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Twisted and intermediate eigenvalue problems in shape optimization

Problèmes de valeurs propres tordues et intermédiaires dans l'optimisation de formes

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Declaration of Authorship

I hereby declare that this doctoral dissertation does not compromise in any way the rights of third parties, including those relating to the security of personal data. Its content is the product of my own research work carried out during my doctoral period in collaboration with several researchers: Beniamin Bogosel (University Aurel Vlaicu), Dorin Bucur (Université Savoie Mont Blanc), and Davide Zucco (Università di Torino).

Part of the results presented herein are contained in

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Abstract

This thesis is devoted to the study of *twisted* and *intermediate* eigenvalue problems in the context of shape optimization. These problems generalize the classical Dirichlet eigenvalue problems on bounded (quasi-)open sets with additional orthogonality constraints. On the one hand, several well-known properties of the Dirichlet setting, such as the L^∞-bounds for the eigenfunctions à la Davies, extend to this more general framework. On the other hand, not all classical results persist: for instance a Courant-type nodal domain theorem does not hold in general, contrary to prevailing expectations in the mathematical community. Indeed, it is shown a bounded domain (connected open set) whose first twisted (or intermediate) eigenvalue is simple and the corresponding eigenfunction possesses an arbitrary number of nodal domains.

The core of the thesis focuses on the analysis of shape optimization problems for twisted and intermediate eigenvalues. The main results concern both the existence and the non-existence of optimal shapes for the minimization of the k-th twisted and intermediate eigenvalues under volume constraint. Some existence results are obtained in a *local* setting, namely among sets contained in a fixed *box*, while others hold *globally*, over the entire space. These existence results are derived from the introduction of a generalized notion of γ-convergence: while in the twisted case the resulting γ-convergence theory closely parallels the classical Dirichlet setting, in the intermediate case only some of its features persist. Moreover, non-existence results arise from a loss of compactness in the class of admissible sets and can be established via suitable comparison arguments. In addition, a methodology for the numerical computation of optimal shapes within the twisted framework will be presented.

Then, the minimization of the first twisted eigenvalue is studied, not only with respect to the set but also to a single orthogonality function. The minimum value, among sets of fixed volume and functions of given L^∞-bounds, is uniquely reached by a suitable pair of balls and a *bang-bang* function. As a consequence, a one-parameter family of isoperimetric inequalities which, in a sense, interpolates between the classical Faber-Krahn and the Hong-Krahn-Szego inequalities is derived.

Finally, a proof of the Alt–Caffarelli–Friedman monotonicity formula is provided. While this formula is primarily applied in free-boundary problems, it is also relevant for the Lipschitz regularity of (sign-changing) Dirichlet eigenfunctions, thereby suggesting additional applications to the regularity problems considered in this thesis.

Riassunto

Questa tesi è dedicata allo studio di problemi agli autovalori *contorti* ed *intermedi* nel contesto dell'ottimizzazione di forma. Questi problemi generalizzano i classici problemi degli autovalori di Dirichlet su insiemi (quasi-)aperti limitati con ulteriori vincoli di ortogonalità. Da un lato, diverse proprietà ben note nell'ambito Dirichlet, come le stime L^∞ per le autofunzioni à la Davies, si estendono a questo quadro più generale. Dall'altro lato, non tutti i risultati classici persistono: per esempio, un teorema di tipo Courant sui domini nodali non vale in generale, contrariamente alle aspettative diffuse nella comunità matematica. Infatti, si è dimostrata l'esistenza di un dominio (insieme aperto connesso) limitato il cui primo autovalore è semplice e l'autofunzione contorta (o intermedia) corrispondente possiede un numero arbitrario di domini nodali.

Il nucleo della tesi è incentrato sull'analisi dei problemi di ottimizzazione di forma per gli autovalori contorti e intermedi. I risultati principali riguardano sia l'esistenza che la non esistenza di forme ottimali per la minimizzazione del k-esimo autovalore contorto e intermedio soggetto a vincolo di volume. Alcuni risultati di esistenza sono ottenuti in un contesto *locale*, ovvero tra insiemi contenuti in un *dominio* fissato, mentre altri valgono *globalmente*, nell'intero spazio. Questi risultati di esistenza sono derivati tramite l'introduzione di una nozione generalizzata di γ-convergenza: mentre nel caso contorto la teoria della γ-convergenza risultante è molto simile al contesto classico di Dirichlet, nel caso intermedio solo alcune delle sue caratteristiche persistono. I risultati di non esistenza derivano da una perdita di compattezza nella classe degli insiemi ammissibili e possono essere provati tramite opportuni argomenti di confronto. Inoltre verrà presentata, nel contesto contorto, una metodologia per il calcolo numerico delle forme ottimali.

In seguito la minimizzazione del primo autovalore contorto è studiata, non solo rispetto all'insieme ma anche sul vincolo di ortogonalità. Il valore minimo, tra gli insiemi di volume fissato e vincoli di ortogonalità soddisfacenti stime L^∞, è raggiunto unicamente da una coppia adeguata di palle e da una funzione *bang-bang.* Come conseguenza otteniamo una famiglia di disuguaglianze isoperimetriche a un parametro che, in un certo senso, interpola tra le disuguaglianze classiche di Faber-Krahn e Hong-Krahn-Szego.

Infine, viene fornita una dimostrazione della formula di monotonia di Alt-Caffarelli-Friedman. Nonostante questa formula sia utilizzata principalmente nei problemi di frontiera libera, essa è anche utile nel provare la Lipschitzianità delle autofunzioni di Dirichlet (che cambiano segno), suggerendo ulteriori applicazioni ai problemi di regolarità considerati in questa tesi.

Résumé

Cette thèse est consacrée à l'étude des problèmes de valeurs propres *tordues* et *intermédiaires* dans le contexte de l'optimisation de forme. Ces problèmes généralisent les problèmes classiques de valeurs propres de Dirichlet sur des ensembles (quasi-)ouverts bornés avec des contraintes d'orthogonalité supplémentaires. D'une part, plusieurs propriétés bien connues du cadre Dirichlet, telles que les bornes L^∞ pour les fonctions propres *à la Davies*, s'étendent à ce cadre plus général. D'autre part, tous les résultats classiques ne persistent pas: par example, un théorème de domaines nodaux de type Courant ne se vérifie pas en général, contrairement aux attentes dominantes dans la communauté mathématique. En effet, il est démontré l'existence d'un domain (ensemble connexe et ouvert) borné, dont la première valeur propre tordue (ou intermédiaire) est simple et sa fonction propre correspondante possède un nombre arbitraire de domaines nodaux.

Le coeur de la thèse se concentre sur l'analyse des problèmes d'optimisation de forme pour les valeurs propres tordues et intermédiaires. Les principaux résultats concernent à la fois l'existence et la non existence de formes optimales pour la minimisation des k-ième valeurs propres tordues et intermédiaires sous contraintes de volume. Certains résultats d'existence sont obtenus dans un cadre *local*, à savoir parmi des ensembles contenus dans une *boîte* fixe, tandis que d'autres s'appliquent *globalement*, sur l'ensemble de l'espace. Ces résultats d'existence découlent de l'introduction d'une notion généralisée de γ-convergence: alors que dans le cas tordu, la théorie de γ-convergence qui en résulte est très similaire au cadre classique de Dirichlet, dans le cas intermédiaire, seules certaines propriétés persistent. De plus, les résultats de non-existence découlent d'une perte de compacité dans la classe des ensembles admissible et peuvent être établis grace à l'aide d'arguments appropriés de comparaison. En outre, dans le cadre tordu, sera présentée une méthodologie pour le calcul numérique des formes optimales.

Ensuite, on étudie la minimisation de la première valeur propre tordue, non seulement par rapport à l'ensemble, mais aussi par rapport à une seule fonction d'orthogonalité. La valeur minimale, parmi les ensembles de volume fixé et de fonctions ayant des bornes données en L^∞, est atteinte uniquement par une paire de boules appropriée et une fonction *bang-bang*. En conséquence, est derivée une famille à un paramètre d'inégalités isopérimétriques qui, dans un certain sens, interpolent entre les inégalités classiques de Faber-Krahn et celles de Hong-Krahn-Szego.

Enfin, une preuve de la formule de monotonicité d'Alt-Caffarelli-Friedman est fournie. Alors que cet outil soit principalement utilisé dans les problèmes à frontière libre, nous rappelons également qu'il est aussi pertinent pour la régularité Lipschitz des fonctions propres de Dirichlet (à signe variable), suggérant d'autres applications aux problèmes de régularité considérés dans cette thèse.

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Contents

Chapter 1

Introduction

A wide variety of problems, arising from both pure and applied mathematics, involve the optimization of quantities with respect to functions subject to appropriate *orthogonality constraints*. A typical situation occurs with the so-called *min-max principles* ([38, 112]), which provide variational characterizations of the eigenvalues of compact self-adjoint operators on Hilbert spaces: the k-th eigenvalue of such operators can be obtained by minimizing the Rayleigh quotient among functions that are orthogonal to the $(k-1)$ eigenfunctions $u_1, u_2, ..., u_{k-1}$ corresponding to the first $k-1$ eigenvalues. Of course, to compute the k-th eigenvalue in this way, the $k-1$ eigenfunctions must be known beforehand.

Another classical problem, as studied by Lagrange during the 1770s, consists in finding the optimal shape of a column, by optimizing over functions with *zero mean value* (i.e., those that are orthogonal to constant functions in the L^2 sense). Lagrange was interested in the strongest circular elastic solid column with fixed volume and pinned ends (see [83]):

> *Ainsi c'est un Problème de maximis et minimis de déterminer la courbe, qui par sa rotation autour de son axe formera une colonne capable de supporter la plus grande charge possible, la hauteur et la masse de la colonne étant données ; c'est là, ce me semble, le véritable point de vue sous lequel on doit envisager la question du renflement et de la diminution des colonnes.*

The *buckling load of a column*, namely the maximal charge that the column can sustain before it begins to deflect, can be modeled as a one dimensional fourth-order problem involving the minimization of a Rayleigh-type quotient that depends on the column profile. The pinned end condition is equivalent to imposing Dirichlet boundary conditions on both the transversal deflection and its derivative. By focusing on the derivative u of the transversal deflection, such fourth-order problem can be written as a second-order one and in addition to the Dirichlet boundary condition for u, one needs to require that

$$\int_0^1 u(x)\,dx = 0.$$

This orthogonality condition follows from the requirement that the derivative of the transversal deflection also vanishes at the boundary. More details and recent progress on this problem can be found in [48, Introduction], [69, Section 11.4.2], and [74, Section 1.1.1].

More recently, problems with multiple orthogonality constraints are being also studied in the some contexts related hydrodynamics as in [56]. In this case the constraints are infinitely many and are taken with respect to the *harmonic functions*. More generally in [71], where vector fields are considered instead of functions, it is required an orthogonality condition with *harmonic vector fields*.

Even certain topics in Quantum Mechanics, particularly those related to symmetry of solutions of specific problems [92], motivate the study of optimization problems subject to some orthogonality constraints (the next section contains more details and references on this aspect).

The purpose of this thesis is therefore to focus on analogous problems, more closely related to eigenvalue problems, in which an orthogonality constraint may be present either in the functional or in the class of competitors. Two important quantities that fit within this framework are the so-called *twisted* and *intermediate eigenvalues*. In this introduction, we will set up the framework and present the main results of the thesis. The order in which they are presented is chosen solely for clarity of exposition and does not reflect the order in which they will be proved throughout the the thesis.

1.1 Twisted and intermediate eigenvalues

Intermediate eigenvalues were first introduced in 1937 by Alexander Weinstein in his PhD thesis [111]. His main idea was to derive lower bounds for the eigenvalues of biharmonic operators (modeling vibrations of clamped plates or buckling loads) exploiting modified second-order eigenvalue problems involving the Dirichlet Laplacian. This approach was later formalized by Weinstein and Stenger in their 1972 monograph [112]. Over the following decades, several important contributions to this theory appeared, notably those of Weyl [113], Aronszajn [7], Gould [62], and Bazley [14, 15], among others.

With the advent of computers and the rise of numerical methods in engineering, theoretical lower bounds were progressively supplanted by highly accurate numerical computations. Our motivation to study these problems remains primarily theoretical. A number of isoperimetric inequalities à la Faber–Krahn for biharmonic operators remain unsolved, these include the minimization of the first eigenvalue of the clamped plate in dimensions $d \geq 4$ and of the first buckling eigenvalue in dimensions $d \geq 2$, see [69, Chapter 11]. In this context, shape optimization and free boundary formulations of intermediate eigenvalue problems provide, at least *a priori*, a more tractable framework reducing the original fourth-order problem to a second order one and may yield particularly insightful observations.

The rough idea of how intermediate eigenvalues can approximate buckling eigenvalues is as follows (see [112, Chapter 4, Section 12, equations (10) and (11)] for the original description of Weinstein in the planar case). Assume Ω be a smooth open set of $\mathbb{R}^d$ and

$\{g_i\}_{i\in\mathbb{N}}$ be a family of harmonic polynomials, that is dense in $L^2(\partial\Omega)$. Let us pick the first ℓ functions $g_1, \dots, g_\ell$ and consider the following formal eigenvalue problem (with unknowns u, λ, a_i)

$$\begin{cases} -\Delta u = \lambda \left(u - \sum_{i=1}^{\ell} a_i g_i \right) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \\ \Delta u \perp_{L^2(\Omega)} g_i & \text{for every } i = 1, \dots, \ell. \end{cases} \tag{1.1}$$

From the last orthogonality condition, it comes out that $\sum_{i=1}^{\ell} a_i g_i$ is the $L^2(\Omega)$-projection of u on span$\{g_1, \dots, g_l\}$. Applying the Laplace operator to the equation satisfied by u in (1.1), one gets an eigenvalue equation for the biharmonic operator (specifically the buckling load):

$$-\Delta^2 u = \lambda \Delta u.$$

Moreover, the L^2-orthogonality of Δu and g_i together with $u = 0$ on $\partial\Omega$, gives

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} g_i \, dx = 0 \qquad \text{for every } i = 1, \dots, \ell,$$

ν denoting the outward unit normal of $\partial\Omega$. Then, as $\ell \to +\infty$, the boundary conditions expected to be satisfied in the limit are $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and this would mean that the eigenvalues of (1.1) should converge to the so-called buckling eigenvalues (see [11, Section 15.1]). A more detailed treatment of this topic can be found in Section 2.3.

Now, on the contrary, if the orthogonality $\Delta u \perp_{L^2(\Omega)} g_i$ in (1.1) is replaced by

$$u \perp_{L^2(\Omega)} g_i$$

the eigenvalue problem becomes what is known as *twisted* (in French, *tordu*; see [17]). Twisted problems for the sole ($\ell = 1$) orthogonality constraint $g_1 = 1$ have been introduced by Barbosa and Bérard in [12], for studying the stability of *constant mean curvature (CMC) immersions*, and have attracted considerable attention over the past 25 years (see for example [1, 75, 102]). Twisted eigenvalues are closely related to the intermediate ones and share some common properties; in particular, they formally solve a same type of equation

$$-\Delta u = \lambda \left(u - \sum_{i=1}^{\ell} a_i g_i \right), \text{ with } u \in H_0^1(\Omega), \tag{1.2}$$

albeit with different (non-local) definition for the coefficients a_i. From the last orthogonality condition, it comes out that $\sum_{i=1}^{\ell} a_i g_i$ is the $L^2(\Omega)$-projection of Δu on span$\{g_1, \dots, g_l\}$. Moreover, in one dimension ($d = 1$) with Ω an interval and for one ($\ell = 1$) orthogonality constraint $g_1 = 1$ it was shown in [41, Preuve du Lemme 2.3 avec $p = q = 2$, Deuxième étape] that the first twisted and intermediate eigenvalue are the same, but the related eigenspaces not (in particular the intermediate eigenspace can be strictly larger, see Example 2.1.10 in Chapter 2).

We now properly introduce the intermediate and twisted eigenvalue problems. Denote

$$\mathcal{G} = \{g_1, \dots, g_\ell\},\ \mathcal{G} \subset L^2_{\mathrm{loc}}(\mathbb{R}^d),\ \dim(\operatorname{span}\mathcal{G}) = \ell, \tag{1.3}$$

the *orthogonality class* of functions (which should be orthogonal to u or to Δu in the intermediate and twisted settings, respectively). Sometimes, as will be specified throughout the thesis, it will be convenient to consider also the situation where

$$g_i \text{ is a homogeneous harmonic polynomial for every } i \in \{1, 2, \dots, \ell\}. \tag{1.4}$$

If (1.4) holds (in addition to (1.3)), the linear span of the restrictions of the functions of $\mathcal{G}$ to any measurable set of positive measure is also of dimension ℓ. Then, let $\operatorname{span}(\mathcal{G}|_\Omega)$ be the linear span of the restrictions to Ω of the functions of $\mathcal{G}$ and let

$$P^{\mathcal{G}}_\Omega : L^2(\Omega) \to L^2(\Omega), \tag{1.5}$$

be the L^2*-orthogonal projection* onto $\operatorname{span}(\mathcal{G}|_\Omega)$ defined for every $f \in L^2(\Omega)$ as the unique element $P^{\mathcal{G}}_\Omega f \in \operatorname{span}(\mathcal{G}|_\Omega)$ such that

$$\int_\Omega (f(x) - P^{\mathcal{G}}_\Omega f(x)) g(x)\, dx = 0 \ \text{ for every } g \in \operatorname{span}(\mathcal{G}|_\Omega), \tag{1.6}$$

see [26, Corollary 5.4] for more details. In particular it holds the *Pythagorean theorem*

$$||u||^2_{L^2(\Omega)} = ||P^{\mathcal{G}}_\Omega u||^2_{L^2(\Omega)} + ||u - P^{\mathcal{G}}_\Omega u||^2_{L^2(\Omega)}, \tag{1.7}$$

see [82, p. 152, relation (3)].

Dirichlet problems for the Laplace operator are rigorously formulated in Sobolev spaces defined over quasi-open sets (see [29, Section 6.1] or [109, Definition 2.3.15]). Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be a bounded quasi-open set and $\mathcal{G}$ be as in (1.3). Since the embedding of $H^1_0(\Omega)$ in $L^2(\Omega)$ is compact, see [30, Proposition 3.3, condition (7)] (in particular this follows reasoning as in [30, Examples 3.4 and 3.5]). We can consider the following quantities.

Dirichlet Laplacian eigenvalues. The value

$$\lambda^D_k(\Omega) = \min_{S \in \mathcal{S}_k} \max_{\substack{u \in S \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega u(x)^2\, dx}, \tag{1.8}$$

where $\mathcal{S}_k$ is the family of subspaces of dimension k in the space $H^1_0(\Omega)$, is called *k-th (Dirichlet) eigenvalue of* Ω. A function $u \in H^1_0(\Omega)$, critical point for the *Rayleigh quotient* on $H^1_0(\Omega)$, namely a solution of (1.8), is called *eigenfunction corresponding to* $\lambda^D_k(\Omega)$. The corresponding eigenvalue problem formally reads

$$\begin{cases} -\Delta u &= \lambda^D_k(\Omega) u \quad \text{in } \Omega \\ \quad u &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

These values form a countable non-decreasing sequence $(\lambda_n^D(\Omega))_n$, whose elements, ordered with multiplicity, satisfy

$$0 < \lambda_1^D(\Omega) \le \lambda_2^D(\Omega) \le \cdots \le \lambda_k^D(\Omega) \le \ldots,$$

with $\lambda_k^D(\Omega) \to \infty$ as $k \to \infty$.

Intermediate eigenvalues. The value

$$\lambda_k^{I,\mathcal{G}}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{\substack{u \in S \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega (u(x) - P_\Omega^{\mathcal{G}} u(x))^2 \, dx}, \tag{1.10}$$

where $\mathcal{S}_k$ is the family of subspaces of dimension k in the space $H_0^1(\Omega)$ and $P_\Omega^{\mathcal{G}}$ is defined as in (1.5), is called *k-th intermediate eigenvalue of* Ω (the min-max characterization is justified by [9, Theorem 6.1]). A function $u \in H_0^1(\Omega)$, critical point for a *modified Rayleigh quotient* on $H_0^1(\Omega)$, namely a solution of (1.10), is called *(intermediate) eigenfunction corresponding to* $\lambda_k^{I,\mathcal{G}}(\Omega)$. The corresponding eigenvalue problem formally reads

$$\begin{cases} -\Delta u &= \lambda_k^{I,\mathcal{G}}(\Omega)(\mathrm{Id} - P_\Omega^{\mathcal{G}})(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where Id is the identity map. These values form a countable non-decreasing sequence $(\lambda_n^{I,\mathcal{G}}(\Omega))_n$, whose elements, ordered with multiplicity, satisfy

$$0 < \lambda_1^{I,\mathcal{G}}(\Omega) \le \lambda_2^{I,\mathcal{G}}(\Omega) \le \cdots \le \lambda_k^{I,\mathcal{G}}(\Omega) \le \ldots,$$

with $\lambda_k^{I,\mathcal{G}}(\Omega) \to \infty$ as $k \to \infty$.

As we noticed before, a primary motivation for studying these types of eigenvalues comes from their relationship with buckling eigenvalues, see Section 2.3. It is a long-standing open question to prove that, among bounded and open sets in $\mathbb{R}^d$ of unit measure, the (unit) ball B minimizes the first buckling eigenvalue (see also Open problem 1), in formula

$$\Lambda_1(B) = \min\{\Lambda_1(\Omega) : \Omega \subset \mathbb{R}^d, \, \Omega \text{ open}, |\Omega| = 1\},$$

where

$$\Lambda_1(\Omega) = \min_{\substack{u \in H_0^2(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega |\nabla u|^2 \, dx} \tag{1.12}$$

is the *first buckling eigenvalue* (or *buckling load*) of Ω that represents a critical value for the deflection of a clamped plate subjected to compressive forces, see [11, Section 15.1]. Efforts to prove this conjecture began in the 1950s with the paper [106] and the well-known book of Pólya and Szegő, [100, Note F]. Recent developments concerning this problem are contained in the books [11, Chapter 15] and [76, Section 4.6]. The crucial remark is that, if a suitable choice of harmonic functions happens to yield a ball as the minimizer of the first intermediate eigenvalue, then the conjecture for the first buckling eigenvalue would follow as a direct consequence. Therefore, one may take advantage of working with intermediate eigenvalues, which are of lower order (second-order rather than fourth-order). A possible strategy for proving this open problem, would be to study the sequence of local minimizers for the intermediate problem as the number of orthogonality constraints increases (this is somehow similar to the strategy followed by Colbois and El Soufi in [37] for the Pólya conjecture). We point out that in the case $\mathcal{G} = \{1\}$ two results are available in the literature: in [63] Greco and Lucia have proved that the minimizer of the first intermediate eigenvalue is given by the union of two equal disjoint balls; in [31] Bucur, Durus, and Oudet have proved the existence of an optimal shape for the k-th intermediate eigenvalue (among sets contained in a *box*). In this thesis we prove a non-existence result for multiple harmonic constraint, see Theorem 1.4.4.

Twisted eigenvalues. The value

$$\lambda_k^{T,\mathcal{G}}(\Omega) = \min_{S\in\mathcal{S}_{k,\mathcal{G}}} \max_{\substack{u\in S\\ u\not\equiv 0}} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega u(x)^2\,dx}, \tag{1.13}$$

where $\mathcal{S}_{k,\mathcal{G}}$ is the family of subspaces of dimension k in the space $H^1_{0,\mathcal{G}}(\Omega)$, defined as

$$H^1_{0,\mathcal{G}}(\Omega) = \left\{ u \in H^1_0(\Omega) : \int_\Omega u(x)g(x)\,dx = 0 \text{ for every } g \in \mathcal{G} \right\}, \tag{1.14}$$

is called *k-th twisted eigenvalue of* Ω (the min-max characterization is justified the theory developed in Subsection 3.2.1). The space $H^1_{0,\mathcal{G}}(\Omega)$ is closed in $H^1_0(\Omega)$ so that it is separable and the embedding of $H^1_{0,\mathcal{G}}(\Omega)$ in $L^2(\Omega)$ is also compact. Roughly speaking, the space $H^1_{0,\mathcal{G}}(\Omega)$ is defined, as the orthogonal complement of the class $\mathcal{G} \subset L^2_{\text{loc}}(\mathbb{R}^d)$, with respect to the scalar product of $L^2(\mathbb{R}^d)$, within the Sobolev space $H^1_0(\Omega)$, or equivalently as the space of $H^1_0(\Omega)$ functions which are orthogonal in $L^2(\mathbb{R}^d)$ to the functions in $\mathcal{G}$ (as usual, functions in $H^1_0(\Omega)$ are extended by 0 outside Ω). A function $u \in H^1_0(\Omega)$, critical point for the Rayleigh quotient on $H^1_{0,\mathcal{G}}(\Omega)$, namely a solution of (1.13), is called *(twisted) eigenfunction corresponding to* $\lambda_k^{T,\mathcal{G}}(\Omega)$. The corresponding eigenvalue problem formally reads

$$\begin{cases} (\mathrm{Id} - P^{\mathcal{G}}_\Omega)(-\Delta u) &= \lambda_k^{T,\mathcal{G}}(\Omega) u \quad \text{in } \Omega \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Twisted eigenvalues form a countable non-decreasing sequence $(\lambda_n^{T,\mathcal{G}}(\Omega))_n$, whose elements, ordered with multiplicity, satisfy

$$0 < \lambda_1^{T,\mathcal{G}}(\Omega) \le \lambda_2^{T,\mathcal{G}}(\Omega) \le \cdots \le \lambda_k^{I,\mathcal{G}}(\Omega) \le \ldots,$$

with $\lambda_k^{T,\mathcal{G}}(\Omega) \to \infty$ as $k \to \infty$. When the class $\mathcal{G} = \{g\}$ is given by a single function g and $k = 1$ with a little abuse of notation we write $\lambda_1^{T,g}(\Omega)$ instead of $\lambda_1^{T,\{g\}}(\Omega)$ (this convention will always be used throughout the thesis) and (1.15) writes as

$$\begin{cases} -\Delta u = \lambda_1^{T,g}(\Omega)u + \left(\dfrac{\int_\Omega -\Delta u(x)g(x)dx}{\int_\Omega g(x)^2dx}\right) g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

when $g \not\equiv 0$, while it becomes $-\Delta u = \lambda_1^{T,g}(\Omega)u$ in Ω with $u = 0$ on $\partial\Omega$, when $g \equiv 0$ (i.e. u is a Dirichlet eigenfunction). Due to the presence of an average of the Laplacian over Ω, equations like (1.16) are referred to as *non-local* (see, for instance, [23, Introduction] and the references therein). The number of twisted eigenfunctions corresponding to the same twisted eigenvalue $\lambda_1^{T,g}(\Omega)$ is called the *multiplicity* of $\lambda_1^{T,g}(\Omega)$. If the multiplicity is one, the eigenvalue is *simple*; if it is two, the eigenvalue is *double*, and so on. In general, $\lambda_1^{T,g}(\Omega)$ can be *multiple*, meaning that it is not simple (for instance, if Ω is a disk in $\mathbb{R}^2$ and $g \in \{1, u_1\}$, then $\lambda_1^{T,g}(\Omega)$ is double, see [12, Example 3]).

Some possible physical interpretations of these twisted eigenvalues are as follows. In Quantum Mechanics, the eigenvalues of the Dirichlet Laplacian correspond to the energy levels of a particle confined in Ω with infinite walls (i.e., an infinite potential barrier at the boundary), with the corresponding eigenfunctions representing the associated quantum states: $\lambda_1^D(\Omega)$ is the ground-state energy, $\lambda_2^D(\Omega)$ the first excited-state energy after the ground-state energy, and more generally, $\lambda_k^D(\Omega)$ the $(k-1)$-th excited-state energy after the ground-state energy. The eigenvalue $\lambda_k^{T,\mathcal{G}}(\Omega)$ can be regarded as an intermediate energy where the orthogonality constraint required in (1.14) may be related to the so-called *selection* and *superselection rules* (see for instance the rule Ss1 and the relation (7.6.1) in [92, Section 7.7]), which for various reasons (symmetries, external fields, etc...) certain transitions are forbidden and we know a priori that the wave-function of the state must be orthogonal to some subspace: for example $\lambda_1^{T,\mathcal{G}}(\Omega)$ may represent the minimum energy that such a state can achieve. Another interesting motivation comes from the *stability theory* for symmetric solutions of dispersive equations (more specifically, standing waves and solitons for nonlinear Schrödinger equations, or even waves for Korteweg-de Vries-type equations), where one needs to study the spectrum of differential operators on suitable Hilbert spaces that are constrained. This means that one studies the spectrum of a certain operator associated with the quadratic part of the energy (for instance its first eigenvalue, the number of negative eigenvalues, or whether there are any zero eigenvalues) that is subjected to an orthogonality constraint with respect to a given function or a set of functions. Therefore, this constraint is typically expressed through an orthogonality with respect to a given set of functions and it is motivated by the fact that these functions form an element of the kernel, of another operator related to the same linearization of the energy, see for instance [39]. The theory was developed by M. Weinstein and independently by M. Grillakis, J. Shatah and W. Strauss in the late 1980s, and it was subsequently greatly expanded in the following years by numerous authors, with countless applications to linear and orbital stability theory, for more details see [78, Section 5.2] and [96, Section 4.2].

Since the pionnering paper by Barbosa and Bérard [12] a number of works have been devoted to investigating twisted eigenvalues. For instance, Freitas and Henrot showed in [58] that the minimal first twisted eigenvalue for $\mathcal{G} = \{1\}$, under a volume constraint, is attained by the union of two equal disjoint balls (see also [23] for a weighted version of this result). We also point out, in the context of shape optimization, the work of Brandolini, Freitas, Nitsch and Trombetti [22] on the approximation of twisted eigenvalues via penalization of the orthogonality constraint, where the authors notice a surprising saturation phenomenon. Quantities of this type were already studied in the one-dimensional case in [57] and in higher dimensions in [2].

A theoretical motivation for studying twisted eigenvalues stems from the following conjecture: among bounded and open sets in $\mathbb{R}^d$ of unit measure (with $d \geq 2$), the (unit) ball B minimizes the $(d+1)$-th Dirichlet eigenvalue, in formula

$$\lambda^D_{d+1}(B) = \min\{\lambda^D_{d+1}(\Omega) : \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open}, |\Omega| = 1\},$$

see [8, Open problem 7, p. 18] or [70, Open problem 11.4, p. 408]. The problem of minimizing the third Dirichlet eigenvalue (with a measure constraint) arises as a natural question in dimension 2 after the discovery of the optimal shapes for the first two Dirichlet eigenvalues (with a measure constraint), see [69, Theorem 3.2.1] and [69, Theorem 4.1.1]. Existence of an optimal set for the third Dirichlet eigenvalue in any dimension was proved in [32]. Later, almost simultaneously, the existence of an optimal shape for higher Dirichlet eigenvalues was proved by Bucur in [28] and by Mazzoleni and Pratelli in [89]. Numerical computations, see [70, Sections 11.3 and 11.4], support the conjecture. Interestingly, if the ball minimizes the d-th twisted eigenvalue for $\mathcal{G} = \{1\}$, then the conjecture follows, see Section 3.5 and Open problem 2. Why might minimizing the d-th twisted eigenvalue be more tractable than addressing the original problem directly? The key point is that, in the classical Dirichlet setting, one must enforce orthogonality conditions on $d+1$ unknown eigenfunctions, whereas in the twisted case, the orthogonality is required with respect to the constant function 1 and only d unknown eigenfunctions.

We conclude by noticing that these eigenvalues do not change if the functions in $\mathcal{G}$ are modified in a set of measure zero. Moreover, any alteration of the values attained by a function in $\mathcal{G}$ in $\mathbb{R}^d \setminus \Omega$ do not affect them. We choose functions in $L^2_{\text{loc}}(\mathbb{R}^d)$, instead of $g \in L^2(\Omega)$, since in the following we will need to perform scalings and translations. In addition twisted and intermediate eigenvalues share with Dirichlet eigenvalues several properties, such as the *domain monotonicity with respect to the set inclusion*, $\mathcal{C} \in \{I, T\}$, if $\omega \subset \Omega$ then

$$\lambda^{\mathcal{C},\mathcal{G}}_k(\omega) \geq \lambda^{\mathcal{C},\mathcal{G}}_k(\Omega)$$

and they exhibit well-behaved *scaling properties*

$$\lambda^{\mathcal{C},s\mathcal{G}}_k(\Omega) = \lambda^{\mathcal{C},\mathcal{G}}_k(\Omega), \quad \lambda^{\mathcal{C},\mathcal{G}}_k(s\Omega) = \frac{1}{s^2}\lambda^{\mathcal{C},\mathcal{G}_s}_k(\Omega),$$

where $g_s(x) := g(s \cdot x)$ for every $x \in \mathbb{R}^d$ are *inner* scalings of the function g, see Lemmas

2.1.1 and 2.1.2. When $\mathcal{G} = \{0\}$ both intermediate and twisted eigenvalues reduce to the Dirichlet eigenvalues, that is

$$\lambda_k^{D,\mathcal{G}}(\Omega) = \lambda_k^{I,\mathcal{G}}(\Omega) = \lambda_k^{T,\mathcal{G}}(\Omega),$$

while for generic $\mathcal{G}$ one can prove the following chain of inequalities:

$$\lambda_k^{D,\mathcal{G}}(\Omega) \leq \lambda_k^{I,\mathcal{G}}(\Omega) \leq \lambda_k^{T,\mathcal{G}}(\Omega) \leq \lambda_{k+\ell}^{T,\mathcal{G}}(\Omega).$$

Compared to the same index k, the Dirichlet eigenvalue is the smallest one, while the twisted eigenvalue the largest one. These inequalities will be proved in Lemma 2.1.3. Moreover if $\mathcal{G}_1, \mathcal{G}_2$ are as in (1.3) and $\mathcal{G}_1 \subset \mathcal{G}_2$ one has *monotonicity with respect to the orthogonality class*

$$\lambda_k^{\mathcal{C},\mathcal{G}_1}(\Omega) \leq \lambda_k^{\mathcal{C},\mathcal{G}_2}(\Omega).$$

In the next section we will see that not all properties valid for Dirichlet eigenvalues can be extended to the twisted and intermediate settings.

In order to simplify our notation, when $\mathcal{G}$ is fixed and there is no ambiguity, we drop the letter $\mathcal{G}$ in the notation of twisted and intermediate eigenvalues. Generically, we adopt the notation $\lambda_k^{\mathcal{C}}(\Omega)$, where $\mathcal{C} \in \{D, I, T\}$ refers to the *category* of the eigenvalue: classical Dirichlet, intermediate or twisted, respectively. Analogously, when the family $\mathcal{G}$ is fixed and we want to indicate that a function is an eigenfunction corresponding to an eigenvalue of the types above without saying explicitly the number of the eigenvalue we write respectively: a Dirichlet eigenfunction of Ω, an intermediate eigenfunction of Ω and a twisted eigenfunction of Ω.

1.2 On Courant's nodal domain theorem in twisted and intermediate settings

In contrast with the previous section, where we established several properties of Dirichlet eigenvalues that naturally extend to the twisted and intermediate settings, we present here a surprising result that highlights how twisted and intermediate eigenvalues can differ from Dirichlet eigenvalues. This result concerns the number of *nodal domains* (i.e., the connected components of the set where the eigenfunction is not zero) of a given eigenfunction. The celebrated *Courant's nodal domain theorem* says that the number of nodal domains of an eigenfunction associated with a k-th Dirichlet eigenvalue (1.8) should be less than or equal to k. It is a very delicate issue to extend this result in the twisted and intermediate settings. Indeed, we will show that a general form of this nodal domain theorem cannot hold for twisted and intermediate eigenfunctions: there exists a domain Ω such that the k-th eigenfunction of a twisted or intermediate problem may have an arbitrary number of nodal domains. This can be formulated more precisely as in the following result, that will be proved in Section 3.4.

Theorem 1.2.1 (Failure of the Courant's nodal domain theorem)**.** *Let* $\mathcal{G} = \{1\}$ *and* $\mathcal{C} \in \{I, T\}$*. For every* $n \in \mathbb{N}$*, there exists a bounded domain* $\Omega \subset \mathbb{R}^2$ *with Lipschitz boundary such that the eigenvalue* $\lambda_1^{\mathcal{C}}(\Omega)$ *is simple and a eigenfunction* u *corresponding to* $\lambda_1^{\mathcal{C}}(\Omega)$ *has at least* $n+1$ *nodal domains.*

Let us notice that this seems to contradict what stated in [12, Proposition 2.2 (3) and Remark 1], namely a Courant-type theorem for twisted eigenfunctions: a twisted eigenfunction corresponding to $\lambda_k^{T,1}(\Omega)$ has at most $(k+1)$ nodal domains[1].

Related results have also been investigated more recently within a broader generalized setting, including the intermediate eigenvalues corresponding to $\mathcal{G} = \{1\}$. More precisely, in [13] the authors study a constrained differential equation depending on a non-negative parameter and on a smooth non-negative function with strictly positive first derivative. As particular cases, they recover the mean-field equations in dimension two, as well as solutions that are relevant for the analysis of problems arising in plasma physics. They then investigate the associated linearized operator and its eigenfunctions. In this context, the authors establish partial Courant-type results in certain very specific settings (the case of a ball and radial eigenfunctions) and explicitly raise the question of determining upper bounds for the number of nodal domains of an eigenfunction; see [13, p. 5]. We note that, in the case considered in [13, with $\lambda = 0$ and $f(x) = x^+$], the associated linearized operator leads to the intermediate eigenvalue problem with $\mathcal{G} = 1$. Therefore, by Theorem 1.2.1, such an upper bound cannot hold.

A similar phenomenon was recently observed by Enciso, Pistoia and Provenzano in [50] for the eigenfunctions of a first-order non-local operator, the *Dirichlet-to-Neumann operator*. In their proof, which is developed within the framework of Riemannian manifolds, the authors require a specific choice of metric to obtain the result. In contrast, our result does not rely on any particular metric structure, as we work with the standard Euclidean metric. In both cases, this phenomenon appears to be related to the *non-local* nature of the problem, which prevents a direct application of the standard proof of Courant's nodal domain theorem; see [61, p. 347]. We note, however, that this result does not rule out the possibility of an asymptotic bound – namely, that the number of nodal domains could grow as $O(k)$ as $k \to \infty$ once the domain is fixed.

The domain Ω introduced in Theorem 1.2.1 which violates the Courant's nodal domain theorem has its eigenfunction numerically shown (in the twisted case) in Figure 1.1. The main idea consists in connecting a central ball to several smaller (but not too much small) and equal balls through thin objects (such as tubes). This allows the eigenfunction to concentrate on each ball and forces it to change sign from one ball to the others. The

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[1] To the best of our knowledge, we indicate where [12, Proposition 2.2 (3)] has been employed in the literature, especially for two results related to Shape Optimization (in [12] this result has not been used).

- [58, Proof of Proposition 2.2 and p. 1092]. In this case Theorem 1.5.1 shows that the main theorem of [58] continues to hold. Moreover it is possible to restore the original proof of [58] by referring to positivity and negativity sets of an eigenfunction (instead of nodal domains) in [58, Proposition 2.2] and by justifying the displacement of nodal domains in [58, p. 1092] with Theorem 1.2.2.
- [23, Proof of Propositions 4.2, 4.5 and Proof of Theorem 1.1 (Step 1)]. In this case we are unable to determine whether their principal statements remain valid, due to the more intricate nature of the underlying framework.

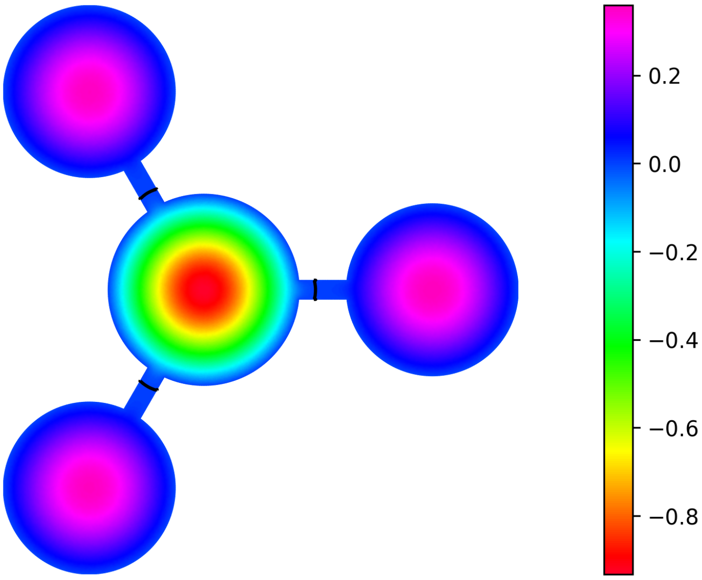

Figure 1.1: Numerical computation of a first twisted eigenfunction with 4 nodal domains (associated to a connected set Ω given by Theorem 1.2.1).

strategy used to construct the counterexample in Theorem 1.2.1 is further developed in Chapter 3 through refinements of classical convergence methods in Shape Optimization (see also Section 1.4).

One can wonder whether for specific (non-pathological) sets the Courant's nodal domain theorem can be recovered. We will prove in the Subsection 2.2.1 that it is possible to determine the exact number of nodal domains of twisted eigenfunctions when Ω is the union of two balls and g is constant on each ball.

Theorem 1.2.2 (Courant's nodal domain theorem)**.** *Let* $\Omega = B_+ \cup B_-$ *where* B_+, B_- *are disjoint balls with* $|B_+| \geq |B_-|$. *Let* $\mathcal{G} = \{g\}$ *with* $g = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$ *and* $0 < \alpha \leq 1$. *If*

$$\lambda_1^{T,\mathcal{G}}(B_+ \cup B_-) < \lambda_2(B_+), \tag{1.17}$$

then the eigenvalue $\lambda_1^{T,\mathcal{G}}(B_+ \cup B_-)$ *is simple, the corresponding eigenfunction* u *is (up to scalar multiples) radially symmetric positive and monotone decreasing (with respect to the distance from the center of* B_+*) in* B_+*, while radially symmetric negative and monotone increasing (with respect to the distance from the center of* B_-*) in* B_-*, with exactly two nodal domains* $\Omega^+ = B_+$ *and* $\Omega^- = B_-$.

The assumption (1.17) is not restrictive, since in the case of equality one knows that the twisted eigenfunction coincides with one corresponding to $\lambda_2^D(B_+)$, see Example 2.1.10.

All in all we have seen that a Courant nodal domain theorem does not hold for some simply connected domains, but can be recovered for specific domains, like the union of two balls. It would be interesting to know whether the Courant's nodal domain theorem could be true under specific geometric assumptions on the domain Ω, such as convexity or star-shapedness.

1.3 Dependence on the orthogonality class

In this section, we begin to study the dependence of twisted eigenvalues with respect to the orthogonality class, at fixed quasi-open set Ω. For specific choices of the class $\mathcal{G}$, twisted eigenvalues (1.13) reduce to well-known quantities.

- If $\mathcal{G}$ is given by ℓ *Dirichlet eigenfunctions*, not including the eigenfunction u_1 corresponding to the first Dirichlet eigenvalue of Ω, then

  $$\lambda_1^{T,\mathcal{G}}(\Omega) = \min_{\substack{u\in H^1_{0,g}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega u(x)^2 dx} = \lambda_1^D(\Omega), \tag{1.18}$$

  where $\lambda_1(\Omega)$ is the *first Dirichlet eigenvalue* of Ω (see [112, Chapter 1, Theorem 1]). When Ω is connected it is always simple (see [19, Theorem 6.34]).

- More generally, if $\mathcal{G}$ is given by ℓ *Dirichlet eigenfunctions*, including all the eigenfunctions corresponding to the first $(k-1)$-th Dirichlet eigenvalues of Ω but not the eigenfunction u_k corresponding to the k-th Dirichlet eigenvalue, then

  $$\lambda_1^{T,\mathcal{G}}(\Omega) = \min_{\substack{u\in H^1_{0,\mathcal{G}}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega u(x)^2 dx} = \lambda_k^D(\Omega), \tag{1.19}$$

  where $\lambda_k^D(\Omega)$ is the *k-th Dirichlet eigenvalue* of Ω (see [112, Chapter 1, Theorem 1]).

- If $\mathcal{G} = \{1\}$ (i.e., the class is given by the constant function identically equal to 1), then

  $$\lambda_1^{T,\mathcal{G}}(\Omega) = \min_{\substack{u\in H^1_{0,1}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega u(x)^2 dx} = \lambda_1^T(\Omega), \tag{1.20}$$

  where $\lambda_1^T(\Omega)$ is the *first (standard) twisted eigenvalue* considered in [12] and [58] (see in particular [12, Proof of Proposition 2.2 (2)]).

A natural question related to these eigenvalues, and one that is also relevant for the applications, is to determine *uniform* bounds that are independent of both the class of functions $\mathcal{G}$ and the set Ω. We focus for simplicity on the dependence on a *single* function g of the class $\mathcal{G}$ and notice that the first Dirichlet eigenvalue, can be characterized as the minimum of twisted eigenvalues

$$\lambda_1^D(\Omega) = \min_{g\in L^2_{\text{loc}}(\mathbb{R}^d)} \lambda_1^{T,g}(\Omega), \tag{1.21}$$

the minimum being realized by a Dirichlet eigenfunction u_k with $k \neq 1$ (and by the function $g = 0$ as well), while the second Dirichlet eigenvalue as the maximum of twisted eigenvalues

$$\lambda_2^D(\Omega) = \max_{g\in L^2_{\text{loc}}(\mathbb{R}^d)} \lambda_1^{T,g}(\Omega), \tag{1.22}$$

(see [112, Chapter 3, Section 2]). In particular, for each $g \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, twisted eigenvalues interlace between the first and second Dirichlet ones with

$$\lambda_1^D(\Omega) \leq \lambda_1^{T,g}(\Omega) \leq \lambda_2^D(\Omega). \tag{1.23}$$

Notice that from (1.18) there exist bounded minimizers of (1.21). The invariance under outer scalings Lemma 2.1.2 allows us to rescale each bounded function in terms of its $\|\cdot\|_\infty$-norm, such that one may consider $|g(x)| \leq 1$ for a.e. $x \in \mathbb{R}^d$. Consequently, the characterization (1.21) still holds, with $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ replaced by the space of *uniformly bounded* functions, see Proposition 4.1.1. After reviewing in Section 2.2 some preliminary results on twisted eigenfunctions, we will prove in Section 4.1, see relation (4.2), that the minimal value in (1.21) can be achieved by working within the more restricted space of *positive*, *uniformly bounded* functions, which we denote by

$$L^\infty_{0,1} := \{g \in L^\infty(\mathbb{R}^d) : 0 < g(x) \leq 1 \text{ for a.e. } x \in \mathbb{R}^d\}.$$

Although the positivity constraint in the class above prevents the function g from being identically zero and from being equal to any Dirichlet eigenfunction u_k with $k \geq 2$, we still have

$$\lambda_1^D(\Omega) = \inf_{g \in L^\infty_{0,1}} \lambda_1^{T,g}(\Omega). \tag{1.24}$$

Here the infimum is indeed not a minimum as it is no longer attained by any function in $L^\infty_{0,1}$. However, minimizing sequences do exists in $L^\infty_{0,1}$, converging to $\lambda_1(\Omega)$, and taking the form (the so-called *bang-bang functions*, see below)

$$g_n := \alpha_n \chi_{\Omega \setminus A_n} + \chi_{\mathbb{R}^d \setminus (\Omega \setminus A_n)},$$

for a suitable sequence of measurable sets A_n and real numbers $0 < \alpha_n \leq 1$ such that $|A_n| \to 0^+$ and $\alpha_n \to 0^+$ as $n \to \infty$ (see Propositions 4.1.2 and 4.1.4 later on).

To track the behaviour of minimizing sequences it is possible to restrict the problem within the smaller class of *uniformly positive* and *uniformly bounded* functions, defined for every $0 < \alpha \leq 1$ as the space

$$L^\infty_{\alpha,1} := \{g \in L^\infty(\mathbb{R}^d) : \alpha \leq g(x) \leq 1 \text{ for a.e. } x \in \mathbb{R}^d\}. \tag{1.25}$$

This class is closed under scalings. Other useful classes can be found in Remark 4.1.5. A particular function in $L^\infty_{\alpha,1}$ is given by a *bang-bang function*

$$\chi_\alpha = \alpha \chi_{\Omega^+} + \chi_{\mathbb{R}^d \setminus \Omega^+},$$

a function defined on two *disjoint* regions (i.e., open bounded sets) $\Omega^+ \cup \Omega^-$, which is $\alpha \in (0,1]$ in Ω^+, 1 in Ω^- and extended by 1 outside $\Omega^+ \cup \Omega^-$. These functions will play a significant role throughout Chapter 4, therefore we reserve the notation χ_α to denote most of them. These bang-bang functions are also of particular importance in Control Theory (see, for instance, [69, Chapters 7, 8, and 9] and the references therein).

As for the upper bound in (1.23), the situation is much simpler. When Ω is either a ball B or the union of two disjoint balls $B^\wedge, B^\vee$ of equal measure, the maximization problem (1.22) is achieved by the constant function $g \equiv 1$. This follows from (1.20) with

$$\lambda_1^T(B) = \lambda_2^D(B) \quad \text{and} \quad \lambda_1^T(B^\wedge \cup B^\vee) = \lambda_2^D(B^\wedge \cup B^\vee),$$

proved, respectively, in [12, Example 3] and [58, Corollary 2.3] (for a more detailed discussion see Examples 2.1.10, 2.1.11, and Remark 2.2.6). Since $1 \in L^\infty_{\alpha,1}$ for all $0 < \alpha \leq 1$, the second Dirichlet eigenvalue can be seen, in a sense, as a sharp upper bound for $\lambda_1^{T,g}(\Omega)$. Actually, in the case of two balls of equal measure, the twisted eigenvalue is independent of g. Indeed, when $\Omega = B^\wedge \cup B^\vee$ then $\lambda_1^D(B^\wedge \cup B^\vee) = \lambda_1^D(B^\wedge) = \lambda_2^D(B^\wedge \cup B^\vee)$ and for every $g \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, the bounds in (1.23) yield

$$\lambda_1^{T,g}(B^\wedge \cup B^\vee) = \lambda_1^D(B^\wedge \cup B^\vee) = \lambda_1^D(B^\wedge) = \lambda_2^D(B^\wedge \cup B^\vee). \tag{1.26}$$

These shape-dependent results naturally raise the question of which properties can be established with respect to the geometry of the set. This motivates a closer study of the shape dependence of twisted and intermediate eigenvalues, which will be addressed in the next section.

1.4 Shape optimization problems for twisted eigenvalues

The aim of this section is to investigate shape optimization problems for the k-th twisted eigenvalue $\lambda_k^{T,\mathcal{G}}(\Omega)$ (the intermediate case is more delicate, and we do not treat it here) with a volume constraint. More precisely, for a *fixed a priori* class $\mathcal{G}$, we are interested in the following shape optimization problem

$$\inf_{\Omega \in \mathcal{QO}} |\Omega|^{2/d} \lambda_1^{T,\mathcal{G}}(\Omega), \tag{1.27}$$

where $\mathcal{QO}$ is the class of bounded quasi-open sets in $\mathbb{R}^d$. As usual, establishing the existence of an optimal shape (possibly in the larger class of quasi-open sets) is not straightforward. In fact, in the twisted setting the problem does not necessarily admit a minimizer, as we will discuss at the end of this section. To prove sufficient conditions for the existence, we develop the theory of γ-convergence, which allows us to establish the existence of optimal shapes when the admissible sets are contained within a fixed box. The situation in the whole space is more delicate and differs from the classical Dirichlet case; in this setting, existence may fail in general in $\mathbb{R}^d$.

Nevertheless, in the particular case $\mathcal{G} = \{1\}$ we will establish a general existence result and derive several qualitative properties of the minimizers, including bounds on the perimeter and the diameter. Numerical approximations of the minimizers are also provided.

Now, with the previous specific choices of $\mathcal{G}$ in (1.18) and (1.19) (with $k = 2$), let us recall the following well-known "isoperimetric" inequalities (that can be read as shape optimization problems). We use the adjective *isoperimetric* as a generic term, following the terminology of [100], even though the perimeter is not explicitly involved.

- The *Faber-Krahn inequality* (see [69, Theorem 3.2.1] and the original sources [52, 79]) states that
$$|\Omega|^{\frac{2}{d}}\lambda_1^D(\Omega) \geq |B|^{\frac{2}{d}}\lambda_1^D(B)\,, \tag{1.28}$$
where B is any ball in $\mathbb{R}^d$. Equality holds if and only if $\Omega = B$.

- The *Hong-Krahn-Szego inequality* (see [69, Theorem 4.1.1] or [72, Theorem 6.4.1 and p.6])[2] states that
$$|\Omega|^{\frac{2}{d}}\lambda_2^D(\Omega) \geq |B^\wedge \cup B^\vee|^{\frac{2}{d}}\lambda_2^D(B^\wedge \cup B^\vee)\,, \tag{1.29}$$
where $B^\wedge$ and $B^\vee$ are any disjoint balls in $\mathbb{R}^d$ of equal measure. Equality holds if and only if $\Omega = B^\wedge \cup B^\vee$.

- The *Freitas-Henrot inequality* (see [58, Theorem 1]) states that
$$|\Omega|^{\frac{2}{d}}\lambda_1^T(\Omega) \geq |B^\wedge \cup B^\vee|^{\frac{2}{d}}\lambda_1^T(B^\wedge \cup B^\vee)\,, \tag{1.30}$$
where $B_\wedge$ and $B_\vee$ are any disjoint balls in $\mathbb{R}^d$ of equal measure. Equality holds if and only if $\Omega = B^\wedge \cup B^\vee$.

The presence of the measure inside (1.28), (1.29), and (1.30) ensures that the inequalities are scale-invariant. An equivalent formulation of these inequalities can be expressed as the minimization of eigenvalues over sets subject to a *measure constraint*. Since these eigenvalues do not change by modifying the set in a region of zero capacity (here the capacity is the one associated to $H_0^1(\Omega)$), the cases of equality (the so-called cases of *rigidity*) stated above (as well as the results in this paper) should be understood, as usual, up to sets of zero capacity (see, for instance, [72, Section 3.3]).

While (1.28) and (1.29) are well-known results, (1.30) is more recent and some generalizations have been proven since then in [23, 31, 40, 63]. All these works provide alternative proofs of (1.30), which, apart for [40], are based on properties of Bessel functions. By allowing more nonlinear degrees of freedom, it has been found in [93] that, for certain configurations, the optimal shape is not symmetric, given by a pair of non-equal balls.

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[2] This "isoperimetric" inequality was first proved by Edgar Krahn in the 1920s in [80, Section 4], but the result was later largely overlooked, since in 1955 in [99, Section 5] George Pólya attributed the observation to Peter Szego (he dropped the "Hungarian umlaut" on the "o" in his last name once he was in the US and during the entirety of his career there). However, almost in the same years as Pólya's paper, the paper [73] by Imsik Hong appeared, providing once again a proof of this result. It should be noted that Hong's paper was published in 1954, just one year before Pólya's. For this reason, we refer to (1.29) as the Hong-Krahn-Szego inequality. We are grateful to Mark S. Ashbaugh for this historical information (for more details and references to these classical works see [42] and also [72, p.6]).

We state four theorems about existence or non-existence of optimal shapes with a fixed family of orthogonality constraints. The first three will be proved in Chapter 3, while the remaining one in Chapter 4. We begin with an existence result for the minimization of the k-th twisted eigenvalue under volume constraint, among quasi-open sets contained in a fixed box.

Theorem 1.4.1. *Let $k \in \mathbb{N}$, $\mathcal{G}$ be as in* (1.3) *and* (1.4)*, $Q \subset \mathbb{R}^d$ be a bounded open set and $0 < m \leq |Q|$. Then the shape optimization problem*

$$\min\{\lambda_k^T(\Omega) : \Omega \subset Q,\ \Omega \text{ quasi-open},\ |\Omega| = m\}$$

has a solution.

This Theorem will be proved in Section 3.2. As preliminary analysis to Theorem 1.4.1, we study the stability of the twisted and intermediate eigenvalues for arbitrary geometric perturbations. While for twisted eigenvalues we prove that the spectrum is stable for the classical γ-convergence, in the case of intermediate eigenvalues stability is achieved imposing further constraints (see Proposition 3.2.14). As a consequence Theorem 1.4.1 can be seen in the framework of Buttazzo-Dal Maso theorem [33].

Remark 1.4.2. The result is slightly more general. For a lower semicontinuous function, $F : \mathbb{R}^k \to \mathbb{R}$ non-decreasing in each variable, the following problem

$$\min\{F(\lambda_1^T(\Omega), \dots, \lambda_k^T(\Omega)) : \Omega \subset Q,\ \Omega \text{ quasi-open},\ |\Omega| = m\}$$

has a solution.

Notice that for intermediate eigenvalues, the existence of a solution falls out from this framework, and this is the reason why we do not treat it here. The main obstacle is that these eigenvalues seem to be continuous only with respect to γ-convergent sequences of sets that preserve the measure in the limit, see Proposition 3.2.14.

Concerning existence in $\mathbb{R}^d$, we recall that, in general, existence fails, see Theorem 1.4.5 below; however, for the particular case of $\mathcal{G} = \{1\}$ we will prove in Section 3.3 the following.

Theorem 1.4.3. *Let $k \in \mathbb{N}$, $\mathcal{G} = \{1\}$ and $m > 0$. The shape optimization problem*

$$\min\{\lambda_k^T(\Omega) : \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open},\ |\Omega| = m\} \tag{1.31}$$

has a solution. Moreover, every solution is a bounded quasi-open set with finite perimeter.

We prove that (local) minimizers are subsolutions of the torsion energy in the spirit of [28], so they have finite perimeter and bounded diameter. This result requires two technical results. First an L^∞-estimate for the eigenfunctions that depends only on the space dimension and on the eigenvalues, but not on the geometry of the set (in the same spirit of Davies's result, see [44]). Moreover a control of the variation of the eigenvalues

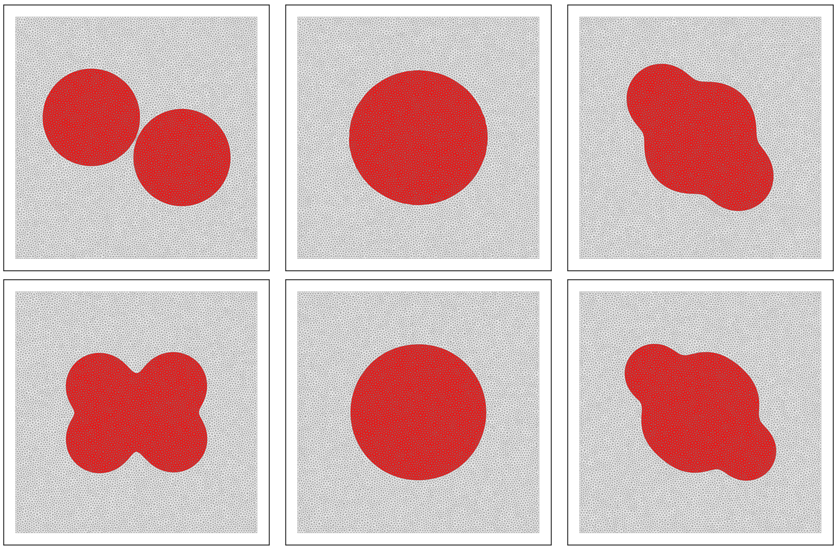

Figure 1.2: From left to right (and then from top to bottom), approximation of the first six twisted optimal sets for $\mathcal{G} = \{1\}$ in dimension $d = 2$.

with respect to inner perturbations of the set in terms of the variation of the torsional rigidities.

Some numerical approximations of the optimal sets are contained in Figure 1.2. These are derived using a method that involves the shape derivative, implemented using the softwares Python and FreeFEM, see Subsection 3.4.1.

In case of multiple harmonic constraints existence may fail.

Theorem 1.4.4 (Non-existence for multiple harmonic constraints)**.** *Let* $m > 0$, $\mathcal{G} = \{1, x, y\}$ *and* $\mathcal{C} \in \{I, T\}$. *Then*

$$\inf\{\lambda_1^{\mathcal{C}}(\Omega) : \Omega \subset \mathbb{R}^2 \text{ quasi-open}, |\Omega| = m\} = \lambda_1^D(B^{\frac{m}{2}}), \tag{1.32}$$

where $B^{\frac{m}{2}}$ *is a disc of measure* $\frac{m}{2}$ *and the shape optimization problem* (1.32) *does not have a solution.*

The proof of this result is contained in Section 3.1. The value in (1.32) coincides also with the minimum attained by choosing $\mathcal{G} = \{1\}$ or $\mathcal{G} = \{1, x\}$ or $\mathcal{G} = \{1, y\}$ and follows from the main results of [63] and [58]. A minimizing sequence for the problem (1.32) is given by sets composed by four balls: two symmetric enlarging big balls with fixed centers and two symmetric shrinking small balls drifting away from each other with centers on the line passing through the centers of the bigger balls, see Figure 3.1. An analogous phenomenon was found for the second eigenvalue of an another *non-local* operator by Brasco and Parini, see [25]. In this case a minimizing sequence is given by a pairs of balls drifting away from each other.

A second non-existence result for the shape optimization problem (1.27) is the following.

Theorem 1.4.5 (Non-existence for one constraint)**.** *If* $\alpha \in (0,1)$ *then there exists* $g \in L^\infty_{\alpha,1}$ *for which problem* (1.27) *has no solution.*

The proof of this result is contained in Section 4.3 and is application of the next Theorems 1.5.1 and 1.5.3. A minimizing sequence is given by a pair of balls: a shrinking small ball with center in the origin and an enlarging big ball drifting away from the origin, see Figure 4.2.

1.5 Isoperimetric inequalities for twisted eigenvalues

The equality (1.24) also implies that a uniform lower bound (both with respect to the function g and to the set Ω) can be directly obtained *via* (1.28). The situation becomes more interesting when one focuses on the double minimization problem

$$\min_{\Omega \in \mathcal{QO}} \min_{g \in L^\infty_{\alpha,1}} |\Omega|^{\frac{2}{d}} \lambda_1^{T,g}(\Omega), \tag{1.33}$$

where $\mathcal{QO}$ is the class of bounded quasi-open sets in $\mathbb{R}^d$ and $0 < \alpha \leq 1$. This optimization problem will be the subject of this section, the corresponding main results are contained in the paper [105]. An analogous question can be formulated for the first intermediate eigenvalue, see (4.32) and Open problem 3. The problem in this framework (that we will not treat here) offers new challenges: some tools available in the twisted case, such as the continuity of the eigenvalue with respect to γ-convergent sequences of sets, can not be employed.

The first result in this setting is an isoperimetric inequality for twisted eigenvalues: for every $0 < \alpha \leq 1$ the unique minimizer of (1.33) is given by the union of two disjoint balls with orthogonality constraint of bang-bang type.

Theorem 1.5.1 (Isoperimetric inequality for twisted eigenvalues)**.** *Fix* $0 < \alpha \leq 1$*. There exists a* unique *number* $m = m(\alpha)$*, with* $0 < m(\alpha) \leq 1$*, such that for every set* $\Omega \in \mathcal{QO}$ *and every function* $g \in L^\infty_\alpha$

$$|\Omega|^{\frac{2}{d}} \lambda_1^{T,g}(\Omega) \geq |B^\alpha_+ \cup B^\alpha_-|^{\frac{2}{d}} \lambda_1^{T,\chi_\alpha}(B^\alpha_+ \cup B^\alpha_-) =: \lambda(\alpha), \tag{1.34}$$

where B^α_+ *and* B^α_- *are any disjoint balls,* $|B^\alpha_-|/|B^\alpha_+| = m(\alpha)$*, and* $\chi_\alpha = \alpha\chi_{B^\alpha_+} + \chi_{\mathbb{R}^d \setminus B^\alpha_+}$*. Equality holds in* (1.34) *if and only if* $\Omega = B^\alpha_+ \cup B^\alpha_-$ *and* $g = \chi_\alpha$ *a.e. in* $B^\alpha_+ \cup B^\alpha_-$*.*

Moreover, if $d \geq 2$*, for the measure ratio* $m(\alpha)$ *of the optimal balls and for the minimal value* $\lambda(\alpha)$ *in* (1.34) *the following estimates hold:*

$$m(\alpha) \geq \alpha^{\frac{d}{d-1}} \quad \textit{and} \quad \lambda(\alpha) \geq \left(1 + \alpha^{\frac{d}{d-1}}\right)^{\frac{2}{d}} |B|^{\frac{2}{d}} \lambda_1(B), \tag{1.35}$$

where B *is any ball in* $\mathbb{R}^d$*.*

Finally, if u_α *is a twisted eigenfunction corresponding to* $\lambda_1^{\chi_\alpha}(B^\alpha_+ \cup B^\alpha_-)$*, then the*

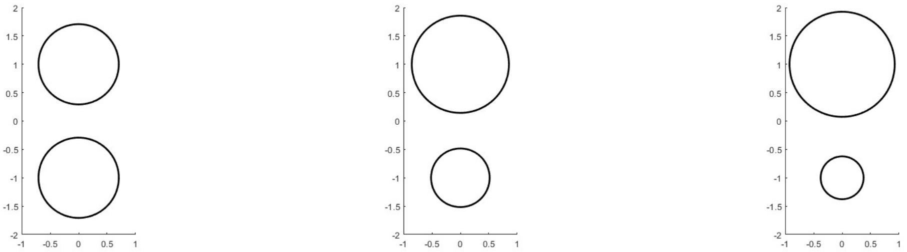

Figure 1.3: The optimal shape (up to scaling) $B_+^\alpha \cup B_-^\alpha$ corresponding to the values $\alpha = 1$, $\alpha = \frac{1}{6}$ and $\alpha = \frac{1}{20}$, in the case $d = 2$. When the balls are different, the optimal bang-bang function χ_α equals α in the larger ball and 1 in the smaller one.

modulus of its gradient $|\nabla u_\alpha|$ *is constant on the boundary* $\partial B_+^\alpha \cup \partial B_-^\alpha$ *with*

$$|\nabla u_\alpha|^2 = \frac{2\lambda_1^{T,\chi_\alpha}(B_+^\alpha \cup B_-^\alpha)}{d|B_+^\alpha \cup B_-^\alpha|} \int_{B_+^\alpha \cup B_-^\alpha} u_\alpha(x)^2 dx, \qquad \text{on } \partial B_+^\alpha \cup \partial B_-^\alpha . \tag{1.36}$$

The proof of this result can be found in Section 4.2, see Figure 1.3 for some plots of the optimal shapes $B_+^\alpha \cup B_-^\alpha$. The estimates (1.35) are derived via *shape derivatives* of twisted eigenvalues. Then, if $\alpha = 1$, by (1.35) we have $m(1) = 1$ and the Freitas-Henrot inequality (1.30) follows, purely from the shape derivative of twisted eigenvalues, without relying on Bessel functions. Potentially, this may have applications to nonlinear eigenvalues, where an explicit representation of the eigenfunctions in terms of Bessel functions is not available (this was already noticed in [40] where a nonlinear variant of the problem was considered, however their proof required a function space larger than $H_0^1(\Omega)$, similar to the one introduced in [63]). The inequality with the lower bound for $\lambda_1^{T,g}(\Omega)$ in (1.23) gives also the Hong-Krahn-Szego inequality.

Corollary 1.5.2 (Hong-Krahn-Szego and Freitas-Henrot inequalities)**.** *If* $\alpha = 1$ *then* (1.30)*, and in particular* (1.29)*, hold, together with their rigidity statements.*

Section 4.2 contains the proof of Theorem 1.5.1 and Corollary 1.5.2. Notice, however, that extending the result to the general case $\alpha < 1$ requires proof strategies that rely on Bessel functions.

As a second topic we study the dependence of the measure ratio $m(\alpha)$ and of the minimal value $\lambda(\alpha)$ in terms of the parameter α (see Figure 1.4 for some plots of these two maps).

Theorem 1.5.3 (Continuous maps connecting the optimal shapes of Dirichlet eigenvalues)**.** *The functions* $m(\alpha)$ *and* $\lambda(\alpha)$ *in Theorem 1.5.1 are continuously differentiable in* $(0, 1)$*, strictly increasing with*

$$\lim_{\alpha \to 0^+} m(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to 1^-} m(\alpha) = m(1) = 1,$$

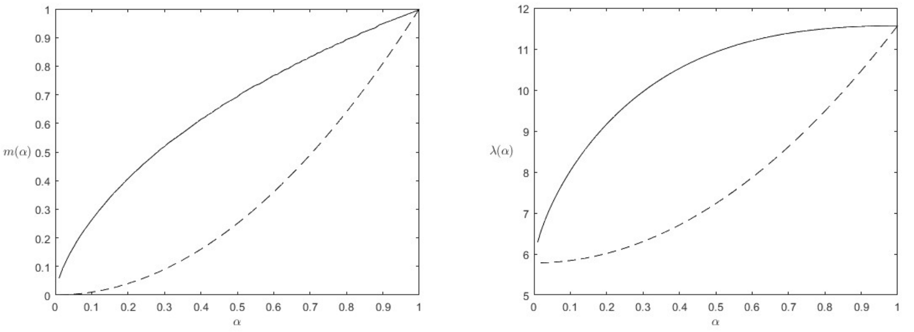

Figure 1.4: Plots of the functions $m(\alpha)$ and $\lambda(\alpha)$ (continuous lines) with their lower bounds in (1.35) (dashed lines), in the case $d = 2$.

and moreover

$$\lim_{\alpha\to 0^+} \lambda(\alpha) = |B^\wedge|^{\frac{2}{d}}\lambda_1(B^\wedge) \text{ and } \lim_{\alpha\to 1^-} \lambda(\alpha) = \lambda(1) = |B^\wedge \cup B^\vee|^{\frac{2}{d}}\lambda_2(B^\wedge \cup B^\vee),$$

where $B^\wedge$ and $B^\vee$ are any disjoint balls in $\mathbb{R}^d$ of equal measure.

Section 4.3 contains the proof of Theorem 1.5.3. Roughly speaking, the double minimization problem (1.33) offers a way to interpolate between the optimal shapes of the first two Dirichlet eigenvalues of the Laplacian, as given in (1.28) and (1.29), through a *continuous* 1-parameter family of optimal shapes that are *union of two disjoint balls*. We are not aware of any results of this type that simultaneously maintain the continuity of the map and ensure that the optimal shapes are unions of balls. We would like to remind the reader that the optimal sets for non-trivial linear combinations of the first two Dirichlet eigenvalues are not represented by unions of balls, see [69, Section 6.4.2, Open problem 20] and also [90, Theorems 1.1 and 1.3]. On the contrary, a connection between the optimal shapes for λ_1 and λ_1^T was found in [22], where the minimizing sets associated with a 1-parameter family of eigenvalues are studied. This problem exhibits a saturation behavior: the minimal eigenvalue increases with the parameter up to a finite critical value, after which it remains constant. This critical point marks the transition between one ball to two equal balls as the optimal shapes. Therefore, the map that associates the parameter to the optimal shape is not continuous in the topology induced by the Hausdorff distance (see [45] for similar questions). We notice that another 1-parameter family of balls gives the solution of an optimization problem involving the *twisted Cheeger constant*, see [21].

Now, we write a comment on the role of the dimension. On the one hand, the main results presented in this section – such as the isoperimetric inequalities – also hold in the one-dimensional case ($d = 1$), where the proofs simplify and may even yield stronger results. On the other hand, certain arguments encounter technical difficulties in this setting. For instance, the estimate (1.35) becomes ill-defined in dimension one, and it is unclear how to recover an appropriate analogue. In this particular case we present in Section 4.2.1 another proof of the uniqueness of the optimal shape that does not rely on

Bessel functions for every $\alpha \in (0, 1]$.

1.6 The Alt-Caffarelli-Friedman monotonicity formula

In this section we consider the ball of radius two $B_2 \subset \mathbb{R}^d$, with center in the origin $\mathbf{0}$, and two non-negative functions $u_+, u_- \in H^1(B_2)$ that satisfy

$$\begin{cases} \Delta u_+ \geq 0 & \text{in } B_2, \\ \Delta u_- \geq 0 & \text{in } B_2, \\ \int_{\mathbb{R}^d} u_+(x) u_-(x)\, dx = 0, \end{cases} \tag{1.37}$$

where the first two inequalities hold in the sense of distributions (see Figure 1.5). Results relying on these assumptions are already present in the literature, see for example [108, Theorem 1.3]. The following *Alt-Caffarelli-Friedman monotonicity formula* (in short ACF formula) was first introduced in [6, Lemma 5.1] as a tool designed to deal with the regularity of solutions of a particular two-phase free boundary problem. In particular, the theory developed in [6] has been designed to understand the model studied in [5], which involves the irrotational flow of two ideal (incompressible) fluids. In terms of their (suitably normalized) stream functions, these properties translate into a null divergence condition. This requirement is generalized by the first two relations in (1.37).

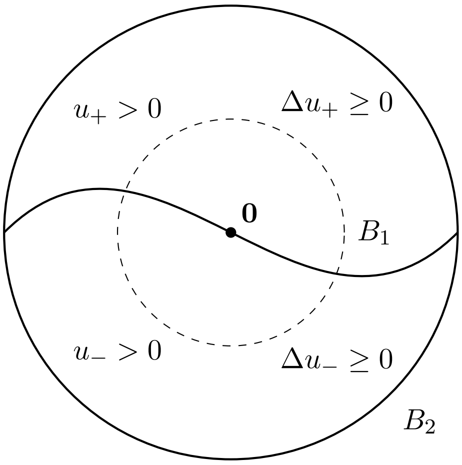

Figure 1.5: A pair u_+, u_- satisfying (1.37) in $\mathbb{R}^2$.

Theorem 1.6.1 (Alt-Caffarelli-Friedman monotonicity formula)**.** *Let u_+, u_- be as in* (1.37)*. The function $J : (0, 1) \to \mathbb{R}$, defined by*

$$J(s) = \frac{1}{s^4} \int_{B_s} \frac{|\nabla u_+(x)|^2}{|x|^{d-2}}\, dx \int_{B_s} \frac{|\nabla u_-(x)|^2}{|x|^{d-2}}\, dx \tag{1.38}$$

for every $s \in (0, 1)$, is finite and increasing.

A crucial issue in two-phase free boundary problems is to establish the optimal regularity of solutions across the free boundary. In this setting, the ACF formula is a valuable device, together with regularity techniques (growth estimates, improvement of flatness), to provide estimates for the behavior of the gradient of a solution of the problem in a point of the free boundary, taking into account of the contribution of the two phases. As an example one can consider [46, Proposition 2.1] where Lipschitz regularity and non-degeneracy of local minimizers for the two-phase Bernoulli problem are obtained.

This formula also proves to be useful for determining the Lipschitz regularity of Dirichlet eigenfunctions that change sign (see for example [70, Chapter 3, Introduction, Step 2]) and so it could be useful in determining the regularity of optimal shapes for twisted and intermediate eigenvalues treated in the thesis (see Remark 1.4.2 and Theorem 1.4.3). More recently analogous formulae have been used to tackle blow-ups related to spectral optimal partition problems with volume and inclusion constraints, see [88, Section 5]. Blow-ups represent also a valuable source of questions, see Open problem 4.

Despite being a frequently used and very well-known object in the field of free boundary problems, in the wide literature concerning this subject, we were not able to find a self-contained comprehensive proof. This probably happened because the central fact necessary to obtain monotonicity, i.e., the *Friedland-Hayman inequality*, obtained as a corollary of [59, Theorem 3], had not at that time been demonstrated in a totally analytical way. This inequality takes the form

$$\left(\sqrt{\left(\frac{d-2}{2}\right)^2+\lambda_1^D(\Gamma_+)}+\sqrt{\left(\frac{d-2}{2}\right)^2+\lambda_1^D(\Gamma_-)}\right)-(d-2)\geq 2, \tag{1.39}$$

for any pair of disjoint measurable sets $\Gamma_+, \Gamma_- \subset \partial B_1$ (where $\lambda_1^D(\Gamma)$ denotes the first Dirichlet eigenvalue of the set Γ). This result is a sharp inequality concerning the growth rates of two homogeneous harmonic functions with Dirichlet boundary conditions on disjoint cones of the Euclidean space, see (5.29). It plays a central role in the proof of the ACF formula, since it is possible to associate u_+ and u_- with two such functions and to obtain through the inequality a lower bound on the growth rate of the ACF formula. The original proof of this powerful tool relies on a result achieved using numerical techniques, see [66], which involves *Hermite's functions*. A way to analytically complete this proof of the Friedland-Hayman inequality is provided in [49, Section 8], which, however, contains rather involved calculations.

A more detailed version of the proof of the ACF formula was given, following a different approach that exploits the one-dimensional *Gaussian measure*, in [35, Chapter 12]. Indeed, an easier proof of the two-dimensional case is provided, while in dimension greater than or equal to three a more accurate analysis and refined tools are needed. However, a part of this proof relies on the unpublished paper [16] (the contents of this work are sketched in [34, Section 2.4]).

In Chapter 5 we give a self-contained and comprehensive proof of the ACF formula in the case $d \geq 3$, with a different approach to the Friedland-Hayman inequality. This proof is contained in the paper [104]. It is *not original*, its structure is the same as [94]. We exploit the content of [77] to obviate a flaw present in the proof of a convexity property (we notice that some computations in our proof differ a little from the ones of [77]). In particular instead of [94, Teorema 4.5] we use Proposition 5.2.1, which corrects this result to the extent necessary for our purposes.

We have deliberately decided to avoid technicalities related to some regularity issues in order to make the presentation more immediate, while providing adequate references when necessary.

1.7 Structure and notation of the thesis

Structure

The thesis is organized as follows. Chapter 2 is devoted to preliminary results such as: ordering between the different eigenvalues, Euler-Lagrange equations and eigenspaces properties. Chapter 3 deals with the existence of optimal shapes in the twisted setting. Moreover there are shown non-existence of optimal shapes and the failure of Courant's nodal domain theorem in both twisted and intermediate frameworks. In Chapter 4 it is derived a one parameter family of isoperimetric inequalities for the first twisted eigenvalue. In addition we provide a description of the behavior of the optimal shapes in terms of the parameter. Chapter 5 revolves around the Alt-Caffarelli-Friedman monotonicity formula and a convexity property of the first Dirichlet eigenvalue on spherical caps. Appendix A contains basic properties of Bessel functions, parametrizations, and rearrangements.

Notation

We denote by $\mathbb{N}$ the set of natural numbers starting at one. The value $d \in \mathbb{N}$ will denote the dimension of the space $\mathbb{R}^d$. Let $k \in \mathbb{N}$ be fixed, this value will denote the number of the eigenvalue. Let $(\mathcal{Q})\mathcal{O}$ be the class of bounded (quasi-)open sets in $\mathbb{R}^d$. We use the term *box* with the symbol D to denote a bounded connected set of $\mathbb{R}^d$. Let $r \in \mathbb{R}$, we denote B_r the ball of radius r that, unless otherwise specified, is centered in the origin. The characteristic function χ_Ω of the set Ω is defined as usual by $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ otherwise. We sometimes abstain from writing the arguments of the functions contained in an integral. Unless otherwise specified, we denote a(n entry of a) vector in $\mathbb{R}^\ell$ and a matrix in $\mathbb{R}^{\ell \times \ell}$ as $(\cdot)_i$ and as $(\cdot)_{ij}$, respectively. Let δ_{ij} be the *Kronecker delta* defined as

$$\delta : \mathbb{N} \times \mathbb{N} \to \{0, 1\}, \quad (i, j) \mapsto \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

We use the shorthand notation *a.e.* to say *almost everywhere* with respect to the Lebesgue measure and *w.r.t.* to say *with respect to.* Given $a \in \mathbb{R}$ and $f \in L^2_{\text{loc}}(\mathbb{R}^d)$, for the sake of brevity, we simply write $f \geq a$ to denote $f \geq a$ a.e. in $\mathbb{R}^d$ (and similarly for the inequalities $>, <, \leq$ or the equality $=$). The symbol δ_d denotes the volume of the ball of unit radius in

$\mathbb{R}^d$. Given $p \in [1, +\infty]$ the symbol $||\cdot||_p$ is the L^p-norm of a function defined in $\mathbb{R}^d$. When there is no ambiguity, given a set of class C^1 we denote by ν its outward unit normal, defined on each point of its boundary. A connected open set will be called *domain*.

Generally we let Ω be a quasi-open set and $\mathcal{G}$ satisfy (1.3). Let $\ell \in \mathbb{N}$ be fixed, this value will denote the number of elements of the family $\mathcal{G}$. In order to simplify the notation, when $\mathcal{G}$ is fixed and there is no ambiguity, we drop the letter $\mathcal{G}$ in the notation of twisted, intermediate eigenvalues and for the orthogonal projection (namely in Section 2.1 and Chapter 3). We adopt the notation $\lambda_k^{\mathcal{C}}(\Omega)$, where $\mathcal{C} \in \{D, I, T\}$ refers to the *category* of the eigenvalue: classical Dirichlet, intermediate or twisted, respectively. In the case of twisted eigenvalues with a family given by exactly one orthogonality constraint $\mathcal{G} = \{g\}$ we set $\lambda_k^g(\Omega) := \lambda_k^{T,g}(\Omega)$ (when $g \equiv 0$ we omit the apex). In this case, unless otherwise specified, we always mean Ω bounded open set and $0 < \alpha \leq 1$ (namely in Section 2.2 and Chapter 4). Let $u \in H^1(\Omega)$, we use the shorthands

$$\mathcal{R}_\Omega(u) = \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega u(x)^2 \, dx} \quad , \quad \mathscr{R}_{\Omega,\mathcal{G}}(u) = \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega (u(x) - P_\Omega u(x))^2 \, dx}.$$

The first quantity above is the so-called *Rayleigh quotient.* If $\mathcal{G} = \{0\}$ we have $\mathscr{R}_{\Omega,\mathcal{G}}(u) = \mathcal{R}_\Omega(u)$. When the subscript is not necessary we omit it. For any continuous function $u \in H_0^1(\Omega)$ we associate the positivity and negativity sets $\Omega^+ = \{u > 0\}$ and $\Omega^- = \{u < 0\}$, and define the positive and negative parts of u as $u^+ = u|_{\Omega^+}$ and $u^- = -u|_{\Omega^-}$, respectively (so that $u = u^+ - u^-$). This should not be confused with the notation B_+ and B_- (here $+$ and $-$ are subscripts) which always refers to two disjoint *non-empty* balls. As usual, functions in $H_0^1(\Omega)$ are extended by 0 outside Ω. When there is no ambiguity, given two bounded open sets A_1, A_2 we associate the bang-bang function χ_α, defined as $\chi_\alpha := \alpha\chi_{A_1} + \chi_{\mathbb{R}^d \setminus A_1}$. If there is no misunderstanding, χ_α is the function equal to α in A_1 and 1 in A_2 (and extended to 1 elsewhere). By *the* first Dirichlet eigenfunction u_1 of a simple eigenvalue $\lambda_1(\Omega)$ (e.g., when Ω is connected) we always refer to the *positive* and *normalized in* $L^2(\Omega)$, namely $\int_\Omega u_1(x)^2 dx = 1$, eigenfunction corresponding to $\lambda_1(\Omega)$. We denote w_Ω the *torsion function* of the set Ω, namely the function $w_\Omega \in H_0^1(\Omega)$ solving in the weak sense

$$\begin{cases} -\Delta w_\Omega = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is

$$\int_\Omega \nabla w_\Omega(x) \nabla \varphi(x) \, dx = \int_\Omega \varphi(x) \, dx, \quad \text{for every } \varphi \in H_0^1(\Omega).$$

Chapter 2

Preliminary results

In the first part of the chapter we extend some classical results of Dirichlet eigenvalues – such as domain monotonicity and Euler-Lagrange equations – to the twisted and intermediate settings considered in this thesis. We also establish ordering relations between twisted, intermediate, and Dirichlet eigenvalues, together with some connections between the corresponding eigenspaces.

In the second part we focus on a specific case: the twisted setting with a single orthogonality function. In this framework, we are able to prove a Courant's nodal domain theorem when the set consists of two balls and the orthogonality function is given by a bang-bang function (i.e., constant on each of the balls). We then present a Pohozaev-type identity.

In the last part we provide the link between intermediate and buckling eigenvalues, then we introduce a related open problem.

2.1 Generic properties in twisted and intermediate settings

In this section we consider a quasi-open set Ω and a family of linearly independent functions $\mathcal{G} = \{g_1, \dots, g_\ell\}$ satisfying (1.3). We prove some useful basic facts on twisted and intermediate eigenvalues and eigenfunctions.

2.1.1 Monotonicity and scaling of twisted and intermediate eigenvalues

We extend the well-known property regarding monotonicity with respect to set inclusion of Dirichlet eigenvalues, see also [63, Proposition 2.2 (d)].

Lemma 2.1.1. *Let $k \in \mathbb{N}$ and $\mathcal{C} \in \{I, T\}$. If ω and Ω are bounded quasi-open sets with $\omega \subset \Omega$, then*

$$\lambda_k^{\mathcal{C}}(\omega) \geq \lambda_k^{\mathcal{C}}(\Omega).$$

Proof. Let $\mathcal{C} = I$ and $u \in H_0^1(\omega)$. Since $\chi_\omega \cdot P_\Omega u \in \mathcal{G}|_\omega$, by the *projection theorem in*

Hilbert spaces, see [26, Theorem 5.2], we have

$$||u - P_\omega u||^2_{L^2(\omega)} \leq ||u - \chi_\omega \cdot P_\Omega u||^2_{L^2(\omega)}.$$

Enlarging the domain of integration yields

$$||u - \chi_\omega \cdot P_\Omega u||^2_{L^2(\omega)} \leq ||u - P_\Omega u||^2_{L^2(\Omega)},$$

and in particular $\mathscr{R}_\omega(u) \geq \mathscr{R}_\Omega(u)$. By (1.10), since $H^1_0(\omega) \subset H^1_0(\Omega)$, we obtain the desired conclusion.

Let $\mathcal{C} = T$. We notice that $H^1_{0,\mathcal{G}}(\omega) \subset H^1_{0,\mathcal{G}}(\Omega)$ and we conclude by (1.13). □

Let $s > 0$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, we define the *rescaled* function $g_s \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ as $g_s(x) := g(sx)$ for every $x \in \mathbb{R}^d$. We extend the well-known scaling properties of Dirichlet eigenvalues, see also [63, Proposition 2.2 (e)].

Lemma 2.1.2. *Let $k \in \mathbb{N}$ and $\mathcal{C} \in \{I, T\}$. Let $s > 0$, $s\mathcal{G} = \{sg_1, \ldots, sg_\ell\}$ and $\mathcal{G}_s = \{(g_1)_s, \ldots, (g_\ell)_s\}$. Then following relations hold:*

- *(a) outer scalings $\lambda_k^{\mathcal{C},s\mathcal{G}}(\Omega) = \lambda_k^{\mathcal{C},\mathcal{G}}(\Omega)$.*
- *(b) inner scalings $\lambda_k^{\mathcal{C},\mathcal{G}}(s\Omega) = \frac{1}{s^2}\lambda_k^{\mathcal{C},\mathcal{G}_s}(\Omega)$.*

Proof. For the outer scalings $s\mathcal{G}$, taking into account (1.10) and (1.13), it is sufficient to note that $\mathrm{span}(s\mathcal{G}) = \mathrm{span}(\mathcal{G})$ in the case $\mathcal{C} = I$ and $H^1_{0,s\mathcal{G}}(\Omega) = H^1_{0,\mathcal{G}}(\Omega)$ in the case $\mathcal{C} = T$.

For the inner scalings we consider first the case $\mathcal{C} = T$. Let u be an eigenfunction corresponding to $\lambda_k^{T,\mathcal{G}}(s\Omega)$. By the change of variables formula and (1.13) we have

$$\lambda_k^{T,\mathcal{G}}(s\Omega) = \mathcal{R}(u) = \frac{1}{s^2}\mathcal{R}(u_s) \geq \frac{1}{s^2}\lambda_k^{T,\mathcal{G}_s}(\Omega). \tag{2.1}$$

On the other hand in the case $\mathcal{C} = I$, consider the map $M_s : L^2(s\Omega) \to L^2(\Omega)$ defined as

$$(M_s g)(x) := s^{d/2} g(sx)$$

for every $g \in L^2(s\Omega)$ and $x \in \Omega$. The map M_s is an isometry (and so a unitary operator). We recall that given an orthogonal projection P and a unitary operator M, the operator MPM^{-1} is the orthogonal projection onto the subspace $M(\mathrm{Ran}\ P)$, where Ran indicates the *range* of an operator. In particular it holds

$$M_s P^{\mathcal{G}}_{s\Omega} = P^{M_s\mathcal{G}}_{\Omega} M_s,$$

moreover we have $P^{M_s\mathcal{G}}_{\Omega} = P^{\mathcal{G}}_{\Omega}$. Let v be an eigenfunction corresponding to $\lambda_k^{I,\mathcal{G}}(s\Omega)$. By the change of variables formula, the relations above and (1.10) we have

$$\lambda_k^{I,\mathcal{G}}(s\Omega) = \mathscr{R}_{s\Omega,\mathcal{G}}(v) = \frac{1}{s^2}\mathscr{R}_{\Omega,\mathcal{G}_s}(v_s) \geq \frac{1}{s^2}\lambda_k^{I,\mathcal{G}_s}(\Omega). \tag{2.2}$$

Noticing that $\Omega = \frac{1}{s}(s\Omega)$, the conclusion follows by (2.1) and (2.2). □

2.1.2 Ordering between Dirichlet, intermediate, and twisted eigenvalues

We give a relation among the classical Dirichlet, the intermediate, and the twisted eigenvalues, see also [63, Proposition 2.2 (f)].

Lemma 2.1.3. *Let $k, \ell \in \mathbb{N}$ and Ω be a quasi-open set of $\mathbb{R}^d$. Then*

$$\lambda_k^D(\Omega) \le \lambda_k^I(\Omega) \le \lambda_k^T(\Omega) \le \lambda_{k+\ell}^D(\Omega). \tag{2.3}$$

Proof. Let u be a function in $H_0^1(\Omega)$, for the first inequality it is enough to notice that (1.7) yields

$$||u||^2_{L^2(\Omega)} \ge ||u - P_\Omega u||^2_{L^2(\Omega)},$$

which in turn gives $\mathcal{R}(u) \le \mathscr{R}_\Omega(u)$. We conclude comparing (1.8) with (1.10).

Let v be a function in $H^1_{0,\mathcal{G}}(\Omega)$, for the second inequality it is enough to notice that $\mathscr{R}_\Omega(v) = \mathcal{R}(v)$, since $P_\Omega v = 0$. We conclude comparing (1.10) with (1.13), recalling that $H^1_{0,\mathcal{G}}(\Omega) \subset H^1_0(\Omega)$.

For the third inequality let $S_{k+\ell}$ be a subspace of dimension $k+\ell$ of $H_0^1(\Omega)$. Taking linear combinations of the elements of $S_{k+\ell}$ it is possible to build a subspace S_k of dimension k of $H^1_{0,\mathcal{G}}(\Omega)$ such that $S_k \subset S_{k+\ell}$. Then

$$\max_{u \in S_k} \mathcal{R}(u) \le \max_{u \in S_{k+\ell}} \mathcal{R}(u)$$

and, by comparing (1.8) with (1.13), (2.3) follows. □

Remark 2.1.4. The lower and upper bounds in (2.3) are sharp. More precisely, let $\{u_1, \dots, u_{\ell+1}\}$ be a family of orthonormal eigenfunctions corresponding to $\lambda_1^D(\Omega), \dots, \lambda_{\ell+1}^D(\Omega)$. By taking $\mathcal{G} = \{u_2, \dots, u_{\ell+1}\}$ we have $\lambda_1^T(\Omega) = \lambda_1^D(\Omega)$. Moreover, by taking $\mathcal{G} = \{u_1, \dots, u_\ell\}$, from [111, Chapter 3, Section 2] we have $\lambda_1^T(\Omega) = \lambda_{\ell+1}^D(\Omega)$.

2.1.3 Twisted and intermediate Euler-Lagrange equations

We provide rigorous justifications for the formal relations (1.11) and (1.15) given in the introduction. We present the weak forms of the equations satisfied by twisted and intermediate eigenfunctions, see also [12, Proposition 2.2, relation (2.13)]. We use tools naturally employed in functional optimization, see [86, Chapter 7].

Lemma 2.1.5. *Let u be an eigenfunction corresponding to $\lambda_k^I(\Omega)$. Then it satisfies* (1.11) *in the weak sense, namely*

$$\int_\Omega \nabla u \nabla \psi \, dx = \lambda_k^I(\Omega) \left(\int_\Omega (u - P_\Omega(u))(\psi - P_\Omega(\psi)) \, dx \right), \quad \text{for every } \psi \in H_0^1(\Omega), \tag{2.4}$$

or equivalently

$$\int_\Omega \nabla u \nabla \psi \, dx = \lambda_k^I(\Omega) \left(\int_\Omega u\psi \, dx - \int_\Omega P_\Omega(u)\psi \, dx \right), \quad \textit{for every } \psi \in H_0^1(\Omega). \tag{2.5}$$

Moreover, if Ω *is open and* $u \in H^2(\Omega)$*, then*

$$\int_\Omega \Delta u(x) g(x) \, dx = 0, \quad \textit{for every } g \in \mathcal{G}. \tag{2.6}$$

Proof. Let $\psi \in H_0^1(\Omega)$, then

$$\mathscr{R}_\Omega(u + \epsilon\psi) = \frac{\int_\Omega [|\nabla u|^2 + 2\varepsilon \nabla u \nabla \psi + O(\varepsilon^2)] \, dx}{\int_\Omega [(u - P_\Omega u)^2 + 2\varepsilon(u - P_\Omega u)(\psi - P_\Omega \psi) + O(\varepsilon^2)] \, dx},$$

as $\varepsilon \to 0$. By (1.6) we have $\int_\Omega (u - P_\Omega u) P_\Omega \psi \, dx = 0$, so we can compute the derivative with respect to ε at $\varepsilon = 0$ as

$$\frac{d}{d\epsilon} \left(\mathscr{R}_\Omega(u + \epsilon\psi) \right) \Big|_{\epsilon=0} = \frac{2}{\int_\Omega (u - P_\Omega u)^2 \, dx} \left(\int_\Omega \nabla u \nabla \psi \, dx - \mathscr{R}_\Omega(u) \int_\Omega [u\psi - P_\Omega(u)\psi] \, dx \right). \tag{2.7}$$

Since by (1.10) the function u is a critical point of $\mathscr{R}_\Omega$, we find (2.5) or equivalently (2.4). Assume that Ω is open, $u \in H^2(\Omega)$, and let $(\psi_n) \subset H_0^1(\Omega)$ be a sequence converging in $L^2(\Omega)$ to $g \in \mathcal{G}$ as $n \to +\infty$. Since $u - P_\Omega u$ is orthogonal to g in $L^2(\Omega)$ by (1.6), an integration by parts in the left-hand side of (2.5), the *Cauchy-Schwartz inequality* and letting $\psi = \psi_n$ yield

$$\left| \int_\Omega (\Delta u) \psi_n \, dx \right| = \left| \int_\Omega (u - P_\Omega u) \psi_n \, dx \right| \leq \|u - P_\Omega u\|_{L^2(\Omega)} \|\psi_n - g\|_{L^2(\Omega)}.$$

The relation (2.6) follows letting $n \to +\infty$. ☐

Lemma 2.1.6. *Let* u *be an eigenfunction corresponding to* $\lambda_k^T(\Omega)$*. Then it satisfies*

$$\int_\Omega \nabla u \nabla \phi \, dx = \lambda_k^T(\Omega) \int_\Omega u\phi \, dx, \quad \textit{for every } \phi \in H_{0,\mathcal{G}}^1(\Omega) \tag{2.8}$$

and

$$\int_\Omega \nabla u \nabla \psi \, dx = \lambda_k^T(\Omega) \int_\Omega u\psi \, dx + \int_\Omega P_\Omega(\xi_T)\psi \, dx, \quad \textit{for every } \psi \in H_0^1(\Omega), \tag{2.9}$$

for some $\xi_T \in L^2(\Omega)$*, possibly depending on* u *and* $\mathcal{G}$*. If* Ω *is open and* $u \in H^2(\Omega)$*, then it satisfies* (1.15) *in the weak sense, namely* (2.9) *with*

$$P_\Omega(\xi_T) = P_\Omega(-\Delta u). \tag{2.10}$$

Proof. Consider the functional

$$\mathcal{R} : H_0^1(\Omega) \to \mathbb{R}, \, \varphi \mapsto \mathcal{R}(\varphi).$$

Recall that the functionals $\mathcal{R}$ and P_Ω are regular in $H_0^1(\Omega)$ so that their Gâteaux and Fréchet differential are the same, see [86, Section 7.2]. Let $\psi \in H^1_{0,\mathcal{G}}(\Omega)$, the derivative of $\mathcal{R}(u+\varepsilon\psi)$ with respect to ε at $\varepsilon = 0$ is

$$\frac{d}{d\epsilon}\left(\mathcal{R}(u+\epsilon\psi)\right)\Big|_{\epsilon=0} = \frac{2}{\int_\Omega u^2\,dx}\left(\int_\Omega \nabla u \nabla\psi\,dx - \mathcal{R}(u)\int_\Omega u\psi\,dx\right). \tag{2.11}$$

Since by (1.13) the function u is a critical point of the functional $\mathcal{R}|_{H^1_{0,\mathcal{G}}(\Omega)}$, namely the restriction of $\mathcal{R}$ to $H^1_{0,\mathcal{G}}(\Omega)$, we find (2.8). On the other hand, by (1.13) u can be seen as a critical point of $\mathcal{R}$ under the constraint $P_\Omega\varphi = 0$. By the *Lagrange multiplier theorem in Banach spaces* [86, Section 9.3, Theorem 1, p. 243], the relations (2.11) and

$$\frac{d}{d\epsilon}\left(P_\Omega(u+\epsilon\psi)\right)\Big|_{\epsilon=0} = P_\Omega(\psi),$$

and the fact that P_Ω is selfadjoint we obtain (2.9).

Assume Ω is open and $u \in H^2(\Omega)$, let $i \in \{1, 2, \dots, \ell\}$ and $(\psi_n^i)_n \subset H_0^1(\Omega)$ be a sequence converging in $L^2(\Omega)$ to g_i as $n \to +\infty$. From an integration by parts in the left-hand side of (2.9), then letting first $\psi = \psi_n^i$ and finally $n \to +\infty$ we find

$$\int_\Omega P_\Omega(\xi_T) g_i\,dx = \int_\Omega P_\Omega(-\Delta u) g_i\,dx.$$

Since $P_\Omega(\xi_T) \in \mathcal{G}|_\Omega$ and the functions $g_1, \dots, g_\ell$ are linearly independent by (1.3), we obtain (2.10). □

To obtain more information, we need to state some regularity results for eigenfunctions, depending on the regularity of Ω and of the elements of $\mathcal{G}$.

Lemma 2.1.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Let u be an eigenfunction corresponding to $\lambda_k^{\mathcal{C}}(\Omega)$ with $\mathcal{C} \in \{D, I, T\}$.*

- *(a) If $\mathcal{G} \subset L^\infty(\mathbb{R}^d)$, then u is continuous in Ω.*
- *(b) If the elements of $\mathcal{G}$ are analytic in Ω, then u is analytic in Ω.*
- *(c) If the elements of $\mathcal{G}$ belong to $C^\infty(\overline{\Omega})$ and Ω is a set of class C^∞, then $u \in C^\infty(\overline{\Omega})$. In particular if $\mathcal{C} = D$, $\mathcal{C} = I$ or $\mathcal{C} = T$ then u satisfies –in the classical sense– (1.9), (1.11) or (1.15), respectively.*
- *(d) If Ω is convex or is a set of class $C^{1,1}$, then $u \in H^2(\Omega)$.*

Proof. All items follow from classical elliptic regularity results, see for instance [18, 51, 60]. More precisely, from Lemmas 2.1.5 and 2.1.6 an eigenfunction solves the corresponding Euler-Lagrange equation (2.5) in the case $\mathcal{C} = I$ and (2.9) in the case $\mathcal{C} = T$ (the case $\mathcal{C} = D$ can be retrieved by taking $\mathcal{G} = \{0\}$), so by [60, Theorem 8.22] there holds item (a). Moreover, by [18, Analyticity Theorem, p. 136 and Theorem 8, p. 200] and [51, Theorem 2, p. 273] it follow the remaining items (b) and (c). Finally, by [64, Theorem 3.2.1.2] and [64, Theorem 2.4.2.5] there holds item (d). □

2.1.4 Twisted and intermediate eigenspaces

We present some results regarding eigenspaces and eigenfunctions (a related problem in the presence of the specific constraint $\mathcal{G} = \{1\}$ has been analyzed in [12, Proposition 2.4]).

Lemma 2.1.8. *Let u be an eigenfunction corresponding to $\lambda_k^T(\Omega)$ with*

$$\lambda_k^T(\Omega) = \lambda_k^I(\Omega). \tag{2.12}$$

Then u is a Dirichlet and intermediate eigenfunction corresponding to $\lambda = \lambda_k^I(\Omega) = \lambda_k^T(\Omega)$.

Proof. Let $u_1, \dots, u_{k-1}, u$ be orthonormal eigenfunctions corresponding to $\lambda_1^T(\Omega), \dots, \lambda_k^T(\Omega)$ and $S = \operatorname{span}\{u_1, \dots, u_{k-1}, u\}$. Since $S \subset H^1_{0,\mathcal{G}}(\Omega)$, from the relation (2.8) and (2.12) we obtain

$$\max_{v \in S} \mathscr{R}_\Omega(v) = \max_{v \in S} \mathcal{R}(v) = \mathcal{R}(u) = \lambda_k^T(\Omega) = \lambda_k^I(\Omega).$$

By (1.10) we obtain that u is an (intermediate) eigenfunction corresponding to $\lambda_k^I(\Omega)$. Since u is a twisted eigenfunction we have $P_\Omega(u) = 0$. By plugging this relation in (2.5) we obtain that u satisfies in the weak sense

$$\int_\Omega \nabla u \nabla \psi \, dx = \lambda \int_\Omega u \psi \, dx, \quad \text{for every } \psi \in H_0^1(\Omega),$$

which means that u is also a Dirichlet eigenfunction. ☐

Let $\mathcal{C} \in \{D, I, T\}$. We define the *eigenspace corresponding to* $\lambda_k^{\mathcal{C}}(\Omega)$ to be the linear span of the eigenfunctions corresponding to $\lambda_k^{\mathcal{C}}(\Omega)$, and we denote it $E_k^{\mathcal{C}}(\Omega)$.

Remark 2.1.9 (Eigenspaces inclusions)**.** Notice that when $\lambda_k^T(\Omega) = \lambda_k^I(\Omega)$, by Lemma 2.1.8 we have the inclusion between eigenspaces $E_k^T(\Omega) \subset E_k^I(\Omega)$. On the other hand if $\lambda_k^T(\Omega)$ is also a Dirichlet eigenvalue of Ω, different from $\lambda_k^D(\Omega)$, an analogous result does not hold: for example both the inclusions $E_k^T(\Omega) \subset E_{k+1}^D(\Omega)$ and $E_k^T(\Omega) \supset E_{k+1}^D(\Omega)$ can hold (also in the strict sense), see [12, Proposition 2.4].

We present a few eigenspaces for some explicit domains. We show an example where a twisted eigenspace coincide with a Dirichlet eigenspace but is stricly included in an intermediate eigenspace, and another Dirichlet eigenspace is strictly included in a twisted eigenspace.

Example 2.1.10 (A ball)**.** Let B be a ball of unit radius in $\mathbb{R}^2$ and $\mathcal{G} = \{1\}$. It holds

$$\lambda_1^I(B) = \lambda_1^T(B) = \lambda_2^D(B).$$

Let u_2 and v_2 be the two eigenfunctions corresponding to $\lambda_1^T(B)$ given by [12, 2.4.3 Example 3, Table 2 with $k = 1$]. We note that one is the rotation of the other and also they

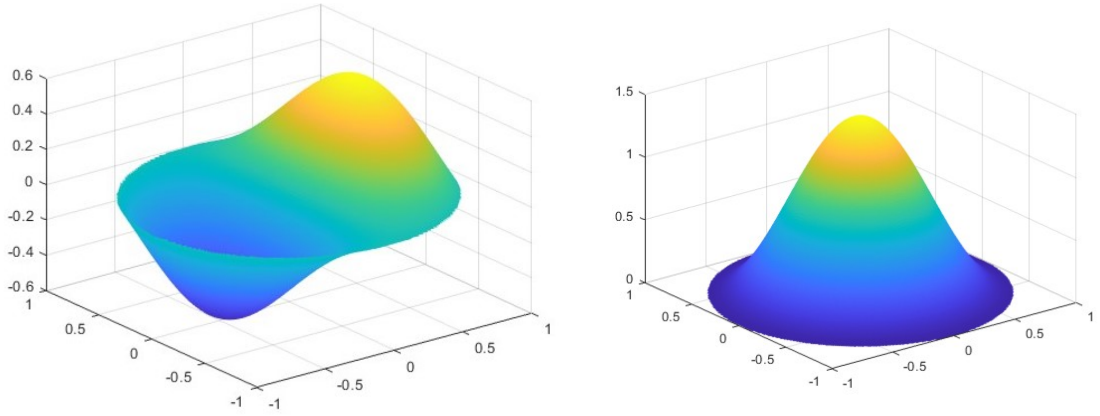

Figure 2.1: The functions u_2 (right) and $u_{\mathcal{N}} - c_{\mathcal{N}}$ (left).

are eigenfunctions corresponding to $\lambda_2^D(B)$. We have

$$E_1^T(B) = E_2^D(B) = \operatorname{span}\{u_2, v_2\}.$$

Let $u_{\mathcal{N}}$ be a radial non-constant Neumann eigenfunction on B with the smallest corresponding eigenvalue (among the radial non-constant Neumann eigenfunction). The function $u_{\mathcal{N}}$ is radially monotone (see Figure 2.1), let $c_{\mathcal{N}}$ be the value attained by $u_{\mathcal{N}}$ on ∂B. By [63, Proposition 3.4, with $N = 2$], recalling that the study of [63] is made in a space isometric to $H_0^1(B)$, see [63, p.765, equation (1.8)], we obtain

$$E_1^I(B) = \operatorname{span}\{u_2, v_2, u_{\mathcal{N}} - c_{\mathcal{N}})\}.$$

In particular we have $E_1^I(B) \supsetneq E_1^T(B) = E_2^D(B)$. An analogous reasoning holds in higher dimensions and in dimension $d = 1$, in this last case we refer to [12, 2.4.1. Example 1] and [63, Beginning of Section 3]. Finally, as observed in [12, 2.4.3 Example 3] we have $\lambda_3^T(B) = \lambda_4^D(B)$ and $E_3^T(B) \supsetneq E_4^D(B)$.

Now, we present an example where twisted and intermediate eigenspace coincide that are strictly included in a Dirichlet eigenspace.

Example 2.1.11 (Two equal balls)**.** Let $B^\wedge$ and $B^\vee$ be two disjoint balls in $\mathbb{R}^d$ with $|B^\wedge| = |B^\vee|$ equal to half the measure of the d-dimensional ball of unit radius and $\mathcal{G} = \{1\}$. It holds

$$\lambda_1^D(B^\wedge \cup B^\vee) = \lambda_1^I(B^\wedge \cup B^\vee) = \lambda_1^T(B^\wedge \cup B^\vee) = \lambda_2^D(B^\wedge \cup B^\vee).$$

Let $u^\wedge$ and $u^\vee$ be the first Dirichlet eigenfunctions of $B^\wedge$ and $B^\vee$, respectively (which are assumed to be positive and L^2-normalized). From [58, equation (19), p.1093] and [63, Proposition 4.4] we have

$$E_1^I(B^\wedge \cup B^\vee) = E_1^T(B^\wedge \cup B^\vee) = \operatorname{span}\{u^\wedge - u^\vee\}.$$

Moreover, by [69, Remark 1.2.4] we obtain

$$E_2^D(B^\wedge \cup B^\vee) = \operatorname{span}\{u^\wedge, u^\vee\}.$$

In particular it holds $E_1^I(B^\wedge \cup B^\vee) = E_1^T(B^\wedge \cup B^\vee) \subsetneq E_2^D(B^\wedge \cup B^\vee)$.

In contrast to the previous two examples, we provide an example where the first twisted and intermediate eigenvalues corresponding to the same domain differ (and the corresponding twisted and intermediate eigenspaces differ as well).

Example 2.1.12 (Two different balls)**.** Let B° and $B^\bullet$ be two disjoint balls in $\mathbb{R}^d$ with $|B^\circ| < |B^\bullet|$, $|B^\circ \cup B^\bullet|$ equal the measure of the d-dimensional ball of unit radius and $\mathcal{G} = \{1\}$. By [63, Proposition 4.4] we have that $\lambda_1^I(B^\circ \cup B^\bullet)$ is simple. Moreover, it holds

$$\lambda_1^I(B^\circ \cup B^\bullet) < \lambda_1^T(B^\circ \cup B^\bullet).$$

Indeed, if by contradiction $\lambda_1^I(B^\circ \cup B^\bullet) = \lambda_1^T(B^\circ \cup B^\bullet)$ then Lemma 2.1.8 would imply $\lambda_1^I(B^\circ \cup B^\bullet) = \lambda_2^D(B^\circ \cup B^\bullet)$ (because $\lambda_1^D(B^\circ \cup B^\bullet)$ is simple and its corresponding eigenfunction does not change sign). This is in contradiction with [63, relation (4.10)], stating that $\lambda_1^I(B^\circ \cup B^\bullet) < \lambda_2^D(B^\circ \cup B^\bullet)$. As a by-product, an intermediate eigenfunction v corresponding to $\lambda_1^I(B^\circ \cup B^\bullet)$ satisfies the nonzero mean value condition $\int_{B^\circ \cup B^\bullet} v(x)\, dx \neq 0$, then v is not a twisted eigenfunction.

We conclude with a result involving harmonic polynomials.

Remark 2.1.13. Let $\Omega \subset \mathbb{R}^d$ be a bounded disconnected open set of class C^∞, ω be one of its connected components. Let $\mathcal{G} = \{g_1, \dots, g_\ell\}$ be a family of functions that satisfies (1.3) and (1.4), namely made up of linearly independent harmonic functions belonging to $L^2(\Omega)$. Let u be an eigenfunction corresponding to $\lambda_k^{\mathcal{C}}(\Omega)$ with $\mathcal{C} \in \{I, T\}$, by item (b) of Lemma 2.1.7 we obtain that u is analytic in Ω. If $u \equiv 0$ on ω (or on an open subset of ω) exploiting the analyticity (and more generally the *unique continuation property*) by (1.2) we have $a_1 g_1 + \cdots + a_l g_l \equiv 0$ on ω. The linear independence of the elements of $\mathcal{G}$ yields $a_1 = \cdots = a_l = 0$. Consequently, u is a Dirichlet eigenfunction of Ω (which vanishes on ω). In other words a twisted or an intermediate eigenfunction cannot vanish on a connected component, unless it is of Dirichlet type.

Note that this argument also applies in the case the elements of $\mathcal{G}$ only satisfies the unique continuation property in Ω (one can take for example the torsion function w_Ω).

2.2 Results in the twisted setting with one orthogonality constraint

In this section we consider the case $\mathcal{G} = \{g\}$. We prove some properties of twisted eigenfunctions corresponding to $\lambda_1^g(\Omega)$.

We start by deriving again the Euler-Lagrange equation satisfied by twisted eigenfunctions in this simpler framework without using the *Lagrange multiplier Theorem* in infinite dimension, as in the proof of Lemma 2.1.6. To this purpose we need to construct a function in $H^1_{0,g}(\Omega)$ as a suitable linear combination of functions in $H^1_0(\Omega)$. Since this construction will be useful in several proofs, we state it separately in the following.

Lemma 2.2.1. *Let $v_1, v_2 \in H^1_0(\Omega)$ and consider their linear combination v defined by*

$$v := \begin{cases} v_1 - \gamma v_2 & \text{if } v_2 \notin H^1_{0,g}(\Omega), \\ v_2 & \text{if } v_2 \in H^1_{0,g}(\Omega), \end{cases} \qquad \text{with } \gamma := \frac{\int_\Omega v_1(x)g(x)dx}{\int_\Omega v_2(x)g(x)dx}. \tag{2.13}$$

Then $v \in H^1_{0,g}(\Omega)$. If moreover v_1 and v_2 are linearly independent *then $v \not\equiv 0$.*

Proof. Linear combinations of $H^1_0(\Omega)$ functions are in $H^1_0(\Omega)$. By definition of v and the linearity of the integral, $\int_\Omega v(x)g(x)dx = 0$, thus $v \in H^1_{0,g}(\Omega)$. Moreover, if the functions v_1, v_1 are linearly independent, then $v_1 \not\equiv 0$, $v_2 \not\equiv 0$, and also $v \not\equiv 0$. ☐

Proposition 2.2.2. *Let u be a minimizer of*

$$\lambda_1^g(\Omega) := \min_{\substack{u \in H^1_{0,g}(\Omega) \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega u(x)^2 dx}. \tag{2.14}$$

Then

$$\int_\Omega \nabla u \nabla \phi \, dx = \lambda_1^g(\Omega) \int_\Omega u\phi \, dx, \quad \text{for every } \phi \in H^1_{0,g}(\Omega) \tag{2.15}$$

and

$$\int_\Omega \nabla u \nabla \psi \, dx = \lambda_1^g(\Omega) \int_\Omega u\psi \, dx + \xi \int_\Omega g\psi \, dx, \quad \text{for every } \psi \in H^1_0(\Omega), \tag{2.16}$$

for some $\xi \in \mathbb{R}$ (possibly depending on u and g, i.e., $\xi = \xi(u,g)$). If Ω is open, $g \not\equiv 0$ and $u \in H^2(\Omega)$, then it satisfies (1.16) *in the weak sense, namely* (2.16) *with*

$$\xi = \frac{\int_\Omega (-\Delta u(x))g(x)dx}{\int_\Omega g(x)^2 dx}. \tag{2.17}$$

Proof. Let $\lambda = \lambda_1^g(\Omega)$. The existence of a minimizer follows by the Direct Methods of the Calculus of Variations. Indeed, we take a minimizing sequence $(u_n)_n$ whose elements are normalized in $L^2(\Omega)$. Recall that the space $H^1_{0,g}(\Omega)$ is compact with respect to the weak convergence in $H^1(\Omega)$ (notice that the L^2-orthogonality condition with respect to g is preserved by this convergence) and the Rayleigh quotient $\mathcal{R}$ is lower semicontinuous with respect to this convergence. So there exists $u \in H^1_{0,g}(\Omega) \setminus \{0\}$ (weak limit in $H^1(\Omega)$ of the minimizing sequence) such that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega u(x)^2 dx},$$

hence the minimum in (2.14) is attained.

Now, if $g \equiv 0$ there is nothing to prove, since λ is the first Dirichlet eigenvalue. Let $g \not\equiv 0$ and $u \in H^1_{0,g}(\Omega) \setminus \{0\}$ be a minimizer of (2.14) then

$$\frac{d}{d\epsilon}\left(\mathcal{R}(u+\epsilon\phi)\right)\Big|_{\epsilon=0} = 0, \quad \text{for every } \phi \in H^1_{0,g}(\Omega),$$

and (2.15) follows. For $\psi \in H^1_0(\Omega)$ and $\varphi \in H^1_0(\Omega) \setminus H^1_{0,g}(\Omega)$, by Lemma 2.2.1, we can consider the function

$$\phi := \psi - \frac{\int_\Omega g(x)\psi(x)dx}{\int_\Omega g(x)\varphi(x)dx}\varphi \in H^1_{0,g}(\Omega),$$

which plugged into (2.15) yields

$$\int_\Omega \nabla u(x)\nabla\psi(x)dx = \lambda\int_\Omega u(x)\psi(x)dx + \xi\int_\Omega g(x)\psi(x)dx, \quad \text{for every } \psi \in H^1_0(\Omega),$$

with

$$\xi = \xi(u,g,\varphi) = \frac{\int_\Omega \nabla u(x)\nabla\varphi(x)dx - \lambda\int_\Omega u(x)\varphi(x)dx}{\int_\Omega g(x)\varphi(x)dx}.$$

This is equation (2.16) and it remains to prove that the quantity ξ does not depend on φ. To this purpose, let $\varphi_1, \varphi_2, \psi_0 \in H^1_0(\Omega) \setminus H^1_{0,g}(\Omega)$. Subtracting the equation above with $\varphi = \varphi_1$ and $\psi = \psi_0$ from the one with $\varphi = \varphi_2$ and $\psi = \psi_0$ we obtain

$$[\xi(u,g,\varphi_1) - \xi(u,g,\varphi_2)]\int_\Omega g(x)\psi_0(x)dx = 0.$$

We chose ψ_0 such that $\int_\Omega g(x)\psi_0(x)dx \neq 0$, hence $\xi(u,g,\varphi_1) = \xi(u,g,\varphi_2) = \xi(u,g)$.

Finally, when Ω is open, $g \not\equiv 0$ and u also belongs to $H^2(\Omega)$, an integration by parts yields

$$\int_\Omega \nabla u\nabla\psi dx = \lambda\int_\Omega u\psi dx + \frac{\int_\Omega(-\Delta u)\varphi dx - \lambda\int_\Omega u\varphi dx}{\int_\Omega g\varphi dx}\int_\Omega g\psi dx, \tag{2.18}$$

for every $\varphi \in C^\infty_c(\Omega) \setminus H^1_{0,g}(\Omega)$ and $\psi \in H^1_0(\Omega)$. To finish the proof we would take $\varphi = g$ in (2.18), which is clearly not possible, since g could not belong to $C^\infty_c(\Omega)$, the space of smooth functions with compact support in Ω. However, this can be justified with a limiting argument: by density, there exists a sequence $(\varphi_n)_n$ in $C^\infty_c(\Omega)$ such that $\varphi_n \to g$ in $L^2(\Omega)$ as $n \to +\infty$. Moreover, for every n sufficiently large such that $\|\varphi_n - g\|_2 < \|\varphi_n\|_2$ (this is always possible since the left-hand side converges to 0 while the right-hand side goes to $\|g\|_2 > 0$ being $g \not\equiv 0$) it follows that

$$2\int_\Omega g(x)\varphi_n(x)dx = \|\varphi_n\|_2^2 + \|g\|_2^2 - \|\varphi_n - g\|_2^2 > \|g\|_2^2 > 0,$$

that is there exists a sequence $(\varphi_n)_n$ in $C^\infty_c(\Omega) \setminus H^1_{0,g}(\Omega)$ converging to g in $L^2(\Omega)$. By letting first $\varphi = \varphi_n$ in (2.18) and then $n \to +\infty$ we find that a minimizer solves (1.16) in the weak sense. □

We aim to characterize the sign of ξ in (2.16). A preliminary result is the following.

Remark 2.2.3. Let u be a twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$. If u is not a Dirichlet eigenfunction and g does not vanish on any open subset of Ω, then u also cannot vanish on any open subset of Ω. Indeed, assume that $u \equiv 0$ in some open set $A \subset \Omega$. Then, since g is not identically zero in A there exists $\psi_0 \in H_0^1(A)$ such that $\int_A \psi_0 g dx \neq 0$ and by plugging ψ_0 in (2.16) one has $\xi \equiv 0$. Therefore, u satisfies (2.16) with $\xi = 0$ and it is a Dirichlet eigenfunction of Ω (which vanishes in A), that proves what claimed above.

A particular case is when Ω is not connected with A one of its connected components and g piecewise constant on the connected components.

Remark 2.2.4. Let u be a twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$. We can always consider the positive and negative parts u^+ and u^- of u with their positivity and negativity sets Ω^+ and Ω^-. Of particular importance is the role played by the space of *positive* uniformly bounded functions $L_{0,1}^\infty$. If $g \in L_{0,1}^\infty$ then every $u \in H_{0,g}^1(\Omega)$ must change sign in Ω, that is neither u^+ nor u^- is identically zero, since $\int_\Omega gu\, dx = 0$. Suppose that Ω is an open set, by item (a) of Lemma 2.1.7 every twisted eigenfunction has at least two nodal domains. Recall that the *nodal domains* of a continuous function u defined over Ω are the connected components of the open set $\Omega \setminus \{u = 0\}$ (since u is continuous, then $\{u = 0\}$ is closed).

We first characterize the sign of ξ in (2.16) in terms of the magnitude of the Rayleigh quotients of twisted eigenfunctions restricted to the positivity and negativity sets.

Proposition 2.2.5. *If $g \in L_{0,1}^\infty$ and u is a twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$ then*

$$\mathcal{R}(u^+) \le \mathcal{R}(u^-) \quad \textit{if and only if} \quad \xi \le 0.$$

The equality $\mathcal{R}(u^+) = \mathcal{R}(u^-)$ holds if and only if $\xi = 0$ (i.e., u is a Dirichlet eigenfunction).

Proof. By letting $\psi = u^+/\|u^+\|_2^2$ in (2.16), the function u^+ satisfies

$$\mathcal{R}(u^+) = \lambda_1^g(\Omega) + \frac{\xi}{||u^+||_2^2} \int_\Omega g(x)u^+(x)dx,$$

while, by letting $\psi = u^-/\|u^-\|_2^2$ in (2.16), the function u^- satisfies

$$\mathcal{R}(u^-) = \lambda_1^g(\Omega) - \frac{\xi}{||u^-||_2^2} \int_\Omega g(x)u^-(x)dx.$$

Subtracting the second equation from the first, and using the identity $\int_\Omega gu^+dx = \int_\Omega gu^-dx$, which follows from the condition $u \in H_{0,g}^1(\Omega)$, yields

$$\mathcal{R}(u^+) - \mathcal{R}(u^-) = \xi \int_\Omega g(x)u^+(x)dx \left(\frac{1}{||u^+||_2^2} + \frac{1}{||u^-||_2^2}\right),$$

and the thesis follows from the positivity of g and u^+. □

Remark 2.2.6. Let $\Omega = B_+ \cup B_-$ where B_+, B_- are disjoint balls with $|B_+| \geq |B_-|$ and $g = \chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$. If a twisted eigenfunction u corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ is a Dirichlet eigenfunction (recall $\xi = 0$ in (2.16)), by [69, Remark 1.2.4] and (1.23)

$$\lambda_1(B_+) \leq \lambda_1^{\chi_\alpha}(B_+ \cup B_-) \leq \min\{\lambda_1(B_-), \lambda_2(B_+)\},$$

and Proposition 2.2.5 only two situations may occur:

(i) either $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ equals $\lambda_1(B_+)$ or $\lambda_1(B_-)$ which, by the orthogonality constraint, implies that B_+ and B_- have the same measure $|B_+| = |B_-|$, $\lambda_1^{\chi_\alpha}(B_+ \cup B_-) = \lambda_2(B_+ \cup B_-)$ and u (up to scaling) is equal to $u_1^+ - \alpha u_1^-$ (with $u_1^\pm$ the first Dirichlet eigenfunctions in $B_\pm$);

(ii) or $\lambda_1^{\chi_\alpha}(B_+ \cup B_-) = \lambda_2(B_+)$ and then u (up to scaling) is equal to a Dirichlet eigenfunction u_2^+ corresponding to $\lambda_2(B_+)$.

We recall the following

Lemma 2.2.7. *Let $a_1, a_2, b_1, b_2 > 0$ with $a_1/a_2 \geq b_1/b_2$. The function $Q\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by*

$$Q(t) = \frac{a_1 t + b_1}{a_2 t + b_2}$$

is non-decreasing in $\mathbb{R}^+$. The function Q is strictly increasing if and only if $a_1/a_2 > b_1/b_2$ (otherwise it is constant). In particular, there hold

$$\frac{b_1}{b_2} \leq \frac{a_1 + b_1}{a_2 + b_2} \leq \frac{a_1}{a_2},$$

with equalities if and only if $a_1/a_2 = b_1/b_2$.

2.2.1 Proof of Theorem 1.2.2

Now, we need a result *à la* Courant, concerning the number of nodal domains of twisted eigenfunctions (see Remark 2.2.4). This is a very delicate issue and we are able to determine the exact number of nodal domains of twisted eigenfunctions when Ω is the union of two balls and g is of bang-bang type. The strategy needed to prove this result will be enriched in Chapter 3 to obtain the counterexample contained in Theorem 1.2.1.

Theorem 1.2.2. Let u be an eigenfunction corresponding to $\lambda := \lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ and denote $u_\pm := u|_{B_\pm}$ the restrictions of u to each ball. By Remark 2.2.6 we know that u is not a Dirichlet eigenfunction. We divide the proof in several steps.

Step 1 (u has a nodal domains in B_+ and another in B_-). Let $u_1^\pm$ denote the first Dirichlet eigenfunctions on $B_\pm$, and recall that, by the convention on the sign of $u_1^\pm$ it holds $\int_{B_\pm} u_1^\pm g dx > 0$. By [69, Remark 1.2.4] $\lambda_1(B_+ \cup B_-) = \lambda_1(B_+)$ and $\lambda_1(B_+ \cup B_-)$ is simple with u_1^+ the eigenfunction corresponding to $\lambda_1(B_+ \cup B_-)$ then

$$\lambda_1(B_+) = \lambda_1(B_+ \cup B_-) < \mathcal{R}(u) = \lambda. \tag{2.19}$$

Now, consider the linear combination v defined as in (2.13) with $v_1 = u_1^+$ and $v_2 = u_1^-$. By Lemma 2.2.1 the function v can be plugged into (2.14) to obtain

$$\lambda \leq \mathcal{R}(v) < \lambda_1(B_-), \tag{2.20}$$

where the strict inequality follows now from Lemma 2.2.7 (recall that the assumption $|B_+| > |B_-|$ guarantees that $\lambda_1(B_+) < \lambda_1(B_-)$). Therefore, by combining (2.19) and (2.20) with (1.17), and recalling (1.23), u can not be a Dirichlet eigenfunction corresponding to $B_+ \cup B_-$. Then, by Remark 2.2.3, it does not vanish identically on any of the connected components of $B_+ \cup B_-$. In particular u has at least two nodal domains.

Step 2 (u is radially symmetric w.r.t. the each center of the balls B_+ and B_-). By item (b) of Lemma 2.1.7 u is C^∞ in $B_+ \cup B_-$ and it satisfies

$$-\Delta u = \lambda u + \xi \chi_\alpha \quad \text{in } B_+ \cup B_-. \tag{2.21}$$

Let I_+ be an isometry of $\mathbb{R}^d$ with the same center of B_+. Let v_+ be the function obtained by moving isometrically the function u with I_+ on the ball B_+, that is

$$v_+(x) = u_+(I_+(x)) + u_-(x), \quad \text{for every } x \in B_+ \cup B_-.$$

Since $v_+ \in H^1_{0,g}(\Omega)$ and $\mathcal{R}(v_+) = \mathcal{R}(u)$ then v_+ is an eigenfunction corresponding to λ that satisfies the equation (2.21) with the same ξ. If, by contradiction, $u - v_+ \not\equiv 0$ in $B_+ \cup B_-$, then by (2.21) $u - v_+$ is a Dirichlet eigenfunction of $B_+ \cup B_-$ with corresponding eigenvalue λ. In particular $u_+ - (u_+ \circ I_+)$ is a Dirichlet eigenfunction of B_+ with corresponding eigenvalue λ, a contradiction with (1.17) and (2.19). This implies that u_+ is radially symmetric in B_+. A similar argument applies for u_-, using the relation (2.20) at the end.

Step 3 (u has nodal domains B_+ and B_-). Suppose by contradiction that u_+ has not constant sign in B_+. By radiality and smoothness, *Rolle's theorem* implies the existence of an $r > 0$ such that on the sphere $\partial B_r \subset B_+$ (with $B_r \subset B_+$ the ball of radius r with same center of B_+) the radial derivative of u_+ is equal to 0. Now we consider the function w_+ – the restriction to $\overline{B_r}$ of a particular vertical displacement of the function u_+ – defined as

$$w_+(x) = u_+(x) - u_+(x_r) \quad \text{for every } x \in \overline{B_r},$$

where x_r is a point on ∂B_r. By definition, w_+ is analytic in B_r and then, by (2.21) and the construction of r, it satisfies

$$\begin{cases} \Delta^2 w_+ = -\lambda \Delta w_+, & \text{in } B_r, \\ w_+ = (\partial w_+)/(\partial \nu) = 0, & \text{on } \partial B_r. \end{cases}$$

Therefore, λ is a *buckling eigenvalue* of B_r, see [11, Section 15.1, equations (15.1.1)-(15.1.2)]. We can use the *Payne inequality*, see [11, Section 15] or [95, Section 2, Part A], to obtain

$$\lambda \geq \Lambda_1(B_r) \geq \lambda_2(B_r) > \lambda_2(B_+),$$

where $\Lambda_1(B_r)$ is the first buckling eigenvalue of B_r. This last inequality contradicts the assumption (1.17) and thus u_+ does not change sign in B_+. Note that this argument also prevents the function u_+ to attain the value 0, without changing sign, inside B_+. An analogous strategy applies for u_-, using the relation (2.20) at the end. Therefore we have that u_+ and u_- have constant sign in B_+ and B_- respectively.

By Remark 2.2.4 the function u has to change sign in $B_+ \cup B_-$, so we can assume that u is positive on B_+ and negative on B_-. In particular, this means that λ is a simple eigenvalue. ☐

As a consequence of Remark 2.2.6 and Theorem 1.2.2 we have

Corollary 2.2.8. *Let $\Omega = B_+ \cup B_-$ where B_+ and B_- are disjoint balls with $|B_+| \geq |B_-|$ and $g = \chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$. There exists a twisted eigenfunction u corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ with exactly two nodal domains Ω^+, Ω^-. Moreover, if $|B_+| = |B_-|$ then the (unique) twisted eigenfunction u corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ is given by a linear combination of the first Dirichlet eigenfunctions of B_+ and B_- scaled in such a way (see Lemma 2.2.1) to satisfy the orthogonality constraint.*

We conclude this section with an interesting result that allows us to determine the sign of ξ in (2.16) when the set Ω is the union of two disjoint balls and the function g is of bang-bang type.

Proposition 2.2.9. *Let $\Omega = B_+ \cup B_-$ where B_+, B_- are disjoint balls with $|B_+| \geq |B_-|$ and $g = \chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$. If u denote a twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$ with two nodal domains Ω^+, Ω^-, then*

$$|\Omega^+| \geq |\Omega^-| \quad \text{if and only if} \quad \mathcal{R}(u^+) \leq \mathcal{R}(u^-).$$

The equality $|\Omega^+| = |\Omega^-|$ holds if and only if $\mathcal{R}(u^+) = \mathcal{R}(u^-)$ (i.e., u is a Dirichlet eigenfunction).

Proof. We first prove the cases of equality. By Proposition 2.2.5 we know that the equality $\mathcal{R}(u^+) = \mathcal{R}(u^-)$ holds if and only if the twisted eigenfunction is a Dirichlet eigenvalue. We first assume $|\Omega^+| = |\Omega^-|$. If $\Omega^+, \Omega^- \subset B_+$, then from [12, Example 3] we have $\lambda_1^{\chi_\alpha}(B_+ \cup B_-) = \lambda_2(B_+)$ and, by [69, Remark 1.2.4], u is a Dirichlet eigenfunction. If instead $\Omega^+ \subset B_+$ while $\Omega^- \subset B_-$, then item (b) of Lemma 2.1.7 yields $|B_+| = |\Omega^+| = |\Omega^-| = |B_-|$, therefore, from the second part in Corollary 2.2.8 u is given by a linear combination of Dirichlet eigenfunctions (by (1.26) and [69, Remark 1.2.4] we obtain that u is an eigenfunction corresponding to $\lambda_1(B_+ \cup B_-)$). On the contrary, assume that $\mathcal{R}(u^+) = \mathcal{R}(u^-)$, that is u is a Dirichlet eigenfunction. From [69, Remark 1.2.4] and (1.23) either u is an eigenfunction corresponding to $\lambda_2(B_+)$ or it is an eigenfunction corresponding to $\lambda_1(B_+ \cup B_-) = \lambda_2(B_+ \cup B_-)$. In all cases it turns out that $|\Omega^+| = |\Omega^-|$.

Now, in order to prove the double implication for the strict inequalities in the statement, we may assume $|\Omega^+| \neq |\Omega^-|$. This gives, up to exchanging the sets Ω^+ and Ω^-, $\Omega^+ \subset B_+$ and $\Omega^- \subset B_-$. In this case the analyticity of u, as stated in item (b) of Lemma 2.1.7, implies that $\Omega^+ = B_+$ and $\Omega^- = B_-$; that is u is non-zero in both balls. The strategy of the proof is to test (2.14) with suitable combinations of u^+ or u^-. We proceed in two steps.

Step 1 (upper bounds for the twisted eigenvalue). We first use a suitable combination of u^- in the two balls, where on B_+ it is appropriately (translated and) scaled so as to belong to $H^1_{0,\chi_\alpha}(B_+ \cup B_-)$, and hence is admissible for (2.14). Let c_+ and c_- be the centers of the balls B_+ and B_-. For every $x \in B_+$ define the function $v^+(x) := u^-((x - c_+ + c_-)/a)$ with $a := (|B_+|/|B_-|)^{1/d}$ (i.e., v^+ is u^- rescaled by the factor a and translated so as to belong to $H^1_0(B_+)$). By using Lemma 2.2.1 with $v_1 = v^+$ and $v_2 = u^-$, then $\gamma = \alpha a^d$ and the linear combination $v := v^+ - \alpha a^d u^- \in H^1_{0,\chi_\alpha}(B_+ \cup B_-)$ can be plugged into (2.14) to yield

$$\begin{aligned}\lambda_1^{\chi_\alpha}(B_+ \cup B_-) &\leq \frac{\int_{B_+} |\nabla v^+|^2 dx + \alpha^2 a^{2d} \int_{B_-} |\nabla u^-|^2 dx}{\int_{B_+} (v^+)^2 dx + \alpha^2 a^{2d} \int_{B_-} (u^-)^2 dx} \\ &= \frac{a^{d-2} \int_{B_-} |\nabla u^-|^2 dx + \alpha^2 a^{2d} \int_{B_-} |\nabla u^-|^2 dx}{a^d \int_{B_-} (u^-)^2 dx + \alpha^2 a^{2d} \int_{B_-} (u^-)^2 dx} = \frac{a^{-2} + \alpha^2 a^d}{1 + \alpha^2 a^d} \mathcal{R}(u^-),\end{aligned}$$

where the first equality is obtained by changing variable with scaling factor a. Similarly, we may use $v := u^+ - \alpha a^d v^-$ with $v^-(x) := u^+(a(x - c_- + c_+))$ for every $x \in B_-$ to obtain

$$\lambda_1^{\chi_\alpha}(B_+ \cup B_-) \leq \frac{1 + \alpha^2 a^{d+2}}{1 + \alpha^2 a^d} \mathcal{R}(u^+). \tag{2.22}$$

Step 2 (double implication for strict inequalities). If $|B_+| > |B_-|$, namely $a > 1$, then from the previous inequality $\lambda_1^{\chi_\alpha}(B_+ \cup B_-) < \mathcal{R}(u^-)$. The estimates in Lemma 2.2.7 yield

$$\min\{\mathcal{R}(u^+), \mathcal{R}(u^-)\} \leq \lambda_1^{\chi_\alpha}(B_+ \cup B_-) \leq \max\{\mathcal{R}(u^+), \mathcal{R}(u^-)\}, \tag{2.23}$$

where, in Lemma 2.2.7, $\mathcal{R}(u^+) = b_1/b_2 < a_1/a_2 = \mathcal{R}(u^-)$. On the other hand, if $\mathcal{R}(u^+) < \mathcal{R}(u^-)$, then (2.23) is equivalent to $\mathcal{R}(u^+) < \lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ and combined with (2.22) yields $(1 + \alpha^2 a^d) < (1 + \alpha^2 a^{d+2})$, that is $a > 1$. □

By combining Propositions 2.2.5 with 2.2.9 one may deduce the sign of ξ in (2.16) for an eigenfunction with exactly two nodal domains, that is when Ω is the union of two balls and g is of bang-bang type.

Corollary 2.2.10. *Let $\Omega = B_+ \cup B_-$ where B_+ and B_- are disjoint balls with $|B_+| \geq |B_-|$ and $g = \chi_\alpha = \alpha \chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$. If u is a twisted eigenfunction corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ with two nodal domains Ω^+, Ω^- such that $|\Omega^+| \geq |\Omega^-|$, then $\xi \leq 0$.*

2.2.2 Pohozaev identity and shape derivative in the twisted setting

In this section, we recall two standard results for twisted eigenvalues and twisted eigenfunctions. Given two disjoint non-empty balls $B_+ \subset \{x_1 > 1\}$, $B_- \subset \{x_1 < -1\}$ with $|B_+| \geq |B_-|$ we consider in the whole subsection

$$\Omega_0 = B_+ \cup B_-, \qquad \chi_\alpha = \alpha\chi_{\{x_1>0\}} + \chi_{\{x_1\leq 0\}}$$

and a twisted eigenfunction u corresponding to $\lambda = \lambda_1^{\chi_\alpha}(\Omega_0)$ which, by Lemma 2.1.7, solves, in the classical sense, the following system

$$\begin{cases} -\Delta u = \lambda u + \xi\chi_\alpha & \text{in } \Omega_0, \\ u = 0 & \text{on } \partial\Omega_0, \\ \int_{\Omega_0} u(x)\chi_\alpha(x)dx = 0, \\ \int_{\Omega_0} u(x)^2 dx = 1, \end{cases} \tag{2.24}$$

with ξ as in (2.17) (the latter condition is the normalization of the eigenfunction). The following Pohozaev-type identity for twisted eigenfunctions holds.

Proposition 2.2.11. *Every solution u of* (2.24) *satisfies*

$$2\lambda = \int_{\partial\Omega_0} |\nabla u(x)|^2 (X\cdot\nu)(x)\, d\mathcal{H}^{d-1}(x),$$

with $X = (x_1, x_2, \dots, x_d)$.

Proof. As in [101, equation (9.1.5)] the following integration-by-parts formula holds

$$\int_{\partial\Omega_0} |\nabla u|^2 (X\cdot\nu) d\mathcal{H}^{d-1} + (2-d)\int_{\Omega_0} u\Delta u dx = 2\int_{\Omega_0} (\nabla u \cdot X)\Delta u dx. \tag{2.25}$$

The equation in the system (2.24), combined with the orthogonality and normalization conditions satisfied by u, allow to compute the integral in the second term of (2.25) as follows

$$\int_{\Omega_0} u(x)\Delta u(x)dx = -\lambda\int_{\Omega_0} u(x)^2 dx - \xi\int_{\Omega_0} u(x)\chi_\alpha(x)dx = -\lambda, \tag{2.26}$$

and similarly, for the integral in the third term of (2.25)

$$\int_{\Omega_0} (\nabla u(x)\cdot X)\Delta u(x)dx = -\lambda\int_{\Omega_0} (\nabla u(x)\cdot X)u(x)dx - \xi\int_{\Omega_0} (\nabla u(x)\cdot X)\chi_\alpha(x)dx.$$

By making use of the boundary condition $u = 0$ on $\partial\Omega_0$, and the normalization condition $||u||_2 = 1$ again, an integration by parts yields

$$-\int_{\Omega_0} (\nabla u(x)\cdot X)u(x)dx = -\int_{\Omega_0} \operatorname{div}\left(\frac{1}{2}u(x)^2 X\right) dx + \int_{\Omega_0} \frac{d}{2}u(x)^2 dx = \frac{d}{2},$$

and moreover, since $\nabla\chi_\alpha = 0$ in Ω_0 and u solves (2.24),

$$\int_{\Omega_0} (\nabla u(x) \cdot X)\chi_\alpha(x)dx = \int_{\Omega_0} \operatorname{div}(u(x)\chi_\alpha(x)X)\, dx = 0.$$

From the last two relations we may rewrite the third term in (2.25) as follows

$$2\int_{\Omega_0} (\nabla u(x) \cdot X)\Delta u(x)dx = d\lambda,$$

which together with (2.26) yields the thesis. □

Now, we derive the expression of the shape derivative of twisted eigenvalues. Let $T > 0$ and consider a family of maps $\mathfrak{F}(t)$ that satisfy

$$\mathfrak{F} : t \in [0, T) \to W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ differentiable at 0 with } \mathfrak{F}(0) = I,\ \mathfrak{F}'(0) = W,$$

where I is the identity map from $\mathbb{R}^d$ to $\mathbb{R}^d$ and W is a smooth vector field from $\mathbb{R}^d$ to $\mathbb{R}^d$. We denote by $\Omega_t = \mathfrak{F}(t)(\Omega_0)$ and for t sufficiently small we may assume $\mathfrak{F}(t)(B_+) \subset \{x_1 > 0\}$ and $\mathfrak{F}(t)(B_-) \subset \{x_1 < 0\}$. We consider $u_t \in H^1_{0,\chi_\alpha}(\Omega_t)$ a twisted eigenfunction corresponding to $\lambda_t = \lambda_1^{\chi_\alpha}(\Omega_t)$ normalized so that

$$\int_{\Omega_t} u_t(x)\chi_\alpha(x)dx = 0 \quad \text{and} \quad \int_{\Omega_t} u_t(x)^2 dx = 1. \tag{2.27}$$

Let $\xi_t = \xi(u_t, \chi_\alpha)$, defined as in Proposition 2.2.2, and denote by u', λ' and ξ' the derivatives at 0 of the functions that associates $t \mapsto u_t$, $t \mapsto \lambda_t$, and $t \mapsto \xi_t$, respectively.

Proposition 2.2.12. *Let $\Omega_0 = B_+ \cup B_-$ with $|\Omega_0|^{2/d}\lambda < |B|^{2/d}\lambda_2(B)$ with B any ball in $\mathbb{R}^d$ and u be an eigenfunction corresponding to λ. Then*

$$\lambda' = -\int_{\partial\Omega_0} |\nabla u(x)|^2 (W \cdot \nu)(x)\, d\mathcal{H}^{d-1}(x),$$

where W is a smooth vector field from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Proof. By Theorem 1.2.2 λ is simple. As it is often the case, the shape derivative of a simple eigenvalue can be obtained through a formal calculation, which can be rigorously justified *a posteriori* as in [72, Section 5.3]. For every sufficiently small $t \in [0, T)$ consider the family of maps $\mathfrak{F}(t)$ introduced prior to the statement of the Proposition. In particular, the function u_t solves

$$\begin{cases} -\Delta u_t = \lambda_t u_t + \xi_t \chi_\alpha & \text{in } \Omega_t, \\ u_t = 0 & \text{on } \partial\Omega_t, \end{cases}$$

thus by differentiating with respect to t the equations in the previous system as well as

(2.27), and evaluating at $t = 0$, we obtain

$$\begin{cases} -\Delta u' = \lambda' u + \lambda u' + \xi' \chi_\alpha & \text{in } \Omega_0, \\ u' = -|\nabla u|\, W \cdot \nu & \text{on } \partial\Omega_0, \\ \int_{\Omega_0} u'(x)\chi_\alpha(x)dx = 0, \\ \int_{\Omega_0} u(x)u'(x)dx = 0. \end{cases} \tag{2.28}$$

By multiplying the first equation in (2.28) by u and integrating over Ω_0, then performing an integration by parts and using the orthogonality of u to u' and χ_α (see (2.28) and (2.24)), we obtain

$$\int_{\Omega_0} \nabla u'(x)\nabla u(x)dx = \lambda' \int_{\Omega_0} u(x)^2 dx + \lambda \int_{\Omega_0} u'(x)u(x)dx + \xi' \int_{\Omega_0} u(x)\chi_\alpha(x)dx = \lambda'.$$

Another integration by parts, applied to the integral in the left-hand side of the previous equality, yields

$$\int_{\Omega_0} \nabla u'(x)\nabla u(x)dx = \lambda \int_{\Omega_0} u'(x)u(x)dx + \xi \int_{\Omega_0} u'(x)\chi_\alpha(x)dx - \int_{\partial\Omega_0} u'\,(\nabla u \cdot \nu)d\mathcal{H}^{d-1}(x),$$

which combined with the boundary and orthogonality conditions satisfied by u' in (2.28), gives the thesis. ☐

2.3 An open problem related to buckling eigenvalues

We provide here the link between the first intermediate eigenvalues for harmonic polynomials and the first buckling eigenvalue (see (1.12)). We denote with $\mathscr{H}(\Omega)$ the space of the harmonic functions on the set Ω that are also in $H^1(\Omega)$. We begin by recalling an useful fact.

Remark 2.3.1. Let Ω be a set of class C^∞ and $\bar{f} \in H^{1/2}(\partial\Omega) \subset L^2(\Omega)$, namely the space of admissible trace for an harmonic function. By classical elliptic theory there exists a unique harmonic function $\bar{u}$ defined in Ω such that $\bar{u}|_{\partial\Omega} = \bar{f}$. Since $C^\infty(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ we have that $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$. Moreover let $\mathcal{T}$ be a dense set in $L^2(\partial\Omega)$, as a by-product of the maximum principle the set

$$\{\bar{u} \,:\, \bar{u} \text{ is harmonic in } \Omega \text{ and } \bar{u}|_{\partial\Omega} = \bar{f}, \text{ with } \bar{f} \in \mathcal{T}\}$$

is dense in $\mathscr{H}(\Omega)$ with respect to the L^2-norm.

We present the main result of the section.

Lemma 2.3.2 (Buckling as limit of intermediate)**.** *Let Ω be a bounded open set of class C^∞ and $\mathcal{G}_n = \{h_1, \dots, h_n\}$, where h_i is an harmonic function, such that the family $\bigcup_{n\in\mathbb{N}} \mathcal{G}_n|_{\partial\Omega}$*

is dense in $L^2(\partial\Omega)$ *and* $\mathcal{G}_n \subset \mathcal{G}_{n+1}$. *Then*

$$\lim_{n\to\infty} \lambda_1^{I,\mathcal{G}_n}(\Omega) = \lambda_1^{I,\mathscr{H}(\Omega)}(\Omega) = \Lambda_1(\Omega), \tag{2.29}$$

where $\Lambda_1(\Omega)$ *is the first buckling eigenvalue of* Ω, *defined in* (1.12).

Proof. To simplify the notation we write the projections omitting the set Ω since it is fixed.

Step 1 (existence of the limit). Let u be an eigenfunction corresponding to $\Lambda := \Lambda_1(\Omega)$, namely a minimizer $u \not\equiv 0$ of (1.12), see also [11, Section 15.1, equations (15.1.1)-(15.1.2)]. From the smoothness of Ω we have $u \in H_0^2(\Omega) \cap C^\infty(\bar{\Omega})$ and

$$\Delta^2 u = -\Lambda \Delta u \quad \text{in } \Omega.$$

The relation above can be rewritten as

$$\Delta(\Delta u + \Lambda u) = 0 \quad \text{in } \Omega,$$

namely there exists an harmonic function $h \in \mathscr{H}(\Omega)$ such that

$$-\Delta u = \Lambda(u - h) \quad \text{in } \Omega. \tag{2.30}$$

Let $\lambda_n := \lambda_1^{I,\mathcal{G}_n}(\Omega)$, since $\mathcal{G}_n \subset \mathscr{H}(\Omega)$ we obtain

$$||u - P^{\mathcal{G}_n} u||^2_{L^2(\Omega)} \geq ||u - P^{\mathscr{H}(\Omega)} u||^2_{L^2(\Omega)},$$

then testing with u in (1.10) yields

$$\lambda_n \leq \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega (u - P^{\mathcal{G}_n} u)^2\, dx} \leq \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega (u - P^{\mathscr{H}(\Omega)} u)^2\, dx}. \tag{2.31}$$

Let $h' \in \mathscr{H}(\Omega)$, multiplying (2.30) by h' and integrating over Ω, after two integration by parts it follows that

$$\int_\Omega (u - h) h'\, dx = \frac{1}{\Lambda} \int_\Omega (-\Delta u) h'\, dx = 0$$

where we have used $u \in H_0^2(\Omega)$. So by the projection definition (1.6) and the arbitrariness of h' in $\mathscr{H}(\Omega)$ we deduce that $h = P^{\mathscr{H}(\Omega)} u$. Moreover, multiplying (2.30) by u, integrating by parts over Ω and using again (1.6) we obtain

$$\Lambda = \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega (u - P^{\mathscr{H}(\Omega)} u)^2\, dx}.$$

In particular (2.31) combined with the above equation yields $\lambda_n \leq \Lambda$. It turns out that $(\lambda_n)_n$ is a bounded and increasing (from $\mathcal{G}_n \subset \mathcal{G}_{n+1}$) sequence, and then $(\lambda_n)_n$ has a finite limit as $n \to \infty$.

Step 2 (a lower bound for the limit). Let $u_n \in H_0^1(\Omega)$ be an eigenfunction corresponding to λ_n such that

$$\int_\Omega |\nabla u_n|^2\, dx \leq \Lambda \quad \text{and} \quad \int_\Omega (u_n - P^{\mathcal{G}_n} u_n)^2\, dx = 1.$$

By Rellich-Kondrachov compactness theorem $u_n \to v$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, moreover by (1.6) it holds

$$\int_\Omega (u_n - P^{\mathcal{G}_n} u_n)^2\, dx = \int_\Omega (u_n - P^{\mathcal{G}_n} u_n) u_n\, dx. \tag{2.32}$$

Now we determine the weak limit in $L^2(\Omega)$ of the sequence $(P^{\mathcal{G}_n} u_n)_n$. Let $f_n := P^{\mathcal{G}_n} u_n$, from $||P^{\mathcal{G}_n}||_{\mathcal{L}(L^2(\Omega))} \leq 1$ we have

$$||f_n||_{L^2(\Omega)} \leq ||P^{\mathcal{G}_n}||_{\mathcal{L}(L^2(\Omega))} ||u_n||_{L^2(\Omega)} \leq 2||v||_{L^2(\Omega)}$$

for n large. So there exists $f \in L^2(\Omega)$ such that $f_n \to f$ weakly in $L^2(\Omega)$. By Remark 2.3.1 the family $\bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is dense in $\mathscr{H}(\Omega)$, then there exists a sequence $(h'_n)_n \subset L^2(\Omega)$ with $h'_n \in \mathcal{G}_n$ such that

$$\lim_{n\to\infty} \int_\Omega (h' - h'_n)^2\, dx = 0.$$

By (1.6) it holds

$$\int_\Omega (u_n - f_n) h'\, dx = \int_\Omega (u_n - f_n)(h' - h'_n)\, dx,$$

then letting $n \to \infty$ from the Cauchy-Schwartz inequality we obtain

$$\int_\Omega (v - f) h'\, dx = 0.$$

The projection characterization (1.6) and the arbitrariness of h' in $\mathscr{H}(\Omega)$ yields $f = P^{\mathscr{H}(\Omega)} v$. Now by letting $n \to \infty$ in (2.32) and using (1.6) we obtain

$$\lim_{n\to\infty} \int_\Omega (u_n - P^{\mathcal{G}_n} u_n)^2\, dx = \int_\Omega (v - P^{\mathscr{H}(\Omega)} v) v\, dx = \int_\Omega (v - P^{\mathscr{H}(\Omega)} v)^2\, dx.$$

In particular by the weak lower semicontinuity of the H^1-norm and the previous identity we obtain

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{\int_\Omega |\nabla u_n|^2\, dx}{\int_\Omega (u_n - P^{\mathcal{G}_n} u_n)^2\, dx} \geq \frac{\int_\Omega |\nabla v|^2\, dx}{\int_\Omega (v - P^{\mathscr{H}(\Omega)} v)^2\, dx}. \tag{2.33}$$

Step 3 (quantification of the limit). Now we define

$$\tilde{\Lambda} := \lambda_1^{I,\mathscr{H}(\Omega)}(\Omega) = \min_{v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\, dx}{\int_\Omega (v - P^{\mathscr{H}(\Omega)} v)^2\, dx}, \tag{2.34}$$

reasoning (even if the family $\mathscr{H}(\Omega)$ is not finite) as in Lemmas 2.1.5 and 2.1.7 item (b) one can show that there exists an analytic minimizer $\tilde{v} \in H_0^1(\Omega)$ for the function in the right hand side of the above equation satisfying in the strong sense

$$-\Delta\tilde{v} = \tilde{\Lambda}(\tilde{v} - P^{\mathscr{H}(\Omega)}\tilde{v}) \quad \text{in } \Omega. \tag{2.35}$$

Multiplying (2.35) by h' and integrating by parts we obtain

$$0 = \int_\Omega (-\Delta\tilde{v})h'\,dx = \int_\Omega \frac{\partial\tilde{v}}{\partial\nu}h'\,dx,$$

where we have used (1.6), thus Remark 2.3.1 and the du Bois-Reymond Lemma yields $\frac{\partial\tilde{v}}{\partial\nu} = 0$ on $\partial\Omega$. In particular by applying the Laplacian to (2.35) we have that $\tilde{v}$ solves

$$\begin{cases} -\Delta^2\tilde{v} = \tilde{\Lambda}\Delta\tilde{v} & \text{in } \Omega, \\ \tilde{v} = \frac{\partial\tilde{v}}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, $\tilde{\Lambda}$ is a *buckling eigenvalue* of Ω, see [11, Section 15.1, equations (15.1.1)-(15.1.2)], and then $\tilde{\Lambda} \geq \Lambda$. By combining (2.33), (2.34) and $\tilde{\Lambda} \geq \Lambda$ we obtain (2.29). □

We notice that the hypothesis of Lemma 2.3.2 requires the existence of a family that is dense in $L^2(\partial\Omega)$. However this assumption is not suitable to face (in an easy way) a shape optimization problem on $\mathbb{R}^d$ because the given family depends from the set (and it is defined only on that set and not globally on $\mathbb{R}^d$).

Open problem 1. *Is there an increasing (with respect to set inclusion) sequence $(\mathcal{G}_n)_n$ of families of harmonic functions on $\mathbb{R}^d$ such that for every smooth set Ω it holds*

$$\lim_{n\to\infty} \lambda_1^{I,\mathcal{G}_n}(\Omega) = \lambda_1^{I,\mathscr{H}(\Omega)}(\Omega) = \Lambda_1(\Omega)\,?$$

In addition, we ask whether this relation is true when we take

$$\mathcal{G}_n = \{h \,:\, h \textit{ is a homogeneous harmonic polynomial of degree } \leq n\}.$$

Chapter 3

Shape optimization for twisted and intermediate eigenvalues

In this chapter, we discuss shape optimization problems related to twisted and intermediate eigenvalues. We focus on proving not only existence, but also some non-existence results, of optimal shapes of the k-th twisted and intermediate eigenvalues. Indeed, we first exhibit a shape optimization problem that admits no optimal shape in either framework (twisted or intermediate), a fact established *via* a comparison argument. We subsequently introduce a (generalized) notion of γ-convergence adapted to the twisted setting and employ it to establish an existence result of optimal sets contained in a *box* (with box we always mean a bounded open set of $\mathbb{R}^d$). In the standard twisted setting (i.e., $\mathcal{G} = \{1\}$) we prove a global version of an existence result, that is of optimal sets for λ_k^T among sets in the whole $\mathbb{R}^d$. In addition, we discuss in this setting a methodology for the numerical computation of the optimal shapes. Moreover, we construct a bounded, connected, and open set whose first twisted (or intermediate) eigenfunction possesses an arbitrary number of nodal domains. More specifically, we show that a Courant's nodal domains result does not hold in general, contrary to prevailing expectations in the mathematical community. Finally we discuss an application of twisted eigenvalues to an open optimization problem involving Dirichlet eigenvalues.

3.1 Proof of the Theorem 1.4.4

We first prove a non existence result for the first twisted an intermediate eigenvalue with a particular family of constraints $\mathcal{G}$. The proof is only base on a comparison argument.

Proof of Theorem 1.4.4. The statement holds for both twisted and intermediate eigenvalues with the same value of the infimum. However the proof of the non-existence is different. In particular the minimizing sequence of sets will be the same, but the computations are slightly dissimilar.

Let $\Omega \subset \mathbb{R}^2$ be a quasi-open set, $\mathcal{G} = \{1, x, y\}$ and $\lambda_1^{\mathcal{C}}(\Omega) := \lambda_1^{\mathcal{C},\mathcal{G}}(\Omega)$. Notice that,

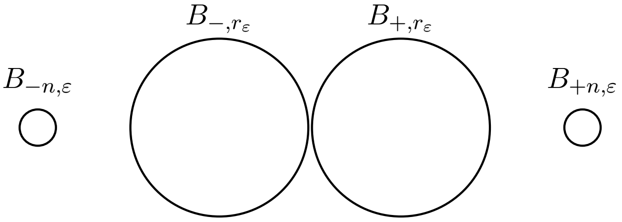

Figure 3.1: The set $\Omega_{n,\varepsilon}$ with $n = 3$ and $\varepsilon = 0.2$.

given $s > 0$, we have $\mathrm{span}(\mathcal{G}_s|_\Omega) = \mathrm{span}(\mathcal{G}|_\Omega)$, therefore Lemma 2.1.2 item (b) yields

$$\lambda_1^{\mathcal{C}}(s\Omega) = \frac{1}{s^2}\lambda_1^{\mathcal{C}}(\Omega).$$

So, without loss of generality, we may fix $m = 2\pi$ and define

$$\lambda^{xy} := \inf\{\lambda_1^{\mathcal{C}}(\Omega) : \Omega \subset \mathbb{R}^2 \text{ quasi-open}, |\Omega| = 2\pi\}.$$

By inclusion of the functional spaces, that follows from $\{1\} \subset \mathcal{G}$, we get

$$\lambda_1^{\mathcal{C}}(\Omega) \geq \lambda_1^{\{1\},\mathcal{C}}(\Omega). \tag{3.1}$$

By the results [58, Theorem 1] and [63, Theorem 1.1] for the family $\{1\}$, taking the infimum in (3.1) we obtain

$$\lambda^{xy} \geq \lambda_1^{\{1\},\mathcal{C}}(B^\wedge \cup B^\vee) = \lambda_1^D(B^\wedge), \tag{3.2}$$

where $B^\wedge, B^\vee$ are two disjoint disks of measure π.

Suppose by contradiction that the value in (1.32) is achieved by a set Ω. By the rigidity statement in [58, Theorem 1] and [63, Theorem 1.1], we have $\Omega = B^\wedge \cup B^\vee$ and an eigenfunction corresponding to $\lambda_1^{\{1\},\mathcal{C}}(\Omega)$ has to be orthogonal to $1, x, y$ (this is immediate when $\mathcal{C} = T$, while for $\mathcal{C} = I$ we would obtain a strict inequality in (3.1) if this were not the case). Since such an eigenfunction is given (up to scalar multiples) by the difference of the first Dirichlet eigenfunctions of $B^\wedge$ and $B^\vee$, see Example 2.1.11, then both $B^\wedge$ and $B^\vee$ need to have the center at the origin, namely they coincide, a contradiction.

Let $\varepsilon > 0$ and $n \in \mathbb{N} \cup \{0\}$ we introduce the set

$$\Omega_{n,\varepsilon} := B_{-n,\varepsilon} \cup B_{-,r_\varepsilon} \cup B_{+,r_\varepsilon} \cup B_{+n,\varepsilon},$$

where $B_{\pm n,\varepsilon} := B((\pm n, 0), \varepsilon)$ is the ball with center $(\pm n, 0)$ and radius ε, $B_{\pm,r_\varepsilon} := B((\pm 1, 0), r_\varepsilon)$ is the ball with center $(\pm 1, 0)$ and radius r_ε, and r_ε is such that $|\Omega_{n,\varepsilon}| = 2\pi$, see Figure 3.1. We denote by $v_\varepsilon^\pm$ and $v_{n,\varepsilon}^\pm$ the first Dirichlet eigenfunctions of $B_{\pm,r_\varepsilon}$ and $B_{\pm n,\varepsilon}$, respectively.

Let us now prove that the infimum is asymptotically attained. For the remainder of the proof, all integrals are taken with respect to the two coordinates of $\mathbb{R}^2$; therefore, we omit the variables of integration.

Step 1 (case $\mathcal{C} = I$). We define the function

$$u_{n,\varepsilon} := v_\varepsilon^- - v_\varepsilon^+.$$

Observe that $u_{n,\varepsilon}$ is orthogonal in $L^2(\Omega_{n,\varepsilon})$ to 1 and y. In addition, note that $1, x, y$ are pairwise orthogonal in $L^2(\Omega_{n,\varepsilon})$. So if we explicitly write the orthogonal projection as

$$P^{\mathcal{G}}_{\Omega_{n,\varepsilon}} u_{n,\varepsilon} = a_{n,\varepsilon} + b_{n,\varepsilon} x + c_{n,\varepsilon} y,$$

we obtain $a_{n,\varepsilon} = c_{n,\varepsilon} = 0$ and the symmetries of $u_{n,\varepsilon}$ yield

$$b_{n,\varepsilon} = -\frac{\int_{B_{+,r_\varepsilon}} x v_\varepsilon^+}{\int_{B_{+,r_\varepsilon}} x^2 + \int_{B_{+n,\varepsilon}} x^2}.$$

In addition it holds

$$\lim_{n\to+\infty} \frac{1}{n^2} \int_{B_{+n,\varepsilon}} x^2 = |B_{0,\varepsilon}|,$$

which gives $b_{n,\varepsilon} \sim n^{-2}$ as $n \to \infty$. From (3.2) and by taking $u_{n,\varepsilon}$ as a test in (1.10) we obtain (by making use of the symmetries of $u_{n,\varepsilon}$)

$$\lambda_1^D(B^\wedge) \le \lambda_1^{\mathcal{C}}(\Omega_{n,\varepsilon}) \le \frac{\int_{B_{+,r_\varepsilon}} |\nabla v_\varepsilon^+|^2}{\int_{B_{+,r_\varepsilon}} |v_\varepsilon^+ - b_{n,\varepsilon} x|^2 + b_{n,\varepsilon}^2 \int_{B_{+n,\varepsilon}} x^2}.$$

Recall that $\int_{B_{+n,\varepsilon}} x^2 \sim n^2$ as $n \to \infty$. Let us define the set $\Omega_n := \Omega_{n,n^{-1/2}}$, then we have

$$\lambda^{xy} \le \lim_{n\to\infty} \lambda_1^{\mathcal{C}}(\Omega_n) = \lambda_1^D(B^\wedge),$$

which combined with (3.2) yields the desired conclusion.

Step 2 (case $\mathcal{C} = T$). We define the function

$$u_{n,\varepsilon} := -t_{n,\varepsilon} v_{n,\varepsilon}^- + v_\varepsilon^- - v_\varepsilon^+ + t_{n,\varepsilon} v_{n,\varepsilon}^+,$$

where $t_{n,\varepsilon} > 0$ is chosen such that $u_{n,\varepsilon} \in H^1_{0,\mathcal{G}}(\Omega_{n,\varepsilon})$. Note that $u_{n,\varepsilon}$ is orthogonal in $L^2(\Omega_{n,\varepsilon})$ to 1 and y, so this last request amounts to

$$\int_{\Omega_{n,\varepsilon}} x u_{n,\varepsilon} = 0,$$

that employing the symmetries of $u_{n,\varepsilon}$ is equivalent to

$$\int_{B_{+,r_\varepsilon}} x v_\varepsilon^+ = t_{n,\varepsilon} \int_{B_{+n,\varepsilon}} x v_{n,\varepsilon}^+.$$

We notice that

$$\lim_{n\to\infty} \frac{1}{n} \int_{B_{+n,\varepsilon}} x v_{n,\varepsilon}^+ = \int_{B_{0,\varepsilon}} v_{0,\varepsilon}^+,$$

that in turn gives

$$\lim_{n\to\infty} nt_{n,\varepsilon} = \frac{\int_{B_{+,r_\varepsilon}} xv_\varepsilon^+}{\int_{B_{0,\varepsilon}} v_{0,\varepsilon}^+},$$

and in particular $t_{n,\varepsilon} \sim n^{-1}$ as $n \to \infty$. From (3.2) and by taking $u_{n,\varepsilon}$ as a test in (1.13) we obtain (making use of the symmetries of $u_{n,\varepsilon}$)

$$\lambda_1^D(B^\wedge) \le \lambda_1^{\mathcal{C}}(\Omega_{n,\varepsilon}) \le \frac{\int_{B_{+,r_\varepsilon}} |\nabla v_\varepsilon^+|^2 + t_{n,\varepsilon}^2 \int_{B_{0,\varepsilon}} |\nabla v_{0,\varepsilon}^+|^2}{\int_{B_{+,r_\varepsilon}} |v_\varepsilon^+|^2 + t_{n,\varepsilon}^2 \int_{B_{0,\varepsilon}} |v_{0,\varepsilon}^+|^2}.$$

Recall that $||\nabla v_{0,\varepsilon}^+||^2_{L^2(B_{0,\varepsilon})} \sim \varepsilon^{-2}$ as $\varepsilon \to 0^+$. Let us define the set $\Omega_n := \Omega_{n,n^{-1/2}}$, then we have

$$\lambda^{xy} \le \lim_{n\to\infty} \lambda_1^{\mathcal{C}}(\Omega_n) = \lambda_1^D(B^\wedge),$$

which combined with (3.2) yields the desired conclusion. ☐

In the following sections we present the notion of γ-convergence for twisted eigenvalues and its application to shape optimization problems.

3.2 γ-convergence and proof of Theorem 1.4.1

Assume $Q \subset \mathbb{R}^d$ is a bounded open set. In this section we will always deal with a sequence $(\Omega_n)_n$ of quasi-open subsets of Q, generic geometric perturbation of a quasi-open subset Ω of Q. We aim to generalize the notion of γ-convergence. We refer the reader for example to [29, Chapter 4] for more details. We say that Ω_n γ-converges to Ω if

$$R^D_{\Omega_n} \to R^D_\Omega \quad \text{in } \mathcal{L}(L^2(Q)),$$

where $R^D_\Omega : L^2(Q) \to L^2(Q)$ is the resolvent operator of the classical Dirichlet problem. This operator is defined as $R^D_\Omega(f) = u$, where u solves $u \in H^1_0(\Omega)$ and $-\Delta u = f$ in the weak sense. The γ-convergence is largely understood and enjoys several characterizations.

We recall an useful approximation result, see [103, Lemma 2.6].

Lemma 3.2.1 (Approximation with smooth sets)**.** *Let $\Omega \subset Q$ be a quasi-open set. Then there is a sequence $(\Omega_n)_n$ of open sets of class C^∞ contained in D, that γ-converges to Ω.*

In this section we consider $\mathcal{G} = \{g_1, \dots, g_\ell\} \subset L^2(Q)$ such that $\mathcal{G}|_\Omega$ is as in (1.3), namely made up of linearly independent functions.

3.2.1 γ-convergence in the twisted setting

We study how the twisted eigenvalues behave with respect to the γ-convergence following the approach of [72, Section 3.5], where the classical case $\mathcal{G} = \{0\}$ has been developed. In the following it will be useful this specialization of the definition of *Mosco convergence*, see [72, Definition 3.5.4].

Definition 3.2.2. We say that $H^1_{0,\mathcal{G}}(\Omega_n)$ *Mosco converges* to $H^1_{0,\mathcal{G}}(\Omega)$ if the two following conditions hold:

$(M1)_{\mathcal{G}}$ For every $u \in H^1_{0,\mathcal{G}}(\Omega)$, there exists a sequence $(u_n)_n$ with $u_n \in H^1_{0,\mathcal{G}}(\Omega_n)$ and such that $u_n \to u$ strongly in $H^1_0(Q)$.

$(M2)_{\mathcal{G}}$ For any sequence $(v_{n_m})_m$, with $v_{n_m} \in H^1_{0,\mathcal{G}}(\Omega_{n_m})$, that converges weakly in $H^1_0(Q)$ to some v, then $v \in H^1_{0,\mathcal{G}}(\Omega)$.

We will refer to these two condition with $(M1)_{\mathcal{G}}$ and $(M2)_{\mathcal{G}}$.

We give a sufficient condition for which the Mosco convergence of twisted spaces holds.

Proposition 3.2.3. *If $H^1_0(\Omega_n)$ Mosco converges to $H^1_0(\Omega)$, then $H^1_{0,\mathcal{G}}(\Omega_n)$ Mosco converges to $H^1_{0,\mathcal{G}}(\Omega)$.*

Proof. Let $\varepsilon > 0$ and $i, j \in \{1, \dots, \ell\}$. From the hypothesis (1.3) on the linear independence of the elements of $\mathcal{G}|_{\Omega}$, we have that the matrix

$$\mathcal{H} = \left(\int_{\Omega} g_i g_j \, dx \right)_{ij}$$

is invertible. Notice that there exist functions $\bar{u}^1, \dots, \bar{u}^\ell$ belonging to $H^1_0(\Omega)$ such that the matrix

$$\overline{\mathcal{H}} = \left(\int_{\Omega} \bar{u}^i g_j \, dx \right)_{ij}$$

is invertible. Indeed, by the density of $H^1_0(\Omega)$ in $L^2(\Omega)$ we can choose $\bar{u}^i$ such that $||\bar{u}^i - g_i||_{L^2(\Omega)} < \epsilon$. From the *Cauchy-Schwartz inequality* we have

$$|(\overline{\mathcal{H}})_{ij} - (\mathcal{H})_{ij}| = \left| \int_{\Omega} (\bar{u}^i - g_i) g_j \, dx \right| \leq ||\bar{u}^i - g_i||_{L^2(\Omega)} ||g_j||_{L^2(\Omega)},$$

which yields

$$||\overline{\mathcal{H}} - \mathcal{H}||_{\mathbb{R}^{\ell \times \ell}} = O(\varepsilon).$$

By choosing ϵ sufficiently small we obtain that the matrix $\overline{\mathcal{H}}$ is invertible, being a small perturbation of the invertible matrix $\mathcal{H}$.

We prove the property $(M1)_{\mathcal{G}}$ in Definition 3.2.2. Let $u \in H^1_{0,\mathcal{G}}(\Omega)$, from the property $(M1)_0$ there exists a sequence $(\widetilde{u}_n)_n$ with $\widetilde{u}_n \in H^1_0(\Omega_n)$ and such that $\widetilde{u}_n \to u$ strongly in $H^1_0(Q)$. Moreover, exploiting the same property, there exists a sequence $(\bar{u}^i_n)_n$ with $\bar{u}^i_n \in H^1_0(\Omega_n)$ such that $\bar{u}^i_n \to \bar{u}^i$ strongly in $H^1_0(Q)$.

Claim: there exists a $n_* \in \mathbb{N}$ and a unique vector $(\alpha^1_n, \dots, \alpha^\ell_n) \in \mathbb{R}^\ell$ such that the function

$$u_n = \widetilde{u}_n + \sum_{i=1}^{\ell} \alpha^i_n \bar{u}^i_n \tag{3.3}$$

belongs to $H^1_{0,\mathcal{G}}(\Omega_n)$ for every $n > n_*$.

This claim is equivalent, by multiplying (3.3) for g_j, to show that the matrix

$$\overline{\mathcal{H}}_n = \left(\int_{\Omega_n} \bar{u}_n^i g_j \, dx \right)_{ij}$$

is invertible for $n > n_*$. By the *Cauchy-Schwartz inequality* there holds

$$|(\overline{\mathcal{H}}_n)_{ij} - (\overline{\mathcal{H}})_{ij}| = \left| \int_Q (\bar{u}_n^i - \bar{u}^i) g_j \, dx \right| \leq ||\bar{u}_n^i - \bar{u}^i||_{L^2(Q)} ||g_j||_{L^2(Q)},$$

then the strong convergence $\bar{u}_n^i \to \bar{u}^i$ in $L^2(Q)$ yields

$$||\overline{\mathcal{H}}_n - \overline{\mathcal{H}}||_{\mathbb{R}^{\ell \times \ell}} = O(1/n)$$

for $n > n_*$. So we obtain that the matrix $\overline{\mathcal{H}}_n$ is invertible for $n > n_*$, being a small perturbation of the invertible matrix $\overline{\mathcal{H}}$, which proves the claim.

Noticing that $\overline{\mathcal{H}}_n \to \overline{\mathcal{H}}$ in $\mathbb{R}^{\ell \times \ell}$ (this is equivalent to the convergence in $\mathbb{R}^{\ell^2}$) and

$$\int_D (u_n - \widetilde{u}_n) g_i \, dx \to 0 \quad \text{for every } i = 1, \dots, \ell$$

we obtain $(\alpha_n^1, \dots, \alpha_n^\ell) \to (0, \dots, 0)$. Therefore, for $n > n_*$ we have that $u_n \in H^1_{0,\mathcal{G}}(\Omega_n)$and $u_n \to u$ strongly in $H^1_0(Q)$.

We prove the property $(M2)_{\mathcal{G}}$ in Definition 3.2.2. Let $(v_{n_m})_m$ be a sequence, with $v_{n_m} \in H^1_{0,\mathcal{G}}(\Omega_{n_m})$, such that $v_{n_m} \to v$ weakly in $H^1_0(Q)$. From the property $(M2)_0$ we have that $v \in H^1_0(\Omega)$. Moreover, $v_{n_m} \to v$ strongly in $L^2(Q)$ and it follows that

$$\int_D vg \, dx = \lim_{m \to +\infty} \int_D v_{n_m} g \, dx = 0,$$

which yields $v \in H^1_{0,\mathcal{G}}(\Omega)$. The proposition is proved ☐

Since $H^1_{0,\mathcal{G}}(\Omega) \subset H^1_0(\Omega)$, from the *Poincaré inequality* we have that the inner product

$$(u, v)_{H^1_{0,\mathcal{G}}(\Omega)} = \int_\Omega \nabla u(x) \nabla v(x) \, dx,$$

defined for every $u, v \in H^1_{0,\mathcal{G}}(\Omega)$, induces on $H^1_{0,\mathcal{G}}(\Omega)$ a norm that is equivalent to the standard one of $H^1_0(\Omega)$. Let $H^{-1}_{\mathcal{G}}(\Omega)$ be the dual space of $H^1_{0,\mathcal{G}}(\Omega)$. We consider the operator $-\Delta^T_\Omega$, called *twisted (Dirichlet) Laplacian*, with domain of definition $H^1_{0,\mathcal{G}}(\Omega)$, defined as

$$\langle -\Delta^T_\Omega u, v \rangle_{H^{-1}_{\mathcal{G}}(\Omega) \times H^1_{0,\mathcal{G}}(\Omega)} = \int_\Omega \nabla u(x) \nabla v(x) \, dx$$

for every $u, v \in H^1_{0,\mathcal{G}}(\Omega)$, where $\langle \cdot, \cdot \rangle_{H^{-1}_{\mathcal{G}}(\Omega) \times H^1_{0,\mathcal{G}}(\Omega)}$ is the duality pairing between $H^{-1}_{\mathcal{G}}(\Omega)$ and $H^1_{0,\mathcal{G}}(\Omega)$. One can prove that $-\Delta^T_\Omega$ is a positive self-adjoint operator, by [19, Theorem 5.15] and (1.13) we have that its corresponding sequence of eigenvalues is $(\lambda^T_n(\Omega))_n$. An eigenfunction of $-\Delta^T_\Omega$ with associated eigenvalue $\lambda^T_k(\Omega)$ satisfy the equation (2.8), so it is a twisted eigenfunction corresponding to $\lambda_k^T(\Omega)$ and vice-versa.

Extending the approach of the classical theory we define the *resolvent of the twisted (Dirichlet) Laplacian* on Ω as

$$\begin{aligned} R_\Omega^T : H_{\mathcal{G}}^{-1}(\Omega) &\to H_{0,\mathcal{G}}^1(\Omega) \\ f \quad &\mapsto u_{\Omega,T}^f, \end{aligned} \tag{3.4}$$

where $u_{\Omega,T}^f \in H_{0,\mathcal{G}}^1(\Omega)$ is the unique solution (by the *Riesz representation theorem*) of

$$\int_\Omega \nabla u_{\Omega,T}^f(x) \nabla \varphi(x)\, dx = \langle f, \varphi \rangle_{H_{\mathcal{G}}^{-1}(\Omega) \times H_{0,\mathcal{G}}^1(\Omega)} \quad \text{for every } \varphi \in H_{0,\mathcal{G}}^1(\Omega). \tag{3.5}$$

In particular, if $f \in L_{\mathcal{G}}^2(\Omega)$ this relation reads as

$$\int_\Omega \nabla u_{\Omega,T}^f(x) \nabla \varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx \quad \text{for every } \varphi \in H_{0,\mathcal{G}}^1(\Omega). \tag{3.6}$$

Since $H_{0,\mathcal{G}}^1(\Omega) \subset L^2(\Omega)$ by duality we have $L^2(\Omega) \subset H_{\mathcal{G}}^{-1}(\Omega)$, so we can restrict the resolvent operator to act on $L^2(\Omega)$, namely we consider $R_\Omega^T : L^2(\Omega) \to L^2(\Omega)$. This operator is linear, bounded, self-adjoint, positive and compact.

We also extend the notion of γ-convergence, see [72, Definition 3.5.1].

Definition 3.2.4. We say that the sequence $(\Omega_n)_n$ γ_T*-converges* to Ω (and we write $\Omega_n \xrightarrow{\gamma_T} \Omega$) if for every $f \in H_{\mathcal{G}}^{-1}(Q)$ then $u_{\Omega_n,T}^f \to u_{\Omega,T}^f$ strongly in $H_0^1(Q)$.

Remark 3.2.5. In the classical case $\mathcal{G} = \{0\}$, the γ-convergence of Ω_n to Ω is equivalent to the Mosco convergence of $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$, see [72, Proposition 3.5.5].

Lemma 3.2.6. *Let $f \in H_{\mathcal{G}}^{-1}(Q)$. There exists a constant C depending only on Q such that for every quasi-open subset $\Omega \subset Q$, the solution of* (3.5) *satisfies*

$$||u_{\Omega,T}^f||_{H_0^1(Q)} \leq C||f||_{H_{\mathcal{G}}^{-1}(Q)}$$

Proof. Let $u = u_{\Omega,T}^f$. From (3.5), exploiting the duality pairing, we obtain

$$\int_\Omega |\nabla u(x)|^2\, dx \leq ||f||_{H_{\mathcal{G}}^{-1}(Q)} ||u||_{H_0^1(Q)}.$$

By the *Poincaré inequality*, there exists a constant $\bar{C} > 0$ depending only on Q such that

$$\bar{C}||u||_{H_0^1(Q)}^2 \leq \int_D |\nabla u(x)|^2\, dx,$$

since $\Omega \subset Q$ we reach the desired inequality. □

We present a partial extension of Remark 3.2.5.

Proposition 3.2.7. *If $H_{0,\mathcal{G}}^1(\Omega_n)$ Mosco converges to $H_{0,\mathcal{G}}^1(\Omega)$, then $\Omega_n \xrightarrow{\gamma_T} \Omega$.*

Proof. Let $f \in L^2(Q)$ and $u_n = u^f_{\Omega_n,T}$. By Lemma 3.2.6 the sequence $(u_n)_n$ is bounded in $H^1_0(Q)$, then there exists $u^* \in H^1_0(Q)$ such that $u_n \to u^*$ weakly in $H^1_0(Q)$. Since $u_n \in H^1_0(\Omega_n)$, from $(M2)_\mathcal{G}$ the function $u^* \in H^1_0(\Omega)$.

Let $\psi \in H^1_{0,\mathcal{G}}(\Omega)$, by $(M1)_\mathcal{G}$ there exists a sequence $(\psi_n)_n$ with $\psi_n \in H^1_{0,\mathcal{G}}(\Omega_n)$ such that $\psi_n \to \psi$ strongly in $H^1_0(Q)$. By testing in (3.5) with $u = u_n, \varphi = \psi_n$ and then passing to the limit as $n \to +\infty$ we obtain

$$\int_D \nabla u^* \nabla \psi \, dx = \lim_{n\to+\infty} \int_D \nabla u_n \nabla \psi_n \, dx = \lim_{n\to+\infty} \int_D f\psi_n \, dx = \int_D f\psi \, dx.$$

This means that u^* solves (3.5) and by uniqueness of the solution we obtain $u^* = u^f_{\Omega,T}$. By testing in (3.5) with $u = u_n, \varphi = u_n$ and then passing to the limit, since $u_n \to u^*$ strongly in $L^2(Q)$, we have

$$\lim_{n\to+\infty} \int_D |\nabla u_n|^2 \, dx = \lim_{n\to+\infty} \int_D f u_n \, dx = \int_D f u^* \, dx = \int_D |\nabla u^*|^2 \, dx.$$

This combined with the strong convergence in $L^2(Q)$ of u_n to u^* yields $u_n \to u^*$ strongly in $H^1_0(Q)$. This proves the proposition. ☐

We present an extension [72, Lemma 4.7.3].

Proposition 3.2.8. *If* $\Omega_n \xrightarrow{\gamma_T} \Omega$, *then* $R^T_{\Omega_n} \to R^T_\Omega$ *in the uniform operator topology of* $\mathcal{L}(L^2(Q))$, *namely*

$$\lim_{n\to+\infty} \sup_{||f||_{L^2(Q)}\le 1} ||R^T_{\Omega_n}(f) - R^T_\Omega(f)||_{L^2(Q)} = 0.$$

Proof. The operators $R_n := R^T_{\Omega_n}$ and $R := R^T_\Omega$ are continuous from $H^{-1}_\mathcal{G}(Q)$ to $L^2(Q)$, with norm bounded by the constant C of Lemma 3.2.6.

There exists f^n in the unit ball of $L^2(Q)$ such that

$$\sup_{||f||_{L^2(Q)}\le 1} ||R_n f - Rf||_{L^2(Q)} = ||R_n f^n - Rf^n||_{L^2(Q)}. \tag{3.7}$$

Indeed, let $(f_k)_k$ be a maximizing sequence in $L^2(Q)$, since it is bounded there exists a subsequence weakly converging in $L^2(Q)$ to some f^n that belongs to the unit ball of $L^2(Q)$. The embedding of $L^2(Q)$ in $H^{-1}_\mathcal{G}(Q)$ is compact, because it is the adjoint of a compact embedding. Therefore letting $k \to +\infty$ we deduce the existence of a function achieving the supremum as in (3.7).

Similarly there exists a function f in the unit ball of $L^2(Q)$ such that $(f^n)_n$ (or a subsequence) converges weakly in $L^2(Q)$ and strongly in $H^{-1}_\mathcal{G}(Q)$ to f. Given $\varepsilon > 0$, by Definition 3.2.4 there exists $\bar{n} \in \mathbb{N}$ such that for $n \ge \bar{n}$ we have

$$||f^n - f||_{H^{-1}_\mathcal{G}(Q)} \le \frac{\varepsilon}{4C} \quad \text{and} \quad ||R_n f - Rf||_{L^2(Q)} \le \frac{\varepsilon}{2}, \tag{3.8}$$

where C is the constant in the statement of Lemma 3.2.6. Eventually we obtain

$$\begin{aligned}||R_n f^n - R f^n||_{L^2(Q)} &\leq ||R_n f - Rf||_{L^2(Q)} + ||R_n(f^n - f) - R(f^n - f)||_{L^2(Q)} \\ &\leq \frac{\varepsilon}{2} + ||R_n - R||_{\mathcal{L}(H_{\mathcal{G}}^{-1}(Q), H^1_{0,\mathcal{G}}(Q))} \cdot ||f^n - f||_{H_{\mathcal{G}}^{-1}(Q)} \\ &\leq \varepsilon,\end{aligned}$$

where the inequalities follows from the boundedness of the operators R_n, R and (3.8). Letting $n \to +\infty$, from the arbitrariness of ε, the convergence of R_n to R in the uniform topology follows. ☐

Since the resolvent R^T_Ω is a compact, selfadjoint, and positive operator, by standard spectral theory, see for example [19, Section 4.4.1], we have a countable sequence $(\lambda_n(R^T_\Omega))_n$ of corresponding eigenvalues. By definition of R^T_Ω we have that it is the inverse operator of $-\Delta^T_\Omega$, therefore it holds

$$\lambda_k(R^T_\Omega) = \frac{1}{\lambda^T_k(\Omega)}. \tag{3.9}$$

An eigenfunctions corresponding to $\lambda_k(R^T_\Omega)$ is also corresponding to $\lambda^T_k(\Omega)$, and vice-versa. We conclude with an application of γ-convergence to continuity of eigenvalues.

Corollary 3.2.9. *If* $\Omega_n \xrightarrow{\gamma} \Omega$*, then*

$$\lim_{n\to+\infty} \lambda^T_k(\Omega_n) = \lambda^T_k(\Omega).$$

Moreover, any sequence of L^2*-normalized eigenfunctions associated to* $\lambda^T_k(\Omega_n)$ *strongly converge (up to a subsequence) in* $H^1_0(Q)$ *to an eigenfunction associated to* $\lambda^T_k(\Omega)$*. If* $\lambda^T_k(\Omega)$ *is simple, the full sequence converges.*

Proof. This is a consequence of the convergence of the resolvent operators, see [47, Corollary 4 (a), p. 1090]. ☐

We are now ready to prove the existence of an optimal shape in a box, a result that extends [33, Example 2.6].

Proof of Theorem 1.4.1. We recall that twisted eigenvalues are monotone with respect to set inclusion, see Lemma 2.1.1. The hypothesis (1.4) allows us to be in the setting of Section 3.2. Moreover, by Corollary 3.2.9 the twisted eigenvalues are continuous with respect to γ-convergence, then from the *Buttazzo-Dal Maso theorem*, see [33, Theorem 2.5], the shape optimization problem

$$\min\{\lambda^T_k(\Omega) : \Omega \subset Q,\ \Omega \text{ quasi-open},\ |\Omega| = m\}$$

has a solution. ☐

3.2.2 γ-convergence in the intermediate setting

In this section we study the behavior of intermediate eigenvalues with respect to the γ-convergence. The case of $\mathcal{G} = \{1\}$ was analyzed in [31, Section 2] where it was observed that, contrary to twisted eigenvalues, the γ-convergence alone does not imply the stability (continuity) of intermediate eigenvalues.

Proposition 3.2.14 below extends [31, Theorems 2 and 3]. The key difference with the twisted case is that the convergence of measures is also required in order to get some information on the convergence of eigenvalues. It is well known that γ-convergence alone does not imply the convergence of measures. Indeed let $n \in \mathbb{N}$ and define

$$\Omega_n := [0,1]^2 \setminus \bigcup_{k=1}^{n} \left(\left\{ \frac{k}{n} \right\} \times \left[\frac{1}{2}, 1 \right] \right).$$

We have that Ω_n γ-converges to Ω, where $\Omega := (0,1) \times \left(0, \frac{1}{2}\right)$, see Figure 3.2, but

$$\lim_{n \to +\infty} |\Omega_n| = 1 \neq \frac{1}{2} = |\Omega|.$$

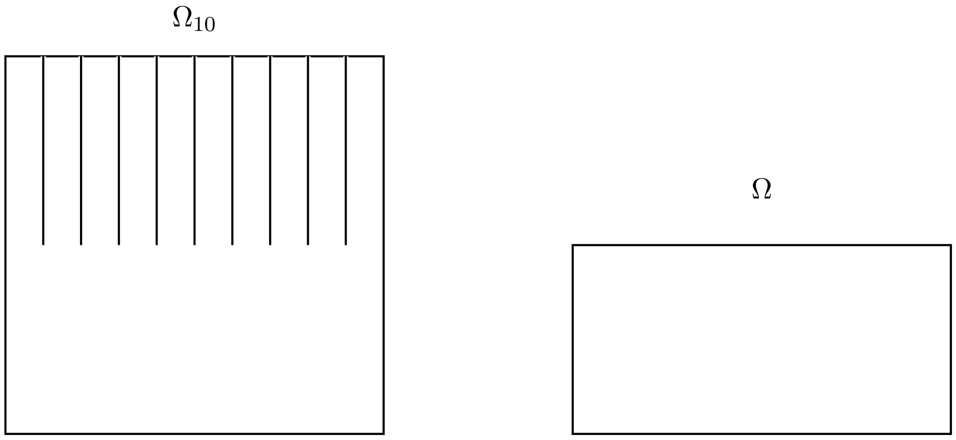

Figure 3.2: The sets Ω_{10} and Ω.

We recall the notion of weak γ-convergence of sets from [29, Definition 5.3.1].

Definition 3.2.10. We say that the sequence $(\Omega_n)_n$ *weakly γ-converge* to Ω (and we write $\Omega_n \xrightarrow{w\gamma} \Omega$), if w_{Ω_n} (the torsion function of the set Ω_n) converges weakly in $H_0^1(Q)$ to a function $w \in H_0^1(Q)$ such that $\Omega = \{w > 0\}$.

Remark 3.2.11. Recall that if $\Omega_n \xrightarrow{\gamma} \Omega$, then $\Omega_n \xrightarrow{w\gamma} \Omega$, see [29, Proposition 5.3.3].

Definition 3.2.12. Let $f \in L^2(\Omega)$. We say that f is *P_Ω-normalized* if

$$\int_{\Omega} (f - P_\Omega f)^2 \, dx = 1. \tag{3.10}$$

Let $n \in \mathbb{N}$. A family of functions $\{f_1, \dots, f_n\} \subset L^2(\Omega)$ is called P_Ω-*orthonormal* if

$$\int_\Omega (f_i - P_\Omega f_i)(f_j - P_\Omega f_j)\, dx = \delta_{ij}, \tag{3.11}$$

for every $i, j \in \{1, \dots, n\}$, where δ_{ij} is the *Kronecker delta.*

We begin with a technical result.

Lemma 3.2.13. *Let $(v_n)_n \subset H^1_0(Q)$ be a bounded sequence in $H^1(Q)$, such that $v_n \in H^1_0(\Omega_n)$ and v_n satisfies* (3.10) *in Ω_n. If $\Omega_n \xrightarrow{w\gamma} \Omega$ and $|\Omega_n| \to |\Omega|$, then there exists a function $v \in H^1_0(\Omega)$ that satisfies* (3.10) *such that $v_n \to v$ weakly in $H^1_0(Q)$.*

Proof. By the boundedness of the sequence $(v_n)_n$ we have $v_n \to v$ weakly in $H^1_0(Q)$ and strongly in $L^2(Q)$. Moreover, from the weak γ-convergence hypothesis we obtain $v \in H^1_0(\Omega)$, see [33, Lemma 3.2]. So it remains to show that v satisfies (3.10).

The hypothesis $|\Omega_n| \to |\Omega|$ and [29, Proposition 5.3.6] yield

$$\chi_{\Omega_n} \to \chi_\Omega \text{ strongly in } L^1(Q),$$

since Q is bounded and the sequence $(\chi_{\Omega_n})_n$ is uniformly bounded in $L^\infty(Q)$, by the dominated convergence Theorem, the convergence of χ_{Ω_n} also holds in $L^2(Q)$. We claim that $P_{\Omega_n} v_n \to P_\Omega v$ strongly in $L^2(Q)$. Indeed, let $n \in \mathbb{N}$ sufficiently large, there exist $a_n = (a_n^1, \dots, a_n^\ell) \in \mathbb{R}^\ell$ such that

$$P_{\Omega_n} v_n = \sum_{i=1}^{\ell} a_n^i g_i.$$

The vector a_n is the unique solution of the linear system $\mathcal{H}_n a_n = \mathcal{V}_n$, where

$$\mathcal{H}_n = \left(\int_{\Omega_n} g_i g_j\, dx \right)_{ij} \quad \text{and} \quad \mathcal{V}_n = \left(\int_{\Omega_n} v_n g_i\, dx \right)_i.$$

Similarly the vector $a = (a^1, \dots, a^\ell) \in \mathbb{R}^\ell$ defined by

$$P_\Omega v = \sum_{i=1}^{\ell} a^i g_i,$$

is the unique solution of the linear system $\mathcal{H} a = \mathcal{V}$, where

$$\mathcal{H} = \left(\int_\Omega g_i g_j\, dx \right)_{ij} \quad \text{and} \quad \mathcal{V} = \left(\int_\Omega v g_i\, dx \right)_i.$$

The weak convergence of $(v_n)_n$ in $H^1(Q)$ and the dominated convergence theorem yields

$$\lim_{n \to +\infty} \int_{\Omega_n} v_n g_i\, dx = \int_\Omega v g_i\, dx \quad \text{and} \quad \lim_{n \to +\infty} \int_{\Omega_n} g_i g_j\, dx = \int_\Omega g_i g_j\, dx,$$

so that $a_n \to a$ in $\mathbb{R}^\ell$. In particular the sequence $(P_{\Omega_n} v_n)_n$ converges strongly in $L^2(Q)$ to

$P_\Omega v$, giving as a by-product $\int_\Omega (v - P_\Omega v)^2\, dx = 1$. This proves the lemma. $\square$

We prove a result that is related to [31, Theorem 2].

Proposition 3.2.14. *If* $\Omega_n \xrightarrow{w\gamma} \Omega$ *and* $|\Omega_n| \to |\Omega|$, *then*

$$\liminf_{n\to+\infty} \lambda_k^I(\Omega_n) \geq \lambda_k^I(\Omega). \tag{3.12}$$

If, in addition to the previous hypotheses, $\Omega_n \xrightarrow{\gamma} \Omega$, *then*

$$\lim_{n\to+\infty} \lambda_k^I(\Omega_n) = \lambda_k^I(\Omega). \tag{3.13}$$

Proof. We first prove the inequality (3.12). Let $n \in \mathbb{N}$ and $\{u_n^1, \dots, u_n^k\}$ be a family of eigenfunctions corresponding to $\lambda_1^I(\Omega_n), \dots, \lambda_k^I(\Omega_n)$ satisfying (3.11) in Ω_n. The existence of such a family follows from (2.4) and a reasoning analogous to the *Gram–Schmidt orthonormalization process.* If $\liminf\limits_{n\to+\infty} \lambda_k^I(\Omega_n) = +\infty$, then there is nothing to prove. If not, let $j \in \{1, \dots, k\}$, up to subsequences, we can assume that

$$\int_Q |\nabla u_n^j(x)|^2\, dx \leq M,$$

where M is a positive constant independent of j and n. By Sobolev embedding theorem we have that there exists $u^j \in H_0^1(\Omega)$ such that $u_n^j \to u^j$ weakly in $H_0^1(Q)$ and strongly in $L^2(Q)$. Moreover, byLemma 3.2.13, $\{u^1, \dots, u^k\}$ satisfy (3.11), therefore

$$\operatorname{span}(\{u^1 - Pu^1, \dots, u^k - Pu^k\})$$

is of dimension k. It follows that $\operatorname{span}\{u^1, \dots, u^k\}$ has dimension k, since the linear independence of the projected vectors implies the linear independence of the original vectors. Let $a = (a^1, \dots, a^k) \in \mathbb{R}^k$, we define

$$\psi_n := \sum_{i=1}^k a^i u_n^i,$$

by (3.11) in Ω_n, (2.4) and Lemma 2.2.7 we have $\mathscr{R}_{\Omega_n}(\psi_n) \leq \lambda_k^I(\Omega_n)$. Let

$$\psi := \sum_{i=1}^k a^i u^i,$$

by construction we have $\psi_n \to \psi$ weakly in $H_0^1(Q)$ and strongly in $L^2(Q)$, so that

$$\mathscr{R}_\Omega(\psi) \leq \liminf_{n\to+\infty} \mathscr{R}_{\Omega_n}(\psi_n) \leq \liminf_{n\to+\infty} \lambda_k^I(\Omega_n).$$

Since ψ is obtained as a limit with respect to n of a linear combination with coefficients independent from n, then $\psi \in \operatorname{span}(\{u^1, \dots, u^k\})$. from the arbitrariness of ψ in span($\{u^1,\dots,u^k\}$), by (1.10) we obtain (3.12).

Assume now that $\Omega_n \xrightarrow{\gamma} \Omega$, we prove (3.13). Let $\{v^1,\dots,v^k\}$ be a family of eigenfunctions corresponding to $\lambda_1^I(\Omega),\dots,\lambda_k^I(\Omega)$ that satisfies (3.11). From Remark 3.2.5, there exist a sequence $(v_n^j)_n$ such that $v_n^j \in H_0^1(\Omega_n)$ and $v_n^j \to v^j$ strongly in $H_0^1(Q)$. Moreover, for n sufficiently large, analogously to the *Gram–Schmidt orthonormalization process*, the family $\{v_n^1,\dots,v_n^k\}$ can be chosen to satisfy (3.11) in Ω_n. Let $b_n = (b_n^1,\dots,b_n^k)$ be a vector of unit norm in $\mathbb{R}^k$, we define the function

$$\phi_n(b_n) := \sum_{i=1}^k b_n^i v_n^i.$$

By compactness of the unit sphere of $\mathbb{R}^k$ there exists a unit vector $\mathbf{b}_n = (\mathbf{b}_n^1,\dots,\mathbf{b}_n^k)$ such that

$$\max_{\substack{b\in\mathbb{R}^k\\ |b|=1}} \mathscr{R}_{\Omega_n}(\phi_n(b_n)) = \mathscr{R}_{\Omega_n}(\phi_n(\mathbf{b}_n)).$$

We define $\varphi_n := \phi_n(\mathbf{b}_n)$, the variational characterization (1.10) yields $\lambda_k^I(\Omega_n) \le \mathscr{R}_{\Omega_n}(\varphi_n)$. Again, by compactness of the unit sphere, there exists a unit vector $\mathbf{b} = (\mathbf{b}^1,\dots,\mathbf{b}^k)$, such that, up to subsequences $\mathbf{b}_n \to \mathbf{b}$ in $\mathbb{R}^k$. We define the function

$$\varphi := \sum_{i=1}^k \mathbf{b}^i v^i,$$

from (3.11), (2.4) and Lemma 2.2.7 we obtain $\mathscr{R}_\Omega(\varphi) \le \lambda_k^I(\Omega)$. By construction we have $\varphi_n \to \varphi$ strongly in $H_0^1(Q)$ and in particular letting $n \to +\infty$ we obtain

$$\limsup_{n\to+\infty} \lambda_k^I(\Omega_n) \le \limsup_{n\to+\infty} \mathscr{R}_{\Omega_n}(\varphi_n) = \mathscr{R}_\Omega(\varphi) \le \lambda_k^I(\Omega),$$

that yields (3.13). ☐

3.3 Proof of Theorem 1.4.3

In order to follow the strategy outlined in [28] that allows to obtain the global existence of an optimal shape for the k-th twisted eigenvalue we need two fundamental results:

- an uniform L^∞ estimate of the eigenfunctions;
- a control on the variation of the eigenvalue with respect to domain perturbations, by means of the torsion energy.

3.3.1 Uniform L^∞ bounds of the eigenfunctions.

The following inequality provides a uniform L^∞ bound for twisted and intermediate eigenfunctions. It is similar (same statement) to the one of Davies [44, Lemma 3.1] for the

Dirichlet Laplacian. However, it is not a direct consequence of it, since twisted and intermediate problems have a non-local character and do not fit into the classical framework.

Proposition 3.3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Let u be an eigenfunction corresponding to $\lambda = \lambda_k^{\mathcal{C}}(\Omega)$ with $\mathcal{C} \in \{I, T\}$ and $\mathcal{G} = \{1\}$. There exists a dimensional constant C_d such that*

$$\|u\|_\infty \leq C_d \lambda^{\frac{d}{4}} \|u\|_{L^2(\Omega)}. \tag{3.14}$$

Proof. We can assume without loosing generality that Ω is smooth. Indeed, a non-smooth domain can be approximated by γ-convergence, see Lemma 3.2.1. By Lemmas 2.1.5, 2.1.6 and 2.1.7, there exists $c = c(u, \mathcal{C}) \in \mathbb{R}$ such that $u \in H_0^1(\Omega)$ satisfies

$$\begin{cases} -\Delta u = \lambda u + c \quad \text{in } \Omega, \text{ in the strong sense} & \text{if } \mathcal{C} = I \\ -\Delta u = \lambda u + c \quad \text{in } \Omega, \ u = 0 \text{ on } \partial\Omega, \text{ in the weak sense} & \text{if } \mathcal{C} = T. \end{cases} \tag{3.15}$$

Integrating in Ω when $\mathcal{C} = I$ and testing with $\psi = w_\Omega$ when $\mathcal{C} = T$, where $w_\Omega \in H_0^1(\Omega)$ is the *torsion function* of the set Ω, since by (2.6) we have

$$\int_\Omega \Delta u(x)\, dx = 0 \ \text{ if } \mathcal{C} = I, \qquad \int_\Omega u(x)\, dx = 0 \ \text{ if } \mathcal{C} = T,$$

we obtain

$$c = c(u, \mathcal{C}) = \begin{cases} -\lambda \dfrac{\int_\Omega u\, dx}{|\Omega|} & \text{if } \mathcal{C} = I, \\ -\lambda \dfrac{\int_\Omega u w_\Omega\, dx}{\int_\Omega w_\Omega\, dx} & \text{if } \mathcal{C} = T. \end{cases} \tag{3.16}$$

We prove the inequality in two steps following the scheme developed by Velichkov in [109, Proposition 3.4.37].

Step 1 (the eigenfunction $u \in L^\infty(\Omega)$). Note that u is continuous (analytic) in Ω, as a consequence of Lemma 2.1.7. Let $\Omega^+ := \{u > 0\}$ and $\Omega^- := \{u > 0\}$, we have $u = u^+ - u^- := u\chi_{\Omega^+} - u\chi_{\Omega^-}$ and in particular $u^+ \in H_0^1(\Omega^+)$ solves

$$-\Delta u^+ = \lambda u^+ + c \quad \text{in } \Omega^+.$$

Let u_+^* be the (spherically) symmetric decreasing rearrangement (the so-called *Schwartz symmetrization*) of the function u^+, see [76, Section 1.3]. Let B be the ball centered in the origin of $\mathbb{R}^d$ with $|B| = |\Omega^+|$ and $v \in H_0^1(B)$ be the function satisfying

$$-\Delta v = \lambda u_+^* + c \quad \text{in } B. \tag{3.17}$$

Since $u^+ \geq 0$ in Ω^+, we can apply *Talenti's theorem*, see [107, Theorem 1] or [76, Section 3.1], and obtain

$$u_+^* \leq v \quad \text{in } B. \tag{3.18}$$

Let $f = \lambda u_+^* + c$, since $v \in L^{p_0}(B)$ with $p_0 = 2$, by (3.18) we have $f \in L^{p_0}(B)$. We now describe the first step of an iterative argument:

- since it holds (3.17) and $f \in L^{p_0}(B)$ by elliptic regularity theory, see [60, Theorem 9.15], we obtain $v \in W^{2,p_0}(B)$, where $W^{2,p_0}(B)$ is the second-order Sobolev space with exponent p_0;
- from the Sobolev embedding theorem $v \in L^{p_1}(B)$ with $p_1 := \left(\frac{d}{d-2}\right) p_0$;
- the inequality (3.18) yields $u_+^* \in L^{p_1}(B)$ and therefore $f \in L^{p_1}(B)$.

We continue the procedure and at the n-th iteration, we obtain $v \in L^{p_n}(B)$ with $p_n := \left(\frac{d}{d-2}\right)^n p_0$. In particular there exists $\bar{n} \in \mathbb{N}$ such that $p_{\bar{n}} > d/2$, then by the Sobolev embedding theorem $v \in L^\infty(B)$. So the inequality (3.18) yields $u_+^* \in L^\infty(B)$ and finally $u^+ \in L^\infty(B)$. The same reasoning applies for u^-, and then $u \in L^\infty(\Omega)$.

Step 2 (L^∞-estimate for the function u). Suppose $\|u\|_{L^2(\Omega)} = 1$, otherwise we normalize u in $L^2(\Omega)$. Let us define $M := \lambda\|u^+\|_\infty + |c|$. Assuming that u^+ is extended by 0 on $\mathbb{R}^d \setminus \Omega$, on the whole Euclidean space it holds $-\Delta u^+ \leq M$ in $\mathcal{D}'(\mathbb{R}^d)$, namely in the sense of distributions. Let $x_0 \in \Omega$, the function in $L^1_{\text{loc}}(\Omega)$ defined as

$$x \to u^+(x) + M\frac{\|x - x_0\|^2}{2d}$$

is subharmonic in $\mathbb{R}^d$ in the sense of distribution (from now on we replace it with the representative defined in [85, Theorem 9.3]). Let $R > 0$ and B_R the ball with radius R with center in x_0, by the *mean value inequality* (see [85, Theorem 9.3, third black point]) we have

$$u^+(x_0) \leq \frac{1}{|B_R|}\int_{B_R}\left[u^+(x) + M\frac{\|x - x_0\|^2}{2d}\right] dx.$$

Let C_d be a dimensional constant (which from now on in this proof could vary from line to line), by the *Cauchy-Schwartz inequality* we can estimate

$$u^+(x_0) \leq C_d\left(\|u^+\|_{L^2(B_R)}R^{-\frac{d}{2}} + MR^2\right).$$

Since $\|u^+\|_{L^2(B_R)} \leq 1$, by minimizing the right-hand side with respect to R, we have

$$u^+(x_0) \leq C_d M^{\frac{d}{d+4}}.$$

From the arbitrariness of x_0 in Ω, we get

$$\|u^+\|_\infty \leq C_d(\lambda\|u^+\|_\infty + |c|)^{\frac{d}{d+4}},$$

moreover by (3.16) we obtain $|c| \leq \lambda\|u\|_\infty$, which in turn gives

$$\|u^+\|_\infty \leq C_d\lambda^{\frac{d}{4}}.$$

The same reasoning applies for u^-, the relation $\|u\|_\infty \leq \max\{\|u^+\|_\infty, \|u^-\|_\infty\}$ and the normalization of u yield (3.14). $\square$

3.3.2 Estimate of twisted eigenvalues by torsion energies

We begin with an elementary fact that will be used only in this subsection.

Lemma 3.3.2. *Let A be a set. If $f_1 : A \to \mathbb{R}$ and $f_2 : A \to \mathbb{R}$ are bounded functions, then*

$$\left| \inf_{x\in A} f_1(x) - \inf_{x\in A} f_2(x) \right| \leq \sup_{x\in A} |f(x) - g(x)| \,.$$

The next result is fundamental for the proof of Theorem 1.4.3, which will be shown following the ideas of [28], it has a function similar to [28, Lemma 3].

Theorem 3.3.3. *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ be two quasi-open sets of finite measure and $\mathcal{G} = \{1\}$. There exists a dimensional constant C_d such that*

$$0 \leq \frac{1}{\lambda_k^T(\Omega_2)} - \frac{1}{\lambda_k^T(\Omega_1)} \leq C_d k^2 \lambda_k^T(\Omega_2)^{\frac{d}{2}} \int_{\Omega_2} w_{\Omega_2}(x) - w_{\Omega_1}(x)\, dx, \tag{3.19}$$

where $w_{\Omega_2}, w_{\Omega_1}$ are the torsion function of Ω_1, Ω_2 respectively.

Proof. By Lemma 3.2.1 we can assume that Ω_1 and Ω_2 are smooth open sets. Recall that the connected components of an open set in $\mathbb{R}^d$ are at most countable. We define the connected set Ω_2^n by attaching thin tubes, of total measure $1/n$, between the connected components of Ω_2. The sequence $(\Omega_2^n)_n$ of connected sets γ-converges to Ω_2. So we can assume that Ω_2 is connected.

Step 0 (preliminary notions). Let us consider a family of $L^2(\Omega_2)$-orthonormal eigenfunctions $\mathcal{U} := \{u_1, \dots, u_k\}$ corresponding to $\lambda_1^T(\Omega_2), \dots, \lambda_k^T(\Omega_2)$. By Proposition 3.3.1 we have that $\mathcal{U} \subset L^\infty(\Omega_2)$. From Lemma 2.1.7 any element of $\mathcal{U}$ is analytic in Ω_2.

Let Ω be a quasi-open set. Instead of evaluating the twisted eigenvalue $\lambda_k^T(\Omega)$ directly exploiting the Rayleigh quotient, we shall rather estimate its reciprocal. From (3.9), this quantity correspond to $\lambda_k(R_\Omega^T)$, an eigenvalue of the associated resolvent operator $R_\Omega^T : L^2(\Omega) \to L^2(\Omega)$, introduced in (3.4). Let $\psi \in H_0^1(\Omega)$, by plugging

$$\varphi = \psi - \frac{\int_\Omega \psi\, dx}{\int_\Omega w_\Omega\, dx} w_\Omega$$

in (3.6) one notice that this operator can be equivalently defined by

$$R_\Omega^T(f) = u,$$

where $u \in H_0^1(\Omega)$ solves in the weak sense

$$-\Delta u = f + c \text{ in } \Omega, \text{ with } c = -\frac{\int_\Omega f w_\Omega\, dx}{\int_\Omega w_\Omega\ dx}. \tag{3.20}$$

From the *Courant-Fischer theorem*, see [19, Theorem 4.22] we have that

$$\lambda_k(R_\Omega^T) = \max_{S\in\mathcal{S}_k} \min_{\substack{\phi\in S\\ \phi\neq 0}} \frac{(R_\Omega^T\phi,\phi)_{L^2(\Omega)}}{||\phi||_{L^2(\Omega)}}, \tag{3.21}$$

where $\mathcal{S}_k$ is the family of subspaces of dimension k in the space $L^2(\Omega)$. Let $j\in\{1,2\}$, to simplify the notation we adopt the following conventions: we denote the torsion function $w_j = w_{\Omega_j}$, the resolvent operator $R_j = R^T_{\Omega_j}$, the inner product $(\cdot,\cdot)_j = (\cdot,\cdot)_{L^2(\Omega_j)}$ and the corresponding norm $||\cdot||_j = ||\cdot||_{L^2(\Omega_j)}$.

Step 1 (control with the resolvent operators). We show that

$$\lambda_k(R_2) - \lambda_k(R_1) \le k \max_{1\le i\le k} ||u_i||_\infty \sum_{i=1}^k \int_{\Omega_2} |R_2u_i - R_1u_i|\,dx. \tag{3.22}$$

Let us consider the set

$$E := \text{span}\ \{u_1|_{\Omega_1},\dots,u_k|_{\Omega_1}\}.$$

By analyticity and connectedness of Ω_2 we obtain $\dim(E) = k$ in $L^2(\Omega_1)$, otherwise span($\mathcal{U}$) would have dimension strictly smaller than k. Indeed, suppose by contradiction that $\dim(E) < k$, then (up to relabeling the functions u_i) there exist real numbers $b_1,\dots,b_{k-1}$ such that

$$u_k(x) = \sum_{i=1}^{k-1} b_iu_i(x) \quad \text{for every } x\in\Omega_1.$$

Let us define the analytic function

$$\hat{u}(x) := u_k(x) - \sum_{i=1}^{k-1} b_iu_i(x) \quad \text{for every } x\in\Omega_2.$$

Since $\hat{u}$ is analytic in the connected set Ω_2 and $\hat{u} = 0$ in the open set Ω_1, then $\hat{u}=0$ in Ω_2, a contradiction with $\dim(\text{span}\mathcal{U}) = k$.

Testing (3.21) with $S = E$ in we obtain

$$\lambda_k(R_1) \ge \min_{\phi\in E} \frac{(R_1\phi,\phi)_1}{||\phi||_1}.$$

Since the function in the right-hand side of the above equation is scale-invariant, it is equivalent to work with

$$\phi = a_1u_1 + \cdots + a_ku_k,$$

where $a = (a_1,\dots,a_k)$ is of unit norm in $\mathbb{R}^k$. From now (until the end of the proof) on when the expressions involve operations in Ω_1 and a function η defined on Ω_2, we refer to $\eta\big|_{\Omega_1}$, with a little abuse, using η (similarly we treat the extension by 0 of a function defined on Ω_1 to Ω_2). In particular, from the orthonormality of the elements of $\mathcal{U}$, we have

$||\phi||_1 \leq 1$. So enlarging the domain of integration yields

$$\frac{(R_1\phi,\phi)_1}{||\phi||_1} \geq (R_1\phi,\phi)_2,$$

and by taking the minimum with respect to a we obtain

$$\lambda_k(R_1) \geq \min_{\substack{a\in\mathbb{R}^k\\|a|=1}} (R_1\phi,\phi)_2.$$

Therefore, from (3.21) and Lemma 3.3.2 we have

$$\lambda_k(R_2)-\lambda_k(R_1) \leq \min_{\substack{a\in\mathbb{R}^k\\|a|=1}} (R_2\phi,\phi)_2 - \min_{\substack{a\in\mathbb{R}^k\\|a|=1}} (R_1\phi,\phi)_2 \leq \max_{\substack{a\in\mathbb{R}^k\\|a|=1}} |(R_2\phi-R_1\phi,\phi)_2|.$$

It is possible to control the function in the right-hand side of the above chain of inequality as

$$\left|\int_{\Omega_2}(R_2\phi-R_1\phi)\phi\,dx\right| \leq ||\phi||_\infty \int_{\Omega_2}|R_2\phi-R_1\phi|\,dx.$$

Since a is a unit vector in $\mathbb{R}^k$ we have

$$||\phi||_\infty \leq k \max_{1\leq i\leq k} ||u_i||_\infty,$$

moreover the linearity of the resolvent operators and the triangular inequality yield

$$|R_2\phi-R_1\phi| \leq \sum_{i=1}^k |R_2u_i-R_1u_i|,$$

combining these relations we obtain (3.22).

Step 2 (estimate involving the torsion functions). We show that (3.19) holds.

Let $i\in\{1,\dots,k\}$, to simplify the notation we write $\lambda_i=\lambda_i^T(\Omega_2)$. From the definition of eigenfunction of the resolvent we have $u_i=\lambda_iR_2u_i$ and, by (3.20) it solves in the weak sense

$$-\Delta u_i = \lambda_iu_i + c_i \text{ in } \Omega_2, \text{ with } c_i = -\lambda_i\frac{\int_{\Omega_2}u_iw_2\,dx}{\int_{\Omega_2}w_2\,dx}.$$

Analogously the function $v_i:=\lambda_iR_1u_i$, belongs to $H_0^1(\Omega_1)$ and solves in the weak sense

$$-\Delta v_i = \lambda_iu_i + \tilde{c}_i \text{ in } \Omega_1, \text{ with } \tilde{c}_i = -\lambda_i\frac{\int_{\Omega_1}u_iw_1\,dx}{\int_{\Omega_1}w_1\,dx}.$$

Let us introduce the auxiliary functions $\tilde{v}_i,\overline{v}_i\in H_0^1(\Omega_1)$, that satisfy in the weak sense

$$-\Delta\tilde{v}_i = \lambda_iu_i + c_i \text{ in } \Omega_1,\ -\Delta\overline{v}_i = \tilde{c}_i - c_i \text{ in } \Omega_1.$$

By definition of v_i we have

$$\int_{\Omega_2} |R_2 u_i - R_1 u_i| \, dx \leq \frac{1}{\lambda_i} \int_{\Omega_2} |u_i - v_i| \, dx, \tag{3.23}$$

moreover, since by construction $v_i = \tilde{v}_i + \overline{v}_i$, we obtain

$$\int_{\Omega_2} |u_i - v_i| \, dx \leq \int_{\Omega_2} |u_i - \tilde{v}_i| \, dx + \int_{\Omega_2} |\overline{v}_i| \, dx.$$

From the *weak maximum principle*, see for example [109, Proposition 2.1.13, item (ii)], we have

$$|u_i - \tilde{v}_i| \leq (\lambda_i ||u_i||_\infty + |c_i|)(w_2 - w_1) \quad \text{and} \quad |\overline{v}_i| \leq |\tilde{c}_i - c_i| w_1.$$

The definition of c_i yields $|c_i| \leq \lambda_i ||u_i||_\infty$, on the other hand it holds

$$\begin{aligned} |\tilde{c}_i - c_i| &= \lambda_i \frac{\left| \int_{\Omega_1} w_1 \, dx \int_{\Omega_2} u_i (w_2 - w_1) \, dx - \int_{\Omega_1} u_i w_1 \, dx \int_{\Omega_2} (w_2 - w_1) \, dx \right|}{\int_{\Omega_1} w_1 \, dx \int_{\Omega_2} w_2 \, dx} \\ &\leq \lambda_i \frac{2||u_i||_\infty}{\int_{\Omega_2} w_2 \, dx} \int_{\Omega_2} (w_2 - w_1) \, dx. \end{aligned}$$

From the estimates above, since $w_1 \leq w_2$ in $\mathbb{R}^d$, we obtain

$$\int_{\Omega_2} |u_i - v_i| \, dx \leq 4\lambda_i ||u_i||_\infty \int_{\Omega_2} (w_2 - w_1) \, dx.$$

By combining (3.22), (3.23) and this last estimate we find

$$\lambda_k(R_2) - \lambda_k(R_1) \leq 4k^2 \max_{1 \leq i \leq k} ||u_i||_\infty^2 \int_{\Omega_2} (w_2 - w_1) \, dx,$$

the conclusion follows applying Proposition 3.3.1. □

Now we connect the problem (1.31), involving a measure constrain, to a problem that contains a penalization term given by the measure.

Remark 3.3.4. Note that the proof of Theorem 1.4.1 applies as well for penalized problems like

$$\min\{\lambda_k^T(\Omega) + |\Omega| : \Omega \subset Q \text{ quasi-open}\}, \tag{3.24}$$

since the measure is lower semi-continuous for the γ-convergence (see for example [29, Proposition 5.3.6]).

Fix a constant $m > 0$, the existence of solutions to the shape optimization problem (1.31) is equivalent, by the homogeneity of twisted eigenvalues with respect to scalings (see Lemma 2.1.2), to determine optimal sets for

$$\min\{\lambda_k^T(\Omega) + |\Omega| : \Omega \subset \mathbb{R}^d \textit{ quasi-open}\}. \tag{3.25}$$

Our path follows the following rough idea: if a shape is optimal for a general spectral functional, it may satisfy some sub-optimality conditions for the torsion energy.

We recall that the *torsion energy* $T(\Omega)$ of a quasi-open set Ω is given by

$$T(\Omega) := \min_{u \in H^1_0(\Omega)} \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega u \, dx.$$

The function achieving the minimum is the torsion w_Ω, it holds $T(\Omega) = -\int_\Omega w_\Omega(x) \, dx$. Now we need to introduce a fundamental concept, taken from [28, equation (2)].

Definition 3.3.5. We say that a quasi-open set of finite measure Ω is a *shape subsolution* for the torsion energy if there exists $\Lambda > 0$ such that

$$T(\Omega) + \Lambda|\Omega| \le T(\widetilde{\Omega}) + \Lambda|\widetilde{\Omega}|$$

for every quasi-open set $\widetilde{\Omega}$ with $\widetilde{\Omega} \subset \Omega$.

Actually we need a weaker notion.

Definition 3.3.6. We say that a quasi-open set of finite measure Ω is a *local shape subsolution* for the torsion energy if there exist $\Lambda > 0$ and $\delta > 0$ such that

$$T(\Omega) + \Lambda|\Omega| \le T(\widetilde{\Omega}) + \Lambda|\widetilde{\Omega}|$$

for every quasi-open set $\widetilde{\Omega}$ with $\widetilde{\Omega} \subset \Omega$ and $\int_\Omega |w_\Omega - w_{\widetilde{\Omega}}| \, dx \le \delta$.

The strategy we employ is to prove that any local optimizer (in a bounded box) given by Remark 3.3.4 is a local shape subsolution for the torsion energy. This allows us to get (uniform) control of the diameter and perimeter of the optimal sets. In such a way, enlarging the box we obtain global existence.

Proposition 3.3.7. *There exist $n_0, d_0, p_0 > 0$ such that for every $R \ge 1$ any solution of the problem*

$$\min\{\lambda_k^T(\Omega) + |\Omega| : \Omega \subset B_R, \ \Omega \text{ quasi-open}\} \tag{3.26}$$

has at most n_0 quasi-connected components, each one of diameter not larger than d_0 and with total perimeter not larger than p_0.

Proof. Reasoning as in [28, Theorem 2] and using Theorem 3.3.3 instead of [28, Lemma 3] we obtain that a solution Ω_R^* of problem (3.26) is a local shape subsolution for the torsion energy. Then by [28, Theorem 1] we obtain that Ω_R^* is bounded with finite perimeter and finite number of connected components. ☐

We are now ready to prove global existence of optimal shapes for twisted eigenvalues.

Proof of Theorem 1.4.3. Let Ω_R^* be a solution of problem (3.25), by [28, Remark 1] we have an uniform bound (independent on R) on the diameter. Passing to the limit $R \to$

$+\infty$ we obtain a solution of (3.25) which is bounded, is madeup of a finite number of quasi-connected components and has finite perimeter. Since determining the existence of minimal set for this problem is equivalent to find optimal shapes for problem (1.31) we reach the desired conclusion. $\square$

3.4 Proof of Theorem 1.2.1

This result is somehow surprising and, as said in the Section 1.2, seems to be in contradiction with [12, Proposition 2.2 (3) and Remark 1]. Our proof relies on the strong form of the equations, so the intermediate and twisted case present lots of similarities, but also some subtle differences. We start with the following fact that highlights the presence of "xenomorphic" eigenfunctions.

Remark 3.4.1. A twisted eigenfunction corresponding to $\lambda_k^T(\Omega)$ necessarily changes sign. On the other hand, from its explicit expression of [63, Proposition 3.4], the eigenfunction $u_{\mathcal{N}} - c_{\mathcal{N}}$, introduced in Example 2.1.10 and corresponding to $\lambda_1^I(B)$, where B is a ball of unit radius, does not change sign. Moreover its normal derivative vanishes on ∂B.

Recall that the *nodal domains* of a continuous function u defined over the open set Ω are the connected components of the open set $\Omega \setminus \{u \neq 0\}$.

Proof of Theorem 1.2.1. Let us fix $\varepsilon > 0$ such that

$$\lambda_1^D(B_{1-\varepsilon}) < \lambda_2^D(B_1). \tag{3.27}$$

Step 0 (estimates on a non-connected set). Let us define in $\mathbb{R}^2$ the set

$$\Omega = B_1(X_0) \cup \bigcup_{i=1}^{n} B_{1-\varepsilon}(X_i)$$

where $X_0 = (0,0)$ and $X_i = n(\cos\frac{i2\pi}{n}, \sin\frac{i2\pi}{n})$ for $i \in \{1,2,\dots,n\}$ (from now on and until the end of the proof we will use only the index i for elements of this set), see Figure 3.3.

These points satisfy $\|X_j - X_{j'}\| > 2$ for $j, j' \in \{0,1,\dots,n\}$ and $j \neq j'$ (from now on and until the end of the proof we will use only the index j for elements of this set).

In particular the set Ω is of class C^∞ and has $n+1$ connected components. To ease the notation we define

$$H := \begin{cases} H_0^1(\Omega) & \text{if } \mathcal{C} = I \\ H_{0,\mathcal{G}}^1(\Omega) & \text{if } \mathcal{C} = T, \end{cases}$$

$\lambda := \lambda_1^{\mathcal{C}}(\Omega)$ and write Ω^j to denote the ball with center X_j connected component of Ω.

Note that $\lambda_1^D(\Omega^0)$ is simple. Let u^0 be a corresponding eigenfunction, we have that $P_\Omega u^0 \neq 0$. From [69, Remark 1.2.4] we have that $\lambda_1^D(\Omega) = \lambda_1^D(\Omega^0)$ is simple and u^0 is a

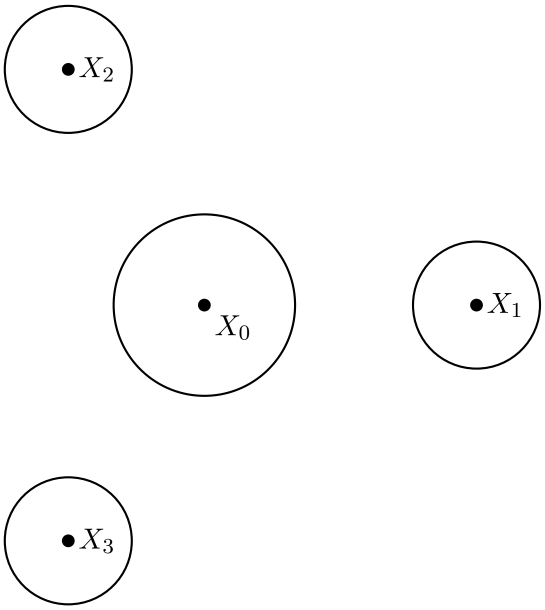

Figure 3.3: The set Ω with $n = 3$ and $\varepsilon = 0.3$, $X_0 = (0, 0), X_1 = (3, 0)$.

corresponding eigenfunction. Let u be an eigenfunction corresponding to $\lambda_1^I(\Omega)$, it holds

$$\lambda_1^D(\Omega) = \mathcal{R}(u^0) \leq \mathcal{R}(u) \leq \mathscr{R}(u) = \lambda_1^I(\Omega),$$

where the last inequality comes from (1.7). If $u \in \operatorname{span}\{u^0\}$, then the second inequality above is strict, while if $u \notin \operatorname{span}\{u^0\}$, then the first inequality above is strict. Therefore, by Lemma 2.1.3 we obtain

$$\lambda_1^D(\Omega) < \lambda. \tag{3.28}$$

Let $\widehat{u} \in H$ be the function obtained as a linear combination of the first Dirichlet eigenfunctions on the balls Ω^j, such that $\widehat{u} > 0$ on Ω^0, $\widehat{u} < 0$ on the balls Ω^i, $\int_\Omega \widehat{u}\, dx = 0$ and $\|\widehat{u}\|_{L^2(\Omega)} = 1$. Since $\mathcal{R}(\widehat{u}|_{\Omega^0}) < \mathcal{R}(\widehat{u}|_{\Omega^i})$, by Lemma 2.2.7 it holds

$$\lambda \leq \mathscr{R}(\widehat{u}) = \mathcal{R}(\widehat{u}) < \lambda_1^D(B_{1-\varepsilon}). \tag{3.29}$$

Step 1 (example with a non-connected set). Let u be an eigenfunction corresponding to λ, we define $u_j := u|_{\Omega^j}$. From the estimates on λ (3.27), (3.28), (3.29), and the behavior of the Dirichlet eigenvalue for a disconnected set, see [69, Remark 1.2.4], we have that u is not a Dirichlet eigenfunction of Ω. By item (b) of Lemma 2.1.7 it satisfies

$$-\Delta u = \lambda u + \xi(u) \quad \text{in } \Omega, \tag{3.30}$$

with

$$\xi(u) = \begin{cases} -\lambda P_\Omega(u) & \text{if } \mathcal{C} = I, \\ P_\Omega(-\Delta u) & \text{if } \mathcal{C} = T. \end{cases} \tag{3.31}$$

Since u is not a Dirichlet eigenfunction of Ω, by Remark 2.1.13, it does not vanish identically on any of the connected components of Ω. In particular u has at least $n+1$ nodal domains. Let $u_{i,i'}$ be the function obtained by switching the values of u between the balls Ω^i and $\Omega^{i'}$, namely

$$u_{i,i'} = u_0 + u_1 + \ldots + u_i(\cdot + X_i - X_{i'}) + \ldots + u_{i'}(\cdot + X_{i'} - X_i) + \ldots + u_n.$$

Since $u_{i,i'} \in H$, $\mathcal{R}(u_{i,i'}) = \mathcal{R}(u)$ and $\mathscr{R}(u_{i,i'}) = \mathscr{R}(u)$ we have that it is an eigenfunction corresponding to λ. In particular there hold (3.30), (3.31) with $u = u_{i,i'}$ and $\xi(u_{i,i'}) = \xi(u)$.

Suppose by contradiction that $u - u_{i,i'} \not\equiv 0$ in Ω. We have that $u - u_{i,i'}$ is a Dirichlet eigenfunction of Ω with corresponding eigenvalue λ. In particular $u_i - u_{i'}(\cdot + X_{i'} - X_i)$ is a Dirichlet eigenfunction of Ω^i with corresponding eigenvalue λ, a contradiction with (3.29). Therefore we have

$$u_i = u_1(\cdot + X_1 - X_i) \quad \text{for every } i \in \{1, 2, \ldots, n\},$$

namely u has the same profile on all balls of radius $1 - \varepsilon$.

Let R_j be a rotation of $\mathbb{R}^2$ with center X_j. Let u_{R_j} be the function obtained by rotating with R_j the function u on the ball Ω^j, namely

$$u_{R_j} = u_0 + \ldots + u_i \circ R_j + \ldots + u_n.$$

Since $u_{R_j} \in H$, $\mathcal{R}(u_{R_j}) = \mathcal{R}(u)$ and $\mathscr{R}(u_{i,i'}) = \mathscr{R}(u)$ we have that it is an eigenfunction corresponding to λ. In particular there hold (3.30), (3.31) with $u = u_{R_j}$ and $\xi(u_{R_j}) = \xi(u)$. Suppose by contradiction that $u - u_{R_j} \not\equiv 0$ in Ω. We have that $u - u_{R_j}$ is a Dirichlet eigenfunction of Ω with corresponding eigenvalue λ. In particular $u_j - u_j \circ R_j$ is a Dirichlet eigenfunction of Ω^j with corresponding eigenvalue λ, a contradiction with (3.28) or (3.27) and (3.29). Therefore we have that u_j is a radial function with respect to X_j for every $j \in \{0, 1, \ldots, n\}$.

Suppose by contradiction that there exists j such that the function u_j has not constant sign. Because of its radiality and smoothness (*Rolle's theorem*), there exists an $r > 0$ such that on the circle $\partial B_r(X_j) \subset \Omega^j$, the radial derivative of u_j is equal to 0. In particular there exist $c \in \mathbb{R}$ such that $(u_j + c)|_{B_r(X_j)} \in H^2_0(B_r(X_j))$, moreover by (3.30) it holds

$$\Delta^2(u + c) = -\lambda\Delta(u + c) \quad \text{in } B_r(X_j).$$

So λ is a *buckling eigenvalue* of B_r (this quantity is invariant with respect to translations), see [11, Section 15.1, equations (15.1.1)-(15.1.2)]. By the *Payne inequality*, see [11, Section 15] or [95, Section 2, Part A], we have

$$\lambda \geq \Lambda_1(B_r) \geq \lambda_2^D(B_r) > \lambda_2^D(\Omega^j),$$

where $\Lambda_1(B_r)$ is the first buckling eigenvalue of B_r a contradiction with (3.27) and (3.29).

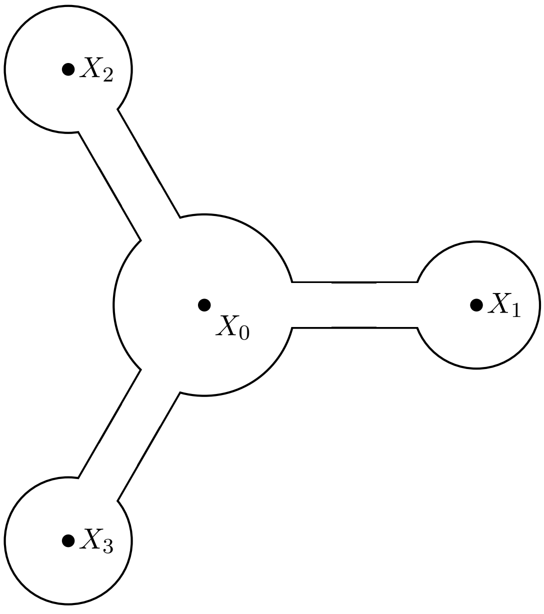

Figure 3.4: The set Ω_δ with $n = 3$, $\varepsilon = 0.3$, $\delta = 0.25$, $X_0 = (0,0), X_1 = (3,0)$.

Note that this argument also prevents the function u_j to attain the value 0, without changing sign, inside Ω^j. Therefore we have that u_j has constant sign in the ball Ω^j for every $j \in \{0, 1, \ldots, n\}$.

When $\mathcal{C} = T$ the function u has to change sign. When $\mathcal{C} = I$, suppose by contradiction that u does not change sign. Consider the function $u_\pm = u|_{\Omega^0} - u|_{\Omega\setminus\Omega^0}$. We have that $u_\pm \in H$ and $|\int_\Omega u_\pm \, dx| < |\int_\Omega u \, dx|$, yielding $\lambda = \mathscr{R}(u) > \mathscr{R}(u_\pm)$, a contradiction. So we can assume that u is positive on Ω^0 and negative on the balls $\Omega^1, \ldots, \Omega^n$. In particular, this means that λ is a simple eigenvalue.

Step 2 (example with a connected set). Let $\delta > 0$ and denote with $[X_0, X_i]_\delta$ the *δ-neighborhood* of the segment $[X_0, X_i]$. We define the open connected set

$$\Omega_\delta = \Omega \cup \bigcup_{i=1}^{\ell} [X_0, X_i]_\delta.$$

It is the set obtained joining the ball Ω^0 with each ball Ω^i by means of "tubes" of width 2δ having extrema in their respective centers, see Figure 3.4.

If $\delta < 1$, there exists a bounded open set Q, independent of δ, such that $\Omega_\delta \subset Q$. We claim that for $\delta \ll 1$ (small), the first intermediate and twisted eigenvalues of Ω_δ are simple and the corresponding eigenfunctions has at least $n + 1$ nodal domains.

Let $(\delta_m)_m$ be a positive sequence decreasing to 0 and define (with a little abuse) $\Omega_m := \Omega_{\delta_m}$. The sequence of sets $(\Omega_m)_m$ converges to the set Ω in the sense of Hausdorff (for open sets) as $m \to \infty$, see [72, Definition 2.2.8]. By *Sverak's theorem*, see [72, Theorem 3.4.14], we have that the sequence of sets Ω_m γ-converges to the set Ω as $m \to \infty$. The measure is stable in the limit process, namely

$$\lim_{m\to\infty} |\Omega_m| = |\Omega|.$$

Therefore, for $m \gg 1$ (namely $\delta_m \ll 1$), the first eigenvalue $\lambda_m := \lambda_1^{\mathcal{C}}(\Omega_m)$ is simple. Indeed suppose by contradiction that there exists a sequence (denoted with a little abuse) $(\Omega_m)_m$ such that λ_m is multiple for every $m \in \mathbb{N}$. Then there exist two eigenfunctions v_m^1, v_m^2 corresponding to λ_m that satisfy (3.10) when $\mathcal{C} = I$ while are orthonormal in $L^2(Q)$ when $\mathcal{C} = T$. From the boundedness of these subsequences in $H^1(Q)$ we have that $v_m^1 \to v^1$ and $v_m^2 \to v^2$ weakly in $H^1(Q)$. Moreover, the results on the continuity of the eigenvalues, see Proposition 3.2.14 when $\mathcal{C} = I$ and Corollary 3.2.9 when $\mathcal{C} = T$, and the γ-convergence of the sequence $(\Omega_m)_m$ to Ω yield that v_1 and v_2 are eigenfunctions corresponding to $\lambda_1^{\mathcal{C}}(\Omega)$ that satisfy (3.10) when $\mathcal{C} = I$ while are orthonormal in $L^2(Q)$ when $\mathcal{C} = T$, a contradiction with Step 1.

Let u be the eigenfunction corresponding to λ, such that satisfies (3.10) when $\mathcal{C} = I$ or it is normalized in $L^2(Q)$ when $\mathcal{C} = T$, assumed to be positive on Ω^0 and negative on the balls $\Omega^1, \dots, \Omega^n$. By Proposition 3.2.14 when $\mathcal{C} = I$ and Corollary 3.2.9 when $\mathcal{C} = T$, there exists a sequence of eigenfunctions $(u_m)_m$ corresponding to λ_m, satisfying (3.10) in Ω_m when $\mathcal{C} = I$ or normalized in $L^2(Q)$ when $\mathcal{C} = T$, that converges weakly in $H_0^1(Q)$, strongly in $L^2(Q)$, and a.e. in Q (up to subsequence and change of sign) to u. In particular u_m solves (3.30), (3.31) with $\Omega = \Omega_m$, $u = u_m$ and $\xi_m := \xi(u_m)$. So, for $m \gg 1$, there exist points $\widetilde{X}_j \in \Omega^j$ such the function u_m achieves a positive value in the point $\widetilde{X}_0$ and some negative values in each point $\widetilde{X}_i$. The function u_m is analytic in Ω_m by item (a) of Lemma 2.1.7.

Assume by contradiction that the nodal domains of u_m are less than or equal to n. From the values achieved by u_m in each point $\widetilde{X}_j$ there exists a connected component C_m of $\{u_m < 0\}$ and $i \neq i'$, such that $\{\widetilde{X}_i, \widetilde{X}_{i'}\} \subset C_m$. In particular, since open connected sets in $\mathbb{R}^2$ are path connected, there exists a curve $\tau_m \subset C_m$ with endpoints $\widetilde{X}_i$ and $\widetilde{X}_{i'}$. We note that there exists a line S passing through the origin of $\mathbb{R}^2$ such that for $m \gg 1$ we have $S \cap \Omega_m \subset \Omega^0$ and $\widetilde{X}_i, \widetilde{X}_{i'}$ are contained in the two distinct hyperplanes individuated by S. Moreover there exists $\sigma > 0$ such that for $m \gg 1$ it holds $S_\sigma \cap \Omega_m \subset \Omega^0$, where S_σ is the σ-neighborhood of the set S, see Figure 3.5.

Let $\eta \in C_c^\infty(S_\sigma)$, $\eta = 1$ on $S_{\sigma/2}$, $\eta = 0$ on $\mathbb{R} \setminus S_\sigma$ and $A := \Omega^0 \cap S_\sigma$. The analytic function $v \in H_0^1(A)$ defined by

$$v_m(x,y) := \eta(x,y)u_m(x,y)$$

for every $(x,y) \in A$, solves (in the strong sense)

$$\begin{aligned} -\Delta v_m &= -u_m\Delta\eta - 2\nabla\eta\nabla u_m - \eta\Delta u_m \\ &= -u_m\Delta\eta - 2\nabla\eta\nabla u_m + \eta(\lambda_m u_m + \xi_m) \text{ in } A. \end{aligned} \tag{3.32}$$

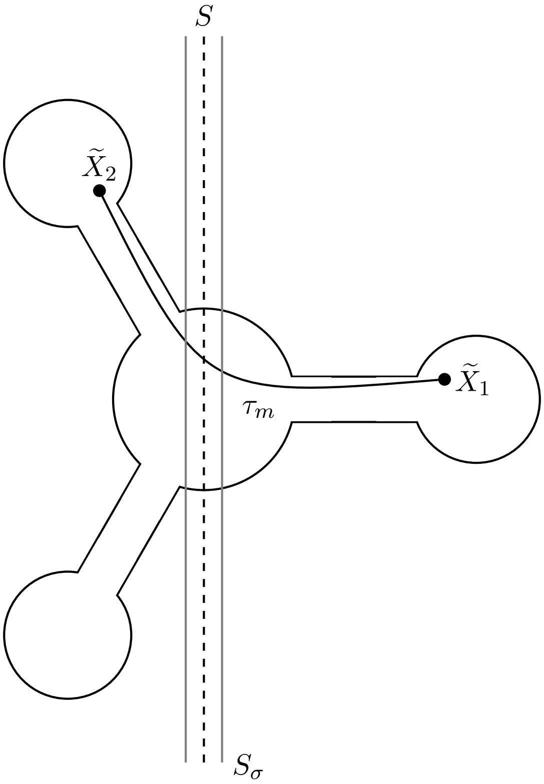

Figure 3.5: The curve τ_m with $i = 1$ and $i' = 2$, the line S and the set S_σ with $\sigma = 0.2$.

Recall that when $\mathcal{C} = I$, by testing with $u = u_m$ and $\psi = w_{\Omega_m}$ in (2.5) we obtain

$$P_{\Omega_m} u_m = \frac{\lambda_m \int_{\Omega_m} (u_m - P_{\Omega_m} u_m) w_{\Omega_m}\, dx}{|\Omega_m|},$$

then the *Cauchy-Schwartz inequality* yields

$$||P_{\Omega_m} u_m||_2^2 = \frac{\lambda_m ||w_{\Omega_m}||_2^2}{|\Omega_m|}.$$

Since A is an open bounded convex set and the right hand side of (3.32) is uniformly bounded in $L^2(A)$, by [64, Theorem 3.2.1.2] the sequence $(v_m)_m$ is uniformly bounded in $H^2(A)$. So u_m is uniformly bounded in $H^2(A \cap S_{\sigma/2})$. Repeating the argument with the smaller strips $S_{\sigma/2}$ and $S_{\sigma/4}$ we obtain that u_m is uniformly bounded in $H^3(A \cap S_{\sigma/4})$. The continuous embedding of $H^3(A \cap S_{\sigma/4})$ in $C^1(A \cap S_{\sigma/4})$ yields $u_m \to u$ in $C^1(A \cap S_{\sigma/4})$. In particular from $u_m \to u$ in $C^0(A \cap S_{\sigma/4})$ we obtain $u_m(X_0) > 0$ for $m \gg 1$.

Let $\{P, P'\} = S \cap \partial\Omega^0$. By *Rolle's theorem*, up to subsequences and possibly exchanging P with P', on the radius $[P, X_0]$ joining the point X_0 with a point P, there is a point q_m such that $u_m(q_m) < 0$ and $\partial_r u_m(q_m) = 0$, where $\partial_r u_m$ is the radial derivative of u_m. Since $u_m \to u$ in $C^1(A \cap S_{\sigma/4})$ we obtain $\partial_r u(P) = 0$. Since $u_0 = u|_{\Omega^0}$ is radial we obtain

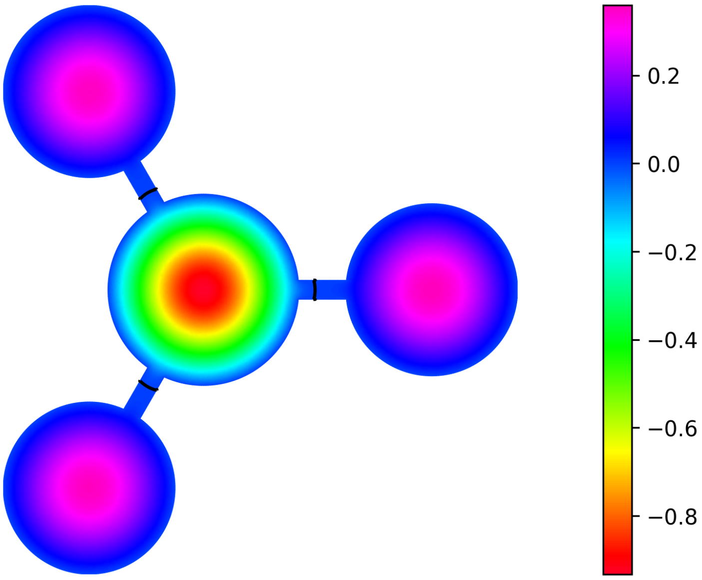

Figure 3.6: Numerical computation of the first twisted eigenfunction of the set Ω_δ with $n = 3$, $\varepsilon = 0.1$, $\delta = 0.1$, $X_0 = (0, 0), X_1 = (2.4, 0)$. The nodal lines are highlighted in black.

$u_0 \in H_0^2(\Omega^0)$ and by (3.30) it holds

$$\Delta^2 u_0 = -\lambda \Delta u_0 \quad \text{in } \Omega^0.$$

In particular λ is a buckling eigenvalue of Ω^0 and thus by Payne's inequality we have

$$\lambda \geq \Lambda_1(\Omega^0) \geq \lambda_2^D(\Omega^0),$$

where $\Lambda_1(\Omega^0)$ is the first buckling eigenvalue of Ω^0, a contradiction with (3.27) and (3.29).

So there exists an $m \gg 1$ (namely $\delta_m \ll 1$) such that the nodal domains of u_m are at least $n + 1$, yielding the desired conclusion (see Figure 3.6 for an example). $\square$

3.4.1 Numerical approximation of optimal shapes for twisted eigenvalues

Throughout this section we assume $\Omega \subset \mathbb{R}^d$ is a smooth, bounded, open set. To perform numerical computations necessary to determine the optimal shape for the problem 1.31 we need two main theoretical tools: shape derivative and optimality conditions. Then we use various software to numerically compute the eigenvalues, perform the optimization procedure by a gradient method and produce numerically accurate pictures of the optimal shapes.

We shall justify the differentiability of simple eigenvalues for twisted problems. As we shall notice, the expression of the shape derivative for the twisted eigenvalue coincides with the expression of the classical Dirichlet eigenvalue. We follow the strategy and notations from [72, Chapter 5].

We introduce the following notations. Given $u \in L^2(\Omega)$, let $P_\Omega(u) = \sum_{i=1}^{\ell} \alpha_i g_i$ and denote

$$\mathcal{H}_\Omega = \Big(\int_\Omega g_i g_j \, dx \Big)_{i,j}, \mathcal{V}_\Omega(u) = \Big(\int_\Omega u g_i \, dx \Big)_i .$$

Then the coefficients α_i in the previous decomposition of $P_\Omega(u)$ solve the linear system

$$\mathcal{H}_\Omega \cdot (\alpha_i)_{i=1}^{\ell} = \mathcal{V}_\Omega(u). \tag{3.33}$$

Although $g_1, \dots, g_l$ are fixed, independent on Ω, note that $\mathcal{H}_\Omega$ and $\mathcal{V}_\Omega$ depend implicitly on Ω since they are defined through integrals on Ω.

Shape derivative. Let (λ, u) be an eigenpair for the twisted problem on Ω (namely u is a twisted eigenfunction corresponding to λ), such that λ is simple and u is normalized in $L^2(\Omega)$. Since $u \in H^2(\Omega)$, the equation satisfied by (λ, u)

$$-\Delta u = \lambda u + \sum_{i=1}^{n} \alpha_i g_i \tag{3.34}$$

holds in a strong form on Ω. The coefficients α_i are given by (3.33).

Let $W \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ and $I \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ defined as $I(x) = x$ for all $x \in \mathbb{R}^d$. For t in a neighborhood of 0, we denote $\Omega_t = (I + tW)(\Omega)$ and (λ_t, u_t) the corresponding eigenpair. Like in [72, Chapter 5, Theorem 5.7.1], it can be shown using the implicit function theorem that the eigenvalue λ and the eigenfunction u are differentiable with respect to t at 0.

Differentiating (3.34) with respect to t gives at 0

$$-\Delta u' = \lambda' u + \lambda u' + \sum_{i=1}^{\ell} \alpha_i' g_i. \tag{3.35}$$

Differentiating the orthogonality constraints $\int_\Omega u g_i \, dx = 0$ gives

$$\int_\Omega u' g_i \, dx = 0.$$

Differentiating the boundary condition $u = 0$ gives

$$u' = -\frac{\partial u}{\partial \nu} W \cdot \nu.$$

Thus, multiplying (3.35) with the eigenfunction u, integrating on Ω and applying Green's formula twice gives (exploiting the orthogonality of u' on g_i on Ω)

$$\lambda' := \lambda'(\Omega)(W) = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 W \cdot \nu \, d\mathcal{H}^{d-1}. \tag{3.36}$$

Remark 3.4.2. Observe the similarity between the previous formula and the classical one for the k-th eigenvalue of the classical Dirichlet eigenavlue. This is not surprising since conceptually the k-th eigenvalue is the twisted eigenvalue associated to the space $\mathcal{G}$

containing the first $k-1$ eigenfunctions.

Optimality conditions and numerical approximation. We focus on the two dimensional case (even though the extension to dimension three is straightforward, but more computationally intensive).

Consider Ω discretized using a triangular mesh $\mathcal{T}_h$. Consider $\mathbb{P}_1$ piecewise affine finite elements with canonical basis $(\varphi_i)_{i=1}^N$, piecewise affine functions taking value 1 at one node of the mesh and zero for all other nodes. Consider the rigidity and mass matrices

$$\mathcal{K} = \left(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx\right)_{1\leq i,j\leq N}, \quad \mathcal{M} = \left(\int_\Omega \varphi_i\varphi_j \, dx\right)_{1\leq i,j\leq N}.$$

Functions $g_i \in \mathcal{G}$ are discretized using vectors $G_i \in \mathbb{R}^N$, containing their values at the nodes of the mesh. The discrete version of the condition $\int_\Omega u g_i \, dx = 0$ becomes $U^T M G_i = 0$, where $U \in \mathbb{R}^N$ contains the values of u at the nodes of the mesh. Given $k \geq 1$ and and a mesh with maximal triangle size sufficiently small, we approximate the k-th twisted eigenvalue by solving

$$\lambda_k = \inf_{\substack{\dim S = k \\ S \perp \mathcal{G}}} \max_{\substack{U\in S \\ U\neq 0}} \frac{U^T\mathcal{K}U}{U^T\mathcal{M}U}.$$

It is possible to use projectors on the orthogonal of the space $(G_i)_{i=1,m}$ to characterize λ_k, however, these generate full matrices. We choose instead to work with the formulation generated by the *Karush–Kuhn–Tucker (KKT) optimality conditions*

$$\begin{pmatrix} \mathcal{K} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} U \\ \Lambda \end{pmatrix} = \lambda_k \begin{pmatrix} \mathcal{M} & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} U \\ \Lambda \end{pmatrix}. \tag{3.37}$$

If $\varepsilon = 0$, the eigensystem above contains all the eigenvalues λ_k. In numerical simulations we take $\varepsilon = 10^{-6}$ to avoid working with singular matrices.

The numerical shape optimization framework uses the level set method with exact remeshing of the interface. The implementation uses Python for handling the files, nullspace algorithm for handling the constraints and optimization process [54] and FreeFEM [67] for solving PDEs. We use the companion code associated with the publication [43] found at `https://github.com/dapogny/sotuto`. We modify the computation of the objective function associated to twisted eigenvalues as described previously. Computations are made in dimension two, using area constraints on the sets, see Figure 1.2.

3.5 An open problem related to Dirichlet eigenvalues

We discuss here a conjecture that we believe useful for the future developments of the theory: it is a natural guess for the minimizing set given by Theorem 1.4.3.

Remark 3.5.1. Let $B \subset \mathbb{R}^d$ be a ball with $|B| = 1$ and $\mathcal{G} = \{1\}$. Suppose that

$$\lambda_d^T(B) = \min\{\lambda_d^T(\Omega) : \Omega \subset \mathbb{R}^d \text{ quasi-open }, |\Omega| = 1\}. \tag{3.38}$$

Let $\Omega \subset \mathbb{R}^d$ be a quasi-open set with $|\Omega| = 1$. We have

$$\lambda_{d+1}^D(B) = \lambda_d^T(B) \leq \lambda_d^T(\Omega) \leq \lambda_{d+1}^D(\Omega), \tag{3.39}$$

where the first equality is a consequence of [12, 2.4.3 Example 3], while the last inequality follows from Lemma 2.1.3. In particular the ball is an optimal shape for the $(d+1)$-th Dirichlet eigenvalue in dimension d.

Inspired by the assumption of the preceding remark, and supported by the numerical evidence of Figure 1.2 in dimension $d = 2$, we state the following.

Open problem 2. *Let $B \subset \mathbb{R}^d$ be a ball with $|B| = 1$ and $\mathcal{G} = \{1\}$. We ask whether the equality* (3.38) *holds. More generally we ask to determine the optimal shapes for the d-th twisted eigenvalue in dimension d.*

Chapter 4

An isoperimetric inequality for twisted eigenvalues

In this chapter, we consider the case where Ω is an open set and $\mathcal{G} = \{g\}$. We retrieve the results in the framework of quasi-open sets by an approximation argument and continuity of twisted eigenvalues with respect to γ-convergence, see Lemma 3.2.1 and Corollary 3.2.9. We present the main results concerning a double minimization problem for the first twisted eigenvalue with a single orthogonality function. Some of the results presented here are contained in the paper [105]. We first analyze the dependence of these eigenvalues either on the set Ω and on the orthogonality function g. Then, we prove some isoperimetric inequalities, by showing that the minimum value of the first twisted eigenvalue (under a constraint on the measure of Ω and on the L^∞-norm of the orthogonality function) is reached by a suitable pair of balls and a bang-bang function. In the particular one-dimensional setting, we provide an alternative proof of this result that does not rely on Bessel functions. As a consequence, we identify a one-parameter family of isoperimetric inequalities that interpolates, in a certain sense, between the Faber-Krahn and the Hong-Krahn-Szego inequalities. Finally an open problem concerning intermediate eigenvalues is discussed.

4.1 Estimates of twisted eigenvalues with Dirichlet eigenvalues

We prove bounds for twisted eigenvalues in terms of Dirichlet ones. We separately analyze the dependence on the function g and on the shape Ω.

4.1.1 Estimates in terms of the function g

We give necessary and sufficient conditions on the function g under which the lower bound for $\lambda_1^g(\Omega)$ in (1.23) is attained. We start with a sufficient condition.

Proposition 4.1.1. *For all* $a_1, a_2 \in (0, +\infty]$ *there exists* $g \in L^\infty(\mathbb{R}^d)$ *with* $|\{g > 0\}| = a_1$ *and* $|\{g < 0\}| = a_2$ *such that*

$$\lambda_1^g(\Omega) = \lambda_1(\Omega).$$

Proof. If Ω is connected then consider the first Dirichlet eigenfunction u_1 be of Ω and two disjoint open sets $A_1, A_2 \subset \mathbb{R}^d$ such that $|A_1| = a_1$, $|A_1 \cap \Omega| > 0$, $|A_2| = a_2$, $|A_2 \cap \Omega| > 0$. If

$$g = \Big(\int_{A_2 \cap \Omega} u_1(x) dx \Big) \chi_{A_1} - \Big(\int_{A_1 \cap \Omega} u_1(x) dx \Big) \chi_{A_2} \,,$$

then $u_1 \in H^1_{0,g}(\Omega)$. Plugging u_1 into (2.14) and using (1.23) gives the thesis.

If Ω is not connected, by [69, Remark 1.2.4], an analogous argument applies by restricting to a connected component whose first Dirichlet eigenvalue equals $\lambda_1(\Omega)$. ☐

We discuss a necessary condition in the case of connected sets.

Proposition 4.1.2. *If* Ω *is connected and* $g \in L^\infty_{0,1}$, *then*

$$\lambda_1^g(\Omega) > \lambda_1(\Omega).$$

Proof. We proceed by contradiction, assuming that $\lambda_1^g(\Omega) = \lambda_1(\Omega)$. Let u_1 be the first Dirichlet eigenfunction of Ω, and let u be a normalized in $L^2(\Omega)$ twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$, chosen such that $\int_\Omega u(x)u_1(x)dx \geq 0$. Since the first Dirichlet eigenvalue is simple on the connected set Ω, and by the assumption $\lambda_1^g(\Omega) = \lambda_1(\Omega)$, it follows that $u = u_1$ (recall the choice $\int_\Omega uu_1 dx \geq 0$). Moreover, it is known that u_1 does not change sign (see [19, Theorem 6.34]). Therefore,

$$\int_\Omega u(x)g(x)dx = \int_\Omega u_1(x)g(x)dx > 0,$$

which contradicts the fact that $\int_\Omega u(x)g(x)dx = 0$. The thesis follows from (1.23). ☐

Remark 4.1.3. Proposition 4.1.2 is no longer true if the connectedness assumption on Ω is removed: there exists a disconnected set $\Omega \in \mathcal{O}$ and $g \in L^\infty_{0,1}$, for which the lower bound for $\lambda_1^g(\Omega)$ in (1.23) is attained. For instance, one may consider $\Omega = B^\wedge \cup B^\vee$, where $B^\wedge, B^\vee$ are two disjoint balls of equal measure and then (1.26) holds.

Proposition 4.1.1 shows that the lower bound for $\lambda_1^g(\Omega)$ in (1.23) can be attained by a bounded function g that is allowed to change sign. In view of Proposition 4.1.2, one may wonder whether the infimum of $\lambda_1^g(\Omega)$, when restricted to positive functions $g \in L^\infty_{0,1}$, is strictly greater that $\lambda_1(\Omega)$. The following proposition addresses this question and shows that the answer is negative.

Proposition 4.1.4. *Given* $\epsilon > 0$, *there exist* $\alpha > 0$ *and* $g \in L^\infty_\alpha$ *such that*

$$\lambda_1^g(\Omega) \leq \lambda_1(\Omega) + \epsilon \,.$$

Proof. Assume that Ω is connected. For a given $x_0 \in \partial\Omega$ consider the open ball $B(r)$ centered at x_0 of radius $r > 0$ sufficiently small such that

$$\lambda_1(\Omega \setminus \overline{B}(r)) \leq \lambda_1(\Omega) + \epsilon/2\,, \tag{4.1}$$

(by the continuity of the first Dirichlet eigenvalue, see for instance [72, Theorem 3.2.7 and Corollary 4.7.4], $\lambda_1(\Omega \setminus \overline{B}(\rho)) \to \lambda_1(\Omega)$ as $\rho \to 0^+$ and such an r always exists). Let u_r^+ and u_r^- be, respectively, two first Dirichlet eigenfunctions corresponding to $\lambda_1(\Omega \setminus \overline{B}(r))$ and $\lambda_1(\Omega \cap B(r))$. For a given $\alpha > 0$, let $g = \alpha\chi_{\Omega\setminus\overline{B}(r)} + \chi_{\mathbb{R}^d\setminus(\Omega\setminus\overline{B}(r))}$ and consider as in (2.13) the linear combination $u_r := u_r^+ - \gamma u_r^-$ with $\gamma = \alpha\|u_r^+\|_1/\|u_r^-\|_1$. By Lemma 2.2.1 $u_r \in H^1_{0,g}(\Omega)$, thus it can be used as test function in (2.14) yielding

$$\lambda_1^g(\Omega) \leq \mathcal{R}(u_r) = \frac{\int_\Omega |\nabla u_r^+(x)|^2 dx + \gamma^2 \int_\Omega |\nabla u_r^-(x)|^2 dx}{\int_\Omega u_r^+(x)^2 dx + \gamma^2 \int_\Omega u_r^-(x)^2 dx}.$$

As $\alpha \to 0^+$, by definition $\gamma \to 0^+$, and the quantity in the right-hand side tends to $\lambda_1(\Omega \setminus \overline{B}(r))$. Therefore, there exists a sufficiently small $\alpha > 0$ such that

$$\lambda_1^g(\Omega) \leq \lambda_1(\Omega \setminus \overline{B}(r)) + \epsilon/2.$$

One obtains the thesis by combining the previous inequality with (4.1).

As in the proof of the previous Proposition 4.1.2, if Ω is not connected, by [69, Remark 1.2.4], an analogous argument applies by restricting to a connected component whose first Dirichlet eigenvalue equals $\lambda_1(\Omega)$. ☐

As a consequence of the arbitrariness of ϵ in Proposition 4.1.4 it follows that

$$\inf_{g \in L^\infty_{0,1}} \lambda_1^g(\Omega) = \lambda_1(\Omega). \tag{4.2}$$

Moreover, by Proposition 4.1.2, if Ω is connected, the infimum is not a minimum – that is, it is not attained by any bounded positive function. This equality also implies that a uniform lower bound for $\lambda_1^g(\Omega)$, including with respect to variations of Ω, can be obtained via (1.28). The situation becomes more interesting when the infimum is restricted to the class $L^\infty_{\alpha,1}$ of *uniformly positive* and *uniformly bounded* functions, see (1.25).

Remark 4.1.5. In order to carry out a finer analysis, to determine when the minimum is not attained in (1.33), it could be useful to consider the constrained formulation and consider the class

$$\{g \in L^\infty_{\alpha,1} \,:\, |\{g = \alpha\}| \leq c\},$$

for some $c > 0$. Notice that this family is not closed under scalings, in contrast to $L^\infty_{\alpha,1}$.

We conclude this subsection with a monotonicity result involving bang-bang functions as orthogonality constraints in the class $L^\infty_{\alpha,1}$.

Proposition 4.1.6. *Let $g \in L^\infty_{\alpha,1}$ and u be a twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$. Then there exist $\alpha_1, \alpha_2 \in [\alpha, 1]$ such that*

$$\lambda_1^g(\Omega) \geq \lambda_1^\chi(\Omega),$$

where $\chi = \alpha_1\chi_{\Omega^+} + \alpha_2\chi_{\mathbb{R}^d\setminus B_+\setminus\Omega^+} \in L^\infty_{\alpha,1}$. Equality holds if and only if u is a twisted eigenfunction corresponding to $\lambda_1^\chi(\Omega)$.

If moreover $\mathcal{R}(u^+) \leq \mathcal{R}(u^-)$, then

$$\lambda_1^g(\Omega) \geq \lambda_1^{\chi_\alpha}(\Omega),$$

where $\chi_\alpha = \alpha\chi_{\Omega^+} + \chi_{\Omega^-}$. Equality holds only if either $g = \chi_\alpha$ a.e. in Ω or $\mathcal{R}(u^+) = \mathcal{R}(u^-)$.

Proof. Let $\alpha_1, \alpha_2 \in [\alpha, 1]$ be such that $\|u^+g\|_1 = \alpha_1\|u^+\|_1$ and $\|u^-g\|_1 = \alpha_2\|u^-\|_1$, and define $\chi = \alpha_1\chi_{\Omega^+} + \alpha_2\chi_{\mathbb{R}^d\setminus B_+\setminus\Omega^+} \in L^\infty_{\alpha,1}$. Then $u \in H^1_{0,\chi}(\Omega)$ and it can be used in (2.14) to obtain

$$\lambda_1^\chi(\Omega) \leq \frac{\int_{\Omega^+} |\nabla u^+(x)|^2 dx + \int_{\Omega^-} |\nabla u^-(x)|^2 dx}{\int_{\Omega^+} u^+(x)^2 dx + \int_{\Omega^-} u^-(x)^2 dx} = \lambda_1^g(\Omega),$$

which proves the first inequality. Now, assume $\mathcal{R}(u^+) \leq \mathcal{R}(u^-)$ and define the linear combination v as in (2.13) with $v_1 = u^+$, $v_2 = u^-$, and $g = \chi_\alpha$ there. By Lemma 2.2.1 the function $v \in H^1_{0,\chi_\alpha}(\Omega)$ and it can be used in (2.14) to obtain

$$\lambda_1^{\chi_\alpha}(\Omega) \leq \frac{\int_{\Omega^+} |\nabla u^+(x)|^2 dx + \gamma^2 \int_{\Omega^-} |\nabla u^-(x)|^2 dx}{\int_{\Omega^+} u^+(x)^2 dx + \gamma^2 \int_{\Omega^-} u^-(x)^2 dx} = Q(\gamma^2),$$

with Q as in Lemma 2.2.7 and $a_1/a_2 = \mathcal{R}(u^-) \geq \mathcal{R}(u^+) = b_1/b_2$ by hypothesis, so that

$$\lambda_1^{\chi_\alpha}(\Omega) \leq Q(\gamma^2) \leq Q(1) = \lambda_1^g(\Omega).$$

If $\lambda_1^{\chi_\alpha}(\Omega) = \lambda_1^g(\Omega)$ then all the previous inequalities are equalities, in particular $Q(\gamma^2) = Q(1)$ means $\mathcal{R}(u^+) = \mathcal{R}(u^-)$ or $\gamma = 1$, namely $g = \chi_\alpha$ a.e. in Ω. ☐

Remark 4.1.7. Notice that the equality condition in Proposition 4.1.6 is only a *necessary condition* but not a sufficient one. However, if $\mathcal{R}(u^+) < \mathcal{R}(u^-)$, then the equality $\lambda_1^g(\Omega) = \lambda_1^{\chi_\alpha}(\Omega)$ holds if and only if $g = \chi_\alpha$ a.e. in Ω.

4.1.2 Estimates also in terms of the set Ω

As a consequence of the results in the previous sections we now focus on the shape dependence.

Lemma 4.1.8. *If $\Omega = B_+ \cup B_-$ where B_+ and B_- are disjoint balls with $|B_+| \geq |B_-|$, and u is a twisted eigenfunction corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ with $\chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d\setminus B_+}$, then*

$$|B_+|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(B_+ \cup B_-) \leq |B|^{\frac{2}{d}} \lambda_2(B),$$

where B is any ball in $\mathbb{R}^d$.

Proof. Recall that the Dirichlet eigenfunction u_2 corresponding to $\lambda_2(B_+)$ has vanishing mean value, so it belongs to $H^1_{0,\chi_\alpha}(B_+) = H^1_{0,1}(B_+)$. By testing (2.14) with $u_2 \in H^1_{0,\chi_\alpha}(B_+ \cup B_-)$ it follows that $\lambda_1^{\chi_\alpha}(B_+ \cup B_-) \leq \lambda_2(B_+)$. The scaling property in Lemma 2.1.2 for Dirichlet eigenvalues provides the bound. □

Remark 4.1.9. In the case of equality, the twisted eigenvalue is not simple: there exist at least d orthogonal twisted eigenfunctions, which are the ones corresponding to the second Dirichlet eigenvalue on a ball B, see [12, Example 3].

By testing with a suitable linear combination of the first Dirichlet eigenfunctions on two disjoint balls we also have an upper bound for twisted eigenvalues with orthogonality constraint of bang-bang type.

Proposition 4.1.10. *If* $\Omega = B_+ \cup B_-$ *where* B_+ *and* B_- *are two disjoint balls with* $|B_+| \geq |B_-|$, *and* $\chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$, *then*

$$\frac{|B_+|^{-1} + \alpha^2|B_-|^{-1}}{|B_+|^{-1-\frac{2}{d}} + \alpha^2|B_-|^{-1-\frac{2}{d}}}\lambda_1^{\chi_\alpha}(B_+ \cup B_-) \leq |B|^{\frac{2}{d}}\lambda_1(B),$$

where B is any ball in $\mathbb{R}^d$.

Proof. Let c, c_+ and c_- be the centers of the balls B, B_+ and B_-. Consider the first Dirichlet eigenfunction u_1 of B for some ball B in $\mathbb{R}^d$ (translate it) and scale it by the factor $a_\pm := (|B_\pm|/|B|)^{1/d}$ to belong to $H^1_0(B_\pm)$, namely define $u_\pm(x) := u_1((x - c_\pm + c)/a_\pm)$ for every $x \in B_\pm$. Now, the function $v := |B_-|u_+ - \alpha|B_+|u_-$ satisfies

$$\begin{aligned}\int_{B_+ \cup B_-} v\chi_\alpha dx &= \alpha|B_-|\int_{B_+} u_+ dx - \alpha|B_+|\int_{B_-} u_- dx \\ &= \alpha|B_-|a_+^d \int_B u_1 dx - \alpha|B_+|a_-^d \int_B u_1 dx = 0\end{aligned}$$

and thus, since $v \in H^1_{0,\chi_\alpha}(B_+ \cup B_-)$, one may test (2.14) with v to obtain

$$\begin{aligned}\lambda_1^{\chi_\alpha}(B_+ \cup B_-) &\leq \frac{|B_-|^2 \int_{B_+} |\nabla u_+|^2 dx + \alpha^2|B_+|^2 \int_{B_-} |\nabla u_-|^2 dx}{|B_-|^2 \int_{B_+} u_+^2 dx + \alpha^2|B_+|^2 \int_{B_-} u_-^2 dx} \\ &= \frac{|B_-|^2 a_+^{d-2} \int_B |\nabla u_1|^2 dx + \alpha^2|B_+|^2 a_-^{d-2} \int_B |\nabla u_1|^2 dx}{|B_-|^2 a_+^d \int_B u_1^2 dx + \alpha^2|B_+|^2 a_-^d \int_B u_1^2 dx} \\ &= \left(\frac{|B_-|^2 a_+^{d-2} + \alpha^2|B_+|^2 a_-^{d-2}}{|B_-|^2 a_+^d + \alpha^2|B_+|^2 a_-^d}\right)\frac{\int_B |\nabla u_1|^2 dx}{\int_B u_1^2 dx},\end{aligned}$$

where the first equality holds by (a translation and) a change of variable with $a_\pm$. Replacing the expressions for $a_\pm$ yields the desired bound. □

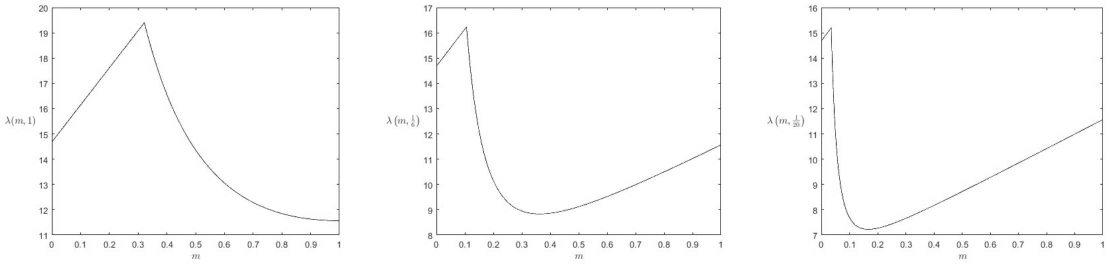

Figure 4.1: Plots (for $d=2$) of $\lambda(m,\alpha)=|B_+\cup B_-|^{\frac{2}{d}}\lambda_1^{\chi_\alpha}(B_+\cup B_-)$ in terms of the variable $m=|B_-|/|B_+|$, at fixed $\alpha=1,1/6,1/20$ (continuous lines) with their upper bounds in Proposition 4.1.10 (dashed lines).

Corollary 4.1.11. *If $\alpha<1$ there exist two disjoint balls B_+,B_-, $|B_+|>|B_-|$ and*

$$|B_+\cup B_-|^{\frac{2}{d}}\lambda_1^{\chi_\alpha}(B_+\cup B_-)<|B^\wedge\cup B^\vee|^{\frac{2}{d}}\lambda_2(B^\wedge\cup B^\vee),$$

where $B_\wedge,B_\vee$ are any disjoint balls of equal measure.

Proof. By denoting $m=|B_-|/|B_+|$, the bound in Proposition 4.1.10 reads as follows

$$|B_+\cup B_-|^{\frac{2}{d}}\lambda_1^{\chi_\alpha}(B_+\cup B_-)\leq\frac{m^{1+\frac{2}{d}}+\alpha^2}{m+\alpha^2}\left(1+\frac{1}{m}\right)^{\frac{2}{d}}|B|^{\frac{2}{d}}\lambda_1(B)=:f(m).$$

The function f just defined is continuously differentiable in $(0,1)$ with

$$f(1)=|B^\wedge\cup B^\vee|^{\frac{2}{d}}\lambda_1(B^\wedge\cup B^\vee)\quad\text{and}\quad\lim_{m\to1^-}f'(m)=\frac{4^{\frac{1}{d}}(1-\alpha^2)}{d(1+\alpha^2)}|B|^{\frac{2}{d}}\lambda_1(B)>0.$$

Thus, there exists $m<1$, sufficiently close to 1, such that $f(m)<f(1)$. ☐

Lemma 4.1.8 with Corollary 4.1.11 provide bounds on the value $\lambda_1^{\chi_\alpha}(B_+\cup B_-)$ in terms of the ratio $m=|B_-|/|B_+|$; see Figure 4.1 for some of these curves. The following theorem is crucial for proving the main results of this chapter: this enables the reduction of the minimization problem to a family of pairs of disjoint balls and bang-bang functions.

Theorem 4.1.12. *Let Ω be such that one of the following conditions holds:*

(i) $\alpha\in(0,1)$ *and* $|\Omega|^{\frac{2}{d}}\lambda_1^g(\Omega)<|B^\wedge\cup B^\vee|^{\frac{2}{d}}\lambda_2(B^\wedge\cup B^\vee)$*;*

(ii) $\alpha=1$ *and* $|\Omega|^{\frac{2}{d}}\lambda_1^g(\Omega)<|B^\wedge|^{\frac{2}{d}}\lambda_2(B^\wedge)$*;*

where $B^\wedge,B^\vee$ denote disjoint balls of equal measure and $g\in L^\infty_{\alpha,1}$. Then there exist B_+, B_- disjoint balls with $|B_+|\geq|B_-|$ such that

$$|\Omega|^{\frac{2}{d}}\lambda_1^g(\Omega)\geq|B_+\cup B_-|^{\frac{2}{d}}\lambda_1^{\chi_\alpha}(B_+\cup B_-),\tag{4.3}$$

where $\chi_\alpha = \alpha\chi_{B_+} + \chi_{B_-}$ *and* B_+, B_- *are the nodal domains of the twisted eigenfunction corresponding to* $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$*. Equality holds in* (4.3) *if and only if* $\Omega = B_+ \cup B_-$ *and* $g = \chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$ *a.e. in* Ω*.*

Proof. By Remark 2.2.4 there exists a twisted eigenfunction corresponding to $\lambda_1^g(\Omega)$ that changes sign in Ω. By the first statement in Proposition 4.1.6 there exist $\alpha_1, \alpha_2 \in [\alpha, 1]$ such that

$$\lambda_1^g(\Omega) \geq \lambda_1^\chi(\Omega), \tag{4.4}$$

where $\chi = \alpha_1\chi_{\Omega^+} + \alpha_2\chi_{\Omega^-} \in L^\infty_{\alpha,1}$. By using Remark 2.2.4 again, there exists a twisted eigenfunction $u = u^+ - u^-$ corresponding to $\lambda_1^\chi(\Omega)$ that changes sign in Ω, which, without loss of generality, satisfies $|\Omega^+| \geq |\Omega^-|$. Then consider the *(spherically) symmetric decreasing rearrangements* (the so-called *Schwarz symmetrizations*) u_+^* and u_-^* of u^+ and u^-, respectively. The functions u_+^*, u_-^* are positive and radially symmetric. In particular $\Omega_+^* := \{u_+^* > 0\}$ and $\Omega_-^* := \{u_-^* > 0\}$ are two disjoint balls satisfying $|\Omega_+^*| = |\Omega^+|$, $|\Omega_-^*| = |\Omega^-|$, $|\Omega_+^*| \geq |\Omega_-^*|$, and eventually $|\Omega_+^* \cup \Omega_-^*| \leq |\Omega|$. Moreover the functions u_+^*, u_-^* are decreasing with respect to the distance from the centers of Ω_+^*, Ω_-^* and belong to the spaces $H_0^1(\Omega_+^*), H_0^1(\Omega_-^*)$, respectively. From [69, Theorems 2.1.2 and 2.1.3], it is well-known that this rearrangement preserves the L^p-norms of a function, while decreasing the L^p-norm of its gradient (the so-called *Pólya-Szegő inequality*). Therefore,

$$\begin{aligned}\lambda_1^\chi(\Omega) &= \frac{\int_{\Omega^+} |\nabla u^+(x)|^2 dx + \int_{\Omega^-} |\nabla u^-(x)|^2 dx}{\int_{\Omega^+} u^+(x)^2 dx + \int_{\Omega^-} u^-(x)^2 dx} \\ &\geq \frac{\int_{\Omega_+^*} |\nabla u_+^*(x)|^2 dx + \int_{\Omega_-^*} |\nabla u_-^*(x)|^2 dx}{\int_{\Omega_+^*} u_+^*(x)^2 dx + \int_{\Omega_-^*} u_-^*(x)^2 dx},\end{aligned} \tag{4.5}$$

The function $u^* := u_+^* - u_-^* \in H_0^1(\Omega_+^* \cup \Omega_-^*)$ satisfies

$$\begin{aligned}\int_{\Omega_+^* \cup \Omega_-^*} u^* \chi dx = \alpha_1 \int_{\Omega_+^*} u_+^* dx - \alpha_2 \int_{\Omega_-^*} u_-^* dx &= \alpha_1 \int_{\Omega^+} u^+ dx - \alpha_2 \int_{\Omega^-} u^- dx \\ &= \int_{\Omega^+ \cup \Omega^-} u\chi dx = 0,\end{aligned}$$

hence $u^* \in H_{0,\chi}^1(\Omega_+^* \cup \Omega_-^*)$ is admissible in (2.14) and

$$\frac{\int_{\Omega_+^*} |\nabla u_+^*(x)|^2 dx + \int_{\Omega_-^*} |\nabla u_-^*(x)|^2 dx}{\int_{\Omega_+^*} u_+^*(x)^2 dx + \int_{\Omega_-^*} u_-^*(x)^2 dx} \geq \lambda_1^\chi(\Omega_+^* \cup \Omega_-^*). \tag{4.6}$$

By Corollary 2.2.8, Proposition 2.2.9 and the second statement in Proposition 4.1.6 we obtain

$$\lambda_1^\chi(\Omega_+^* \cup \Omega_-^*) \geq \lambda_1^{\chi_\alpha}(\Omega_+^* \cup \Omega_-^*), \tag{4.7}$$

where $\chi_\alpha = \alpha\chi_{\Omega_+^*} + \chi_{\Omega_-^*}$. Overall, we obtain

$$|\Omega|^{\frac{2}{d}} \lambda_1^g(\Omega) \geq |\Omega|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(\Omega_+^* \cup \Omega_-^*) \geq |\Omega_+^* \cup \Omega_-^*|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(\Omega_+^* \cup \Omega_-^*), \tag{4.8}$$

and (4.3) follows.

For the rigidity, the "if" part is trivial. Let us prove the "only if" part, assuming all the inequalities above to be equalities so that from the hypothesis and (1.29)

$$|\Omega_+^* \cup \Omega_-^*|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(\Omega_+^* \cup \Omega_-^*) = |\Omega_+^* \cup \Omega_-^*|^{\frac{2}{d}} \lambda_1^{\chi}(\Omega_+^* \cup \Omega_-^*) = |\Omega|^{\frac{2}{d}} \lambda_1^{g}(\Omega) < |B^\wedge|^{\frac{2}{d}} \lambda_2(B^\wedge)$$

and we can apply Theorem 1.2.2 and Lemma 2.1.7 to infer that $\lambda_1^{\chi_\alpha}(\Omega_+^* \cup \Omega_-^*)$, $\lambda_1^{\chi}(\Omega_+^* \cup \Omega_-^*)$ are simple and their unique (up to scalings) corresponding eigenfunctions $v = v^+ - v^-$, $w = w^+ - w^-$ are given by the difference of two strictly positive analytic functions v^+, w^+ and v^-, w^- defined in Ω_+^* and Ω_-^*, respectively. Notice that v is also an eigenfunction corresponding to $\lambda_1^{\hat{g}}(\hat{\Omega})$, for every $\hat{\Omega}$ that coincides with $\Omega_+^* \cup \Omega_-^*$ up to a set of zero capacity and $\hat{g} = \chi_\alpha$ a.e. in $\hat{\Omega}$. When $\alpha = 1$, (4.7) and (4.4) reduce to equalities, while when $\alpha < 1$, the equality in (4.7) by Proposition 4.1.6, either $\chi = \chi_\alpha$ or $\mathcal{R}(w^+) = \mathcal{R}(w^-)$. The latter case $\mathcal{R}(w^+) = \mathcal{R}(w^-)$ is not possible, since by Proposition 2.2.9, it follows that $|\Omega_+^*| = |\Omega_-^*|$ and by (1.26) we obtain

$$|\Omega_+^* \cup \Omega_-^*|^{\frac{2}{d}} \lambda_1^{\chi}(\Omega_+^* \cup \Omega_-^*) = |\Omega_+^* \cup \Omega_-^*|^{\frac{2}{d}} \lambda_2(\Omega_+^* \cup \Omega_-^*) = |B^\wedge \cup B^\vee|^{\frac{2}{d}} \lambda_2(B^\wedge \cup B^\vee),$$

which contradicts the hypothesis on $|\Omega|^{\frac{2}{d}} \lambda_1^{g}(\Omega)$. Hence, for $\alpha \in (0,1]$ $\chi = \chi_\alpha$ a.e. and $v = w$ (up to scalings). From the equality in (4.6) and the simplicity of $\lambda_1^{\chi_\alpha}(\Omega_+^* \cup \Omega_-^*)$ we also deduce $w = u^*$ (up to scalings).

Moreover, from the equality in (4.5) and since $u^* = w$ is non vanishing and analytic, by [91, Proposition 1] we can apply [27, Theorem 1.1], to deduce that (up to translations) $u^+ = u_+^*$ and $u^- = u_-^*$ a.e. in $\mathbb{R}^d$. In particular, since u and u^* are analytic $u^+ = u_+^*$ and $u^- = u_-^*$ (hence a twisted eigenfunction $u = u^*$, i.e, is radially symmetric and strictly decreasing in each ball), with $\Omega^+ = \Omega_+^*$ and $\Omega^- = \Omega_-^*$ two balls. From the equality in (4.8), $\Omega_+^* \cup \Omega_-^* \subset \Omega$ and Ω open we have $\Omega \subset \overline{\Omega_+^* \cup \Omega_-^*}$, and thus $\Omega = \Omega_+^* \cup \Omega_-^*$. From the equality in (4.4), by Proposition 4.1.6, there exists $\tilde{u}$ eigenfunction corresponding to $\lambda_1^{g}(\Omega_+^* \cup \Omega_-^*)$ and $c \in \mathbb{R} \setminus \{0\}$ such that $cu = \tilde{u}$. Let $\tilde{\Omega} \subset \Omega_+^* \cup \Omega_-^*$ be the set such that $g \neq \chi_\alpha$ in $\tilde{\Omega}$. Since $g \in L^\infty_{\alpha,1}$ we have that $g > \alpha$ in $\tilde{\Omega} \cap \Omega_+^*$ or $g < 1$ in $\tilde{\Omega} \cap \Omega_-^*$, this yields

$$\begin{aligned}\int_{\Omega_+^* \cup \Omega_-^*} \tilde{u} g \, dx &= c \int_{\Omega_+^* \cup \Omega_-^*} u g \, dx \\ &= c\Big(\int_{\Omega_+^* \cup \Omega_-^*} u \chi_\alpha \, dx + \int_{\tilde{\Omega} \cap \Omega_+^*} u^+ (g - \alpha) \, dx + \int_{\tilde{\Omega} \cap \Omega_-^*} u^- (1 - g) \, dx \Big).\end{aligned}$$

Since $u \in H^1_{0,\chi_\alpha}(\Omega_+^* \cup \Omega_-^*)$, $\tilde{u} \in H^1_{0,g}(\Omega_+^* \cup \Omega_-^*)$, $u^+ > 0$ in Ω_+^*, $u^- > 0$ in Ω_-^* and $g \in L^\infty_{\alpha,1}$ we obtain $|\tilde{\Omega}| = 0$. The definition of $\tilde{\Omega}$ yields $g = \chi_\alpha$ a.e. in $\Omega_+^* \cup \Omega_-^*$ and this concludes the proof. $\square$

4.2 Proof of Theorem 1.5.1 and Corollary 1.5.2

Proof of Theorem 1.5.1. Fix $0 < \alpha \leq 1$ and, without loss of generality, we can assume B_+, B_- and χ_α be as in Appendix 2.2.2. By combining Theorem 4.1.12 with the scale invariance in Lemma 2.1.2 and the (partial) translation invariance of $|B_+ \cup B_-|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(B_+ \cup B_-)$, problem (1.33) is equivalent to the minimization of twisted eigenvalues over the class made up of two disjoint *non-empty* balls of total measure equal to δ_d, contained in a fixed domain $D \subset \mathbb{R}^d$ (the *box*), with orthogonality constraint of bang-bang type, namely one reduces to study

$$\lambda(\alpha) = \delta_d^{2/d} \min_{\Omega \in \mathcal{O}_1(D)} \lambda_1^{\chi_\alpha}(\Omega) \tag{4.9}$$

with

$$\mathcal{O}_1(D) := \{B_+ \cup B_- \subset D : \ B_+, B_- \text{ disjoint balls, } |B_+| \geq |B_-|, \ |B_+| + |B_-| = \delta_d\}$$

and the bang-bang function $\chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}^d \setminus B_+}$. In other words, proving Theorem 1.5.1 is equivalent to establishing its properties for the minimizers of (4.9). The proof is divided into four steps.

Step 1 (existence of a minimizer). We claim that there exists a pair of disjoint *non-empty* balls minimizing (4.9). The existence of such a minimizer is standard and follows the Direct Methods of the Calculus of Variations. Let $\{\Omega_n = B_+^n \cup B_-^n\}_n \subset \mathcal{O}_1(D)$ be a minimizing sequence for (4.9) and define $g_n := \alpha\chi_{B_+^n} + \chi_{\mathbb{R}^d \setminus B_+^n}$. By compactness – since we are working with pairs of balls of fixed volume – up to subsequences, the sequence $\{\Omega_n\}_n$ converges (e.g., in terms of the radii of the balls) to some limit set $\Omega = B_+ \cup B_-$. Note that the limit may degenerate into a single ball of measure δ_d, in the case $B_- = \emptyset$. Moreover, $g_n \to \chi_\alpha = \alpha\chi_{B_+} + \chi_{B_-}$ a.e. in D. Now, let u_n be a twisted eigenfunction corresponding to $\lambda_1^{g_n}(\Omega_n)$ normalized in $L^2(\Omega_n)$. By minimality, the sequence $\{u_n\}_n$ is uniformly bounded in $H_0^1(D)$. Therefore, up to extracting further subsequences, we may assume that $\{u_n\}_n$ converges weakly in $H_0^1(D)$, strongly in $L^2(D)$, and a.e. in D to some function $u \in H_0^1(D)$ such that

$$\int_D u(x)^2 dx = \lim_{n \to +\infty} \int_D u_n(x)^2 dx = 1, \int_D u(x)\chi_\alpha(x) dx = \lim_{n \to +\infty} \int_D u_n(x) g_n(x) dx = 0.$$

Since $u \equiv 0$ a.e in $D \setminus \Omega$ and Ω is a Lipschitz set, it follows from [72, Remarks 3.4.8 and 3.4.9] that $u \in H_0^1(\Omega)$. Combined with the previous equality, this implies that $u \in H_{0,g}^1(\Omega)$. By lower semicontinuity of the H^1-norm w.r.t. the weak convergence in H^1

$$\lambda_1^{\chi_\alpha}(\Omega) \leq \int_D |\nabla u(x)|^2 dx \leq \liminf_{n \to \infty} \int_D |\nabla u_n(x)|^2 dx = \frac{\lambda(\alpha)}{\delta_d^{2/d}},$$

hence Ω is a minimizer for (4.9) and u is a twisted eigenfunction corresponding to $\lambda_1^{\chi_\alpha}(\Omega)$.

By (1.26) and the rigidity statement of (1.29) we have

$$\lambda(\alpha) \leq |B^\wedge \cup B^\vee|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(B^\wedge \cup B^\vee) = |B^\wedge \cup B^\vee|^{\frac{2}{d}} \lambda_2(B^\wedge \cup B^\vee) < |B|^{\frac{2}{d}} \lambda_2(B), \tag{4.10}$$

for any ball B in $\mathbb{R}^d$. This shows that the minimizer $\Omega \in \mathcal{O}_1(D)$, thus by Theorem 1.2.2 we obtain that every twisted eigenfunction corresponding to $\lambda_1^{\chi_\alpha}(\Omega)$ has nodal domains $\Omega^+ = B_+$ and $\Omega^- = B_-$.

Step 2 (the optimality condition). Let $\Omega_\alpha = B_+ \cup B_- \in \mathcal{O}_1(D)$ be a minimizer of (4.9). By (4.10) we can apply Proposition 2.2.12 to Ω_α and by minimality of Ω_α we may infer the existence of a constant μ (the Lagrange multiplier for the volume constrained problem (4.9), see [72, Section 5.9.3, p. 243]) such that

$$\int_{\partial\Omega_\alpha} \left(|\nabla u|^2 - \mu \right) W \cdot \nu \, d\mathcal{H}^{d-1} = 0, \tag{4.11}$$

for every smooth vector field W from $\mathbb{R}^d$ to $\mathbb{R}^d$ and, by the arbitrariness of W,

$$|\nabla u(x)|^2 = \mu \quad \text{for a.e. } x \in \partial\Omega_\alpha. \tag{4.12}$$

By taking $W(x) = x$ in (4.11), and then by using Proposition 2.2.11, it follows that

$$2\lambda_1^{\chi_\alpha}(\Omega_\alpha) = \int_{\partial\Omega_\alpha} |\nabla u|^2 \, x \cdot \nu \, d\mathcal{H}^{d-1} = \mu \int_{\partial\Omega_\alpha} x \cdot \nu \, d\mathcal{H}^{d-1} = \mu \int_{\Omega_\alpha} \mathrm{div}(x) \, dx = \mu d |\Omega_\alpha| \, ,$$

which gives the explicit value of the Lagrange multiplier in (4.12), and (1.36) follows.

Step 3 (bounds on measure ratio and eigenvalues of optimal shapes). Let $\Omega_\alpha = B_+ \cup B_- \in \mathcal{O}_1(D)$ be a minimizer of (4.9) and u be a twisted eigenfunction corresponding to $\lambda_1^{\chi_\alpha}(\Omega_\alpha)$. By plugging the expression (2.17) into Corollary 2.2.10 yields

$$0 \geq - \int_{B_+ \cup B_-} \Delta u(x) \chi_\alpha(x) dx = -\alpha \int_{B_+} \Delta u^+(x) dx + \int_{B_-} \Delta u^-(x) dx.$$

An integration by parts with the optimality condition (4.12) allows to compute the integrals in the right-hand side of the previous inequality and rewrite it as follows

$$0 \geq \sqrt{\mu} \left(\alpha |B_+|^{\frac{d-1}{d}} - |B_-|^{\frac{d-1}{d}} \right),$$

where we also used the fact that for every ball B its perimeter $P(B)$ satisfies $P(B) = d\delta_d^{1/d} |B|^{\frac{d-1}{d}}$. By solving this inequality with respect to $m(\alpha)$ the first bound in (1.35) follows. By applying (1.23), [69, Remark 1.2.4] and Lemma 2.1.2 we obtain

$$\begin{aligned} \lambda(\alpha) &= |B_+^\alpha \cup B_-^\alpha|^{\frac{2}{d}} \lambda_1^{\chi_\alpha}(B_+^\alpha \cup B_-^\alpha) \geq |B_+^\alpha \cup B_-^\alpha|^{\frac{2}{d}} \lambda_1(B_+^\alpha \cup B_-^\alpha) \\ &= |B_+^\alpha \cup B_-^\alpha|^{\frac{2}{d}} \lambda_1(B_+^\alpha) = (1 + m(\alpha))^{\frac{2}{d}} |B|^{\frac{2}{d}} \lambda_1(B), \end{aligned}$$

that proves the second bound in (1.35). By letting $\alpha = 1$ in (1.35), and recalling (4.10), we obtain a direct proof of Corollary 1.5.2 without relying on Bessel functions.

Step 4 (uniqueness of the minimizer). We proceed by contradiction, assuming the existence of two pairs of disjoint balls $B_+ \cup B_-$ of radii R_+, R_- and $B_\oplus \cup B_\ominus$ of radii $R_\oplus, R_\ominus$ with $R_+ < R_\oplus < 1$ and such that

$$\lambda_1^{\chi_\alpha}(B_+ \cup B_-) = \lambda_1^{\overline{\chi}_\alpha}(B_\oplus \cup B_\ominus) = \lambda_1^{\chi_\alpha}(\Omega_\alpha) = \lambda,$$

with $\chi_\alpha = \alpha\chi_{B_+} + \chi_{B_-}$ and $\overline{\chi}_\alpha = \alpha\chi_{B_\oplus} + \chi_{B^d_{\mathbb{R}} \setminus B_\oplus}$. Notice that the volume constraint in (4.9) implies $R_+^d + R_-^d = 1$ and $R_\oplus^d + R_\ominus^d = 1$. If $u = u^+ - u^-$ and $v = u^\oplus - u^\ominus$ are twisted eigenfunctions corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ and to $\lambda_1^{\overline{\chi}_\alpha}(B_\oplus \cup B_\ominus)$ (here $u^\oplus$ and $u^\ominus$ denote positive and negative part of v), then by Theorem 4.1.12 we know they are radially symmetric (with respect to the center of each ball). Let $u_\pm$ denote the 1-dimensional functions describing the radial profiles (centered at the origin) of $u^\pm$, which by (2.24) and the radiality property satisfy the system

$$\begin{cases} u_\pm''(r) + \frac{d-1}{r} u_\pm'(r) + \lambda u_\pm(r) = \xi_\pm & \text{for } r \in (0, R_\pm), \\ u_\pm(R_\pm) = 0\,,\ u_\pm'(0) = 0\,, \end{cases} \tag{4.13}$$

where $\xi_+ = -\alpha\xi$, $\xi_- = \xi$ for some $\xi \in \mathbb{R}$, and in particular $\xi_+ = -\alpha\xi_-$. From the optimality condition (4.12) there holds

$$\left|\frac{u_+'(R_+)}{u_-'(R_-)}\right| = \left|\frac{u_\oplus'(R_\oplus)}{u_\ominus'(R_\ominus)}\right| = 1. \tag{4.14}$$

On the other hand, by Proposition A.1.5 solutions to (4.13) read as follows

$$u_\pm(r) = s_\pm \left(r^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega r) - R_\pm^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_\pm) \right) \quad \text{for } r \in (0, R_\pm) \tag{4.15}$$

with $s_\pm \in \mathbb{R}$ and $\omega = \sqrt{\lambda}$. By inserting the explicit expressions (4.15) for $u_\pm$ into the orthogonality condition $\int_{B_+ \cup B_-} u(x)\chi_\alpha(x)dx = 0$, and writing the integral in polar coordinates, it follows that

$$\begin{aligned} 0 = \int_{B_+ \cup B_-} u(x)\chi_\alpha(x)dx = &\alpha s_+ \Big(d\delta_d \int_0^{R_+} r^{\frac{d}{2}} J_{\frac{d}{2}-1}(\omega r)\, dr - \delta_d R_+^{\frac{d}{2}+1} J_{\frac{d}{2}-1}(\omega R_+) \Big) \\ &- s_- \Big(d\delta_d \int_0^{R_-} r^{\frac{d}{2}} J_{\frac{d}{2}-1}(\omega r)\, dr - \delta_d R_-^{\frac{d}{2}+1} J_{\frac{d}{2}-1}(\omega R_-) \Big), \end{aligned}$$

where $d\delta_d$ is the $(d-1)$-measure of the unit sphere in $\mathbb{R}^d$. Relying on (A.3) and (A.1) we obtain

$$\alpha s_+ R_+^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(\omega R_+) - s_- R_-^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(\omega R_-) = 0 \tag{4.16}$$

and, in particular, we may take

$$s_+ = R_-^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(\omega R_-) \quad \text{and} \quad s_- = \alpha R_+^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(\omega R_+). \tag{4.17}$$

By differentiating (4.15) with respect to r, and by using the second relation in (A.2), we obtain

$$u'_\pm(r) = -s_\pm \omega r^{1-\frac{d}{2}} J_{\frac{d}{2}}(\omega r) \quad \text{for every } r \in (0, R_\pm)$$

which combined with (4.17) allows to write *the optimality condition* (4.14), as an implicit equation relating α, ω and the radii $R_\pm$

$$\left|\frac{u'_+(R_+)}{u'_-(R_-)}\right| = \frac{1}{\alpha}\frac{\Phi(\omega R_-)}{\Phi(\omega R_+)} = 1, \quad \text{with } \Phi(x) := x^d \frac{J_{\frac{d}{2}+1}(x)}{J_{\frac{d}{2}}(x)}. \tag{4.18}$$

Similarly an equation as the previous one holds for a twisted eigenfunction $v = u^\oplus - u^\ominus$ corresponding to $\lambda_1^{\chi_\alpha}(B_\oplus \cup B_\ominus)$. From the hypotheses, the scaling-invariance and (4.10)

$$0 < \omega R_\ominus < \omega R_- \le \omega R_+ < \omega R_\oplus \le |\Omega_\alpha|^{\frac{1}{d}} \sqrt{\lambda_1^{\chi_\alpha}(\Omega_\alpha)} < |B|^{\frac{1}{d}} \sqrt{\lambda_2(B)} = j_{\frac{d}{2},1},$$

and since by Proposition A.1.2 the function Φ is strictly increasing in $(0, j_{\frac{d}{2},1})$, we obtain

$$\left|\frac{u'_+(R_+)}{u'_-(R_-)}\right| = \frac{1}{\alpha}\frac{\Phi(\omega R_-)}{\Phi(\omega R_+)} > \frac{1}{\alpha}\frac{\Phi(\omega R_\ominus)}{\Phi(\omega R_\oplus)} = \left|\frac{u'_\oplus(R_\oplus)}{u'_\ominus(R_\ominus)}\right|,$$

that is in contradiction with (4.14) and the uniqueness follows. ☐

Remark 4.2.1. In dimension $d = 2$, for the values of α considered in Figure 1.3 we have

α	$\lambda(\alpha)$	$m(\alpha)$
1	$\pi \cdot j_{0,1}^2 \approx 36.3368$	1
1/6	27.7534	0.3635
1/20	22.7001	0.1664

Remark 4.2.2. Let $\omega = \sqrt{\lambda}$, where λ is defined as in (2.24). By (A.5) and $\xi_+ = -\alpha\xi_-$, a configuration solving of (2.24) satisfies

$$s_+ R_+^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_+) = -\alpha s_- R_-^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_-),$$

hence one may also take

$$s_+ = -\alpha R_-^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_-) \quad \text{and} \quad s_- = R_+^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_+).$$

By inserting these coefficients into the *orthogonality condition* (4.16) we obtain another *implicit equation* relating α, ω and the radii $R_\pm$

$$R_-^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(\omega R_-) R_+^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_+) + \alpha^2 R_+^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(\omega R_+) R_-^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(\omega R_-) = 0.$$

If $\omega R_+ \notin \{j_{\frac{d}{2}-1,\sigma}\}_\sigma$ (and so, by *Porter's theorem* [110, Section 15.22, p. 480] $\omega R_- \notin \{j_{\frac{d}{2}-1,\sigma}\}_\sigma$), namely the twisted eigenfuctions corresponding to this configuration are not

Dirichlet ones, it can be rewritten as

$$\phi(\omega R_-) + \alpha^2 \phi(\omega R_+) = 0, \quad \phi(x) := x^d \frac{J_{\frac{d}{2}+1}(x)}{J_{\frac{d}{2}-1}(x)}. \tag{4.19}$$

If $\alpha < 1$, by Corollary 4.1.11, (4.10) and the equality case in Proposition 2.2.9, twisted eigenfunctions are not Dirichlet ones, hence in this case (4.19) holds.

Remark 4.2.3. If $\alpha = 1$ then Corollary 1.5.2 follows directly from Theorem 1.5.1 (Step 2 is sufficient in this case and the proof does not rely on Bessel functions. Notice also that, the uniqueness of the minimizer proved in Step 4, together with the optimality condition (4.12), gives another proof of Corollary 1.5.2 (in this case it uses Bessel functions). Indeed, let $B^\wedge$ and $B^\vee$ be any pair of disjoint balls in $\mathbb{R}^d$ of equal measure and u_1 and v_1 be the first Dirichlet eigenfunctions of $B^\wedge$ and $B^\vee$, respectively. The twisted eigenfunction $u_1 - v_1$ corresponding to $\lambda_1^T(B^\wedge \cup B^\vee)$ has constant modulus on $\partial B^\wedge \cup \partial B^\vee$.

Remark 4.2.4. The *optimality condition* (4.18) can also be obtained using the Lagrange multiplier methods on the relation (4.19) with a volume constraint, as in [63, Proposition 4.4]. This method, alternative to the shape derivative, guarantees a unique minimizer but does not lead to equation (1.36). This strategy of proof relies on Bessel functions.

4.2.1 Another proof of Theorem 1.5.1 in the one-dimensional case

In order to prove uniqueness of the optimal set in Theorem 1.5.1, without relying on Bessel functions, we need a fact analogous to Proposition A.1.2. In particular, in the proof of this result we deliberately avoid solving an ordinary differential equation explicitly, showing that it is not necessary to do so.

Proposition 4.2.5. *Let u be the analytic extension to $\mathbb{R}$ of a Dirichlet eigenfunction corresponding to $\lambda_1((-1,1))$. Let $I : (0,2) \to \mathbb{R}$ be the analytic function defined by*

$$I(\eta) := \int_0^1 u(\eta x) - u(\eta)\, dx \tag{4.20}$$

for every $\eta \in (0,2)$. Then, the analytic function $\widetilde{\Phi} : (0,2) \to \mathbb{R}$ defined by

$$\widetilde{\Phi}(\eta) := \frac{\eta I(\eta)}{-u'(\eta)}$$

for every $\eta \in (0,2)$ is positive and increasing.

Proof. A Dirichlet eigenfunction u corresponding to the eigenvalue $\lambda = \lambda_1((-1,1))$ solves

$$\begin{cases} -u''(x) = \lambda u(x) & \text{for } x \in (-1,1) \\ u(-1) = u(1) = 0, \end{cases} \tag{4.21}$$

is even, strictly monotone in $(0,1)$ and has constant sign in $(-1,1)$. The function u can be extended to $\mathbb{R}$ analytically, continuing to satisfy the differential equation in (4.21), since the solutions of this second order linear ordinary differential equation are defined on $\mathbb{R}$. In particular, the extended function, that we will also denote u, is 4-periodic and can be defined in $(-1,3)$ as

$$\begin{cases} u(x) & \text{for } x \in (-1,1) \\ -u(x-2) & \text{for } x \in (1,3), \end{cases}$$

which makes it even. The function $\widetilde{\Phi}$ is well defined because it is independent of the normalization of u.

Consider a u that is positive in $(-1,1)$, then u is strictly decreasing in $(0,2)$ and in particular $\widetilde{\Phi}$ is positive. By the Fundamental Theorem of Calculus, $u'(0) = 0$ and (4.21) we have

$$-u'(\eta) = \lambda \int_0^\eta u(x)\,dx,$$

on the other hand, an integration by parts yields

$$\eta \int_0^1 u'(\eta x)x\,dx = u(\eta) - \int_0^1 u(\eta x)\,dx,$$

therefore we obtain

$$\eta I'(\eta) = -I(\eta) - \eta u'(\eta). \tag{4.22}$$

Combining (4.21) and (4.22), the derivative of $\widetilde{\Phi}$ is given by

$$\widetilde{\Phi}'(\eta) = \frac{\eta[(-u'(\eta))^2 - I(\eta)u(\eta)\lambda]}{(-u'(\eta))^2}. \tag{4.23}$$

To conclude the proof it is sufficient to show that $\widetilde{\Phi}'$ is positive. To do so we define the function (the part with unknown sign in the expression of $\widetilde{\Phi}'$)

$$N(\eta) = (-u'(\eta))^2 - I(\eta)u(\eta)\lambda, \tag{4.24}$$

then differentiating with respect to η and using (4.21) and (4.22) we obtain

$$N'(\eta) = \lambda\left[u(\eta)(-u'(\eta)) + \frac{I(\eta)u(\eta)}{\eta} + I(\eta)(-u'(\eta))\right]. \tag{4.25}$$

Eventually, combining the relations (4.20), (4.21), (4.24), and (4.25) we have

$$\begin{aligned}(\eta N(\eta))' = N(\eta) + \eta N'(\eta) &= (-u'(\eta))^2 + \lambda\eta(-u'(\eta))[u(\eta) + I(\eta)] \\ &= (-u'(\eta))^2 + (-u'(\eta))\int_0^\eta \lambda u(x)\,dx \\ &= 2(-u'(\eta))^2 > 0.\end{aligned}$$

Since the function $\eta N(\eta)$ is equal to 0 in $\eta = 0$, then it is positive strictly increasing for

$\eta \in (0,2)$ and thus $\widetilde{\Phi}$ is strictly increasing. $\square$

In the one-dimensional case, the correct rigidity statement of Theorem 1.5.1 needs the addition of the interval as a set for which equality holds. This variation comes from a careful inspection of equality case in Theorem 4.1.12, in the situation of adjacent positivity and negativity sets of the eigenfunction considered.

Now we are ready to proceed with the proofs of the main theorems.

Proof of Theorem 1.5.1. In the case $\alpha = 1$ a proof of this fact can be retrieved (in an easier way) by taking the minimizers of Wirtinger inequality (see [65, Section 7.7, p. 185, 258]) with zero boundary conditions.

Fix $0 < \alpha < 1$. As in the proof of Theorem 1.5.1, the problem (1.33) is equivalent to the minimization of twisted eigenvalues over the class made up of two disjoint *non-empty* intervals, of total measure equal to 2, contained in a fixed interval $D \subset \mathbb{R}$ (the *box*, independent from α), with orthogonality constraint of bang-bang type, namely one reduces to study

$$\lambda(\alpha) = 4 \min_{\Omega \in \mathcal{O}_1(D)} \lambda_1^{\chi_\alpha}(\Omega) \tag{4.26}$$

with

$$\mathcal{O}_1(D) := \{B_+ \cup B_- \subset D : \ B_+, B_- \text{ disjoint intervals, } |B_+| \geq |B_-|, \, |B_+| + |B_-| = 2\}$$

and the bang-bang function $\chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}\setminus B_+}$. In other words, proving Theorem 1.5.1 is equivalent to establishing its properties for the minimizers of (4.26). The main difference with the proof of Theorem 1.5.1 is the last step.

Step 1 and Step 2, which can be performed as in proof of Theorem 1.5.1, without relying on Bessel function, give the existence of a minimizer $\Omega_\alpha = B_+ \cup B_-$ for (4.26), where B_+ and B_- are two disjoint intervals of total measure 2. The eigenvalue $\lambda_1^{\chi_\alpha}(\Omega_\alpha)$ is simple, $\lambda_1^{\chi_\alpha}(\Omega_\alpha) < \lambda_1(B)$ where B is an interval of measure 2 and (1.36) holds. In particular the normal derivative of an eigenfunction u corresponding to $\lambda_1^{\chi_\alpha}(\Omega_\alpha)$ is constant on $\partial\Omega_\alpha$, namely there exists $\mu > 0$ (depending on the normalization of u) such that

$$|u'(x)| = \sqrt{\mu} \quad \text{for } x \in \partial\Omega_\alpha. \tag{4.27}$$

Moreover the positivity and negativity sets of u can be taken as B_+ and B_-, respectively. These properties also hold for the eigenfunctions corresponding to every other minimizing sets. On the other hand Step 3 does not give any information. Now we are going to prove uniqueness of the minimizing configuration without using Bessel functions.

Step 4' (uniqueness of the minimizer). We proceed by contradiction, assuming the existence of two pairs of disjoint intervals $B_+ \cup B_-$ of radii R_+, R_- and $B_\oplus \cup B_\ominus$ of radii $R_\oplus, R_\ominus$ with $R_+ < R_\oplus < 1$ and such that

$$\lambda_1^{\chi_\alpha}(B_+ \cup B_-) = \lambda_1^{\overline{\chi}_\alpha}(B_\oplus \cup B_\ominus) = \lambda_1^{\chi_\alpha}(\Omega_\alpha) = \lambda,$$

with $\chi_\alpha = \alpha\chi_{B_+} + \chi_{\mathbb{R}\setminus B_+}$ and $\overline{\chi}_\alpha = \alpha\chi_{B_\oplus} + \chi_{\mathbb{R}\setminus B_\oplus}$. Notice that the volume constraint in (4.26) implies $R_+ + R_- = 1$ and $R_\oplus + R_\ominus = 1$. Consider $u = u_+ - u_-$ and $v = u_\oplus - u_\ominus$ twisted eigenfunctions corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ and to $\lambda_1^{\overline{\chi}_\alpha}(B_\oplus \cup B_\ominus)$ (here $u_\oplus$ and $u_\ominus$ denote positive and negative part of v). Since these sets are meant to be disjoint, With a little abuse, we can identify

$$B_+ \cup B_- = (-R_+, R_+) \cup (-R_-, R_-) \quad \text{and} \quad B_\oplus \cup B_\ominus = (-R_\oplus, R_\oplus) \cup (-R_\ominus, R_\ominus),$$

and from the optimality condition (4.27) there holds

$$\left|\frac{u_+'(R_+)}{u_-'(R_-)}\right| = \left|\frac{u_\oplus'(R_\oplus)}{u_\ominus'(R_\ominus)}\right| = 1. \tag{4.28}$$

On the other hand let $R \in \mathbb{R}$ and denote with u_R a Dirichlet eigenfunction corresponding to $\lambda_R = \lambda_1((-R, R))$, it solves

$$\begin{cases} -u_R''(x) = \lambda_R u_R(x) & \text{for } x \in (-R, R) \\ u_R(R) = u_R(-R) = 0, \end{cases} \tag{4.29}$$

and can be extended analytically to an even function defined on $\mathbb{R}$ that satisfies the same differential equation. Consider the function

$$u = s_+ u_{T_+} - s_- u_{T_-},$$

where $s_+, s_- \in \mathbb{R}$ and

$$u_{T_\pm}(x) = u_{R_\pm}(\eta_\pm x) - u_{R_\pm}(\eta_\pm R_\pm) \quad \text{for } x \in [-R_\pm, R_\pm]$$

with

$$\eta_\pm = \sqrt{\frac{\lambda}{\lambda_{R_\pm}}} = \sqrt{\frac{\lambda}{\lambda_1}} R_\pm.$$

Notice that u is defined with a little abuse, since the supports of u_{T_+} and u_{T_-} are meant to be disjoint, this has been done to lessen technical difficulties. The relation (4.29) yields

$$\begin{cases} -u''(x) = \lambda u(x) \pm s_\pm \lambda u_{R_\pm}(\eta_\pm R_\pm) & \text{for } x \in (-R_\pm, R_\pm) \\ u(R_\pm) = u(-R_\pm) = 0. \end{cases} \tag{4.30}$$

Consider s_+, s_- such that (1.16) holds with $\Omega = B_+ \cup B_-$ and $g = \chi_\alpha$, by this choice u is an eigenfunction corresponding to $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ (the unique one, up to scaling, since $\lambda_1^{\chi_\alpha}(B_+ \cup B_-)$ is simple). Let $\eta \in \mathbb{R}$ and define

$$I(\eta, R) = \int_0^R u_R(\eta x) - u_R(\eta R)\, dx,$$

from the orthogonality condition $\int_{B_+ \cup B_-} u(x)\chi_\alpha(x)dx = 0$ we have

$$\alpha s_+ I(\eta_+, R_+) - s_- I(\eta_-, R_-) = 0$$

hence we can take

$$s_+ = I(\eta_-, R_-) \quad \text{and} \quad s_- = \alpha I(\eta_+, R_+).$$

Consider the Dirichlet eigenfunctions defined in (4.28) to be normalized in $L^\infty(\mathbb{R})$, then they satisfy

$$u_R(x) = u_1(x/R) \quad \text{and} \quad u_R'(x) = \frac{u_1'(x/R)}{R},$$

a change of variable yields $I(\eta, R) = R\, I(\eta)$. Moreover, if

$$\eta = \sqrt{\frac{\lambda}{\lambda_R}} = \sqrt{\frac{\lambda}{\lambda_1}} R,$$

these scaling property give

$$\frac{I(\eta, R)}{\eta u_R'(\eta R)} = \frac{\lambda_1}{\lambda} \widetilde{\Phi}(\eta).$$

Finally, the optimality condition (4.27) yields

$$\left| \frac{u_+'(R_+)}{u_-'(R_-)} \right| = \frac{1}{\alpha} \left| \frac{I(\eta_-, R_-)\, \eta_+\, u_{R_+}'(\eta_+ R_+)}{I(\eta_+, R_+)\, \eta_-\, u_{R_-}'(\eta_- R_-)} \right| = \frac{1}{\alpha} \frac{\widetilde{\Phi}(\eta_-)}{\widetilde{\Phi}(\eta_+)}.$$

Similarly an equation as the previous one holds for a twisted eigenfunction $v = u_\oplus - u_\ominus$ corresponding to $\lambda_1^{\chi_\alpha}(B_\oplus \cup B_\ominus)$. Recall that from (1.28), (1.30) and the hypotheses

$$\lambda \leq \frac{\lambda(1)}{4} = \lambda_1^T((-1, 1)) = 4\lambda_1,$$

therefore, from $R_\oplus < 1$ we obtain

$$0 < \eta_\ominus < \eta_- \leq \eta_+ < \eta_\oplus = \sqrt{\frac{\lambda}{\lambda_1}} R_\oplus < 2,$$

the strict monotonicity of $\widetilde{\Phi}$ given by Proposition 4.2.5 yields

$$\left| \frac{u_+'(R_+)}{u_-'(R_-)} \right| = \frac{1}{\alpha} \frac{\widetilde{\Phi}(\eta_-)}{\widetilde{\Phi}(\eta_+)} > \frac{1}{\alpha} \frac{\widetilde{\Phi}(\eta_\ominus)}{\widetilde{\Phi}(\eta_\oplus)} = \left| \frac{u_\oplus'(R_\oplus)}{u_\ominus'(R_\ominus)} \right|,$$

that is in contradiction with (4.28) and the uniqueness follows. □

4.3 Proof of Theorem 1.5.3 and Theorem 1.4.5

Proof of Theorem 1.5.3. The proof is an application of the implicit function theorem for vector-valued maps. Although Theorem 1.5.1 already ensures the existence of the functions

$m(\alpha)$ and $\lambda(\alpha)$ in terms of the variable α, the use of the implicit function theorem is needed to establish their continuous differentiability. To this end, it is convenient to consider a vectorial-valued map F defined on α and on two *auxiliary* variables: a scaled measure $M = R_+^d$ of the largest ball B_+ of an optimal pair and the frequency $\omega = \sqrt{\lambda}$. As in Theorem 1.5.1 we consider optimal sets of measure δ_d. These are related to the original variables of the theorem through the identities $\lambda(\alpha) = \delta_d^{2/d}\omega(\alpha)^2$ and

$$m(\alpha) = \frac{|B_-^\alpha|}{|B_+^\alpha|} = \frac{\delta_d - |B_+^\alpha|}{|B_+^\alpha|} = \frac{1}{R_+(\alpha)^d} - 1 = \frac{1}{M(\alpha)} - 1,$$

where $R_+(\alpha)$ is the radius of the ball $|B_+^\alpha|$. Once the continuous differentiability and the monotonicity are established for these auxiliary variables, the continuous differentiability and the monotonicity of m and λ follows directly from the previous identities. The proof is divided into two steps.

Step 1 (continuous differentiability of the functions). Consider $F\colon X \subset \mathbb{R}^3 \to \mathbb{R}^2$ with $X = (0,1) \times (1/2, 1) \times (j_{d/2-1,1}, j_{d/2,1})$ and $F = (f_1, f_2)$ and $f_1, f_2\colon X \to \mathbb{R}$ scalar functions defined by

$$f_1(\alpha, M, \omega) := \phi(\omega(1-M)^{\frac{1}{d}}) + \alpha^2\phi(\omega M^{\frac{1}{d}}), \quad \text{with } \phi(x) := x^d \frac{J_{\frac{d}{2}+1}(x)}{J_{\frac{d}{2}-1}(x)}$$

and

$$f_2(\alpha, M, \omega) := \Phi(\omega(1-M)^{\frac{1}{d}}) - \alpha\Phi(\omega M^{\frac{1}{d}}), \quad \text{with } \Phi(x) := x^d \frac{J_{\frac{d}{2}+1}(x)}{J_{\frac{d}{2}}(x)}.$$

Notice that by Remark 4.2.2 and (4.18) there holds $F(\alpha, M(\alpha), \omega(\alpha)) = (0,0)$ for every $\alpha \in (0,1)$. Therefore, we may consider a point $x_0 = (\alpha_0, M_0, \omega_0) \in X$ such that $M_0 = M(\alpha_0)$ and $\omega_0 = \omega(\alpha_0)$, then $F(x_0) = (0,0)$. Since $f_1(x_0) = 0$ by Proposition A.1.1 ϕ is negative at $\omega_0 M_0^{1/d} \in (j_{d/2-1}, j_{d/2})$ while positive at $\omega_0(1-M_0)^{1/d} \in (0, j_{d/2-1})$. By definition F is continuously differentiable in x_0 with respect to all the three variables. Therefore, to apply the implicit function theorem to F, see for instance [81], it remains to show that the Jacobian of F at x_0 with respect the variables to be made explicit –namely, M and ω– is invertible. We compute first the partial derivatives of f_1:

$$\frac{\partial f_1}{\partial M}(\alpha_0, M_0, \omega_0) = \frac{\omega_0}{d}\left(-(1-M_0)^{\frac{1}{d}-1}\phi'(\omega_0(1-M_0)^{\frac{1}{d}}) + \alpha_0^2 M_0^{\frac{1}{d}-1}\phi'(\omega_0 M_0^{\frac{1}{d}})\right)$$

and

$$\frac{\partial f_1}{\partial \omega}(\alpha_0, M_0, \omega_0) = (1-M_0)^{\frac{1}{d}}\phi'(\omega_0(1-M_0)^{\frac{1}{d}}) + \alpha_0^2 M_0^{\frac{1}{d}}\phi'(\omega_0 M_0^{\frac{1}{d}}).$$

Now, for f_2 we have

$$\frac{\partial f_2}{\partial M}(\alpha_0, M_0, \omega_0) = -\frac{\omega_0}{d}\left((1-M_0)^{\frac{1}{d}-1}\Phi'(\omega_0(1-M_0)^{\frac{1}{d}}) + \alpha_0 M_0^{\frac{1}{d}-1}\Phi'(\omega_0 M_0^{\frac{1}{d}})\right)$$

and

$$\frac{\partial f_2}{\partial \omega}(\alpha_0, M_0, \omega_0) = (1-M_0)^{\frac{1}{d}}\Phi'(\omega_0(1-M_0)^{\frac{1}{d}}) - \alpha_0 M_0^{\frac{1}{d}}\Phi'(\omega_0 M_0^{\frac{1}{d}}).$$

The determinant detJF of the Jacobian matrix of F (computed with respect to the variables M and ω) evaluated at x_0 is given by

$$\begin{aligned}\left|\frac{\partial f_1}{\partial M}\frac{\partial f_2}{\partial \omega} - \frac{\partial f_1}{\partial \omega}\frac{\partial f_2}{\partial M}\right| &= \frac{1}{d\omega_0}\left(-\frac{x_-\phi'(x_-)}{1-M_0} + \frac{\alpha_0^2 x_+\phi'(x_+)}{M_0}\right)(x_-\Phi'(x_-) - \alpha_0 x_+\Phi'(x_+)) \\ &\quad + \frac{1}{d\omega_0}\left(x_-\phi'(x_-) + \alpha_0^2 x_+\phi'(x_+)\right)\left(\frac{x_-\Phi'(x_-)}{1-M_0} + \frac{\alpha_0 x_+\Phi'(x_+)}{M_0}\right) \\ &= \frac{\alpha_0 x_- x_+}{d\omega_0 M_0(1-M_0)}\left(\phi'(x_-)\Phi'(x_+) + \alpha_0\phi'(x_+)\Phi'(x_-)\right),\end{aligned}$$

where we denoted by $x_- = \omega_0(1-M_0)^{\frac{1}{d}}$ and $x_+ = \omega_0 M_0^{\frac{1}{d}}$. By Propositions A.1.1 and A.1.2 we deduce that the determinant detJF of the Jacobian matrix of F (computed with respect to the variables M and ω) evaluated at x_0 is strictly positive, namely the Jacobian matrix is invertible. Therefore, the implicit function theorem [81, Theorem 3.3.1] applies and there uniquely exist two functions, that we already denoted as $M(\alpha)$ and $\omega(\alpha)$, that are continuously differentiable for every $\alpha \in (0,1)$.

Step 2 (monotonicity of the functions). Let $0 < \alpha_1 < \alpha_2 < 1$ and $\Omega_{\alpha_1}, \Omega_{\alpha_2}$ be the sets with measure δ_d achieving the equality in (1.34) for $\alpha = \alpha_1, \alpha = \alpha_2$, respectively. By Remark 4.2.2 the eigenfunction u_{α_2} is not a Dirichlet eigenfunction, therefore combining Propositions 2.2.5 and 4.1.6 we obtain

$$\omega(\alpha_2)^2 = \lambda_1^{\chi_{\alpha_2}}(\Omega_{\alpha_2}) > \lambda_1^{\chi_{\alpha_1}}(\Omega_{\alpha_2}) \geq \lambda_1^{\chi_{\alpha_1}}(\Omega_{\alpha_1}) = \omega(\alpha_1)^2.$$

Thus the function ω is strictly increasing in $(0,1)$ and in particular $\omega'(\alpha_0) \geq 0$ for every $\alpha_0 \in (0,1)$. The function $f\colon (0,1) \to \mathbb{R}$ defined by

$$\alpha \mapsto f(\alpha) = f_2(\alpha, M(\alpha), \omega(\alpha))$$

is such that $f(\alpha_0) = 0$ for every $\alpha_0 \in (0,1)$ and differentiable with respect to α at α_0 with

$$f'(\alpha_0) = \frac{\partial f_2}{\partial \alpha}(\alpha_0, M_0, \omega_0) + M'(\alpha_0)\frac{\partial f_2}{\partial M}(\alpha_0, M_0, \omega_0) + \omega'(\alpha_0)\frac{\partial f_2}{\partial \omega}(\alpha_0, M_0, \omega_0) = 0.$$

Solving this equation in terms of the unknown M' we have that

$$M'(\alpha_0) = \frac{\frac{\partial f_2}{\partial \alpha}(\alpha_0, M_0, \omega_0) + \omega'(\alpha_0)\frac{\partial f_2}{\partial \omega}(\alpha_0, M_0, \omega_0)}{-\frac{\partial f_2}{\partial M}(\alpha_0, M_0, \omega_0)} < 0,$$

since the partial derivative w.r.t. α is $(\partial f_2/\partial \alpha)(\alpha_0, M_0, \omega_0) = -\Phi(\omega_0 M_0^{\frac{1}{d}}) < 0$, moreover

$(\partial f_2/\partial M)(\alpha_0, M_0, \omega_0) < 0$ by the sign of Φ', see Proposition A.1.2, while

$$\begin{aligned}\frac{\partial f_2}{\partial \omega}(\alpha_0, M_0, \omega_0) &= (1 - M_0)^{\frac{1}{d}}\Phi'(\omega_0(1 - M_0)^{\frac{1}{d}}) - \alpha_0 M_0^{\frac{1}{d}}\Phi'(\omega_0 M_0^{\frac{1}{d}}) \\ &= \Phi(\omega_0(1 - M_0)^{\frac{1}{d}})(\Upsilon(\omega_0(1 - M_0)^{\frac{1}{d}}) - \Upsilon(\omega_0 M_0^{\frac{1}{d}})) < 0,\end{aligned}$$

where we used that $f_2(\alpha_0, M_0, \omega_0) = 0$, the positivity of Φ in Proposition A.1.2 with Remark A.1.4 and Proposition A.1.3 to rewrite and deduce the sign of the partial derivative of f_2 w.r.t. ω. Combining Proposition 4.1.4 with $\Omega = B$ and (1.28) we obtain $\lim_{\alpha\to 0^+} \lambda(\alpha) = |B|^{\frac{2}{d}}\lambda_1(B)$, where B is any ball in $\mathbb{R}^d$. Suppose by contradiction that $\lim_{\alpha\to 0^+} m(\alpha) = \overline{m} > 0$, then by (1.23) and [69, Remark 1.2.4] we have

$$|B_+^\alpha \cup B_-^\alpha|^{\frac{2}{d}}\lambda_1^{\chi_\alpha}(B_+^\alpha \cup B_-^\alpha) \geq |B_+^\alpha \cup B_-^\alpha|^{\frac{2}{d}}\lambda_1(B_+^\alpha) > (1 + \overline{m})^{\frac{2}{d}}|B_+^\alpha|^{\frac{2}{d}}\lambda_1(B_+^\alpha),$$

therefore $\lim_{\alpha\to 0^+} \lambda(\alpha) \geq (1 + \overline{m})^{\frac{2}{d}}|B|^{\frac{2}{d}}\lambda_1(B)$, a contradiction. Moreover, by (1.35) in Theorem 1.5.1 letting $\alpha \to 1^-$ we deduce $m(1) = 1$ and $\lambda(1) = |B^\wedge \cup B^\vee|^{\frac{2}{d}}\lambda_2(B^\wedge \cup B^\vee)$ and the theorem is proved. $\square$

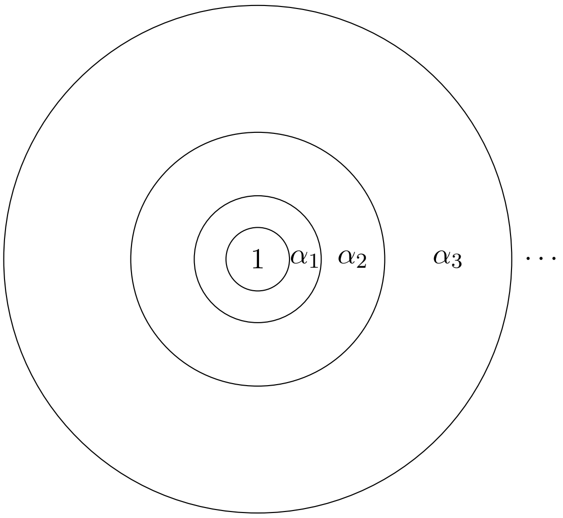

Figure 4.2: The function $g \in L^\infty_{\alpha,1}$ that proves the non-existence of optimal shapes in Theorem 1.4.5

We now prove a result on the non-existence of optimal shapes for a shape optimization problem involving twisted eigenvalues under a single orthogonality constraint.

Proof of Theorem 1.4.5. Let $(\alpha_n)_n \subset \mathbb{R}$ be a strictly decreasing sequence bounded by 1 with $\alpha_n \to \alpha$, as $n \to \infty$, and consider $g \in L^\infty_{\alpha,1}$ defined by

$$g = \chi_{B_1} + \sum_{n=1}^{\infty} \alpha_n \chi_{B_{2n}\setminus B_{2n-1}},$$

where B_r is the ball of radius $r > 0$ centered at the origin (g is a function that decreases as the spherical shell expands, see figure 4.2). For every (sufficiently large) $n \in \mathbb{N}$, consider a pair of balls $B_-^{\alpha_n}$ and $B_+^{\alpha_n}$ included, respectively, in the ball B_1 and in the spherical shell $B_{2^n} \setminus B_{2^{n-1}}$, and such that $|B_-^{\alpha_n}| = m(\alpha_n)|B_+^{\alpha_n}|$, where $m(\alpha_n)$ is given by Theorem 1.5.1 with α_n as parameter (the existence of such a pair of balls is always guaranteed, at least for sufficiently large n). By definition of $\lambda(\alpha)$, as infimum also over $L^\infty_{\alpha,1}$, and (1.34) we obtain

$$\lambda(\alpha) \leq \inf_{\Omega \in \mathcal{O}} |\Omega|^{\frac{2}{d}} \lambda_1^g(\Omega) \leq |B_+^{\alpha_n} \cup B_-^{\alpha_n}|^{\frac{2}{d}} \lambda_1^g(B_+^{\alpha_n} \cup B_-^{\alpha_n}) = \lambda(\alpha_n).$$

Letting $n \to \infty$ above, since Theorem 1.5.3 gives $\lambda(\alpha_n) \to \lambda(\alpha)$, we deduce

$$\lambda(\alpha) = \inf_{\Omega \in \mathcal{O}} |\Omega|^{\frac{2}{d}} \lambda_1^g(\Omega),$$

that is $\lambda(\alpha)$ is the infimum of the shape optimization problem. Now, assume by contradiction the existence of an optimal shape $\Omega_* \in \mathcal{O}$ (i.e., an open and bounded set in $\mathbb{R}^d$) reaching the previous infimum. By boundedness of Ω_* there exists $n_* \in \mathbb{N}$ such that $\Omega_* \subset B_{2^{n_*}}$ and in particular $g \in L^\infty_{\alpha_{n_*},1}$. This combined with the definition and the strict monotonicity of the function $\lambda(\cdot)$, see Theorem 1.5.3, imply

$$\lambda(\alpha) = |\Omega_*|^{\frac{2}{d}} \lambda_1^g(\Omega_*) \geq \lambda(\alpha_{n_*}) > \lambda(\alpha), \tag{4.31}$$

which is a contradiction. An optimal shape does not exist for (1.27). □

4.4 An open problem for intermediate eigenvalues

We consider the double minimization problem

$$\min_{\Omega \in \mathcal{QO}} \min_{g \in L^\infty_{\alpha,1}} |\Omega|^{\frac{2}{d}} \lambda_1^{I,\{g\}}(\Omega), \tag{4.32}$$

with $0 < \alpha \leq 1$, analogous to (1.33).

Open problem 3. *Determine the optimal shapes for* (4.32)*. In case an optimal shape is given by a pair of two disjoint balls, we ask to compare it with the solution of* (1.33)*.*

Chapter 5

The Alt-Caffarelli-Friedman monotonicity formula

We provide a self-contained proof of the Alt–Caffarelli–Friedman monotonicity formula, namely Theorem 1.6.1 (we adopt the notation introduced in Appendix A.2 for parametrizations and rearrangements). Some of the results presented are contained in the paper [104]. We then recall the notion of the first Dirichlet eigenvalue for sets lying on a sphere, together with the associated characteristic constant. Next, we establish the convexity of the first eigenvalue of a spherical cap with respect to its colatitude. This property that plays a crucial role in the proof of the Friedland–Hayman inequality, a sharp result concerning the growth rates of two homogeneous harmonic functions satisfying Dirichlet boundary conditions on disjoint cones in Euclidean space, see (5.29). With these elements at hand, the monotonicity formula is obtained as a direct consequence. Finally, this tool can be employed to study the Lipschitz regularity of sign-changing Dirichlet eigenfunctions (see for example [70, Chapter 3, Introduction, Step 2]), suggesting potential applications to the problems addressed in this thesis.

5.1 The first Dirichlet eigenvalue of a set liying on a sphere

For the first part of this section we follow [36], where a regularity hypothesis on the boundary of a set and connectedness of the set are also required, see [36, p. 1]. Since the results and the definitions we borrow are valid even without these assumptions (with the appropriate modifications), we continue to refer to [36].

Let $\Gamma \subset \partial B_1$ be a measurable set, the *first Dirichlet eigenvalue of* Γ, see [36, p. 17], is

$$\lambda(\Gamma) = \min \left\{ \frac{\displaystyle\int_\Gamma |\nabla_\phi u(\phi)|^2 \, d\sigma}{\displaystyle\int_\Gamma u(\phi)^2 \, d\sigma} \ : \ u \in H^1(\partial B_1),\, \mathcal{H}^{d-1}(\{u \neq 0\} \setminus \Gamma) = 0 \right\} \tag{5.1}$$

A function u achieving equality in (5.1) is called *(first Dirichlet) eigenfunction corresponding to* $\lambda(\Gamma)$. The eigenfunctions are, as usual, extended by zero in $\partial B_1 \setminus \Gamma$.

Assume that Γ is an open set, this notion reduces to

$$\lambda(\Gamma) = \min_{\substack{u \in H_0^1(\Gamma) \\ u \not\equiv 0}} \frac{\displaystyle\int_\Gamma |\nabla_\phi u(\phi)|^2 \, d\sigma}{\displaystyle\int_\Gamma u(\phi)^2 \, d\sigma}. \tag{5.2}$$

By regularity theory (combining (A.17) and [18, Analyticity Theorem, p. 136]), we have that an eigenfunction u corresponding to $\lambda(\Gamma)$ is analytic in Γ and solves

$$-\Delta_\phi u(\phi) = \lambda(\Gamma) u(\phi) \quad \text{for every } \phi \in \Gamma. \tag{5.3}$$

By the *Courant's nodal domain Theorem*, see [36, p. 19], an eigenfunction corresponding to $\lambda(\Gamma)$ has only a nodal domain, namely the set $\Gamma \setminus \{u = 0\}$ has only a connected component. In particular, when Γ is connected, the eigenfunction corresponding to $\lambda(\Gamma)$ is unique, up to scalar multiples (in this case the eigenvalue is called *simple*). In addition, let $\psi : \partial B_1 \to \partial B_1$ be an isometry (of ∂B_1), we notice that $\lambda(\Gamma) = \lambda(\psi(\Gamma))$.

A central role in the proof of the ACF formula will be played by the properties of the function describing the first Dirichlet eigenvalue of a spherical cap in terms of its colatitude, so we define $\lambda(\theta_0) := \lambda(\Gamma(\theta_0))$. In the case $\theta_0 = \pi/2$, by (A.16), we notice that the function $u : \Gamma(\pi/2) \to \mathbb{R}$, defined by

$$u_{\mathscr{S}}(\theta, \xi) = \cos(\theta),$$

for every $(\theta, \xi) \in \mathscr{S}^{-1}(\Gamma(\pi/2))$, is the restriction to the sphere ∂B_1 of the positive part of the first euclidean coordinate. In particular u is positive in $\Gamma(\pi/2)$ and vanishes on $\partial\Gamma(\pi/2)$. By (A.15) we have that u solves (5.3) with

$$\lambda(\pi/2) = d - 1, \tag{5.4}$$

so it is an eigenfunction corresponding to $\lambda(\pi/2)$.

We present two useful facts regarding homogeneous functions. To do so we define the *cone generated by* Γ with vertex in the origin (of $\mathbb{R}^d$) as the set

$$\{x \in \mathbb{R}^d : x = r\phi, \text{ with } \phi \in \Gamma \text{ and } r \in \mathbb{R}^+\},$$

see Figure 5.1.

Proposition 5.1.1. *Let* Γ *be a measurable set,* $\alpha > 0$*,* u *be an eigenfunction corresponding to* $\lambda(\Gamma)$ *and* $w : \mathbb{R}^d \to \mathbb{R}$ *be the function defined by*

$$w_{\mathscr{P}}(r, \phi) = r^\alpha u(\phi),$$

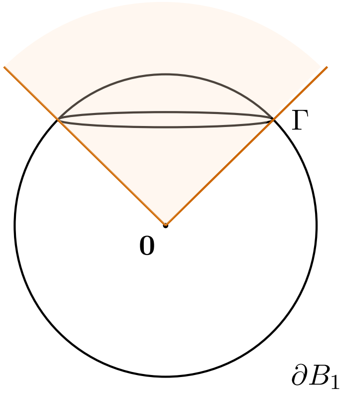

Figure 5.1: The cone generated by a set $\Gamma \subset \partial B_1$ with vertex in the origin (of $\mathbb{R}^3$).

for every $(r, \phi) \in \mathbb{R}^+ \times \partial B_1$*. Define the quantity*

$$\alpha(\Gamma) := \sqrt{\left(\frac{d-2}{2}\right)^2 + \lambda(\Gamma)} - \frac{d-2}{2}. \tag{5.5}$$

Assume that Γ *is open. Then the value* $\alpha = \alpha(\Gamma)$*, called the characteristic constant of the set* Γ*, is the unique positive value such that* w *is harmonic in the cone generated by* Γ *with vertex in the origin.*

Proof. Requiring w to be harmonic, by (A.11) we have

$$0 = r^{\alpha-2}\big[\alpha(\alpha-1) + \alpha(d-1)\big]u(\phi) + r^{\alpha-2}\Delta_\phi u(\phi)$$

for every $(r, \phi) \in \mathbb{R}^+ \times \partial B_1$, that is equivalent by (5.3) to

$$\alpha^2 + (d-2)\alpha - \lambda(\Gamma) = 0,$$

giving the desired conclusion. □

As a consequence of (A.9) and (A.10) we have the following result.

Lemma 5.1.2. *Let* $\alpha > 0$*,* $u : \partial B_1 \to \mathbb{R}$ *be a differentiable function and* $w : \mathbb{R}^d \to \mathbb{R}$ *be the function defined by*

$$w_{\mathscr{P}}(r, \phi) = r^\alpha u(\phi)$$

for every $(r, \phi) \in \mathbb{R}^+ \times \partial B_1$*. Then*

$$\int_{B_1} \frac{|\nabla w(x)|^2}{|x|^{d-2}}\, dx = \frac{1}{2\alpha} \int_{\partial B_1} \left[\alpha^2 u(\phi)^2 + |\nabla_\phi u(\phi)|^2\right] d\sigma.$$

Proposition 5.1.3. *Let* $\Gamma \subset \partial B_1$ *be an open set, then*

$$\alpha(\Gamma) \geq \alpha(\Gamma^\#),$$

where $\Gamma^{\#}$ *is a spherical cap with* $|\Gamma^{\#}| \leq |\Gamma|$*. In particular if* Γ *is a spherical cap, then an eigenfunction* w *corresponding to* $\lambda(\Gamma)$ *is radially symmetric with respect to the center of* Γ*. Namely there exist an isometry* $\psi : \partial B_1 \to \partial B_1$ *and a function* $w^{\mathscr{R}} : [0,\pi) \to \mathbb{R}$ *such that*

$$(w \circ \psi)_{\mathscr{S}}(\theta, \xi) = w^{\mathscr{R}}(\theta),$$

for every $(\theta, \xi) \in (0,\pi) \times \mathbb{S}^{d-2}$*.*

Proof. Let u be a non-negative eigenfunction corresponding to $\lambda(\Gamma)$ and define the open set

$$\Gamma^+ := \{\phi \in \partial B_1 \, : \, u(\phi) > 0\}.$$

We have that $|\Gamma^+| \leq |\Gamma|$ and u^+ is a non-negative eigenfunction corresponding to $\lambda(\Gamma^+)$. Consider $u^{\#}$, the symmetric decreasing rearrangement of u^+, and define the open set

$$\Gamma^{\#} := \{\phi \in \partial B_1 \, : \, u^{\#}(\phi) > 0\}.$$

It is a spherical cap, moreover u and $u^{\#}$ are equidistributed, see [10, Proposition 1.30 (a) and p. 219], and in particular it holds $|\Gamma^{\#}| = |\Gamma^+|$. Therefore, by the so-called *Pólya-Szegő inequality on the sphere*, see [10, Theorem 7.4], we have $u^{\#} \in H^1_0(\Gamma^{\#})$ and from (5.1) we obtain

$$\lambda(\Gamma) = \lambda(\Gamma^+) = \frac{\displaystyle\int_{\Gamma^+} |\nabla_\phi u(\phi)|^2 \, d\sigma}{\displaystyle\int_{\Gamma^+} u(\phi)^2 \, d\sigma} \geq \frac{\displaystyle\int_{\Gamma^{\#}} |\nabla_\phi u^{\#}(\phi)|^2 \, d\sigma}{\displaystyle\int_{\Gamma^{\#}} u^{\#}(\phi)^2 \, d\sigma} \geq \lambda(\Gamma^{\#}), \tag{5.6}$$

by (5.5) we retrieve the first statement.

For the second statement we notice that if Γ is a spherical cap, then there exists an isometry $\psi : \partial B_1 \to \partial B_1$ such that $\Gamma = \psi(\Gamma^{\#})$. Since the inequalities in (5.6) are equalities we have that $u^{\#}$ and $u \circ \psi$ are eigenfunction corresponding to $\lambda(\Gamma^{\#})$. By the connectedness of $\Gamma^{\#}$ and the fact that $u^{\#}$ and u are equidistributed we obtain $u^{\#} = u \circ \psi$, which gives the desired thesis. ☐

In the following the function $V : (0,\pi) \to \mathbb{R}$ defined by

$$V(\theta) = \left(\frac{d-2}{2}\right)\left(\left(\frac{d-4}{2}\right)(\cot\theta)^2 - 1\right)$$

for every $\theta \in (0,\pi)$, where $\cot(\cdot)$ is the *cotangent function*, will play a central role. Since the minimum in (5.1) when Γ is a spherical cap is reached by a function that is radially symmetric with respect to the center of Γ, we can narrow down the set of optimal maps to one-variable (the colatitude) functions.

Lemma 5.1.4. *Let $\theta_0 \in (0,\pi)$, then*

$$\lambda(\theta_0) = \min_{\substack{v \in H_0^1((0,\theta_0)) \\ v \not\equiv 0}} \frac{\displaystyle\int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2\right] d\theta}{\displaystyle\int_0^{\theta_0} v(\theta)^2 \, d\theta}. \tag{5.7}$$

There is a unique non-negative function v normalized in $L^2((0,\theta_0))$ achieving the minimum in (5.7)*. This function satisfies*

$$v(\theta) = w^{\mathscr{R}}(\theta)(\sin\theta)^{\frac{d-2}{2}}$$

for every $\theta \in [0,\theta_0]$, where w is an eigenfunction corresponding to $\lambda(\theta_0)$ and $w^{\mathscr{R}}$ is defined as in Proposition 5.1.3. Moreover it holds $v \in C^\infty([0,\theta_0])$, $v(0) = 0$, $v(\theta_0) = 0$ and $v(\theta) > 0$ for every $\theta \in (0,\theta_0)$.

Proof. Let $v \in H_0^1((0,\theta_0))$ such that the function in the right-hand side of (5.7) is finite. Recall that by the *Sobolev embedding Theorem* $v \in C([0,\theta_0])$, moreover it is a.e. differentiable in $(0,\theta_0)$ and by [51, Theorem 2, p. 273] we have $v(\theta_0) = 0$. Let $u : [0,\theta_0] \to \mathbb{R}$ be the function defined by

$$v(\theta) = u(\theta)(\sin\theta)^{\frac{d-2}{2}}$$

for every $\theta \in [0,\theta_0]$, we have $u(\theta_0) = 0$. Differentiating it we obtain

$$u'(\theta)(\sin\theta)^{\frac{d-2}{2}} = v'(\theta) - \left(\frac{d-2}{2}\right)\cot(\theta)v(\theta) \quad \text{for a.e. } \theta \in (0,\theta_0).$$

An integration by part yields

$$\int_0^{\theta_0} (2v'(\theta)v(\theta))\cot(\theta)\, d\theta = \int_0^{\theta_0} \frac{v(\theta)^2}{(\sin\theta)^2} d\theta,$$

therefore by combining these relations we obtain

$$\frac{\displaystyle\int_0^{\theta_0} u'(\theta)^2(\sin\theta)^{d-2}\, d\theta}{\displaystyle\int_0^{\theta_0} u(\theta)^2(\sin\theta)^{d-2}\, d\theta} = \frac{\displaystyle\int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2\right] d\theta}{\displaystyle\int_0^{\theta_0} v(\theta)^2\, d\theta}.$$

Let $w : \overline{\Gamma(\theta_0)} \to \mathbb{R}$ be the function, radially symmetric with respect to p, defined by

$$w_{\mathscr{S}}(\theta,\xi) = u(\theta)$$

for every $(\theta,\xi) \in [0,\theta_0] \times \mathbb{S}^{d-2}$, we have $w = 0$ on $\partial\Gamma(\theta_0)$. By (A.13) and (A.14) we obtain

$$\frac{\displaystyle\int_{\Gamma(\theta_0)} |\nabla_\phi w(\phi)|^2\, d\sigma}{\displaystyle\int_{\Gamma(\theta_0)} w(\phi)^2\, d\sigma} = \frac{\displaystyle\int_0^{\theta_0} u'(\theta)^2(\sin\theta)^{d-2}\, d\theta}{\displaystyle\int_0^{\theta_0} u(\theta)^2(\sin\theta)^{d-2}\, d\theta},$$

therefore $w \in H_0^1(\Gamma(\theta_0))$. By combining these relations and (5.1) we have

$$\frac{\displaystyle\int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2\right] \, d\theta}{\displaystyle\int_0^{\theta_0} v(\theta)^2 \, d\theta} = \frac{\displaystyle\int_{\Gamma(\theta_0)} |\nabla_\phi w(\phi)|^2 \, d\sigma}{\displaystyle\int_{\Gamma(\theta_0)} w(\phi)^2 \, d\sigma} \geq \lambda(\theta_0),$$

equality holds if and only if w is an an eigenfunction corresponding to $\lambda(\theta_0)$.

Assume that w is an an eigenfunction corresponding to $\lambda(\theta_0)$, by Proposition 5.1.3 and the above relations we have

$$v(\theta) = w^{\mathscr{R}}(\theta)(\sin\theta)^{\frac{d-2}{2}} \tag{5.8}$$

for every $\theta \in [0, \theta_0]$. Since $\Gamma(\theta_0)$ is a connected set, the eigenvalue $\lambda(\theta_0)$ is simple and $w \neq 0$ in $\Gamma(\theta_0)$. Moreover, since $\Gamma(\theta_0)$ is a set of class C^∞, by [36, Theorem 1, p. 8] we obtain that $w \in C^\infty(\overline{\Gamma(\theta_0)})$. By imposing the conditions of non-negativity and normalization on v, using (5.8) and the analyticity of $\mathscr{S}$ we retrieve the desired thesis. □

5.2 The convexity of the first eigenvalue of spherical caps

This section is dedicated to the introduction of the main result exploited in the proof of the Friedland-Hayman inequality: the convexity of the one-dimensional function that describes the first eigenvalue of a spherical caps in terms of its colatitude. This fact was proved in a more general setting (and using more sophisticated tools) in [24, Corollary 1.15]. We follow the approach of [77]. It is sufficient to retrieve it in dimension greater than or equal to 5, since it will be exploited in a limiting process in Section 5.3.

Proposition 5.2.1. *The function $\lambda : (0, \pi) \to \mathbb{R}$, defined by* (5.7) *for every $\theta_0 \in (0, \pi)$, is twice differentiable, strictly decreasing and*

$$\lim_{\theta_0 \to 0^+} \lambda(\theta_0) = +\infty. \tag{5.9}$$

Moreover if $d \geq 5$, it is a strictly convex function.

Proof. Let $\theta_0 \in (0, \pi)$ and denote $v = v_{\theta_0}$ the unique non-negative function normalized in $L^2((0, \theta_0))$ that achieves the minimum in (5.7), it holds

$$\lambda(\theta_0) = \int_0^{\theta_0} \left[v'(\theta)^2 + V(\theta)v(\theta)^2\right] \, d\theta.$$

By Lemma 5.1.4 we have $v \in C^\infty([0, \theta_0])$, $v(0) = 0$, $v(\theta_0) = 0$ and $v(\theta) > 0$ for every $\theta \in (0, \theta_0)$, which yields

$$v'(0) > 0 \quad , \quad v'(\theta_0) < 0. \tag{5.10}$$

The function v is the unique non-negative solution of the boundary value problem

$$\begin{cases} -v''(\theta) + V(\theta)v(\theta) = \lambda(\theta_0)v(\theta) & \text{for every } \theta \in (0, \theta_0), \\ v(0) = v(\theta_0) = 0, \\ \int_0^{\theta_0} v(\theta)^2 \, d\theta = 1. \end{cases} \tag{5.11}$$

The first relation in (5.11) is the so-called *Euler-Lagrange equation* associated to (5.7). Consider the set

$$T = \{(\theta, \theta_0) \subset [0, \pi] \times (0, \pi) \, : \, \theta \leq \theta_0\},$$

it is possible define a composite function as

$$v : T \to \mathbb{R}, \quad (\theta, \theta_0) \mapsto v(\theta, \theta_0) = v_{\theta_0}(\theta).$$

We indicate its derivative in the first argument as v', while the one in the second argument as $\dot{v}$. This new function satisfies

$$\begin{cases} -v''(\theta, \theta_0) + V(\theta)v(\theta, \theta_0) = \lambda(\theta_0)v(\theta, \theta_0) & \text{for every } (\theta, \theta_0) \in \overset{\circ}{T}, \\ v(0, \theta_0) = v(\theta_0, \theta_0) = 0 & \text{for every } \theta_0 \in (0, \pi), \\ \int_0^{\theta_0} v(\theta, \theta_0)^2 \, d\theta = 1 & \text{for every } \theta_0 \in (0, \pi), \end{cases} \tag{5.12}$$

and it holds

$$\lambda(\theta_0) = \int_0^{\theta_0} \left[v'(\theta, \theta_0)^2 + V(\theta) v(\theta, \theta_0)^2 \right] d\theta \quad \text{for every } \theta_0 \in (0, \pi). \tag{5.13}$$

We denote by $\dot{\lambda}$ and $\ddot{\lambda}$ the first and second derivative of the function λ, respectively. In the following we will always evaluate the function v and its derivatives at a fixed second coordinate θ_0, so, whenever present, the argument of these functions will always correspond to the first one. The proof is divided in five steps.

Step 1 (regularity of λ). We claim that $\lambda \in C^\infty(0, \pi)$. By direct calculation we have $\mathscr{X}(\Gamma(\theta_0)) = B_\wedge$, where $B_\wedge$ is the ball of radius $\tan(\theta_0/2)$ in $\mathbb{R}^{d-1}$ with center in the origin. Let $\tilde{u} : \mathbb{R}^{d-1} \to \mathbb{R}$ be a solution of the eigenvalue problem

$$\begin{cases} \dfrac{1}{\Upsilon^2} \left(\Delta \tilde{u} + (d-2) \nabla \tilde{u} \cdot \nabla \log \Upsilon \right) + \lambda(\theta_0) \tilde{u} = 0 & \text{in } B_\wedge, \\ \tilde{u} = 0 & \text{on } \partial B_\wedge. \end{cases} \tag{5.14}$$

By (A.17) we notice that the function $\tilde{u}_{\mathscr{X}^{-1}}$ is an eigenfunction corresponding to $\lambda(\theta_0)$. Let u_{θ_0} be the unique non-negative solution of (5.14) satisfying the normalization condition

$$\int_{B_\wedge} u_{\theta_0}(\hat{x})^2 \, d\hat{x} = 1,$$

we define $\mathbf{u}_{\theta_0} := (u_{\theta_0})_{\mathscr{X}^{-1}}$. Let $\delta \in \mathbb{R}$ such that $|\delta| \ll \theta_0$, we define the function h_δ :

$\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ as

$$h_\delta(\hat{x}) = \frac{\tan\left(\dfrac{\theta_0+\delta}{2}\right)}{\tan(\theta_0/2)}\hat{x}$$

for every $\hat{x} \in \mathbb{R}^{d-1}$. We notice that $h_\delta(B_\wedge) = B_\wedge^\delta$, where $B_\wedge^\delta$ is the ball of radius $\tan(\theta_0/2)+\delta$ in $\mathbb{R}^{d-1}$ with center in the origin. Since the operator in (5.14) has analytic coefficients, is uniformly elliptic on bounded sets, $\lambda(\theta_0)$ is simple and $B_\wedge$ is a set of class C^∞, it is possible to follow the same strategy of [68, Example 3.2, p. 33] with the family $\{h_\epsilon\}_{\epsilon\in(-\delta,\delta)}$ to conclude that $\lambda \in C^\infty(0,\pi)$.

Step 2 (regularity of v). We claim that $v \in C^\infty(T)$. Consider the set

$$\mathcal{T} = \{(\hat{x},\theta_0) \subset \mathbb{R}^{d-1}\times(0,\pi) \,:\, |\hat{x}| \le \tan(\theta_0/2)\},$$

it is possible define a composite function as

$$u : \mathcal{T} \to \mathbb{R}, \quad (\hat{x},\theta_0) \mapsto u(\hat{x},\theta_0) = u_{\theta_0}(\hat{x}).$$

Let $\underline{u}_{\theta_0} : \mathbb{R}^{d-1} \to \mathbb{R}$ be the extension of u_{θ_0} defined in [68, Example 3.2, p. 33] exploiting [68, Theorem 1.9], we have $\underline{u}_{\theta_0}(\hat{x}) = u_{\theta_0}(\hat{x})$ for every $\hat{x} \in \overline{B_\wedge}$. The construction of Step 1 allows also to conclude that for every $i \in \mathbb{N}$ the map

$$(-\delta,\delta) \to H^i(\mathbb{R}^{d-1}), \quad \epsilon \mapsto \underline{u}_{\theta_0+\epsilon} \tag{5.15}$$

is C^∞. We use the symbol $\dot{}$ to denote the derivative with respect to the 1-dimensional parameter used to describe the family $\{h_\epsilon\}_{\epsilon\in(-\delta,\delta)}$. By the *Sobolev embedding Theorem* the functions $\underline{u}_{\theta_0}$ and $\dot{\underline{u}}_{\theta_0}$ belong to $C^\infty(B_\wedge^\delta)$. Moreover, from the regularity of the map (5.15) it holds

$$\frac{\underline{u}_{\theta_0+\epsilon}(\hat{x}) - \underline{u}_{\theta_0}(\hat{x})}{\epsilon} = \dot{\underline{u}}_{\theta_0}(\hat{x}) + \tau_\epsilon(\hat{x})$$

for every $(\hat{x},\epsilon) \in B_\wedge^\delta \times (-\delta,\delta)$, where the differentiable function $\tau_\epsilon : B_\wedge^\delta \to \mathbb{R}$ satisfies $\tau_\epsilon \to 0$ in $C^1(B_\wedge^\delta)$ as $\epsilon \to 0$. Differentiating with respect to the coordinates of $\mathbb{R}^{d-1}$ and then letting $\epsilon \to 0$ we obtain

$$(\dot{\nabla\underline{u}_{\theta_0}})(\hat{x}) = \nabla\dot{\underline{u}}_{\theta_0}(\hat{x})$$

for every $\hat{x} \in B_\wedge^\delta$. We notice that a similar reasoning can also be applied to higher-order derivatives, therefore restricting ourselves to $\overline{B_\wedge}$ we obtain that $u \in C^\infty(\mathcal{T})$. As a by-product, since $\mathscr{S}$ and $\mathscr{X}$ are analytic, we have that the function

$$\mathbf{u}^{\mathscr{R}} : T \to \mathbb{R}, \quad (\theta,\theta_0) \mapsto \mathbf{u}^{\mathscr{R}}(\theta,\theta_0) = \mathbf{u}^{\mathscr{R}}_{\theta_0}(\theta)$$

belongs to $C^\infty(T)$. Recall that by Lemma 5.1.4 we have

$$v(\theta,\theta_0) = w^{\mathscr{R}}_{\theta_0}(\theta)(\sin\theta)^{\frac{d-2}{2}} \tag{5.16}$$

for every $(\theta, \theta_0) \in T$, where w_{θ_0} is a non-negative eigenfunction corresponding to $\lambda(\theta_0)$. Since both $\mathbf{u}_{\theta_0}$ and w_{θ_0} are non-negative eigenfunctions corresponding to $\lambda(\theta_0)$, then they differs by a positive multiplicative factor. In particular the third condition in (5.11) yields

$$w_{\theta_0}^{\mathscr{R}} = \left(\int_0^{\theta_0} \mathbf{u}_{\theta_0}^{\mathscr{R}}(\theta)^2 (\sin\theta)^{d-2}\, d\theta \right)^{-1/2} \mathbf{u}_{\theta_0}^{\mathscr{R}},$$

that combined with (5.16) gives $v \in C^\infty(T)$.

Step 3 (monotonicity and limit behavior of λ). We claim that the function λ is strictly decreasing and that (5.9) holds. Differentiating the second and the third relation in (5.12) with respect to θ_0 we have

$$\dot v(0) = 0, \quad v'(\theta_0) = -\dot v(\theta_0)\,, \quad \int_0^{\theta_0} v(\theta)\dot v(\theta)\, d\theta = 0. \tag{5.17}$$

Differentiating (5.13) we obtain

$$\dot\lambda(\theta_0) = v'(\theta_0)^2 + \int_0^{\theta_0} \left[2v'(\theta)\dot v'(\theta) + 2V(\theta)v(\theta)\dot v(\theta)\right]\, d\theta,$$

integrating by parts, from the first two relations in (5.17), we have

$$\dot\lambda(\theta_0) = -v'(\theta_0)^2 + 2\int_0^{\theta_0} \dot v(\theta)[-v''(\theta) + V(\theta)v(\theta)]\, d\theta.$$

Thence exploiting the first relation in (5.11) and the third relation in (5.17) we find

$$\dot\lambda(\theta_0) = -v'(\theta_0)^2, \tag{5.18}$$

that combined with the second relation in (5.10) gives the desired strict monotonicity for the map λ. The relation (5.9) follows from [36, equation (36), p. 318].

Step 4 (nodal domains of $\dot v$). We claim that the function $\dot v$ has exactly two nodal domains, namely that the connected components of the set $(0, \theta_0) \setminus \{\dot v = 0\}$ are two. We introduce the function

$$q : (0, \theta_0) \to \mathbb{R} \quad , \quad q = \frac{\dot v}{v},$$

whose derivatives are

$$q' = \frac{\dot v'}{v} - \frac{v'}{v} q$$

and

$$\begin{aligned} q'' &= \frac{\dot v''}{v} - \frac{\dot v' v'}{v^2} - \frac{v''}{v} q + \left(\frac{v'}{v}\right)^2 q - \frac{v'}{v} q' \\ &= \frac{\dot v''}{v} - \frac{v''}{v} q - 2\frac{v'}{v} q'. \end{aligned}$$

By the first relation in (5.11) and the derivative of the first relation in (5.12) with respect to the second variable restricted to the set $(0, \theta_0) \times \{\theta_0\}$, namely

$$-\dot{v}''(\theta) + V(\theta)\dot{v}(\theta) = \dot{\lambda}(\theta_0)v(\theta) + \lambda(\theta_0)\dot{v}(\theta) \quad \text{for every } \theta \in (0, \theta_0),$$

it is possible to infer

$$q'' = -\dot{\lambda}(\theta_0) - 2\frac{v'}{v}q'$$

and thus

$$(v^2 q')' = -\dot{\lambda}(\theta_0)v^2.$$

In particular, by Step 3, the function $v^2 q' : [0, \theta_0] \to \mathbb{R}$ is strictly increasing, so it attains its minimum at 0. Since $v \in C^\infty(T)$, the second relation in (5.11) and the first relation in (5.17) give

$$\dot{v}'(0), v'(0) < \infty \quad \text{and} \quad \dot{v}(0) = v(0) = 0,$$

therefore we have

$$(v^2 q')(\theta) > (v^2 q')(0) = \dot{v}'(0)v(0) - \dot{v}(0)v'(0) = 0 \quad \text{for every } \theta \in (0, \theta_0).$$

So q is a strictly increasing function and thus it has at most one zero in $(0, \theta_0)$. This latter fact holds also for $\dot{v}$, since v is a positive function in $(0, \theta_0)$. By the third relation in (5.17) we notice that $\dot{v}$ is orthogonal to v in $L^2((0, \theta_0))$, therefore $\dot{v}$ has non-constant sign in $(0, \theta_0)$. In particular, since $\dot{v}$ is continuous, there exists a unique value $\bar{\theta} \in (0, \theta_0)$ such that $\dot{v}(\bar{\theta}) = 0$. At this point we sketch the structure of $\dot{v}$. By the first two relations in (5.17) and the second relation in (5.10) we have

$$\dot{v}(0) = 0, \quad \dot{v}(\bar{\theta}) = 0, \quad \dot{v}(\theta_0) = -v'(\theta_0) > 0,$$

so

$$\begin{aligned} \dot{v}(\theta) < 0 \quad &\text{for every } \theta \in \big(0, \bar{\theta}\big), \\ \dot{v}(\theta) > 0 \quad &\text{for every } \theta \in \big(\bar{\theta}, \theta_0\big), \end{aligned} \tag{5.19}$$

and therefore

$$\dot{v}'(0) \le 0. \tag{5.20}$$

Step 5 (convexity of λ). We claim that the function λ is strictly convex if $d \ge 5$. We notice that it is possible to express the derivative of the map λ in another way. Multiplying the first relation in (5.11) by v' and integrating, by the third relation in (5.17) we find

$$-v'(\theta_0)^2 = -\int_0^{\theta_0} V(\theta)\left(v(\theta)^2\right)' d\theta - v'(0)^2,$$

that integrating by parts and using (5.18) yields

$$\dot{\lambda}(\theta_0) = \int_0^{\theta_0} V'(\theta)v(\theta)^2\,d\theta - v'(0)^2. \tag{5.21}$$

Differentiating (5.21), by the third relation in (5.17), we find

$$\ddot{\lambda}(\theta_0) = 2\int_0^{\theta_0} \Big[V'(\theta) - V'(\bar{\theta})\Big]\, v(\theta)\dot{v}(\theta)\,d\theta - 2v'(0)\dot{v}'(0) > 0.$$

The last inequality is obtained combining – recall that $\cot(\cdot)^2$ is a strictly convex function in $(0,\pi)$ – the strict convexity of V in $(0,\pi)$ for $d \geq 5$, the information contained in (5.19), the first relation in (5.10) and (5.20), and gives the desired strict convexity for the map λ. ☐

5.3 Proof of Theorem 1.6.1

Proof of Theorem 1.6.1. The proof is divided in four steps. For the first two steps we follow the original strategy of [6, Lemma 5.1], while for the remaining ones, which involve the proof of the Friedland-Hayman inequality, we rely on [94, Section 4.3].

Step 1 (finiteness and reduction to an inequality). We claim that the function J is finite and to obtain its monotonicity it is sufficient to prove an inequality. Let $u \in H^1(B_2)$ be a function satisfying

$$\begin{cases} \Delta u \geq 0 & \text{in } B_2, \\ u \text{ is non-negative.} \end{cases} \tag{5.22}$$

where the first inequality holds in the sense of distribution. Below, we will repeatedly use (A.9) without explicitly stating it. For every $0 < s \leq 1$, when integrals over subsets of ∂B_s arise, for brevity we write u instead of $u_{\mathscr{P}}$ and omit the dependence on polar coordinates. Define the map $I : (0,1) \to \mathbb{R}$ as

$$I(s) = \int_{B_s} \frac{|\nabla u(x)|^2}{|x|^{d-2}}\,dx$$

for every $s \in (0,1)$, when we want to emphasize the dependence on the function we write $I(\cdot, u)$ instead of $I(\cdot)$. Let $\varepsilon > 0$ and $u_\varepsilon = \phi_\varepsilon * u$, where $\phi_\varepsilon \in C_c^\infty(B_\varepsilon)$ is a standard mollifier. Then $u_\varepsilon \in C^\infty(B_2)$, $\Delta u_\varepsilon \geq 0$ in B_{2s}, and $u_\varepsilon \to u$ strongly in $H^1(B_2)$ as $\varepsilon \to 0$. By (5.22) we have

$$\Delta u_\varepsilon^2 \geq 2|\nabla u_\varepsilon|^2 \quad \text{in } B_{2s} \tag{5.23}$$

in the sense of distributions. Let $\psi \in C_c^\infty(B_{2s})$ be a radially decreasing function such that

$\psi \equiv 1$ in B_s, $\psi \equiv 0$ in $\mathbb{R}^d \setminus B_{2s}$ with $r < 1$. By (5.23) we obtain

$$\begin{aligned}
2\int_{B_{2s}} \frac{\psi|\nabla u_\varepsilon|^2}{|x|^{d-2}}\,dx &\le \int_{B_{2s}} u_\varepsilon^2 \Delta\left(\frac{\psi}{|x|^{d-2}}\right)dx \\
&= \int_{B_{2s}} \Big[u_\varepsilon^2\psi\Delta\Big(|x|^{2-d}\Big) + 2u_\varepsilon^2\nabla\psi\cdot\nabla(|x|^{2-d}) + u_\varepsilon^2|x|^{2-d}\Delta\psi\Big]\,dx \\
&\le \bar{C}\left(-u_\varepsilon^2(\mathbf{0}) + s^{-d}\int_{B_{2s}\setminus B_s} u_\varepsilon^2\,dx\right) \\
&\le \bar{C}s^{-d}\int_{B_{2s}\setminus B_s} u_\varepsilon^2\,dx,
\end{aligned}$$

where $\bar{C}$ is a positive constant depending on the dimension d and the penultimate inequality follows from the radial decreasing behavior of ψ. Letting $\epsilon \to 0$ we obtain

$$I(s) \le \bar{C}s^{-d}\int_{B_{2s}\setminus B_s} u(x)^2\,dx$$

for every $s \in (0,1)$. This shows immediately that the value $J(s)$ is finite for every $s \in (0,1)$. So the map I is differentiable a.e. in $(0,1)$, passing to polar coordinates and exploiting (A.10) we have

$$I'(s) = s^{2-d}\int_{\partial B_s}\left[u_r^2 + \frac{1}{s^2}|\nabla_\phi u|^2\right]d\sigma$$

for a.e. $s \in (0,1)$. Differentiating (1.38) and evaluating it at one of these points $S \in (0,1)$ for u^+ and u^- we obtain

$$J'(S) = I(S,u_+)I(S,u_-)S^{-5}\Big(S\Big(\frac{I'(S,u_+)}{I(S,u_+)} + \frac{I'(S,u_-)}{I(S,u_-)}\Big) - 4\Big).$$

Consider the map $u_{\boxed{S}} : B_2 \to \mathbb{R}$ defined as $u_{\boxed{S}}(x) := \frac{u(Sx)}{S}$ for every $x \in B_2$, it satisfies (5.22) with $u = u_{\boxed{S}}$ and we can write

$$\frac{I'(S,u)}{I(S,u)} = \frac{1}{S}\frac{\displaystyle\int_{\partial B_1}\Big[((u_{\boxed{S}})_r)^2 + |\nabla_\phi(u_{\boxed{S}})|^2\Big]\,d\sigma}{\displaystyle\int_{B_1}\frac{|\nabla u_{\boxed{S}}(x)|^2}{|x|^{d-2}}\,dx}. \tag{5.24}$$

In particular to obtain the desired monotonicity it is sufficient to show that

$$\frac{I'(1,u_+)}{I(1,u_+)} + \frac{I'(1,u_-)}{I(1,u_-)} - 4 \ge 0 \tag{5.25}$$

for functions u_+, u_- satisfying (1.37).

Step 2 (reduction to the Friedland-Hayman inequality). We claim that to obtain the monotonicity of the function J it is sufficient to prove the Friedland-Hayman inequality. Define the measurable set

$$\Gamma = \{u > 0\} \cap \partial B_1.$$

Since u is subharmonic from (5.22), then by an integration by parts we obtain

$$I(1) \leq \int_\Gamma u u_r \, d\sigma + \frac{d-2}{2} \int_{B_1} \frac{\nabla u(x)^2 \cdot x}{|x|^d} \, dx.$$

Again the subharmonicity of u and an integration by part yield

$$\int_{B_1} \frac{\nabla u(x)^2 \cdot x}{|x|^d} \, dx \leq \int_\Gamma u^2 \, d\sigma,$$

so, in particular, it holds

$$I(1) \leq \int_\Gamma \left[u u_r + \frac{d-2}{2} u^2 \right] \, d\sigma.$$

Consequently by (5.24) we have

$$\frac{I'(1)}{I(1)} \geq \frac{\displaystyle\int_\Gamma \left[u_r^2 + |\nabla_\phi u|^2 \right] \, d\sigma}{\displaystyle\int_\Gamma \left[u u_r + \frac{d-2}{2} u^2 \right] \, d\sigma}. \tag{5.26}$$

Let $t \in [0, 1]$ and denote $\lambda = \lambda(\Gamma)$, by (5.1) and the *Young's inequality* for products we have

$$\int_\Gamma \left[u_r^2 + |\nabla_\phi u|^2 \right] \, d\sigma \geq 2 \Big(\int_\Gamma u_r^2 \, d\sigma \Big)^{\frac{1}{2}} \Big(t\lambda \int_\Gamma u^2 \, d\sigma \Big)^{\frac{1}{2}} + (1-t)\lambda \int_\Gamma u^2 \, d\sigma,$$

on the other hand by *Hölder's inequality* we obtain

$$\int_\Gamma u u_r \, d\sigma + \frac{d-2}{2} \int_\Gamma u^2 \, d\sigma \leq \Big(\int_\Gamma u_r^2 \, d\sigma \Big)^{\frac{1}{2}} \Big(\int_\Gamma u^2 \, d\sigma \Big)^{\frac{1}{2}} + \frac{d-2}{2} \int_\Gamma u^2 \, d\sigma.$$

Therefore setting

$$z = \frac{\Big(\displaystyle\int_\Gamma u^2 \, d\sigma \Big)^{\frac{1}{2}}}{\Big(\displaystyle\int_\Gamma u_r^2 \, d\sigma \Big)^{\frac{1}{2}}},$$

it holds

$$\frac{I'(1)}{I(1)} \geq \frac{2(t\lambda)^{\frac{1}{2}} + \lambda(1-t)z}{1 + \frac{d-2}{2} z},$$

moreover it is possible to estimate

$$\frac{2(t\lambda)^{\frac{1}{2}} + \lambda(1-t)z}{1 + \frac{d-2}{2} z} \geq 2 \min \left\{ (t\lambda)^{\frac{1}{2}}, \frac{\lambda}{d-2}(1-t) \right\}.$$

At this point, we choose t such that these two lower bounds are equal, namely

$$t\lambda + (d-2)(t\lambda)^{\frac{1}{2}} - \lambda = 0, \tag{5.27}$$

this is equivalent to require

$$\sqrt{t} = \frac{\sqrt{4\lambda}}{(d-2) + \sqrt{(d-2)^2 + 4\lambda}}.$$

In particular there exists a unique such $t \in [0,1]$, therefore by (5.5) and (5.27) we have $t\lambda = \alpha(\Gamma)^2$ and

$$\frac{I'(1)}{I(1)} \geq 2\alpha(\Gamma). \tag{5.28}$$

Combining (5.25) and (5.28) we are reduced to show that

$$\alpha(\Gamma_+) + \alpha(\Gamma_-) \geq 2, \tag{5.29}$$

for any pair of disjoint measurable sets $\Gamma_+, \Gamma_- \subset \partial B_1$, i.e., the Friedland-Hayman inequality.

Step 3 (a monotonicity property of characteristic constants). We claim that the characteristic constant of a spherical cap of fixed colatitude is monotonically decreasing with respect to its dimension. Fix $\theta_0 \in (0,\pi)$, we denote $\Gamma_d(\theta_0)$ the spherical cap of colatitude θ_0 with center p in $\partial B_1 \subset \mathbb{R}^d$. Let $w : \mathbb{R}^d \to \mathbb{R}$ be the positive homogeneous function defined by

$$w_{\mathscr{P}}(r,\phi) = r^{\alpha(\theta_0,d)}u(\phi),$$

for every $(r,\phi) \in \mathbb{R}^+ \times \partial B_1$, where $\alpha(\theta_0,d)$ is the characteristic constant of the set $\Gamma_d(\theta_0)$, i.e, $\alpha(\theta_0,d) := \alpha(\Gamma_d(\theta_0))$, and u is a positive eigenfunction corresponding to $\lambda(\Gamma_d(\theta_0))$. By Proposition 5.1.1, the function w is harmonic in $\{w > 0\}$, this set is the cone generated by $\Gamma_d(\theta_0)$ with vertex in the origin of $\mathbb{R}^d$, in particular w satisfies (5.22). Recall that it is possible to embed the space $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ using the map

$$(x_1,\dots,x_d) \to (x_1,\dots,x_d,0),$$

in this way the we have $\partial B_1 \subset \mathbb{S}^d$, where $\mathbb{S}^d$ is the sphere of unit radius in $\mathbb{R}^{d+1}$ with center in the origin. We define the function $\tilde{w} : \mathbb{R}^{d+1} \to \mathbb{R}$ as

$$\tilde{w}(x_1,\dots,x_d,x_{d+1}) = w(x_1,\dots,x_d),$$

it has the same homogeneity degree of w and is harmonic in $\{\tilde{w} > 0\}$. We notice that it holds

$$\{\tilde{w} > 0\} = \{w > 0\} \times \mathbb{R},$$

and so this set is the cone generated by $(\{w > 0\} \times \mathbb{R}) \cap \mathbb{S}^d$ with vertex in the origin of

$\mathbb{R}^{d+1}$.

Moreover we can write

$$\tilde{w}_{\tilde{\mathscr{S}}}(\tilde{r},\tilde{\phi}) = \tilde{r}^{\alpha(\theta_0,d)}\tilde{u}(\tilde{\phi}),$$

for every $(\tilde{r},\tilde{\phi}) \in \mathbb{R}^+ \times \mathbb{S}^d$, where $\tilde{\mathscr{S}}$ is the polar parametrization of $\mathbb{R}^{d+1}$ (and $(\tilde{r},\tilde{\phi})$ the corresponding polar coordinates) and $\tilde{u} : \mathbb{S}^d \to \mathbb{R}$ is a function. We observe that $\tilde{w}$ satisfies the analogous of (5.22) in $\mathbb{R}^{d+1}$, so by Step 2, see (5.28), we obtain

$$\frac{I'(1,\tilde{w})}{I(1,\tilde{w})} \geq 2\alpha(\theta_0, d+1),$$

on the other hand by (5.24) and Lemma 5.1.2 we have

$$\frac{I'(1,\tilde{w})}{I(1,\tilde{w})} = \frac{I'(1,w)}{I(1,w)} = 2\alpha(\theta_0, d),$$

which yields

$$\alpha(\theta_0, d) \geq \alpha(\theta_0, d+1). \tag{5.30}$$

Step 4 (the Friedland-Hayman inequality). We claim that the Friedland-Hayman inequality holds. Consider (5.29), by Proposition 5.1.3 it is sufficient to prove that it holds for all pairs (Γ_+, Γ_-) of disjoint spherical caps in ∂B_1. From the strict monotonicity statement of Proposition 5.2.1, (an isometry) and (5.5) we can take these sets to be complementary in ∂B_1, namely this is equivalent to prove that

$$\min_{\theta_0 \in (0,\pi)} \alpha(\theta_0, d) + \alpha(\pi - \theta_0, d) \geq 2.$$

By (5.9) and the differentiability statement of Proposition 5.2.1 there exists a value $\theta_d \in (0,\pi)$ such that the minimum is achieved. Define the function $\beta : \{m \in \mathbb{N} : m \geq 3\} \to \mathbb{R}$ as

$$\beta(d) = \alpha(\theta_d, d) + \alpha(\pi - \theta_d, d),$$

for $d \in \{m \in \mathbb{N} : m \geq 3\}$, by (5.30) it is monotone non-increasing.
Suppose by way of contradiction that exists an $d_0 \in \mathbb{N}$ and $\delta > 0$ such that $\beta(d_0) < 2 - \delta$, then $\beta(d) < 2 - \delta$ for all $d \geq d_0$ (this value will be increased in the following in order to satisfy more conditions and ease the notation). Therefore by minimality of θ_d we have

$$\alpha(\theta_d, d), \alpha(\pi - \theta_d, d) < 2,$$

that by (5.5) gives

$$\lambda(\theta_d), \lambda(\pi - \theta_d) < 2d. \tag{5.31}$$

We study the behavior of the sequence

$$\left\{\beta(d) = \left(\frac{d-2}{2}\right)\left(\sqrt{1 + \frac{4\lambda(\theta_d)}{(d-2)^2}} - 1 + \sqrt{1 + \frac{4\lambda(\pi - \theta_d)}{(d-2)^2}} - 1\right)\right\}_d$$

as $d \to +\infty$, we can estimate

$$\beta(d) \geq \frac{\lambda(\theta_d) + \lambda(\pi - \theta_d)}{d-2} - \frac{\lambda(\theta_d)^2 + \lambda(\pi - \theta_d)^2}{(d-2)^3}$$

for all $d \geq d_0$, up to taking a bigger d_0, since the third term in the Taylor formula – the first one not appearing here – is positive. By (5.31) we can estimate

$$-\frac{\lambda(\theta_d)^2 + \lambda(\pi - \theta_d)^2}{(d-2)^3} \geq -\frac{c}{d},$$

where c is a positive constant (not depending on d). Consider the function $\gamma_d : (0, \pi) \to \mathbb{R}$ defined as

$$\gamma_d(\theta) := \frac{\lambda(\theta) + \lambda(\pi - \theta)}{d-2},$$

for every $\theta \in (0, \pi)$. By Proposition 5.2.1, for $d \geq 5$, γ_d is a strictly convex function, evenly symmetric with respect to the point $\pi/2$, that is its unique minimum. So by (5.4) we obtain

$$\gamma_d(\theta_d) \geq \gamma_d\left(\frac{\pi}{2}\right) = \frac{2d-2}{d-2} > 2$$

for all $d \geq 5$, therefore by combining the estimates above we have

$$\beta(d) > 2 - \frac{c}{d} \geq 2 - \delta,$$

for all $d \geq d_0$, up to taking a bigger d_0, a contradiction. This proves the Friedland-Hayman inequality and therefore, by Step 2, the desired monotonicity of J. ☐

5.4 An open problem on uniqueness of blow-up limits

We present an open question taken from [3, Introduction].

Consider the family of *two-planes solutions*, namely

$$\left\{ \begin{array}{l} \ell : \mathbb{R}^d \to \mathbb{R} \\ \qquad x \mapsto c_+(x \cdot \nu)^+ + c_-(x \cdot \nu)^- \end{array} \;\middle|\; c_+, c_- > 0 \text{ and } \nu \in \partial B_1 \right\}.$$

A two-planes solution is a function made up of a pair of positive linear functions defined on two disjoint complementary half-spaces (see Figure 5.2), vanishing on the common boundary that contains the origin, they satisfy (1.37). Let u_+, u_- be two functions satisfying (1.37) and

$$\lim_{s \to 0^+} J(s) > 0. \tag{5.32}$$

Consider $\{s_k\}_k \subset (0,1)$ a *sequence of radii* decreasing to 0, and the associated *blow-up sequences*

$$\left\{\frac{u_+(s_k \,\cdot)}{s_k}\right\}_k \quad \text{and} \quad \left\{\frac{u_-(s_k \,\cdot)}{s_k}\right\}_k,$$

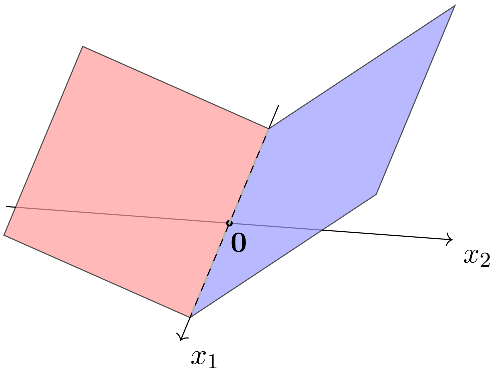

Figure 5.2: An example of two-planes solution in $\mathbb{R}^2$.

restricted to B_1. As in [6, Section 6], up to subsequence, it is possible to show that they converge, in an appropriate way, to the the functions

$$x \mapsto c_+(x \cdot \nu)^+ \quad \text{and} \quad x \mapsto c_-(x \cdot \nu)^-,$$

where $c_+, c_- > 0$ and $\nu \in \partial B_1$, defined for $x \in B_1$. This pair is called *blow-up limit*. Observe that the values c_+, c_-, ν, depend *a priori* on the sequence $\{s_k\}_k$. The result of the blow-up procedure described above is a pair of functions representing one of the possible behaviors that the original pair exhibits when approaching the origin. In the theory of free boundary problems, classifying the possible blow-up limits is useful to obtain information about the regularity of the free boundary, see for example [46, proof of Theorem 1.1] for a related result.

It has been shown in [3, Theorem 1.3] that it is possible to construct two functions $\tilde{u}_+$ and $\tilde{u}_-$ satisfying (1.37), that admit multiple different blow-up limits, depending on the chosen sequence of radii. In particular the positivity sets of these functions wrap around the origin. More precisely for all of the possible blow-up limits, $\tilde{c}_+$ and $\tilde{c}_-$ are the same, while $\tilde{\nu}$ can be different, depending on the particular subsequence of radii.

Open problem 4 ([3]). *Is it possible to find a pair u_+, u_- satisfying* (1.37) *and* (5.32) *such that for all of the blow-up limits, ν is the same, while the pair (c_+, c_-) can be different, depending on the particular subsequence of radii?*

Appendix A

Auxiliary facts

We collect several standard results required for the proofs of the main theorems. The first section is devoted to properties of Bessel functions that are essential for the arguments developed in Chapter 4, while the second section recalls basic facts on parametrizations and rearrangements, which are used primarily in Chapter 5.

A.1 Some results on Bessel functions

In this subsection, we recall some facts about Bessel functions of arbitrary order. For $\sigma \in \mathbb{R}$ and $m \in \mathbb{N}$ let J_σ denote the *Bessel function of the first kind* of order σ (it is an analytic function of real variable, defined for positive values) and $j_{\sigma,m}$ its m-th zero. If $\sigma > -1$ by [110, Section 15.22, p. 479] $j_{\sigma,1} < j_{\sigma+1,1}$ and moreover the Bessel function J_σ is positive in $(0, j_{\sigma,1})$. By [110, Section 3.2, p. 45], for every $x > 0$, the following *recurrence formulae* hold:

$$\frac{2\sigma}{x} J_\sigma(x) - J_{\sigma-1}(x) = J_{\sigma+1}(x), \tag{A.1}$$

and

$$(x^\sigma J_\sigma(x))' = x^\sigma J_{\sigma-1}(x), \quad (x^{-\sigma} J_\sigma(x))' = -x^{-\sigma} J_{\sigma+1}(x). \tag{A.2}$$

In particular, given $R, \omega > 0$, from the first formula for the derivative in (A.2) we derive the following *integral representation* of Bessel functions

$$\int_0^R r^\sigma J_{\sigma-1}(\omega r)\, dr = \frac{R^\sigma}{\omega} J_\sigma(\omega R). \tag{A.3}$$

Proposition A.1.1. *For $\sigma \geq 1/2$, the analytic function $\phi_\sigma \colon (0, j_{\sigma-1,1}) \cup (j_{\sigma-1,1}, j_{\sigma,1}) \to \mathbb{R}$ defined by*

$$\phi_\sigma(x) := x^{2\sigma} \frac{J_{\sigma+1}(x)}{J_{\sigma-1}(x)}, \quad \textit{for every } x \in (0, j_{\sigma-1,1}) \cup (j_{\sigma-1,1}, j_{\sigma,1}),$$

is positive in $(0, j_{\sigma-1,1})$ and negative in $(j_{\sigma-1,1}, j_{\sigma,1})$. Moreover, ϕ_σ is strictly increasing in $(0, j_{\sigma-1,1}) \cup (j_{\sigma-1,1}, j_{\sigma,1})$.

Proof. The sign of ϕ_σ follows from the signs of the Bessel functions $J_{\sigma+1}$ and $J_{\sigma-1}$ in the domain of definition of ϕ_σ (see Porter's theorem above). By (A.2) and (A.1) we have

$$\begin{aligned}\phi'_\sigma(x) &= \left(\frac{x^{\sigma+1}J_{\sigma+1}(x)}{x^{-\sigma+1}J_{\sigma-1}(x)}\right)' = \frac{x^2 J_\sigma(x)(J_{\sigma-1}(x)+J_{\sigma+1}(x))}{x^{-2\sigma+2}J_{\sigma-1}(x)^2} \\ &= x^{2\sigma}\frac{J_\sigma(x)}{J_{\sigma-1}(x)}\left(\frac{J_{\sigma-1}(x)+J_{\sigma+1}(x)}{J_{\sigma-1}(x)}\right) = 2\sigma x^{2\sigma-1}\frac{J_\sigma(x)^2}{J_{\sigma-1}(x)^2},\end{aligned}$$

which implies the positivity of ϕ'_σ on its domain of definition. □

Proposition A.1.2. *For $\sigma \geq 1/2$, the analytic function $\Phi_\sigma \colon (0, j_{\sigma,1}) \to \mathbb{R}$ defined by*

$$\Phi_\sigma(x) := x^{2\sigma}\frac{J_{\sigma+1}(x)}{J_\sigma(x)}, \quad \textit{for every } x \in (0, j_{\sigma,1}),$$

is positive, strictly increasing with

$$\Phi'_\sigma(x) = x^{2\sigma} - x^{-1}\Phi_\sigma(x) + x^{-2\sigma}\Phi_\sigma(x)^2, \quad \textit{for every } x \in (0, j_{\sigma,1}). \tag{A.4}$$

Proof. By the *Mittag-Leffler expansion*, see [110, Section 15.41, p. 498, equation (1)], we have

$$\frac{J_{\sigma+1}(x)}{J_\sigma(x)} = 2x\sum_{m=1}^{\infty}\frac{1}{(j_{\sigma,m})^2 - x^2}.$$

So, since it is a pointwise convergent sum of positive strictly increasing functions, the function $J_{\sigma+1}(x)/J_\sigma(x)$ is positive and strictly increasing for every $x \in (0, j_{\sigma,1})$. Therefore, the function Φ_σ is positive and strictly increasing, as it is the product of two positive, strictly increasing functions. Now, by (A.2) then

$$\Phi'_\sigma(x) = \left(\frac{x^{\sigma+1}J_{\sigma+1}(x)}{x(x^{-\sigma}J_\sigma(x))}\right)' = x^{2\sigma} - \frac{x^{2\sigma-1}J_{\sigma+1}(x)}{J_\sigma(x)} + \frac{x^{2\sigma}J_{\sigma+1}(x)^2}{J_\sigma(x)^2},$$

and also (A.4) follows. □

Proposition A.1.3. *For $\sigma \geq 1/2$, the analytic function $\Upsilon_\sigma \colon (0, j_{\sigma,1}) \to \mathbb{R}$ defined by*

$$\Upsilon_\sigma(x) := x\left(\frac{J_{\sigma+1}(x)}{J_\sigma(x)} + \frac{J_\sigma(x)}{J_{\sigma+1}(x)}\right) = 2(\sigma+1) + x\left(\frac{J_{\sigma+1}(x)}{J_\sigma(x)} - \frac{J_{\sigma+2}(x)}{J_{\sigma+1}(x)}\right)$$

for every $x \in (0, j_{\sigma,1})$, is strictly increasing.

Proof. First notice that the last equality in the statement is obtained appealing to (A.1). By the *Mittag-Leffler expansion*, [110, Section 15.41, p. 498, equation (1)], we have

$$\frac{J_{\sigma+1}(x)}{J_\sigma(x)} - \frac{J_{\sigma+2}(x)}{J_{\sigma+1}(x)} = 2x\sum_{m=1}^{\infty}\left(\frac{1}{(j_{\sigma,m})^2 - x^2} - \frac{1}{(j_{\sigma+1,m})^2 - x^2}\right) \quad \text{in } (0, j_{\sigma,1}).$$

So it is sufficient to show that the function

$$x \mapsto \sum_{m=1}^{\infty} \frac{(j_{\sigma+1,m})^2 - (j_{\sigma,m})^2}{[(j_{\sigma,m})^2 - x^2][(j_{\sigma+1,m})^2 - x^2]}$$

is positive and strictly increasing in $(0, j_{\sigma,1})$. Let $n \in \mathbb{N}$, define the function

$$\theta_n(x) := \frac{1}{[(j_{\sigma,n})^2 - x^2][(j_{\sigma+1,n})^2 - x^2]} \quad \text{in } (0, j_{\sigma,1}),$$

differentiating it we obtain

$$\theta_n'(x) = \frac{2x}{[(j_{\sigma,n})^2 - x^2]^2[(j_{\sigma+1,n})^2 - x^2]} + \frac{2x}{[(j_{\sigma,n})^2 - x^2][(j_{\sigma+1,n})^2 - x^2]^2} > 0.$$

Eventually, the sequence of positive strictly increasing functions

$$\left\{x \mapsto \sum_{m=1}^{n} [(j_{\sigma+1,m})^2 - (j_{\sigma,m})^2]\theta_m(x)\right\}_{n\in\mathbb{N}}$$

pointwise converge in $(0, j_{\sigma,1})$ to

$$x \mapsto \sum_{m=1}^{\infty} [(j_{\sigma+1,m})^2 - (j_{\sigma,m})^2]\theta_m(x),$$

therefore this function is positive strictly increasing in $(0, j_{\sigma,1})$. □

Remark A.1.4. By (A.4) for every $x \in (0, j_{\sigma,1})$ there holds

$$\Upsilon_\sigma(x) = x\left(\frac{x^{2\sigma}}{\Phi_\sigma(x)} + x^{-2\sigma}\Phi_\sigma(x)\right) = \frac{x\Phi_\sigma'(x)}{\Phi_\sigma(x)} + 1.$$

Proposition A.1.5. *Let* $R > 0$ *and* $\xi \in \mathbb{R}$. *The solutions*

$$(u, \lambda) \in (C^2(0,R) \cap C^1([0,R])) \times \mathbb{R}^+$$

of the eigenvalue problem

$$\begin{cases} u''(r) + \frac{d-1}{r}u'(r) + \lambda u(r) = \xi & \text{for } r \in (0,R), \\ u(R) = 0, \ u'(0) = 0, \end{cases}$$

are obtained by taking u *in the family*

$$\left\{u_s(r) = s\left(r^{1-\frac{d}{2}}J_{\frac{d}{2}-1}(\sqrt{\lambda}r) - R^{1-\frac{d}{2}}J_{\frac{d}{2}-1}(\sqrt{\lambda}R)\right), \ s \in \mathbb{R}\right\},$$

and λ *one of the countably many positive solutions of the equation*

$$\xi = -s\lambda R^{1-\frac{d}{2}}J_{\frac{d}{2}-1}(\sqrt{\lambda}R). \tag{A.5}$$

Proof. Let Y_n be the *Bessel function of the second kind* of order n, see [20, p. 116]. By applying [20, equations (6.80), (6.81) and (6.82)] (with $\alpha = 1 - \frac{d}{2}, \gamma = 1, n = \frac{d}{2} - 1$ and $\beta^2 = \lambda$) we deduce that the general solution of the differential equation

$$u''(r) + \frac{2n+1}{r} u'(r) + \lambda u(r) = \xi \,,$$

is given by

$$u(r) = \begin{cases} r^{-n}(aJ_n(\sqrt{\lambda} r) + bY_n(\sqrt{\lambda} r)) + \xi/\lambda & \text{if } n \in \mathbb{Z}, \\ r^{-n}(aJ_n(\sqrt{\lambda} r) + bJ_{-n}(\sqrt{\lambda} r)) + \xi/\lambda & \text{if } n \notin \mathbb{Z}, \end{cases} \tag{A.6}$$

for every $a, b \in \mathbb{R}$. Since, by [20, below equation (6.74) p. 116] and [20, equations (6.2) and (6.3) p. 87] we have

$$\lim_{r \to 0^+} \left| \left(r^{-n} Y_n(\sqrt{\lambda} r) \right)' \right| = +\infty \quad , \quad \lim_{r \to 0^+} \left(r^{-n} J_n(\sqrt{\lambda} r) \right)' = 0 \quad \text{if } n \in \mathbb{Z},$$

and by [20, equation (6.5) p. 88] we have

$$\lim_{r \to 0^+} \left| \left(r^{-n} J_{-n}(\sqrt{\lambda} r) \right)' \right| = +\infty \quad , \quad \lim_{r \to 0^+} \left(r^{-n} J_n(\sqrt{\lambda} r) \right)' = 0 \quad \text{if } n \notin \mathbb{Z},$$

then from $u'(0) = 0$ we obtain $b = 0$ in (A.6). Thus we can restrict to the class of solutions of the form

$$u(r) = ar^{-n} J_n(\sqrt{\lambda} r) + \xi/\lambda$$

for $a \in \mathbb{R}$. The condition $u(R) = 0$ is equivalent to find the $\sqrt{\lambda}$ that solve

$$\lambda J_n(\sqrt{\lambda} R) = -\frac{\xi}{a} R^n. \tag{A.7}$$

Consider the function $f \colon \mathbb{R}^+ \to \mathbb{R}$ defined by $f(\lambda) := \lambda J_n(\sqrt{\lambda} R)$ and since by truncating *Hankel's expansion*[1], see [110, Section 7.1, p. 195], we have

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\Big(x - \frac{2n+1}{4} \pi \Big) + O\Big(\frac{1}{x^{3/2}} \Big) \quad \text{as } x \to \infty \,,$$

there exist two strictly increasing sequences of positive numbers λ_n^+ and λ_n^- such that

$$\lim_{n \to +\infty} f(\lambda_n^+) = +\infty \quad , \quad \lim_{n \to +\infty} f(\lambda_n^-) = -\infty.$$

Since f is an analytic function of real variable we have $\mathrm{Im}(f) = \mathbb{R}$, moreover $f^{-1}(-\xi R^n / a)$ has countably many elements and (A.7) can be rewritten as $\xi = -a\lambda R^{-n} J_n(\sqrt{\lambda} R)$, that is the last relation. $\square$

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[1] This particular formula was first shown by Jacobi, see [110, Section 7.1, p. 195]

A.2 Parametrizations and rearrangements

Consider the manifold $(\mathbb{R}^d, g_{\mathbb{R}^d})$, where $g_{\mathbb{R}^d}$ denotes the (standard) *flat* Euclidean metric. We have that the $(d-1)$-dimensional sphere of unit radius ∂B_1 can be endowed with the Riemannian metric inherited from $(\mathbb{R}^d, g_{\mathbb{R}^d})$, that we call *round* and denote $g_{\partial B_1}$. So it is possible to define the notions of *Riemannian gradient* ∇_ϕ and *Laplace-Beltrami operator* Δ_ϕ on $(\partial B_1, g_{\partial B_1})$, see [36, Definitions 1 and 3, p. 2-3].

The *polar parametrization* of $\mathbb{R}^d$, with respect to the origin $\mathbf{0}$, is given by the function

$$\begin{aligned} \mathscr{P} : \mathbb{R}^+ \times \partial B_1 &\to \mathbb{R}^d \\ (r, \phi) &\mapsto x = r\phi, \end{aligned} \tag{A.8}$$

we call the parameters in $\mathbb{R}^+ \times \partial B_1$ *polar coordinates* and the first one *radial coordinate*. It is know that $(\mathbb{R}^d \setminus \{\mathbf{0}\}, g_{\mathbb{R}^d})$ is isometric to $(\mathbb{R}^+ \times \partial B_1, g_{\mathbb{R}^+} + r^2 g_{\partial B_1})$, see [97, Section 1.4.4]. Let F be a function with domain of definition $\mathbb{R}^d$, we define $F_{\mathscr{P}} := F \circ \mathscr{P}$. Let $v \in L^1(\mathbb{R}^d)$ by the changes of variables formula we have

$$\int_{\mathbb{R}^d} v(x)\, dx = \int_0^{+\infty} \int_{\partial B_1} v_{\mathscr{P}}(r, \phi)\, r^{d-1}\, d\sigma_\phi\, dr, \tag{A.9}$$

where $d\sigma_\phi$ is $(d-1)$-dimensional Hausdorff measure on ∂B_1, see [55, Theorem 2.49]. From now on we will write $d\sigma$ instead of $d\sigma_\phi$. Let $\mathbf{r}$ be the vector of the orthonormal frame of $\mathbb{R}^+ \times \partial B_1$ corresponding to the radial coordinate. Let us also assume that v is a twice differentiable function. We can express its gradient in polar coordinates via an orthogonal decomposition (that relies on the orthonormal frame) as

$$(\nabla v)_{\mathscr{P}} = (v_{\mathscr{P}})_r\, \mathbf{r} + \frac{1}{r} \nabla_\phi (v_{\mathscr{P}}), \tag{A.10}$$

where the subscript r denotes the differentiation with respect to the radial coordinate, see [53, equation (1.4.6)]. Similarly its Laplacian can be written as

$$(\Delta v)_{\mathscr{P}} = (v_{\mathscr{P}})_{rr} + \frac{d-1}{r} (v_{\mathscr{P}})_r + \frac{1}{r^2} \Delta_\phi (v_{\mathscr{P}}), \tag{A.11}$$

see [53, Lemma 1.4.1].

The *hypershperical parametrization* of ∂B_1, with respect to the point $p := (1, 0, \dots, 0)$, called *north pole*, is given by the function

$$\begin{aligned} \mathscr{S} : (0, \pi) \times \mathbb{S}^{d-2} &\to \partial B_1 \\ (\theta, \xi) &\mapsto \phi = (\cos(\theta), \sin(\theta)\xi), \end{aligned} \tag{A.12}$$

where we define $\mathbb{S}^{d-2} := \partial B_1 \cap \{x_1 = 0\}$, see Figure A.1.

We call the parameters in $(0, \pi) \times \mathbb{S}^{d-2}$ *hypershperical coordinates* and the first one *colatitude coordinate*. Notice that it is possible to obtain an explicit expression for the

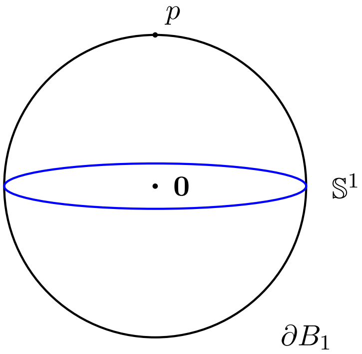

Figure A.1: The circumference of unit radius $\mathbb{S}^1$, in $\mathbb{R}^3$ (in blue).

colatitude coordinate, that from now on we will call colatitude, namely

$$\theta = \arccos(p \cdot \phi).$$

This quantity, geometrically, represents the amplitude (in radiant) of the angle between the two radii of ∂B_1 connecting the origin $\mathbf{0}$ with the points p and ϕ, respectively (see Figure A.2).

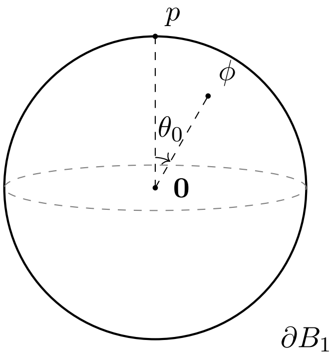

Figure A.2: A point ϕ of colatitude $\theta_0 \approx \pi/6$, in $\partial B_1 \subset \mathbb{R}^3$.

Let $\theta_0 \in (0, \pi)$, we define the open set

$$\Gamma(\theta_0) := \{\phi \in \partial B_1 \,:\, 0 \leq \arccos(p \cdot \phi) < \theta_0\},$$

see Figure A.3. A *spherical cap* of colatitude θ_0 (with center $\psi(p)$, in $\partial B_1 \subset \mathbb{R}^d$) is a set of the type $\psi(\Gamma(\theta_0))$, where $\psi : \partial B_1 \to \partial B_1$ is an isometry of ∂B_1. Recall that the isometry group of ∂B_1 is given by *the orthogonal group* $O(d)$, that is made up of composition of the so-called *rotations* and *reflections*, see [84, Problem 5-8, p. 88].

It is know that $(\partial B_1, g_{\partial B_1})$ is isometric to $((0,\pi) \times \mathbb{S}^{d-2}, g_{(0,\pi)} + (\sin\theta)^2 g_{\mathbb{S}^{d-2}})$, where $g_{\mathbb{S}^{d-2}}$ is the round metric on $\mathbb{S}^{d-2}$, see [97, Example 1.4.6]. It is possible to define the notions of Riemannian gradient ∇_ξ and Laplace-Beltrami operator Δ_ξ on $(\mathbb{S}^{d-2}, g_{\mathbb{S}^{d-2}})$. Let f be a function with domain of definition ∂B_1, we define $f_{\mathscr{S}} = f \circ \mathscr{S}$. Let $u \in L^1(\partial B_1)$

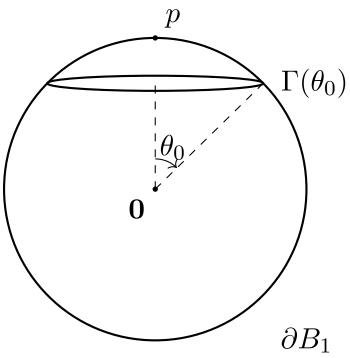

Figure A.3: The set $\Gamma(\theta_0)$ with $\theta_0 \approx \pi/4$ (in $\partial B_1 \subset \mathbb{R}^3$).

by the changes of variables formula we have that it holds

$$\int_{\partial B_1} u(\phi)\, d\sigma = \int_0^\pi \int_{\mathbb{S}^{d-2}} u_{\mathscr{S}}(\theta,\xi) \sin^{d-2}(\theta)\, d\sigma_\xi\, d\theta, \tag{A.13}$$

where $d\sigma_\xi$ is $(d-2)$-dimensional Hausdorff measure on $\mathbb{S}^{d-2}$, see [53, equation (1.5.4)]. Let $\boldsymbol{\theta}$ be the vector of the orthonormal frame of $(0,\pi)\times\mathbb{S}^{d-2}$ corresponding to the colatitude coordinate. Let us also assume that u is a twice differentiable function. We can express its gradient in hypershperical coordinates via an orthogonal decomposition (that relies on the orthonormal frame) as

$$(\nabla u)_{\mathscr{S}} = (u_{\mathscr{S}})_\theta\, \boldsymbol{\theta} + \frac{1}{\sin(\theta)} \nabla_\xi (u_{\mathscr{S}}), \tag{A.14}$$

where the subscript θ denotes the differentiation with respect to the colatitude. Similarly its Laplace-Beltrami operator can be written as

$$(\Delta_\phi u)_{\mathscr{S}} = \frac{1}{(\sin\theta)^{d-2}} \Big((\sin\theta)^{d-2} (u_{\mathscr{S}})_\theta \Big)_\theta + \frac{1}{(\sin\theta)^2} \Delta_\xi (u_{\mathscr{S}}), \tag{A.15}$$

see [53, Lemma 1.4.2].

Combining (A.8) and (A.12) it is possible to find another parametrization of $\mathbb{R}^d$, given by the function

$$\begin{aligned} &\mathbb{R}^+ \times (0,\pi)\times \mathbb{S}^{d-2} \to \mathbb{R}^d \\ &\qquad (r,\theta,\xi) \mapsto \mathscr{P}(r,\mathscr{S}(\theta,\xi)) = (r\cos(\theta), r\sin(\theta)\xi). \end{aligned} \tag{A.16}$$

The *stereographic projection* of ∂B_1, with respect to the *south pole* $-p$, is given by the function

$$\begin{aligned} &\mathscr{X} : \partial B_1 \setminus \{-p\} \to \mathbb{R}^{d-1} \\ &\qquad \phi = (\hat{\phi}, \phi_d) \mapsto \hat{x} = \frac{\hat{\phi}}{1+\phi_d}. \end{aligned}$$

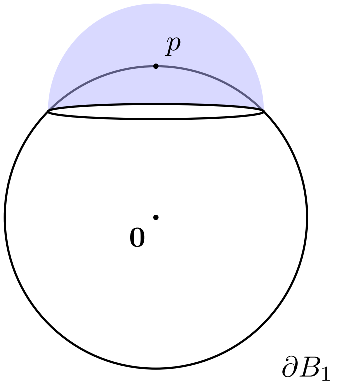

Figure A.4: The graph of a function that is radially symmetric with respect to p (in violet), decreasing as the colatitude increases.

It is know that $(\partial B_1 \setminus \{-p\}, g_{\partial B_1})$ is isometric to $(\mathbb{R}^{d-1}, \Upsilon^2 g_{\mathbb{R}^{d-1}})$, where the analytic function $\Upsilon : \mathbb{R}^{d-1} \to \mathbb{R}$ is defined by

$$\Upsilon(\hat{x}) = \frac{2}{1 + |\hat{x}|^2}$$

for every $\hat{x} \in \mathbb{R}^{d-1}$, see [84, Proof of Lemma 3.4, p. 37]. Let $v : \mathbb{R}^{d-1} \to \mathbb{R}$ be a function and define $v_{\mathscr{X}^{-1}} := v \circ \mathscr{X}^{-1}$. Suppose in addition that v is twice differentiable and denote Δ_Υ the Laplace-Beltrami operator induced on $\mathbb{R}^{d-1}$ by the metric $\Upsilon^2 g_{\mathbb{R}^{d-1}}$, it holds

$$\Delta_\Upsilon v = \frac{1}{\Upsilon^2} \left(\Delta v + (d-2) \nabla v \cdot \nabla \log \Upsilon \right). \tag{A.17}$$

We retrieve this relation by direct calculation exploiting [36, equation (33), p. 5].

In the following it will be useful to produce functions on ∂B_1 that are radially symmetric (with respect to a point of ∂B_1), namely whose superlevel sets are spherical caps with the same center, see [10, Sections 1.1, 1.2 and 7.1] and Figure A.4. We notice that each of these functions can be obtained by as a composition of a map that is radially symmetric with respect to p and an isometry of ∂B_1. We denote $|\cdot|$, with a little abuse with respect to the norm of a vector, the $(d-1)$-dimensional Hausdorff measure on ∂B_1. We recall that, given a non-negative measurable function $u : \partial B_1 \to \mathbb{R}$, its *distribution function*, i.e., the map that describes the measure of superlevel sets, $\mu_u : [0, \infty) \to [0, |\partial B_1|]$ is defined as

$$\mu_u(t) = |\{\phi \in \partial B_1 \,:\, u(\phi) > t\}|$$

for every $0 \leq t < +\infty$. In addition we can define its *decreasing rearrangement* $u^* : [0, |\partial B_1|] \to [0, \infty)$ as

$$u^*(s) = \inf\{t \geq 0 \,:\, \mu_u(t) \leq s\}$$

for every $s \in [0, |\partial B_1|]$. Now it is useful to introduce the map $M : \partial B_1 \to [0, |\partial B_1|]$, defined as

$$M(\phi) = |\Gamma(\arccos(p \cdot \phi))| = |\Gamma(\theta)|$$

for every $\phi \in \partial B_1$. Finally, the *symmetric decreasing rearrangement* $u^\# : \partial B_1 \to (0, +\infty)$ is defined as

$$u^\#(\phi) = (u^* \circ M)(\phi)$$

for every $\phi \in \partial B_1$ (extended by 0 where u^* is not defined). The function $u^\#$ is radially symmetric with respect to p and decreases as the colatitude increases.

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