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The wave function of stabilizer states and the Wehrl conjecture

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ABSTRACT

We focus on quantum systems represented by a Hilbert space $L^2(A)$, where A is a locally compact Abelian group that contains a compact open subgroup. We examine two interconnected issues related to Weyl-Heisenberg operators. First, we provide a complete and elegant solution to the problem of describing the stabilizer states in terms of their wave functions — an issue that arises in quantum information theory. Subsequently, we demonstrate that the stabilizer states are exactly the minimizers of the Wehrl entropy, thereby solving the Wehrl-type entropy conjecture for any such group (in particular, for finite-dimensional vector spaces over non-Archimedean local fields). Additionally, we construct a moduli space for the set of stabilizer states, that is, a parametrization of this set, that endows it with a natural algebraic structure, and we derive a formula for the number of stabilizer states when A is finite. Indeed, these results are novel even for finite Abelian groups.

RÉSUMÉ

Nous intéressons aux systèmes quantiques représentés par un espace de Hilbert $L^2(A)$, où A est un groupe abélien localement compact qui contient un sous-groupe ouvert compact. Nous examinons deux questions interconnectées relatives aux opérateurs de Weyl-Heisenberg. Tout d'abord, nous donnons une solution complète et élégante au problème de la description des états stabilisateurs en termes de leurs fonctions d'onde — une question qui se pose en théorie de l'information quantique. Ensuite, nous montrons que les états stabilisateurs sont exactement les minimiseurs de l'entropie de Wehrl, résolvant ainsi la conjecture de Wehrl pour tout tel groupe (en particulier, pour les espaces vectoriels de dimension finie sur des corps locaux non archimédiens). De plus, nous construisons un espace de modules pour l'ensemble des états stabilisateurs, c'est-à-dire une paramétrisation de cet ensemble qui lui confère une structure algébrique naturelle, et nous dérivons une formule pour le nombre d'états stabilisateurs lorsque A est fini. En effet, ces résultats sont nouveaux même pour les groupes abéliens finis.

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1. Introduction and discussion of the main results

Consider a Hilbert space $\mathcal{H}$ together with a "representation", i.e. a unitary operator $U : \mathcal{H} \to L^2(A)$, where A is a locally compact Abelian (LCA) group. For $|\psi\rangle \in \mathcal{H}$ we denote by $\psi = U|\psi\rangle \in L^2(A)$ its *wave function*. We write $\widehat{A}$ for the group of continuous characters of A, namely the continuous homomorphisms $\xi : A \to U(1)$ (the multiplicative group of complex numbers of modulus 1). For $x \in A$, $\xi \in \widehat{A}$ we denote by $\xi(x) \in U(1)$ the value of ξ at x. In addition, the group laws in A and $\widehat{A}$ are written additively. We will interpret A as the classical configuration space, and the group product $A \times \widehat{A}$ as the classical phase space associated with $\mathcal{H}$.

Given $(x, \xi) \in A \times \widehat{A}$, we define the phase space shift $\pi(x, \xi) : L^2(A) \to L^2(A)$ by

$$\pi(x,\xi)\psi(y) = \xi(y)\psi(y - x) \qquad y \in A,\ \psi \in L^2(A). \tag{1.1}$$

We also consider the Weyl-Heisenberg operators $W(x, \xi) : \mathcal{H} \to \mathcal{H}$, defined as

$$W(x,\xi) = U^\dagger \pi(x,\xi) U \qquad (x,\xi) \in A \times \widehat{A}. \tag{1.2}$$

These operators are widely used in both mathematics and physics, and their importance can hardly be overrated (see, e.g., [5,6,16,32]). For a spinless particle in $\mathbb{R}^d$, in the Schrödinger representation, hence $A = \mathbb{R}^d$, they are generated by the usual position and momentum operators. The following is an illustration in a finite-dimensional context.

Example 1.1. Suppose that $\mathcal{H}$ has finite dimension, and let $\{|x\rangle\}_{x\in A}$ be an orthonormal basis, labeled by elements of the (finite) Abelian group A. This is a common framework for most systems occurring in quantum information, where such a fixed basis is referred to as the computational basis. For example, in the case of n qudits, we have $A = \mathbb{Z}_d^n$. The product $A = \mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_n}$ arises similarly for multipartite systems; see [2,43] and the references therein. Observe that every finite Abelian group is isomorphic to the direct sum of finite cyclic groups and, therefore, occurs in this way. Here, the representation $U : \mathcal{H} \to L^2(A)$, by definition, maps the vector $|\psi\rangle$ into its wave function $\psi(x) := \langle x|\psi\rangle$.

For $(x, \xi) \in A \times \widehat{A}$, the Weyl-Heisenberg operator $W(x, \xi) : \mathcal{H} \to \mathcal{H}$ acts on the computational basis as

$$W(x,\xi)|y\rangle = \xi(y + x)|y + x\rangle \qquad y \in A. \tag{1.3}$$

In the case of one qubit, with $A = \mathbb{Z}_2 = \{0, 1\}$, the shift operator $W(1, 0)$ is known as the Pauli gate X, while the operator $W(0, 1)$ is the Pauli gate Z; see [2,3,17,22,47], the monographs [20,21], [37, Section 10.5] and the references therein.

In this note we study two interconnected problems related to the Weyl-Heisenberg operators, that is, we offer a thorough and elegant solution to the problem of describing the so-called stabilizer states in terms of their wave functions, and we show that such states are exactly the minimizers of the Wehrl entropy.

1.1. Stabilizer states

Given A as above, consider the corresponding *Heisenberg group*, defined as

$$\mathbb{H}_A = \{u\, W(x,\xi) :\ u \in U(1),\ (x,\xi) \in A \times \widehat{A}\}. \tag{1.4}$$

It is a noncommutative locally compact and Hausdorff topological group, endowed with the topology induced by the natural bijection $\mathbb{H}_A \simeq U(1)\times A\times \widehat{A}$ (equivalently, the strong operator topology of bounded operators in $\mathcal{H}$; see, e.g., [23, Lemma 1]). When A is finite, say of cardinality N, we also consider the subgroup of $\mathbb{H}_A$, known as the *Pauli group*, defined as

$$\mathbb{P}_A = \{\zeta^k W(z) : \; k \in \mathbb{Z}, \; z \in A \times \widehat{A}\}, \tag{1.5}$$

where $\zeta = \exp(\pi i/N)$.

It is well known that, if A is finite, of cardinality N, and $\mathcal{G} \subset \mathbb{P}_A$ is an Abelian subgroup of cardinality N that does not contain multiples of the identity other than the identity itself, there exists exactly one state that is stabilized by $\mathcal{G}$, that is, a common eigenstate, with eigenvalues 1, of all elements of $\mathcal{G}$. The study of these particular states began with [14], motivated by the realization that the most interesting states in quantum information theory are frequently more manageable via their stabilizer group $\mathcal{G}$ rather than through direct examination. Since then, these states have become an essential resource in the development of quantum error correction codes; see [37, Chapter 10] and the references therein.

Not every state is a stabilizer state. Hence, one faces the problem of describing these states more explicitly in terms of their wave function. This issue was addressed in [8] and [22] for a system of n qubits ($A = \mathbb{Z}_2^n$) and n qudits ($A = \mathbb{Z}_d^n$) respectively. There, the authors presented a clever algorithm to construct the wave function of the stabilizer state from a given group $\mathcal{G}$ as above, requiring certain matrix manipulations in modular arithmetic, the search for a minimal set of generators of $\mathcal{G}$, the reduction to a Smith normal form and finally the resolution of a Diophantine system (see also [17] for the case where $A = \mathbb{Z}_d^n$ and d is an odd integer, in which the analysis is substantially simplified due to the fact that multiplication by 2 acts as an isomorphism of $\mathbb{Z}_d$, in this case). While this method could be extended, in principle, to finite Abelian groups (albeit with additional algebraic complexities), it seems that a more conceptual and systematic understanding has not yet been developed. In the first part of this note we will bridge this deficiency by introducing an elegant and general rule to derive the wave function of a stabilizer state from the group $\mathcal{G}$. Building on the foundational work [40,45] on Weyl-Heisenberg operators, we shall consider a general locally compact Abelian group A (containing a compact open subgroup; see below). Our analysis will be "coordinate-free", based solely on the structure of A as an LCA group, without depending on other choices, such as a set of generators. This could also offer significant practical advantages, as highlighted in [43]. We stress that the derived formula is novel even for finite Abelian groups and that our analysis is *not* a modification of the approach discussed in the aforementioned references.

To properly state our results, we introduce some terminology. We recall that, according to [45], a *character of second degree* (alias *quadratic form*) of an LCA group A is a continuous function $h : A \to U(1)$ such that

$$h(x + y) = h(x)h(y)\,\beta(x)(y) \qquad x, y \in A \tag{1.6}$$

for some continuous symmetric homomorphism $\beta : A \to \widehat{A}$; hence $\beta(x)(y) = \beta(y)(x)$ for $x, y \in A$. The set of characters of second degree of A is easily seen to be a group that will be denoted by $\mathrm{Ch}_2(A)$. Moreover, we will denote by $\mathrm{Sym}(A)$ the group of continuous symmetric homomorphisms β as above. It is clear that $\widehat{A} \subset \mathrm{Ch}_2(A)$ and it is known that $\mathrm{Ch}_2(A)/\widehat{A} \simeq \mathrm{Sym}(A)$ (algebraic isomorphism).

We also say that a function $h : A \to \mathbb{C}$ is a *subcharacter of second degree* of A if there exists a compact open subgroup $H \subset A$ such that $h(x) = 0$ for $x \in A \setminus H$ and the restriction of h to H is a character of second degree of H; we refer to Section 2 for details and for an explicit description (construction) of all characters of second degree of a finite Abelian group.

We now select a special class of state vectors.

Definition 1.2 *(S-state).* We say that $|\phi\rangle \in \mathcal{H}$ is an S-state vector if its wave function has the form

$$\phi(x) = c\, h_0(x - y) \tag{1.7}$$

for some $c \in \mathbb{C} \setminus \{0\}$, $y \in A$ and some subcharacter h_0 of second degree of A. We denote by $\mathfrak{S}$ the set of corresponding S-states, that is, states represented by an S-state vector.

We also give the following definition. We endow $A \times \widehat{A}$ with the Radon product measure, where the Haar measures on A and $\widehat{A}$ are normalized so that the Fourier inversion formula holds with constant 1.

Definition 1.3. Let A be any LCA group and let $\mathbb{H}_A$ be the corresponding Heisenberg group. Consider the collection $\mathfrak{G}$ of the compact subgroups $\mathcal{G} \subset \mathbb{H}_A$ such that the homomorphism

$$\mathcal{G} \to A \times \widehat{A} \qquad u\,W(z) \mapsto z \tag{1.8}$$

is injective (hence $\mathcal{G}$ is Abelian) and has an image of measure 1 in $A \times \widehat{A}$.

We say that $|\psi\rangle \in \mathcal{H} \setminus \{0\}$ is a stabilizer state vector if there exists $\mathcal{G} \in \mathfrak{G}$, such that $|\psi\rangle$ is an eigenvector, with eigenvalue 1, of all elements of $\mathcal{G}$. In this case, we will say that $\mathcal{G}$ stabilizes $|\psi\rangle$. A stabilizer state is a state represented by a stabilizer state vector.

We refer to Remark 3.1 and Proposition 3.2 below for a discussion and some results that justify the conditions that appear in the definition of the class $\mathfrak{G}$. Here we limit ourselves to pointing out that $\mathfrak{G}$ is nonempty exactly when A contains a compact open subgroup and that this scenario turns out to be the natural framework for a general stabilizer formalism (incidentally, if A lacks any compact open subgroup, the statements below are vacously true). We also anticipate that, as a consequence of Theorem 1.4 below, every $\mathcal{G} \in \mathfrak{G}$ stabilizes exactly one state.

By assumption, for every $\mathcal{G} \in \mathfrak{G}$ the homomorphism in (1.8) is injective. This implies that $\mathcal{G}$ can be regarded as the graph of some function, that is, it has necessarily the form

$$\mathcal{G} = \{\alpha(z)W(z) : \ z \in K\}, \tag{1.9}$$

for some compact subgroup $K \subset A \times \widehat{A}$ of measure $|K| = 1$ and some function $\alpha : K \to U(1)$. Since $\mathcal{G}$ is compact, α is continuous (see Proposition 3.3).

We can now state our first result. We recall that the annihilator of a subgroup $H \subset A$ is the subgroup $H^\perp \subset \widehat{A}$ given by

$$H^\perp = \{\xi \in \widehat{A} : \ \xi(x) = 1 \text{ for all } x \in H\}. \tag{1.10}$$

Theorem 1.4. *Any S-state vector $|\phi\rangle$ (Definition 1.2) is stabilized by only one $\mathcal{G} \in \mathfrak{G}$ (Definition 1.3), which can be obtained as follows. Let $\phi \in L^2(A)$ be the wave function of $|\phi\rangle$, hence of the form (1.7) for some $y \in A$ and some subcharacter h_0 of second degree of A. Let $H = \{x \in A : \ h_0(x) \neq 0\}$, and let $h = h_0|_H \in \mathrm{Ch}_2(H)$ be associated with a continuous symmetric homomorphism $\beta \in \mathrm{Sym}(H)$ as in (1.6) (hence, $\beta(x) \in \widehat{H}$ for $x \in H$). Then $\mathcal{G}$ has the form (1.9) where*

$$K = \{(x, \xi) \in A \times \widehat{A} : \ x \in H, \ \xi|_H = \beta(x)\} \tag{1.11}$$

and

$$\alpha(x, \xi) = \overline{h_0(-x)\xi(y)} \qquad (x, \xi) \in K. \tag{1.12}$$

Conversely, given any subgroup $\mathcal{G} \in \mathfrak{G}$, there is only one corresponding stabilizer state. It is an S-state, that is, its wave function is given by the formula (1.7), where h_0 and y are given as follows.

(i) *With $\mathcal{G}$ as in (1.9), let H be the image of K under the canonical projection $A \times \widehat{A} \to A$. Then $H^\perp = \{\xi \in \widehat{A} : (0,\xi) \in K\}$ and the map $\xi \mapsto \alpha(0,\xi)$ is a continuous character of $H^\perp$. Therefore there exists $y \in A$ such that*

$$\xi(y) = \overline{\alpha(0,\xi)} \quad \textit{for } \xi \in H^\perp.$$

(ii) *The function $h : H \to U(1)$ given by*

$$h(-x) = \overline{\alpha(x,\xi)\xi(y)} \quad \textit{for } x \in H \textit{ and any } \xi \in \widehat{A} \textit{ with } (x,\xi) \in K$$

is well defined and belongs to $\mathrm{Ch}_2(H)$.

(iii) *Let $h_0 : A \to \mathbb{C}$ be defined by $h_0(x) = h(x)$ for $x \in H$ and $h_0(x) = 0$ for $x \in A \setminus H$. Then for these h_0 and y, the wave function $\phi(x) := h_0(x - y)$ defines the stabilizer state of $\mathcal{G}$.*

As a consequence, a state is a stabilizer state if and only if it is an S-state.

Theorem 1.4 suggests a method to build a *moduli space* for the set of stabilizer states, that is, a set that parametrizes the collection of stabilizer states and endows it with some inherent nontrivial algebraic structure. Indeed, we will see (Corollary 4.7) that for the collections $\mathfrak{G}$ and $\mathfrak{S}$ of stabilizer subgroups and S-states, respectively (Definitions 1.3 and 1.2) there are natural bijections

$$\mathfrak{G} \simeq \mathfrak{S} \simeq \bigsqcup_{H \subset A} A \times_H \mathrm{Ch}_2(H)$$

where the first identification is given by Theorem 1.4. The disjoint union is taken over all compact open subgroups $H \subset A$ and the set $A \times_H \mathrm{Ch}_2(H)$ will be suggestively constructed as "a bundle over A/H associated with the H-principal bundle $A \to A/H$, with typical fiber $\mathrm{Ch}_2(H)$". Moreover, each fiber has a *natural* structure of Abelian group. Hence, the sets $\mathfrak{G}$ and $\mathfrak{S}$, as disjoint unions of groups, are naturally groupoids.

As a consequence of the above construction, we obtain a formula for the number of stabilizer states for an arbitrary finite Abelian group, that is

$$\#\mathfrak{G} = \#\mathfrak{S} = \#A \cdot \sum_{H \subset A} \#\mathrm{Sym}(H).$$

Explicit formulas were previously known for certain cases, specifically for $A = (\mathbb{F}_q)^n$ [17] ($\mathbb{F}_q$ being the finite field with q elements), and for $A = \mathbb{Z}_d^n$, with an arbitrary d — a case settled only recently in [41] (see also the references therein).

1.2. The Wehrl entropy

We now consider the problem of minimizing the Wehrl entropy for a system as above.

For any "reference" ket $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, consider the corresponding *Weyl-Heisenberg coherent states*

$$|\phi_z\rangle := W(z)|\phi\rangle \qquad z \in A \times \widehat{A}. \tag{1.13}$$

We recall that, given $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, and a density operator ρ (a compact nonnegative operator in $\mathcal{H}$ with trace 1), the corresponding *Husimi function* is given by

$$u_{\phi,\rho}(z) := \langle \phi_z|\rho|\phi_z\rangle \qquad z \in A \times \widehat{A}. \tag{1.14}$$

If $G : [0,1] \to \mathbb{R}$ is a concave function, the (generalized) Wehrl entropy of a density operator ρ, with respect to a reference state $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, is then defined as

$$\mathcal{E}_G(\phi,\rho) := \int\limits_{A\times\widehat{A}} G(u_{\phi,\rho}(x,\xi))\, dx\, d\xi \tag{1.15}$$

provided the integral makes sense. In the following theorem we assume that $G(0) = 0$, which ensures that the negative part of the integrand function is summable. Hence, the above integral is well defined, although it might be $+\infty$ (see Remark 5.2).

The next result provides a full characterization of the minimal entropy states.

Theorem 1.5. *Let $G : [0,1] \to \mathbb{R}$ be a concave function with $G(0) = 0$. For every $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, and density operator ρ, we have*

$$\mathcal{E}_G(\phi,\rho) \geq G(1). \tag{1.16}$$

Moreover, if G is not linear (that is, not of the form $G(\tau) = \alpha\tau$ for any $\alpha \in \mathbb{R}$) the following facts are equivalent.

- (i) *Equality occurs in (1.16) for a pair $|\phi\rangle$, ρ as above.*
- (ii) *The Husimi function $u_{\phi,\rho}$ is the indicator function of a subset $\mathcal{K} \subset A \times \widehat{A}$ of measure $|\mathcal{K}| = 1$.*
- (iii) *$|\phi\rangle$ is an S-state (Definition 1.2), $\rho = |\psi\rangle\langle\psi|$ is a pure state, and there exist $z \in A \times \widehat{A}$, $\theta \in \mathbb{R}$ such that*

$$|\psi\rangle = e^{i\theta} W(z)|\phi\rangle.$$

Whereas (1.16) and the equivalence $(i) \iff (ii)$ are easy and also well known in special cases (see, e.g., [48]), *the characterization in (iii) — hence in terms of the wave functions — is definitively nontrivial and new.* Indeed, it is new even in the case of finite Abelian groups.

In Theorem 1.5, instead of S-states we could equivalently refer to stabilizer states, in view of Theorem 1.4. Also, note that Theorem 1.5 is relevant for groups A that have a compact open subgroup. Indeed, if this condition is not met, by the equivalence $(i) \iff (iii)$ we see that equality in (1.16) never occurs (if G is not linear). Hence, the above result is in a sense *complementary* to that for a spinless particle in $\mathbb{R}^d$ in the Schrödinger representation, therefore $A = \mathbb{R}^d$, originally studied by Wehrl [44] when $G(\tau) = -\tau \log \tau$. There, it was conjectured that if $|\phi\rangle$ represents the ground state of the harmonic oscillator, the entropy $\mathcal{E}_G(\phi,\rho)$ is minimized by the pure states $\rho = |\phi_z\rangle\langle\phi_z|$. This conjecture was proved in [28], whereas the uniqueness of the minimizers was subsequently established in [4], that is, the Wehrl entropy (with $|\phi\rangle$ as specified above) is minimized exclusively by these states. The analogous conjecture for $SU(2)$ and certain $SU(N)$ coherent states was proved in [30] (see also [13]), and [31], respectively, whereas the case of the group $SU(1,1)$ was settled in [26]. The uniqueness of the minimizers in the $SU(2)$ case has been obtained simultaneously and independently in [11] and [27], whereas the $SU(N)$ case was resolved only recently in [35] (see also [12] for a quantitative version of the lower bound and [36] for closely related estimates).

Hence, Theorem 1.5 (equivalence $(i) \iff (iii)$) solves the Wehrl-type entropy conjecture for any LCA group that contains a compact open subgroup. This covers the case of finite Abelian groups, of course, but also compact or discrete Abelian groups, or finite-dimensional vector spaces over local fields other than $\mathbb{R}$ and $\mathbb{C}$ (see [38,46]) — hence also settling the non-Archimedean counterpart of the Wehrl conjecture.

A characterization of the maximizers of the Wehrl entropy in the finite-dimensional case and further results are presented in Section 6.

We are confident that the methods discussed in this note and the characterization of stabilizer states in Theorem 1.4 can be utilized to investigate optimizers of other entropic inequalities in quantum information theory (see [7] for an extensive review of current advances of this issue in the case of Gaussian channels; hence $\mathcal{H} \simeq L^2(\mathbb{R}^d)$). We will postpone the examination of these topics to subsequent investigations.

2. Tools from harmonic analysis

2.1. Notation

As explained in the introduction, we denote by A a locally compact Abelian (LCA) group, and by $\widehat{A}$ its dual group, whose elements are the continuous characters $\xi : A \to U(1)$. Endowed with the topology of the uniform convergence on compact subsets, $\widehat{A}$ becomes an LCA group ([18, Theorems 23.13 and 23.15]). We endow A and $\widehat{A}$ with Haar measures normalized so that the Fourier inversion formula holds with constant 1. We denote by $|H|$ the measure of a subset H of A, or $\widehat{A}$ or $A \times \widehat{A}$ (the latter endowed with the *Radon product* measure; see [18, Section 13]). The inner product in $L^2(A)$ is defined as

$$\langle \phi|\psi\rangle_{L^2(A)} = \int_A \overline{\phi(x)}\psi(x)\,dx,$$

where dx is the Haar measure on A (similarly, we write $d\xi$ for the Haar measure on $\widehat{A}$).

We recall that the Haar measure on any LCA group assigns a positive measure to nonempty open subsets; see, e.g., [10, Proposition 2.19]. Conversely, if $H \subset A$ is a compact subgroup of measure $|H| > 0$ then H is open. Indeed, the set where $\chi_H * \chi_H > 0$ (χ_H denoting the indicator function of H) is open (because $\chi_H * \chi_H$ is continuous) and nonempty if $|H| > 0$, and is easily seen to be contained in H. Hence, H is open since it has a nonempty interior.

For a subgroup $H \subset A$, the annihilator of H is the closed subgroup of $\widehat{A}$ given by

$$H^\perp = \{\xi \in \widehat{A} :\ \xi(x) = 1 \text{ for all } x \in H\}$$

(cf. [18, Definition 23.23]). We recall that if $H \subset A$ is a compact and open subgroup, the same holds for $H^\perp \subset \widehat{A}$ ([39, Remark 4.2.22]). Moreover, as a consequence of the Fourier inversion formula, we have

$$|H||H^\perp| = 1$$

(see [39, Formula (4.4.6)]). We also recall that a subgroup $H \subset A$ is open if and only if G/H is discrete ([18, Theorem 5.21]).

We observe that, given a closed subgroup $H \subset A$, we have a natural isomorphism of LCA groups ([18, Theorem 24.11])

$$\widehat{H} \to \widehat{A}/H^\perp \qquad \xi \mapsto \underline{\xi} \tag{2.1}$$

with $\underline{\xi} = \xi' + H^\perp$, where $\xi' \in \widehat{A}$ is any extension of $\xi \in \widehat{H}$, that is, $\xi'|_H = \xi$.

We refer to [18] for a comprehensive introduction to the theory of LCA groups.

Definition 2.1. A continuous homomorphism $\beta : A \to \widehat{A}$ is called symmetric if $\beta(x)(y) = \beta(y)(x)$ for every $x, y \in A$. The set of continuous symmetric homorphisms $\beta : A \to \widehat{A}$ will be denoted by $\mathrm{Sym}(A)$. It is easily seen to be a group with respect to the product given by

$$(\beta_1\beta_2)(x) := \beta_1(x) + \beta_2(x) \qquad x \in A$$

(the sum being understood in $\widehat{A}$).

Observe that a homomorphism $\beta \in \mathrm{Sym}(A)$ can be identified with the continuous symmetric bicharacter $A \times A \to U(1)$ given by $\beta(x, y) := \beta(x)(y)$.

2.2. Isotropic subgroups

For an LCA group A as above, the phase space $A \times \widehat{A}$ is also an LCA group. It is naturally endowed with a symplectic structure given by the bicharacter

$$\sigma : (A \times \widehat{A}) \times (A \times \widehat{A}) \to U(1)$$

defined as

$$\sigma((x, \xi), (y, \eta)) = \xi(y)\overline{\eta(x)} \qquad (x, \xi), (y, \eta) \in A \times \widehat{A}.$$

Definition 2.2. A subgroup $K \subset A \times \widehat{A}$ is called *isotropic* if $\sigma(z, w) = 1$ for every $z, w \in K$.

The next result from [34, Propositions 3.5 and 3.6] will be used later.

Proposition 2.3. *A compact open isotropic subgroup $K \subset A \times \widehat{A}$ necessarily has measure $|K| \leq 1$. Moreover $|K| = 1$ if and only if it is a maximal compact open isotropic subgroup, that is, if it is not strictly contained in any compact open isotropic subgroup.*

The class of maximal compact open isotropic subgroups plays a role analogous to that of Lagrangian subspaces in the symplectic geometry over $\mathbb{R}$. A class of examples is given by subgroups of product type, that is, $K = H \times H^\perp$ for any compact open subgroup $H \subset A$. The following result from [34, Propositions 3.1, 3.5 and 3.6] describes, in fact, all the maximal compact open isotropic subgroups. We will make use of the notation ξ and $\underline{\xi}$ for the correspondence given in (2.1). Hence, if $\beta \in \mathrm{Sym}(H)$ and $x \in H$, we have $\beta(x) \in \widehat{H}$ and $\underline{\beta(x)} \in \widehat{A}/H^\perp$.

Theorem 2.4. *There is a one-to-one correspondence between the set of maximal compact open isotropic subgroups $K \subset A \times \widehat{A}$ and the set of pairs (H, β), where $H \subset A$ is a compact open subgroup and $\beta \in \mathrm{Sym}(H)$. Exactly, every such K can be uniquely written in the form*

$$\begin{aligned} K &= \{(x, \xi) \in A \times \widehat{A} : \ x \in H, \ \xi \in \underline{\beta(x)}\} \\ &= \{(x, \xi) \in A \times \widehat{A} : \ x \in H, \ \xi|_H = \beta(x)\}. \end{aligned}$$

Moreover $H^\perp = \{\xi \in \widehat{A} : \ (0, \xi) \in K\}$.

Observe that $\underline{\beta(x)} \in \widehat{A}/H^\perp$ is a subset of $\widehat{A}$ — a coset of $H^\perp$ in $\widehat{A}$. Hence H is the image of K under the canonical projection $A \times \widehat{A} \to A$ and the fiber in K over x is identified with $\underline{\beta(x)}$. In algebraic terms, K is regarded as an extension of H by $H^\perp$, that is, we have an exact sequence of Abelian groups

$$0 \to H^\perp \to K \to H \to 0,$$

where the second arrow is the map $\xi \mapsto (0,\xi)$ and the third arrow is the restriction to K of the canonical projection $(x,\xi) \mapsto x$.

We observe that, with the above notation, if $x, y \in H$ then

$$\beta(x)(y) = \xi(y) \qquad \text{for every } \xi \in \underline{\beta(x)}. \tag{2.2}$$

Indeed, if $x, y \in H$ and $\underline{\beta(x)} = \xi' + H^\perp$, where $\xi' \in \widehat{A}$ extends $\beta(x) \in \widehat{H}$, and $\xi = \xi' + \eta$, with $\eta \in H^\perp$ we have

$$\beta(x)(y) = \xi'(y) = \xi'(y)\eta(y) = (\xi' + \eta)(y) = \xi(y), \tag{2.3}$$

where we used that $\eta(y) = 1$, since $\eta \in H^\perp$ and $y \in H$.

2.3. Coherent state transform

We have already defined in (1.1) the phase-space shift $\pi(x,\xi) : L^2(A) \to L^2(A)$, for $(x,\xi) \in A \times \widehat{A}$. It is easy to check that, for $x, y \in A$, $\xi, \eta \in \widehat{A}$ we have

$$\pi(x,\xi)\pi(y,\eta) = \overline{\eta(x)}\pi(x+y,\xi+\eta), \tag{2.4}$$

whence the following commutation relations follow:

$$\pi(x,\xi)\pi(y,\eta) = \xi(y)\overline{\eta(x)}\pi(y,\eta)\pi(x,\xi). \tag{2.5}$$

Definition 2.5. Given $\phi \in L^2(A)$, the *coherent state transform* (alias *short-time Fourier transform*) of a function $\psi \in L^2(A)$ with "window" ϕ is the function

$$V_\phi\psi(z) := \langle \pi(z)\phi, \psi\rangle_{L^2(A)} \qquad z \in A \times \widehat{A}.$$

This phase-space transform is widely used in harmonic analysis and signal processing [16], as well as in mathematical physics [5,6,29]. We list here some basic properties whose proof can be found (in the context of a general LCA group) in [15,34].

It is known (see, e.g., [34, Proposition 2.1]) that $V_\phi\psi$ is a continuous function that vanishes at infinity, i.e., for every $\epsilon > 0$ there exists a compact subset $K \subset A \times \widehat{A}$ such that $|V_\phi\psi(z)| < \epsilon$ for $z \in A \setminus K$.

As a consequence of the commutation relations (2.5) we have the following covariance-type properties, for $x, y \in A$, $\xi, \eta \in \widehat{A}$:

$$V_\phi(\pi(x,\xi)\psi)(y,\eta) = \xi(x)\overline{\eta(x)}V_\phi\psi(y-x,\eta-\xi) \tag{2.6}$$

and

$$V_{\pi(x,\xi)\phi}(\pi(x,\xi)\psi)(y,\eta) = \xi(y)\overline{\eta(x)}V_\phi\psi(y,\eta). \tag{2.7}$$

By the Cauchy-Schwarz inequality we also have

$$|V_\phi\psi(x,\xi)| \le \|\psi\|_{L^2(A)}\|\phi\|_{L^2(A)} \qquad x \in A,\ \xi \in \widehat{A}. \tag{2.8}$$

Moreover, as a consequence of the Parseval Formula for the Fourier transform on A, we have the following Parseval-type formula for the coherent state transform:

$$\int\limits_{A\times\widehat{A}} |V_\phi\psi(x,\xi)|^2\,dx\,d\xi = \|\phi\|^2_{L^2(A)}\|\psi\|^2_{L^2(A)}. \tag{2.9}$$

The following result from [34, Proposition 4.1] is less known. It provides some information on the subset of $A\times\widehat{A}$ where the function $V_\psi\psi$ achieves its maximum value. First we observe that if $\phi\in L^2(A)$, with $\|\phi\|_{L^2(A)}=1$ then by (2.8) we have

$$|V_\phi\phi(z)|\le V_\phi\phi(0)=1 \qquad z\in A\times\widehat{A}.$$

Proposition 2.6. *Let $\phi\in L^2(A)$, with $\|\phi\|_{L^2(A)}=1$. The set*

$$\{z\in A\times\widehat{A}:\ |V_\phi\phi(z)|=1\}$$

is a compact isotropic subgroup of $A\times\widehat{A}$.

Finally we recall from [34, Proposition 2.3] the following uniqueness result (see [16, Section 4.2] for the case $A=\mathbb{R}^d$).

Proposition 2.7. *Let $\phi,\psi\in L^2(A)$, with $\|\phi\|_{L^2}=\|\psi\|_{L^2}$. Then*

$$V_\phi\phi(z)=V_\psi\psi(z)\quad \textit{for every } z\in A\times\widehat{A}$$

if and only if $\psi=e^{i\theta}\phi$ for some $\theta\in\mathbb{R}$.

2.4. Characters of second degree

The notion of *character of second degree* was introduced explicitly in [45], motivated by algebraic number theory, and found applications in harmonic analysis [38], the theory of theta functions [24] and quantum information theory, where these characters are generally referred to as *quadratic forms*, see, e.g., [22,43].

Definition 2.8. Consider a symmetric homomorphism $\beta\in\mathrm{Sym}(A)$ (Definition 2.1). A continuous function $h:A\to U(1)$ is a character of second degree of A, associated with β, if

$$h(x+y)=h(x)h(y)\,\beta(x)(y) \qquad x,y\in A.$$

The set of characters of second degree of A, endowed with the product defined by $(h_1h_2)(x)=h_1(x)h_2(x)$, $x\in A$, forms an Abelian group denoted by $\mathrm{Ch}_2(A)$.

We have the following important existence and uniqueness result.

Theorem 2.9. *For every symmetric homomorphism $\beta\in\mathrm{Sym}(A)$ there exists a corresponding character of second degree, and two characters of second degree associated with the same β differ by the multiplication by a character of A. In other terms, we have the following short exact sequence of Abelian groups:*

$$0\to\widehat{A}\to\mathrm{Ch}_2(A)\to\mathrm{Sym}(A)\to 0.$$

In fact, the existence is nontrivial (in the present generality) and was first proved in [23, Lemma 6], while the uniqueness (up to multiplication by a character) is an easy consequence of the definition, as already observed in [45, page 146].

The case of finite groups is particularly important in quantum information theory. The corresponding characters of second degree can be described explicitly as follows.

Explicit construction for finite Abelian groups

Consider a finite Abelian group A (endowed with the discrete topology). The following construction of characters of second degree, associated with a given symmetric homomorphism $\beta \in \mathrm{Sym}(A)$, is inspired by that given in [38] in the case where A is a vector space over a local field. We use the fact that every finite Abelian group is isomorphic to the direct sum of a finite number of cyclic groups.

Case $A = \mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$. We identify $\widehat{\mathbb{Z}_d}$ with $\mathbb{Z}_d$ via the pairing $(x, y) \mapsto \exp(2\pi ixy/d)$. Then a (symmetric) homomorphism $\beta : \mathbb{Z}_d \to \widehat{\mathbb{Z}_d}$ is just the multiplication $x \mapsto px$, for some $p \in \{0, \dots, d-1\}$ (we emphasize that p is here an integer). We consider the function

$$h(x) = \exp\Big(\pi i\, p\, \tilde{x}^2(d+1)/d\Big) \quad x = \tilde{x} + d\mathbb{Z} \in \mathbb{Z}_d.$$

It is easy to see that $h(x)$ is well defined, i.e. the result does not depend on the representative $\tilde{x} \in \mathbb{Z}$ that we choose within its residue class $x \in \mathbb{Z}_d$, and that it is indeed a character of second degree associated with β (see, e.g., [9,33]).

Case $A = A_1 \oplus \dots \oplus A_n$, with A_j cyclic. Each $x \in A$ can be uniquely decomposed as $x = x_1 + \dots + x_n$, with $x_j \in A_j$. Given a symmetric homomorphism $\beta : A \to \widehat{A}$, consider the symmetric bicharacter $\beta(x, y) = \beta(x)(y)$, and let β_j be its restriction to $A_j \times A_j$. Let h_j be the corresponding character of second degree of A_j, as constructed above. Then it is easy to check by induction on n that the function

$$h(x) := \prod_{j=1}^{n} h_j(x_j) \prod_{1 \le j < k \le n} \beta(x_j, x_k)$$

is a character of second degree associated with β.

Hence, we have constructed characters of second degree associated with any given $\beta \in \mathrm{Sym}(A)$. By Theorem 2.9 every character of second degree is obtained from these by multiplying by a character of A.

We observe that the characters of second degree of any finite Abelian group were already constructed in [25], relying on the Smith normal form of A, namely the fact that A is isomorphic to the direct sum $\mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_n}$ for some $d_1, d_2, \dots, d_n$ with $d_1|d_2|...|d_n$. The above construction offers several advantages over the one in [25]: it does not require knowing that particular normal form, but rather just the knowledge of an isomorphism of A into a direct sum of cyclic groups, and it avoids any matrix manipulation in modular arithmetic.

Remark 2.10. If $x \in A$, $\xi \in \widehat{A}$, $n \in \mathbb{Z}$, we have $\xi(nx) = \xi(x)^n$. Hence, if A is finite, say of cardinality N, every character $\xi \in \widehat{A}$ takes values in the multiplicative subgroup of $U(1)$ generated by $\exp(2\pi i/N)$. Similarly, by the above construction (or by a direct inspection of the very definition, cf. [Lemma 5 (f)] [43]) we see that any $h \in \mathrm{Ch}_2(A)$ takes values in the multiplicative subgroup generated by $\zeta = \exp(\pi i/N)$. This fact is related to the definition of the Pauli group (1.5); see Proposition 3.3.

Connection with the Clifford group

Consider a character of second degree $h \in \mathrm{Ch}_2(A)$, associated with a symmetric homomorphism $\beta \in \mathrm{Sym}(A)$ (Definition 2.8). Let $C_h : L^2(A) \to L^2(A)$ be the unitary operator given by the pointwise multiplication by h, that is, $C_h\psi = h\psi$, for $\psi \in L^2(A)$. Then a direct inspection shows that

$$C_h \pi(z) C_h^\dagger = \overline{h(-x)}\pi(Sz), \quad z = (x, \xi) \in A \times \widehat{A}, \quad S := \begin{pmatrix} I & 0 \\ \beta & I \end{pmatrix},$$

where the matrix S is regarded as a homomorphism $A \times \widehat{A} \to A \times \widehat{A}$. We notice that S is a symplectic matrix, that is, $\sigma(Sz, Sw) = \sigma(z, w)$ for every $z, w \in A \times \widehat{A}$, because β is symmetric. Hence C_h is an element of the so-called *Clifford* (alias *metaplectic*) group of A. Indeed, it intertwines the projective unitary representations $\pi(z)$ and $\pi(Sz)$ (see [38,45]).

2.5. Subcharacters of second degree

The notion of subcharacter of second degree was introduced in [34] and will play a key role in the following.

Definition 2.11. A function $h_0 : A \to \mathbb{C}$ is called a subcharacter of second degree if there exists a compact open subgroup $H \subset A$ such that $h_0(x) = 0$ for $x \in A \setminus H$ and the restriction of h_0 to H is a character of second degree of H.

This terminology was inspired by that of *subcharacter*, that is a function $A \to \mathbb{C}$ that vanishes outside some compact open subgroup H and whose restriction to H is a character of H; see [19, Definition 43.3].

Definition 2.12. Let h_0 be a subcharacter of second degree of A; hence $H := \{x \in A : \ h_0(x) \neq 0\}$ is an open compact subgroup of A and $h := h_0|_H \in \mathrm{Ch}_2(H)$. Consider the symmetric homomorphism $\beta \in \mathrm{Sym}(H)$ associated with h according to Definition 2.8. Let $K \subset A \times \widehat{A}$ be the unique maximal compact open isotropic subgroup corresponding to the pair (H, β), according to Theorem 2.4. Then we will say that the subgroup K is associated with the subcharacter h_0.

The following result from [34, Proposition 4.4, Theorem 4.5, Theorem 5.2], provides an analytic counterpart of the above algebraic constructions. We denote by χ_K the indicator function of a subset $K \subset A \times \widehat{A}$.

Theorem 2.13.

(i) *Let h_0 be a subcharacter of second degree of A associated with a maximal compact open isotropic subgroup $K \subset A \times \widehat{A}$ in the sense of Definition 2.12. Then*

$$V_{h_0}h_0(x,\xi) = |H|\overline{h_0(-x)}\chi_K(x,\xi) \qquad (x,\xi) \in A \times \widehat{A}.$$

Moreover, the function $|H|^{-1}V_{h_0}h_0$, restricted to K, is a character of second degree of K associated with the symmetric homomorphism $\beta' \in \mathrm{Sym}(K)$ given by $\beta'(x,\xi)(y,\eta) = \overline{\eta(x)}$, that is, for $(x,\xi), (y,\eta) \in K$,

$$V_{h_0}h_0(x+y, \xi+\eta) = |H|^{-1}V_{h_0}h_0(x,\xi)V_{h_0}h_0(y,\eta)\overline{\eta(x)}.$$

(ii) *Let $\phi \in L^2(A) \setminus \{0\}$. Then $K := \{z \in A \times \widehat{A} : \ V_\phi\phi(z) \neq 0\}$ has measure $|K| \geq 1$. Moreover $|K| = 1$ if and only if*

$$\phi(x) = c\, h_0(x-y)$$

for some $c \in \mathbb{C} \setminus \{0\}$, and some subcharacter h_0 of second degree. In that case, K is the maximal compact open isotropic subgroup associated with h_0 (Definition 2.12) and the function $\|\phi\|^{-2}_{L^2(A)}V_\phi\phi$, restricted to K, is a character of second degree of K associated with the homomorphism $\beta' \in \mathrm{Sym}(K)$ given in (i).

(iii) *Let $\phi, \psi \in L^2(A) \setminus \{0\}$. Then $\mathcal{K} := \{z \in A \times \widehat{A} : \ V_\phi\psi(z) \neq 0\}$ has measure $|\mathcal{K}| \geq 1$. Moreover $|\mathcal{K}| = 1$ if and only if*

$$\phi(x) = c\, h_0(x - y) \qquad \psi(x) = c'\, \pi(x', \xi')\phi(x) \qquad x \in A \tag{2.10}$$

where $c, c' \in \mathbb{C} \setminus \{0\}$, $y, x' \in A$, $\xi' \in \widehat{A}$ and h_0 is a subcharacter of second degree of A. In this case, $\mathcal{K}$ is a coset in $A \times \widehat{A}$ of the maximal compact open isotropic subgroup associated with h_0 (Definition 2.12).

Remark 2.14. In Theorem 2.13, given a function ϕ in the form (1.16), we have mentioned the maximal compact isotropic subgroup K associated with h_0 in the sense of Definition 2.12. Although h_0 is not uniquely determined by ϕ, it follows from the results in that theorem that K is, in fact, uniquely determined by ϕ; the same remark applies to the subgroup K in (1.11). See also Lemma 4.1.

Remark 2.15. The results in Theorem 2.13 (iii) were formulated in [34, Theorem 5.2] in a slightly different (but equivalent) way. Precisely, instead of $\phi(x) = c\, h_0(x - y)$, there one finds that $\phi = c\, \pi(y, \eta)h_0$ for some $(y, \eta) \in A \times \widehat{A}$. This amounts to the same thing, because $\pi(y, \eta)h_0(x) = c'\, h_0'(x - y)$ where $c' \in \mathbb{C} \setminus \{0\}$ and $h'(x) := \eta(x)h_0(x)$ is still a subcharacter of A associated (in the sense of Definition 2.12) with the same maximal compact open isotropic subgroup $K \subset A \times \widehat{A}$ as h_0, by Theorem 2.9.

Remark 2.16. The above results are presented in terms of the phase space shifts $\pi(x, \xi)$ (see (1.1)) and the coherent state transform $V_\phi \psi$ (Definition 2.5), hence in terms of wave functions in $L^2(A)$. We can trivially translate everything in terms of state vectors in $\mathcal{H}$. Indeed, a unitary operator $U = \mathcal{H} \to L^2(A)$ is given, and the Weyl-Heisenberg operators $W(x, \xi)$ are defined by $W(x, \xi) = U^\dagger \pi(x, \xi)U$, see (1.2). Hence, for example,

$$\langle \psi | W(x, \xi) | \phi \rangle = \langle \psi | \pi(x, \xi)\phi \rangle_{L^2(A)} = \overline{V_\phi \psi(x, \xi)}.$$

In the following, when necessary, we will freely use the above results rephrased in terms of state vectors and Weyl-Heisenberg operators.

3. Stabilizer states

In this section we describe the stabilizer states in terms of their wave function, as explained in the introduction. In particular, we prove Theorem 1.4. We begin with a number of remarks and a result that clarify the choice of the class $\mathfrak{G}$ in Definition 1.3.

Remark 3.1.

1) If $\mathcal{G} \subset \mathbb{H}_A$ (see (1.4)) is a subgroup such that all elements of $\mathcal{G}$ admit a common eigenvector, with eigenvalue 1, then it is clear that the homomorphism in (1.8) must be injective. We will see that if, moreover, $\mathcal{G}$ is locally compact, that is, closed in $\mathbb{H}_A$, then $\mathcal{G}$ is necessarily compact (see Proposition 3.2).
2) The requirement, in Definition 1.3, that the image of the homomorphism (1.8) has measure 1, should be interpreted as a maximality condition (see Proposition 3.3). When A is finite, say of cardinality N, that condition is equivalent to stating that $\mathcal{G}$ has cardinality N. Indeed, we can consider the counting measure as the Haar measure on A. Then the corresponding Haar measure on $\widehat{A}$ (for which the Fourier inversion formula holds with constant 1) will be the counting measure multiplied by N^{-1}.
3) Since a compact subgroup of an LCA group has a positive measure if only if it is open, we see that $\mathfrak{G}$ is nonempty only if A contains a compact open subgroup (we do not assume this condition explicitly in Theorem 1.4, because the statement is vacuously true when it is not satisfied). Interestingly, we will show that, in a sense, there is no loss of generality in assuming that A contains a compact open subgroup (see Proposition 3.2 and the sentence before its statement).

4) The following are examples of LCA groups that contain a compact open subgroup: finite Abelian groups, or more generally, compact or discrete Abelian groups, finite-dimensional vector spaces over local fields other than $\mathbb{R}$ and $\mathbb{C}$ (see [46, page 20]) and products of such groups.

In the following, we will make use of the coherent state transform defined in Definition 2.5 (see also Remark 2.16), and in particular of the relationship

$$V_\phi \phi(z) = \langle \pi(z)\phi|\phi\rangle_{L^2(A)} = \overline{\langle \phi|W(z)|\phi\rangle} \qquad z \in A \times \widehat{A}. \tag{3.1}$$

Also, to put the following result into perspective, we recall that, if A is any LCA group, there exist an integer $d \geq 0$ and an LCA group A_0, containing a compact and open subgroup, such that $A \simeq \mathbb{R}^d \times A_0$ (topological isomorphism); see, e.g., [18, Theorem 24.30].

Proposition 3.2. *Let $\mathcal{G} \subset \mathbb{H}_A$ be a closed subgroup such that there exists a common eigenvector of all the elements of $\mathcal{G}$. Then $\mathcal{G}$ is Abelian and compact.*

Moreover, if $A \simeq \mathbb{R}^d \times A_0$ (topological isomorphism), where $d \geq 0$ is an integer and A_0 is an LCA group, then, regarding $\mathbb{H}_{A_0}$ as a subgroup of $\mathbb{H}_A$, we have $\mathcal{G} \subset \mathbb{H}_{A_0}$, and the image of $\mathcal{G}$ in $A_0 \times \widehat{A_0}$ under the canonical projection $\mathbb{H}_{A_0} \simeq U(1) \times A_0 \times \widehat{A_0} \to A_0 \times \widehat{A_0}$ is a compact isotropic subgroup $K \subset A_0 \times \widehat{A_0}$ of measure $|K| \leq 1$ (where the Haar measure of $A_0 \times \widehat{A_0}$ is understood).

Proof. Let $|\psi\rangle$ be a common normalized eigenvector of the elements of $\mathcal{G}$ (hence with eigenvalues of modulus 1) and let $\psi(x)$ be its wave function. Let $K \subset A \times \widehat{A}$ be the image of $\mathcal{G}$ in $A \times \widehat{A}$ under the canonical projection $\mathbb{H}_A \simeq U(1) \times A \times \widehat{A} \to A \times \widehat{A}$. Since $\mathcal{G}$ is closed and $U(1)$ is compact, K is closed. In addition, by (3.1),

$$|V_\psi \psi(z)| = |\langle \pi(z)\psi|\psi\rangle_{L^2(A)}| = |\langle \psi|W(z)|\psi\rangle| = 1 \qquad z \in K.$$

Hence

$$K \subset S := \{z \in A \times \widehat{A} : \ |V_\psi \psi(z)| = 1\}.$$

By Proposition 2.6, S is a compact isotropic subgroup of $A \times \widehat{A}$. This implies that K is compact and isotropic as well and, therefore, $\mathcal{G}$ is Abelian (cf. (1.2) and (2.5)). In addition, $\mathcal{G}$ is compact, because, as assumed, it is a closed subset of $U(1) \times A \times \widehat{A}$ and is contained in the compact subset $U(1) \times K$.

Since $\mathcal{G}$ is compact, with the identification $\mathbb{H}_A \simeq U(1) \times A \times \widehat{A} \simeq U(1) \times \mathbb{R}^d \times A_0 \times \widehat{\mathbb{R}^d} \times \widehat{A_0}$, we see that $\mathcal{G} \subset U(1) \times \{0\} \times A_0 \times \{0\} \times \widehat{A_0} \simeq \mathbb{H}_{A_0}$. That is, $K \subset \{0\} \times A_0 \times \{0\} \times \widehat{A_0}$.

Finally, regarded as subgroup of $A_0 \times \widehat{A_0}$, K is still compact and isotropic. Moreover, if $|K| > 0$ then K is also open in A_0 (as already observed, a compact subgroup of an LCA group has a positive measure if and only if it is open) and therefore, by Proposition 2.3, $|K| \leq 1$. In all cases, we have $|K| \leq 1$. $\square$

Proposition 3.3. *A subset $\mathcal{G} \subset \mathbb{H}_A$ belongs to $\mathfrak{G}$ (Definition 1.3) if and only if*

$$\mathcal{G} = \{\alpha(z)W(z) : \ z \in K\}, \tag{3.2}$$

where

(i) *$K \subset A \times \widehat{A}$ is a maximal compact open isotropic subgroup;*
(ii) *$\alpha \in \mathrm{Ch}_2(K)$ is associated (in the sense of Definition 2.8) with the symmetric homomorphism $\beta' \in \mathrm{Sym}(K)$ given by*

$$\beta'(x,\xi)(y,\eta) = \overline{\eta(x)} \qquad (x,\xi),\, (y,\eta) \in K,$$

that is, $\alpha : K \to U(1)$ is continuous and satisfies

$$\alpha(x+y,\xi+\eta) = \alpha(x,\xi)\alpha(y,\eta)\overline{\eta(x)} \qquad (x,\xi),\, (y,\eta) \in K. \tag{3.3}$$

Moreover, if A is finite, every subgroup $\mathcal{G} \in \mathfrak{G}$ is in fact a subgroup of the Pauli group $\mathbb{P}_A$ (see (1.5)).

Proof. Let $\mathcal{G}$ be as in the statement. Hence $\mathcal{G}$ has the form in (3.2) for some maximal compact open isotropic subgroup $K \subset A \times \widehat{A}$ and some $\alpha \in \mathrm{Ch}_2(K)$ satisfying (3.3). It is clear that the map (1.8) is injective. Moreover, since $K \subset A \times \widehat{A}$ is a subgroup and the function $\alpha : K \to U(1)$ satisfies (3.3), using (1.2) and (2.4) we see that $\mathcal{G}$ is a subgroup of $\mathbb{H}_A$. Also, recall that $\mathbb{H}_A$ is identified with $U(1) \times A \times \widehat{A}$, as a topological space, and $\mathcal{G}$ in (3.2) is consequently identified with the graph of the function α, that is, $\{(\alpha(z), z) :\ z \in K\} \subset U(1) \times A \times \widehat{A}$. Since $U(1)$ is Hausdorff and α is continuous by assumption, its graph is closed in $U(1) \times K$, and therefore is compact. To conclude that $\mathcal{G} \in \mathfrak{G}$ it remains to check that $|K| = 1$. But this follows from the characterization of the maximal compact open isotropic subgroups of $A \times \widehat{A}$ in Proposition 2.3.

Conversely, let $\mathcal{G} \in \mathfrak{G}$. Since the homomorphism (1.8) is injective, and $\mathcal{G}$ is compact, we see that $\mathcal{G}$ has the form (3.2) for some function $\alpha : K \to U(1)$, where the subgroup $K \subset A \times \widehat{A}$ is compact. Moreover, since $\mathcal{G}$ is Abelian, by (1.2) and (2.5) we see that K is isotropic. Also, K has measure $|K| = 1$ by assumption, and therefore is open in $A \times \widehat{A}$. As a consequence of Proposition 2.3, it is a maximal compact open isotropic subgroup of $A \times \widehat{A}$. Moreover, the function α satisfies (3.3) because $\mathcal{G}$ is a subgroup of $\mathbb{H}_A$. It remains to prove that α is continuous. As already observed, its graph is identified with the subgroup $\mathcal{G}$, which is compact by assumption, and therefore it is closed as a subset of $U(1) \times K$. Since $U(1)$ is compact and Hausdorff, we deduce that α is continuous, by the closed graph theorem in point-set topology.

Finally, let A be a finite Abelian group, say of cardinality N. Since $|K| = 1$, from Remark 3.1 2) we see that K has cardinality N. Hence, by Remark 2.10 we have that $\alpha \in \mathrm{Ch}_2(K)$ takes values in the multiplicative group generated by $\zeta = \exp(\pi i/N)$. Therefore, $\mathcal{G} \subset \mathbb{P}_A$. □

Proposition 3.4. *Let $\mathcal{G} \in \mathfrak{G}$ be as in Proposition 3.3, that is, in the form (3.2) where $K \subset A \times \widehat{A}$ is a maximal compact open isotropic subgroup and $\alpha \in \mathrm{Ch}_2(K)$ satisfies (3.3).*

A vector $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, is a common eigenvector, with eigenvalue 1, of all elements of $\mathcal{G}$ if and only if

$$V_\phi\phi(z) = \begin{cases} \alpha(z) & z \in K \\ 0 & z \notin K. \end{cases} \tag{3.4}$$

Proof. Suppose that $\phi \in L^2(A)$, with $\|\phi\|_{L^2(A)} = 1$, satisfies (3.4). Then, by (3.1), we have

$$\langle\phi|W(z)|\phi\rangle = \overline{\alpha(z)} \qquad z \in K.$$

Since $|\alpha(z)| = 1$, this means that, if $z \in K$, ϕ and $W(z)\phi$ are linearly dependent and that, in fact,

$$\alpha(z)W(z)|\phi\rangle = |\phi\rangle \qquad z \in K.$$

Conversely, it is clear that the latter formula implies that $V_\phi\phi(z) = \alpha(z)$ for $z \in K$. Moreover $|K| = 1$ by Proposition 2.3. Hence, $V_\phi\phi(z) = 0$ for almost every $z \in A \times \widehat{A} \setminus K$, by (2.9). Since $A \times \widehat{A} \setminus K$ is open and $V_\phi\phi$ is continuous, we see that $V_\phi\phi(z) = 0$ for every $z \in A \times \widehat{A} \setminus K$. Hence (3.4) is satisfied. □

We can now prove Theorem 1.4.

Proof of Theorem 1.4. We use the description of the subgroups $\mathcal{G} \in \mathfrak{G}$ given in Proposition 3.3. It is clear from Proposition 3.4 that every $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, is stabilized by *at most* one subgroup $\mathcal{G} \in \mathfrak{G}$, because the relation (3.4) determines uniquely K and α, and therefore $\mathcal{G}$, from ϕ. On the other hand we see that every $\mathcal{G}$ stabilizes *at most* one state, again by the relationship (3.4) and the uniqueness result in Proposition 2.7.

Now, let $|\phi\rangle$ be an S-state vector, with $\||\phi\rangle\| = 1$. Hence

$$\phi(x) = c\, h_0(x - y)$$

for some $c \in \mathbb{C} \setminus \{0\}$, $y \in A$ and some subcharacter of second degree h_0. Since ϕ is normalized, with the notation in the statement, $|c| = |H|^{-1/2}$. Let us prove that $|\phi\rangle$ is stabilized by the subgroup $\mathcal{G} \in \mathfrak{G}$ given in the statement. We have to prove that the pair K, α given in (1.11) and (1.12) satisfies the conditions in Proposition 3.3 and that (3.4) is satisfied. It is clear from Theorem 2.4 that the set K in (1.11) is a maximal compact open isotropic subgroup of $A \times \widehat{A}$. In fact, it is the subgroup associated with the subcharacter h_0 in the sense of Definition 2.12. Moreover, by the covariance property (2.7) and Theorem 2.13 (i) we have

$$V_\phi \phi(x,\xi) = |H|^{-1}\overline{\xi(y)} V_{h_0} h_0(x,\xi) = \overline{h_0(-x)\xi(y)}\chi_K(x,\xi). \tag{3.5}$$

This implies that (3.4) is satisfied with the function α defined in (1.12). Moreover, Theorem 2.13 (ii) tells us that the function $V_\phi\phi$, restricted to K, that is α, is a character of second degree of K associated with the homomorphism $\beta' \in \mathrm{Sym}(K)$ given in Proposition 3.3. This concludes the proof of the first part of the statement.

Now, consider a subgroup $\mathcal{G} \in \mathfrak{G}$. We know from Proposition 3.3 that $\mathcal{G}$ has the form in (3.2), where $K \subset A \times \widehat{A}$ is a maximal compact open isotropic subgroup and $\alpha \in \mathrm{Ch}_2(K)$ satisfies (3.3). Moreover K has necessarily the form given in Theorem 2.4 for some compact open subgroup $H \subset A$ (the image of K under the canonical projection $A \times \widehat{A} \to A$) and some symmetric homomorphism $\beta \in \mathrm{Sym}(H)$. Let us prove that there exists an S-state that is stabilized by $\mathcal{G}$. The proof is constructive and follows steps (i)—(iii) in the statement.

(i) We know from Theorem 2.4 that $H^\perp = \{\xi \in \widehat{A} : \ (0,\xi) \in K\}$. For $\xi, \eta \in H^\perp$, by (3.3) we have

$$\alpha(0, \xi + \eta) = \alpha(0,\xi)\alpha(0,\eta)\eta(0) = \alpha(0,\xi)\alpha(0,\eta).$$

Hence the map $H^\perp \ni \xi \mapsto \alpha(0,\xi)$ is a continuous character of $H^\perp$. The continuous character $H^\perp \ni \xi \mapsto \overline{\alpha(0,\xi)}$ admits an extension to a continuous character of $\widehat{A}$ (see, e.g., [18, Corollary 24.12]), that is an element of the dual group of $\widehat{A}$, which is canonically identified with A (see, e.g., [18, Theorem 24.8]). Therefore there exists $y \in A$ such that $\xi(y) = \overline{\alpha(0,\xi)}$ for every $\xi \in H^\perp$.

(ii) First, let us prove that the function $h : H \to U(1)$ in the statement is well defined. If $(x,\xi), (x,\eta) \in K$, then $\eta' := \eta - \xi \in H^\perp$ by Theorem 2.4. Hence by (3.3),

$$\begin{aligned}\alpha(x,\eta)\eta(y) = \alpha(x,\xi+\eta')(\xi+\eta')(y) &= \alpha(x,\xi)\alpha(0,\eta')\overline{\eta'(x)}\xi(y)\eta'(y)\\ &= \alpha(x,\xi)\xi(y),\end{aligned}$$

because $\eta'(x) = 1$ ($x \in H$ and $\eta' \in H^\perp$) and $\eta'(y) = \overline{\alpha(0,\eta')}$ by the point (i).

Let us prove that $h \in \mathrm{Ch}_2(H)$, and precisely that h is associated (in the sense of Definition 2.8) with the aforementioned symmetric homomorphism β. We have $h(x) = \overline{\alpha(-x,-\xi)}\xi(y)$ for $(x,\xi) \in K$. Hence by (3.3), if $(x,\xi), (x',\xi') \in K$,

$$\begin{aligned}
h(x+x') &= \overline{\alpha(-x-x',-\xi-\xi')}(\xi+\xi')(y)\\
&= \overline{\alpha(-x,-\xi)\alpha(-x',-\xi')}\xi'(x)\xi(y)\xi'(y)\\
&= h(x)h(x')\xi'(x)\\
&= h(x)h(x')\beta(x)(x').
\end{aligned}$$

The last equality is true because $\beta(x)(x') = \beta(x')x = \xi'(x)$ by (2.3).

It remains to prove that h is continuous. It suffices to prove that there exists a continuous section of K, that is, a continuous map $\gamma : H \to \widehat{A}$ such that $(x, \gamma(x)) \in K$ for every $x \in H$. The continuity of h then follows, because $h(-x) = \overline{\alpha(x, \gamma(x))\gamma(x)(y)}$ and $\alpha : K \to U(1)$ is continuous. To construct such a section, observe that, since $H^\perp$ is open, $\widehat{H} \simeq \widehat{A}/H^\perp$ is discrete, and $\beta(H) \subset \widehat{H}$ is compact, therefore finite. Let $\beta(H) = \{\eta_1, \dots, \eta_N\}$, and set $H_j := \beta^{-1}(\{\eta_j\}) \subset H$, for $j = 1, \dots, N$. Then each H_j is open and the collection of the H_j's is a partition of H. Now, for each $j = 1, \dots, N$, pick an element $\xi_j \in \widehat{A}$ such that $\xi_j|_H = \eta_j$ (such an extension always exists, see [18, Corollary 24.12]). Define $\gamma : H \to \widehat{A}$ by setting $\gamma(x) = \xi_j$ if $x \in H_j$. It is clear that γ is continuous and that $(x, \gamma(x)) \in K$ for every $x \in H$ by construction (recall that K and β are related as in Theorem 2.4).

(iii) We now extend h to a function $h_0 : A \to \mathbb{C}$ by setting $h_0(x) = h(x)$ for $x \in H$ and $h_0(x) = 0$ for $x \in A \setminus H$. Hence h_0 is a subcharacter of second degree associated with K in the sense of Definition 2.12. For the (normalized) function $\phi(x) = |H|^{-1/2} h_0(x - y)$ we see, arguing as above, that (3.5) holds, and therefore (3.4) is satisfied because $\overline{h(-x)\xi(y)} = \alpha(x, \xi)$ for $(x, \xi) \in K$, by the very definition of h. By Proposition 3.4, this implies that $|\phi\rangle$ is stabilized by the group $\mathcal{G} \in \mathfrak{G}$. This concludes the proof. □

Remark 3.5. As a consequence of Theorem 1.4, the support of the wave function ϕ of a stabilizer state is a coset of a compact open subgroup $H \subset A$ and ϕ has constant modulus on its support. When A is finite these properties also follow from general results concerning *monomial stabilizer states*; see [42,43].

4. A moduli space for the set of stabilizer states

In this section we provide a natural parametrization of the set of stabilizer states (Definition 1.3). We know from Theorem 1.4 that the stabilizer states are exactly the S-states (Definition 1.2). Their wave functions therefore have the form in (1.7) for some $y \in A$ and some subcharacter of second degree h_0 of A. However, different pairs y, h_0 might give rise to the same state. This fact is clarified by the following result.

Lemma 4.1. *Let*

$$\phi(x) = c\, h_0(x - y) \qquad x \in A \tag{4.1}$$

and

$$\phi'(x) = c\, h_0'(x - y') \qquad x \in A, \tag{4.2}$$

where $c, c' \in \mathbb{C} \setminus \{0\}$, $y, y' \in A$ and h_0, h_0' are subcharacters of second degree of A, hence $H := \{x \in A : h_0(x) \neq 0\}$ and $H' := \{x \in A : \ h_0'(x) \neq 0\}$ are compact open subgroups of A, and $h := h_0|_H \in \mathrm{Ch}_2(H)$, $h' := h_0'|_{H'} \in \mathrm{Ch}_2(H')$. Suppose that h is associated with $\beta \in \mathrm{Sym}(H)$ and h' is associated with $\beta' \in \mathrm{Sym}(H')$ in the sense of Definition 2.8.

Then $\phi(x)$ and $\phi'(x)$ are linearly dependent if and only if

$$\begin{cases} H = H' \\ y - y' \in H \\ h(x) = h'(x)\beta'(y-y')(x) \qquad x \in H. \end{cases} \tag{4.3}$$

In this case, we also have $\beta = \beta'$.

Proof. Suppose that ϕ and ϕ' are linearly dependent. Then $\{x \in A : \ \phi(x) \neq 0\} = \{x \in A : \ \phi'(x) \neq 0\}$, which implies $H = H'$.

Moreover, for some $c'' \in \mathbb{C} \setminus \{0\}$,

$$h_0(x-y) = c'' h_0'(x-y') \quad x \in A,$$

that is,

$$h_0(x) = c'' h_0'(x+y-y') \qquad x \in A.$$

This implies that $y - y' \in H$ and that

$$h(x) = c'' h'(x+y-y') \qquad x \in H,$$

whence

$$h(x) = c'' h'(x) h'(y-y') \beta'(y-y')(x).$$

Setting $x = 0$ we see that $c'' h'(y-y') = 1$ and therefore

$$h(x) = h'(x)\beta'(y-y')(x) \qquad x \in H.$$

Since h and h' differ by the multiplication by a character, by Theorem 2.9 we have $\beta = \beta'$.

Conversely it is clear, by reversing the above reasoning, that condition (4.3) implies that ϕ and ϕ' are linearly dependent. $\square$

We now fix a compact open subgroup $H \subset A$. Using the notation of Lemma 4.1, and inspired by that result, we give the following definition.

Definition 4.2. In the set $A \times \mathrm{Ch}_2(H)$ we say that a pair (y, h) is equivalent to (y', h'), and we write $(y, h) \sim (y', h')$, if $y - y' \in H$ and $h(x) = h'(x)\beta'(y-y')(x)$ for $x \in H$. We denote by

$$A \times_H \mathrm{Ch}_2(H) := A \times \mathrm{Ch}_2(H)/\sim$$

the corresponding quotient set.

Indeed, it is easy to see that the above is an equivalence relation.

Remark 4.3. The above construction admits a suggestive interpretation. That is, $A \times_H \mathrm{Ch}_2(H)$ can be regarded as "a bundle over A/H, associated with the H-principal bundle $A \to A/H$, with typical fiber $\mathrm{Ch}_2(H)$". Indeed, we know that H acts naturally on A via the map $y \mapsto u+y$, $u \in H$, $y \in A$, and this action is free and transitive on the fibers of the map $A \to A/H$. Moreover, H also acts on $\mathrm{Ch}_2(H)$ by $u \cdot h = h_u$, with $h_u(x) = h(x)\beta(u)(x)$, for $u \in H$, $x \in A$. We now consider the diagonal action on $A \times \mathrm{Ch}_2(H)$, given by

$$u \cdot (y, h) = (y + u, h_u) \qquad u \in H,\ y \in A,\ h \in \mathrm{Ch}_2(H).$$

Then $A \times_H \mathrm{Ch}_2(H)$ is just the set of the orbits of this action.

As a consequence of Lemma 4.1 we have the following result.

Proposition 4.4. *Let $H \subset A$ be a compact open subgroup. There is a natural one-to-one correspondence between the set $A \times_H \mathrm{Ch}_2(H)$ and that of S-states (Definition 1.2) whose wave functions have a coset of H as support. Exactly, with $[(y,h)] \in A \times_H \mathrm{Ch}_2(H)$ it is associated the S-state represented by the wave function $\phi(x) = h_0(x - y)$, where $h_0(x) = h(x)$ for $x \in H$ and $h_0(x) = 0$ for $x \in A \setminus H$.*

We now consider the natural surjective map

$$A \times_H \mathrm{Ch}_2(H) \to A/H \qquad [(y, h)] \mapsto [y], \tag{4.4}$$

where the square brackets represent the residue classes in the corresponding quotient spaces. It is easy to see that this map is well defined, and we now consider its fibers. Given $h, h' \in \mathrm{Ch}_2(H)$ we denote by β and $\beta' \in \mathrm{Sym}(H)$ the corresponding symmetric homomorphisms, as in Definition 2.8.

Proposition 4.5. *The fibers of the map (4.4) have a natural structure of Abelian group, where the group law is defined as follows. If $[(y, h)], [(y', h')] \in A \times_H \mathrm{Ch}_2(H)$, with $y - y' \in H$,*

$$[(y, h)] \cdot [(y', h')] := [(y, h'')]$$

where

$$h''(x) = h(x)h'(x)\beta'(y - y')(x).$$

Furthermore, each fiber is isomorphic to $\mathrm{Ch}_2(H)$, although not canonically.

Proof. It is easy to see, by direct inspection, that the above product is well defined and endows each fiber with a structure of Abelian group. One has to use the fact that, with the notation in the statement, $hh' \in \mathrm{Ch}_2(H)$ is associated with the symmetric homomorphism $\beta\beta' \in \mathrm{Sym}(H)$, in the sense of Definition 2.8, where $\beta\beta'(x) := \beta(x) + \beta'(x)$ for $x \in H$ (the sum being understood in $\widehat{H}$).

Concerning the second part of the statement, we observe that, for any given $y \in A$, the map

$$\mathrm{Ch}_2(H) \to A \times_H \mathrm{Ch}_2(H) \qquad h \mapsto [(y, h)]$$

is a group isomorphism onto its image, and that the latter is the fiber of the map (4.4) over $[y]$. With the interpretation in Remark 4.3, injectivity follows from the fact that the action of H on the coset $y + H$ (that is, the fiber of the map $A \to A/H$ over $[y]$) is free, whereas surjectivity is a consequence of that action being transitive. □

Remark 4.6. The group law introduced in Proposition 4.5 on the fibers of the map (4.4) can be interpreted in terms of pointwise multiplication of wave functions. Namely, according to Proposition 4.4, to the pairs $[(y, h)]$ and $[(y', h')]$ in the same fiber (hence $y - y' \in H$) there correspond two wave functions ϕ and ϕ'

(defined up to nonzero multiplicative constants) with the same support — a coset of H in A. Then, with the notation of Lemma 4.1 we have, for some constant $c \neq 0$ and every $x \in A$,

$$\begin{aligned}\phi(x)\phi'(x) &= c\, h_0(x-y)h_0'(x-y') \\ &= c\, h_0(x-y)h_0'(x-y)\beta'(y-y')(x) \\ &= C(y,y')h_0(x-y)h_0'(x-y)\beta'(y-y')(x-y)\end{aligned}$$

for some number $C(y,y') \in \mathbb{C} \setminus \{0\}$ depending on y and y'. The latter expression is the wave function of an S-state associated, in the sense of Proposition 4.4, with $[(y,h)] \cdot [(y',h')]$ (in fact, one could also interpret this product in terms of composition of shift and Clifford operators; cf. Section 2).

We can summarize the above results and Theorem 1.4 as follows.

Corollary 4.7. *Let $\mathfrak{G}$ and $\mathfrak{S}$ be the collections of stabilizer subgroups $\mathcal{G} \subset \mathbb{H}_A$ and S-states, respectively (see Definitions 1.3 and 1.2). Then we have natural bijections*

$$\mathfrak{G} \simeq \mathfrak{S} \simeq \bigsqcup_{H \subset A} A \times_H \mathrm{Ch}_2(H)$$

where the first correspondence is given in Theorem 1.4, while the second is described in Proposition 4.4, and the disjoint union is taken over all compact open subgroups $H \subset A$.

Moreover, if A is finite, the cardinality of these sets is given by

$$\#\mathfrak{G} = \#\mathfrak{S} = \#A \cdot \sum_{H \subset A} \#\mathrm{Sym}(H). \tag{4.5}$$

Proof. It remains only to prove the last part of the statement, when A is finite. We clearly have

$$\#\mathfrak{G} = \#\mathfrak{S} = \sum_{H \subset A} \#A \times_H \mathrm{Ch}_2(H).$$

On the other hand, by Proposition 4.5,

$$\#A \times_H \mathrm{Ch}_2(H) = \#A/H \cdot \#\mathrm{Ch}_2(H)$$

and by Theorem 2.9 we have

$$\#\mathrm{Ch}_2(H) = \#\widehat{H} \cdot \#\mathrm{Sym}(H).$$

Since $\#A/H = \#A/\#H = \#A/\#\widehat{H}$, we obtain the desired conclusion. □

5. Minimizing the Wehrl entropy

In this section we prove Theorem 1.5. We preliminarily observe that, by the spectral theorem, a density operator ρ in $\mathcal{H}$ (that is, a compact nonnegative operator with trace 1) can always be written as

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \tag{5.1}$$

where $p_j > 0$, with $\sum_j p_j = 1$, and the $|\psi_j\rangle$'s are an *orthonormal system* of $\mathcal{H}$. The set of the indices j is at most countable, even if $\mathcal{H}$ is not separable. Indeed, we have the orthogonal decomposition in ρ-invariant subspaces $\mathcal{H} = \mathrm{Ker}\,\rho \oplus \overline{\rho(\mathcal{H})}$, and $\overline{\rho(\mathcal{H})}$ is separable, since ρ is compact.

As a consequence of (5.1), for the corresponding Husimi function in (1.14) we have

$$u_{\phi,\rho}(z) = \sum_j p_j |V_\phi \psi_j|^2, \tag{5.2}$$

for any $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, where $V_\phi \psi_j$ denotes the coherent state transform of ψ_j with window ϕ (see Definition 2.5).

Lemma 5.1. *Let $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$ and let ρ be a density operator in $\mathcal{H}$. The Husimi function $u_{\phi,\rho}$ satisfies*

$$0 \le u_{\phi,\rho}(z) \le 1 \qquad z \in A \times \widehat{A} \tag{5.3}$$

and

$$\int\limits_{A\times\widehat{A}} u_{\phi,\rho}(x,\xi)\, dx\, d\xi = 1. \tag{5.4}$$

Moreover $u_{\phi,\rho}$ is a continuous function in $A \times \widehat{A}$, which tends to zero at infinity. As a consequence, the set $\{z \in A \times \widehat{A} :\ u_{\phi,\rho}(z) > 0\}$ is σ-compact.

Proof. We use the formula (5.2). Then (5.3) and (5.4) are immediate consequence of (2.8) and (2.9). Moreover, we know that the functions $V_\phi \psi_j$ are continuous in $A \times \widehat{A}$ and tend to zero at infinity (see Section 2). The same holds true for $u_{\phi,\rho}$, because the series (possibly a finite sum) (5.2) converges uniformly in $A \times \widehat{A}$ by the Weierstrass M-test. $\square$

The following remark clarifies the hypothesis "$G(0) = 0$" in Theorem 1.5.

Remark 5.2.

1) The assumption "$G(0) = 0$" in Theorem 1.5 ensures that the integral in (1.15) is well defined. Indeed, this property implies that $G(\tau) \ge \tau G(1)$ for $0 \le \tau \le 1$; in particular, $G \ge 0$ if $G(1) \ge 0$. Instead, if $G(1) < 0$ then $G(u_{\rho,\phi})_-$, the negative part of $G(u_{\rho,\phi})$, satisfies

$$0 \le G(u_{\rho,\phi})_- \le |G(1)| u_{\rho,\phi},$$

and is therefore summable by (5.4).

2) The following example shows that for certain LCA groups A and concave functions G, with $G(0) \ne 0$, the integral in (1.15) is not well defined.
Let $\mathcal{H} = L^2(A)$, with $A = U(1)$, equipped with the normalized Haar measure. Then $\widehat{A} \simeq \mathbb{Z}$, endowed with the counting measure as Haar measure. Let $\phi(x) = 1$ and $\psi_j(x) = x^j$, for $x \in U(1)$, $j = 1, 2, \ldots$. An explicit computation shows that $V_\phi \psi_j(x,\xi)$ is the indicator function of the set $U(1) \times \{j\} \subset U(1) \times \mathbb{Z}$. Setting

$$\rho = \sum_{j=1}^{\infty} 2^{-j} |\psi_j\rangle\langle\psi_j|,$$

we see that $u_{\phi,\rho} = \sum_{j=1}^{\infty} 2^{-j} |V_\phi \psi_j|^2$ vanishes on a subset of $U(1) \times \mathbb{Z}$ of infinite measure, and for every $\epsilon > 0$, we have $0 < u_{\phi,\rho} < \epsilon$ on a subset of infinite measure. Consider therefore any concave function

$G : [0,1] \to \mathbb{R}$ with $G(0) < 0$ and $\lim_{\tau\to 0^+} G(\tau) > 0$. Then both the positive and negative parts of the function $G(u_{\phi,\rho}(z))$ have an infinite integral on $U(1) \times \mathbb{Z}$. Hence, the integral in (1.15) would not make sense in this case.

3) To ensure that the integral in (1.15) is well defined, we could even consider concave functions $G : [0,1] \to \mathbb{R}$ continuous at 0 (without assuming that $G(0) = 0$). However, when $|A| = \infty$, that integral would always be $+\infty$ if $G(0) > 0$ or always $-\infty$ if $G(0) < 0$, due to Lemma 5.1.

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality we can assume that $G(1) = 0$. Indeed, if the theorem has been proved in this case, given a function G as in the statement, we can then apply the theorem to the function $G(\tau) - \tau G(1)$ and use (5.4).

Hence, thereafter we assume that $G(1) = 0$. The formula (1.16) is then clear, because G is nonnegative (recall that G is concave and $G(0) = 0$ by assumption).

Concerning the remaining part of the statement, we observe that the assumption that G is not linear means (for our concave function G satisfying $G(0) = G(1) = 0$) that $G(\tau) > 0$ for $0 < \tau < 1$. Under this assumption, we now prove the following chain of implications between the points in the statement.

$(i) \implies (ii)$ Since $G(0) = G(1) = 0$ and $G(\tau) > 0$ for $0 < \tau < 1$ we see that if equality occurs in (1.16) for some ϕ and ρ then the subset of $A \times \widehat{A}$ where $0 < u_{\phi,\rho} < 1$ has measure zero. But this subset is open, because $u_{\phi,\rho}$ is continuous (Lemma 5.1), and therefore it is empty. It follows that $u_{\phi,\rho}$ is the indicator function of some subset of $\mathcal{K} \subset A \times \widehat{A}$, necessarily having measure $|\mathcal{K}| = 1$, by (5.4).

$(ii) \implies (iii)$ Using the spectral decomposition (5.1) for ρ, and therefore (5.2), we see that

$$\sum_j p_j |V_\phi \psi_j|^2 = \chi_{\mathcal{K}},$$

where $|\mathcal{K}| = 1$. Since $p_j > 0$ for all j, and $\sum_j p_j = 1$, we have

$$|V_\phi \psi_j| = \chi_{\mathcal{K}} \quad \text{for every } j. \tag{5.5}$$

By Theorem 2.13 (iii) and (ii) we deduce that $|\phi\rangle$ is an S-state,

$$\psi_j = e^{i\theta_j} \pi(z_j)\phi, \tag{5.6}$$

namely

$$|\psi_j\rangle = e^{i\theta_j} W(z_j)|\phi\rangle,$$

for some $\theta_j \in \mathbb{R}$ and $z_j \in A \times \widehat{A}$ (see (1.2)), and that $\mathcal{K}$ is a coset in $A \times \widehat{A}$ of the maximal compact open isotropic subgroup $K \subset A \times \widehat{A}$ given by

$$K = \{z \in A \times \widehat{A} : V_\phi \phi(z) \neq 0\}. \tag{5.7}$$

We also know that the $|\psi\rangle_j$'s are pairwise orthogonal. Hence, if there exist two indices j, k with $j \neq k$ in the sum (5.1), we have

$$|V_\phi \phi(z_j - z_k)| = |\langle \phi | W(-z_k) W(z_j) | \phi \rangle| = |\langle \psi_k | \psi_j \rangle| = 0,$$

that is, z_j and z_k belong to different cosets of K. By (5.6), (5.7) and the covariance property (2.6) we deduce that $z_j + K = \{z \in A \times \widehat{A} : V_\phi \psi_j(z) \neq 0\}$ and $z_k + K = \{z \in A \times \widehat{A} : V_\phi \psi_k(z) \neq 0\}$, which contradicts (5.5). In summary, ρ is a pure state and the desired properties of $|\phi\rangle$ and ρ have been proved.

$(iii) \Longrightarrow (i)$ It follows from Theorem 2.13 (ii) that $|V_\phi \phi| = \chi_K$ for some maximal compact open isotropic subgroup $K \subset A \times \widehat{A}$, hence with $|K| = 1$ by Theorem 2.4. By the covariance property (2.6) we see that

$$u_{\phi,\rho} = |V_\phi \psi|^2 = \chi_{\mathcal{K}}$$

for some subset $\mathcal{K} \subset A \times \widehat{A}$ — a coset of K in $A \times \widehat{A}$ — of measure $|\mathcal{K}| = 1$. As a consequence

$$\mathcal{E}_G(\phi, \rho) = \int_{A\times\widehat{A}} G(|V_\phi \psi(x,\xi)|^2)\, dx\, d\xi = G(1)|\mathcal{K}| = G(1). \quad \square$$

6. Further results for finite-dimensional systems

In this section we suppose that A is a finite Abelian group, say of cardinality

$$\#A = N.$$

Hence $\mathcal{H} \simeq L^2(A)$ has dimension N. We prove some results for the Wehrl entropy in (1.15) that are, at least in part, specific for this finite-dimensional setting. Observe that now we have (cf. Remark 3.1 2))

$$\mathcal{E}_G(\phi, \rho) = \frac{1}{N} \sum_{z \in A\times\widehat{A}} G(u_{\phi,\rho}(z)). \tag{6.1}$$

6.1. Berezin-Lieb inequality

Let ρ be a density operator in $\mathcal{H}$. Let

$$\rho = \sum_{j=1}^N p_j |\psi_j\rangle\langle\psi_j| \tag{6.2}$$

be a spectral decomposition of ρ. Hence $p_j \geq 0$ for $j = 1, \ldots, N$, $\sum_{j=1}^N p_j = 1$ and the $|\psi_j\rangle$'s define now an *orthonormal basis* of $\mathcal{H}$. Given a function $G : [0, 1] \to \mathbb{R}$, we define the operator

$$G(\rho) := \sum_{j=1}^N G(p_j)|\psi_j\rangle\langle\psi_j|.$$

Hence

$$\mathrm{Tr}\, G(\rho) = \sum_{j=1}^N G(p_j).$$

If $G(\tau) = -\tau \log \tau$ for $0 < \tau \leq 1$ and $G(0) = 0$, $\mathrm{Tr}\, G(\rho)$ is the von Neumann entropy of ρ; see, e.g., [21, Section 5.2].

Theorem 6.1. *Let $G : [0, 1] \to \mathbb{R}$ be a concave function. For every $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$, and density operator ρ, we have*

$$\mathcal{E}_G(\phi, \rho) \geq \mathrm{Tr}\, G(\rho). \tag{6.3}$$

(i) *Equality occurs in* (6.3) *for every* $|\phi\rangle \in \mathcal{H}$, *with* $\||\phi\rangle\| = 1$ *if* $\rho = N^{-1}I$ *(the so-called chaotic state).*
(ii) *More generally, equality occurs in* (6.3) *if* $|\phi\rangle$ *is an* S*-state vector (Definition 1.2),* ρ *has the form in* (6.2) *with*

$$|\psi_j\rangle = e^{i\theta_j} W(z_j)|\phi\rangle$$

for some $\theta_j \in \mathbb{R}$, *and some* $z_j \in A \times \widehat{A}$, $j = 1, \dots, N$, *belonging each to a different coset in* $A \times \widehat{A}$ *of the maximal isotropic subgroup* $K := \{z \in A \times \widehat{A} : V_\phi \phi(z) \neq 0\}$ *(hence* $\#K = N$*).*
(iii) *If* G *is strictly concave, equality occurs in* (6.3) *for some pair* $|\phi\rangle, \rho$, *with* $\||\phi\rangle\| = 1$ *and* ρ *having distinct eigenvalues, only if* $|\phi\rangle$ *and* ρ *have the form in (ii).*

Proof. Form (6.2) we have (cf. (5.2))

$$u_{\phi,\rho}(z) = \sum_{j=1}^{N} p_j |V_\phi \psi_j(z)|^2 \qquad z \in A \times \widehat{A}. \tag{6.4}$$

Hence

$$\begin{aligned}\mathcal{E}_G(\phi, \rho) &= \frac{1}{N} \sum_{z \in A \times \widehat{A}} G\Big(\sum_{j=1}^{N} p_j |V_\phi \psi_j(z)|^2 \Big) \\ &\geq \frac{1}{N} \sum_{z \in A \times \widehat{A}} \sum_{j=1}^{N} G(p_j) |V_\phi \psi_j(z)|^2 \\ &= \sum_{j=1}^{N} G(p_j),\end{aligned}$$

where the inequality follows from the concavity of G and the fact that, for $z \in A \times \widehat{A}$,

$$\sum_{j=1}^{N} |V_\phi \psi_j(z)|^2 = \sum_{j=1}^{N} |\langle \pi(z)\phi | \psi_j \rangle_{L^2(A)}|^2 = \|\pi(z)\phi\|^2_{L^2(A)} = 1, \tag{6.5}$$

whereas the last equality is a consequence of (2.9), that here reads

$$\frac{1}{N} \sum_{z \in A \times \widehat{A}} |V_\phi \psi_j(z)|^2 = 1 \qquad j = 1, \dots, N. \tag{6.6}$$

By (6.5) it is clear that (i) holds.

From the above argument we see that, more generally, equality occurs in (6.3) if and only if, for every $z \in A \times \widehat{A}$,

$$G\Big(\sum_{j=1}^{N} p_j |V_\phi \psi_j(z)|^2 \Big) = \sum_{j=1}^{N} G(p_j) |V_\phi \psi_j(z)|^2. \tag{6.7}$$

If G is strictly concave and $p_j \neq p_k$ for $j, k \in \{1, \dots, N\}$, $j \neq k$, this happens if and only if, for every $z \in A \times \widehat{A}$ there exists (one and) only one $j \in \{1, \dots, N\}$ such that $V_\phi \psi_j(z) \neq 0$. That is, the functions $V_\phi \psi_j$, $j = 1, \dots, N$, have disjoint supports. By (2.8) and (6.6) such supports have cardinality at least N, and therefore they must have all cardinality N. By Theorem 2.13 (iii) this implies that $|\phi\rangle$ is an S-state vector (Definition 1.2) and there exist $\theta_j \in \mathbb{R}$, $z_j \in A \times \widehat{A}$, with $j = 1, \dots, N$ such that

$$|\psi_j\rangle = e^{i\theta_j} W(z_j)|\phi\rangle.$$

Moreover, it follows from Theorem 2.13 (ii), that the set K in the statement is a maximal isotropic subgroup of $A \times \widehat{A}$. Since the $|\psi_j\rangle$'s are pairwise orthogonal, arguing as in the proof of Theorem 1.5 we see that the z_j's belong to different cosets of K. This proves (iii).

Conversely, it is clear, essentially by reversing this latter reasoning, that for $|\phi\rangle$ and $|\psi_j\rangle$ as above, that is, as in (ii), the vectors $|\psi_j\rangle$ define an orthonormal basis of $\mathcal{H}$, and the functions $V_\phi\psi_j$ have disjoint supports. Hence (6.7) is satisfied for every $z \in A \times \widehat{A}$ regardless of whether G is strictly concave and whether the p_j's are all distinct (indeed, the summations in (6.7) contain only one summand, in this case). Hence, (ii) is proved. □

6.2. Maximizing the Wehrl entropy

We consider the problem of maximizing the Wehrl entropy in (1.15) when A is a finite Abelian group, say of cardinality N.

We recall from [21, Section 12.3] that the *characteristic function* of a density operator ρ is defined by

$$\tilde{\rho}(z) := \operatorname{Tr} \rho W(z) \qquad z \in A \times \widehat{A}. \tag{6.8}$$

When $\rho = |\psi\rangle\langle\psi|$ is a rank one projector, we simply write $\tilde{\psi}(z)$ for $\tilde{\rho}(z)$.

We also consider the Fourier transform on $A \times \widehat{A}$, defined by

$$\mathcal{F}f(\xi', x') = \frac{1}{N} \sum_{(x,\xi)\in A\times\widehat{A}} \overline{\xi'(x)\xi(x')} f(x, \xi) \qquad (\xi', x') \in \widehat{A} \times A.$$

We will need the following useful formula for the Fourier transform of the Husimi function. Related formulas have appeared in various contexts (such as mathematical signal processing [1]) but with different notation and terminology.

Proposition 6.2. *Let $|\phi\rangle \in \mathcal{H}$, with $\||\phi\rangle\| = 1$ and let ρ be a density operator in $\mathcal{H}$. We have*

$$\mathcal{F}u_{\phi,\rho}(\xi, -x) = \tilde{\phi}(x, \xi)\overline{\tilde{\rho}(x, \xi)} \qquad (x, \xi) \in A \times \widehat{A}.$$

Proof. It suffices to prove the desired result when ρ is a rank one projector, therefore $\rho = |\psi\rangle\langle\psi|$. In that case, we have (cf. (6.4))

$$u_{\phi,\rho}(z) = |V_\phi\psi(z)|^2.$$

By direct inspection one sees that

$$\tilde{\phi} = \overline{V_\phi\phi} \qquad \tilde{\rho} = \overline{V_\psi\psi}.$$

As a consequence, we are reduced to prove that

$$\mathcal{F}(|V_\phi\psi|^2)(\xi, -x) = \overline{V_\phi\phi(x, \xi)} V_\psi\psi(x, \xi) \qquad (x, \xi) \in A \times \widehat{A}.$$

This formula was proved in [1, Lemma 2.2] when $A = \mathbb{Z}_d$. The proof given there can be easily adapted to any finite Abelian group (in fact, to any LCA group). □

We also need the following elementary lemma.

Lemma 6.3. *Let* $G : [0,1] \to \mathbb{R}$ *be a concave function and* $N \geq 1$ *be an integer. Let*

$$\mathcal{P} := \Big\{ t = (t_1, \ldots, t_{N^2}) \in [0,1]^{N^2} : \sum_{j=1}^{N^2} t_j = N \Big\},$$

and

$$f(t) := \frac{1}{N} \sum_{j=1}^{N^2} G(t_j) \qquad t \in \mathcal{P}.$$

Then f *attains its maximum value on* $\mathcal{P}$ *at* $\bar{t} := (N^{-1}, \ldots, N^{-1})$.

If G *is strictly concave, then* $\bar{t}$ *is the unique maximum point.*

Proof. For every $t = (t_1, \ldots, t_{N^2}) \in \mathcal{P}$ and every permutation

$$\sigma : \{1, \ldots, N^2\} \to \{1, \ldots, N^2\},$$

consider the point $t_\sigma := (t_{\sigma(1)}, \ldots, t_{\sigma(N^2)}) \in \mathcal{P}$. Using the fact that $\sum_{j=1}^{N^2} t_j = N$ it is easy to check that

$$\bar{t} = \frac{1}{(N^2)!} \sum_\sigma t_\sigma.$$

Indeed, each component of the vector $\sum_\sigma t_\sigma$ is equal to

$$(N^2 - 1)!t_1 + (N^2 - 1)!t_2 + \ldots + (N^2 - 1)!t_{N^2} = (N^2 - 1)!N.$$

Since G is concave, f is concave as well and therefore

$$f(\bar{t}) \geq \frac{1}{(N^2)!} \sum_\sigma f(t_\sigma) = \frac{1}{(N^2)!} \sum_\sigma f(t) = f(t).$$

The last part of the statement is clear, because f is strictly concave whenever G is strictly concave. □

We can now state the desired characterization of the maximizers of the Wehrl entropy. For a function $f : A \times \widehat{A} \to \mathbb{C}$ we set $\operatorname{supp} f = \{z \in A \times \widehat{A} : \ f(z) \neq 0\}$.

Theorem 6.4. *Let* $G : [0,1] \to \mathbb{R}$ *be a concave function. Let* $|\phi\rangle \in \mathcal{H}$, *with* $\||\phi\rangle\| = 1$ *and let* ρ *be a density operator in* $\mathcal{H}$. *Then*

$$\mathcal{E}_G(\phi, \rho) \leq N\, G(1/N). \tag{6.9}$$

Moreover, if G *is strictly concave, equality occurs in* (6.9) *if and only if*

$$\operatorname{supp} \tilde{\phi} \cap \operatorname{supp} \tilde{\psi} = \{0\}. \tag{6.10}$$

Proof. The upper bound (6.9) follows from (6.1) and Lemma 6.3.

Suppose that G is strictly concave. Then it follows from Lemma 6.3 that equality occurs in (6.9) if and only if

$$u_{\phi,\rho}(z) = \frac{1}{N} \qquad z \in A \times \widehat{A}.$$

Taking the Fourier transform of both sides, we see that this is equivalent to

$$\mathcal{F}u_{\phi,\rho}(\xi, x) = \begin{cases} 1 & (\xi, x) = 0 \\ 0 & (\xi, x) \neq 0. \end{cases}$$

Using Proposition 6.2, this amounts to (6.10), because $\tilde{\rho}(0) = \operatorname{Tr}\rho = 1$ and, similarly, $\tilde{\phi}(0) = 1$. □

Remark 6.5. If $G(\tau) = -\tau \log \tau$ for $0 < \tau \leq 1$ and $G(0) = 0$, as already observed, $\operatorname{Tr} G(\rho)$ is the von Neumann entropy of ρ. In this case, by combining Theorems 6.1 and 6.4 we obtain that

$$\operatorname{Tr} G(\rho) \leq \mathcal{E}_G(\phi, \rho) \leq \log N.$$

It is well known and easy to check that $\operatorname{Tr} G(\rho) = \log N$ if and only if $\rho = N^{-1}I$. Theorems 6.1 and 6.4 provide information on the cases where the two above inequalities hold separately as an equality.

Declaration of competing interest

We declare no conflicts of interest.

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Data availability

The present paper has no associated data.

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