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Preservation of ILR and IFR aging classes in sums of dependent random variables

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Abstract
Closure of aging classes with respect to sums is a relevant topic in different areas of applied probability. In the case of independent random variables (i.e., for convolutions) this preservation property has been proved in the literature for a number of classes such as the increasing in likelihood ratio (ILR) and the increasing failure rate (IFR) classes. These results were applied, for example, to sums of life lengths in reliability theory and to sums of incomes/returns in economic/risk studies. However, in many practical situations the independence assumption is unrealistic. In the present paper, we provide conditions such that these closure properties are satisfied by the ILR and IFR classes when the assumption of independence is dropped. The classical copula approach is used to model the dependence structure, but other dependence models, such as relevation transforms and load sharing models, are considered as well. Several illustrative examples and counterexamples show how to use the presented theoretical results and which preservation results do not hold.

KEYWORDS
C-convolutions, copulas, load sharing models, relevation transforms

1 | INTRODUCTION

In different applied probability fields, like reliability theory or actuarial sciences, various concepts of aging have been proposed and studied to characterize and classify (in terms of "aging classes") the distributions of random lifetimes of components, devices or individuals. Apart for the usefulness in the understanding of the intrinsic meaning of aging and its different effects on lifetimes, these classes serve for a number of purposes. They can be used, for example, to provide bounds for survival probabilities or expected residual lifetimes, or to define optimal replacement policies under different constrains. They can also serve to evaluate risks in insurance contracts. We refer the reader to the monograph Barlow and Proschan[1] for a detailed description of the main aging classes and their applications in reliability modeling. Some among the most well-known of these classes are recalled in the next section.

Because of their usefulness in the applicative fields mentioned above, one can find in the literature a list of papers which study these classes and their properties. Several of them deal with their preservations under different stochastic operations, such as the closures with respect to construction of coherent systems, mixtures or sums. Concerning sums of random lifetimes, it is well known that the increasing failure rate (IFR) aging class is preserved under convolutions, that is, if X and Y are two independent random variables with distributions in the IFR class, then also $X+Y$ has an

IFR distribution (see, e.g., Barlow and Proschan[1(p. 100)]). On the contrary, the decreasing failure rate (DFR) class is not preserved under convolutions. These properties can be different for other transformations. For example, the DFR class is preserved under the formation of mixtures but the IFR class is not, and both of them are not preserved under the formation of coherent systems with i.i.d. components. However, the New Better than Used (NBU) class is preserved for these systems (see Barlow and Proschan[1(p. 187)]), but it is not preserved when the independence assumption is not satisfied (see Navarro et al.[2]). Analogously, the increasing likelihood ratio (ILR) class (equivalent to logconcavity of the densities) is preserved (with some conditions) by the formation of parallel and series systems (see Navarro and Shaked[3]), while the preservation with respect to convolutions was proved by Karlin and Proschan[4] (see also Belzunce et al.[5(p. 7)]). Conditions for such preservations in distorted distributions and systems were recently obtained in Navarro et al.,[2] and additional results for convolutions can be found in Alimohammadi et al.[6] Properties of conditional distributions of sums are given in Pellerey and Navarro.[7]

It must be pointed out that for almost all the results available in the literature dealing with preservation of aging classes under sums, the independence among summands is required. However, since in general the sums reduce the uncertainty, it is expectable to observe sums that are "more" IFR (or more ILR) than their summands, also when the assumption of independence is unsatisfied (in particular, this intuition seems especially true for negatively dependent random variables).

The purpose of the present paper is to investigate this insight in more detail, in order to provide new conditions for preservation of some of the most well known aging classes under sums of dependent random variables. The new statements can be added to similar ones recently obtained in Navarro and Sarabia[8] for the IFR/DFR classes. The results described here are mainly based on preservation properties for distorted distributions and C-convolutions, which are copula representations for the distributions of sums of dependent random variables (see Section 2). In particular, such representations, as well as other tools, are used here to provide new conditions for the closure under sums of IFR and ILR aging classes. Some properties for the limiting behavior of the failure rate of the sum are provided as well, together with new conditions for the closure under maximum of dependent IFR and ILR lifetimes.

The rest of the paper is structured as follows. Basic definitions and some preliminary results are introduced in Section 2. The main results for ILR and IFR classes are placed in Sections 3 and 4, respectively. Illustrative examples and counterexamples dealing with a specific kind of sums modeled with relevation transforms are given in Section 5. Similar results, but dealing with preservation of ILR and IFR aging classes with respect to maximum of non-independent variables, are presented in Section 6.

Throughout the paper the notions increasing and decreasing are used in a wide sense, that is, they mean non-decreasing and non-increasing, respectively.

2 | PRELIMINARIES

2.1 | Aging classes

First we recall the following definition.

Definition 1. We say that $f : \mathbb{R}^n \to \mathbb{R}^+$ is logconcave (logconvex) if

$$\log f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq (\leq) \lambda \log f(\mathbf{x}) + (1-\lambda)\log f(\mathbf{y})$$

for all $\lambda \in (0,1)$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Given a nonnegative random variable (lifetime) X with distribution function $F(t) = \Pr(X \leq t)$ and survival (reliability) function $\overline{F}(t) = 1 - F(t) = \Pr(X > t)$, let us denote with $f(t) = F'(t)$ for $t \geq 0$ (zero elsewhere) its probability density function (pdf), and with $r(t) = f(t)/\overline{F}(t)$, for $t \geq 0$, its failure (or hazard) rate function, assuming they exist (from now on, whenever we use f or r, we are tacitly assuming that X has an absolutely continuous distribution). Then we say that:

- X (or F) is ILR (DLR) if $f(t)$ is logconcave (logconvex) for $t \geq 0$;
- X (or F) is IFR (DFR) if $\overline{F}(t)$ is logconcave (logconvex) for $t \geq 0$;
- X (or F) is NBU (NWU) if $\overline{F}(t)\overline{F}(s) \geq \overline{F}(t+s)$ for all $s, t \geq 0$.

The IFR (DFR) property is equivalent to have an increasing (decreasing) failure rate function. It is also equivalent to have $X_s \geq_{ST} X_t$ ($\leq_{ST}$) for all $t \geq s \geq 0$, where $\geq_{ST}$ represents the usual stochastic order (see Shaked and Shanthikumar[9])

TABLE 1 Relationships among positive aging classes

LFR	$\Rightarrow$	*ILR*	$\Rightarrow$	*IFRA*
$\Downarrow$		$\Downarrow$		$\Downarrow$
LCFR	$\Rightarrow$	*IFR*	$\Rightarrow$	*NBU*

and $X_t = (X - t|X > t)$ represents the residual lifetime of X at time t. Analogously, the NBU (NWU) property is equivalent to have $X \geq_{ST} X_t$ $(\leq_{ST})$ for all $t \geq 0$. The ILR (DLR) class is equivalent to have $X_s \geq_{LR} X_t$ $(\leq_{LR})$ for all $t \geq s \geq 0$, where $\geq_{LR}$ represents the likelihood ratio order (see Shaked and Shanthikumar[9]). The ILR and DLR classes are also equivalent to the respective monotonicity properties of the function $\eta(t) = -f'(t)/f(t)$ for $t \geq 0$. Clearly, since the likelihood ratio order implies the usual stochastic order, $ILR \Rightarrow IFR \Rightarrow NBU$ and, when the support of X is (l, ∞), $DLR \Rightarrow DFR \Rightarrow NWU$.

The IFR, NBU, and ILR classes represent the usual (positive or natural) aging properties where the older units have worse performance than the younger ones. The reverse scenario holds for the dual classes DFR, NWU, and DLR.

It must be pointed out that, even if some notions like IFR and NBU usually refer to non-negative variables, on the contrary the ILR notion also applies to variables that are not necessarily positive, and some of the statements described below hold true even in this case.

Some additional (related) properties, defined here, will be considered through the following sections. Given a non-negative random variable X, and denoted with $R(t) = -\log \overline{F}(t) = \int_0^t r(s)ds$ its cumulative failure rate (or hazard rate) function, then we say that:

- X (or F) is IFRA (increasing in failure rate average) if $R(t)/t$ is increasing for $t \geq 0$;
- X (or F) is LFR (logconcave in the failure rate) if $r(t)$ is logconcave for $t \geq 0$;
- X (or F) is LCFR (logconcave in the cumulative failure rate) if $R(t)$ is logconcave for $t \geq 0$.

While the IFRA class is well-known, being the largest class closed with respect to construction of coherent systems (see Barlow and Proschan[1]), the LCFR and LFR classes have been considered few times in the literature. Their usefulness in describing aging properties has been pointed out in Pellerey et al.,[10] where they have been used to describe sufficient conditions for a nonhomogeneous Poisson processes to satisfy specific monotonicity properties. Note that, in Pellerey et al.,[10] the LFR and LCFR classes were denoted as (A.3) and (A.2), respectively, and their relationships with the ILR and IFR classes have been fully described. These relationships are summarized in Table 1. The dual classes (describing negative aging rather positive aging) can be considered as well.

Apart for applications in reliability theory, aging classes can be of interest also in other contexts. The following simple result shows the relevance of the ILR property for the estimation of the parameter in a scale family $\mathcal{F}_G$ of a nonnegative random variable. Recall that, given a distribution function G, we say that F belongs to the *scale family of distributions* $\mathcal{F}_G$ determined by G if $F(t) = G(\theta t)$ for all t, where $\theta > 0$ is the parameter of the model. Some examples are the exponential, the Weibull (with a fixed shape parameter) and the Pareto (with a fixed shape parameter) distributions.

Proposition 1. *If X is a nonnegative random variable with a distribution function $F \in \mathcal{F}_G$ and G is absolutely continuous and ILR, then the maximum likelihood estimator (MLE) for θ is unique.*

Proof. If $X_1, \dots, X_n$ are i.i.d. with distribution function F, then the likelihood function for θ is

$$\ell(\theta) = f(x_1)\dots f(x_n) = \theta^n g(\theta x_1)\dots g(\theta x_n),$$

where g is the pdf of G. To maximize ℓ in $(0, \infty)$ is equivalent to maximize

$$\tilde{\ell}(\theta) = \log(\ell(\theta)) = n\log(\theta) + \sum_{i=1}^{n} \log(g(\theta x_i))$$

for $\theta > 0$. Then

$$\tilde{\ell}'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} x_i \frac{g'(\theta x_i)}{g(\theta x_i)}.$$

As G is ILR, then g'/g is decreasing and so $\ell'(\theta)$ is strictly decreasing in $(0, \infty)$ for all $x_1, \dots, x_n > 0$. Therefore, $\ell(\theta)$ is strictly concave. Thus it has a unique maximum value in $(0, \infty)$. ∎

The class of ILR distributions contains many of the most commonly used parametric distributions, thus it is a rich and flexible nonparametric class of distributions. Further, the MLE for the density exists and can be computed with readily available algorithms (see, e.g., the review in Walther[11]).

The following statement, which is theorem 1 in Block et al.,[12] will be also used in the following sections. It essentially affirms that, in the limit, the hazard rate of a convolution converges to the smallest of the hazards of the summands.

Proposition 2 (Block et al.[12]). *Let X and Y be two independent random variables with hazard rates r_X and r_Y, respectively. Assume that* $\lim_{t\to\infty} r_X(t) = a$, $\lim_{t\to\infty} r_Y(t) = b$ *with* $0 < a < b < \infty$ *and that r_X is bounded. Then the failure rate r_S of the convolution $S = X + Y$ satisfies* $\lim_{t\to\infty} r_S(t) = a$.

2.2 | C-convolutions

First, we recall here the notion of C-convolutions, introduced in Cherubini et al.,[13] that will be used in the following sections to prove some of the main results. Let (X, Y) be two possibly dependent random variables with joint distribution

$$\mathbf{F}(x, y) = \Pr(X \le x, Y \le y).$$

From copula theory (see, e.g., Durante and Sempi[14] or Nelsen[15]), it is well known that this joint distribution can be written as

$$\mathbf{F}(x, y) = C(F(x), G(y)),$$

where $F(x) = \Pr(X \le x)$ and $G(y) = \Pr(Y \le y)$ are the marginal distribution functions of X and Y, respectively, and where C is a copula function. This representation (by means of a copula C) is unique when F and G are continuous (by the Sklar's theorem, see e.g., Durante and Sempi[14]). Also note that given a copula C, the right-hand side of this expression provides a proper bivariate distribution function for any marginal distribution functions F and G.

Then, from Cherubini et al.[13] (see also Navarro and Sarabia[8]), the distribution function H of the sum $S = X + Y$ can be obtained as

$$H(t) = \int_{-\infty}^{\infty} f(x)\ \Pr(Y \le t - x | X = x) dx = \int_{-\infty}^{\infty} f(x)\ \partial_1 C(F(x), G(t - x)) dx,$$

where $\partial_1 C$ represents the partial derivative of C with respect to its first argument. If X and Y are nonnegative, then the limits in the integral reduces to 0 and t, respectively. If F^{-1} is the inverse function of F, then this expression can be rewritten as

$$H(t) = \int_0^1 \partial_1 C(u, G(t - F^{-1}(u))) du.$$

This distribution is called the *C-convolution* and it is represented as $H = F \overset{C}{*} G$ in Cherubini et al.[13] If C is the product copula, we get the usual expression for the convolution (i.e., the distribution of the sum of two independent random variables).

In some models it is better to use survival functions instead of distribution functions. For them, we have the expression

$$\overline{H}(t) = \Pr(S > t) = \int_{-\infty}^{\infty} f(x)\ \partial_1 \widehat{C}(\overline{F}(x), \overline{G}(t - x)) dx, \tag{1}$$

where $\overline{H} = 1 - H$, $\overline{F} = 1 - F$, and $\overline{G} = 1 - G$ are the survival functions of S, X, and Y, respectively, and where $\widehat{C}$ is a copula, called *survival copula*, that allows to represent the joint survival function of (X, Y) as

$$\overline{\mathbf{F}}(x, y) = \Pr(X > x, Y > y) = \widehat{C}(\overline{F}(x), \overline{G}(y)).$$

In particular, for nonnegative random variables Equation (1) reduces to

$$\overline{H}(t) = \overline{F}(t) + \int_0^t f(x)\, \partial_1 \widehat{C}(\overline{F}(x), \overline{G}(t-x))dx. \tag{2}$$

The exponential distribution plays a central role in reliability theory and survival analysis, representing units without aging. If both marginal distributions coincide and are exponentials, that is, if $\overline{F}(t) = \overline{G}(t) = e^{-\lambda t}$ for $t \geq 0$ and $\lambda > 0$, then Equation (2) reduces to

$$\overline{H}(t) = e^{-\lambda t} + \int_{e^{-\lambda t}}^1 \partial_1 \widehat{C}(v, e^{-\lambda t}/v)dv = \overline{q}(e^{-\lambda t}) \tag{3}$$

for $t \geq 0$, where

$$\overline{q}(u) = u + \int_u^1 \partial_1 \widehat{C}\left(v, \frac{u}{v}\right) dv$$

for $u \in [0, 1]$. This kind of representations are called *distorted distributions*, and they are studied in the following subsection. A similar representation can be obtained when just G is an exponential distribution (see theorem 2.2 in Navarro and Sarabia[8]). In the bivariate case such a representation can be stated as follows.

Proposition 3. *If (X, Y) has survival copula $\widehat{C}$, $\overline{F}$ is strictly decreasing in its support $(0, \infty)$ and $\overline{G}(t) = \exp(-\lambda t) = \psi(\overline{F}(t))$ for $t \geq 0$ and $\lambda > 0$, where $\psi(u) = \exp(-\lambda \overline{F}^{-1}(u))$ for $0 < u < 1$, then the survival function of $S = X + Y$ can be written as $\overline{H}(t) = \overline{q}(\overline{F}(t))$ with*

$$\overline{q}(u) = u + \int_u^1 \partial_1 \widehat{C}\left(v, \frac{\psi(u)}{\psi(v)}\right) dv \tag{4}$$

for $u \in [0, 1]$.

Proof. First we note that $\overline{F}$ can be written as $\overline{F}(t) = \psi^{-1}(\exp(-\lambda t))$ for all $t \geq 0$, where $\psi^{-1}(u) = \overline{F}(-\log(u)/\lambda)$ is strictly increasing for $u \in (0, 1)$. Hence, $\exp(-\lambda t) = \psi(\overline{F}(t))$ for $t \geq 0$, where ψ is a strictly increasing distortion function. Then, from (2), we get

$$\begin{aligned}
\overline{H}(t) &= \overline{F}(t) + \int_0^t f(x)\, \partial_1 \widehat{C}(\overline{F}(x), \exp(\lambda x - \lambda t))dx \\
&= \overline{F}(t) + \int_0^t f(x)\, \partial_1 \widehat{C}\left(\overline{F}(x), \frac{\psi(\overline{F}(t))}{\psi(\overline{F}(x))}\right) dx
\end{aligned}$$

for all $t \geq 0$. Finally, by doing the change $v = \overline{F}(x)$, we conclude the proof. ■

Remark 1. The representation based on function ψ considered above is important in order to get formula (4) for the distortion function $\overline{q}$ (see next subsection). For specific copulas and functions ψ, we can obtain explicit expressions for $\overline{q}$ useful to study the preservation of IFR and ILR aging classes (see Example 4), while numerical approximations of the solutions of (4) can be used in other cases. Note that the distortion representation obtained in the preceding proposition can also be expressed in terms of $\overline{G}$ (exponential model) if we are able to get an explicit expression of ψ^{-1} (the inverse function of ψ). Thus, $\overline{F}(t) = \psi^{-1}(\exp(-\lambda t))$ and

$$\overline{H}(t) = \overline{q}(\overline{F}(t)) = \overline{q}(\psi^{-1}(\exp(-\lambda t))) = \overline{q}_2(\overline{G}(t))$$

for $t \geq 0$, where $\overline{q}_2(u) = \overline{q}(\psi^{-1}(u))$ for $u \in [0, 1]$. This approach is used in Example 4.

2.3 | Distorted distributions

The distorted distributions were introduced in the theory of choice under risk (see Yaari[16]). They will be used in the next sections to prove the main results on closure with respect to sums of dependent random lifetimes. This is their formal definition.

Definition 2. We say that F_q is a distorted distribution function from a distribution function F if $F_q(t) = q(F(t))$ for all t, where $q : [0,1] \to [0,1]$ is a distortion function, that is, q is continuous, increasing, and satisfies $q(0) = 0$ and $q(1) = 1$.

We will use the notation $F_q = q \circ F$ for the composition. Note that the respective survival functions satisfy a similar relationship, that is, one can write $\overline{F}_q = \overline{q} \circ \overline{F}$, where $\overline{q}(u) = 1 - q(1-u)$ for $u \in [0,1]$ is another distortion function called *dual distortion function*.

Clearly, the representations in (3) and in Proposition 3 are distortion representations for $\overline{H}$. There are others well known models that can be represented as distorted distribution, like the proportional hazard rate (PHR) Cox model with $\overline{q}(u) = u^\theta$ for $\theta > 0$, or the proportional reversed hazard rate (PRHR) model with $q(u) = u^\theta$ for $\theta > 0$. It can also be proved that the distributions of coherent systems with identically distributed components, are also distortions of the common components' distribution function (see, e.g., Navarro et al.[2]).

We say that an aging class C is preserved by a distortion function q if $F_q = q \circ F$ belongs to C for all F in C. For example, it is easy to see that the PHR model preserves the IFR and DFR classes. The preservation of aging classes under distortions, such as for the ILR/DLR, IFR/DFR, NBU/NWU, and IFRA/DFRA classes, were studied in Navarro et al.[2] In particular, in that paper (see remark 2.3) it is proved that if the IFR class is preserved, then the NBU and IFRA classes are preserved as well. An extension of the result obtained there for the IFR/DFR classes is stated in the following proposition.

Proposition 4. *If $F_q = q \circ F$ for a differentiable distortion function q and an absolutely continuous distribution function F, and $\alpha(u) = u\overline{q}'(u)/\overline{q}(u)$ for $u \in [0,1]$, then the following conditions are equivalent:*

i) The IFR (DFR) class is preserved by q;
ii) The IFR (DFR) class is preserved by q for the standard exponential distribution;
iii) The function α is decreasing (increasing) in $(0,1)$.

Proof. We are going to prove the proposition for the IFR class (the proof for the DFR class is analogous). First, we note that the hazard rate function r_q of F_q can be obtained as

$$r_q(t) = \frac{\overline{q}'(\overline{F}(t))}{\overline{q}(\overline{F}(t))} f(t) = \alpha(\overline{F}(t)) r(t), \tag{5}$$

where $r = f/\overline{F}$ is the hazard rate of F. Clearly, (i) implies (ii).

To prove that (ii) implies (iii), we note that for the standard exponential distribution $r(t) = 1$ for $t \geq 0$. Then we know that

$$r_q(t) = \alpha(\overline{F}(t)) = \alpha(e^{-t})$$

is increasing for $t \geq 0$. Therefore, α is decreasing in $[0,1]$.

Finally, to prove that (iii) implies (i), we note that if α is decreasing, then $\alpha(\overline{F}(t))$ is increasing in t. This function is also nonnegative. Therefore, if r is increasing, then r_q is increasing as well, since from (5) it is the product of two nonnegative increasing functions. ■

The preceding proposition shows that the IFR and DFR classes are both preserved with respect to a distortion q if and only if the corresponding α is constant, which leads to the PHR model. In the other cases, if the IFR is preserved, then the DFR class is not (and vice versa). It may also happen that neither of them are preserved (when α is not monotone).

For the ILR class we have the following new result, to be added to the one already described in Navarro et al..[2] To provide its statement, we need the following definition.

Definition 3. We say that a function $\phi : \mathbb{R} \to \mathbb{R}^+$ preserves logconcave functions if $\phi \circ f$ is logconcave for any logconcave function f.

For example, the functions $\phi_c(t) = t^c$ preserve logconcave functions for any $c > 0$.

Proposition 5. *If $F_q = q \circ F$ for a differentiable distortion function q and $\overline{q}'$ preserves logconcave survival functions, then the ILR class is preserved by q.*

Proof. First, we note that $\log f_q$ can be written as

$$\log(f_q(t)) = \log(\overline{q}'(\overline{F}(t))) + \log(f(t))$$

for all t in the support of F.

Note that if F is ILR, then F is also IFR, that is, $\overline{F}$ is logconcave. Hence, we have that $\overline{q}' \circ \overline{F}$ is logconcave. As, by assumption, f is logconcave, we have that f_q is logconcave (since $\log f_q$ is the sum of two concave functions). ■

It should be noted here that $\overline{q}'$ is not a distortion function. Also note that we do not have necessary and sufficient conditions for the preservation of the ILR class (as we have for the IFR/DFR classes). However, Proposition 5 holds for not necessarily positive variables too. Other sufficient conditions for the preservation of ILR class can be found in Navarro et al..[2]

Similar conditions can be obtained for the classes LFR and LCFR. They can be stated as follows. The proofs are analogous, and therefore omitted.

Proposition 6. *If $F_q = q \circ F$ for a differentiable distortion function q and an absolutely continuous distribution function F, and if the function α in Proposition 4 preserves logconcave survival functions, then the LFR class is preserved by q.*

Proposition 7. *If $F_q = q \circ F$ for a distortion function q and an IFR distribution function F and the function $\beta = -\log \overline{q}$ preserves logconcave survival functions, then F_q is LCFR. In particular, under this condition for β, the LCFR class is preserved by q.*

We must say that it is not easy to determine if a function ϕ preserves logconcave functions. A result similar to Proposition 4 can be obtained for increasing functions ϕ.

The distorted distributions can also be used to get stochastic comparisons, see for example, Navarro et al..[17] They can be stated as follows. The definitions and main properties of the stochastic (ST), hazard rate (HR), reversed hazard rate (RHR), and likelihood ratio (LR) orders can be found in Belzunce et al.[5] and Shaked and Shanthikumar.[9]

Proposition 8. *If T_i has the survival function $\overline{q}_i(\overline{F}(t))$ and the distribution function $q_i(F(t))$ for $i = 1, 2$, then the following properties hold:*

(i) $T_1 \leq_{ST} T_2$ *for all F if and only if* $\overline{q}_2 \geq \overline{q}_1$ *(or $q_2 \leq q_1$) in* $(0, 1)$*;*
(ii) $T_1 \leq_{HR} T_2$ *for all F if and only if* $\overline{q}_2/\overline{q}_1$ *decreases in* $(0, 1)$*;*
(iii) $T_1 \leq_{RHR} T_2$ *for all F if and only if* q_2/q_1 *increases in* $(0, 1)$*;*
(iv) $T_1 \leq_{LR} T_2$ *for all absolutely continuous distribution functions F if and only if* $\overline{q}_2'/\overline{q}_1'$ *decreases (or q_2'/q_1' increases) in* $(0, 1)$.

In particular, from (5), the condition in (ii) can be replaced by $\alpha_2(u) \leq \alpha_1(u)$ for all $u \in (0, 1]$, where $\alpha_i(u) = u\overline{q}_i'(u)/\overline{q}_i(u)$ for $i = 1, 2$.

3 | PRESERVATION RESULTS FOR THE ILR CLASS

In general, if $X_1, \ldots, X_n$ are ILR, then one cannot affirm that also $S = X_1 + \cdots + X_n$ is ILR (see, e.g., Sections 4 and 5.1 below). Conditions for this preservation property are discussed here. In the first result we study the preservation of the ILR class when $(X_1, \ldots, X_n)$ is a random vector having a logconcave probability density function. To this purpose we need Prékopa's theorem (see Prékopa[18]) which is stated in the following lemma.

Lemma 1 (Prékopa's theorem). *Suppose that $g : \mathbb{R}^m \times \mathbb{R}^k \to [0, \infty)$ is a logconcave function and that*

$$h(\mathbf{x}) = \int_{\mathbb{R}^k} g(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

is finite for each $\mathbf{x} \in \mathbb{R}^m$. Then h is logconcave on $\mathbb{R}^m$.

Now we are ready to state the following preservation result.

Theorem 1. *Let $(X_1, \dots, X_n)$ be a random vector having a logconcave pdf f. Then $S = X_1 + \cdots + X_n$ is ILR.*

Proof. Let us consider the random vector $(Y_1, \dots, Y_n)$ with $Y_1 = S$ and $Y_j = X_j$ for $j = 2, \dots, n$. Then the pdf g of $(Y_1, \dots, Y_n)$ is

$$g(y_1, \dots, y_n) = f(y_1 - y_2 - \cdots - y_n, y_2, \dots, y_n)$$

for all $y_1, \dots, y_n$. Therefore, g is also logconcave, since

$$\begin{aligned}
\log\ g(\lambda \mathbf{y} + (1-\lambda)\mathbf{z}) &= \log g(\lambda y_1 + (1-\lambda) z_1, \dots, \lambda y_n + (1-\lambda) z_n) \\
&= \log f(\lambda(y_1 - y_2 - \cdots - y_n) + (1-\lambda)(z_1 - z_2 - \cdots - z_n), \\
&\quad \lambda y_2 + (1-\lambda) z_2, \dots, \lambda y_n + (1-\lambda) z_n) \\
&\geq \lambda \log f(y_1 - y_2 - \cdots - y_n, y_2, \dots, y_n) + (1-\lambda) \log f(z_1 - z_2 - \cdots - z_n, z_2, \dots, z_n) \\
&= \lambda \log g(\mathbf{y}) + (1-\lambda) \log g(\mathbf{z})
\end{aligned}$$

for all $\lambda \in (0, 1)$ and all $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Finally, we note that the pdf h of S can be obtained as

$$h(x) = \int_{\mathbb{R}^{n-1}} g(x, y_2, \dots, y_n) dy_2 \dots dy_n,$$

thus, from Lemma 1, we have that h is logconcave. ■

It must be pointed out that Theorem 1, as well as the following two statements, applies to any vector of random variables, not necessarily non-negative. Multivariate normal, Wishart, Dirichlet, uniform distributions over compact and convex sets and elliptically contoured distributions are examples of logconcave densities (see Hu and Li[19] and Walther[11]).

As an immediate consequence of the preceding theorem we obtain the following result, proved in Karlin and Proschan.[4]

Corollary 1. *If $X_1, \dots, X_n$ are independent and ILR, then $X_1 + \cdots + X_n$ is ILR.*

The preceding theorem can be extended to linear combinations as follows.

Theorem 2. *Let $(X_1, \dots, X_n)$ be a random vector having a logconcave pdf f. Then $S = a_1 X_1 + \cdots + a_n X_n$ is ILR for all $a_1, \dots, a_n \in \mathbb{R}$.*

Proof. The result is trivial if $a_1 = \cdots = a_n = 0$. Let us assume that $a_1 \neq 0$ (the proof is similar in the other cases) and let us consider the random vector $(Y_1, \dots, Y_n)$ with $Y_1 = S$ and $Y_j = X_j$ for $j = 2, \dots, n$. Then the pdf g of $(Y_1, \dots, Y_n)$ is

$$g(y_1, \dots, y_n) = \frac{1}{|a_1|} f\left(\frac{y_1 - a_2 y_2 - \cdots - a_n y_n}{a_1}, y_2, \dots, y_n \right).$$

Therefore, g is also logconcave (see the similar proof in the preceding theorem). Finally, we note that the pdf h of S can be obtained as

$$h(x) = \int_{\mathbb{R}^{n-1}} g(x, y_2, \dots, y_n) dy_2 \dots dy_n$$

and so, from Lemma 1, we have that h is logconcave. ■

In particular, this result proves that any sum of the X_i is ILR when $(X_1, \dots, X_n)$ has a logconcave joint pdf. Note that this property can be applied to L-statistics, that is, to linear combinations of order statistics, when its joint pdf is logconcave. When the order statistics are obtained from i.i.d. random variables with a common univariate pdf f (the usual case), then it is well known that its joint pdf is logconcave whenever f is logconcave, see proposition 5 in An.[20] This property can be extended as follows. If $(X_1, \dots, X_n)$ have a joint logconcave pdf f_n and they are exchangeable (i.e., f_n is permutation invariant), then the joint pdf of the order statistics g_n is

$$g_n(x_1, \dots, x_n) = n!\, f_n(x_1, \dots, x_n)$$

for $x_1 < \cdots < x_n$ (zero elsewhere), and so it is logconcave as well. This is an important result since the distribution of any L-statistic obtained from them is ILR. Hence its pdf can be estimated by using MLE, see Walther.[11]

In particular, we can apply this property to a single order statistic and therefore to the extremes $\min(X_1, \dots, X_n)$ and $\max(X_1, \dots, X_n)$. Note that this result is related to proposition 3 in Navarro and Shaked,[3] which proves that if $(X_1, \dots, X_n)$ is exchangeable (i.e., it is permutation symmetric in law) and it has a logconcave pdf, then all the minima (series systems) and maxima (parallel systems) formed with the X_i are also ILR. A similar result is obtained in proposition 1 of that article for the IFR/DFR classes (without the assumption of exchangeability).

In a similar way, this property can also be applied to linear combinations of generalized order statistics and the related concepts obtained from them (e.g., generalized spacings). Here we first need to prove the logconcave property for their joint pdf which is given by

$$g_n(x_1, \dots, x_n) = k\gamma f(x_n)\overline{F}^{k-1}(x_n)\prod_{i=1}^{n-1} f(x_i)\overline{F}^{m_i}(x_i),$$

where $\gamma, k > 0$ (see, e.g., Cramer and Kamps[21]). Hence, clearly, it is logconcave when f is logconcave and $m_i \geq 0$ for $i = 1, \dots, n-1$. More conditions to get this property were obtained in Chen et al..[22] The same property holds for linear combinations of progressively censored order statistics, see (1) in Cramer.[23]

4 | PRESERVATION RESULTS FOR THE IFR CLASS

The following example shows that IFR and ILR classes are not always preserved with respect to sums of dependent random variables.

Example 1. Let us consider the random vector (X, Y) with uniform distribution over the set $S = [0, 1/2]^2 \cup [1/2, 1]^2$, that is, $f(x, y) = 2$ for $(x, y) \in S$ (zero elsewhere). Note that X and Y have a positive dependence and that the marginal distributions are uniform over $(0, 1)$ (i.e., they define a copula). Hence X and Y are both *IFR* and *ILR*. A straightforward calculation shows that the survival function $\overline{H}$ of $S = X + Y$ is

$$\overline{H}(s) = \begin{cases} 1 - s^2, & \text{for } 0 \leq s \leq 1/2; \\ 3/4 - (s - 1/2)^2, & \text{for } 1/2 < s \leq 1; \\ 1/2 - (s - 1)^2, & \text{for } 1 < s \leq 3/2; \\ (2 - s)^2, & \text{for } 3/2 < s \leq 2; \end{cases}$$

and its pdf h is

$$h(s) = \begin{cases} 2s, & \text{for } 0 \leq s \leq 1/2; \\ 2 - 2s, & \text{for } 1/2 < s \leq 1; \\ 2s - 2, & \text{for } 1 < s \leq 3/2; \\ 4 - 2s, & \text{for } 3/2 < s \leq 2; \end{cases}$$

(zero elsewhere). Note that h is not logconcave (see Figure 1, left), that is, S is not ILR.

The hazard rate of S is

$$r_S(s) = \begin{cases} \frac{2s}{1-s^2}, & \text{for } 0 \leq s \leq 1/2; \\ \frac{2-2s}{3/4-(s-1/2)^2}, & \text{for } 1/2 < s \leq 1; \\ \frac{2s-2}{1/2-(s-1)^2}, & \text{for } 1 < s \leq 3/2; \\ \frac{4-2s}{(2-s)^2}, & \text{for } 3/2 < s \leq 2. \end{cases}$$

It is plotted in Figure 1, right, and there we can see that it is not increasing, that is, S is not IFR.

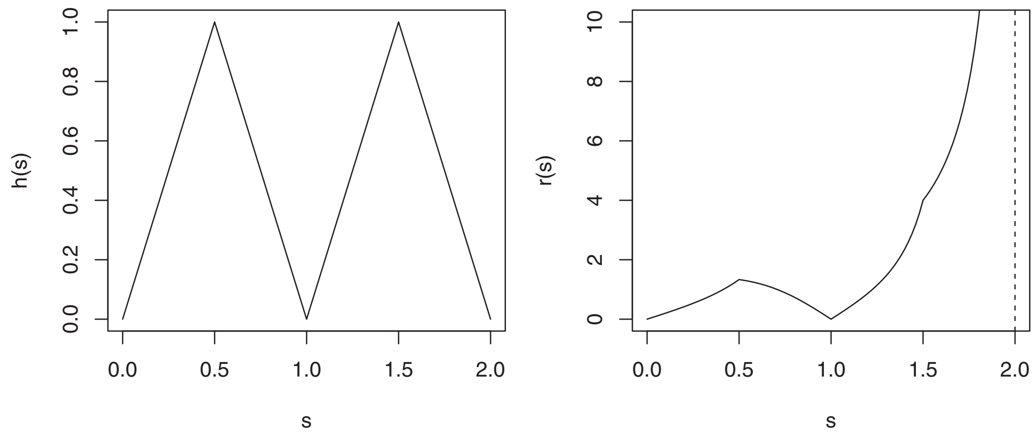

FIGURE 1 Plots of the pdf h (left) and the hazard rate r_S for the sum of two dependent random variables with uniform distributions over the interval $(0, 1)$ (see Example 1)

However, the IFR class is preserved in many models, even if, unfortunately, it does not seem possible to find a very general result such as Theorem 1 for the ILR class. Nevertheless, there are cases in which something can be proved, as in the following example, which deals with a model that can describe both positive or negative dependence. As the IFR class is preserved for this model, then the DFR class is not preserved (from Proposition 4).

Example 2. Let us consider a random vector (X, Y) with a Farlie-Gumbel-Morgenstern (FGM) survival copula defined by

$$\widehat{C}(u, v) = uv[1 + \theta(1 - u)(1 - v)] \tag{6}$$

for $u, v \in [0, 1]$ and $-1 \leq \theta \leq 1$. The independent case is represented by $\theta = 0$ while $\theta > 0$ (resp. $<$) represents a positive (negative) dependence (see, e.g., Nelsen[15]).

If we also assume that X and Y have a common exponential distribution with hazard rate $\lambda > 0$, then a straightforward calculation from (3) gives $\overline{H}(t) = \overline{q}_\theta(\overline{F}(t))$ with $\overline{F}(t) = e^{-\lambda t}$ for $t \geq 0$ and

$$\overline{q}_\theta(u) = u - u \log u + \theta u \left[-3 + 3u - \log u - 2u \log u\right]$$

for $u \in [0, 1]$ (see also Navarro and Sarabia[8]). As expected, in the independent case ($\theta = 0$), we obtain $\overline{q}_0(u) = u - u \log u$ for $u \in [0, 1]$.

Hence

$$\gamma_\theta(u) = \overline{q}'_\theta(u) = -\log u - \theta \left[4 - 4u + \log u + 4u \log u\right]$$

and

$$\alpha_\theta(u) = \frac{u\overline{q}'_\theta(u)}{\overline{q}_\theta(u)} = \frac{-\log u - \theta \left[4 - 4u + \log u + 4u \log u\right]}{1 - \log u + \theta \left[-3 + 3u - \log u - 2u \log u\right]}$$

for $u \in [0, 1]$. A straightforward calculation shows that both functions are decreasing for any $\theta \in [-1, 1]$. They are plotted in Figure 2 for $\theta = -1, -0.5, 0, 0.5, 1$. As γ_θ is decreasing, we have $X \leq_{LR} X + Y$ from Proposition 8, (iv). Also, as α_θ is strictly decreasing, from Proposition 4 we have that $X + Y$ is IFR and not DFR for all $\lambda > 0$ and all $\theta \in [-1, 1]$.

In Figure 3 we plot $\log h$ (left) and the hazard rate r_S of the sum (right) for $\theta = -1, -0.5, 0, 1, 0.5, 1$. In this figure we can see that IFR and ILR classes are preserved. As a consequence, NBU and IFRA classes are preserved as well (see Navarro et al.[2]). However, as mentioned above, the DFR class is not preserved. Moreover, from (5), we have that

$$\lim_{u \to 0^+} \alpha_\theta(u) = 1 \quad \text{and} \quad \lim_{t \to \infty} r_S(t) = \lambda$$

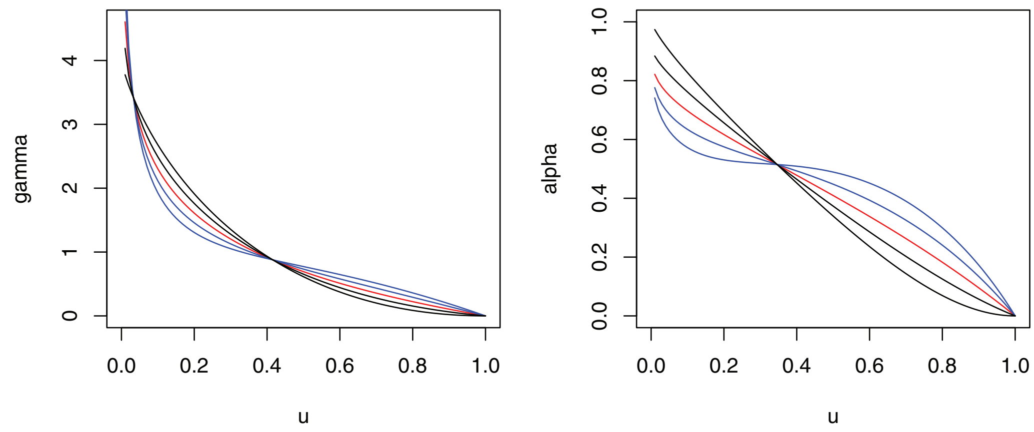

FIGURE 2 Plots of $\gamma_\theta(u) = \overline{q}'_\theta(u)$ (left) and $\alpha_\theta(u) = u\overline{q}'_\theta(u)/\overline{q}_\theta(u)$ for the sum $S = X + Y$ of two exponential distributions with a FGM survival copula with $\theta = -1, -0.5$ (black), 0 (red, convolution), and 0.5,1 (blue)

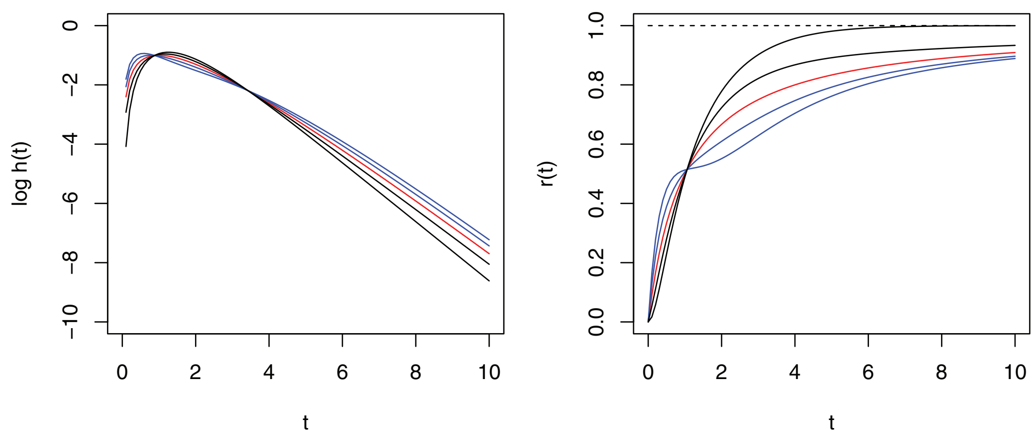

FIGURE 3 Plots of $\log h$ (left) and the hazard rate r_S (right) for the sum $S = X + Y$ of two dependent standard exponential distributions with a FGM survival copula with $\theta = -1, -0.5$ (black), 0 (red, convolution), and 0.5,1 (blue). The dashed line represents the common hazard rate of X and Y and the limiting behavior of r_S

for all $\theta \in [-1, 1]$. Therefore, Proposition 2 for convolutions ($\theta = 0$) can be extended for the specific C-convolution defined by the FGM survival copula for all $-1 \le \theta \le 1$ when they have a common exponential distribution.

It can be seen that this sum does not preserve LFR for $\theta = 1$ (see Figure 4, left). However, the plot of $\log R_S$ (see Figure 4, right), where

$$R_S(t) = -\log \overline{H}(t) = \int_0^t r_S(x)dx,$$

for $t \ge 0$, shows that LCFR is preserved in this dependence model.

Alternative conditions for a sum of two dependent lifetimes to be IFR are described in the next statement.

Theorem 3. *Let (X, Y) be a random vector with survival copula $\widehat{C}$ and marginal survival functions $\overline{F}$ and $\overline{G}$. Let $S = X + Y$.*

i) If X is ILR and

$$\Delta_1(x, s) = \partial_1 \widehat{C}(\overline{F}(x), \overline{G}(s - x))$$

is logconcave, then S is IFR.

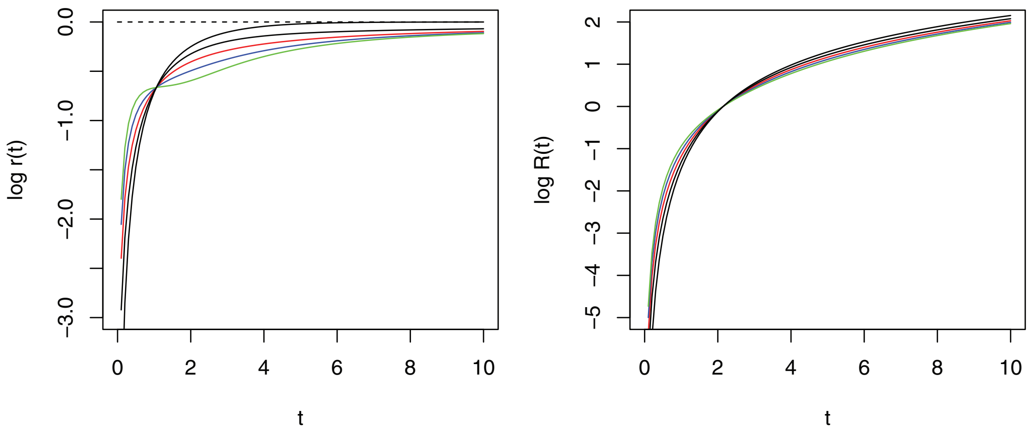

FIGURE 4 Plots of $\log r_S$ (left) and $\log R_S$ (right) for the sum $S = X + Y$ of two dependent standard exponential distributions with a FGM survival copula with $\theta = -1, -0.5$ (black), 0 (red, convolution), 0.5 (blue) and 1 (green). The dashed line represents the log of the common hazard rate of X and Y and the limiting behavior of $\log r_S$

ii) If Y is ILR and

$$\Delta_2(y, s) = \partial_2 \widehat{C}(\overline{F}(s - y), \overline{G}(y))$$

is logconcave, then S is IFR.

Proof. Let us prove (i) (the proof of (ii) is symmetric). From (2), the survival function of S can be written as

$$\overline{H}(s) = \int_{-\infty}^{\infty} f(x)\ \partial_1 \widehat{C}(\overline{F}(x), \overline{G}(s - x))dx = \int_{-\infty}^{\infty} f(x)\Delta_1(x, s)dx \tag{7}$$

It is easy to see that if $\Delta_1(x, s)$ is logconcave and X is ILR, then $f(x)\Delta_1(x, s)$ is also logconcave. Then the proof is completed by applying Prékopa's theorem (i.e., Lemma 1) to (7). ∎

The following numerical example shows how to use Theorem 3 to study the preservation of the IFR class.

Example 3. Let us consider the sum of two dependent random variables X and Y having exponential and Rayleigh (Weibull) distributions with $\overline{F}(t) = \exp(-t)$ and $\overline{G}(t) = \exp(-t^2)$ for $t \geq 0$ and with the FGM survival copula $\widehat{C}$ defined in (6). Its partial derivative is

$$\partial_1 \widehat{C}(u, v) = v\,[1 + \theta(1 - 2u)(1 - v)]\,. \tag{8}$$

Hence

$$L(x, s) = \log \Delta(x, s) = \log(\overline{G}(s - x)) + \log\left(1 + \theta(1 - 2\overline{F}(x))(1 - \overline{G}(s - x))\right),$$

that is,

$$L(x, s) = -(s - x)^2 + \log\left(1 + \theta(1 - 2\exp(-x))(1 - \exp(-(s - x)^2))\right),$$

for $0 \leq x \leq s$. Its 3D plot for $\theta = 1$ is shown in Figure 5, left. There, we can see that, in this case, $L(x, s)$ is concave. Moreover, the exponential distribution is ILR. Hence, by applying Theorem 3, (i), we obtain that $S = X + Y$ is IFR for $\theta = 1$. On the contrary, for $\theta = -1$ the function $L(x, s)$ is not concave, being, for example, $L(0.5, 0) = -0.06845$, $L(0.5, 0.15) = -0.05031$ and $L(0.5, 0.3) = -0.02315$. The 3D plot for $\theta = -1$ is shown in Figure 5, right. Thus, Theorem 3 cannot be applied to this case.

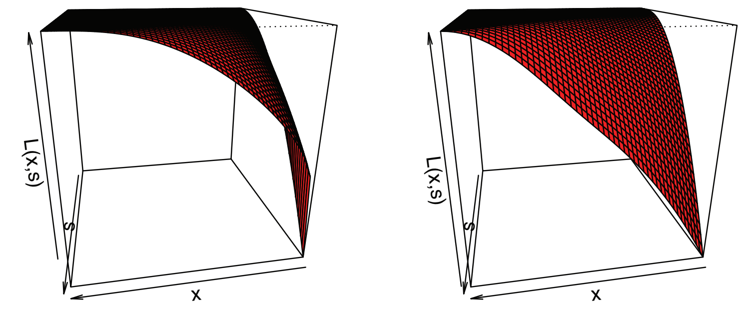

FIGURE 5 Plots of the function $L(x, s)$ in Example 3 for $\theta = 1$ (left) and $\theta = -1$ (right)

In the following example we show how to use Propositions 3 and 4 to check the preservation of the IFR class in sums of dependent random variables when just one of them has an exponential distribution.

Example 4. Let us consider a random vector (X, Y) with the FGM survival copula defined in (6). Let us assume that Y has a standard exponential distribution and that the survival function of X is

$$\overline{F}(t) = 1 - \sqrt{1 - \exp(-t)}$$

for $t \geq 0$. Note that the survival function of Y can be represented as

$$\overline{G}(t) = \exp(-t) = \psi(\overline{F}(t))$$

for $\psi(u) = 2u - u^2$. Hence, the survival function of $S = X + Y$ can be written as $\overline{H}(t) = \overline{q}_1(\overline{F}(t))$, where $\overline{q}_1$ is obtained from (4) as

$$\overline{q}_1(u) = u + \int_u^1 \partial_1 \widehat{C}\left(v, \frac{\psi(u)}{\psi(v)}\right) dv$$

for $u \in [0, 1]$. In particular, for the FGM copula we get the expression in Equation (8) and

$$\overline{q}_1(u) = u + \psi(u)\int_u^1 \frac{1}{\psi(v)}dv + \theta\psi(u)\int_u^1 \frac{1-2v}{\psi(v)}dv + \theta\psi^2(u)\int_u^1 \frac{1-2v}{\psi^2(v)}dv$$

for $u \in [0, 1]$. A straightforward calculation leads to

$$\begin{aligned}\overline{q}_1(u) = u + \frac{1}{2}\psi(u)\log\left(\frac{2}{u} - 1\right) - \frac{1}{2}\theta\psi(u)\log(u(2-u)^3) \\ - \theta\psi(u)\left[\frac{1+u}{2} - \psi(u) + \frac{1}{4}\psi(u)\log\left(\frac{u}{2-u}\right)\right]\end{aligned}$$

for $u \in [0, 1]$. By plotting $\alpha_1(u) = u\overline{q}_1'(u)/\overline{q}_1(u)$ we see that it is decreasing for all $\theta \in [0, 1]$. However, we cannot apply Proposition 4 to this representation since $\overline{F}$ is not IFR (actually it is strictly DFR). Instead, we are going to use a representation based on $\overline{G}$ which is IFR. Note that $\overline{F}(t) = \beta(\overline{G}(t))$ for $\beta(u) = 1 - \sqrt{1-u}$ for $u \in [0, 1]$. Then the survival function of S can be obtained as

$$\overline{H}(t) = \overline{q}_1(\overline{F}(t)) = \overline{q}_1(\beta(\overline{G}(t))) = \overline{q}_2(\overline{G}(t)),$$

where $\overline{q}_2(u) = \overline{q}_1(\beta(u))$ for $u \in [0, 1]$. Hence

$$\alpha_2(u) = \frac{u\overline{q}_2'(u)}{\overline{q}_2(u)} = \frac{u\overline{q}_1'(\beta(u))\beta'(u)}{\overline{q}_1(\beta(u))}$$

for $u \in (0, 1]$. This function is plotted in Figure 6, left, for a sample of possible values of θ, that is, $\theta = -1, 0.5, 0, 0.5, 1$. There we can see that α_2 is decreasing for such values of θ, except for the case $\theta = -1$. Then, from Proposition 4, S is IFR

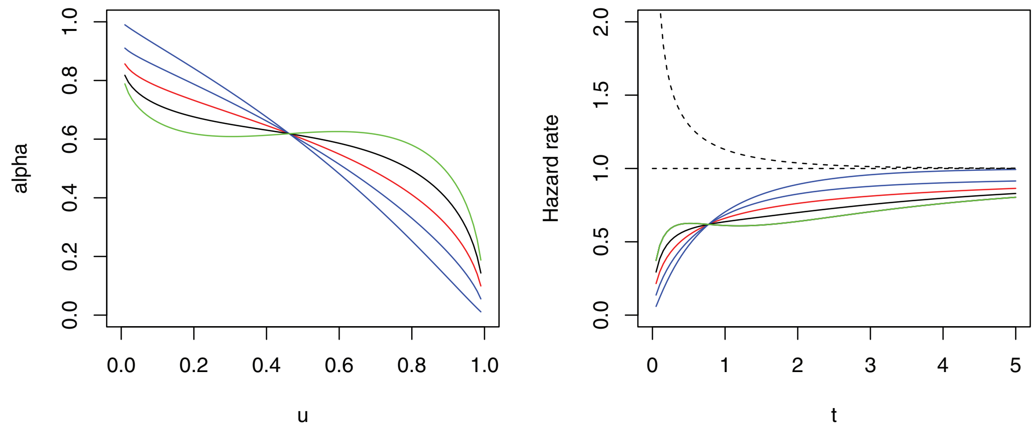

FIGURE 6 Plots of the function α_2 (left) and the hazard rate function r_S (right) for the sum $S = X + Y$ in Example 4 and $\theta = -1$ (green), -0.5 (black), 0 (red, convolution), 0.5,1 (blue). The dashed lines represent the hazard rate functions of X (decreasing line) and Y (constant line)

for $\theta = 0.5, 0, 0.5, 1$. The hazard rate functions of X and Y (dashed lines) and S (continuous lines) are plotted in Figure 6, right. There we can see that S is IFR for such a sample of values for θ, except for $\theta = -1$ (green line).

5 | RELEVATION TRANSFORM MODELS

A model of interest in reliability theory which deals with sums of not independent lifetimes is the relevation transformation. This model is considered here as an example where the results described in previous sections can be applied, but it is of interest also because it provides meaningful counterexamples. Under a *relevation transform* (see, e.g., Krakowski[24] or Belzunce et al.[25]), the unit with lifetime X is replaced (or repaired) when it fails at a time $x \geq 0$ by a unit Y having survival function $\overline{G}$ but with the same age as X, that is, with survival function

$$\overline{G}_x(y) = \Pr(Y > y|X = x) = \Pr(Y - x > y|Y > x) = \frac{\overline{G}(x+y)}{\overline{G}(x)}$$

for $y \geq 0$. Then X and Y are dependent and the survival function $\overline{H}(t) = \Pr(S > t)$ of $S = X + Y$ can be obtained as

$$\overline{H}(t) = \overline{F}(t) + \int_0^t \overline{G}_x(t-x)f(x)dx = \overline{F}(t) + \int_0^t \frac{\overline{G}(t)}{\overline{G}(x)}f(x)dx. \tag{9}$$

This is formula (1.1) in Krakowski,[24] in which the notation $\overline{H} = \overline{F}\#\overline{G}$ is introduced. In a relevation transform with *perfect repair*, the unit with lifetime X is replaced when it fails at a time x by an independent new unit with lifetime Y having survival function $\overline{G}_x(y) = \overline{G}(y)$. Then

$$\overline{H}(t) = \overline{F} * \overline{F}(t) = \overline{F}(t) + \int_0^t \overline{G}(t-x)f(x)dx, \tag{10}$$

which is the survival function of the sum of two independent random variables (convolution). More realistic cases are considered in the following subsections.

5.1 | Relevation transform with minimal repair

In a relevation transform with *minimal repair*, the unit X is replaced when it fails at a time x by a unit $Y_x = (Y - x|Y > x)$ having survival function $\overline{F}$ but with the same age as X, that is, $\overline{G}_x(y) = \overline{F}(x+y)/\overline{F}(x)$ for $y \geq 0$. This is equivalent to

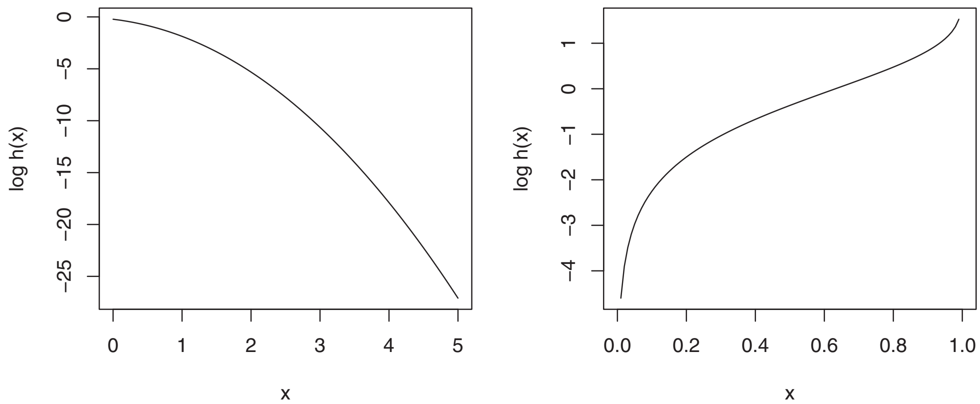

FIGURE 7 Plots of $\log h$ for the relevation transform under minimal repair for a truncated standard Gaussian (normal) distribution (left) and for a uniform distribution over $(0, 1)$ (right)

minimally repair the broken unit to be like it was before its failure. In this case, from (9),

$$\overline{H}(t) = \overline{F}\#\overline{F}(t) = \overline{F}(t) + \int_0^t \frac{\overline{F}(t)}{\overline{F}(x)} f(x)dx = \overline{F}(t) - \overline{F}(t)\log \overline{F}(t).$$

This is formula (3.1) in Krakowski[24] with the notation $\overline{F}\#\overline{F}$ for the relevation transform under minimal repair. Note that for the exponential distribution, this expression coincides with the one obtained for the usual convolutions from (10).

Note that it is a distorted distribution with dual distortion $\overline{q}(u) = u - u\log u$ for $u \in [0, 1]$. The pdf of S is $h(t) = -f(t)\log \overline{F}(t)$ and its hazard rate is

$$r_S(t) = \frac{h(t)}{\overline{H}(t)} = -\frac{\log \overline{F}(t)}{1 - \log \overline{F}(t)} r(t)$$

for $t \geq 0$, where $r = f/\overline{F}$ is the hazard rate of $\overline{F}$. Hence $\overline{q}'(u) = -\log u$ and $\alpha(u) = -\log u/(1 - \log u)$ for $u \in (0, 1)$. As α is decreasing in $(0, 1)$, the IFR class is preserved by the relevation transform with minimal repair, which is a well known result (see the references in Pellerey et al.[10]). As the IFR class is preserved, so are the NBU and IFRA classes. As $\alpha \leq 1$, then $S = X + Y \geq_{HR} X$ for all $\overline{F}$. Moreover, from (5), we get

$$\lim_{t\to\infty} \frac{r_S(t)}{r(t)} = \lim_{u\to 0^+} \alpha(u) = 1.$$

Therefore, Proposition 2, which deals with the limit of the hazard rate of the sum of independent variables (thus, with convolutions), can be extended for the specific C-convolution defined by the relevation transform.

However, $\overline{q}'(u) = -\log u$ does not always preserve logconcave survival functions. Thus the ILR class is not always preserved. Sufficient conditions for that preservation in this model were obtained in Pellerey et al.,[10] where it is proved that the ILR class is preserved when F is also LCFR.

For example, we observe that the ILR class is preserved for a positively truncated standard normal (Gaussian) distribution as can be seen in Figure 7, left. Note that the normal distribution satisfies condition LFR, which implies condition LCFR. However, this is not the case for a uniform distribution over $(0, 1)$ (see Figure 7, right) since the uniform distribution is not LCFR. Therefore, Theorem 1 cannot be extended to arbitrary vectors, that is, the ILR class is not always preserved.

5.2 | Relevation transform with imperfect repair

In a relevation transform with *imperfect repair*, the unit X is replaced when it fails at a time x by a unit $Y_x = (Y - x|Y > x)$ having survival function $\overline{G} = \overline{F}^{\theta}$ for $\theta > 0$ and $\theta \neq 1$ but with the same age as X (see Brown and Proschan[26] and

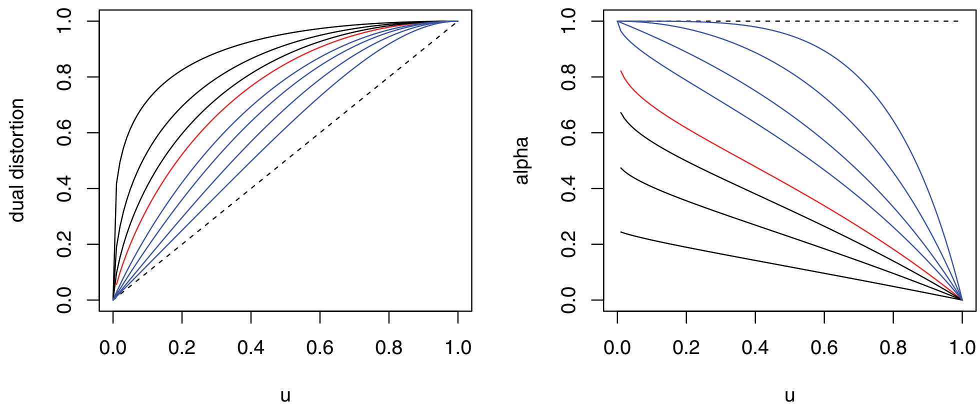

FIGURE 8 Plots of $\overline{q}_\theta$ (left) and α_θ (right) for a relevation transform with imperfect repair and $\theta = 0.25, 0.5, 0.75$ (black), $\theta = 1$ (red, minimal repair) and $\theta = 1.5, 2, 3, 5$ (blue)

Bhattacharjee[27] for details). In this case, from (9),

$$\overline{H}_\theta(t) = \overline{F}(t) + \int_0^t \frac{\overline{F}^\theta(t)}{\overline{F}^\theta(x)} f(x)dx = \overline{F}(t) + \overline{F}^\theta(t)\left(\frac{\overline{F}^{1-\theta}(t)}{\theta-1} - \frac{1}{\theta-1}\right)$$

for $t \geq 0$. Note that it is also a distorted distribution with dual distortion

$$\overline{q}_\theta(u) = \frac{\theta}{\theta-1}u - \frac{1}{\theta-1}u^\theta$$

for $u \in [0,1]$. Also note that it is a negative mixture of $\overline{F}$ and $\overline{F}^\theta$ and that $\lim_{\theta\to\infty} \overline{q}_\theta(u) = u$, $\lim_{\theta\to 0^+} \overline{q}_\theta(u) = 1$, $\lim_{\theta\to 1} \overline{q}_\theta(u) = u - u\log u$ for $u \in (0,1)$ (i.e., a relevation transform with minimal repair, as expected). Their plots for different values of θ can be seen in Figure 8, left.

The pdf of S in this model is

$$h_\theta(t) = \frac{\theta}{\theta-1}\left(1 - \overline{F}^{\theta-1}(t)\right) f(t)$$

and its hazard rate is $r_\theta(t) = \alpha_\theta(\overline{F}(t))r(t)$, where $r = f/\overline{F}$ is the hazard rate of $\overline{F}$ and

$$\alpha_\theta(u) = \theta\frac{1-u^{\theta-1}}{\theta-u^{\theta-1}}$$

for $u \in [0,1]$. The plots of α_θ can be seen in Figure 8, right. As they are ordered, the respective sums are ordered in hazard rate sense and, as they are strictly decreasing, from Proposition 4 it follows that the IFR class is preserved (and the DFR class is not preserved). This is a well-known property, proved in Brown and Proschan.[26] As a consequence, NBU and IFRA classes are preserved as well. Even more, since $\alpha_\theta \leq 1$, then $S \geq_{HR} X$ for all $\overline{F}$ and all $\theta > 0$. Moreover, from (5),

$$\lim_{t\to\infty}\frac{r_\theta(t)}{r(t)} = \lim_{u\to 0^+}\alpha_\theta(u) = \begin{cases} 1 & \text{for } \theta > 1; \\ \theta & \text{for } \theta < 1. \end{cases}$$

Therefore, Proposition 2 for convolutions can be extended for the specific C-convolution defined by the relevation transform with imperfect repair and $\theta > 1$. However, this property does not hold when $0 < \theta < 1$. The respective hazard rates for a standard exponential distribution can be seen in Figure 9, right. Note that they have different limiting behavior.

However,

$$\overline{q}'_\theta(u) = \frac{\theta}{\theta-1}\left(1-u^{\theta-1}\right)$$

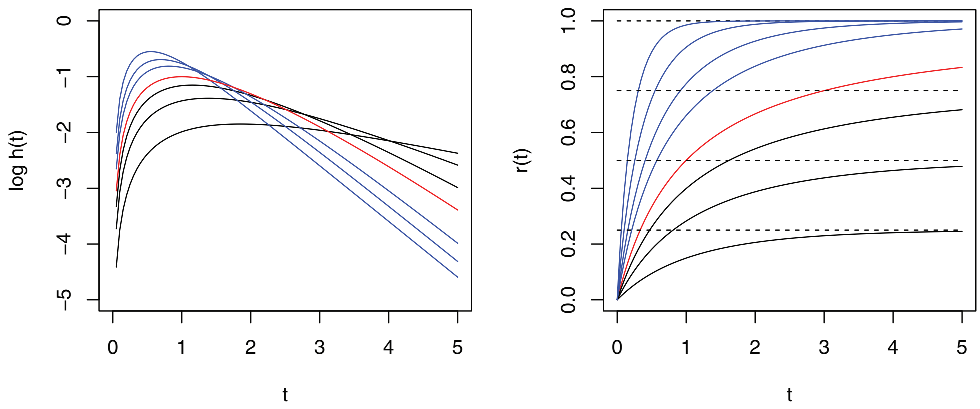

FIGURE 9 Plots of $\log h_\theta$ (left) and hazard rate r_θ (right) for a relevation transform with imperfect repair with a baseline standard exponential distributions and $\theta = 0.25, 0.5, 0.75$ (black), $\theta = 1$ (red, minimal repair) and $\theta = 1.5, 2, 3, 5$ (blue). The dashed lines represent the limiting behavior for $t \to \infty$

does not always preserve logconcave functions. So we do not know if the ILR class is preserved. For example, in the case of a standard exponential distribution, the ILR class is preserved (see Figure 9, left). As $\overline{q}'_\theta(u)$ is decreasing in u, in this dependence model, from Proposition 8, (iv), we also get $S \geq_{LR} X$ for all $\overline{F}$ and all $\theta > 0$.

Remark 2. The sequential order statistics fit to a relevation transform model with different distributions, see Cramer and Kamps.[21] However, they do not fit to a minimal (or imperfect) repair model. In the case of a PHR model for the baseline distributions, they can be written as distortions. This case was studied in Burkschat and Navarro.[28] So we can also apply to them the new results for distortions obtained here.

6 | CLOSURES WITH RESPECT TO MAXIMUM

The closure properties of maximum also have interest in several applicative areas. Some general results for IFR/DFR and ILR/DLR classes were obtained in Hu and Li[19] and Navarro and Shaked.[3] The preservation for other classes, such as NBU, DMRL and NBUE, was studied in Abouammoh and El-Neweihi[29] and Navarro.[30] By using Proposition 4 we can obtain the following result, dealing with the preservation of the IFR class in maxima of identically distributed (i.d.) components.

Theorem 4. *If $X_1, \dots, X_n$ are i.d. with copula C, then the following conditions are equivalent:*

i) *The IFR (DFR) class is preserved by $M = \max(X_1, \dots, X_n)$;*
ii) *The IFR (DFR) class is preserved by $M = \max(X_1, \dots, X_n)$ for the standard exponential distribution;*
iii) *The function $\alpha(u) = u\delta'(1-u)/(1-\delta(1-u))$ is decreasing (increasing) in $(0,1)$, where $\delta(u) = C(u, \dots, u)$ is the diagonal section of C.*

Proof. The distribution function of M is

$$F_M(t) = \Pr(X_1 \leq t, \dots, X_n \leq t) = C(F(t), \dots, F(t)) = \delta(F(t)),$$

where F is the common distribution function of $X_1, \dots, X_n$. Hence, its survival function is

$$\overline{F}_M(t) = 1 - \delta(F(t)) = 1 - \delta(1 - \overline{F}(t)) = \overline{q}(\overline{F}(t))$$

for $\overline{q}(u) = 1 - \delta(1-u)$. Then the proof is completed by applying Proposition 4. ■

It is well known that the IFR property is preserved in maxima of i.i.d. random variables (see Esary and Proschan[31]), that is, the property in (iii) above holds for the product copula. This property can be extended for k-out-of-n systems (order statistics) with i.i.d. components (see Esary and Proschan[31]). It is also true for other copulas (see next example). However, it can be easily verified that this preservation property is not always true.

Example 5. Let $M = \max(X_1, X_2)$ with (X_1, X_2) having the same marginal distribution function F and the following Clayton-Oakes copula

$$C(u, v) = \frac{uv}{u + v - uv} \tag{11}$$

for $u, v \in [0, 1]$. Its diagonal section is

$$\delta(u) = C(u, u) = \frac{u}{2 - u}$$

for $u \in [0, 1]$. Hence the dual distortion function of M is

$$\overline{q}(u) = 1 - \delta(1 - u) = 1 - \frac{1 - u}{1 + u} = \frac{2u}{1 + u}$$

and the associated α function is

$$\alpha(u) = \frac{1}{1 + u}$$

for $u \in [0, 1]$. As α is strictly decreasing, the IFR class is preserved by M but the DFR class is not. Then the NBU and IFRA classes are preserved as well. Moreover, the limiting behavior of the hazard rate of M can be determined as

$$\lim_{t\to\infty} \frac{r_M(t)}{r(t)} = \lim_{u\to 0^+} \alpha(u) = \lim_{u\to 0^+} \frac{1}{1 + u} = 1.$$

Also note that $X_i \leq_{HR} M$ holds for any F (since $\alpha(u) \leq 1$ for all $u \in [0, 1]$). Even more, as $\overline{q}'$ is decreasing, then $X_i \leq_{LR} M$ holds for any F from Proposition 8, (iv).

It is easy to verify that results similar to those described in Example 5 can be provided considering the FGM copula as the connecting copula of (X_1, X_2) instead of the Clayton-Oakes copula.

For non i.d. components we have the following result. The proof is immediate from proposition 2.5 in Navarro et al..[2] The similar results for other aging classes such as DMRL/IMRL, NBU/NWU, and IFRA/DFRA can be obtained from the results for distortions given in Navarro[30] and Navarro et al..[2] Results for NBUE/NWUE aging classes are still an open question.

Theorem 5. *If $X_1, \dots, X_n$ have distribution functions $F_1, \dots, F_n$ and copula C, and if the following conditions hold:*

i) $X_1, \dots, X_n$ are IFR (DFR);
ii) The functions

$$\alpha_i(u_1, \dots, u_n) = \frac{u_i \partial_i C(1 - u_1, \dots, 1 - u_n)}{1 - C(1 - u_1, \dots, 1 - u_n)}$$

are decreasing (increasing) in $(0, 1)^n$ for $i = 1, \dots, n$;

then $M = \max(X_1, \dots, X_n)$ is IFR (DFR).

For the ILR class we have the following result, that follows from Proposition 5.

Theorem 6. *Let $(X_1, \dots, X_n)$ be a random vector with copula C and ILR margins X_i having the same distribution F. Let $\delta(u) = C(u, \dots, u)$ be the diagonal section of C. If δ' preserves logconcave distribution functions, then $M = \max(X_1, \dots, X_n)$ is ILR.*

Proof. Observe that the distribution function of M can be written as $F_M(t) = \delta(F(t))$, for all t. Thus its density f_M can be written as $f_M(t) = \delta'(F(t))f(t)$, so that

$$\log f_M(t) = \log(\delta'(F(t))) + \log f(t)$$

for all t. The term $\log f$ is concave by assumption (being F ILR). Moreover, since logconcavity of f implies logconcavity of F (see, e.g., Barlow and Proschan[1(p. 77)]), if δ' preserves logconcave distribution functions, then the first term is also concave, and the assertion follows. ∎

Note that non-negativity of $X_1, \ldots, X_n$ is not required in Theorem 6. Unfortunately, as stated already above, it is not easy to check if a function preserves logconcavity. However, Theorem 6 provides an immediate tool to check when the maximum of dependent and ILR variables is still ILR, knowing the copula C and the distribution F. For example, if C is the Clayton copula with parameter $\theta \in (0, \infty)$, so that $\delta(u) = (2u^{-\theta} - 1)^{-1/\theta}$ for $u \in [0, 1]$, and the X_i are exponentially distributed with the same rate λ, then one can easily verify that the ratio

$$\frac{\delta'(F(t+s))}{\delta'(F(t))}$$

is decreasing in $t \geq 0$ for all $s \geq 0$, that is, that $\delta'(F(t))$ is logconcave. Here $\delta'(u) = 2u^{-1-\theta}(2u^{-\theta} - 1)^{-(1+\theta^{-1})}$ for $u \in [0, 1]$. Thus, the maximum is ILR.

Similarly, if the connecting copula C is an AMH copula (see (4.2.3) in Nelsen[15(p. 116)]) with parameter $\theta \in [-1, 1)$, so that $\delta(u) = u^2/(1 - \theta(1 - u)^2)$, and the X_i are exponentially distributed with the same rate, then one can easily verify that the ratio above is decreasing in $t \geq 0$ for all $s \geq 0$, that is, that $\delta'(F(t))$ is logconcave. Here $\delta'(u) = (2u(1 - \theta(1 - u)^2) - 2\theta(1 - u)t^2)/(1 - \theta(1 - u)^2)^4$. Thus, again the maximum is ILR.

The preceding theorem can also be applied to copulas having diagonal sections $\delta(u) = u^c$ for $1 < c \leq 2$ (since, as mentioned above, the functions $\overline{q}'(u) = \delta'(u) = u^{c-1}$ preserve logconcave functions for $c > 1$). For example, this property holds for the Cuadras-Augé family of copulas, see Nelsen.[15(p. 15)]

6.1 | Load-sharing models with time-homogeneous failure rates

Here we consider a particular model dealing with the maximum of dependent variables. In the case of absolute continuity, the joint probability law of n non-negative random variables $X_1, \ldots, X_n$ can be described by means of the family of its *Multivariate Conditional Hazard Rates* (MCHR) functions. Such a description is alternative but mathematically equivalent to the one expressed in terms of the joint density function of $(X_1, \ldots, X_n)$ (see, e.g., the review paper Shaked and Shanthikumar[32] and the references cited therein).

In a sense, the MCHR functions arise as direct extensions of the univariate concept of hazard rate function for a single non-negative random variable X. In the bivariate case the family of the MCHR functions is $\mathcal{L} = \{\lambda_1^{(0)}(t), \lambda_2^{(0)}(t), \lambda_1^{(1)}(t, x), \lambda_2^{(1)}(t, x)\}$, where

$$\lambda_1^{(0)}(t) = \lim_{\Delta t \to 0^+} \frac{\Pr[X_1 > t + \Delta t | X_1 > t, X_2 > t]}{\Delta t},$$

$$\lambda_2^{(0)}(t) = \lim_{\Delta t \to 0^+} \frac{\Pr[X_2 > t + \Delta t | X_1 > t, X_2 > t]}{\Delta t},$$

and, for $0 < x < t$,

$$\lambda_1^{(1)}(t, x) = \lim_{\Delta t \to 0^+} \frac{\Pr[X_1 > t + \Delta t | X_1 > t, X_2 = x]}{\Delta t},$$

$$\lambda_2^{(1)}(t, x) = \lim_{\Delta t \to 0^+} \frac{\Pr[X_2 > t + \Delta t | X_2 > t, X_1 = x]}{\Delta t}.$$

The functions $\lambda_1^{(0)}(t)$, $\lambda_2^{(0)}(t)$, $\lambda_1^{(1)}(t, x)$, and $\lambda_2^{(1)}(t, x)$ belonging to $\mathcal{L}$ can be obtained in terms of the joint density function f_{X_1, X_2} of (X_1, X_2). One can also check that, vice versa, f_{X_1, X_2} can be recovered from the knowledge of the family $\mathcal{L}$.

In particular, special models arise from the condition that $\lambda_1^{(1)}(t, x)$ and $\lambda_2^{(1)}(t, x)$ do not depend on x. The corresponding models were called *Load-Sharing models*, see, for example, Spizzichino[33] and references cited therein for more details and further remarks. They were also called *models in which the failure rates depend on the working set* by some authors, such as in Ross.[34] The term load-sharing model is inspired by the model in which $L(t)$ has the meaning of a level of load to be shared by the surviving components at time $t > 0$, and by letting $L(t) = \lambda_1^{(0)}(t) + \lambda_2^{(0)}(t)$ if both components are alive at time t, or $L(t) = \lambda_i^{(1)}(t)$ if only component i is alive at time t.

Note that if X_1 and X_2 are exchangeable, so that $\lambda_1^{(0)}(t) = \lambda_2^{(0)}(t) = \lambda^{(0)}(t)$ and $\lambda_1^{(1)}(t) = \lambda_2^{(1)}(t) = \lambda^{(1)}(t)$, then, in agreement to what is stated above, it must be $\lambda^{(1)}(t) = 2\lambda^{(0)}(t)$. However, we will consider here the more general case where $\lambda^{(1)}(t) = \theta\lambda^{(0)}(t)$ for some $\theta > 0$.

Let us thus consider a vector (X_1, X_2) with distribution defined as in the model above. One can be interested in the distribution of the sum $S = X_1 + X_2$, having survival function $\overline{H}_\theta$ depending on the parameter θ. In this case, with straightforward calculations, one can verify that

$$\overline{H}_\theta(t) = \overline{F}(t) + \int_0^t \frac{\overline{F}^\theta(t)}{\overline{F}^\theta(x)} f(x)dx$$

for $t \geq 0$, where $\overline{F}(t)$ is the survival function corresponding to the hazard rate $\lambda^{(0)}(t)$, while $\overline{F}^\theta(t)$ is the survival function corresponding to the hazard rate $\lambda^{(1)}(t)$. Thus, the study of the sum of two lifetimes in a load-sharing model reduces to the case of minimal repair models (see Section 5.1).

However, dealing with these models, one is mainly interested in the total time the components can deal with the load to be shared, that is, in $M = \max(X_1, X_2)$. Letting $\overline{H}_\theta$ be the survival function of M, one has

$$\overline{H}_\theta(t) = \Pr(X_1 > t, X_2 > t) + \int_0^t \Pr(X_1 = x, X_2 > t)dx + \int_0^t \Pr(X_1 > t, X_2 = x)dx$$

that is,

$$\overline{H}_\theta(t) = \overline{F}^2(t)(1 - 2\log \overline{F}(t)) \quad \text{if } \theta = 2,$$

and

$$\overline{H}_\theta(t) = \frac{1}{2-\theta}\left(2\overline{F}^\theta(t) - \theta\overline{F}^2(t)\right) \quad \text{if } \theta \neq 2$$

for $t \geq 0$. In both cases we get distorted distributions.

Thus, for $\theta = 2$, one has $\overline{q}_2(u) = u^2(1 - 2\log u)$ and $\overline{q}_2'(u) = -4u\log u$. It follows that

$$\alpha_2(u) = -4\frac{\log u}{1 - 2\log u}$$

which is decreasing in $(0, 1)$. Therefore, by Proposition 4, if $\overline{F}$ is IFR (i.e., if $\lambda^{(0)}(t)$ is increasing), then M is IFR as well.

For $\theta \neq 2$ one has $\overline{q}_\theta(u) = \frac{1}{2-\theta}(2u^\theta - \theta u^2)$ and $\overline{q}_\theta'(u) = \frac{1}{2-\theta}(2\theta u^{\theta-1} - 2\theta u)$. It follows that

$$\alpha_\theta(u) = \frac{2\theta u^\theta - 2\theta u^2}{2u^\theta - \theta u^2},$$

which is also decreasing in $(0, 1)$ for every strictly positive θ, with $\theta \neq 2$. Therefore, again by Proposition 4, if $\overline{F}$ is IFR (i.e., if $\lambda^{(0)}(t)$ is increasing), then M is IFR as well.

It is interesting to observe that this property is satisfied also for $\theta \in (0, 1)$ even if in this case the marginal distributions of X_1 and X_2 can be not IFR (for example, if $\lambda^{(0)}(t) = \lambda \in \mathbb{R}^+$, then at the failure of one component the hazard of the remaining one reduces to $\theta\lambda$, which is smaller than λ).

For some specific choices of $\lambda^{(0)}(t)$, and therefore of $\overline{F}$, one can verify that M is also ILR. For example, if $\theta = 2$ and $\lambda^{(0)}(t) = bkt^{k-1}$ for $t \geq 0$ (which is the case of a Weibull distribution), then

$$\log \overline{q}_2'(\overline{F}(t)) = \log(4bt^k \exp(-bt^k)) = \log(4b) + k\log t - bt^k$$

for $t \geq 0$. Since this function is concave for any positive b and k then, by Proposition 5, one has that M is also ILR.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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