Abstract

In this thesis, we study some geometric problems depending on the curvatures, with a specific emphasis on their applications to biological membranes and geometric flows.

The modeling of biological membranes has become a dynamic topic in recent years, with numerous applications in biophysics, biology, and materials science. Biological membranes are modeled as regular surfaces in the three-dimensional space, and their equilibrium configurations are associated with the minima of the Canham-Helfrich functional. This functional takes into account the bending and stretching of the membrane and is formulated in terms of the curvatures of the membrane surface.

In the thesis, we investigate the existence of minimizers of this functional on generalized Gauss graphs, a class of currents which include regular surfaces as a special case. As a first step, we extend the Canham-Helfrich energy, usually defined on regular surfaces, to generalized Gauss graphs, and then we prove lower semicontinuity and compactness, under a suitable condition on the bending constants ensuring coerciveness. Finally, we show the existence of a minimizer by applying the direct method of the Calculus of Variations.

Geometric flows are evolution equations that describe the behavior of sets as they evolve over time based on their geometry. The velocity at which each set moves is determined by its geometric attributes, such as curvature. The equations that arise from this study are, in a suitable sense, parabolic differential equations that have broad applications across various fields, including materials science, computer vision, image processing, and physics. In particular, we focus on two examples of geometric flows which depend on the mean curvature: the volume-preserving mean curvature flow and the surface diffusion flow, which describe the evolution of surfaces under the influence of surface tension and mass diffusion, respectively.

We establish the stability of strictly stable critical sets of the perimeter in the flat torus $\mathbb{T}^N$ for the volume-preserving mean curvature flow, both in the continuous time and discrete time cases, and for the surface diffusion flow, only in the continuous time case. Moreover, for the time-discrete volume-preserving mean curvature flow we completely characterize, in dimension two, the asymptotic behavior starting from any initial set. In every cases, we show an exponential rate of convergence by establishing a sharp quantitative estimate of Alexandrov type for periodic strictly stable constant mean curvature hypersurfaces.

We also develop the long-time convergence analysis for the time-discretization of the fractional volume-preserving mean curvature flow, proving that the flow converges exponentially fast to a single ball under suitable hypotheses on the dimension N and on the fractional exponent s, namely: $N \leq 7$ and $s \approx 1$, or $N = 2$ and $s \in (0,1)$.