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***Abstract*—We give a description in terms of phase and amplitude deviation for networks of nonlinear oscillators with noise. The case of white Gaussian noise is considered. The equations for the amplitude and the phase are rigorous, and their validity is not limited to the weak noise limit. We show that using Floquet theory, a partial decoupling between the amplitude and the phase is obtained. The decoupling can be exploited to describe the oscillator's dynamics solely by the phase variable. We discuss to what extent the reduced model is appropriate and some implications on the role of noise on the frequency and the synchronization of the oscillators.**

Synchronization of coupled oscillators is a paradigm for complexity in many areas of science and engineering [1]–[3]. Any realistic network model should include noise effects [4], [5].

A network composed of N weakly coupled nonlinear oscillators with noise can be described by the set of differential equations

$$\dot{\boldsymbol{x}}_i = \boldsymbol{a}_i(\boldsymbol{x}_i) + \varepsilon \boldsymbol{c}_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{\zeta}(t)) \qquad i = 1, \ldots, N \quad (1)$$

where $\boldsymbol{x}_i$ is the state of the ith oscillator, ε is the coupling intensity and $\boldsymbol{\zeta}(t)$ is the vector of the noise sources. Linearizing around the noiseless state yields the description of the network in terms of stochastic differential equations

$$\begin{aligned} d\boldsymbol{X}_i = &\left[\boldsymbol{a}_i(\boldsymbol{X}_i) + \varepsilon \boldsymbol{c}_i(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N)\right] dt \\ &+ \varepsilon \boldsymbol{B}_i(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N)\, d\boldsymbol{W}_i \quad i = 1, \ldots, N \end{aligned} \quad (2)$$

where $\boldsymbol{B}_i : \mathbb{R}^{n \cdot N} \mapsto \mathbb{R}^{n,m}$ is a $n \times m$ diffusion matrix, and $\boldsymbol{W}_i : \mathbb{R} \mapsto \mathbb{R}^m$ is a vector of Wiener processes (the integral of a white noise). For the sake of simplicity, in equation (2) we assume that all oscillators are of the same order ($\boldsymbol{X}_i \in \mathbb{R}^n$, for all i), but we allow the interaction to vary for each oscillator both in the modulating matrix $\boldsymbol{B}_i$ and in the random fluctuation $\boldsymbol{W}_i$.

For $\varepsilon = 0$, the SDE (2) reduce to an ordinary differential equation (ODE) describing N independent, noiseless oscillators. The i–th oscillator is described by the ODE

$$\frac{d\boldsymbol{x}_i(t)}{dt} = \boldsymbol{a}_i(\boldsymbol{x}_i(t)) \quad (3)$$

We assume that the ODE (3) admits an asymptotically stable T_i–periodic solution, represented by a limit cycle $\boldsymbol{x}_{S_i}(t)$ in its state space. For each oscillator we define the vector

$$\boldsymbol{u}_{1_i}(t) = \frac{\boldsymbol{a}_i(\boldsymbol{x}_{S_i}(t))}{|\boldsymbol{a}_i(\boldsymbol{x}_{S_i}(t))|} \quad (4)$$

$\boldsymbol{u}_{1_i}(t)$ is the unit vector that at each time instant is tangent to the limit cycle $\boldsymbol{x}_{S_i}(t)$. Together with $\boldsymbol{u}_{1_i}(t)$ we consider other $n-1$ vectors $\boldsymbol{u}_{2_i}(t), \ldots, \boldsymbol{u}_{n_i}(t)$, such that the set $\{\boldsymbol{u}_{1_i}(t), \ldots, \boldsymbol{u}_{n_i}(t)\}$ is a basis for $\mathbb{R}^n$ for all t.

A crucial concept to be defined in the analysis of synchronization of oscillators is the phase concept. A phase function is intended to represent the projection of the oscillator's state onto a reference trajectory, normally the unperturbed limit cycle. For each oscillator we introduce a phase function $\theta_i : \mathbb{R}^n \mapsto [0, T_i)$, interpreted as an elapsed time from an initial reference point. Together with the phase function we shall consider an amplitude deviation function $\boldsymbol{R}_i : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$, with $\theta_i, \boldsymbol{R}_i \in \mathcal{C}^m(R^n)$, $m \geq 2$.

The following theorem establishes the amplitude and phase equation for the network

Theorem 1: Consider the Itô diffusion (2), and consider the coordinate transformation

$$\boldsymbol{x}_i = \boldsymbol{h}_i(\theta_i, \boldsymbol{R}_i) = \boldsymbol{x}_{S_i}(\theta_i(t)) + \boldsymbol{Y}_i(\theta_i(t))\, \boldsymbol{R}_i(t) \quad (5)$$

Then in a neighborhood of the limit cycle $\boldsymbol{x}_{S_i}$ the phase $\theta_i(t)$ and the amplitude $\boldsymbol{R}_i(t)$ are Itô processes and satisfy

$$\begin{aligned} d\theta_i = &\big[1 + a_{\theta_i}(\theta_i, \boldsymbol{R}_i) + \varepsilon^2\, \hat{a}_{\theta_i}(\theta_1 \ldots \boldsymbol{R}_N) \\ &+ \varepsilon c_{\theta_i}(\theta_1 \ldots \boldsymbol{R}_N)\big] dt + \varepsilon \boldsymbol{B}_{\theta_i}(\theta_1 \ldots \boldsymbol{R}_N)\, d\boldsymbol{W}_i \end{aligned} \quad (6a)$$

$$\begin{aligned} d\boldsymbol{R}_i = &\big[\boldsymbol{L}_i(\theta_i)\boldsymbol{R}_i + \boldsymbol{a}_{\boldsymbol{R}_i}(\theta_i, \boldsymbol{R}_i) + \varepsilon^2 \hat{\boldsymbol{a}}_{\boldsymbol{R}_i}(\theta_1 \ldots \boldsymbol{R}_N) \\ &+ \varepsilon \boldsymbol{c}_{\boldsymbol{R}_i}(\theta_1 \ldots \boldsymbol{R}_N)\big] dt + \varepsilon \boldsymbol{B}_{\boldsymbol{R}_i}(\theta_1 \ldots \boldsymbol{R}_N)\, d\boldsymbol{W}_i \end{aligned} \quad (6b)$$

where $(\theta_1 \ldots \boldsymbol{R}_N)$ is a shorthanded notation for $(\theta_1, \boldsymbol{R}_1, \ldots, \theta_N, \boldsymbol{R}_N)$. The explicit expression of all the terms in equations (6a)-(6b) is here omitted, but it can be found in [8]. The important point is that they admit analytical expressions in terms of the unperturbed limit cycles $\boldsymbol{x}_{S_i}$ and the basis vectors $\boldsymbol{u}_{2_i}, \ldots, \boldsymbol{u}_{n_i}$.

The amplitude and phase equations (6a) and (6b) are exact, since no approximation is involved in their derivation, and they are valid for any value of the noise intensity ε as long as the Jacobian matrices $D\boldsymbol{h}_i$ are regular. The amplitude and phase equations obtained crucially depends on the choice of the basis vectors $\boldsymbol{u}_{2_i}, \ldots, \boldsymbol{u}_{n_i}$.

In general, the equations for the two Itô processes for the phase and for the amplitude are coupled together, but it is possible to show that making use of Floquet theory, a partial decoupling between the phase and the amplitude dynamics is obtained. Before introducing the theorem we recall the main results of the Floquet theory [6], [7]. Let $\boldsymbol{A}_i(t) = \frac{\partial \boldsymbol{a}_i(\boldsymbol{x}_{S_i})}{\partial \boldsymbol{x}_i}$ be the Jacobian matrix of the i–th oscillator evaluated on the limit cycle $\boldsymbol{x}_{S_i}(t)$, and let $\Phi_i(t)$ be the fundamental matrix of the variational equation

$$\frac{d\boldsymbol{y}_i(t)}{dt} = \boldsymbol{A}_i \boldsymbol{y}_i(t).$$

Thus, from Floquet theory we get:

$$\Phi_i(t) = \boldsymbol{P}_i(t) e^{\boldsymbol{D}t} \boldsymbol{P}_i^{-1}(0), \tag{7}$$

where $\boldsymbol{P}_i(t)$ is a T_i–periodic matrix, and $\boldsymbol{D}_i = \mathrm{diag}[\nu_{1_i}, \dots, \nu_{n_i}]$ is a diagonal matrix whose diagonal entries are the Floquet characteristic exponents [6], [7].

Theorem 2: If the vectors $\boldsymbol{u}_{2_i}(t), \dots, \boldsymbol{u}_{n_i}(t)$ are chosen such that

$$[r_i \boldsymbol{u}_{1_i}(t), \boldsymbol{u}_{2_i}(t), \dots, \boldsymbol{u}_{n_i}(t)] = \boldsymbol{P}_i(t),$$

then the Itô processes (6a) and (6b) become

$$\begin{aligned} d\theta_i =& \big[1 + \tilde{a}_{\theta_i}(\theta_i, \boldsymbol{R}_i) + \varepsilon^2 \hat{a}_{\theta_i}(\theta_1 \dots \boldsymbol{R}_N) \\ &+ \varepsilon\, c_{\theta_i}(\theta_1 \dots \boldsymbol{R}_N)\big] dt + \varepsilon \boldsymbol{B}_{\theta_i}(\theta_1 \dots \boldsymbol{R}_N)\, d\boldsymbol{W}_i \end{aligned} \tag{8a}$$

$$\begin{aligned} d\boldsymbol{R}_i =& \big[\widetilde{\boldsymbol{D}}_i \boldsymbol{R}_i + \tilde{\boldsymbol{a}}_{\boldsymbol{R}_i}(\theta_i, \boldsymbol{R}_i) + \varepsilon^2 \hat{\boldsymbol{a}}_{\boldsymbol{R}_i}(\theta_1 \dots \boldsymbol{R}_N) \\ &+ \varepsilon\, \boldsymbol{c}_{\boldsymbol{R}_i}(\theta_1 \dots \boldsymbol{R}_N)\big] dt + \varepsilon \boldsymbol{B}_{\boldsymbol{R}_i}(\theta_1 \dots \boldsymbol{R}_N)\, d\boldsymbol{W}_i, \end{aligned} \tag{8b}$$

where $\widetilde{\boldsymbol{D}}_i = \mathrm{diag}[\nu_{2_i}, \dots, \nu_{n_i}]$ and the Taylor series of $\tilde{a}_{\theta_i}(\theta_i, \boldsymbol{R}_i)$ and $\tilde{\boldsymbol{a}}_{\boldsymbol{R}_i}(\theta_i, \boldsymbol{R}_i)$ do not contain linear terms in $\boldsymbol{R}_i$.

Another major advantage of using Floquet basis is that the resulting phase functions are locally coincident with asymptotic phase introduced in [1], [2].

As an example we consider the following system composed by two second order ($N = 2$, $n = 2$, and $m = 2$) oscillators written in polar coordinates

$$d\rho_i = \rho_i\,(1 - \rho_i)\, dt + \varepsilon\, \rho_i dW_{\rho_i} \tag{9a}$$

$$d\phi_i = [\nu_i \rho_i + \varepsilon(\phi_j - \phi_i)]\, dt + \varepsilon \rho_j\, dW_{\phi_i} \tag{9b}$$

for $i, j = 1, 2$, and $j \neq i$. The real parameters ν_i define the free running frequency of the oscillators in absence of noise. By using a Floquet basis, the related amplitude and phase equations can be derived:

$$\begin{aligned} d\theta_i =& \left\{1 - R_i^2 + \varepsilon \left[\frac{\nu_j}{\nu_i}(\theta_j - R_j) - (\theta_i - R_i)\right]\right\} dt \\ &+ \varepsilon \left[\mu_i (1 + R_i) dW_{\rho_i} + \frac{1 + R_j}{\nu_i} dW_{\phi_i}\right] \end{aligned} \tag{10a}$$

$$dR_i = -\,[R_i\,(1 + R_i)]\, dt + \varepsilon \mu_i (1 + R_i) dW_{\rho_i} \tag{10b}$$

We remark that, according to Theorem 2, eq. (10a) has a drift coefficient that starts with a quadratic term in R_i. Moreover, it is possible to show that asymptotically the two oscillators synchronize with a phase difference

$$\psi = \frac{\nu_i - \nu_j}{2\varepsilon} \tag{11}$$

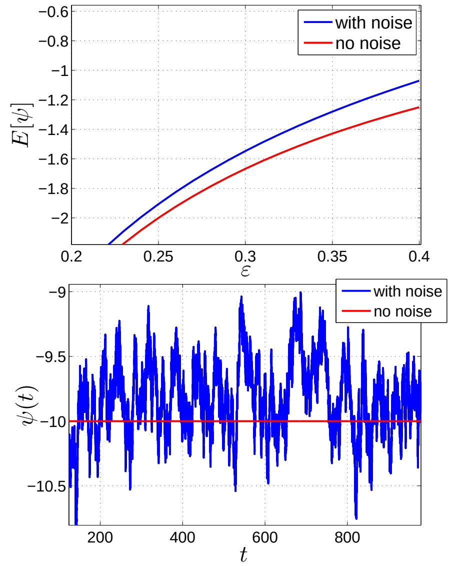

Fig. 1. Top: Phase difference given by (11) versus the noise intensity ε for two oscillators with different free running angular frequencies, $\nu_1 = 1$ and $\nu_2 = 2$. Bottom: Phase difference for two oscillators (free running angular frequencies are $\nu_1 = 1$ and $\nu_2 = 2$ respectively), as a function of time for a specific realization of the noise. The noise intensity is set to $\varepsilon = 0.05$. The phase difference in absence of noise is shown for reference.

The phase difference in presence of noise is compared with that obtained without noise in figure 1. On the top we can see the asymptotic expected phase difference versus the noise intensity, while on the bottom it is shown the phase difference versus time for a specific realization of the noise process. It can be seen how noise operates to actively reduce the phase difference between the oscillators.

It is worth noting that the amplitude and phase description highlights the influence of noise on the phases of the oscillators. Therefore, it represents a good starting point for the analysis of the role of noise on synchronization.

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