta\chi - \delta\boldsymbol{F}\rangle + \delta\mu\, t_{\mathrm{c}}\mathcal{C} + \langle \delta\boldsymbol{T}|\boldsymbol{\mathcal{C}}_{\mathrm{a}}\rangle\big\}, \tag{10b}$$

$$\mathcal{W}_{\mathrm{ext}} := \int_{\mathscr{B}} \big\{\langle \boldsymbol{f}|\delta\chi\rangle + \langle \boldsymbol{Z}|\boldsymbol{K}^{-1}\delta\boldsymbol{K}\rangle\big\} + \int_{\partial_{\mathrm{N}}^{\chi}\mathscr{B}} \langle \boldsymbol{\tau}|\delta\chi\rangle. \tag{10c}$$

Here, $\boldsymbol{P}$ is the first Piola–Kirchhoff stress tensor; μ is the real-valued Lagrange multiplier associated with the Chetaev condition (7); $\delta\mu$ and $\delta\boldsymbol{T}$ are the virtual variations of μ and $\boldsymbol{T}$; t_{c} is a strictly positive characteristic time introduced to make the term $\delta\mu\, t_{\mathrm{c}}\mathcal{C}$ dimensionally homogeneous with all the other addends of Equation (10b); $\boldsymbol{f}$ and $\boldsymbol{\tau}$ represent the external body forces and the external surface forces dual to $\delta\chi$. Note that $\boldsymbol{\tau}$ is defined over the Neumann portion of $\partial\mathscr{B}$, denoted by $\partial_{\mathrm{N}}^{\chi}\mathscr{B}$. In the sequel, $\partial\mathscr{B}$ is partitioned as $\partial\mathscr{B} = \partial_{\mathrm{D}}^{\chi}\mathscr{B} \sqcup \partial_{\mathrm{N}}^{\chi}\mathscr{B}$, where $\partial_{\mathrm{D}}^{\chi}\mathscr{B}$ is referred to as Dirichlet boundary.

The strong form of the dynamic problem associated with Equations (9) and (10a)–(10c) is given by the set of equations

$$\mathrm{Div}\boldsymbol{T} + \boldsymbol{f} = \boldsymbol{0}, \qquad \text{in } \mathscr{B}, \tag{11a}$$

$$\boldsymbol{\tau} - \boldsymbol{T}\boldsymbol{N} = \boldsymbol{0}, \qquad \text{on } \partial_{\mathrm{N}}^{\chi}\mathscr{B}, \tag{11b}$$

$$\boldsymbol{P} = \boldsymbol{T}, \qquad \text{in } \mathscr{B}, \tag{11c}$$

$$\boldsymbol{Y}_{\mathrm{u}} + \mu\boldsymbol{I}^{\mathrm{T}} - \boldsymbol{Z} = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{11d}$$

$$\boldsymbol{\mathcal{C}}_{\mathrm{a}} = D\chi - \boldsymbol{F} = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{11e}$$

$$\mathcal{C} = \langle \boldsymbol{K}^{-\mathrm{T}}|\dot{\boldsymbol{K}}\rangle - R = 0, \qquad \text{in } \mathscr{B}, \tag{11f}$$

which has to be completed with Dirichlet boundary conditions on χ, and with all the necessary initial conditions. As for the Hu–Washizu method (see e.g. [14]), Equation (11c) identifies $\boldsymbol{T}$ with $\boldsymbol{P}$.

Because of the difference of formulation with respect to the one recently presented in [47, 50], the physical quantities featuring in Equations (11a)–(11f) can be grouped as follows: 21 kinematic variables χ, $\boldsymbol{F}$, and $\boldsymbol{K}$; 10 Lagrange multipliers μ and $\boldsymbol{T}$; 18 constitutive functions $\boldsymbol{P}$ and $\boldsymbol{Y}_{\mathrm{u}}$; 15 external forces $\boldsymbol{f}$, $\boldsymbol{\tau}$, and $\boldsymbol{Z}$. The kinematic variables and the Lagrange multipliers yield a set of 31 scalar unknowns to be determined by solving the 31 scalar equations given by the balance laws (11a), (11c), (11d), and by the constraints (11e), (11f).

The presentation of the full boundary value problem (11a)–(11f) serves as comparison for the dynamic equations that will be determined in the sequel.

2.3.2 The Hyperelastic Case and the Extended Hamilton Method

Although some biological tissues show viscoelastic behavior in certain dynamic regimes [10, 39], a case of particular interest is when the growing medium under study can be assumed to be hyperelastic [3–5, 27, 29, 32, 53, 73, 84]. Then, the constitutive representation of $\boldsymbol{P}$ can be obtained as

$$\boldsymbol{P} = \frac{\partial \hat{\Psi}}{\partial \boldsymbol{F}} \circ (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) = J_{\boldsymbol{K}} \left[\frac{\partial \hat{\Psi}_\nu}{\partial \boldsymbol{F}_{\mathrm{e}}} \circ (\boldsymbol{F}_{\mathrm{e}}; \mathcal{X}, \mathcal{T}) \right] \boldsymbol{K}^{-\mathrm{T}}, \tag{12}$$

where $\Psi := \hat{\Psi} \circ (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) = J_{\boldsymbol{K}}[\hat{\Psi}_\nu \circ (\boldsymbol{F}_{\mathrm{e}}; \mathcal{X}, \mathcal{T})]$ is the strain energy density of the medium per unit volume of its reference placement; $\Psi_\nu := \hat{\Psi}_\nu \circ (\boldsymbol{F}_{\mathrm{e}}; \mathcal{X}, \mathcal{T})$ is the same physical quantity, but expressed per unit volume of the medium's natural state. Note that the arguments of $\hat{\Psi}$ are the same as the collection $\natural_\gamma$, so that we can write $\Psi = \hat{\Psi} \circ \natural_\gamma$.

Typically, the growth of a biological medium, such as a tumor or a cellular aggregate, occurs over time scales that allow to neglect its kinetic energy. However, nothing forbids, in principle, to consider the "classical" kinetic energy density, $\mathcal{K} = \frac{1}{2} J_{\boldsymbol{K}} \varrho_\nu \|\dot{\chi}\|^2$, and define the *medium's Lagrangian density function* $\mathcal{L}_{\mathrm{b}} := \mathcal{K} - \Psi$ [32].

The function $\mathcal{L}_{\mathrm{b}}$ can be generalized by including other interactions that the medium can experience. These could be represented by the kinetic energy density associated with $\dot{\boldsymbol{K}}$ [105], and potential densities that may depend both on χ and on $\boldsymbol{K}$. Therefore, to include such contributions, we consider, in lieu of $\mathcal{L}_{\mathrm{b}}$, a more general Lagrangian density function, defined as $\mathcal{L} := \hat{\mathcal{L}} \circ \natural$. However, in the hyperelastic case, $\dot{\boldsymbol{F}}$ is ignorable, since it holds that $\partial_{\dot{\boldsymbol{F}}} \hat{\mathcal{L}} \circ \natural = \boldsymbol{O}$. We also notice that $\hat{\mathcal{L}}$ can be assumed to be formally independent of $D\chi$, since the dependence on the deformation gradient tensor is already accounted for through $\boldsymbol{F}$, which is one of the arguments of $\hat{\Psi}$. Hence, we set $\partial_{D\chi} \hat{\mathcal{L}} \circ \natural = \boldsymbol{O}$.

To imitate the Hu–Washizu formulation of the PVW [14] of Sect. 2.3.1, it is convenient to augment $\mathcal{L}$ with $\langle \boldsymbol{T} \,|\, \boldsymbol{F} - D\chi \rangle$, so that one finds

$$\mathcal{L}_{\mathrm{a}} \equiv \hat{\mathcal{L}}_{\mathrm{a}} \circ (\natural; \boldsymbol{T}) = \hat{\mathcal{L}} \circ \natural + \langle \boldsymbol{T} \,|\, \boldsymbol{F} - D\chi \rangle, \tag{13a}$$

$$\mathcal{A}_{\mathrm{a}}(\chi, \boldsymbol{F}, \boldsymbol{K}; \boldsymbol{T}) := \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \left\{ \int_{\mathscr{B}} \hat{\mathcal{L}}_{\mathrm{a}}(\natural(X, t); \boldsymbol{T}(X, t)) \mathrm{dV}(X) \right\} \mathrm{d}t. \tag{13b}$$

The Lagrange multiplier $\boldsymbol{T}$ has to be included among the arguments both of the *augmented Lagrangian density function* $\hat{\mathcal{L}}_{\mathrm{a}}$ and of the *augmented action* $\mathcal{A}_{\mathrm{a}}$, obtained by integration over the time interval $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$. Moreover, since $\boldsymbol{F}$ is regarded here as an independent kinematic variable, $\mathcal{A}_{\mathrm{a}}$ has to be defined as a functional of $\boldsymbol{F}$ as well as of χ and $\boldsymbol{K}$. Note that, because of the hypothesis of hyperelastic material, $\hat{\mathcal{L}}_{\mathrm{a}}$ and $\hat{\mathcal{L}}$ are independent of $\dot{\overline{D\chi}}$, that is, $\partial_{\dot{\overline{D\chi}}} \hat{\mathcal{L}}_{\mathrm{a}} \circ (\natural; \boldsymbol{T}) = \partial_{\dot{\overline{D\chi}}} \hat{\mathcal{L}} \circ \natural = \boldsymbol{O}$.

No matter how accurate $\mathcal{L}_{\mathrm{a}}$ can be, it is not sufficient, in general, to provide a comprehensive description of a growing medium. There are at least two reasons for this. First, one should expect the presence of generalized forces for which no potential density exists. Second, in the *classical formulation* of Variational Calculus, one cannot attach the nonholonomic constraint (4) to $\mathcal{L}$ or $\mathcal{L}_{\mathrm{a}}$, as one could instead do if the constraint were holonomic (see e.g. [46, 67, 68, 90, 93]). Still, to obtain the dynamic equations of the problem under study, one may have recourse to *Extended Hamilton's Principle* [11, 67]. To this end, we introduce a smallness parameter $\varepsilon \in \mathscr{Y}(\varepsilon_0)$, where $\mathscr{Y}(\varepsilon_0)$ is an open neighborhood of zero with radius $\varepsilon_0 > 0$, and we define the homotopies

$$(\natural(X,t);\, \boldsymbol{T}(X,t)) \;\mapsto\; (\tilde{\natural}(X,t,\varepsilon);\, \tilde{\boldsymbol{T}}(X,t,\varepsilon)), \tag{14a}$$

$$\tilde{\natural}(X,t,\varepsilon) = \tilde{\natural}(X,t,0) + \partial_\varepsilon \tilde{\natural}(X,t,0)\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0, \tag{14b}$$

$$\tilde{\boldsymbol{T}}(X,t,\varepsilon) = \tilde{\boldsymbol{T}}(X,t,0) + \partial_\varepsilon \tilde{\boldsymbol{T}}(X,t,0)\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0, \tag{14c}$$

with $\tilde{\natural}(X,t,0) = \natural(X,t)$, $\tilde{\boldsymbol{T}}(X,t,0) = \boldsymbol{T}(X,t)$.

Since the Extended Hamilton Principle relies on "classical variations" [11, 67], we proceed as follows. By indicating with φ a generic variable of $\natural$, $\varphi(X,t)$ is varied into $\tilde{\varphi}(X,t,\varepsilon)$, and we denote by $\boldsymbol{\eta}_\varphi(X,t) := \partial_\varepsilon \tilde{\varphi}(X,t,0)$ the entity defining the direction (in a generalized sense) along which the infinitesimal first-order variation of $\varphi(X,t)$ is computed, that is, $\delta\varphi(X,t,\varepsilon) \equiv \varepsilon \boldsymbol{\eta}_\varphi(X,t)$. However, with a slight abuse of terminology, from here on we shall refer to $\boldsymbol{\eta}_\varphi(X,t)$ and $\boldsymbol{\eta}_\varphi$ as the first-order *increment* and *incremental field* associated with $\varphi(X,t)$ and φ, respectively (in fact, since in the sequel these quantities are attributed only to first-order variations, we shall omit the specification "first-order"). Analogously, we call $\boldsymbol{\eta}_{\boldsymbol{T}}(X,t) = \partial_\varepsilon \tilde{\boldsymbol{T}}(X,t,0)$ the increment associated with $\boldsymbol{T}(X,t)$. If φ is the time derivative of another variable ψ of $\natural$, i.e., if $\varphi = \dot{\psi}$, then the hypothesis of "classical variations" implies the commutative relationship $\boldsymbol{\eta}_{\dot{\psi}} = \dot{\boldsymbol{\eta}}_\psi$. Moreover, it holds that $\boldsymbol{\eta}_{D\chi} = \mathrm{Grad}\boldsymbol{\eta}_\chi$, and $\boldsymbol{\eta}_{\dot{\overline{D\chi}}} = \dot{\boldsymbol{\eta}}_{D\chi} = \mathrm{Grad}\dot{\boldsymbol{\eta}}_\chi$. The incremental fields $\boldsymbol{\eta}_\chi$ and $\boldsymbol{\eta}_{\boldsymbol{K}}$ are required to vanish at t_{in} and t_{fin}, while $\boldsymbol{\eta}_\chi$ must be null also on the Dirichlet boundary of $\mathscr{B}$. Finally, we clarify that the incremental fields associated with $\mathcal{X}$ and $\mathcal{T}$ are taken to be null.

Now, we let $\boldsymbol{f}_{\mathrm{np}}$ and $\mathfrak{S}_{\mathrm{np}}$ be the non-potential forces dual to $\boldsymbol{\eta}_\chi$ and $\boldsymbol{\eta}_{\boldsymbol{K}}$, respectively. The former represents all the non-potential contributions to the balance of *"deformational forces"* [54], while the latter collects the non-potential contributions to the quantity $\boldsymbol{K}^{-\mathrm{T}}[\boldsymbol{Z} - \boldsymbol{Y}_{\mathrm{u}}]$ that can be defined from Equation (11d). In addition, following Lanczos' approach [67] to nonholonomic systems, we consider also the constraint (4), which contributes to the overall virtual work through the term $\langle \mu \boldsymbol{K}^{-\mathrm{T}} \,|\, \boldsymbol{\eta}_{\boldsymbol{K}} \rangle$, with $\mu \boldsymbol{K}^{-\mathrm{T}}$ acquiring the meaning of the associated reactive force, up to the sign. Therefore, by integrating the virtual work produced by $\boldsymbol{f}_{\mathrm{np}}$, $\mathfrak{S}_{\mathrm{np}}$, and $\mu \boldsymbol{K}^{-\mathrm{T}}$ over $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$, we write Extended Hamilton's Principle [11, 24, 67] as

$$\begin{aligned}\frac{\mathrm{d}\tilde{\mathcal{A}}_{\mathrm{a}}}{\mathrm{d}\varepsilon}(0) = &- \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \left\{ \int_{\mathscr{B}} \left[\langle \boldsymbol{f}_{\mathrm{np}} | \boldsymbol{\eta}_\chi \rangle + \langle \mathfrak{S}_{\mathrm{np}} - \mu \boldsymbol{K}^{-\mathrm{T}} \,|\, \boldsymbol{\eta}_{\boldsymbol{K}} \rangle \right] \mathrm{dV} \right\} \mathrm{d}t \\ &- \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \left\{ \int_{\partial_{\mathrm{N}}^{\chi}\mathscr{B}} \langle \boldsymbol{\tau} | \boldsymbol{\eta}_\chi \rangle \,\mathrm{dA} \right\} \mathrm{d}t,\end{aligned} \tag{15}$$

with $\tilde{\mathcal{A}}_{\mathrm{a}}(\varepsilon) := \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \left\{ \int_{\mathscr{B}} \hat{\mathcal{L}}_{\mathrm{a}}(\tilde{\natural}(X,t,\varepsilon);\, \tilde{\boldsymbol{T}}(X,t,\varepsilon)) \mathrm{dV}(X) \right\} \mathrm{d}t$.

Computing the left-hand side of Equation (15), grouping together the terms dual to the same variation, and localizing the resulting expression yield

$$\mathcal{E}_\chi \hat{\mathcal{L}} + \mathrm{Div}\boldsymbol{T} = -\boldsymbol{f}_{\mathrm{np}}, \qquad \text{in } \mathscr{B}, \tag{16a}$$

$$\boldsymbol{T}\boldsymbol{N} = \boldsymbol{\tau}, \qquad \text{on } \partial_{\mathrm{N}}^{\chi}\mathscr{B}, \tag{16b}$$

$$\mathcal{E}_{\boldsymbol{F}} \hat{\mathcal{L}} + \boldsymbol{T} = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{16c}$$

$$\boldsymbol{F} - D\chi = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{16d}$$

$$\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}} = -\mathfrak{S}_{\text{np}} + \mu \boldsymbol{K}^{-\text{T}}, \qquad \text{in } \mathscr{B}, \tag{16e}$$

$$\langle \boldsymbol{K}^{-\text{T}} | \dot{\boldsymbol{K}} \rangle - R = 0, \qquad \text{in } \mathscr{B}, \tag{16f}$$

where, for a given variable φ of $\natural$, $\mathcal{E}_{\varphi}\hat{\mathcal{L}} := \partial_{\varphi}\hat{\mathcal{L}} \circ \natural - \partial_t(\partial_{\dot{\varphi}}\hat{\mathcal{L}} \circ \natural)$ denotes the associated Euler–Lagrange operator applied to $\mathcal{L} = \hat{\mathcal{L}} \circ \natural$. In particular, we obtain $\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}} = \partial_{\boldsymbol{F}}\hat{\mathcal{L}} \circ \natural$, since $\dot{\boldsymbol{F}}$ is ignorable for $\hat{\mathcal{L}}$, and, thus, $\partial_{\dot{\boldsymbol{F}}}\hat{\mathcal{L}} \circ \natural = \boldsymbol{O}$.

The sets of Equations (16a)–(16f) and (11a)–(11f) are equivalent to each other, and the following identifications can be made:

$$\boldsymbol{f} = \mathcal{E}_{\chi}\hat{\mathcal{L}} + \boldsymbol{f}_{\text{np}}, \tag{17a}$$

$$\boldsymbol{Z} - \boldsymbol{Y}_{\text{u}} = \boldsymbol{K}^{\text{T}}[\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}} + \mathfrak{S}_{\text{np}}]. \tag{17b}$$

In the sequel, when we speak of *"Traditional NonHolonomic Method"* (TNHM) [95], we refer equivalently either to Equations (11a)–(11f) or to (16a)–(16f).

2.3.3 A Note on the Configurational Generalized Forces Associated with Growth

In the paradigm of the Principle of Virtual Work [33, 42], the configurational forces $\boldsymbol{Y}_{\text{u}}$, $\mu \boldsymbol{I}^{\text{T}}$, and $\boldsymbol{Z}$ are the entities that represent the real-valued linear functional defining the growth part of the virtual work [29], i.e.,

$$\delta\boldsymbol{K} \mapsto \mathscr{W}_{\text{g}}(\delta\boldsymbol{K}) := \int_{\mathscr{B}} \langle \mathfrak{F} | \boldsymbol{K}^{-1}\delta\boldsymbol{K} \rangle = 0, \qquad \mathfrak{F} \equiv \boldsymbol{Z} - \boldsymbol{Y}_{\text{u}} - \mu \boldsymbol{I}^{\text{T}} = \boldsymbol{O}. \tag{18}$$

By its own definition, $\mathscr{W}_{\text{g}}(\cdot)$ is dual to $\delta\boldsymbol{K}$ (viewed as a *test* tensor field), while $\mathfrak{F}$ is dual to the tensor field given by $\boldsymbol{K}^{-1}\delta\boldsymbol{K}$.

Since growth requires an irreversible expenditure of energy, $\boldsymbol{Y}_{\text{u}}$ must feature a dissipative contribution, which we denote by $\boldsymbol{Y}_{\text{ud}}$. If $\boldsymbol{Y}_{\text{ud}}$ can be expressed constitutively (see, e.g., [5, 15, 28, 29, 47, 50, 51]); if the chosen constitutive representation is continuous and differentiable in all the values of $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$ in which it is defined, including $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = \boldsymbol{O}$; if it vanishes for $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = \boldsymbol{O}$; and if one is interested in the evolution of $\boldsymbol{K}$ only in a small neighborhood of $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = \boldsymbol{O}$, then $\boldsymbol{Y}_{\text{ud}}$ can be taken *linear* in $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$. This yields a relation of the type $\boldsymbol{Y}_{\text{ud}} = \mathbb{T}[\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}]$, where $\mathbb{T}$ is a positive semi-definite fourth-order tensor field enjoying the major symmetry, i.e., such that $\langle \mathbb{T}[\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}] | \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} \rangle \geq 0$, for all $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, and $\mathbb{T} = \mathbb{T}^{\text{T}}$. Clearly, $\mathbb{T}$ may depend on $\boldsymbol{F}$, $\boldsymbol{K}$, and other variables, apart from $\dot{\boldsymbol{K}}$. In fact, $\mathbb{T}$ represents a generalized tensorial viscosity independent of $\dot{\boldsymbol{K}}$ [18, 19, 47–49, 92].

The study of dissipation permits to conclude that $\boldsymbol{Y}_{\text{u}}$ consists also of a non-dissipative term, given by the Eshelby stress tensor $\boldsymbol{H}$. This is found also in many other approaches to growth [3, 29, 32, 41, 72] and to configurational mechanics in general. It descends from $\boldsymbol{H}$ being naturally conjugate to the kinematic variables describing the structural transformations of a body, such as plastic distortions [15, 22, 31, 56] and remodeling of biological media [23, 43]. In all these situations, the pairing $\langle \boldsymbol{H} | \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} \rangle$ occurs, and, in the examined case of growth, it holds that $\boldsymbol{Y}_{\text{u}} = \boldsymbol{H} + \boldsymbol{Y}_{\text{ud}}$. In more general frameworks, tensors similar to $\boldsymbol{H}$ arise, e.g., in the evolution of interfaces [25, 54] and of the chemical composition of mixtures [51, 99].

For hyperelastic bodies, and in the quasi-static case, $\boldsymbol{H}$ can be expressed by differentiation of the body's strain energy density with respect to $\boldsymbol{K}$ (see Equation (82a) below). Thus, the physical meaning of $\boldsymbol{Y}_{\mathrm{ud}}$ and $\boldsymbol{H}$ is embedded in the constitutive relations by which they are determined. In particular, since $\boldsymbol{Y}_{\mathrm{ud}}$ is a non-potential force, it contributes to the overall non-potential force $\mathfrak{S}_{\mathrm{np}}$.

Quite differently, since $\boldsymbol{Z}$ is external, it may feature, in general, inertial-like terms as well as other contributions that can be assigned phenomenologically either through the differentiation of the Lagrangian density function with respect to $\boldsymbol{K}$ or as generalized forces that generally do not admit a potential. The latter ones, in particular, should capture, at least, how the most relevant chemo-mechanical processes occurring at lower scales influence growth at the scale of the body as a whole [29, 47]. Following [48, 49] (see Remark 2), when the inertial-like terms are negligible, and the contributions that stem from the Lagrangian density function are disregarded, one can supply $\boldsymbol{Z}$ as (in the sequel, $\boldsymbol{C}$ is the right Cauchy–Green deformation tensor induced by $\boldsymbol{F}$)

$$\boldsymbol{Z} = \tfrac{1}{3} J_{\boldsymbol{K}} \beta R \boldsymbol{I}^{\mathrm{T}} + J_{\boldsymbol{K}} Q_{\nu\mathrm{t}} \|\mathrm{Grad}\mathfrak{c}\|_{\boldsymbol{C}^{-1}} \boldsymbol{I}^{\mathrm{T}} + J_{\boldsymbol{K}} [Q_{\nu\ell} - Q_{\nu\mathrm{t}}] \frac{\mathrm{Grad}\mathfrak{c} \otimes \boldsymbol{C}^{-1}\mathrm{Grad}\mathfrak{c}}{\|\mathrm{Grad}\mathfrak{c}\|_{\boldsymbol{C}^{-1}}}. \tag{19}$$

The first term on the right-hand side of Equation (19) is a purely volumetric contribution that directly induces the mass variation within the body ($\beta > 0$ gives the correct physical dimensions); the second and third terms represent a configurational force driving the evolution of $\boldsymbol{K}$ in response to the material gradient of the nutrients' concentration. In Equation (19), the parameters $Q_{\nu\ell}$ and $Q_{\nu\mathrm{t}}$ are strictly positive, and presumed, since we are aware of no experiment determining them. For any material co-vector field $\boldsymbol{\Phi}$, $\|\boldsymbol{\Phi}\|_{\boldsymbol{C}^{-1}} := \sqrt{\boldsymbol{\Phi}\boldsymbol{C}^{-1}\boldsymbol{\Phi}}$.

3 Quasi-Velocities and Transpositional Relations

In Analytical Mechanics, the terminology *quasi-velocities* refers to the result of a "change of variables" in the collection of the generalized velocities of a given mechanical system [58, 71, 89, 90]. It is done to describe the system's kinematics in a way that best "fits" the system at hand [82]. This is because many important features of the constraints are made explicit by the most appropriate choice of quasi-velocities.

Before studying how quasi-velocities work in growth mechanics, we refer the reader to Supplementary Material for further details on their employment for discrete mechanical systems.

Within our context, the generalized velocities are the time derivatives that feature in the list of variables $\natural$ of Equation (1), i.e., $\dot{\chi}$, $\mathrm{Grad}\dot{\chi}$, $\dot{\boldsymbol{F}}$, and $\dot{\boldsymbol{K}}$. However, only $\dot{\boldsymbol{K}}$ is constrained through Equation (4), while all the other ones are unconstrained. In fact, the holonomic condition $D\chi = \boldsymbol{F}$ implies *a posteriori* that $\mathrm{Grad}\dot{\chi} = \dot{\boldsymbol{F}}$, but these velocities are not restricted *a priori*, and also $D\chi$ and $\boldsymbol{F}$, in spite of their being constrained to be equal to each other, are varied independently of one another at the price of introducing the tensorial Lagrange multiplier $\boldsymbol{T}$. For these reasons, and on the basis of the rationale outlined above, there is no physical advantage in transforming $\dot{\chi}$, $\mathrm{Grad}\dot{\chi}$, and $\dot{\boldsymbol{F}}$. On the contrary, it is meaningful to transform $\dot{\boldsymbol{K}}$.

3.1 Quasi-Velocities

Let us denote by $\Omega^{\alpha}{}_{A}$ the generic component of the tensor of quasi-velocities $\boldsymbol{\Omega}$, which represents a "change of variables" performed on $\dot{\boldsymbol{K}}$, and let us define it through the transformation (cf. Supplementary Material)

$$\Omega^{\alpha}{}_{A} = \hat{\Omega}^{\alpha}{}_{A} \circ (\boldsymbol{F}, \boldsymbol{K}; \dot{\boldsymbol{K}}; \mathcal{X}, \mathcal{T}) \equiv \hat{\Omega}^{\alpha}{}_{A} \circ \natural_{\mathrm{c}}, \qquad \alpha, A = 1, 2, 3, \tag{20}$$

where the arguments of $\hat{\Omega}^{\alpha}{}_{A}$ are the same as those of $\hat{\mathcal{C}}$.

Together with the quasi-velocities, we introduce the *quasi-coordinates* and their virtual incremental fields (cf. Supplementary Material) [58, 90]

$$\dot{\Theta}^{\alpha}{}_{A} := \hat{\Omega}^{\alpha}{}_{A} \circ \natural_{\mathrm{c}}, \tag{21a}$$

$$[\boldsymbol{\eta}_{\boldsymbol{\Theta}}]^{\alpha}{}_{A} := \left(\frac{\partial \hat{\Omega}^{\alpha}{}_{A}}{\partial \dot{K}^{\beta}{}_{B}} \circ \natural_{\mathrm{c}} \right) [\boldsymbol{\eta}_{\boldsymbol{K}}]^{\beta}{}_{B} \equiv (\hat{\mathbb{J}}^{\alpha}{}_{A\beta}{}^{B} \circ \natural_{\mathrm{c}}) [\boldsymbol{\eta}_{\boldsymbol{K}}]^{\beta}{}_{B}. \tag{21b}$$

Equation (21b) is, for each pair of indices α and A, a linear differential form in the *virtual incremental field* $\boldsymbol{\eta}_{\boldsymbol{K}}$ of $\boldsymbol{K}$, which defines *explicitly* the new virtual incremental field $[\boldsymbol{\eta}_{\boldsymbol{\Theta}}]^{\alpha}{}_{A}$. Equation (21a), instead, defines quasi-coordinates *implicitly*, i.e., as the functions $\Theta^{\alpha}{}_{A}$ that solve the differential equations that it represents. Hence, for given α and A, there exists a function $\Theta^{\alpha}{}_{A}$ that satisfies Equation (21a), and is, thus, a primitive of $\Omega^{\alpha}{}_{A} \equiv \hat{\Omega}^{\alpha}{}_{A} \circ \natural_{\mathrm{c}}$ in the sense of the Fundamental Theorem of Calculus, i.e.,

$$\Theta^{\alpha}{}_{A}(X, t) = \Theta^{\alpha}{}_{A}(X, t_{\mathrm{in}}) + \int_{t_{\mathrm{in}}}^{t} \Omega^{\alpha}{}_{A}(X, s) \mathrm{d}s, \qquad \forall X \in \mathscr{B}, \tag{22}$$

provided $\Omega^{\alpha}{}_{A}$ is continuous in time. However, as expanded in the Supplementary Material, this does not imply a representation of $\Theta^{\alpha}{}_{A}$ of the form $\hat{\Theta}^{\alpha}{}_{A} \circ (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T})$.

The functions $\dot{\Theta}^{\alpha}{}_{A}$ and $[\boldsymbol{\eta}_{\boldsymbol{\Theta}}]^{\alpha}{}_{A}$ can be identified with the components of the two two-point second-order tensor fields defined by the right-hand sides of Equations (21a) and (21b), respectively.

The non-singularity of the transformation (20) requires that the collection of functions

$$\mathbb{J}^{\alpha}{}_{A\beta}{}^{B} = \hat{\mathbb{J}}^{\alpha}{}_{A\beta}{}^{B} \circ \natural_{\mathrm{c}} := \frac{\partial \hat{\Omega}^{\alpha}{}_{A}}{\partial \dot{K}^{\beta}{}_{B}} \circ \natural_{\mathrm{c}}, \qquad \alpha, A, \beta, B = 1, 2, 3, \tag{23}$$

gives the components of the non-singular fourth-order tensor $\mathbb{J}$ that represents the Jacobian of the transformation itself. Hence, in compact notation, we write $\boldsymbol{\eta}_{\boldsymbol{\Theta}} = \mathbb{J}[\boldsymbol{\eta}_{\boldsymbol{K}}]$, with $\mathbb{J} = \hat{\mathbb{J}} \circ \natural_{\mathrm{c}} = \partial_{\dot{\boldsymbol{K}}} \hat{\boldsymbol{\Omega}} \circ \natural_{\mathrm{c}}$.

3.2 Transpositional Relations and Choice of the Quasi-Velocities

As anticipated in the previous section, the choice of the most appropriate system of quasi-velocities permits to compute the associated "transpositional relations" (see, e.g., [58, 71, 90]), which express the fact that, in general, the operations of "virtual variation" and of "time differentiation" are not commutative when nonholonomic constraints are featured [90].

To compute the transpositional relations characterizing our problem of growth mechanics, we adapt a procedure reported in a work by Jarzębowska [58], and recently reviewed in [95], that is based on the quasi-velocities and on the variations of the quasi-coordinates introduced in Equations (20) and (21b). To recall the main steps of such procedure, we refer the reader to the Supplementary Material, in which we explain it for the case of a generic discrete mechanical problem. For our problem of growth mechanics, we introduce the homotopies

$$\boldsymbol{F}(X, t) \mapsto \tilde{\boldsymbol{F}}(X, t, \varepsilon) = \boldsymbol{F}(X, t) + \boldsymbol{\eta}_{\boldsymbol{F}}(X, t)\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0, \tag{24a}$$

$$\boldsymbol{K}(X, t) \mapsto \tilde{\boldsymbol{K}}(X, t, \varepsilon) = \boldsymbol{K}(X, t) + \boldsymbol{\eta}_{\boldsymbol{K}}(X, t)\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0, \tag{24b}$$

$$\dot{\boldsymbol{K}}(X, t) \mapsto \tilde{\boldsymbol{V}}(X, t, \varepsilon) = \dot{\boldsymbol{K}}(X, t) + \boldsymbol{\eta}_{\dot{\boldsymbol{K}}}(X, t)\varepsilon + o(\varepsilon), \qquad \varepsilon \to 0, \tag{24c}$$

$$\mathcal{X}(X,t) \mapsto \tilde{\mathcal{X}}(X,t,\varepsilon) = \mathcal{X}(X,t) = X, \qquad \forall\, \varepsilon \in \mathscr{Y}(\varepsilon_0), \tag{24d}$$

$$\mathcal{T}(X,t) \mapsto \tilde{\mathcal{T}}(X,t,\varepsilon) = \mathcal{T}(X,t) = t, \qquad \forall\, \varepsilon \in \mathscr{Y}(\varepsilon_0), \tag{24e}$$

where, as before, $\mathscr{Y}(\varepsilon_0)$ is an open neighborhood of zero having radius $\varepsilon_0 > 0$, and $\boldsymbol{\eta}_\varphi$ is the incremental field associated with the generic variable φ of $\natural_{\mathrm{c}}$.

Within the present framework, the increment $\boldsymbol{\eta}_{\dot{\boldsymbol{K}}}(X,t)$ on the generalized velocity $\dot{\boldsymbol{K}}(X,t)$ is allowed to be different from $\dot{\boldsymbol{\eta}}_{\boldsymbol{K}}(X,t)$. Hence, in general, we set $\boldsymbol{\eta}_{\dot{\boldsymbol{K}}} \neq \dot{\boldsymbol{\eta}}_{\boldsymbol{K}}$. The homotopies $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ are formal, since material points and time are not transformed. They are introduced with the sole purpose of defining rigorously the variation of a generic function $f = \hat{f} \circ \natural_{\mathrm{c}}$ as $\tilde{f} := \hat{f} \circ \tilde{\natural}_{\mathrm{c}}$, with the understanding that $\tilde{\natural}_{\mathrm{c}} := (\tilde{\boldsymbol{F}}, \tilde{\boldsymbol{K}}; \tilde{\boldsymbol{V}}; \tilde{\mathcal{X}}, \tilde{\mathcal{T}})$, and

$$\tilde{f}(X,t,\varepsilon) = \hat{f}(\tilde{\natural}_{\mathrm{c}}(X,t,\varepsilon)) = \hat{f}(\tilde{\boldsymbol{F}}(X,t,\varepsilon), \tilde{\boldsymbol{K}}(X,t,\varepsilon); \tilde{\boldsymbol{V}}(X,t,\varepsilon); X, t). \tag{25}$$

Equation (25) permits to formalize the homotopy $\tilde{\boldsymbol{\Omega}} := \hat{\boldsymbol{\Omega}} \circ \tilde{\natural}_{\mathrm{c}}$, and to write the increment associated with it, i.e., $\boldsymbol{\eta}_{\boldsymbol{\Omega}}(X,t) := \partial_\varepsilon \tilde{\boldsymbol{\Omega}}(X,t,0)$, as

$$\boldsymbol{\eta}_{\boldsymbol{\Omega}} = \left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \boldsymbol{F}} \circ \natural_{\mathrm{c}}\right)[\boldsymbol{\eta}_{\boldsymbol{F}}] + \left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \boldsymbol{K}} \circ \natural_{\mathrm{c}}\right)[\boldsymbol{\eta}_{\boldsymbol{K}}] + \left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\mathrm{c}}\right)[\boldsymbol{\eta}_{\dot{\boldsymbol{K}}}]. \tag{26}$$

We also compute the time derivative of $\boldsymbol{\eta}_{\boldsymbol{\Theta}}$, i.e.,

$$\dot{\boldsymbol{\eta}}_{\boldsymbol{\Theta}} = \left[\frac{\partial}{\partial t}\left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\mathrm{c}}\right)\right][\boldsymbol{\eta}_{\boldsymbol{K}}] + \left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\mathrm{c}}\right)[\dot{\boldsymbol{\eta}}_{\boldsymbol{K}}], \tag{27}$$

so that the difference $\boldsymbol{\eta}_{\boldsymbol{\Omega}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{\Theta}}$ yields (cf. Supplementary Material)

$$\boldsymbol{\eta}_{\boldsymbol{\Omega}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{\Theta}} = (\mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}})[\boldsymbol{\eta}_{\boldsymbol{F}}] + (\mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}})[\boldsymbol{\eta}_{\boldsymbol{K}}] + \left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\mathrm{c}}\right)[\boldsymbol{\eta}_{\dot{\boldsymbol{K}}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{K}}], \tag{28}$$

where the Euler–Lagrange operators $\mathcal{E}_{\boldsymbol{F}}$ and $\mathcal{E}_{\boldsymbol{K}}$ applied to $\hat{\boldsymbol{\Omega}}$ are given by the following fourth-order tensor fields

$$\mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}} := \frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \boldsymbol{F}} \circ \natural_{\mathrm{c}}, \tag{29a}$$

$$\mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}} := \frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \boldsymbol{K}} \circ \natural_{\mathrm{c}} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\mathrm{c}}\right). \tag{29b}$$

We remark that $\mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}}$ reduces to $\partial_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}} \circ \natural_{\mathrm{c}}$ because the variable $\dot{\boldsymbol{F}}$ is ignorable in the present framework and has thus been excluded from the list $\natural_{\mathrm{c}}$.

According to [46, 58, 90, 93], Equation (28) can be simplified by assuming the vanishing either of $\boldsymbol{\eta}_{\boldsymbol{\Omega}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{\Theta}}$ or of $\boldsymbol{\eta}_{\dot{\boldsymbol{K}}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{K}}$. For our purposes, we consider here the case $\boldsymbol{\eta}_{\boldsymbol{\Omega}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{\Theta}} = \boldsymbol{O}$, and, upon setting $\mathbb{J} = \partial_{\dot{\boldsymbol{K}}} \hat{\boldsymbol{\Omega}} \circ \natural_{\mathrm{c}}$, we achieve the important result (cf. Supplementary Material)

$$\begin{aligned} \boldsymbol{\eta}_{\dot{\boldsymbol{K}}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{K}} &= -(\mathbb{J}^{-1} \diamond \mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}})[\boldsymbol{\eta}_{\boldsymbol{F}}] - (\mathbb{J}^{-1} \diamond \mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}})[\boldsymbol{\eta}_{\boldsymbol{K}}] \\ &= \mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{\eta}_{\boldsymbol{F}}] + \mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{\eta}_{\boldsymbol{K}}], \end{aligned} \tag{30}$$

where the fourth-order tensor fields $\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}$ and $\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}$ are given by

$$\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}} := -\mathbb{J}^{-1} \diamond \mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}}, \qquad \mathbb{W}_{\boldsymbol{K}\boldsymbol{K}} := -\mathbb{J}^{-1} \diamond \mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}}. \tag{31}$$

Equation (30) is referred to as *transpositional relation*, since it shows that, if the quantities $\mathcal{E}_{\boldsymbol{F}}\hat{\boldsymbol{\Omega}}$ and $\mathcal{E}_{\boldsymbol{K}}\hat{\boldsymbol{\Omega}}$ are not null, $\boldsymbol{\eta}_{\dot{\boldsymbol{K}}}$ is different from $\dot{\boldsymbol{\eta}}_{\boldsymbol{K}}$ [58, 90].

3.2.1 Conditions on the Quasi-Velocities

By plugging the last sum on the right-hand side of Equation (30) into Equation (28), grouping together the factors of $\boldsymbol{\eta}_{\boldsymbol{F}}$ and $\boldsymbol{\eta}_{\boldsymbol{K}}$, and setting $\boldsymbol{\eta}_{\boldsymbol{\Omega}} - \dot{\boldsymbol{\eta}}_{\boldsymbol{\Theta}} = \boldsymbol{O}$, we obtain the equality

$$\boldsymbol{O} = \left\{\mathcal{E}_{\boldsymbol{F}}\hat{\boldsymbol{\Omega}} + \mathbb{J} \diamond \mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}\right\}[\boldsymbol{\eta}_{\boldsymbol{F}}] + \left\{\mathcal{E}_{\boldsymbol{K}}\hat{\boldsymbol{\Omega}} + \mathbb{J} \diamond \mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}\right\}[\boldsymbol{\eta}_{\boldsymbol{K}}]. \tag{32}$$

Following the line of thought developed in [71], we can now use Equation (32) as a condition to compute $\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}$ and $\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}$, as done in Supplementary Material for the discrete case, thereby requiring

$$\mathcal{E}_{\boldsymbol{F}}\hat{\boldsymbol{\Omega}} + \mathbb{J} \diamond \mathbb{W}_{\boldsymbol{K}\boldsymbol{F}} = \mathbb{O}, \qquad \mathcal{E}_{\boldsymbol{K}}\hat{\boldsymbol{\Omega}} + \mathbb{J} \diamond \mathbb{W}_{\boldsymbol{K}\boldsymbol{K}} = \mathbb{O}, \tag{33}$$

where $\mathbb{O}$ is the null fourth-order tensor. We remark that this can be done if $\boldsymbol{\eta}_{\boldsymbol{F}}$ and $\boldsymbol{\eta}_{\boldsymbol{K}}$ are linearly independent. In the present framework, they are regarded to be such, because the Lagrange multiplier technique is employed.

The conditions (33) are thus equivalent to those in Equation (31), and are preferable since they allow to determine $\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}$ and $\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}$ without directly inverting $\mathbb{J}$ (as will be seen later). It is also worth to remark that Equations $(33)_1$ and $(33)_2$ can be viewed as characterizing properties for $\hat{\boldsymbol{\Omega}}$ [71].

3.2.2 Best Choice of the Quasi-Velocities

Let us decompose the growth tensor as $\boldsymbol{K} = J_{\boldsymbol{K}}^{1/3}\boldsymbol{K}_{\mathrm{u}}$, where $\det \boldsymbol{K}_{\mathrm{u}} = 1$, and let us write the rate $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$ in the form

$$\boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = \boldsymbol{K}_{\mathrm{u}}^{-1}\dot{\boldsymbol{K}}_{\mathrm{u}} + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})\boldsymbol{I} = \boldsymbol{K}_{\mathrm{u}}^{-1}\dot{\boldsymbol{K}}_{\mathrm{u}} + \tfrac{1}{3}(\dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}})\boldsymbol{I}. \tag{34}$$

Since it holds that $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = \dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}}$, only the spherical part of $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$ is restricted by the constraint (4), which, indeed, becomes $\mathcal{C} = \dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}} - R = 0$. On the other hand, the time derivative of the isochoric part of $\boldsymbol{K}$, i.e., $\dot{\boldsymbol{K}}_{\mathrm{u}}$, is not involved in the constraint. In fact, $\dot{\boldsymbol{K}}_{\mathrm{u}}$ is subjected to no restrictions, except that it has to satisfy the property $\mathrm{tr}(\boldsymbol{K}_{\mathrm{u}}^{-1}\dot{\boldsymbol{K}}_{\mathrm{u}}) = 0$, true by construction, as can be deduced from Equation (34), or, equivalently, from the expression

$$\dot{\boldsymbol{K}}_{\mathrm{u}} = J_{\boldsymbol{K}}^{-1/3}\dot{\boldsymbol{K}} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})J_{\boldsymbol{K}}^{-1/3}\boldsymbol{K}. \tag{35}$$

Hence, $\dot{\boldsymbol{K}}_{\mathrm{u}}$ has only 8 independent tensor components. On the basis of these considerations, we choose as quasi-velocities the constraint (4) itself and eight independent components of $\dot{\boldsymbol{K}}_{\mathrm{u}}$. Thus, we set (cf. Supplementary Material)

$$\Omega^1{}_1 = \hat{\Omega}^1{}_1 \circ \natural_{\mathrm{c}} := \hat{\mathcal{C}} \circ \natural_{\mathrm{c}} = \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) - \hat{R} \circ \natural_{\gamma}, \tag{36a}$$

$$\Omega^1{}_B = \hat{\Omega}^1{}_B \circ \natural_{\mathrm{c}} := J_{\boldsymbol{K}}^{-1/3}\left[\dot{K}^1{}_B - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})K^1{}_B\right], \qquad B = 2, 3, \tag{36b}$$

$$\Omega^\beta{}_1 = \hat{\Omega}^\beta{}_1 \circ \natural_{\mathrm{c}} := J_{\boldsymbol{K}}^{-1/3}\left[\dot{K}^\beta{}_1 - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})K^\beta{}_1\right], \qquad \beta = 2, 3, \tag{36c}$$

$$\Omega^\beta{}_B = \hat{\Omega}^\beta{}_B \circ \natural_{\mathrm{c}} := J_{\boldsymbol{K}}^{-1/3}\left[\dot{K}^\beta{}_B - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})K^\beta{}_B\right], \qquad \beta, B = 2, 3. \tag{36d}$$

We now employ the definitions (36a)–(36d) in the conditions (33), which, thus, acquire a block-wise structure. Specifically, Equation $(33)_1$ gives

$$\left(\frac{\partial \hat{\Omega}^1{}_1}{\partial \dot{K}^\lambda{}_L} \circ \natural_c\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^\lambda{}_{La}{}^A = -\frac{\partial \hat{\Omega}^1{}_1}{\partial F^a{}_A} \circ \natural_c$$

$$\Rightarrow \quad (\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{K}^{-\mathrm{T}}])_a{}^A = \frac{\partial \hat{R}}{\partial F^a{}_A} \circ \natural_\gamma, \tag{37a}$$

$$\left(\frac{\partial \hat{\Omega}^1{}_B}{\partial \dot{K}^\lambda{}_L} \circ \natural_c\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^\lambda{}_{La}{}^A = -\frac{\partial \hat{\Omega}^1{}_B}{\partial F^a{}_A} \circ \natural_c$$

$$\Rightarrow \quad J_{\boldsymbol{K}}^{-1/3}\left\{(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^1{}_{Ba}{}^A - \frac{1}{3}K^1{}_B(\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{K}^{-\mathrm{T}}])_a{}^A\right\} = 0, \tag{37b}$$

$$\left(\frac{\partial \hat{\Omega}^\beta{}_1}{\partial \dot{K}^\lambda{}_L} \circ \natural_c\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^\lambda{}_{La}{}^A = -\frac{\partial \hat{\Omega}^\beta{}_1}{\partial F^a{}_A} \circ \natural_c$$

$$\Rightarrow \quad J_{\boldsymbol{K}}^{-1/3}\left\{(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^\beta{}_{1a}{}^A - \frac{1}{3}K^\beta{}_1(\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{K}^{-\mathrm{T}}])_a{}^A\right\} = 0, \tag{37c}$$

$$\left(\frac{\partial \hat{\Omega}^\beta{}_B}{\partial \dot{K}^\lambda{}_L} \circ \natural_c\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^\lambda{}_{La}{}^A = -\frac{\partial \hat{\Omega}^\beta{}_B}{\partial F^a{}_A} \circ \natural_c$$

$$\Rightarrow \quad J_{\boldsymbol{K}}^{-1/3}\left\{(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^\beta{}_{Ba}{}^A - \frac{1}{3}K^\beta{}_B(\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{K}^{-\mathrm{T}}])_a{}^A\right\} = 0, \tag{37d}$$

with $\beta, B = 2, 3$, $\lambda, L = 1, 2, 3$, and $a, A = 1, 2, 3$. The system (37a)–(37d) can be solved by substituting the right-hand side of Equation (37a) in all other equations. This provides all the components of $\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}$ except $(\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}})^1{}_{1a}{}^A$. Hence, by using these results in the first relation of Equation (33), $\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}$ becomes

$$\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}} = \frac{1}{3}\boldsymbol{K} \otimes \left(\frac{\partial \hat{R}}{\partial \boldsymbol{F}} \circ \natural_\gamma\right). \tag{38}$$

Next, we turn to Equation $(33)_2$ to compute $\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}$. Since $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$ in Equation (36a) is the time derivative of $\log(\det \boldsymbol{K})$, it belongs to the kernel of the Euler–Lagrange operator $\mathcal{E}_{\boldsymbol{K}}$, and, thus, we find $\mathcal{E}_{\boldsymbol{K}}\hat{\Omega}^1{}_1 = -\partial_{\boldsymbol{K}}\hat{R} \circ \natural_\gamma$. Analogously, since all the components of $\dot{\boldsymbol{K}}_{\mathrm{u}}$ written explicitly in Equations (36b)–(36d) are time derivatives of functions of $\boldsymbol{K}$, we find $\mathcal{E}_{\boldsymbol{K}}\hat{\Omega}^1{}_B = \boldsymbol{O}$, $\mathcal{E}_{\boldsymbol{K}}\hat{\Omega}^\beta{}_1 = \boldsymbol{O}$, and $\mathcal{E}_{\boldsymbol{K}}\hat{\Omega}^\beta{}_B = \boldsymbol{O}$. Therefore, Equation $(33)_2$ yields

$$\left(\frac{\partial \hat{\Omega}^1{}_1}{\partial \dot{K}^\lambda{}_L} \circ \natural_c\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^\lambda{}_{L\mu}{}^M = -\mathcal{E}_{K^\mu{}_M}\hat{\Omega}^1{}_1$$

$$\Rightarrow \quad (\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{K}^{-\mathrm{T}}])_\mu{}^M = \frac{\partial \hat{R}}{\partial K^\mu{}_M} \circ \natural_\gamma, \tag{39a}$$

$$\left(\frac{\partial \hat{\Omega}^1{}_B}{\partial \dot{K}^\lambda{}_L} \circ \natural_c\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^\lambda{}_{L\mu}{}^M = -\mathcal{E}_{K^\mu{}_M}\hat{\Omega}^1{}_B$$

$$\Rightarrow \quad J_{\boldsymbol{K}}^{-1/3}\left\{(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^1{}_{B\mu}{}^M - \frac{1}{3}K^1{}_B(\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{K}^{-\mathrm{T}}])_\mu{}^M\right\} = 0, \tag{39b}$$

$$\left(\frac{\partial \hat{\Omega}^{\beta}{}_{1}}{\partial \dot{K}^{\lambda}{}_{L}} \circ \natural_{\mathrm{c}}\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^{\lambda}{}_{L\mu}{}^{M} = -\mathcal{E}_{K^{\mu}{}_{M}} \hat{\Omega}^{\beta}{}_{1}$$

$$\Rightarrow \quad J_{\boldsymbol{K}}^{-1/3}\left\{(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^{\beta}{}_{1\mu}{}^{M} - \frac{1}{3}K^{\beta}{}_{1}(\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{K}^{-\mathrm{T}}])_{\mu}{}^{M}\right\} = 0, \tag{39c}$$

$$\left(\frac{\partial \hat{\Omega}^{\beta}{}_{B}}{\partial \dot{K}^{\lambda}{}_{L}} \circ \natural_{\mathrm{c}}\right)(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^{\lambda}{}_{L\mu}{}^{M} = -\mathcal{E}_{K^{\mu}{}_{M}} \hat{\Omega}^{\beta}{}_{B}$$

$$\Rightarrow \quad J_{\boldsymbol{K}}^{-1/3}\left\{(\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}})^{\beta}{}_{B\mu}{}^{M} - \frac{1}{3}K^{\beta}{}_{B}(\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{K}^{-\mathrm{T}}])_{\mu}{}^{M}\right\} = 0, \tag{39d}$$

with $\beta, B = 2, 3$ and $\lambda, \mu, L, M = 1, 2, 3$. Also in this case, substitution of Equation (39a) into (39b)–(39d) and using the second relation of Equation (33) lead to

$$\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}} = \frac{1}{3}\boldsymbol{K} \otimes \left(\frac{\partial \hat{R}}{\partial \boldsymbol{K}} \circ \natural_{\gamma}\right). \tag{40}$$

3.3 Quasi-Coordinates and Their Variation for the Growth Problem

Comparing Equations (36a)–(36d) with the general definitions (21a), we notice that, apart from $\hat{\Omega}^{1}{}_{1} \circ \natural_{\mathrm{c}}$, all the other quasi-velocities are total time derivatives of $\boldsymbol{K}_{\mathrm{u}}$, which is a function of $\boldsymbol{K}$, only. Accordingly, by virtue of the relations

$$\dot{\Theta}^{1}{}_{1} = \Omega^{1}{}_{1} = \mathcal{C} \equiv \hat{\mathcal{C}} \circ \natural_{\mathrm{c}} = \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) - \hat{R} \circ \natural_{\gamma} = 0, \tag{41a}$$

$$\Omega^{1}{}_{B} \equiv \hat{\Omega}^{1}{}_{B} \circ \natural_{\mathrm{c}} = [\partial_t(J_{\boldsymbol{K}}^{-1/3}\boldsymbol{K})]^{1}{}_{B} = \dot{\Theta}^{1}{}_{B}, \qquad B = 2, 3, \tag{41b}$$

$$\Omega^{\beta}{}_{1} \equiv \hat{\Omega}^{\beta}{}_{1} \circ \natural_{\mathrm{c}} = [\partial_t(J_{\boldsymbol{K}}^{-1/3}\boldsymbol{K})]^{\beta}{}_{1} = \dot{\Theta}^{\beta}{}_{1} \qquad \beta = 2, 3, \tag{41c}$$

$$\Omega^{\beta}{}_{B} \equiv \hat{\Omega}^{\beta}{}_{B} \circ \natural_{\mathrm{c}} = [\partial_t(J_{\boldsymbol{K}}^{-1/3}\boldsymbol{K})]^{\beta}{}_{B} = \dot{\Theta}^{\beta}{}_{B}, \qquad \beta, B = 2, 3, \tag{41d}$$

and with appropriate initial conditions, the quasi-coordinates are given by

$$\Theta^{1}{}_{1}(X, t) - \Theta^{1}{}_{1}(X, t_{\mathrm{in}}) = \int_{t_{\mathrm{in}}}^{t} \hat{\mathcal{C}}(\boldsymbol{F}(X, s), \boldsymbol{K}(X, s); \dot{\boldsymbol{K}}(X, s); X, s)\mathrm{d}s = 0, \tag{42a}$$

$$\Theta^{1}{}_{B} = [\boldsymbol{K}_{\mathrm{u}}]^{1}{}_{B}, \quad \Theta^{\beta}{}_{1} = [\boldsymbol{K}_{\mathrm{u}}]^{\beta}{}_{1}, \quad \Theta^{\beta}{}_{B} = [\boldsymbol{K}_{\mathrm{u}}]^{\beta}{}_{B}, \tag{42b}$$

where, as above, β and B take on the values 2, 3, and $\Theta^{1}{}_{1}(X, t_{\mathrm{in}})$ is to be provided by means of suitable initial conditions. The nonholonomic nature of the quasi-velocity $\Omega^{1}{}_{1} = \hat{\mathcal{C}} \circ \natural_{\mathrm{c}}$ implies that the quasi-coordinate $\Theta^{1}{}_{1}$ has to be a functional of $\boldsymbol{F}$ and $\boldsymbol{K}$ and a function of material points and time. Indeed, by exploiting Equation (41a), it follows from (42a) that

$$\begin{aligned}&\Theta^{1}{}_{1}(X, t) - \Theta^{1}{}_{1}(X, t_{\mathrm{in}}) \\ &= \log\left(\frac{J_{\boldsymbol{K}}(X, t)}{J_{\boldsymbol{K}}(X, t_{\mathrm{in}})}\right) - \int_{t_{\mathrm{in}}}^{t} \hat{R}(\boldsymbol{F}(X, s), \boldsymbol{K}(X, s); X, s)\mathrm{d}s = 0.\end{aligned} \tag{43}$$

In addition, the fulfillment of $\mathcal{C} = 0$ implies $\Theta^{1}{}_{1}(X, t) - \Theta^{1}{}_{1}(X, t_{\mathrm{in}}) = 0$, and, thus, that $\Theta^{1}{}_{1}(X, t)$ equals its initial value $\Theta^{1}{}_{1}(X, t_{\mathrm{in}})$ at all times $t \geq t_{\mathrm{in}}$, which returns the formal solution to $\dot{J}_{\boldsymbol{K}} - J_{\boldsymbol{K}} R = 0$, i.e., the constraint itself.

From Equation (43), we notice that, *if* we take $\Theta^1{}_1(X, t_{\rm in}) = \log J_{\boldsymbol{K}}(X, t_{\rm in})$, and *if* we assume that the body finds itself in an initial state for which $J_{\boldsymbol{K}}(X, t_{\rm in}) = 1$, then we obtain $\Theta^1{}_1(X, t) = 0$ at all times $t \geq t_{\rm in}$, and for all $X \in \mathscr{B}$. Thus, if we further suppose $\boldsymbol{K} = J_{\boldsymbol{K}}^{1/3}\boldsymbol{I}$ (some authors [6] refer to this situation as *"spherical growth"*), then Equations (42a) and (42b) give

$$\Theta^1{}_1(X,t) = \Theta^1{}_1(X,t_{\rm in}) = 0, \qquad \Theta^1{}_B(X,t) = 0, \qquad B = 2,3, \tag{44a}$$

$$\Theta^\beta{}_1(X,t) = 0, \quad \beta = 2,3, \qquad \Theta^\beta{}_B(X,t) = \delta^\beta{}_B, \qquad \beta, B = 2,3, \tag{44b}$$

which means that the matrix associated with the quasi-coordinates is *singular* in the just analyzed case (whereas $\boldsymbol{K}$ is non-singular). This result is coherent with the fact that choosing the quasi-velocity $\Omega^1{}_1$ as coincident with the expression of the constraint implies that, if the constraint is fulfilled, the quasi-velocity $\Omega^1{}_1$ is zero. Consequently, the corresponding quasi-coordinate defines a state of rest, which becomes equal to the "origin" $\Theta^1{}_1(X, t_{\rm in}) = 0$ of an appropriate reference frame, if $J_{\boldsymbol{K}}(X, t_{\rm in}) = 1$. On the other hand, if one starts from a perturbed state $0 < J_{\boldsymbol{K}}(X, t_{\rm in}) \neq 1$, then $\Theta^1{}_1(X, t_{\rm in}) = \log J_{\boldsymbol{K}}(X, t_{\rm in})$ is different from zero (it could be negative, if $J_{\boldsymbol{K}}(X, t_{\rm in}) < 1$), and so is also $\Theta^1{}_1(X, t)$ at all times $t \geq t_{\rm in}$. This property descends from the fact that $\Theta^1{}_1(X, t_{\rm in})$ represents $J_{\boldsymbol{K}}(X, t_{\rm in})$ in logarithmic scale.

In conclusion, the representation of quasi-velocities and quasi-coordinates amounts to selecting a frame that co-moves with growth, in which, thus, no growth is perceived. For this reason, the (virtual) incremental field associated with $\Theta^1{}_1$, i.e., $[\boldsymbol{\eta}_{\boldsymbol{\Theta}}]^1{}_1$, must return Equation (7) divided by ε, that is,

$$[\boldsymbol{\eta}_{\boldsymbol{\Theta}}]^1{}_1 = \left(\frac{\partial \hat{\Omega}^1{}_1}{\partial \dot{K}^\beta{}_B} \circ \natural_{\rm c}\right)[\boldsymbol{\eta}_{\boldsymbol{K}}]^\beta{}_B = \left(\frac{\partial \hat{\mathcal{C}}}{\partial \dot{K}^\beta{}_B} \circ \natural_{\rm c}\right)[\boldsymbol{\eta}_{\boldsymbol{K}}]^\beta{}_B = 0, \tag{45}$$

as can be deduced from Equation (21b).

4 Equivalence with the TNHM

In this section, we present the result that we deem most fundamental for our formulation of growth. Specifically, we show that, although being nonholonomic, *the constraint* (4) *can be included in a suitably defined Lagrangian density function of the growing body*. In other words, one is able to handle the constraint "as if" it were holonomic, thereby allowing for a variational study of growth, up to the irreducible non-potential forces $\boldsymbol{f}_{\rm np}$ and $\mathfrak{S}_{\rm np}$.

4.1 Dynamic Equations in the System of the Quasi-Velocities

Written in the system of the quasi-velocities $\boldsymbol{\Omega} \equiv \hat{\boldsymbol{\Omega}} \circ \natural_{\rm c}$, the Lagrangian density function of the growing medium becomes

$$\check{\mathcal{L}} \circ \natural_{\boldsymbol{\Omega}} = \hat{\mathcal{L}} \circ \natural = \mathcal{L}, \tag{46}$$

where $\natural_{\boldsymbol{\Omega}}$ is the same collection of variables as $\natural$, with the exception of $\dot{\boldsymbol{K}}$, which is replaced with $\boldsymbol{\Omega}$.

If φ is a variable of $\natural$ such that neither φ nor $\dot{\varphi}$ are included in $\natural_{\rm c}$, then $\mathcal{E}_\varphi \check{\mathcal{L}}$ is identical to $\mathcal{E}_\varphi \hat{\mathcal{L}}$. However, if ψ is a variable of $\natural$ such that ψ itself and/or $\dot{\psi}$ are included in $\natural_{\rm c}$ (in fact, here $\dot{\psi}$ can be only $\dot{\boldsymbol{K}}$), then the chain rule implies the identities

$$\frac{\partial \hat{\mathcal{L}}}{\partial \psi} \circ \natural = \frac{\partial \check{\mathcal{L}}}{\partial \psi} \circ \natural_{\boldsymbol{\Omega}} + \left(\frac{\partial \check{\mathcal{L}}}{\partial \Omega^{\alpha}{}_{A}} \circ \natural_{\boldsymbol{\Omega}} \right) \left(\frac{\partial \hat{\Omega}^{\alpha}{}_{A}}{\partial \psi} \circ \natural_{\mathrm{c}} \right), \tag{47a}$$

$$\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\psi}} \circ \natural = \left(\frac{\partial \check{\mathcal{L}}}{\partial \Omega^{\alpha}{}_{A}} \circ \natural_{\boldsymbol{\Omega}} \right) \left(\frac{\partial \hat{\Omega}^{\alpha}{}_{A}}{\partial \dot{\psi}} \circ \natural_{\mathrm{c}} \right), \tag{47b}$$

and the corresponding Euler–Lagrange operator $\mathcal{E}_{\psi} \hat{\mathcal{L}}$ transforms accordingly. Since Equations (47a) and (47b) apply to $\boldsymbol{F}$ and $\boldsymbol{K}$ and $\dot{\boldsymbol{K}}$, we obtain

$$\mathcal{E}_{\boldsymbol{F}} \hat{\mathcal{L}} \equiv \frac{\partial \hat{\mathcal{L}}}{\partial \boldsymbol{F}} \circ \natural = \check{\mathcal{E}}_{\boldsymbol{F}} \check{\mathcal{L}} + (\mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}})^{\mathrm{T}} \left[\frac{\partial \check{\mathcal{L}}}{\partial \boldsymbol{\Omega}} \circ \natural_{\boldsymbol{\Omega}} \right], \tag{48a}$$

$$\mathcal{E}_{\boldsymbol{K}} \hat{\mathcal{L}} \equiv \frac{\partial \hat{\mathcal{L}}}{\partial \boldsymbol{K}} \circ \natural - \frac{\partial}{\partial t} \left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\boldsymbol{K}}} \circ \natural \right) = \mathbb{J}^{\mathrm{T}} [\check{\mathcal{E}}_{\boldsymbol{K}} \check{\mathcal{L}}] + (\mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}})^{\mathrm{T}} \left[\frac{\partial \check{\mathcal{L}}}{\partial \boldsymbol{\Omega}} \circ \natural_{\boldsymbol{\Omega}} \right], \tag{48b}$$

where we have exploited the definitions of $\mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}} \equiv \partial_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}} \circ \natural_{\mathrm{c}}$ and $\mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}}$ given in Equations (29a) and (29b), and we have introduced the notation

$$\check{\mathcal{E}}_{\boldsymbol{F}} \check{\mathcal{L}} := \frac{\partial \check{\mathcal{L}}}{\partial \boldsymbol{F}} \circ \natural_{\boldsymbol{\Omega}}, \tag{49a}$$

$$\check{\mathcal{E}}_{\boldsymbol{K}} \check{\mathcal{L}} := (\mathbb{J}^{-1})^{\mathrm{T}} \left[\frac{\partial \check{\mathcal{L}}}{\partial \boldsymbol{K}} \circ \natural_{\boldsymbol{\Omega}} \right] - \frac{\partial}{\partial t} \left(\frac{\partial \check{\mathcal{L}}}{\partial \boldsymbol{\Omega}} \circ \natural_{\boldsymbol{\Omega}} \right) \tag{49b}$$

to denote the transformed Euler–Lagrange operators $\check{\mathcal{E}}_{\boldsymbol{F}}$ and $\check{\mathcal{E}}_{\boldsymbol{K}}$ applied to the Lagrangian density function $\check{\mathcal{L}}$, expressed in the system of the quasi-velocities. On the other hand, the quantity

$$\boldsymbol{\pi}_{\boldsymbol{K}} := \frac{\partial \check{\mathcal{L}}}{\partial \boldsymbol{\Omega}} \circ \natural_{\boldsymbol{\Omega}} \tag{50}$$

can be expressed in terms of the generalized momentum $\boldsymbol{p}_{\boldsymbol{K}} = \partial_{\dot{\boldsymbol{K}}} \hat{\mathcal{L}} \circ \natural$, dual to $\dot{\boldsymbol{K}}$ through $\hat{\mathcal{L}}$, by inverting the relationship

$$\boldsymbol{p}_{\boldsymbol{K}} = \left(\frac{\partial \hat{\boldsymbol{\Omega}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\mathrm{c}} \right)^{\mathrm{T}} [\boldsymbol{\pi}_{\boldsymbol{K}}] = \mathbb{J}^{\mathrm{T}} [\boldsymbol{\pi}_{\boldsymbol{K}}] \quad \Rightarrow \quad \boldsymbol{\pi}_{\boldsymbol{K}} = (\mathbb{J}^{-1})^{\mathrm{T}} [\boldsymbol{p}_{\boldsymbol{K}}]. \tag{51}$$

We could refer to $\boldsymbol{\pi}_{\boldsymbol{K}}$ as *quasi-momentum*, since it is dual to the quasi-velocity $\boldsymbol{\Omega}$ through the duality relationship introduced by $\check{\mathcal{L}}$.

Let us substitute Equation (51) into (48a) and (48b), and let us employ the definitions (31) to see how the Euler–Lagrange operators $\mathcal{E}_{\boldsymbol{F}}$ and $\mathcal{E}_{\boldsymbol{K}}$ transform when switching from the collection $\natural$ to $\natural_{\boldsymbol{\Omega}}$, i.e., to the one of the quasi-velocities, for describing the medium's kinematics. By employing the notation $\diamond$ introduced in 2.1, we find (see also [46, 90])

$$\mathcal{E}_{\boldsymbol{F}} \hat{\mathcal{L}} = \check{\mathcal{E}}_{\boldsymbol{F}} \check{\mathcal{L}} + (\mathbb{J}^{-1} \diamond \mathcal{E}_{\boldsymbol{F}} \hat{\boldsymbol{\Omega}})^{\mathrm{T}} [\boldsymbol{p}_{\boldsymbol{K}}] = \check{\mathcal{E}}_{\boldsymbol{F}} \check{\mathcal{L}} - \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}} [\boldsymbol{p}_{\boldsymbol{K}}], \tag{52a}$$

$$\mathcal{E}_{\boldsymbol{K}} \hat{\mathcal{L}} = \mathbb{J}^{\mathrm{T}} [\check{\mathcal{E}}_{\boldsymbol{K}} \check{\mathcal{L}}] + (\mathbb{J}^{-1} \diamond \mathcal{E}_{\boldsymbol{K}} \hat{\boldsymbol{\Omega}})^{\mathrm{T}} [\boldsymbol{p}_{\boldsymbol{K}}] = \mathbb{J}^{\mathrm{T}} [\check{\mathcal{E}}_{\boldsymbol{K}} \check{\mathcal{L}}] - \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}} [\boldsymbol{p}_{\boldsymbol{K}}], \tag{52b}$$

where $\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}$ and $\mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}$ are given in Equations (38) and (40), respectively.

By virtue of the results (52a) and (52b), we rephrase the dynamic equations (16c) and (16e) in the system of the quasi-velocities as (see [46, 90, 93] for the derivation of similar equations for noholonomic discrete mechanical systems)

$$\check{\mathcal{E}}_{\boldsymbol{F}}\check{\mathcal{L}} - \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}] + \boldsymbol{T} = \boldsymbol{O}, \tag{53a}$$

$$\mathbb{J}^{\mathrm{T}}[\check{\mathcal{E}}_{\boldsymbol{K}}\check{\mathcal{L}}] = -\mathfrak{S}_{\mathrm{np}} + \mu(\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}} \circ \natural_{\mathrm{c}}) + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]. \tag{53b}$$

Before going further, the following remark is in order.

Remark 3 (The origin of the "extra terms") Equations (53a) and (53b) are *Hamel equations* [90], up to the reactive forces $\boldsymbol{T}$ and $\mu(\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}} \circ \natural_{\mathrm{c}})$. With respect to the "classical" Euler–Lagrange equations, they feature the "extra" generalized forces $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$, which both descend from the adoption of the quasi-velocities. In general, these terms are nonzero because the functions $\Omega^{\alpha}{}_{A} = \hat{\Omega}^{\alpha}{}_{A} \circ \natural_{\mathrm{c}}$ are not all the time derivatives of as many functions of $\boldsymbol{F}$, $\boldsymbol{K}$, $\mathcal{X}$, and $\mathcal{T}$ (note that the just mentioned list of variables coincides with $\natural_{\gamma}$). Indeed, if for each pair of indices $\alpha, A = 1, 2, 3$ there existed a C^1 function $\mathcal{F}^{\alpha}{}_{A} := \hat{\mathcal{F}}^{\alpha}{}_{A} \circ \natural_{\gamma}$ such that $\Omega^{\alpha}{}_{A} = \partial_t \mathcal{F}^{\alpha}{}_{A}$, then the quantities $\mathcal{E}_{\boldsymbol{F}}\hat{\boldsymbol{\Omega}}$ and $\mathcal{E}_{\boldsymbol{K}}\hat{\boldsymbol{\Omega}}$ would vanish identically (note that the first condition requires both $\hat{\mathcal{F}}^{\alpha}{}_{A}$ and $\hat{\Omega}^{\alpha}{}_{A}$ to be independent of $\boldsymbol{F}$). In fact, $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ could be viewed as *fictitious "polygenic forces"* [67] generated by the choice of the quasi-velocities. In other words, whereas the Euler–Lagrange dynamic equations are form-invariant in every system of coordinates, they lose this property when one switches to a system of generic quasi-velocities, and, as a result of this loss of invariance, they acquire the extra forces $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$. However, these forces disappear when one goes back to the original system of generalized velocities.

Based on Remark 3, we add $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ to both sides of Equations (16c) and (16e), respectively, thereby finding the *modified dynamic equations*

$$\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}] + \boldsymbol{T} = \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}], \tag{54a}$$

$$\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}] = -\mathfrak{S}_{\mathrm{np}} + \mu(\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}} \circ \natural_{\mathrm{c}}) + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]. \tag{54b}$$

Note that, in doing this, we are taking inspiration from the *"most general formulation of the principle of stationary action"* discussed in [90].

Although Equations (54a) and (54b) are a mere rewriting of (16c) and (16e), they unfold important properties:

P1. While the left-hand sides of Equations (16c) and (16e) are obtained by varying the action functional $\mathcal{A}_{\mathrm{a}}$ according to Hamilton's Principle, the left-hand sides of Equations (54a) and (54b) cannot be retrieved this way because of the extra forces $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$. However, they can be obtained by means of a variational procedure known as *Hamilton–Suslov Principle* (see, e.g., [71]). In brief, the Hamilton–Suslov method computes the variation of a given action functional by hypothesizing that, if φ is a Lagrangian parameter of the theory under consideration, and $\dot{\varphi}$ is its generalized velocity, then the incremental field $\boldsymbol{\eta}_{\dot{\varphi}}$ differs from the time derivative $\dot{\boldsymbol{\eta}}_{\varphi}$ of the variation by the so-called *transpositional relations* (see, e.g., Equation (30)). It can be proven that, if $\dot{\varphi}$ is not involved in any nonholonomic constraint, then the associated transpositional relations are null, and one finds $\boldsymbol{\eta}_{\dot{\varphi}} = \dot{\boldsymbol{\eta}}_{\varphi}$. However, this is not true, in general, when $\dot{\varphi}$ has to comply with a nonholonomic constraint. This is indeed the case for $\dot{\boldsymbol{K}}$ in the growth problem studied here, and the non-vanishing transpositional relations (30) are naturally accounted for by the Hamilton–Suslov method. In particular, they result into the extra forces $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$, which are, thus, an output of the procedure.

P2. Similarly to the right-hand side of Equation (16e), which collects the non-potential forces handled by means of the Extended Hamilton Principle [67],[2] also the right-hand sides of Equations (54a) and (54b) can be framed by "extending" the Hamilton–Suslov method: the variation of $\mathcal{A}_{\mathrm{a}}$ performed with the Hamilton–Suslov variations is set equal to minus the integral in time of the work done on the variations associated with $\boldsymbol{\eta}_{\boldsymbol{F}}$ and $\boldsymbol{\eta}_{\boldsymbol{K}}$ by all the non-potential forces dual to them, including the extra ones $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ that feature on the right-hand sides. In doing this, one has to admit the existence of these fictitious extra forces *a priori*. Although, on the one hand, this may sound artificial, on the other hand, $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ do have a physical meaning. Namely, they represent two rates of momentum introduced in the system by the constraint. In the specific case of growth, upon recognizing that $P_{\boldsymbol{K}} := \frac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\boldsymbol{p}_{\boldsymbol{K}})$ is the spherical part of the fully material generalized momentum $\boldsymbol{K}^{\mathrm{T}}\boldsymbol{p}_{\boldsymbol{K}} = \boldsymbol{K}^{\mathrm{T}}(\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural)$, the results (38) and (40) are such that the extra forces

$$\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}] = \left\{\frac{1}{3}\boldsymbol{K} \otimes \left(\frac{\partial \hat{R}}{\partial \boldsymbol{F}} \circ \natural_{\gamma}\right)\right\}^{\mathrm{T}}[\boldsymbol{p}_{\boldsymbol{K}}] = P_{\boldsymbol{K}}\left(\frac{\partial \hat{R}}{\partial \boldsymbol{F}} \circ \natural_{\gamma}\right), \tag{55a}$$

$$\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}] = \left\{\frac{1}{3}\boldsymbol{K} \otimes \left(\frac{\partial \hat{R}}{\partial \boldsymbol{K}} \circ \natural_{\gamma}\right)\right\}^{\mathrm{T}}[\boldsymbol{p}_{\boldsymbol{K}}] = P_{\boldsymbol{K}}\left(\frac{\partial \hat{R}}{\partial \boldsymbol{K}} \circ \natural_{\gamma}\right) \tag{55b}$$

can be interpreted as momentum rates due to the *coupling* of the mass source R with the system's degrees of freedom represented by $\boldsymbol{F}$ and $\boldsymbol{K}$. Moreover, since the constraint (4) is made nonholonomic by R, the right-hand sides of Equations (55a) and (55b) "measure" how nonholonomic the constraint is.

In the remainder of this section, we show that Equations (54a) and (54b) can be obtained variationally by letting their left-hand sides originate from the Hamilton–Suslov Principle, and interpreting the terms $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ on the right-hand sides as non-potential forces to be accounted for by means of an "Extended Hamilton–Suslov Principle". We accomplish this task by (i) taking the weak forms of Equations (54a) and (54b) through multiplication with $\boldsymbol{\eta}_{\boldsymbol{F}}$ and $\boldsymbol{\eta}_{\boldsymbol{K}}$, respectively; (ii) combining the results with the weak forms of the dynamic equations (16a) and (16b); (iii) integrating over $\mathscr{B}$, $\partial^{\chi}_{\mathrm{N}}\mathscr{B}$, and $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$; and (iv) showing that one achieves the variation of the action according to the Hamilton–Suslov Principle, set equal to minus the work done by the polygenic and the extra forces on their respective dual variations.

If, on the one hand, doing so amounts to going back along the procedure designated by Llibre et al. [71], on the other hand, our result seems to us more general and, to the best of our understanding, capable of explaining when and why the method proposed in [71] is equivalent to the TNHM. This seems to us a very important issue for the following reason.

According to *our* results, it seems that the constraint given in Equation (4) cannot be studied by means of the MVM formulated in [71], since this version of the MVM produces Equations (54a) and (54b) only *up to* the extra forces $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ featuring

[2] With reference to Equation (16e), this means that the variation of the action functional computed with Hamilton's variations, and through the derivative reported in Equation (15), is set equal to the time integral of the negative of the work done by the *"polygenic forces"* [67] dual to $\boldsymbol{\eta}_{\boldsymbol{K}}$ on the variations associated with $\boldsymbol{\eta}_{\boldsymbol{K}}$ itself.

on their right-hand sides. Hence, in our opinion, the MVM of [71] is not equivalent to the TNHM for growth mechanics viewed as a constrained field theory. On the other hand, $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}]$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]$ on the right-hand sides of Equations (54a) and (54b) are necessary to grant the equivalence with the TNHM, and such equivalence is necessary to make the MVM applicable to the mechanics of growth. For this reason, the purpose of the forthcoming calculations is to show how to accommodate for this issue.

4.2 The Lagrangian Density Function of a Growing Medium

We set $\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}} \circ \natural_{\mathrm{c}} = \boldsymbol{K}^{-\mathrm{T}}$ in Equation (53b), as follows from Equation (4). However, the procedure outlined in the sequel applies to any non-singular second-order tensor field $\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}} \circ \natural_{\mathrm{c}}$. Moreover, we write

$$\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}} := \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\boldsymbol{p}_{\boldsymbol{K}}], \qquad \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} := \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\boldsymbol{p}_{\boldsymbol{K}}]. \tag{56}$$

Note that $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}$ and $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$ have components $[\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}]_a{}^A$ and $[\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}]_\alpha{}^A$. It is also convenient to introduce the fully material quantities $\boldsymbol{F}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}$ and $\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$, and, accordingly, to transform Equations (54a) and (54b) as

$$\boldsymbol{F}^{\mathrm{T}}[\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}}] + \boldsymbol{F}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}} + \boldsymbol{F}^{\mathrm{T}}\boldsymbol{T} = \boldsymbol{F}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}, \tag{57a}$$

$$\boldsymbol{K}^{\mathrm{T}}[\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}}] + \boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} = -\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\mathrm{np}} + \mu\boldsymbol{I}^{\mathrm{T}} + \boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}. \tag{57b}$$

In Equation (57b), the term $\mu\boldsymbol{I}^{\mathrm{T}}$ suggests to project the equation itself onto the space of spherical tensors in order to compute μ. To this end, we consider the summand $\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$ on the right-hand side of Equation (57b), and we decompose it as

$$\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}})\,\boldsymbol{I}^{\mathrm{T}} + \mathrm{Dev}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}). \tag{58}$$

Then, by introducing the *rescaled* Lagrange multiplier

$$\kappa := \mu + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}), \tag{59}$$

and recalling that $\boldsymbol{p}_{\boldsymbol{K}} = \partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural$, we rewrite Equation (57b) as

$$\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] = -\mathfrak{S}_{\mathrm{np}} + \kappa\,\boldsymbol{K}^{-\mathrm{T}} + \mathrm{DEV}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}, \tag{60}$$

with $\mathrm{DEV}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} := \boldsymbol{K}^{-\mathrm{T}}\mathrm{Dev}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}) = \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}})\boldsymbol{K}^{-\mathrm{T}}$ being the $\boldsymbol{K}^{-\mathrm{T}}$-deviatoric part of $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$.

By using Equations (54a) and (60) in lieu of (16c) and (16e), respectively, in the system (16a)–(16f), we can write

$$\mathcal{E}_{\chi}\hat{\mathcal{L}} + \mathrm{Div}\boldsymbol{T} = -\boldsymbol{f}_{\mathrm{np}}, \qquad \text{in } \mathscr{B}, \tag{61a}$$

$$\boldsymbol{\tau} - \boldsymbol{T}\boldsymbol{N} = \boldsymbol{0}, \qquad \text{on } \partial^{\chi}_{\mathrm{N}}\mathscr{B}, \tag{61b}$$

$$\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{F}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] + \boldsymbol{T} = \mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}, \qquad \text{in } \mathscr{B}, \tag{61c}$$

$$\boldsymbol{F} - D\chi = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{61d}$$

$$\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{K}\boldsymbol{K}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] = -\mathfrak{S}_{\mathrm{np}} + \kappa\,\boldsymbol{K}^{-\mathrm{T}} + \mathrm{DEV}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}, \qquad \text{in } \mathscr{B}, \tag{61e}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle - R = 0, \qquad \text{in } \mathscr{B}. \tag{61f}$$

Recalling that $\boldsymbol{\eta}_\chi(X,t)=\mathbf{0}$ for $X\in\partial_{\mathrm{D}}^{\chi}\mathscr{B}$ and at all times, we consider the duality pairs between Equations (61a) and $\boldsymbol{\eta}_\chi$, and between (61b) and $\boldsymbol{\eta}_\chi$, we integrate the resulting expressions over $\mathscr{B}$ and $\partial_{\mathrm{N}}^{\chi}\mathscr{B}$, respectively, and we add them together. Then, after some algebraic passages, integrating over the time interval $[t_{\mathrm{in}},t_{\mathrm{fin}}]$, and enforcing the conditions $\boldsymbol{\eta}_\chi(X,t_{\mathrm{in}})=\boldsymbol{\eta}_\chi(X,t_{\mathrm{fin}})=\mathbf{0}$, for all $X\in\mathscr{B}$, we obtain

$$\begin{aligned}&\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\mathscr{B}}\left[\left\langle\frac{\partial\hat{\mathcal{L}}}{\partial\chi}\circ\natural\middle|\boldsymbol{\eta}_\chi\right\rangle+\left\langle\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\chi}}\circ\natural\middle|\boldsymbol{\eta}_{\dot{\chi}}\right\rangle-\langle\boldsymbol{T}\,|\,\mathrm{Grad}\,\boldsymbol{\eta}_\chi\rangle\right]\\&=-\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\mathscr{B}}\langle\boldsymbol{f}_{\mathrm{np}}|\boldsymbol{\eta}_\chi\rangle-\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\partial_{\mathrm{N}}^{\chi}\mathscr{B}}\langle\boldsymbol{\tau}|\boldsymbol{\eta}_\chi\rangle,\end{aligned}\tag{62}$$

where we have used the abbreviated notations $\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\mathscr{B}}[\ldots]\equiv\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\{\int_{\mathscr{B}}[\ldots]\mathrm{dV}\}\mathrm{d}t$ and $\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\partial_{\mathrm{N}}^{\chi}\mathscr{B}}[\ldots]\equiv\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\{\int_{\partial_{\mathrm{N}}^{\chi}\mathscr{B}}[\ldots]\mathrm{dA}\}\mathrm{d}t$, and the identity $\dot{\boldsymbol{\eta}}_\chi=\boldsymbol{\eta}_{\dot{\chi}}$, which is true since the velocity $\dot{\chi}$ is not involved in the constraint.

Next, we take the duality pairs between Equation (61c) and $\boldsymbol{\eta}_{\boldsymbol{F}}$, and between Equation (61e) and $\boldsymbol{\eta}_{\boldsymbol{K}}$, and we integrate over $\mathscr{B}$ and $[t_{\mathrm{in}},t_{\mathrm{fin}}]$. Then, by recalling the identity $\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}}\equiv\partial_{\boldsymbol{F}}\hat{\mathcal{L}}\circ\natural$, the conditions $\boldsymbol{\eta}_{\boldsymbol{K}}(X,t_{\mathrm{in}})=\boldsymbol{\eta}_{\boldsymbol{K}}(X,t_{\mathrm{fin}})=\boldsymbol{O}$ for all $X\in\mathscr{B}$, and the transpositional relation in Equation (30), we find

$$\begin{aligned}&\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\mathscr{B}}\left[\left\langle\frac{\partial\hat{\mathcal{L}}}{\partial\boldsymbol{F}}\circ\natural+\boldsymbol{T}\middle|\boldsymbol{\eta}_{\boldsymbol{F}}\right\rangle+\left\langle\frac{\partial\hat{\mathcal{L}}}{\partial\boldsymbol{K}}\circ\natural\middle|\boldsymbol{\eta}_{\boldsymbol{K}}\right\rangle+\left\langle\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\boldsymbol{K}}}\circ\natural\middle|\boldsymbol{\eta}_{\dot{\boldsymbol{K}}}\right\rangle\right]\\&=\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\mathscr{B}}\left[\langle\mathfrak{S}_{\boldsymbol{KF}}\,|\,\boldsymbol{\eta}_{\boldsymbol{F}}\rangle-\langle\mathfrak{S}_{\mathrm{np}}-\kappa\boldsymbol{K}^{-\mathrm{T}}-\mathrm{DEV}\mathfrak{S}_{\boldsymbol{KK}}\,|\,\boldsymbol{\eta}_{\boldsymbol{K}}\rangle\right].\end{aligned}\tag{63}$$

An important step forward is done if it is possible to find a C^1-function of time, λ, such that $\dot{\lambda}=\kappa$. We assume that this is the case, and, after rewriting $\boldsymbol{K}^{-\mathrm{T}}$ as $\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}}\circ\natural_{\mathrm{c}}$, performing some algebraic calculations, and invoking the relationship (30), we work out the term $\langle\kappa\,\boldsymbol{K}^{-\mathrm{T}}\,|\,\boldsymbol{\eta}_{\boldsymbol{K}}\rangle$ in Equation (63) as

$$\begin{aligned}\langle\kappa\,\boldsymbol{K}^{-\mathrm{T}}\,|\,\boldsymbol{\eta}_{\boldsymbol{K}}\rangle&=\dot{\lambda}\left\langle\frac{\partial\hat{\mathcal{C}}}{\partial\dot{\boldsymbol{K}}}\circ\natural_{\mathrm{c}}\middle|\boldsymbol{\eta}_{\boldsymbol{K}}\right\rangle=\frac{\partial}{\partial t}\left[\lambda\left\langle\frac{\partial\hat{\mathcal{C}}}{\partial\dot{\boldsymbol{K}}}\circ\natural_{\mathrm{c}}\middle|\boldsymbol{\eta}_{\boldsymbol{K}}\right\rangle\right]\\&+\lambda\left\langle\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{C}}+\mathbb{W}_{\boldsymbol{KF}}^{\mathrm{T}}\left[\frac{\partial\hat{\mathcal{C}}}{\partial\dot{\boldsymbol{K}}}\circ\natural_{\mathrm{c}}\right]\middle|\boldsymbol{\eta}_{\boldsymbol{F}}\right\rangle+\lambda\left\langle\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{C}}+\mathbb{W}_{\boldsymbol{KK}}^{\mathrm{T}}\left[\frac{\partial\hat{\mathcal{C}}}{\partial\dot{\boldsymbol{K}}}\circ\natural_{\mathrm{c}}\right]\middle|\boldsymbol{\eta}_{\boldsymbol{K}}\right\rangle\\&-\lambda\left\langle\frac{\partial\hat{\mathcal{C}}}{\partial\boldsymbol{F}}\circ\natural_{\mathrm{c}}\middle|\boldsymbol{\eta}_{\boldsymbol{F}}\right\rangle-\lambda\left\langle\frac{\partial\hat{\mathcal{C}}}{\partial\boldsymbol{K}}\circ\natural_{\mathrm{c}}\middle|\boldsymbol{\eta}_{\boldsymbol{K}}\right\rangle-\lambda\left\langle\frac{\partial\hat{\mathcal{C}}}{\partial\dot{\boldsymbol{K}}}\circ\natural_{\mathrm{c}}\middle|\boldsymbol{\eta}_{\dot{\boldsymbol{K}}}\right\rangle,\end{aligned}\tag{64}$$

where $\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{C}}=\partial_{\boldsymbol{F}}\hat{\mathcal{C}}\circ\natural_{\mathrm{c}}$.

Another important deduction stems from Equations (33), (37a) and (39a). Indeed, by virtue of the identification of $\mathcal{C}$ with the quasi-velocity $\Omega^1{}_1$, and since Equation (64) must hold at this stage for any λ, $\boldsymbol{\eta}_{\boldsymbol{F}}$, and $\boldsymbol{\eta}_{\boldsymbol{K}}$, we obtain

$$\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{C}}+\mathbb{W}_{\boldsymbol{KF}}^{\mathrm{T}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}}\circ\natural_{\mathrm{c}}]=\boldsymbol{O},\qquad\qquad\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{C}}+\mathbb{W}_{\boldsymbol{KK}}^{\mathrm{T}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{C}}\circ\natural_{\mathrm{c}}]=\boldsymbol{O}.\tag{65}$$

Moreover, by integrating Equation (64) over $\mathscr{B}$ and $[t_{\mathrm{in}},t_{\mathrm{fin}}]$, we find

$$\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}}\int_{\mathscr{B}}\langle\kappa\,\boldsymbol{K}^{-\mathrm{T}}\,|\,\boldsymbol{\eta}_{\boldsymbol{K}}\rangle$$

$$= -\int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \lambda\left[\left\langle \frac{\partial \hat{\mathcal{C}}}{\partial \boldsymbol{F}} \circ \natural_{\text{c}} \middle| \boldsymbol{\eta}_{\boldsymbol{F}} \right\rangle + \left\langle \frac{\partial \hat{\mathcal{C}}}{\partial \boldsymbol{K}} \circ \natural_{\text{c}} \middle| \boldsymbol{\eta}_{\boldsymbol{K}} \right\rangle + \left\langle \frac{\partial \hat{\mathcal{C}}}{\partial \dot{\boldsymbol{K}}} \circ \natural_{\text{c}} \middle| \boldsymbol{\eta}_{\dot{\boldsymbol{K}}} \right\rangle\right]. \tag{66}$$

Now, we introduce the *varied constraint* $\tilde{\mathcal{C}} = \hat{\mathcal{C}} \circ \tilde{\natural}_{\text{c}}$, where $\tilde{\natural}_{\text{c}}$ is the collection of the homotopies defined in Equations (24a)–(24e), and we consider also the homotopy $\lambda(X,t) \mapsto \tilde{\lambda}(X,t,\varepsilon) = \lambda(X,t) + \eta_\lambda(X,t)\varepsilon + o(\varepsilon)$, for $\varepsilon \to 0$, such that $\partial_\varepsilon \tilde{\lambda}(X,t,0) = \eta_\lambda(X,t)$ is the increment of $\lambda(X,t)$. Then, by exploiting the fact that the product $-\eta_\lambda[\hat{\mathcal{C}} \circ \natural_{\text{c}}]$ returns the constraint up to the arbitrary factor η_λ, and is, thus, null, we can rephrase Equation (66) as

$$\int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \langle \kappa\, \boldsymbol{K}^{-\text{T}} \,|\, \boldsymbol{\eta}_{\boldsymbol{K}} \rangle = -\int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \frac{\partial\{\tilde{\lambda}[\hat{\mathcal{C}} \circ \tilde{\natural}_{\text{c}}]\}}{\partial \varepsilon}(X,t,0). \tag{67}$$

Granted (67), we introduce the *constrained Lagrangian density function* and the associated *constrained action functional*

$$\mathcal{L}_{\text{c}} := \hat{\mathcal{L}}_{\text{c}} \circ (\natural; \boldsymbol{T}, \lambda) = \hat{\mathcal{L}} \circ \natural + \langle \boldsymbol{T} \,|\, \boldsymbol{F} - D\chi \rangle + \lambda\,[\hat{\mathcal{C}} \circ \natural_{\text{c}}], \tag{68a}$$

$$\mathcal{A}_{\text{c}}(\chi, \boldsymbol{F}, \boldsymbol{K}; \boldsymbol{T}, \lambda) := \int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \hat{\mathcal{L}}_{\text{c}} \circ (\natural; \boldsymbol{T}, \lambda), \tag{68b}$$

and we write the varied action $\tilde{\mathcal{A}}_{\text{c}}$ and its first derivative with respect to ε, evaluated at $\varepsilon = 0$, as

$$\tilde{\mathcal{A}}_{\text{c}}(\varepsilon) := \int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \hat{\mathcal{L}}_{\text{c}}(\tilde{\natural}(X,t,\varepsilon); \tilde{\boldsymbol{T}}(X,t,\varepsilon), \tilde{\lambda}(X,t,\varepsilon)), \tag{69a}$$

$$\frac{\mathrm{d}\tilde{\mathcal{A}}_{\text{c}}}{\mathrm{d}\varepsilon}(0) = \int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \frac{\partial[\hat{\mathcal{L}}_{\text{c}} \circ (\tilde{\natural}; \tilde{\boldsymbol{T}}, \tilde{\lambda})]}{\partial \varepsilon}(X,t,0). \tag{69b}$$

Then, a direct calculation shows that the sum of Equations (62) and (63) is identical to the compact expression

$$\begin{aligned}\frac{\mathrm{d}\tilde{\mathcal{A}}_{\text{c}}}{\mathrm{d}\varepsilon}(0) = &-\int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \langle \boldsymbol{f}_{\text{np}} | \boldsymbol{\eta}_\chi \rangle - \int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\partial_{\text{N}}^{\chi}\mathscr{B}} \langle \boldsymbol{\tau} | \boldsymbol{\eta}_\chi \rangle - \int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \langle \mathfrak{S}_{\text{np}} \,|\, \boldsymbol{\eta}_{\boldsymbol{K}} \rangle \\ &+ \int_{t_{\text{in}}}^{t_{\text{fin}}}\int_{\mathscr{B}} \left\{\langle \mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}} \,|\, \boldsymbol{\eta}_{\boldsymbol{F}} \rangle + \langle \text{DEV}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} \,|\, \boldsymbol{\eta}_{\boldsymbol{K}} \rangle\right\}. \end{aligned} \tag{70}$$

Equation (68a) defines the Lagrangian density function of the growing medium which we were looking for. Furthermore, the dynamics of the growing body is obtained variationally by "extending" the Hamilton–Suslov Principle at the price of considering the extra forces $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}$ and $\text{DEV}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$, due to the nonholonomic nature of the constraint, in addition to the same non-potential forces $\boldsymbol{f}_{\text{np}}$ and $\mathfrak{S}_{\text{np}}$ appearing also in the TNHM. This confirms what has been said at the point P2 of Sect. 4.1. Thus, starting with Equation (70), taking the first-order increment of $\dot{\boldsymbol{K}}$ as $\boldsymbol{\eta}_{\dot{\boldsymbol{K}}}$ (which differs from $\dot{\boldsymbol{\eta}}_{\boldsymbol{K}}$ as specified in the transpositional relation (30)), and going backward until (61a)–(61f) are recovered, the dynamic problem is entirely equivalent to the one stated in Equations (16a)–(16f), deduced from the TNHM. This is, in fact, our reformulation for growth mechanics of the MVM by Llibre et al. [71].

4.3 Implications of Our Formulation of the MVM for Growth Mechanics

The introduction of $\mathcal{L}_{\text{c}}$ and the whole procedure shown in Sects. 4.1 and 4.2 produce some results that we consider noteworthy.

4.3.1 The "True" Dynamic Equations

Since it holds that $\mathfrak{S}_{KF} = \mathbb{W}^{\mathrm{T}}_{KF}[\partial_{\dot{K}}\hat{\mathcal{L}} \circ \natural]$, Equation (61c) returns the well-established result $\boldsymbol{T} = -\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}} = -\partial_{\boldsymbol{F}}\hat{\mathcal{L}} \circ \natural$. Moreover, since Equation (61d) prescribes $\boldsymbol{F} = D\chi$, the replacement of $\boldsymbol{F}$ with $D\chi$ in the arguments of $\hat{\mathcal{L}}$ (redefining $\hat{\mathcal{L}}$ accordingly) permits to write Equations (61a) and (61b) as [85]

$$\frac{\partial \hat{\mathcal{L}}}{\partial \chi} \circ \natural - \frac{\partial}{\partial t}\left(\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\chi}} \circ \natural\right) - \mathrm{Div}\left(\frac{\partial \hat{\mathcal{L}}}{\partial D\chi} \circ \natural\right) = -\boldsymbol{f}_{\mathrm{np}}, \qquad \text{in } \mathscr{B}, \tag{71a}$$

$$\boldsymbol{\tau} = \left(-\frac{\partial \hat{\mathcal{L}}}{\partial D\chi} \circ \natural\right)\boldsymbol{N}, \qquad \text{on } \partial^{\chi}_{\mathrm{N}}\mathscr{B}. \tag{71b}$$

Thus, $\mathfrak{S}_{KF}$ does not contribute to the "true" dynamics of the system, although it is necessary to formulate the variational procedure presented in Sect. 4.2.

We also notice that, if the Lagrangian density function $\hat{\mathcal{L}}$ is defined in such a way that its partial derivative with respect to $D\chi$ equals the negative of the partial derivative of the strain energy density $\hat{\Psi}$ with respect to the same quantity, then, from Equation (12), we obtain that $\boldsymbol{T}$ coincides with the first Piola–Kirchhoff stress tensor of the material, i.e.,

$$\boldsymbol{P} = \frac{\partial \hat{\Psi}}{\partial \boldsymbol{F}} \circ (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) = \boldsymbol{T} = -\frac{\partial \hat{\mathcal{L}}}{\partial D\chi} \circ \natural, \qquad \boldsymbol{F} = D\chi. \tag{72}$$

Equations (71a) and (71b) must be solved together with those for $\boldsymbol{K}$ and the new Lagrange multiplier λ. These, when deduced from Equation (70), are directly given by

$$\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{KK}[\partial_{\dot{K}}\hat{\mathcal{L}} \circ \natural] = -\mathfrak{S}_{\mathrm{np}} + \dot{\lambda}\,\boldsymbol{K}^{-\mathrm{T}} + \mathrm{DEV}\mathfrak{S}_{KK}, \tag{73a}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle - R = 0, \tag{73b}$$

and replace (61e) and (61f) with $\kappa \equiv \dot{\lambda}$. Recall that $\mathfrak{S}_{KK} = \mathbb{W}^{\mathrm{T}}_{KK}[\partial_{\dot{K}}\hat{\mathcal{L}} \circ \natural]$.

Left-multiplying by $\boldsymbol{K}^{\mathrm{T}}$, and extracting the hydrostatic and the volumetric parts of Equation (73a) yield [47–50]

$$\mathrm{Dev}[\boldsymbol{K}^{\mathrm{T}}(\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}})] = -\mathrm{Dev}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\mathrm{np}}], \tag{74a}$$

$$\dot{\lambda} = \tfrac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}(\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}}) + \boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\mathrm{np}}] + \tfrac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{KK}], \tag{74b}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle - R = 0. \tag{74c}$$

Therefore, the deviatoric term $\mathrm{DEV}\mathfrak{S}_{KK}$ does not contribute to the "true" dynamics of the problem, while only $\frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{KK}]$ plays a role in it. However, this generalized force does not influence directly the determination of $\boldsymbol{K}$, and is indeed absorbed in $\dot{\lambda}$, thereby quantifying the entity of the reactive force $\dot{\lambda}\boldsymbol{K}^{-\mathrm{T}}$ predicted by the theory under consideration, and necessary to maintain the constraint. Also, the term $\frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{KK}]$ constitutes a fundamental aspect of the procedure, since it defines the Lagrange multiplier of the MVM, which is inherently different from the one characterizing the TNHM. Yet, Equation (74b) returns the identification (59), and makes the MVM equivalent to the TNHM, provided the equality $\mu = \frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}(\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}}) + \boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\mathrm{np}}]$ holds true, as can be deduced by extracting the hydrostatic part of Equation (16e) left-multiplied by $\boldsymbol{K}^{\mathrm{T}}$. Hence, this yields $\dot{\lambda} = \mu + \frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{KK}]$. To our knowledge, this result represents a novelty in the context of growth mechanics, and generalizes a result reported in [71] for the case of discrete nonholonomic systems.

To gain further physical insight into the MVM, we deem it noteworthy to comment on the explicit expression of the term $\frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}]$. Indeed, looking at the definition of $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$ in Equation (56) and at the result reported in Equation (55b), we find

$$\tfrac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}] = \tfrac{1}{3}P_{\boldsymbol{K}}\mathrm{tr}\big[\boldsymbol{K}^{\mathrm{T}}(\partial_{\boldsymbol{K}}\hat{R}\circ\natural_{\gamma})\big], \tag{75}$$

so that only the hydrostatic part of $\boldsymbol{K}^{\mathrm{T}}(\partial_{\boldsymbol{K}}\hat{R}\circ\natural_{\gamma})$ contributes to $\dot{\lambda}$.

The nine tensor components of $\boldsymbol{K}$ are determined by Equations (74a) and (74c), equivalent to eight and to one scalar equations, respectively (see also [47–50]). In addition, λ is determined by integrating Equation (74b) in time. This requires an initial condition for λ, i.e., $\lambda(X, t_{\mathrm{in}}) = \lambda_{\mathrm{in}}(X)$, for all $X \in \mathscr{B}$, and to assign $\lambda_{\mathrm{in}}(X)$ consistently with the constraint at the initial time. Obtaining the Lagrange multiplier of the theory by solving a Cauchy problem constitutes a difference with respect to the TNHM, in which the Lagrange multiplier μ is computed algebraically, and is a characteristic of the MVM [71].

As noticed in [47–50], Equation (74a) allows to evaluate the "remodeling part" of growth, and frees one from guessing particular forms of $\boldsymbol{K}$, as is often done in tumor growth, growth in anisotropic media, or under the influence of interactions that define preferred growth directions. All these particular cases are physically sound for the problems that they are to model, but they also require to assume that some symmetries are maintained throughout the dynamics of the system. However, when this restriction cannot be guaranteed, and no *a priori* hypotheses are done on the generalized forces $\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}}$ and $\mathfrak{S}_{\mathrm{np}}$, Equations (74a) and (74b) permit to compute $\boldsymbol{K}$.

In conclusion, if the variational procedure defined by our formulation of the MVM is applied to growth mechanics, the dynamic equations to be solved are (71a), (71b), and (74a)–(74c), which have to be equipped with initial and Dirichlet boundary conditions for χ and with initial conditions for $\boldsymbol{K}$ and λ.

4.3.2 Quasi-Static Case: The MVM Seems to Boil down to the TNHM

In almost all the biomechanical problems involving growth, the characteristic time scales of this phenomenon are such that a quasi-static approach is amply justified. Hence, if $\hat{\mathcal{L}}_{\mathrm{c}}$ depends on $\dot{\chi}$ only through the classical kinetic energy density $\mathcal{K} = \frac{1}{2}J_{\boldsymbol{K}}\varrho_{\nu}\|\dot{\chi}\|^2$, which contributes to $\hat{\mathcal{L}}$ in the general setting, one can neglect the inertial force density $-\partial_t[\partial_{\dot{\chi}}\hat{\mathcal{L}}\circ\natural]$ in Equation (71a). In fact, in the quasi-static case, $\hat{\mathcal{L}}$ is defined without $\mathcal{K}$, and, if we assume that Equation (72) applies, the dynamic equations (71a) and (71b) become

$$\frac{\partial\hat{\mathcal{L}}}{\partial\chi}\circ\natural + \mathrm{Div}\boldsymbol{P} = -\boldsymbol{f}_{\mathrm{np}}, \qquad \text{in } \mathscr{B}, \tag{76a}$$

$$\boldsymbol{\tau} = \boldsymbol{P}\boldsymbol{N}, \qquad \text{on } \partial_{\mathrm{N}}^{\chi}\mathscr{B}. \tag{76b}$$

Then, if we indicate with $\boldsymbol{f} := \boldsymbol{f}_{\mathrm{p}} + \boldsymbol{f}_{\mathrm{np}}$ the total body force, where $\boldsymbol{f}_{\mathrm{p}}$ is given here by $\boldsymbol{f}_{\mathrm{p}} = \partial_{\chi}\hat{\mathcal{L}}\circ\natural$, Equation (76a) returns the classical force balance $\boldsymbol{f} + \mathrm{Div}\boldsymbol{P} = \boldsymbol{0}$ of Continuum Mechanics. Note that $\boldsymbol{f}_{\mathrm{np}}$ includes non-potential forces due to growth and describes sources or sinks of linear momentum related to the variation of mass of the body under consideration (see, e.g., [32, 72]). In addition, to maintain the quasi-static hypothesis, $\boldsymbol{f}_{\mathrm{np}}$ and $\boldsymbol{\tau}$ must be taken accordingly. More details on $\boldsymbol{f}_{\mathrm{np}}$ are provided in the Supplementary Material.

Further simplifications are obtained for problems that allow to neglect all body forces, so that Equation (76a) reduces to $\mathrm{Div}\boldsymbol{P} = \boldsymbol{0}$.

Similarly to the conclusions drawn for $\mathcal{K}$, even though it is possible to define a generalized kinetic energy density $\mathcal{K}_{\boldsymbol{K}} := \frac{1}{2}\langle \mathbb{G}[\dot{\boldsymbol{K}}] | \dot{\boldsymbol{K}} \rangle$ associated with $\dot{\boldsymbol{K}}$, where $\mathbb{G}$ is some suitable fourth-order inertial tensor field, the quantity $\mathcal{K}_{\boldsymbol{K}}$ does not contribute to $\hat{\mathcal{L}}$ in the quasi-static regime. Accordingly, if we further hypothesize that $\hat{\mathcal{L}}$ depends on $\dot{\boldsymbol{K}}$ only through $\mathcal{K}_{\boldsymbol{K}}$, then the generalized momentum $\boldsymbol{p}_{\boldsymbol{K}} = \partial_{\dot{\boldsymbol{K}}} \hat{\mathcal{L}} \circ \natural$, which now reads $\boldsymbol{p}_{\boldsymbol{K}} = \mathbb{G}[\dot{\boldsymbol{K}}]$, does not appear at all. Indeed, it disappears from $\mathcal{E}_{\boldsymbol{K}} \hat{\mathcal{L}}$, which reduces to $\partial_{\boldsymbol{K}} \hat{\mathcal{L}} \circ \natural$, and so do also the extra forces $\mathfrak{S}_{\boldsymbol{KF}}$ and $\mathfrak{S}_{\boldsymbol{KK}}$, defined in Equation (56). Consequently, the dynamic equations deduced by means of our formulation of the MVM, i.e., Equations (74a)–(74c), become identical to those obtained with the TNHM in the quasi-static case, the only difference being that μ is replaced by $\dot{\lambda}$.

These results notwithstanding, there remains a methodological difference between the two methods, which is reflected in the way in which the Lagrange multipliers μ and λ are computed. Indeed, as remarked above, the multiplier μ is determined algebraically in the TNHM, while, in the MVM, one has to find λ by solving an ordinary differential equation in time also in the quasi-static case. One may say, in this case, that μ is the rate of λ.

On the basis of the considerations above, in the quasi-static approximation of growth, our formulation of the MVM seems to boil down to the TNHM. However, this is not the case because of an important conceptual and technical difference between the two approaches. Such difference becomes evident by comparing the Extended Hamilton Principle, presented in Equation (15), with Equation (70), which is the "heart" of our formulation of the MVM and, in the quasi-static regime, becomes

$$\frac{\mathrm{d}\tilde{\mathcal{A}}_{\mathrm{c}}}{\mathrm{d}\varepsilon}(0) = -\int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \int_{\mathscr{B}} \left\{ \langle \boldsymbol{f}_{\mathrm{np}} | \boldsymbol{\eta}_{\chi} \rangle + \langle \mathfrak{S}_{\mathrm{np}} | \boldsymbol{\eta}_{\boldsymbol{K}} \rangle \right\} - \int_{t_{\mathrm{in}}}^{t_{\mathrm{fin}}} \int_{\partial_{\mathrm{N}}^{\chi} \mathscr{B}} \langle \boldsymbol{\tau} | \boldsymbol{\eta}_{\chi} \rangle. \tag{77}$$

Indeed, whereas in the TNHM the nonholonomic constraint is handled by regarding the reactive force $\mu \boldsymbol{K}^{-\mathrm{T}}$ as *"polygenic"* [67], thereby giving rise to $\langle \mu \boldsymbol{K}^{-\mathrm{T}} | \boldsymbol{\eta}_{\boldsymbol{K}} \rangle$ in Equation (15), no such term is present in Equation (77), since the constraint is handled variationally although it is nonholonomic.

4.3.3 Explicit Form of the Dynamic Equations in the Quasi-Static Case

Within the quasi-static regime, let us hypothesize that $\hat{\mathcal{L}}$ has expression

$$\hat{\mathcal{L}} \circ \natural = -\hat{\Psi} \circ (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) - \hat{\Psi}_{\mathrm{g}} \circ (\boldsymbol{K}; \mathcal{X}, \mathcal{T}) + \hat{\mathcal{U}} \circ (\chi, \boldsymbol{K}; \mathcal{X}, \mathcal{T}), \tag{78}$$

where $-\hat{\Psi}$ defines the body Lagrangian density function $\mathcal{L}_{\mathrm{b}}$; $\hat{\Psi}_{\mathrm{g}}$ is an energy density depending solely on the growth tensor, material points, and time; while $\hat{\mathcal{U}}$ is the potential density that generates the body forces $\boldsymbol{f}_{\mathrm{p}}$.

The energy density $\hat{\Psi}_{\mathrm{g}}$ is introduced to highlight that a given configuration of the system, defined by the triad $(\chi, \boldsymbol{F} \equiv D\chi, \boldsymbol{K})$, may vary its energetic content in response to variations of $\boldsymbol{K}$, for fixed χ and $\boldsymbol{F}$ (see, e.g., [47, 53], and a discussion on *"Cauchy's gauge"* [30]). We also notice that, in general, $\hat{\mathcal{U}}$ must depend on $\boldsymbol{K}$. Indeed, if $\hat{\mathcal{U}}$ models, e.g., gravity, i.e.,

$$\hat{\mathcal{U}} \circ (\chi, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) = \langle (\det \boldsymbol{K}) \varrho_{\nu} \boldsymbol{a}_{\mathrm{gr}} | \chi - \chi_0 \rangle =: \mathcal{U}, \tag{79}$$

where $\boldsymbol{a}_{\mathrm{gr}}$ is the gravity acceleration (co-)vector, and χ_0 defines a referential position, i.e., $\chi_0(X, t) = x_0$, for all $(X, t) \in \mathscr{B} \times \mathscr{I}$, then the dependence of $\hat{\mathcal{U}}$ on $\boldsymbol{K}$ accounts for the redistribution of the mass density $\varrho_{\mathrm{R}} = (\det \boldsymbol{K})\varrho_\nu$ in the body's reference placement due to growth. Hence, we obtain

$$\partial_\chi \hat{\mathcal{U}} \circ (\chi, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) = \partial_\chi \hat{\mathcal{L}} \circ \natural = (\det \boldsymbol{K})\varrho_\nu \boldsymbol{a}_{\mathrm{gr}} =: \boldsymbol{f}_{\mathrm{p}}, \tag{80a}$$

$$\partial_{\boldsymbol{K}} \hat{\mathcal{U}} \circ (\chi, \boldsymbol{K}; \mathcal{X}, \mathcal{T}) = [\hat{\mathcal{U}} \circ (\chi, \boldsymbol{K}; \mathcal{X}, \mathcal{T})]\boldsymbol{K}^{-\mathrm{T}}, \tag{80b}$$

and Equations (76a) and (76b) take on the form

$$(\det \boldsymbol{K})\varrho_\nu \boldsymbol{a}_{\mathrm{gr}} + \mathrm{Div}\boldsymbol{P} = -\boldsymbol{f}_{\mathrm{np}}, \qquad \text{in } \mathscr{B}, \tag{81a}$$

$$\boldsymbol{\tau} = \boldsymbol{P}\boldsymbol{N}, \qquad \text{on } \partial^\chi_{\mathrm{N}}\mathscr{B}, \tag{81b}$$

with $\boldsymbol{P}$ given in Equation (12) or (72). The coupling with $\boldsymbol{K}$ emerges both through the constitutive representation of $\boldsymbol{P}$ and through the body forces.

Switching to the dynamic sub-problem for the growth tensor, we notice that Equation (78) and the discussion reported in Sect. 2.3.3 lead to

$$\boldsymbol{H} \equiv \boldsymbol{K}^{\mathrm{T}}\big(\partial_{\boldsymbol{K}} \hat{\Psi} \circ (\boldsymbol{F}, \boldsymbol{K}; \mathcal{X}, \mathcal{T})\big) = \Psi \boldsymbol{I}^{\mathrm{T}} - \boldsymbol{F}^{\mathrm{T}}\boldsymbol{P}, \tag{82a}$$

$$\boldsymbol{Z}_{\mathrm{p}} \equiv -\boldsymbol{K}^{\mathrm{T}}\big(\partial_{\boldsymbol{K}} \hat{\Psi}_{\mathrm{g}} \circ (\boldsymbol{K}; \mathcal{X}, \mathcal{T})\big) + \boldsymbol{K}^{\mathrm{T}}\big(\partial_{\boldsymbol{K}} \hat{\mathcal{U}} \circ (\chi, \boldsymbol{K}; \mathcal{X}, \mathcal{T})\big), \tag{82b}$$

$$\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\mathrm{np}} \equiv \boldsymbol{Z}_{\mathrm{np}} - \mathbb{T}[\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}] \equiv \boldsymbol{Z}_{\mathrm{np}} - \boldsymbol{Y}_{\mathrm{ud}}, \tag{82c}$$

where $\boldsymbol{H}$ is the Eshelby stress tensor, $\boldsymbol{Z}_{\mathrm{p}}$ and $\boldsymbol{Z}_{\mathrm{np}}$ are the potential and non-potential contributions to $\boldsymbol{Z} = \boldsymbol{Z}_{\mathrm{p}} + \boldsymbol{Z}_{\mathrm{np}}$ introduced in Equation (10c), while $\mathbb{T}$ and $\boldsymbol{Y}_{\mathrm{ud}}$ are defined in Sect. 2.3.3 (we recall that the inertial terms that, in general, would feature in $\boldsymbol{Z}$ are neglected here). Thus, Equations (74a)–(74c) become

$$\mathrm{Dev}\boldsymbol{Z} - \mathrm{Dev}\boldsymbol{H} = \mathrm{Dev}\big\{\mathbb{T}[\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}]\big\}, \tag{83a}$$

$$\dot{\lambda} = -\tfrac{1}{3}\mathrm{tr}\boldsymbol{H} + \tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\big\{\mathbb{T}[\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}]\big\}, \tag{83b}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle - R = 0. \tag{83c}$$

The generalized force $\boldsymbol{Z}_{\mathrm{np}}$ can be prescribed as in Equation (19). Moreover, if $\mathcal{U}$ is given as in Equation (79), then the second term on the right-hand side of Equation (82b) reduces to $\mathcal{U}\boldsymbol{I}^{\mathrm{T}}$, and does not contribute to $\mathrm{Dev}\boldsymbol{Z}$, but to $\dot{\lambda}$.

Looking at Equation (83b), we notice that, since $\frac{1}{3}\mathrm{tr}\boldsymbol{Z}$ is conjugated to $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$, it does not contribute to trigger the variation of mass. Yet, it does contribute to determine the Lagrange multiplier associated with the constraint. On the other hand, $\mathrm{Dev}\boldsymbol{Z}$ guides, together with $\mathrm{Dev}\boldsymbol{H}$, the evolution of the volume-preserving part of $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, as indicated by Equation (83a).

5 Differences with the MVM of Llibre et al. [71]

There are two differences between the MVM by Llibre et al. [71] and our reformulation of this method. To analyze them, we abandon the quasi-static case, and return to the full system of dynamic equations (61a)–(61f).

The first, and minor, difference is that, in its original conception [71], the MVM does not consider any non-potential force. This amounts to switching off $\boldsymbol{f}_{\mathrm{np}}$ and $\mathfrak{S}_{\mathrm{np}}$ in Equations

(70) and (61a)–(61f). We maintain only $\boldsymbol{\tau}$ for including the case of non-vanishing tractions imposed on the body's boundary. Although eliminating $\boldsymbol{f}_{\mathrm{np}}$ and $\mathfrak{S}_{\mathrm{np}}$ could be unphysical for describing growth, especially for what concerns $\mathfrak{S}_{\mathrm{np}}$, we consider this situation to highlight how our version of the MVM differs from the one by Llibre et al. [71].[3]

The second, and major, difference is that, to the best of our understanding, Llibre et al. [71] do not consider the extra forces $\boldsymbol{\mathfrak{S}_{KF}}$ and DEV$\boldsymbol{\mathfrak{S}_{KK}}$ on the right-hand side of Equation (70), although the variation of their action functional is performed by employing the Hamilton–Suslov variational principle. Hence, with the procedure outlined in [71], our dynamic equations would become

$$\mathcal{E}_\chi \hat{\mathcal{L}} + \mathrm{Div}\boldsymbol{T} = \boldsymbol{0}, \qquad \text{in } \mathscr{B}, \tag{84a}$$

$$\boldsymbol{\tau} - \boldsymbol{T}\boldsymbol{N} = \boldsymbol{0}, \qquad \text{on } \partial_{\mathrm{N}}^{\chi}\mathscr{B}, \tag{84b}$$

$$\mathcal{E}_{\boldsymbol{F}} \hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{KF}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] + \boldsymbol{T} = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{84c}$$

$$\boldsymbol{F} - D\chi = \boldsymbol{O}, \qquad \text{in } \mathscr{B}, \tag{84d}$$

$$\mathcal{E}_{\boldsymbol{K}} \hat{\mathcal{L}} + \mathbb{W}^{\mathrm{T}}_{\boldsymbol{KK}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] = \dot{\lambda}\, \boldsymbol{K}^{-\mathrm{T}}, \qquad \text{in } \mathscr{B}, \tag{84e}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle - R = 0, \qquad \text{in } \mathscr{B}. \tag{84f}$$

This formulation of dynamics produces important consequences, which require conditions for the MVM formulated in [71] to be equivalent to the TNHM and, thus, to our version of the MVM, as presented in Sect. 4. As we shall see in the two following sections, these conditions place restrictions on admissible functional forms of the mass source $\hat{R}$. In this respect, we remark that *no conditions of this type are required in our formulation of the MVM (see Sect. 4), since it is constructed to be always equivalent to the TNHM*.

5.1 First Restriction on the Mass Source Due to Equations (84a)–(84f)

By recalling the identifications $\mathcal{E}_{\boldsymbol{F}}\hat{\mathcal{L}} = \partial_{\boldsymbol{F}}\hat{\mathcal{L}} \circ \natural$ and $\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural = \boldsymbol{p}_{\boldsymbol{K}}$, and using the results (38) and (55a), Equation (84c) implies that $\boldsymbol{T}$ now reads

$$\boldsymbol{T} = -\partial_{\boldsymbol{F}}\hat{\mathcal{L}} \circ \natural - \mathbb{W}^{\mathrm{T}}_{\boldsymbol{KF}}[\boldsymbol{p}_{\boldsymbol{K}}] = \boldsymbol{P} - P_{\boldsymbol{K}}(\partial_{\boldsymbol{F}}\hat{R} \circ \natural_\gamma), \tag{85}$$

where, in the last equality, we have assumed $-\partial_{\boldsymbol{F}}\hat{\mathcal{L}} \circ \natural = \partial_{\boldsymbol{F}}\hat{\Psi} \circ \natural_\gamma = \boldsymbol{P}$, thereby returning the body's first Piola–Kirchhoff stress tensor. Then, by plugging Equation (85) into (84a), one can write the result as

$$\mathcal{E}_\chi \hat{\mathcal{L}} + \mathrm{Div}\boldsymbol{T} \equiv \boldsymbol{f}_{\mathrm{p}} - \mathrm{Div}[P_{\boldsymbol{K}}(\partial_{\boldsymbol{F}}\hat{R} \circ \natural_\gamma)] - \partial_t(\partial_{\dot{\chi}}\hat{\mathcal{L}} \circ \natural) + \mathrm{Div}\boldsymbol{P} = \boldsymbol{0}. \tag{86}$$

Since, in Equation (84c), nothing compensates for $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{KF}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural]$, it goes into Equation (84a), i.e., into the balance of linear momentum, where it generates the extra force $-\mathrm{Div}[P_{\boldsymbol{K}}(\partial_{\boldsymbol{F}}\hat{R} \circ \natural_\gamma)]$. Consequently, for the considered growth problem, the original procedure by Llibre et al. [71] is equivalent to the well-consolidated TNHM *only* if $\hat{R}$ is independent of $\boldsymbol{F}$, i.e., if $\partial_{\boldsymbol{F}}\hat{R} \circ \natural_\gamma = \boldsymbol{O}$, or in the quasi-static approximation, because $P_{\boldsymbol{K}}(\partial_{\boldsymbol{F}}\hat{R} \circ \natural_\gamma)$

[3] In the quasi-static case, and in the absence of $\mathfrak{S}_{\mathrm{np}}$ (specifically, under the assumption that $\boldsymbol{Z}_{\mathrm{np}}$ and $\boldsymbol{Y}_{\mathrm{ud}}$ are both identically null), Equations (83a)–(83c) become $\mathrm{Dev}\boldsymbol{Z}_{\mathrm{p}} - \mathrm{Dev}\boldsymbol{H} = \boldsymbol{O}$; $\dot{\lambda} = -\frac{1}{3}\mathrm{tr}\boldsymbol{H} + \frac{1}{3}\mathrm{tr}\boldsymbol{Z}_{\mathrm{p}}$; and $\langle \boldsymbol{K}^{-\mathrm{T}} | \dot{\boldsymbol{K}} \rangle - R = 0$.

is neglected, provided $\hat{\mathcal{L}}$ depends on $\dot{\boldsymbol{K}}$ only through the generalized kinetic energy density $\mathcal{K}_{\boldsymbol{K}}$.

To us, the comments reported above place the question as to whether the procedure stemming from Theorem 1 of Llibre et al. [71] can be applied, *as is*, to the growth problem, at least for arbitrary $\hat{R}$. The "warning sign" for this applicability issue is the discrepancy of the momentum balance law (86) with Equation (16a), which originates from the TNHM and in which $\boldsymbol{T} = \boldsymbol{P}$.

5.2 Second Restriction on the Mass Source Due to Equations (84a)–(84f)

Let us consider the case in which the generalized kinetic energy density induced by $\dot{\boldsymbol{K}}$ is $\mathcal{K}_{\boldsymbol{K}} = \frac{1}{2}\langle \mathbb{G}[\dot{\boldsymbol{K}}] \,|\, \dot{\boldsymbol{K}}\rangle = \frac{1}{2} J_{\boldsymbol{K}} \mathfrak{m}_\nu \|\boldsymbol{L}_{\boldsymbol{K}}\|^2$, where $\mathfrak{m}_\nu > 0$ is a mass-like material parameter, and $\|\boldsymbol{L}_{\boldsymbol{K}}\|^2 := \mathrm{tr}(\boldsymbol{L}_{\boldsymbol{K}}^{\mathrm{T}} \boldsymbol{G} \boldsymbol{L}_{\boldsymbol{K}} \boldsymbol{G}^{-1})$ is the squared norm of the material rate of $\boldsymbol{K}$, i.e., $\boldsymbol{L}_{\boldsymbol{K}} := \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, while $\boldsymbol{G}$ is the metric tensor associated with the body's reference placement. In this formulation, $\mathbb{G}$ is given by $\mathbb{G} = J_{\boldsymbol{K}} \mathfrak{m}_\nu \, \boldsymbol{b}_{\boldsymbol{K}}^{-1} \underline{\otimes} \boldsymbol{G}^{-1}$, i.e., in components, $\mathbb{G}_\alpha{}^A{}_\beta{}^B = J_{\boldsymbol{K}} \mathfrak{m}_\nu \, [\boldsymbol{b}_{\boldsymbol{K}}^{-1}]_{\alpha\beta} G^{AB}$, with $\boldsymbol{b}_{\boldsymbol{K}} := \boldsymbol{K}\boldsymbol{G}^{-1}\boldsymbol{K}^{\mathrm{T}}$ being the left Cauchy-Green tensor generated by $\boldsymbol{K}$.

If we further hypothesize that $\hat{\mathcal{L}}$ depends on $\dot{\boldsymbol{K}}$ only through $\mathcal{K}_{\boldsymbol{K}}$, then, after differentiating Equation (84f) with respect to time, we can write

$$\mathbb{G}[\ddot{\boldsymbol{K}}] + \boldsymbol{K}^{-\mathrm{T}}\dot{\lambda} = \partial_{\boldsymbol{K}}\hat{\mathcal{L}} \circ \natural - \dot{\mathbb{G}}[\dot{\boldsymbol{K}}] + \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}, \tag{87a}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \ddot{\boldsymbol{K}}\rangle = \mathrm{tr}\boldsymbol{L}_{\boldsymbol{K}}^2 + \dot{R}. \tag{87b}$$

To formally analyze Equations (87a) and (87b), it is convenient to recast them in the form of a block-wise system of second order ordinary differential equations. To this end, we multiply Equation (87a) by $\boldsymbol{b}_{\boldsymbol{K}}$ from the left, and by $\boldsymbol{G}$ from the right, thereby obtaining

$$J_{\boldsymbol{K}} \mathfrak{m}_\nu \ddot{\boldsymbol{K}} + \boldsymbol{K}\dot{\lambda} = \boldsymbol{b}_{\boldsymbol{K}}(\partial_{\boldsymbol{K}}\hat{\mathcal{L}} \circ \natural)\boldsymbol{G} - \boldsymbol{b}_{\boldsymbol{K}}(\dot{\mathbb{G}}[\dot{\boldsymbol{K}}])\boldsymbol{G} + \boldsymbol{b}_{\boldsymbol{K}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}\boldsymbol{G}, \tag{88a}$$

$$\langle \boldsymbol{K}^{-\mathrm{T}} | \ddot{\boldsymbol{K}}\rangle = \mathrm{tr}\boldsymbol{L}_{\boldsymbol{K}}^2 + \dot{R}. \tag{88b}$$

Then, we rewrite the resulting expressions as [95, 111]

$$\begin{bmatrix} \mathsf{M} & \mathsf{N} \\ \mathsf{Q}^{\mathsf{t}} & \mathsf{O} \end{bmatrix} \begin{Bmatrix} \ddot{\mathsf{k}} \\ \dot{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathsf{b} + \sigma \\ \mathsf{c} \end{Bmatrix}, \tag{89}$$

where, by prescribing a convention for converting fourth-order tensors in 9×9 square matrices and second-order tensors either in 9×1 matrices or column vectors (or, depending on the situation, in 1×9 matrices or row vectors), we obtain that $\mathsf{M} = J_{\boldsymbol{K}}\mathfrak{m}_\nu \mathsf{I}$ is proportional to the 9×9 identity matrix I; N is the 9×1 matrix associated with $\boldsymbol{K}$; Q^{t} is the 1×9 matrix representing $\boldsymbol{K}^{-\mathrm{T}}$; $\ddot{\mathsf{k}}$ is the 9×1 column vector representing $\ddot{\boldsymbol{K}}$; O is the 1×1 null matrix; b is the 9×1 column vector that represents the first two addends on the right-hand side of Equation (88a); σ is the 9×1 column vector associated with $\boldsymbol{b}_{\boldsymbol{K}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}\boldsymbol{G}$; finally, c is the 1×1 column vector associated with $\mathrm{tr}\boldsymbol{L}_{\boldsymbol{K}}^2 + \dot{R}$. Note that, although $\dot{\lambda}$ is a scalar, in the system (89) we represent it with the 1×1 column vector $\dot{\lambda}$, and we use the same convention for $\mathrm{tr}\boldsymbol{L}_{\boldsymbol{K}}^2 + \dot{R}$ by renaming it c.

We remark that, although the system (89) is found by following a procedure similar to those outlined in [95, 111], its form is different in that the matrix Q^{t} *is not* the transpose of N.

By means of the Schur complement technique [95, 111], the system (89) becomes

$$\begin{cases} \ddot{\mathsf{k}} = \mathsf{M}^{-1}[\mathsf{I} - \mathsf{N}\mathsf{S}^{-1}\mathsf{Q}^{\mathsf{t}}\mathsf{M}^{-1}]\sigma + \mathsf{M}^{-1}[\mathsf{I} - \mathsf{N}\mathsf{S}^{-1}\mathsf{Q}^{\mathsf{t}}\mathsf{M}^{-1}]\mathsf{b} + \mathsf{M}^{-1}\mathsf{N}\mathsf{S}^{-1}\mathsf{c}, \\ \dot{\lambda} = \mathsf{S}^{-1}\mathsf{Q}^{\mathsf{t}}\mathsf{M}^{-1}\sigma + \mathsf{S}^{-1}\mathsf{Q}^{\mathsf{t}}\mathsf{M}^{-1}\mathsf{b} - \mathsf{S}^{-1}\mathsf{c}, \end{cases} \tag{90}$$

where $\mathsf{S} := \mathsf{Q}^{\mathsf{t}}\mathsf{M}^{-1}\mathsf{N}$ is the Schur matrix. For the problem at hand, S is a 1×1 matrix, and represents the scalar

$$\mathsf{S} = \mathsf{Q}^{\mathsf{t}}\mathsf{M}^{-1}\mathsf{N} = \mathsf{Q}^{\mathsf{t}}\frac{1}{J_{\boldsymbol{K}}\mathfrak{m}_\nu}\mathsf{I}\mathsf{N} = \frac{1}{J_{\boldsymbol{K}}\mathfrak{m}_\nu}\mathsf{Q}^{\mathsf{t}}\mathsf{N} = \frac{1}{J_{\boldsymbol{K}}\mathfrak{m}_\nu}\langle \boldsymbol{K}^{-\mathrm{T}} | \boldsymbol{K}\rangle = \frac{3}{J_{\boldsymbol{K}}\mathfrak{m}_\nu}. \tag{91}$$

Consequently, Equation (90) simplifies to

$$\begin{cases} \ddot{\mathsf{k}} = \frac{1}{J_{\boldsymbol{K}}\mathfrak{m}_\nu}\left[\mathsf{I} - \frac{1}{3}\mathsf{N}\mathsf{Q}^{\mathsf{t}}\right]\sigma + \frac{1}{J_{\boldsymbol{K}}\mathfrak{m}_\nu}\left[\mathsf{I} - \frac{1}{3}\mathsf{N}\mathsf{Q}^{\mathsf{t}}\right]\mathsf{b} + \frac{1}{3}\mathsf{N}\mathsf{c}, \\ \dot{\lambda} = \frac{1}{3}\mathsf{Q}^{\mathsf{t}}\sigma + \frac{1}{3}\mathsf{Q}^{\mathsf{t}}\mathsf{b} - \frac{J_{\boldsymbol{K}}\mathfrak{m}_\nu}{3}\mathsf{c}. \end{cases} \tag{92}$$

Before proceeding, the following remark is in order.

Remark 4 (Consistency with the direct calculation of $\dot{\lambda}$) The products $\mathsf{Q}^{\mathsf{t}}\sigma$ and $\mathsf{Q}^{\mathsf{t}}\mathsf{b}$ are scalars, and admit the identifications

$$\mathsf{Q}^{\mathsf{t}}\sigma = \left\langle \boldsymbol{K}^{-\mathrm{T}} | \boldsymbol{b}_{\boldsymbol{K}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}\boldsymbol{G}\right\rangle = \mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}), \tag{93a}$$

$$\begin{aligned} \mathsf{Q}^{\mathsf{t}}\mathsf{b} &= \left\langle \boldsymbol{K}^{-\mathrm{T}} | \boldsymbol{b}_{\boldsymbol{K}}(\partial_{\boldsymbol{K}}\hat{\mathcal{L}} \circ \natural)\boldsymbol{G}\right\rangle - \left\langle \boldsymbol{K}^{-\mathrm{T}} | \boldsymbol{b}_{\boldsymbol{K}}(\dot{\mathbb{G}}[\dot{\boldsymbol{K}}])\boldsymbol{G}\right\rangle \\ &= \mathrm{tr}\left[\boldsymbol{K}^{\mathrm{T}}(\partial_{\boldsymbol{K}}\hat{\mathcal{L}} \circ \natural)\right] - \mathrm{tr}\left[\boldsymbol{K}^{\mathrm{T}}(\dot{\mathbb{G}}[\dot{\boldsymbol{K}}])\right]. \end{aligned} \tag{93b}$$

Hence, $\dot{\lambda}$ is given by

$$\begin{aligned} \dot{\lambda} = & \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}) + \tfrac{1}{3}\mathrm{tr}\left[\boldsymbol{K}^{\mathrm{T}}\left(\partial_{\boldsymbol{K}}\hat{\mathcal{L}} \circ \natural\right)\right] - \tfrac{1}{3}\mathrm{tr}\left[\boldsymbol{K}^{\mathrm{T}}(\dot{\mathbb{G}}[\dot{\boldsymbol{K}}])\right] \\ & - \tfrac{1}{3}J_{\boldsymbol{K}}\mathfrak{m}_\nu[\mathrm{tr}\boldsymbol{L}_{\boldsymbol{K}}^2 + \dot{R}], \end{aligned} \tag{94}$$

and, by recalling Equation (88b), we obtain

$$\begin{aligned} \dot{\lambda} = & \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}) + \tfrac{1}{3}\mathrm{tr}\left[\boldsymbol{K}^{\mathrm{T}}\left(\partial_{\boldsymbol{K}}\hat{\mathcal{L}} \circ \natural\right)\right] - \tfrac{1}{3}\mathrm{tr}\left[\boldsymbol{K}^{\mathrm{T}}(\dot{\mathbb{G}}[\dot{\boldsymbol{K}}])\right] \\ & - \tfrac{1}{3}J_{\boldsymbol{K}}\mathfrak{m}_\nu\langle \boldsymbol{K}^{-\mathrm{T}} | \ddot{\boldsymbol{K}}\rangle. \end{aligned} \tag{95}$$

Finally, by considering the identity $J_{\boldsymbol{K}}\mathfrak{m}_\nu\langle \boldsymbol{K}^{-\mathrm{T}} | \ddot{\boldsymbol{K}}\rangle \equiv \mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}(\mathbb{G}[\ddot{\boldsymbol{K}}])]$, obtained by assuming $\mathbb{G} = J_{\boldsymbol{K}}\mathfrak{m}_\nu\boldsymbol{b}_{\boldsymbol{K}}^{-1} \underline{\otimes} \boldsymbol{G}^{-1}$, we rewrite the sum of the last two terms of Equation (95) as $-\frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\partial_t(\mathbb{G}[\dot{\boldsymbol{K}}])]$, and, thus, as $-\frac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}\partial_t(\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural)]$. Then, substituting this result into Equation (95), and using the definition of the Euler–Lagrange operator lead to

$$\dot{\lambda} = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}) + \tfrac{1}{3}\mathrm{tr}[\boldsymbol{K}^{\mathrm{T}}(\mathcal{E}_{\boldsymbol{K}}\hat{\mathcal{L}})], \tag{96}$$

which is identical to Equation (74b), up to the presence of $\mathfrak{S}_{\mathrm{np}}$.

We notice that, since σ (representative of $\boldsymbol{b}_{\boldsymbol{K}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}\boldsymbol{G}$) would not appear if the problem were formulated by means of the TNHM, Equations (16e) and (16f), rewritten in matrix formalism, become (if $\mathfrak{S}_{\mathrm{np}}$ is neglected)

$$\begin{cases} \ddot{\mathsf{k}}_{\mathrm{TNHM}} = \frac{1}{J_{\boldsymbol{K}} \mathfrak{m}_\nu} \left[\mathsf{I} - \frac{1}{3}\mathsf{N}\mathsf{Q}^{\mathrm{t}}\right] \mathsf{b} + \frac{1}{3}\mathsf{N}\mathsf{c}, \\ \mu = \frac{1}{3}\mathsf{Q}^{\mathrm{t}}\mathsf{b} - \frac{J_{\boldsymbol{K}} \mathfrak{m}_\nu}{3}\mathsf{c}. \end{cases} \tag{97}$$

Hence, as reported in a theorem of [95] (which rephrases a result given also in [71]), the MVM formulated by Llibre et al. [71] is equivalent to the TNHM if, and only if, it holds that $\mathsf{k} = \mathsf{k}_{\mathrm{TNHM}}$ and $\dot{\lambda} = \mu + \frac{1}{3}\mathsf{Q}^{\mathrm{t}}\sigma$. While the latter condition is a direct consequence of Equations (92) and (97), and is equivalent to Equation (59), provided the substitution $\kappa \equiv \dot{\lambda}$ is done, for the former one to be fulfilled it is necessary and sufficient that

$$\frac{1}{J_{\boldsymbol{K}} \mathfrak{m}_\nu} \left[\mathsf{I} - \frac{1}{3}\mathsf{N}\mathsf{Q}^{\mathrm{t}}\right] \sigma = 0. \tag{98}$$

This means that σ must belong to the kernel of the operator $\mathsf{I} - \frac{1}{3}\mathsf{N}\mathsf{Q}^{\mathrm{t}}$. This operator, in fact, is the matrix representation of the fourth-order tensor

$$\boldsymbol{\delta} \underline{\otimes} \boldsymbol{I}^{\mathrm{T}} - \tfrac{1}{3}\boldsymbol{K} \otimes \boldsymbol{K}^{-\mathrm{T}}, \tag{99}$$

which extracts the $\boldsymbol{K}$-deviatoric part (see Equation (100a) below) of a generic second-order tensor $\boldsymbol{\Phi}$ with components $\Phi^{\alpha}{}_{A}$, and has kernel spanned by all tensors of the type $\boldsymbol{\Phi}_0 = \varphi \boldsymbol{K}$, with $\varphi \in \mathbb{R}$ (in Equation (99), $\boldsymbol{\delta}$ is the identity tensor associated with the body's natural state). These two properties can be verified by a direct calculation, which yields

$$\left\{\boldsymbol{\delta} \underline{\otimes} \boldsymbol{I}^{\mathrm{T}} - \tfrac{1}{3}\boldsymbol{K} \otimes \boldsymbol{K}^{-\mathrm{T}}\right\}[\boldsymbol{\Phi}] = \boldsymbol{\Phi} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{-1}\boldsymbol{\Phi})\boldsymbol{K}, \qquad \forall\, \boldsymbol{\Phi}, \tag{100a}$$

$$\left\{\boldsymbol{\delta} \underline{\otimes} \boldsymbol{I}^{\mathrm{T}} - \tfrac{1}{3}\boldsymbol{K} \otimes \boldsymbol{K}^{-\mathrm{T}}\right\}[\varphi\, \boldsymbol{K}] = \boldsymbol{O}, \qquad \forall\, \varphi. \tag{100b}$$

From these results, it follows that, since the column vector σ represents the second-order tensor $\boldsymbol{b}_{\boldsymbol{K}} \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} \boldsymbol{G}$, Equation (98) requires the $\boldsymbol{K}$-deviatoric part of $\boldsymbol{b}_{\boldsymbol{K}} \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} \boldsymbol{G}$ to be null, i.e.,

$$\left\{\boldsymbol{\delta} \underline{\otimes} \boldsymbol{I}^{\mathrm{T}} - \tfrac{1}{3}\boldsymbol{K} \otimes \boldsymbol{K}^{-\mathrm{T}}\right\}[\boldsymbol{b}_{\boldsymbol{K}} \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} \boldsymbol{G}] = \boldsymbol{O}, \tag{101}$$

which is equivalent to both of the following conditions

$$\mathrm{DEV}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} = \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}})\boldsymbol{K}^{-\mathrm{T}} = \boldsymbol{O}, \tag{102a}$$

$$\mathrm{Dev}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}) = \boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{K}^{\mathrm{T}}\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}})\boldsymbol{I}^{\mathrm{T}} = \boldsymbol{O}. \tag{102b}$$

Equations (102a) and (102b) constitute the most important result of this section, and can be summarized in the following theorem, which particularizes a theorem reported in [95].

Theorem 1 (The MVM by Llibre et al. [71] and the TNHM) *Within the theory of growth under consideration, the MVM formulated in* [71], *which leads to Equations* (84a)–(84f), *is equivalent to the TNHM, and, thus, also to our formulation of the MVM, if, and only if,* $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$ *has vanishing* $\boldsymbol{K}^{-\mathrm{T}}$*-deviatoric part* (*and* $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}} = \boldsymbol{O}$).

Proof To prove the necessary condition, we compare the systems (92) and (97), and we notice that, if they are equivalent to each other, then Equation (98) has to be fulfilled. Since this condition implies Equation (102a), or (102b), $\mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$ must have null $\boldsymbol{K}^{-\mathrm{T}}$-deviatoric part.

To prove the sufficient condition, we assume that Equation (102a), or (102b), holds true. Then, the system (92) returns (97), and, thus, the MVM formulated in [71] is equivalent to the TNHM. □

Hence, if the MVM by Llibre et al. [71] is to be used *as is*, then, in order for it to be equivalent to the TNHM, only tensors $\mathfrak{S}_{KK}$ proportional to $K^{-\mathrm{T}}$ are admissible. More specifically, on account of Equations (55b) and (102a), or (102b), to ensure the equivalence with the TNHM, $\mathfrak{S}_{KK}$ must be such that

$$\mathfrak{S}_{KK} = \tfrac{1}{3} P_{K} \mathrm{tr}\big[K^{\mathrm{T}}(\partial_{K}\hat{R} \circ \natural_{\gamma})\big] K^{-\mathrm{T}}. \tag{103}$$

By virtue of Equations (103), let us write $\mathfrak{S}_{KK} = \frac{1}{3}\mathrm{tr}(K^{\mathrm{T}}\mathfrak{S}_{KK})K^{-\mathrm{T}}$ in Equation (84e). Then, by multiplying it by K^{T}, and projecting it onto the sub-spaces of deviatoric and spherical tensors, we find

$$\mathrm{Dev}[K^{\mathrm{T}}(\mathcal{E}_{K}\hat{\mathcal{L}})] = O, \tag{104a}$$

$$\tfrac{1}{3}\mathrm{tr}[K^{\mathrm{T}}(\mathcal{E}_{K}\hat{\mathcal{L}})] + \tfrac{1}{3}\mathrm{tr}[K^{\mathrm{T}}\mathfrak{S}_{KK}] = \dot{\lambda}, \tag{104b}$$

thereby providing another expression of the fact that the dynamic equations are now equivalent to those obtained via the TNHM or via our formulation of the MVM, up to the non-potential forces.

This result can be formalized in the following Corollary to Theorem 1. Before enunciating it, we emphasize that, also in this case, the restriction that it places on the mass source *is not* required in our formulation of the MVM.

Corollary 1 (Restriction on the constitutive form of the mass source) *In order for $\mathfrak{S}_{KK}$ to be proportional to $K^{-\mathrm{T}}$, the functional form of the mass source, i.e., $\hat{R}$, may depend on K only through $J_{K} = \det K$, i.e., on the volumetric part of the distortions induced by growth.*

Proof To satisfy the first restriction on $\hat{R}$, discussed in Sect. 5.1, let us take $\hat{R}$ independent of F. Then, consistently with the hypothesis of this corollary, we write $R = \hat{R} \circ (K; \mathcal{X}, \mathcal{T}) = \check{R} \circ (J_{K}; \mathcal{X}, \mathcal{T})$, and we calculate $\mathfrak{S}_{KK}$ as prescribed in Equation (55b), i.e.,

$$\mathfrak{S}_{KK} = P_{K}\left(\frac{\partial \hat{R}}{\partial K} \circ (K; \mathcal{X}, \mathcal{T})\right) = J_{K} P_{K}\left(\frac{\partial \check{R}}{\partial J_{K}} \circ (J_{K}; \mathcal{X}, \mathcal{T})\right) K^{-\mathrm{T}}. \tag{105}$$

The tensor $\mathfrak{S}_{KK}$ computed this way is, by construction, proportional to $K^{-\mathrm{T}}$, and has, thus, null $K^{-\mathrm{T}}$-deviatoric part. This completes the proof and makes the growth law $R = \hat{R} \circ (K; \mathcal{X}, \mathcal{T}) = \check{R} \circ (J_{K}; \mathcal{X}, \mathcal{T})$ admissible in the sense of Theorem 1, since the tensor $\mathfrak{S}_{KK}$ in Equation (105) makes the MVM by Llibre et al. [71] equivalent to the TNHM. □

A final remark concerns the fact that, if the dependence of $\hat{\mathcal{L}}$ on $\dot{K}$ is only through the kinetic energy density $\mathcal{K}_{K}$, then, in the quasi-static case, the MVM by Llibre et al. [71] is equivalent to the TNHM because $\partial_{\dot{K}}\hat{\mathcal{L}} \circ \natural$ is neglected and, thus, the extra forces $\mathbb{W}^{\mathrm{T}}_{KF}[\partial_{\dot{K}}\hat{\mathcal{L}} \circ \natural]$ and $\mathbb{W}^{\mathrm{T}}_{KK}[\partial_{\dot{K}}\hat{\mathcal{L}} \circ \natural]$ do not appear on the left-hand sides of Equations (84c) and (84e). However, we deem it important to reformulate this version of the MVM as shown in Sect. 4 because it allows to obtain the quasi-static limit in the correct way. In this regard, we refer the reader to the Supplementary Material for the study of such quasi-static limit for a simple benchmark problem [5, 49].

6 Conclusions

This work is the latest of a series of studies [47–50] devoted to adapt concepts and methods of the analytical mechanics of nonholonomic systems [71, 90, 95] to the continuum mechanics of volumetric growth in biological media.

Our main conclusions can be summarized as follows:

C1. We have developed a *quasi-variational* theory of growth in which the growth law of a body is assigned *a priori*, for example, in compliance with experimental evidences or design choices. This compels to interpret the body's mass balance law as an affine and nonholonomic *constraint* on the time rate of the growth tensor, which is introduced by means of the BKL decomposition. We have *appended* this constraint to the Lagrangian density function of the body, and we have shown that the corresponding dynamic equations can be obtained variationally up to *"polygenic forces"* [67], which have to be considered for a general and physically sound model of growth. To us, this result is noteworthy, because it subsists even though the constraint is nonholonomic. To achieve it, we have started from the dynamic equations supplied by Hamilton's Extended Principle [67] (which are equivalent to those predicted by the Principle of Virtual Work), and we have modified them by introducing the extra forces $\mathfrak{S}_{\boldsymbol{KF}}$ and $\mathfrak{S}_{\boldsymbol{KK}}$ that stem from the transformation rules of the Euler–Lagrange operators when passing from the true velocities to the quasi-velocities of the body. By virtue of these forces, the modified equations remain equivalent to the original ones. Then, we have found that the modified dynamic equations can be made descend from the variational procedure known as Hamilton–Suslov Principle, "corrected" in a fashion similar to Hamilton's Extended Principle, as shown in Sect. 4.2, Equation (70). To do all this, we have taken the *"Modified Vakonomic Method"* MVM by Llibre et al. [71] as a point of departure, and we have reformulated it by considering the extra forces $\mathfrak{S}_{\boldsymbol{KF}}$ and $\mathfrak{S}_{\boldsymbol{KK}}$. These forces must be accounted for to ensure that our formulation of the MVM is consistent with the TNHM [37, 38, 67]. In this regard, we have re-contextualized the work done by Llibre et al. [71] in light of the results provided in [95].

C2. We have compared our reformulation of the MVM with its original version [71] by specializing the latter to the mechanics of a growing medium. Our conclusion is that the original MVM *disagrees* with the TNHM for general growth laws employed in the nonholonomic constraint on the time rate of the growth tensor (see Sect. 5). The discrepancy is due to the fact that, if the procedure by Llibre et al. [71] is strictly followed, the extra forces $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{KF}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] \equiv \mathfrak{S}_{\boldsymbol{KF}}$ and $\mathbb{W}^{\mathrm{T}}_{\boldsymbol{KK}}[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{L}} \circ \natural] \equiv \mathfrak{S}_{\boldsymbol{KK}}$, which are produced by the Hamilton–Suslov procedure [71, 95, 103, 106] (cf. with $\mathsf{W}^{\mathrm{T}}\mathsf{p}$ in [95]), appear only on the left-hand sides of the dynamic equations (84c) and (84e). Therefore, since no other term balances them, they lead, for an arbitrary growth law, to dynamics that cannot be predicted by the TNHM (indeed, no extra force appears in the TNHM). In particular, $\mathfrak{S}_{\boldsymbol{KF}}$ is fully unbalanced, and, although the $\boldsymbol{K}^{-\mathrm{T}}$-spherical part of $\mathfrak{S}_{\boldsymbol{KK}}$ can be incorporated into $\dot{\lambda}$ [71], the $\boldsymbol{K}^{-\mathrm{T}}$-deviatoric part of $\mathfrak{S}_{\boldsymbol{KK}}$ remains unbalanced. Accordingly, to restore the equivalence of the original MVM [71] with the TNHM, $\mathfrak{S}_{\boldsymbol{KF}}$ and $\mathrm{DEV}\mathfrak{S}_{\boldsymbol{KK}}$ must be null. This is the result of our Theorem 1. The requirement that $\mathrm{DEV}\mathfrak{S}_{\boldsymbol{KK}}$ vanishes is coherent with the statement of a theorem [95] that supplies a necessary and sufficient condition for the original MVM [71] to be equivalent to the TNHM for discrete nonholonomic systems. In conclusion, when the Hamilton–Suslov procedure generates extra forces that are both null or that comply with the conditions stated previously, the equivalence between Llibre et al.'s MVM [71] and the TNHM is naturally fulfilled. In fact, the second case is true for the constraints studied in [71, 95], but, in general, not for the constraint considered in this present work.

C3. Since the constraint (4) is made nonholonomic by the mass source R, the equivalence conditions between Llibre et al.'s MVM [71] and the TNHM depend entirely on the functional form of $\hat{R}$. Our corollary 1 states that such conditions require $\hat{R}$ to be independent of $\boldsymbol{F}$ and to depend on $\boldsymbol{K}$ solely through $J_{\boldsymbol{K}}$. The first restriction can be used for some simple benchmark problems, but it is not realistic for many biological problems in which the growth law must depend on mechanical stress and, thus, on $\boldsymbol{F}$, as is the case for tumor growth models involving *"mechanosensing"* [17, 88] and *"mechanotransduction"* [83, 84]. The second restriction, instead, is often imposed when growth is assumed to be isotropic, as in tumors (see, e.g., [6, 44, 84]), but it should be relaxed when growth is not isotropic, as occurs in heart or skin mechanics (see, e.g., [9] for a review), or, more generally, in fiber-reinforced tissues (see, e.g., [73]). Therefore, these situations cannot be modeled by the original MVM. Nevertheless, our reformulation of the MVM, as presented in Sect. 4, can be adopted, since it is equivalent *by construction* with the TNHM, independently of the assigned growth law. Thus, in all the situations mentioned above, our work is able to provide a quasi-variational theory of growth.

C4. Since our theory is of grade zero in $\boldsymbol{K}$, the geometric descriptors induced by $\boldsymbol{K}$ are all determined as *outputs* of the dynamic problem. For example, if one solves Equations (61a)–(61f) in a sufficiently general setting, then, once $\boldsymbol{K}$ is computed, one can determine a Riemannian manifold characterized by the metric tensor associated with $\boldsymbol{K}$, and the corresponding Levi-Civita connection. This connection and the curvature that it produces are consequences of the forces accounted for in the model *and* of the mass source R given in the constraint, and are, thus, directly related to the bio-physics and bio-chemistry of the body [26]. Analogous considerations hold true, for instance, for the affine connection $\Gamma^{A}{}_{BC} = (\boldsymbol{K}^{-1})^{A}{}_{\alpha}(\partial K^{\alpha}{}_{B}/\partial X^{C})$.

It is worth noticing that, for the majority of the cases of biological interest, R is such that the constraint on the evolution of $\boldsymbol{K}$ cannot be reduced to a holonomic condition. This nonholonomicity in time, in turn, induces the nonholonomicity of $\boldsymbol{K}$ in space, and, thus, its incompatibility, especially with respect to its volumetric part, modeled by $J_{\boldsymbol{K}}$. In a previous work of some of us, we have called this type of models *"a priori approach"* [49], since R is prescribed by the phenomenology. Yet, this is not the only possible path. Indeed, one could put no constraints on $\boldsymbol{K}$, and let it evolve according to a dynamic problem in which the force $\boldsymbol{Z}$ contains the biological information necessary to trigger and maintain growth [3, 18, 19, 29, 87]. This is what we called *"a posteriori approach"* [49]. In this case, the geometric descriptors depend on $\boldsymbol{Z}$ and on the constitutive assumptions defining $\boldsymbol{Y}_{\mathrm{ud}}$ and $\boldsymbol{H}$. A similar situation occurs for those models of growth that consider $\boldsymbol{K}$ as an internal variable (see, e.g., [32, 73]), in which $\boldsymbol{K}$ is introduced as the *"implant"* tensor within a theory of uniformity [30–32, 77, 109, 110].

However, a different situation is provided by theories of higher grade [21, 76, 110], in which the geometric descriptors, like affine connection and curvature, are part of the augmented kinematics and, thus, part of the solution of the dynamic problem. In this case, these descriptors can lead to even more complicated manifolds.

As a final remark, we would like to emphasize that our formulation of the MVM can be seen as based on an "Extended" Hamilton–Suslov Principle, since it augments the already existing theory by allowing for non-potential forces (cf. with Extend Hamilton's Principle in Sect. 2.3.2). In particular, the way in which the constraints are handled differs significantly from how Extended Hamilton's Principle operates. Indeed, whereas the latter binds the constraints to merely "polygenic" forces, and equates the first-order variation of the action to the time-integral of the virtual work done by the "reactive" forces, the Extended Hamilton–Suslov Principle directly varies a Lagrangian function that comprehends the constraint by operating variations that are of the Hamilton–Suslov kind [71, 95, 103, 106].

Moreover, it would be interesting to exploit the Extended Hamilton–Suslov Principle to study "broken" symmetries due to growth, and the related loss of conservation laws, by means of an appropriate formulation of Noether's Theorem for nonholonomic systems [34].

Appendix: Glossary of Nonholonomic Mechanics and Notation

In this glossary, we summarize the main concepts of nonholonomic mechanics that are used throughout this work. Moreover, the key variables of our work are reported in Table 1.

Table 1 Main variables used in this work, reported in alphabetical order

Notation	Name
$\boldsymbol{f}$	Body force per unit volume of the reference placement
$\boldsymbol{p}_{\boldsymbol{K}}$	Generalized (tensorial) momentum associated with $\boldsymbol{K}$
$\mathcal{A}$	Action functional
$\mathcal{C} \equiv \hat{\mathcal{C}} \circ \natural_{\mathrm{c}}$	Growth constraint
$\boldsymbol{F}$	Deformation gradient tensor
$\boldsymbol{F}_{\mathrm{e}}$	Tensor of elastic distortions
$\boldsymbol{H}$	Eshelby stress tensor
$\mathbb{J}$	Fourth-order Jacobian tensor of the quasi-velocities
$J_{\boldsymbol{K}}$	Volumetric ratio due to growth
$\boldsymbol{K}$	Growth tensor
$\boldsymbol{K}_{\mathrm{u}}$	Isochoric part of the growth tensor
$\mathcal{L} \equiv \hat{\mathcal{L}} \circ \natural$	Lagrangian density function
$\boldsymbol{T}$	Lagrange multiplier associated with $\boldsymbol{\mathcal{C}}_{\mathrm{a}} = \boldsymbol{0}$
$\boldsymbol{P}$	First Piola–Kirchhoff stress tensor
$R \equiv \hat{R} \circ \natural_{\gamma}$	Source of mass
$\mathbb{W}_{\boldsymbol{K}\boldsymbol{F}}, \mathbb{W}_{\boldsymbol{K}\boldsymbol{K}}$	Fourth-order tensors representing transpositional relations
$\boldsymbol{Y}_{\mathrm{u}}$	Active internal generalized force dual to $\delta\boldsymbol{K}$
$\boldsymbol{Z}$	External generalized force dual to $\delta\boldsymbol{K}$
χ	Motion
λ	Lagrange multiplier associated with $\mathcal{C} = 0$ for the MVM
μ	Lagrange multiplier associated with $\mathcal{C} = 0$ for the TNHM
$\boldsymbol{\tau}$	Contact force
$\boldsymbol{\Omega}$	Quasi-velocities tensor for the growth problem
Ψ	Strain energy density
$\mathfrak{S}_{\boldsymbol{K}\boldsymbol{F}}, \mathfrak{S}_{\boldsymbol{K}\boldsymbol{K}}$	Extra forces due to the MVM for the growth problem
$\boldsymbol{\Theta}$	Quasi-coordinates tensor for the growth problem

Nonholonomic constraint: An *a priori* condition on the generalized velocities of a system that is not a total time derivative of a function of the Lagrangian parameters, space variables, and time, only (note that, in the context of this work of ours we consider solely constraints that are nonholonomic in time).

Chetaev conditions: Conditions fulfilled by the virtual displacements associated with the Lagrangian parameters in order to be *compatible* with the considered nonholonomic constraints.

Traditional Nonholonomic Method (TNHM): A method for obtaining the dynamic equations of mechanical systems subjected to nonholonomic constraints which exploits the fact that reactive forces due to *ideal* constraints produce zero virtual work (power) [67].

Vakonomic method (VM): A method developed by Kozlov [61–64], which aims to obtain variationally the dynamic equations of mechanical systems subjected to nonholonomic constraints. It is based on the application of Hamilton's Principle to a "constrained" Lagrangian function defined by *attaching* the constraints, multiplied by their associated Lagrange multipliers, to the Lagrangian function of the problem at hand.

Quasi-velocities: A system of quasi-velocities is a "change of variable" in the space of the generalized velocities. They are used in nonholonomic mechanics to rephrase the kinematics of a given mechanical system in terms of velocity variables that can automatically satisfy the constraints.

Transpositional relations: Equalities expressing the non-commutativity that, in general, occurs between the operations of "virtual variation" and "time differentiation" when nonholonomic constraints are considered.

Hamilton–Suslov Principle: A generalization of Hamilton's Principle [71, 95] in which the Lagrangian parameters and the generalized velocities are varied according to two distinct families of homotopies, and their first-order variations are reciprocally related through the transpositional relations.

Modified Vakonomic Method (MVM) [71]: A variational method used for obtaining the dynamic equations of mechanical systems subjected to nonholonomic constraints and based on the Hamilton–Suslov Principle applied to a "constrained" Lagrangian [71, 95].

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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