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ASYMPTOTIC ANALYSIS FOR THE STEFAN PROBLEM WITH A DOUBLY NONLINEAR PHASE RELAXATION

VINCENZO RECUPERO, CHIARA ZANINI

Abstract. In this paper we consider the Stefan problem endowed with a doubly nonlinear phase relaxation. More precisely we assume that the rate of convergence of the phase function depends on the temperature via an increasing continuous function, and it depends on the phase via a maximal antimonotone graph. We prove the existence and uniqueness of the solution of the relaxed problem, and we perform the asymptotic analysis toward the Stefan problem as the relaxation parameter approaches zero.

1. Introduction

The *Stefan problem* is a well-known system of partial differential equations formulated in order to model phase-transition phenomena in a substance attaining two phases (e.g. solid and liquid). It is assumed that the substance is contained in a space domain Ω during the time interval $[0,T]$, and the energy balance equation reads

$$\frac{\partial}{\partial t}(\theta+\chi)-\Delta\theta=g \qquad \text{in } Q:=\Omega\times[0,T], \tag{1.1}$$

where for simplicity we have normalized to 1 all the physical constants. The unknowns $\theta=\theta(x,t)$ and $\chi=\chi(x,t)$, $(x,t)\in Q$, represent respectively the temperature and the phase function, i.e. the proportion of phases: for instance, if the substance is solid at x at the time t, then it turns out to be $\chi(x,t)=-1$, while $\chi(x,t)=1$ accounts for a pure liquid state. Intermediate states are represented by the condition $-1<\chi<1$, so that it is allowed the existence of mushy regions where the substance is a mixture of the solid and liquid parts (cf., e.g., [23, p. 99]). Therefore in order to describe the evolution of the system, the following condition relating θ and χ is introduced:

$$\chi\in \operatorname{sign}(\theta) \qquad \text{in } Q, \tag{1.2}$$

where sign denotes the multivalued sign graph (i.e. $\operatorname{sign}(r):=-1$ if $r<0$, $\operatorname{sign}(r):=[-1,1]$ if $r=0$, $\operatorname{sign}(r):=1$ if $r>0$). Formula (1.2) is called *equilibrium condition of Stefan type*, so that $\theta=0$ is the equilibrium temperature at which the two phases can coexist. Therefore problem (1.1)-(1.2) is usually called *Stefan problem*.

Of course the Stefan condition (1.2) could also be written in the equivalent form

$$\operatorname{sign}^{-1}(\chi)\ni\theta \qquad \text{in } Q, \tag{1.3}$$

sign^{-1} being the inverse relation of the multivalued sign graph ($\operatorname{sign}^{-1}(r):=0$ if $r\in\,]-1,1[$, $\operatorname{sign}^{-1}(-1):=\,]-\infty,0]$, $\operatorname{sign}^{-1}(1):=[0,\infty[$).

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If dynamic supercooling or superheating effects are to be taken into account, then condition (1.3) has to be replaced by a dynamic relation, and usually the following *relaxation dynamics* for the phase variable χ is used (cf., e.g., [22, 23] and their references):

$$\varepsilon\frac{\partial\chi}{\partial t} + \operatorname{sign}^{-1}(\chi) \ni \theta \qquad \text{in } Q, \tag{1.4}$$

ε being a small kinetic positive parameter. It is clear that from a mathematical point of view, the relaxation dynamics can also be modeled by the inclusion

$$\varepsilon\frac{\partial\chi}{\partial t} + \chi \in \operatorname{sign}(\theta) \qquad \text{in } Q, \tag{1.5}$$

but this relation is not equivalent to (1.4).

The Stefan problem (1.1)-(1.2) and the Stefan problems with phase relaxation (1.1), (1.4) or (1.1), (1.5) have been extensively studied in the literature. Concerning (1.1)–(1.2), we refer, e.g., to [6, 7, 23], instead one can see [6, 22, 4, 7, 23] for the relaxed problem (1.1), (1.4). In particular in [22], uniqueness and existence of (1.1), (1.4), coupled with suitable initial-boundary conditions, are proved in the framework of Sobolev spaces, and the solution of the relaxed problem is shown to converge to the solution of the problem (1.1)–(1.2) as $\varepsilon \searrow 0$, in a suitable Sobolev topology. The alternative has been studied in [24, 14], in particular in [14] existence, uniqueness, and asymptotic analysis toward the Stefan problem are proved within the Sobolev setting. Let us also observe that the Stefan problem with phase relaxation can also be studied taking into account a hyperbolic energy balance yielding a finite speed of propagation for the temperature field (see, e.g., [22, 20, 21, 5, 15, 16]).

The natural analytic models (1.1), (1.4) and (1.1), (1.5) have some modelling drawbacks. Indeed, as observed in [24], in (1.4) the rate of the phase χ does not depend on χ, because the term $\operatorname{sign}^{-1}(\chi)$ only represents a constraint for the phase function, and in (1.5) the phase depends only on the sign of the temperature θ. It would be more realistic instead that the rate of χ decays as χ approaches 1, and that it also decays as θ tends to 0.

In order to overcome this modelling issue it would be convenient to introduce some more general dependencies on θ and χ in the phase dynamics. A first proposal in this direction can be found in [24], where it is introduced the phase relaxation

$$\varepsilon\frac{\partial\chi}{\partial t} = \psi(\theta,\chi) \qquad \text{in } Q \tag{1.6}$$

for a suitable class of Lipschitz continuous functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ which are increasing in θ, decreasing in χ, and such that $\psi(\theta,\chi) = 0$ if and only if $\chi \in \operatorname{sign}(\theta)$.

Model (1.6) has some very good features, but it requires a very strong regularity of the dependence on the temperature and the phase, namely a Lipschitz continuity in the pair (θ,χ): neither the monotone case, nor the multivalued one are allowed.

Therefore we propose a model which takes a step further in the direction of (1.6). To be more precise we consider the model of phase relaxation

$$\varepsilon\frac{\partial\chi}{\partial t} + \beta(\chi) \ni \gamma(\theta) \qquad \text{in } \Omega\times\,]0,T[\,, \tag{1.7}$$

where γ and β are increasing continuous and β is possibly multivalued function. Of course (1.7) is not a proper generalization of (1.6), but it takes into account much less regular phase relaxation dynamics, and it aims to solve the above mentioned modelling

drawbacks. Let us note that model (1.7) has been considered in the recent paper [19] in the particular case $\beta = \mathrm{sign}^{-1}$, whose only purpose is to provide a constraint for the phase function.

The aim of our present paper is to prove the existence and uniqueness of solutions of (1.1), (1.7), and to perform the asymptotic analysis as ε approaches zero of the model of phase relaxation (1.1), (1.7), in order to understand if the solutions of (1.1), (1.7) converge to the solution of the Stefan problem

$$\frac{\partial}{\partial t}(\theta + \chi) - \Delta\theta = g \qquad \text{in } Q, \tag{1.8}$$

$$\chi \in \alpha(\theta) \qquad \text{in } Q, \tag{1.9}$$

where α is an increasing multivalued function which generalizes the sign function. Of course α cannot be completely arbitrary, but it requires some compatibility conditions with the phase dynamics (1.7), thus in order to perform our asymptotic analysis we will assume that

$$\gamma(\theta) \in \beta(\chi) \quad \Longleftrightarrow \quad \chi \in \alpha(\theta).$$

The generality of (1.7) makes the use of L^2-techniques not particularly useful in order to get enough a priori estimates for the asymptotic analysis as $\varepsilon \searrow 0$. Therefore we need to exploit L^1-techniques, and our analysis is strongly inspired on some L^1-estimates performed by Visintin in [24] (see also [13, 25]), but we have to adapt those arguments to our nonregular multivalued case. Moreover, in [24] the convergence of the relaxed problem (1.1), (1.6) to a suitable weak formulation of the Stefan problem is proved along a subsequence of ε only, because it is not known if this rather weak form of the Stefan problem has a unique solution. Instead in our analysis, by assuming the natural physical assumption that α is bounded, we are able to prove the uniform boundedness of the phase function, and this fact allows us to get some L^2-estimates which make the solution of (1.1), (1.7) converge to the stronger standard formulation of the Stefan problem, having a unique solution. As a result we are able to prove the convergence along the entire index ε.

The phase relaxation (1.7) with γ multivalued would allow to treat with a single model the two phase relaxations (1.4) and (1.5), or even more general relaxation dynamics, but this analysis is considerably more difficult due the lack of uniqueness of the relaxed solution. We will investigate this problem in the future paper [18].

We supply the systems (1.1), (1.2) and (1.1), (1.7) with the rather general initial-boundary conditions described as follows: letting $\{\Gamma_0, \Gamma_1\}$ be a partition of the boundary of Ω into two measurable sets, we take

$$\theta = \theta_D \qquad \text{on } \Gamma_0 \times [0,T], \tag{1.10}$$

$$\partial_{\mathbf{n}}\theta = -\theta_N \qquad \text{on } \Gamma_1 \times [0,T], \tag{1.11}$$

$$\theta(\cdot,0) + \chi(\cdot,0) = \theta_0 + \chi_0 \qquad \text{in } \Omega, \tag{1.12}$$

where θ_D, θ_N, θ_0, χ_0 are given functions and $\mathbf{n}$ is the outward unit vector normal to the boundary of Ω. We assume that θ_D is a sufficiently smooth function defined on the cylinder Q, that $\theta_N : \Gamma_1 \times [0,T] \longrightarrow \mathbb{R}$ is regular enough, and that there is a function $u : Q \longrightarrow \mathbb{R}$ such that $u = \Delta u$ in Q, $u = \theta_D$ on $\Gamma_0 \times [0,T]$, and $-\partial_{\mathbf{n}} u = \theta_N$ on $\Gamma_1 \times [0,T]$ and we set $\tilde{\theta}_0 := \theta_0 - u(\cdot,0)$. Hence we rewrite all the equations in the new unknown $\tilde{\theta} := \theta - u$ so that problem (1.1), (1.7), (1.10)-(1.12) reads, writing again θ instead of $\tilde{\theta}$

for simplicity,

$$\frac{\partial}{\partial t}(\theta+\chi)-\Delta\theta = g-\frac{\partial u}{\partial t}+\Delta u \qquad \text{in } Q, \tag{1.13}$$

$$\varepsilon\frac{\partial \chi}{\partial t}+\beta(\chi)\ni\gamma(\theta+u) \qquad \text{in } Q, \tag{1.14}$$

$$\theta = 0 \qquad \text{on } \Gamma_0\times[0,T], \tag{1.15}$$

$$\partial_{\mathbf{n}}\theta = 0 \qquad \text{on } \Gamma_1\times[0,T], \tag{1.16}$$

$$\theta(\cdot,0)+\chi(\cdot,0)=\theta_0+\chi_0 \qquad \text{in } \Omega. \tag{1.17}$$

This formulation has the advantage that the boundary conditions for θ are homogeneus and the wider generality is incorporated in the u-terms in the right-hand side of the balance equation and in the non-linearity γ.

The outline of the paper is the following. In Section 2.1 we make precise the assumptions on the data and we state our main results. In Section 3 we analyze the relaxed problem (1.1), (1.7). In Section 4 we perform all the necessary a priori estimates independent of ε, and in the final Section 5 we perform the asymptotic analysis as the relaxation parameter ε goes to zero.

2. Main results

2.1. **Preliminaries and notations.** If $d\in\mathbb{N}$, the set of integers greater than or equal to 1, then the d-dimensional Lebesgue measure of an open set $D\subseteq\mathbb{R}^d$ will be denoted by $|D|$. In the following the locutions "almost every" and "almost everywhere" ("a.e.") will always refer to the Lebesgue measure. Given $p\in[1,\infty[$ and a real Banach space B, then $L^p(D;B)$ will denote the space of p-integrable B-valued functions on D; the vector space of essentially bounded B-valued functions on D is denoted by $L^\infty(D;B)$. These spaces will be endowed with their natural norms defined by $\|v\|_{L^p(D;B)} := \left(\int_D \|v(x)\|_B^p\,\mathrm{d}x\right)^{1/p}$ if $p\in[1,\infty[$, and by $\|v\|_{L^\infty(D;B)} := \inf_w \sup_{x\in D}\|w(x)\|_B$, where the infimum is taken over all bounded Lebesgue-measurable functions w equal to v a.e. in D. If $p=2$ and $B=E$ is a Hilbert space then this norm is induced by the inner product $(v_1,v_2)_{L^2(D;E)} = \int_D (v_1(x),v_2(x))_E\,\mathrm{d}x$, where $(\cdot,\cdot)_E$ is the inner product in E. For the theory of integration of vector valued functions we refer, e.g., to [10, Chapter VI]. We set $L^p(D) := L^p(D;\mathbb{R})$ for $p\in[1,\infty]$.

We will use the Sobolev space $H^1(D) := \{v\in L^2(D)\ :\ \partial_i v\in L^2(D),\ i=1,\dots,d\}$, where $\partial_i v$ denotes the partial derivative of v with respect to the i-th variable in the sense of distributions (cf., e.g., [1]). The symbols ∇, div , and Δ denote respectively the distributional gradient, divergence, and laplacian operators. $H^1(D)$ is a real Hilbert space if it is endowed with the inner product $(v_1,v_2)_{H^1(D)} := (v_1,v_2)_{L^2(D)} + (\nabla v_1,\nabla v_2)_{L^2(D;\mathbb{R}^d)}$, $v_1,v_2\in H^1(D)$, which induces the usual norm $\|\cdot\|_{H^1(D)}$. If ∂D is of Lipschitz class, if Γ_0 is open in ∂D, then the restriction operator $C^\infty(\overline{D})\longrightarrow C(\Gamma_0) : v\longmapsto v|_{\Gamma_0}$ can be uniquely continuously extended to the linear continuous operator $\gamma_{\Gamma_0}: H^1(D)\longrightarrow L^2(\Gamma_0)$ where Γ_0 is endowed with the $(n-1)$-dimensional surface (Hausdorff) measure (see, e.g., [12, 9]). The notation $v|_{\Gamma_0} := \gamma_{\Gamma_0}(v)$ is commonly used for functions $v\in H^1(D)$.

If $a,b\in\mathbb{R}$, $a<b$, we set $L^p(a,b;B) := L^p(]a,b[\,;B)$ for $p\in[1,\infty]$ we recall that $W^{1,p}(a,b;B) := \{f\in L^p(a,b;B)\ :\ f'\in L^p(a,b;E)\}$, where g' denotes the distributional derivative of a function $g :]a,b[\longrightarrow B$ and, if $B=E$ is a Hilbert space, we define

$H^1(a,b;E) := W^{1,2}(a,b;E)$. For the main properties of the Sobolev space $W^{1,p}([a,b]\,;B)$ and the space of functions with bounded variation $BV(a,b;B)$ we refer, e.g., to [3, Appendix]. It is also convenient to recall the precise definition of maximal monotonicity (cf. [3]).

Definition 2.1. *If $\mathscr{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, then a "multivalued function" $\zeta : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be* maximal monotone *if, setting $D(\zeta) := \{r \in \mathbb{R} \ : \ \zeta(r) \neq \varnothing\}$, the following two conditions hold:*

$$(\zeta_{r_1} - \zeta_{r_2})(r_1 - r_2) \geq 0 \qquad \forall r_1, r_2 \in D(\zeta),\ \forall \zeta_{r_1} \in \zeta(r_1),\ \forall \zeta_{r_2} \in \zeta(r_2); \tag{2.1}$$

$$(\sigma - \zeta_r)(\rho - r) \geq 0, \quad r \in D(\zeta), \quad \zeta_r \in \zeta(r) \quad \Longrightarrow \quad \sigma \in \zeta(\rho). \tag{2.2}$$

For every $r \in D(\zeta)$ we denote by $\zeta^o(r)$ the number in $\zeta(r)$ with minimal absolute value, i.e. the unique number such that

$$|\zeta^o(r)| = \min\{|s| \ : \ s \in \zeta(r)\}. \tag{2.3}$$

The mapping $\zeta^{-1} : \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ is defined by the relation

$$r \in \zeta^{-1}(s) \Longleftrightarrow s \in \zeta(r)$$

and it is maximal monotone if ζ is maximal monotone.

For a maximal monotone $\zeta : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ we will adopt a slight abuse of notation by writing $\zeta(v)$ instead of $\zeta \circ v$ for any $v : \Omega \times [0,T] \longrightarrow \mathbb{R}$. Thus, for instance, the formula $w \in \zeta(v)$ a.e. in Q means $w(x,t) \in \zeta(v(x,t))$ for a.e. $(x,t) \in \Omega \times [0,T]$.

2.2. **Main results.** Now we can present our set of assumptions.

Assumptions 2.1. The following conditions will be used in the paper.

(A1) $\Omega \subseteq \mathbb{R}^n$ is a bounded open connected set with Lipschitz boundary $\Gamma := \partial\Omega$. Γ_0 and Γ_1 are open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \varnothing$. If $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ denote the closures of Γ_0 and Γ_1 in Γ, then we assume that $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \Gamma$ and that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is of Lipschitz class. We define

$$H := L^2(\Omega), \tag{2.4}$$

$$V := H^1_{\Gamma_0}(\Omega) := \{v \in H^1(\Omega) \ : \ v|_{\Gamma_0} = 0\}, \tag{2.5}$$

where $v \longmapsto v|_{\Gamma_0}$ denotes the trace operator on Γ_0 defined on $H^1(\Omega)$. The space V is endowed with the inner product induced by $H^1(\Omega)$. The duality between V and its topological dual space V' is denoted by ${}_{V'}\langle\cdot,\cdot\rangle_V$, and if we identify H with its dual space, then we have $V \subset H \subset V'$ with dense and compact embeddings, and ${}_{V'}\langle e, v\rangle_V = (e,v)_H$ for every $e \in H$, $v \in V$.

We define the linear continuous operator $A : V \longrightarrow V'$ by

$${}_{V'}\langle Av_1, v_2\rangle_V := \int_\Omega \nabla v_1 \cdot \nabla v_2, \qquad v_1, v_2 \in V. \tag{2.6}$$

The final time of the evolution will be denoted by $T > 0$ and we set $Q := \Omega \times]0,T[$.

(A2) $\beta : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a maximal monotone mapping such that $D(\beta) \neq \varnothing$.

(A3) $\gamma : \mathbb{R} \to \mathbb{R}$ is continuous, increasing (in the sense that $(\gamma(r_1) - \gamma(r_2))(r_1 - r_2) \geq 0$ for every $r_1, r_2 \in \mathbb{R}$), and sublinear, i.e., there exists $C_\gamma > 0$ such that

$$|\gamma(r)| \leq C_\gamma(1 + |r|) \qquad \forall r \in \mathbb{R}. \tag{2.7}$$

(A4) $\alpha : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is maximal monotone, and it is "bounded", i.e. there is a constant $M > 0$ such that

$$|s| \leq M \qquad \forall r \in D(\alpha), \quad \forall s \in \alpha(r). \tag{2.8}$$

(A5) We assume the following "compatibility" conditions between α, β and γ:

$$\alpha(r) \subseteq D(\beta) \quad \forall r \in D(\alpha), \tag{2.9}$$

$$\gamma(r) \in \beta(s) \iff s \in \alpha(r). \tag{2.10}$$

We will prove the following existence and uniqueness result for a doubly nonlinear relaxed Stefan problem.

Theorem 2.1. *Let* (A1)-(A3) *of Assumptions 2.1 be satisfied. Assume that* $\varepsilon > 0$, $f_\varepsilon \in L^1(0,T;H) + L^2(0,T;V')$, $u_\varepsilon \in L^2(Q)$, $\theta_{0\varepsilon} \in L^2(\Omega)$, $\chi_{0\varepsilon} \in L^2(\Omega)$, *and* $\chi_{0\varepsilon}(x) \in \overline{D(\beta)}$ *for a.e.* $x \in \Omega$. *Then there exists a unique pair* $(\theta_\varepsilon, \chi_\varepsilon)$ *such that*

$$\theta_\varepsilon \in L^2(0,T;V) \cap C([0,T];H), \tag{2.11}$$

$$\theta_\varepsilon' \in L^1(0,T;H) + L^2(0,T;V'), \tag{2.12}$$

$$\chi_\varepsilon \in H^1(0,T;H), \tag{2.13}$$

$$\theta_\varepsilon' + \chi_\varepsilon' + A\theta_\varepsilon = f_\varepsilon \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{2.14}$$

$$\varepsilon\chi_\varepsilon' + \beta\big(\chi_\varepsilon\big) \ni \gamma\big(\theta_\varepsilon + u_\varepsilon\big) \qquad \text{a.e. in } Q, \tag{2.15}$$

$$\theta_\varepsilon(0) = \theta_{0\varepsilon}, \quad \chi_\varepsilon(0) = \chi_{0\varepsilon} \qquad \text{a.e. in } \Omega. \tag{2.16}$$

We are interested to perform an asymptotic analysis of the solution of Problem (2.14)-(2.16) as $\varepsilon \searrow 0$. We will prove that this solution converges to the solution of the Stefan problem, whose formulation we recall in the following result (see, e.g., [4, 7, 23]).

Theorem 2.2. *Let* (A1) *and* (A4) *of Assumptions 2.1 be satisfied. Assume that* $f \in L^2(0,T;V') + L^1(0,T;H)$, $\theta_0 \in H$, $\chi_0 \in H$, $u \in L^2(Q)$, $\chi_0 \in \alpha(\theta_0 + u(0))$ *a.e. in* Ω, *and*

$$\exists s_0 \in D(\alpha^{-1}) \quad : \quad \int_{s_0}^{\chi_0} (\alpha^{-1})^o(s)\,\mathrm{d}s \in L^1(\Omega), \tag{2.17}$$

where $(\alpha^{-1})^o(s)$ *is defined according to* (2.3). *Then there exists a unique pair* (θ, χ) *such that*

$$\theta \in L^2(0,T;V) \cap H^1(0,T;V'), \tag{2.18}$$

$$\chi \in L^\infty(Q), \tag{2.19}$$

$$\theta + \chi \in H^1(0,T;V') \tag{2.20}$$

$$(\theta+\chi)'(t) + A\theta(t) = f(t) \qquad \text{in } V', \text{ for a.e. } t \in]0,T[, \tag{2.21}$$

$$\chi \in \alpha\big(\theta + u\big) \qquad \text{a.e. in } Q, \tag{2.22}$$

$$(\theta+\chi)(0) = \theta_0 + \chi_0 \qquad \text{in } V'. \tag{2.23}$$

Here is our main convergence theorem, where the symbols $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ denote respectively the weak convergence and the weak-star convergence.

Theorem 2.3. *Let Assumptions 2.1 be satisfied. Assume that* $f \in L^2(0,T;H)$, $u \in W^{1,1}(0,T;L^1(\Omega))$, $\Delta u \in L^1(Q)$, $\theta_0 \in H$, $\chi_0 \in L^\infty(\Omega)$, $\chi_0 \in \alpha(\theta_0 + u(0))$ *a.e. in* Ω,

and (2.17) *holds. For every* $\varepsilon > 0$ *assume that* $f_\varepsilon \in L^2(0,T;H) \cap BV([0,T];L^1(\Omega))$, $u_\varepsilon \in C([0,T];H^1(\Omega))$, $u'_\varepsilon, \Delta u_\varepsilon \in BV([0,T];L^1(\Omega))$, $u_\varepsilon(t)|_\Gamma = u_\varepsilon(0)|_\Gamma$ *for a.e.* $t \in [0,T]$, $\theta_{0\varepsilon} \in V$, $\Delta\theta_{0\varepsilon} \in L^1(\Omega)$, $\chi_{0\varepsilon} \in L^\infty(\Omega)$, *and* $\chi_{0\varepsilon}(x) \in \overline{D(\beta)}$ *for a.e.* $x \in \Omega$. *Let* $(\theta_\varepsilon, \chi_\varepsilon)$ *be the solution of* (2.14)-(2.16), *and* (θ, χ) *be the solution* (2.18)-(2.23). *If*

$$\sup_{\varepsilon>0}(\|f_\varepsilon\|_{BV([0,T];L^1(\Omega))} + \|u'_\varepsilon\|_{BV([0,T];L^1(\Omega))} + \|\Delta u_\varepsilon\|_{BV([0,T];L^1(\Omega))}) < \infty, \tag{2.24}$$

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^2(Q)} < \infty, \tag{2.25}$$

$$\gamma(\theta_{0\varepsilon}(x) + u_\varepsilon(x,0)) - \beta(\chi_{0\varepsilon}(x)) \subseteq [-\varepsilon,\varepsilon] \qquad \text{for a.e. } x \in \Omega, \tag{2.26}$$

and if

$$f_\varepsilon \to f \qquad \text{in } L^2(0,T;H), \tag{2.27}$$

$$u_\varepsilon \to u \qquad \text{in } W^{1,1}(0,T;L^1(\Omega)), \tag{2.28}$$

$$\Delta u_\varepsilon \to \Delta u \qquad \text{in } L^1(0,T;L^1(\Omega)), \tag{2.29}$$

$$\theta_{0\varepsilon} \to \theta_0, \qquad \text{in } L^2(\Omega), \tag{2.30}$$

$$\chi_{0\varepsilon} \to \chi_0 \qquad \text{in } L^\infty(\Omega), \tag{2.31}$$

then (along the whole set of indexes ε*)*

$$\theta_\varepsilon \to \theta \quad \text{in } L^1(Q), \tag{2.32}$$

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{in } L^2(Q), \tag{2.33}$$

$$\chi_\varepsilon \to \chi \quad \text{in } L^1(Q), \tag{2.34}$$

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(Q). \tag{2.35}$$

3. Analysis of the relaxed Stefan problem

We start by stating a regularity result for the heat equation in its weak formulation.

Lemma 3.1. *Let* (A1) *of Assumptions 2.1 be satisfied. If* $F \in L^2(0,T;H)$ *and* $w_0 \in V$, *then there exists a unique function* w *such that*

$$w \in L^\infty(0,T;V) \cap H^1(0,T;H),\ Aw \in L^2(0,T;H), \tag{3.1}$$

$$Aw = -\Delta w, \tag{3.2}$$

$$w' - \Delta w = F \qquad \text{a.e. in } Q, \tag{3.3}$$

$$w(0) = w_0 \qquad \text{a.e. in } \Omega. \tag{3.4}$$

Proof. Conditions (3.1), (3.4), and the fact that $w' + Aw = F$ in V', a.e. in $]0,T[$, follow from a standard regularity result for parabolic equations. Let us now recall the following generalized Green formula holding for every $v \in H^1(\Omega)$, $\mathbf{v} \in L^2_{\mathrm{div}}(\Omega)^d = \{\mathbf{v} \in L^2(\Omega)^d : \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$ (cf. [8, p. 239-240]):

$$\int_\Omega \mathbf{v}\cdot\nabla v = -\int_\Omega v\,\operatorname{div}\mathbf{v} + {}_{H^{-1/2}(\Gamma)}\langle \gamma_{\mathbf{n}}\mathbf{v}, \gamma_0 v\rangle_{H^{1/2}(\Gamma)},$$

where $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma)$ and $\gamma_{\mathbf{n}} : L^2_{\mathrm{div}}(\Omega) \longrightarrow H^{-1/2}(\Gamma)$ are the unique linear continuous surjective (trace) operators such that $\gamma_0(v) = v|_\Gamma$ for every $v \in C^\infty(\overline{\Omega}) \cap H^1(\Omega)$

and $\gamma_{\mathbf{n}}(\mathbf{w}) = \mathbf{w}|_\Gamma \cdot \mathbf{n}$ for every $w \in C^\infty(\overline{\Omega})^d \cap L^2_{\mathrm{div}}(\Omega)$ (see [1] for the definition of fractional Sobolev spaces). For all $\varphi \in C_c^\infty(\Omega)$ we have therefore

$$\begin{aligned}
{}_{\mathscr{D}'(\Omega)}\langle \Delta w, \varphi\rangle_{\mathscr{D}(\Omega)} &= {}_{\mathscr{D}'(\Omega)}\langle w, \Delta\varphi\rangle_{\mathscr{D}(\Omega)} = \int_\Omega w\Delta\varphi \\
&= -\int_\Omega \nabla w\cdot\nabla\varphi + {}_{H^{-1/2}(\Gamma)}\langle \gamma_{\mathbf{n}}\nabla\varphi, \gamma_0 w\rangle_{H^{1/2}(\Gamma)} \\
&= -\int_\Omega \nabla w\cdot\nabla\varphi = -{}_{V'}\langle Aw,\varphi\rangle_V = -(Aw,\varphi)_H = -\int_\Omega (Aw)\varphi,
\end{aligned}$$

a.e. in $]0,T[$, whence $-Aw$ is equal to the distributional laplacian Δw, and it follows that (3.2), (3.3) hold. □

Now we can prove that problem (2.14)–(2.16) has at most one solution. We will use the auxiliary functions $\mathrm{sign}_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ for $\mu > 0$ and $\mathrm{sign}_0 : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathrm{sign}_\mu(r) = \min\{\max\{r/\mu, -1\}, 1\} \qquad \text{for } \mu > 0, \tag{3.5}$$

$$\mathrm{sign}_0(r) = \lim_{\mu\to 0} \mathrm{sign}_\mu(r). \tag{3.6}$$

Proposition 3.1. *Under the assumptions of Theorem 2.1, there exists at most one pair* $(\theta_\varepsilon, \chi_\varepsilon)$ *such that* (2.11)-(2.16) *hold.*

Proof. Let (θ_i, χ_i) such that (2.11)–(2.16) hold with $(\theta_\varepsilon, \chi_\varepsilon)$ replaced by (θ_i, χ_i), $i = 1, 2$. Let us set $\theta := \theta_1 - \theta_2$, $\chi := \chi_1 - \chi_2$ and observe that by virtue of Lemma 3.1 we have $\theta \in L^\infty(0,T;V) \cap H^1(0,T;H)$, $A\theta = -\Delta\theta \in L^2(Q)$, and

$$\theta'(t) - \Delta\theta(t) = -\chi'(t) \qquad \text{a.e. in } \Omega, \tag{3.7}$$

$$\theta(0) = 0 \qquad \text{a.e. in } \Omega, \tag{3.8}$$

for a.e. $t \in]0,T[$. We also have that there exists $\xi_i \in L^2(Q)$ such that

$$\varepsilon\chi_i'(x,t) + \xi_i(x,t) = \gamma\big(\theta_i(x,t) + u_\varepsilon(x,t)\big) \qquad \text{for a.e. } x \in \Omega, \tag{3.9}$$

$$\xi_i(x,t) \in \beta\big(\chi_i(x,t)\big) \qquad \text{for a.e. } x \in \Omega, \tag{3.10}$$

$$\chi_i(0) = \chi_{0\varepsilon} \qquad \text{a.e. in } \Omega, \tag{3.11}$$

for a.e. $t \in]0,T[$, $i = 1, 2$. Now we fix $t \in]0,T[$ such that (3.7), (3.9) hold and such that $(\partial|\theta|/\partial t)|(x,t)$ and$(\partial|\chi|/\partial t)(x,t)$ exist for a.e. $x \in \Omega$: the set of such t's has full measure in $]0,T[$ by virtue of [26, Theorem 2.1.11, p. 48]. If $\mathrm{sign}_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by (3.5) for $\mu > 0$, then $\mathrm{sign}_\mu(\theta(t)) \in V$ and

$$-\int_\Omega \Delta\theta(t)\mathrm{sign}_\mu(\theta(t)) = \int_\Omega |\nabla\theta(t)|^2 \mathrm{sign}_\mu'(\theta(t)) \geq 0 \qquad \text{for a.e. } t \in]0,T[,$$

therefore if we multiply (3.7) by $\mathrm{sign}_\mu(\theta(t))$ and integrate over Ω, we obtain

$$\int_\Omega \theta'(t)\mathrm{sign}_\mu(\theta(t)) + \int_\Omega \chi'(t)\mathrm{sign}_\mu(\theta(t)) \leq 0. \tag{3.12}$$

Taking the limit as $\mu \searrow 0$ in (3.12) we infer that

$$\int_\Omega \theta'(t)\mathrm{sign}_0(\theta(t)) + \int_\Omega \chi'(t)\mathrm{sign}_0(\theta(t)) \leq 0, \tag{3.13}$$

where sign_0 is defined by (3.6).

Now fix $x \in \Omega$ such that $(\partial\chi/\partial t)(x,t)$ and $(\partial|\chi|/\partial t)(x,t)$ exist: if $\chi(x,t) \neq 0$, then using (3.9) and the monotonicity of β and γ, we infer that

$$\begin{aligned}
&\chi'(x,t)\mathrm{sign}_0(\theta(x,t)) \\
&= \frac{1}{\varepsilon}\Big[\big(\gamma(\theta_1(x,t)) - \gamma(\theta_2(x,t)\big) - \big(\xi_1(x,t) - \xi_2(x,t)\big)\Big]\mathrm{sign}_0(\theta_1(x,t) - \theta_2(x,t)) \\
&\geq \frac{1}{\varepsilon}\Big[\big(\gamma(\theta_1(x,t)) - \gamma(\theta_2(x,t)\big) - \big(\xi_1(x,t) - \xi_2(x,t)\big)\Big]\mathrm{sign}_0(\chi_1(x,t) - \chi_2(x,t)) \\
&= \chi'(x,t)\mathrm{sign}_0(\chi(x,t)) = \frac{\partial|\chi|}{\partial t}(x,t);
\end{aligned} \tag{3.14}$$

if instead $\chi(x,t) = 0$, then it is easy to check that $(\partial|\chi|/\partial t)(x,t) = (\partial\chi/\partial t)(x,t) = 0$ and we have

$$\chi'(x,t)\mathrm{sign}_0(\theta(x,t)) = 0 = \frac{\partial|\chi|}{\partial t}(x,t); \tag{3.15}$$

thus in every case we have

$$\chi'(x,t)\mathrm{sign}_0(\theta(x,t)) \geq \frac{\partial|\chi|}{\partial t}(x,t). \tag{3.16}$$

From (3.13) and (3.16) we therefore get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\theta(t)| + |\chi(t)|) \leq 0 \qquad \text{for a.e. } t \in]0,T[, \tag{3.17}$$

and integrating in time, thanks to (3.8), (3.11), we obtain

$$\int_\Omega (|\theta(t)| + |\chi(t)|) \leq 0 \qquad \forall t \in [0,T]\,. \tag{3.18}$$

Therefore $\|\theta\|_{L^\infty(0,T;L^1(\Omega))} + \|\chi\|_{L^\infty(0,T;L^1(\Omega))} \leq 0$, so that $\theta_1 = \theta_2$, $\chi_1 = \chi_2$ a.e. in Q and we are done. □

We now turn to the question about existence of solutions. We first approximate problem (2.14)-(2.16).

Lemma 3.2. *Let* (A1)-(A3) *of Assumptions 2.1 be satisfied. Fix $\varepsilon > 0$, $\lambda > 0$, and let $\gamma_\lambda := (\lambda + \gamma^{-1})^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ be the λ-Yosida approximation of γ (cf.* [3, Section II.4, p. 28]*). Assume that $f_\varepsilon \in L^1(0,T;H) + L^2(0,T;V')$, $u_\varepsilon \in L^2(Q)$, $\theta_{0\varepsilon} \in L^2(\Omega)$, $\chi_{0\varepsilon} \in L^2(\Omega)$, and $\chi_{0\varepsilon}(x) \in \overline{D(\beta)}$ for a.e. $x \in \Omega$. Then there exist a unique pair $(\theta_\lambda, \chi_\lambda) \in [L^2(0,T;V) \cap C([0,T]\,;H)] \times H^1(0,T;H)$ such that $\theta'_\lambda \in L^1(0,T;H) + L^2(0,T;V')$ and*

$$\theta'_\lambda + \chi'_\lambda + A\theta_\lambda = f_\varepsilon \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{3.19}$$

$$\varepsilon\chi'_\lambda + \beta\big(\chi_\lambda\big) \ni \gamma_\lambda\big(\theta_\lambda + u_\varepsilon\big) \qquad \text{a.e. in } Q, \tag{3.20}$$

$$\theta_\lambda(0) = \theta_{0\varepsilon}, \quad \chi_\lambda(0) = \chi_{0\varepsilon} \qquad \text{a.e. in } \Omega, \tag{3.21}$$

Proof. Fix $\Theta \in L^2(0,T;H)$ and recall that γ_λ is $1/\lambda$-Lipschitz continuous over $\mathbb{R}$. Therefore by [3, Theorem 3.4, Theorem 3.6, Example 2.3.3] (applied with the maximal monotone operator $\mathcal{B}$ defined $\mathcal{B}(v) := \{w \in L^2(\Omega) \ : \ w(x) \in \beta(v(x)) \quad \text{for a.e. } x \in \Omega\}$) there exists a unique $\chi \in H^1(0,T;H)$ such that

$$\varepsilon\chi' + \beta(\chi) \ni \gamma_\lambda(\Theta + u_\varepsilon) \qquad \text{a.e. in } Q, \tag{3.22}$$

$$\chi(0) = \chi_{0\varepsilon} \qquad \text{a.e. in } \Omega, \tag{3.23}$$

thus (cf., e.g., [2, Theorem 3.2]) there exists a unique $\theta \in L^2(0,T;V) \cap C([0,T]\,;H)$ such that $\theta' \in L^1(0,T;H) + L^2(0,T;V')$ and

$$\theta' + A\theta = f_\varepsilon - \chi' \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{3.24}$$

$$\theta(0) = \theta_{0\varepsilon} \qquad \text{a.e. in } \Omega. \tag{3.25}$$

In this way we have defined a mapping $\mathcal{S}_\lambda : L^2(0,T;H) \longrightarrow L^2(0,T;H)$ associating Θ with the unique θ satisfying (3.22), (3.23), (3.24), (3.25). For every $\Theta_k \in L^2(0,T;H)$, $k = 1,2$, set $\theta_k := \mathcal{S}_\lambda(\Theta_k)$, $\widetilde{\Theta} := \Theta_1 - \Theta_2$, and $\widetilde{\theta} := \theta_1 - \theta_2$. Moreover set $\widetilde{\chi} := \chi_1 - \chi_2$ where χ_k is the unique function satisfying (3.22), (3.23) with Θ replaced by Θ_k, $k = 1,2$. Let us fix $t \in]0,T]$. Let us take the difference of the approximated phase relaxations (3.22) with Θ and χ replaced respectively by Θ_k and χ_k, , $k = 1,2$, multiply it by $\widetilde{\chi}$, and integrate over $\Omega \times]0,t[$. Thanks to (3.23), to the maximal monotonicity of β, to the Lipschitz continuity of γ_λ, and to the Young inequality, we obtain

$$\frac{\varepsilon}{2}\|\widetilde{\chi}(t)\|_H^2 \leq \frac{1}{2\lambda^2}\|\widetilde{\Theta}\|^2_{L^2(0,t;H)} + \frac{1}{2}\int_0^t \|\widetilde{\chi}(s)\|_H^2 \,\mathrm{d}s, \tag{3.26}$$

therefore by the Gronwall lemma we infer that there exists a constant $C > 0$ independent of θ (and depending only on T, ε and λ), such that

$$\|\widetilde{\chi}(t)\|_H^2 \leq C\|\widetilde{\Theta}\|^2_{L^2(0,t;H)}. \tag{3.27}$$

Now let us take the difference of the energy balance equations (3.24) with θ and χ replaced respectively by θ_k and χ_k, $k = 1,2$. Integrate in time the resulting equation, and multiply it by $\widetilde{\theta}$. Integrating over $]0,t[\times \Omega$ and using the Young inequality, thanks to (3.25) we infer that

$$\frac{1}{2}\|\widetilde{\theta}\|^2_{L^2(0,t;H)} \leq \frac{1}{2}\int_0^t \|\widetilde{\chi}(s)\|_H^2 \,\mathrm{d}s, \tag{3.28}$$

therefore using (3.27) we infer that

$$\|\widetilde{\theta}\|^2_{L^2(0,t;H)} \leq C\int_0^t \|\widetilde{\Theta}\|^2_{L^2(0,s;H)} \,\mathrm{d}s, \tag{3.29}$$

hence the mapping $\mathcal{S}_\lambda^n$ is a strict contraction for n sufficiently large, and we can infer that $\mathcal{S}_\lambda$ admits a unique fixed point θ_λ. We conclude by observing that $(\theta_\lambda, \chi_\lambda)$ is a solution of (3.19)–(3.21) if and only if θ_λ is a fixed point of $\mathcal{S}_\lambda$. □

Proof of Theorem 2.1. By Proposition 3.1 we only have to prove existence of solutions. For simplicity we omit the subscripts ε. For any $\lambda > 0$ let $\gamma_\lambda := (\lambda + \gamma^{-1})^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ be the λ-Yosida approximation of γ (cf. [3, Section II.4, p.28]). By Lemma 3.2 there exists a unique pair $(\theta_\lambda, \chi_\lambda) \in [L^2(0,T;V) \cap C([0,T]\,;H)] \times H^1(0,T;H)$ such that $\theta'_\lambda \in L^1(0,T;H) + L^2(0,T;V')$ and (3.19)–(3.21) hold. Let us test the energy balance equation (3.19) by θ_λ and integrate over $]0,t[\times \Omega$. Recalling that $f = f_{V'} + f_H$ with $f_{V'} \in$

$L^2(0,T;V')$ and $f_H \in L^1(0,T;H)$, and exploiting the Young inequality, we find

$$\begin{aligned}
&\frac{1}{2}\|\theta_\lambda(t)\|_H^2 + \frac{1}{2}\|\nabla\theta_\lambda\|^2_{L^2(0,t;H^d)} \\
&\le \frac{1}{2}\|\theta_{0\varepsilon}\|_H^2 + \frac{1}{2}\|f_{V'}\|^2_{L^2(0,T;V')} \\
&\quad + \int_0^t \|f_H(s)\|_H\|\theta_\lambda(s)\|_H \,\mathrm{d}s + \frac{\varepsilon+2}{2\varepsilon}\int_0^t \|\theta_\lambda(s)\|_H^2 \,\mathrm{d}s + \frac{\varepsilon}{4}\|\chi'_\lambda\|^2_{L^2(0,t;H)}.
\end{aligned} \tag{3.30}$$

Let $\xi_\lambda \in L^2(0,T;H)$ be such that

$$\xi_\lambda \in \beta(\chi_\lambda) \qquad \text{a.e. in } Q, \tag{3.31}$$

$$\varepsilon\chi'_\lambda + \xi_\lambda = \gamma_\lambda(\theta_\lambda + u) \quad \text{a.e. in } Q. \tag{3.32}$$

Let us recall that $|\gamma_\lambda(r)| \le |\gamma(r)|$ for every $r \in \mathbb{R}$ (cf. e.g., [3, Proposition 2.6]) so that from (2.7) we get that

$$|\gamma_\lambda(r)| \le C_\gamma(1+|r|) \qquad \forall r \in \mathbb{R}. \tag{3.33}$$

Therefore if we multiply the approximated phase relaxation (3.32) by χ'_λ and integrate over $]0,t[\,\times\Omega$, taking some $r_0 \in D(\beta)$ and recalling definition (2.3), we get

$$\begin{aligned}
&\varepsilon\|\chi'_\lambda\|^2_{L^2(0,t;H)} + \int_0^t\int_\Omega (\xi_\lambda(s)) - \beta^o(r_0))\chi'_\lambda(s)\,\mathrm{d}s \\
&= -\int_0^t\int_\Omega \beta^o(r_0)\chi'_\lambda(s)\,\mathrm{d}s + \int_0^t\int_\Omega \gamma_\lambda(\theta_\lambda(s)+u(s))\chi'_\lambda(s)\,\mathrm{d}s \\
&\le \int_0^t\int_\Omega |\beta^o(r_0)||\chi'_\lambda(s)|\,\mathrm{d}s + C_\gamma\int_0^t\int_\Omega (1+|\theta_\lambda(s)+u(s)|)|\chi'_\lambda(s)|\,\mathrm{d}s \\
&\le \frac{\varepsilon}{4}\|\chi'_\lambda\|^2_{L^2(0,t;H)} + \frac{2}{\varepsilon}t|\Omega||\beta^0(r_0)|^2 + \frac{4C_\gamma^2}{\varepsilon}\int_0^t\int_\Omega (1+2|\theta_\lambda(s)|^2+2|u(s)|^2)\,\mathrm{d}s.
\end{aligned} \tag{3.34}$$

Let $\phi_\beta : \mathbb{R} \longrightarrow]0,\infty]$ be the convex lower semicontinuous function such that $\partial\phi_\beta = \beta$ and $\phi_{\beta_\mu}(r_0) = 0$ (cf., e.g., [3] for the definition of the subdifferential $\partial\phi_\beta$), so that

$$\phi_\beta(\chi_\lambda(t)) - \beta^o(r_0)(\xi_\lambda(t) - r_0) \ge 0 \quad \text{for a.e. } t \in [0,T],$$

then by virtue of [3, Lemma 3.3, p. 73]

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi_\beta(\chi_\lambda(t)) - \beta^o(r_0)(\chi_\lambda(t) - r_0)] = [\xi_\lambda(t) - \beta^o(r_0)]\chi'_\lambda(t),$$

thus

$$\int_\Omega \phi_\beta(\chi_\lambda(t)) - \beta^o(r_0)(\xi_\lambda(t) - r_0) = \int_0^t\int_\Omega (\xi_\lambda(s)) - \beta^o(r_0))\chi'_\lambda(s)\,\mathrm{d}s \ge 0$$

for a.e. $t \in [0,T]$, and from (3.34) we infer that

$$\frac{3\varepsilon}{4}\|\chi'_\lambda\|^2_{L^2(0,t;H)} \le \frac{2t|\Omega|(|\beta^o(r_0)|^2 + 2C_\gamma^2)}{\varepsilon} + \frac{8C_\gamma^2}{\varepsilon}\|u\|^2_{L^2(0,t;H)} + \frac{8C_\gamma^2}{\varepsilon}\int_0^t \|\theta_\lambda(s)\|_H^2\,\mathrm{d}s. \tag{3.35}$$

Thus adding (3.30) to (3.35) we deduce that

$$\begin{aligned}&\frac{\varepsilon}{2}\|\chi'_\lambda\|^2_{L^2(0,t;H)}+\frac{1}{2}\|\theta_\lambda(t)\|^2_H+\frac{1}{2}\|\nabla\theta_\lambda\|^2_{L^2(0,t;H^d)}\\&\leq C_1(\varepsilon)\left(1+\int_0^t\|f_H(s)\|_H\|\theta_\lambda(s)\|_H\,\mathrm{d}s+\int_0^t\|\theta_\lambda(s)\|^2_H\,\mathrm{d}s\right),\end{aligned}\tag{3.36}$$

where $C_1(\varepsilon)>0$ is a constant independent of λ, but depending only on ε, T, $|\Omega|$, C_γ, β, r_0, u. Using a generalized version of the Gronwall lemma (see, e.g., [2, Theorem 2.1]), by (3.36) and by a comparison in (3.19), we find a constant $C(\varepsilon)>0$ independent of λ, but depending only on ε, T, $|\Omega|$, C_γ, β, u, such that

$$\|\theta_\lambda\|_{L^2(0,T;V)}+\|\theta'_\lambda\|_{L^2(0,T;V')}+\|A\theta_\lambda\|_{L^2(0,T;V')}+\|\chi_\lambda\|_{H^1(0,T;H)}\leq C(\varepsilon).\tag{3.37}$$

Therefore there exists $\chi\in H^1(0,T;H)$ and $\theta\in L^1(0,T;V)\cap L^2(0,T;H)$ such that $\theta'\in L^2(0,T;V')$ and, at least for a subsequence, the following weak convergences hold:

$$\chi_\lambda\rightharpoonup\chi\qquad\text{in } H^1(0,T;H),\tag{3.38}$$

$$\theta_\lambda\rightharpoonup\theta\qquad\text{in } L^2(0,T;V),\tag{3.39}$$

$$\theta'_\lambda\rightharpoonup\theta',\ A\theta_\lambda\rightharpoonup A\theta\qquad\text{in } L^2(0,T;V').\tag{3.40}$$

By the Aubin-Lions compactness lemma ([11, Theorem 5.1, p. 58]),

$$\theta_\lambda\to\theta\qquad\text{in } L^2(0,T;H),\tag{3.41}$$

thus taking possibly a further subsequence

$$\theta_\lambda(x,t)\to\theta(x,t)\qquad\text{for a.e. } (x,t)\in Q.\tag{3.42}$$

We claim that

$$\gamma_\lambda(\theta_\lambda(x,t)+u(x,t))\to\gamma(\theta(x,t)+u(x,t))\qquad\text{for a.e. } (x,t)\in Q.\tag{3.43}$$

Indeed, fix (x,t) such that (3.42) holds, by (3.33), passing possibly to a further subsequence, we have that there exists $\eta(x,t):=\lim_{\lambda\searrow0}\gamma_\lambda(\theta_\lambda(x,t)+u(x,t))$; moreover, using e.g. [3, Proposition 2.6-(iii), p. 28], we have $\lim_{\lambda\searrow0}\gamma_\lambda(r)=\gamma^o(r)=\gamma(r)$ for every $r\in\mathbb{R}$, thus

$$\begin{aligned}&[\eta(x,t)-\gamma(r)][(\theta(x,t)+u(x,t))-r]\\&\lim_{\lambda\searrow0}[\gamma_\lambda(\theta_\lambda(x,t)+u(x,t))-\gamma_\lambda(r)][(\theta_\lambda(x,t)+u(x,t))-r]\geq0,\end{aligned}\tag{3.44}$$

for every $r\in\mathbb{R}$, hence (3.44) and the monotonicity of γ imply that $\eta(x,t)=\gamma(\theta(x,t)+u(x,t))$. Claim (3.43) is therefore proved, and by the dominated convergence theorem we also have

$$\gamma_\lambda(\theta_\lambda+u)\to\gamma(\theta+u)\qquad\text{in } L^2(0,T;H).\tag{3.45}$$

We take the limit in the energy balance equation (3.30) and find that

$$\theta'+\chi'+A\theta=f\qquad\text{in } V',\text{ a.e. in }]0,T[.\tag{3.46}$$

therefore from (3.37) we infer that there exist $\eta\in L^2(0,T;H)$ such that, at least for a subsequence,

$$\gamma_\lambda(\theta_\lambda+u)\to\gamma(\theta+u)\qquad\text{in } L^2(0,T;H),\tag{3.47}$$

Now let $\xi_\lambda \in L^2(0,T;H)$ be such that (3.31)-(3.32) hold. A comparison in (3.32), together with (3.37), (3.47), yields a constant $C_1 > 0$ independent of λ such that

$$\|\xi_\lambda\|_{L^2(0,T;H)} \leq C_1. \tag{3.48}$$

Thus there exists $\xi \in L^2(0,T;H)$ such that, at least for a subsequence, we have the following weak convergence:

$$\xi_\lambda \rightharpoonup \xi \qquad \text{in } L^2(0,T;H), \tag{3.49}$$

and taking the limit in (3.32) we infer that

$$\varepsilon\chi' + \xi = \gamma(\theta + u) \qquad \text{a.e. in } Q. \tag{3.50}$$

We are left to pass to the limit in the nonlinear operator β. First observe that from (3.38) it is easy to check that

$$\chi_\lambda(t) \rightharpoonup \chi(t) \qquad \text{in } H, \quad \forall t \in [0,T]. \tag{3.51}$$

Thus exploiting (3.32), (3.47) (3.38), (3.51) and (3.50) we infer that

$$\begin{aligned}
\limsup_{\lambda \searrow 0} (\xi_\lambda, \chi_\lambda)_{L^2(Q)} &= \limsup_{\lambda \searrow 0} (\gamma_\lambda(\theta_\lambda + u) - \varepsilon\chi'_\lambda, \chi_\lambda)_{L^2(Q)} \\
&= \limsup_{\lambda \searrow 0} \Big[(\gamma_\lambda(\theta_\lambda + u), \chi_\lambda)_{L^2(Q)} - \frac{\varepsilon}{2}\|\chi_\lambda(T)\|_H + \frac{\varepsilon}{2}\|\chi_{0\varepsilon}\|_H \Big] \\
&\leq (\gamma(\theta + u), \chi)_{L^2(Q)} - \liminf_{\lambda \searrow 0} \frac{\varepsilon}{2}\|\chi_\lambda(T)\|_H + \frac{\varepsilon}{2}\|\chi_{0\varepsilon}\|_H \\
&\leq (\gamma(\theta + u), \chi)_{L^2(Q)} - \frac{\varepsilon}{2}\|\chi(T)\|_H + \frac{\varepsilon}{2}\|\chi_{0\varepsilon}\|_H \\
&= (\gamma(\theta + u) - \varepsilon\chi', \chi)_{L^2(Q)} = (\xi, \chi)_{L^2(Q)},
\end{aligned} \tag{3.52}$$

thus by [3, Proposition 2.5, p. 27] we infer that $\xi \in \beta(\chi)$ a.e. in Q and we are done. □

4. A priori estimates

4.1. **Uniform boundedness of the phase.** In this section we prove that the phase variable χ_ε of the relaxed system is uniformly bounded if it is bounded at the initial time. Before we need the following lemma.

Lemma 4.1. *Let* (A2)-(A5) *of Assumptions 2.1 be satisfied, and let* $\theta \in \mathbb{R}$, $\chi \in D(\beta)$. *Then*

$$\xi > \sup \alpha(\theta) \quad \Longrightarrow \quad \sup[\gamma(\theta) - \beta(\chi)] < 0, \tag{4.1}$$

$$\chi < \inf \alpha(\theta) \quad \Longrightarrow \quad \inf[\gamma(\theta) - \beta(\chi)] > 0. \tag{4.2}$$

Proof. We first prove (4.1). Assume therefore that $\chi > \sup \alpha(\theta)$, pick $\widehat{\chi} \in D(\beta)$ such that

$$\widehat{\chi} \in \alpha(\theta). \tag{4.3}$$

By (2.9) and (2.10) we have that

$$\gamma(\theta) \in \beta(\widehat{\chi}) \tag{4.4}$$

and

$$\inf(\gamma(\theta) - \beta(\widehat{\chi})) = \gamma(\theta) - \sup \beta(\widehat{\chi}) < 0 < \sup(\gamma(\theta) - \beta(\widehat{\chi})) = \gamma(\theta) - \inf \beta(\widehat{\chi}). \tag{4.5}$$

Since $\chi > \sup \alpha(\theta)$, we have that $\widehat{\chi} < \chi$, so that, by the monotonicity of β,

$$\inf \beta(\widehat{\chi}) \leq \sup \beta(\widehat{\chi}) \leq \inf \beta(\chi) \leq \sup \beta(\chi), \tag{4.6}$$

and from assumption (2.10) we infer that $\gamma(\theta) \notin \beta(\chi)$, i.e. either $0 < \inf(\gamma(\theta) - \beta(\chi))$, or $\sup(\gamma(\theta) - \beta(\chi)) < 0$. Thus the proof of (4.1) is completed if we prove that $0 \geq \inf(\gamma(\theta) - \beta(\chi))$: and in order to do this, we assume by contradiction that $0 < \inf(\gamma(\theta) - \beta(\chi)) = \gamma(\theta) - \sup \beta(\chi)$. This condition, together with (4.6), yields

$$\sup \beta(\widehat{\chi}) < \gamma(\theta),$$

contradicting (4.5). The proof of (4.2) is analogous. □

Now we can prove the uniform boundedness of the phase variable by extending the argument of [17, Lemma 4.2] to the nonregular multivalued case.

Proposition 4.1. *Let Assumptions 2.1 be satisfied. If $\varepsilon > 0$, let $a_\varepsilon, b_\varepsilon \in \mathbb{R}$, $\chi_\varepsilon \in H^1(0,T;H)$, $\theta_\varepsilon \in L^2(Q)$, and $u_\varepsilon \in L^2(Q)$ be such that*

$$\varepsilon\chi_\varepsilon'(x,t) \in \gamma\big(\theta_\varepsilon(x,t) + u_\varepsilon(x,t)\big) - \beta\big(\chi_\varepsilon(x,t)\big) \qquad \text{for a.e. } (x,t) \in Q, \tag{4.7}$$

$$\chi_\varepsilon(x,0) \in [a_\varepsilon, b_\varepsilon] \qquad \text{for a.e. } x \in \Omega, \tag{4.8}$$

$$\alpha(r) \subseteq [a_\varepsilon, b_\varepsilon] \qquad \forall r \in D(\alpha). \tag{4.9}$$

Then

$$a_\varepsilon \leq \chi_\varepsilon(x,t) \leq b_\varepsilon \qquad \text{for a.e. } (x,t) \in Q. \tag{4.10}$$

Proof. Let us observe that $\chi_\varepsilon' \in L^1(Q)$, therefore by the Fubini theorem we have that $\chi_\varepsilon'(x,\cdot) \in L^1(0,T)$ for a.e. $x \in \Omega$. If $\varphi \in C_c^\infty(]0,T[)$ and $z \in L^2(\Omega)$, then Fubini theorem also entails that

$$\begin{aligned}
&\int_\Omega z(x) \int_0^T (\chi_\varepsilon(x,t)\varphi'(t) + \chi_\varepsilon'(x,t)\varphi(t))\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^T \int_\Omega z(x)(\chi_\varepsilon(x,t)\varphi'(t) + \chi_\varepsilon'(x,t)\varphi(t))\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T (z, \chi_\varepsilon(t))_H \varphi'(t)\,\mathrm{d}t + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t)\,\mathrm{d}t.
\end{aligned}$$

From the previous chain of equalities, recalling that $\chi_\varepsilon \in H^1(0,T;H)$ and using again the Fubini theorem, we infer that

$$\begin{aligned}
&\int_\Omega z(x) \int_0^T (\chi_\varepsilon(x,t)\varphi'(t) + \chi_\varepsilon'(x,t)\varphi(t))\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^T \Big(z, \chi_\varepsilon(0) + \int_0^t \chi_\varepsilon'(s)\,\mathrm{d}s\Big)_H \varphi'(t)\,\mathrm{d}t + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t)\,\mathrm{d}t \\
&= (z, \chi_\varepsilon(0))_H \int_0^T \varphi'(t)\,\mathrm{d}t + \int_0^T \int_0^t (z, \chi_\varepsilon'(s))_H \varphi'(t)\,\mathrm{d}s\,\mathrm{d}t + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t)\,\mathrm{d}t \\
&= \int_0^T (z, \chi_\varepsilon'(s))_H \int_s^T \varphi'(t)\,\mathrm{d}t\,\mathrm{d}s + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t)\,\mathrm{d}t \\
&= -\int_0^T (z, \chi_\varepsilon'(s))_H \varphi(s)\,\mathrm{d}s + \int_0^T (z, \chi_\varepsilon'(t))_H \varphi(t)\,\mathrm{d}t = 0,
\end{aligned}$$

whence, by the arbitrariness of z, we infer that $\int_0^T (\chi_\varepsilon'(x,t)\varphi(t) + \chi_\varepsilon(x,t)\varphi'(t))\,\mathrm{d}t = 0$ for a.e. $x \in \Omega$, i.e. that $\chi_\varepsilon'(x,\cdot)$ is the distributional derivative of $\chi_\varepsilon(x,\cdot)$ for a.e. $x \in \Omega$.

Therefore there exists a measurable set $A \subseteq \Omega$ such that $|\Omega \setminus A| = 0$ and

$$\chi_\varepsilon(x,t) = \chi_\varepsilon(x,0) + \int_0^t \chi'_\varepsilon(x,s)\,\mathrm{d}s \qquad \forall t \in [0,T]\,, \qquad \forall x \in A.$$

It follows that for every $x \in A$ the function $\chi_\varepsilon(x,\cdot) : [0,T] \longrightarrow \mathbb{R}$ is absolutely continuous. It is not restrictive to assume that $a_\varepsilon \le \chi_\varepsilon(x,0) \le b_\varepsilon$ for every $x \in A$. Let us fix $x \in A$ and prove that $a_\varepsilon \le \chi_\varepsilon(x,t) \le b_\varepsilon$ for every $t \in [0,T]$. Indeed, if by contradiction this were not true, there would exist $t_0 \in \,]0,T]$ such that $\chi_\varepsilon(x,t_0) \notin [a_\varepsilon, b_\varepsilon]$. Let us first assume that $\chi_\varepsilon(x,t_0) > b_\varepsilon$. Then, by continuity, there exists $a_0 \in [0,t_0[$ such that $\chi_\varepsilon(x,a_0) = b_\varepsilon$ and $\chi_\varepsilon(x,t) > b_\varepsilon$ for every $t \in \,]a_0,t_0]$. Therefore by (4.9) we have $\chi_\varepsilon(x,t) > \sup\{\alpha(r) \,:\, r \in D(\alpha)\}$, in particular $\chi_\varepsilon(x,t) > \sup\{\alpha\big(\theta_\varepsilon(x,t) + u_\varepsilon(x,t)\big)\}$ for every $t \in \,]a_0,t_0]$. Hence by Lemma 4.1 we have that $\sup[\gamma(\theta_\varepsilon(x,t) + u_\varepsilon(x,t)) - \beta(\chi_\varepsilon(x,t))] < 0$. It follows that $\chi'_\varepsilon(x,t) < 0$ for a.e. $t \in \,]a_0,t_0]$, therefore, as $\chi_\varepsilon(x,\cdot)$ is absolutely continuous, we infer that $\chi_\varepsilon(x,\cdot)$ is decreasing on $]a_0,t_0]$, a contradiction. The remaining case when $\chi_\varepsilon(x,t_0) < a_\varepsilon$ is dealt in a completely analogous way. □

4.2. **L^2-estimates.** In the reminder of the paper we will need to use the function $\widehat{h} : [0,T] \longrightarrow V'$ defined by

$$\widehat{h}(t) := \int_0^t h(s)\,\mathrm{d}s, \tag{4.11}$$

for every $h \in L^1(0,T;V')$.

Lemma 4.2. *Let Assumptions 2.1 be satisfied. Let $\varepsilon_0 > 0$ be fixed and assume that $0 < \varepsilon \le \varepsilon_0$ and that $f_\varepsilon \in L^2(0,T;H)$, $u_\varepsilon \in L^2(Q)$, $\theta_{0\varepsilon} \in L^2(\Omega)$, $\chi_{0\varepsilon} \in L^2(\Omega)$, and $\chi_{0\varepsilon}(x) \in \overline{D(\gamma)}$ for a.e. $x \in \Omega$. If $(\theta_\varepsilon, \chi_\varepsilon)$ is the solution of (2.11)-(2.16), and if*

$$\sup_{0<\varepsilon\le\varepsilon_0} \|\chi_{0\varepsilon}\|_{L^\infty(\Omega)} < \infty, \tag{4.12}$$

then there exists $C > 0$ independent of ε such that

$$\begin{aligned} &\|\theta_\varepsilon\|_{L^2(0,t;H)} + \|\widehat{\theta}_\varepsilon\|_{L^\infty(0,T;V)} \\ &\le C(1 + \|\theta_{0\varepsilon}\|_H^2 + \|\chi_{0\varepsilon}\|_H^2 + \|\hat{f}_\varepsilon\|^2_{L^2(0,t;H)} + \|u_\varepsilon\|^2_{L^2(Q)}). \end{aligned} \tag{4.13}$$

Proof. Let us first observe that by (4.12) and by Proposition 4.1, χ_ε is uniformly bounded w.r.t. ε by a constant $K > 0$. Therefore integrating the energy balance equation (2.14) in time, testing the resulting equation by θ_ε, and integrating in time and space we get

$$\begin{aligned} &\|\theta_\varepsilon\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla\widehat{\theta}_\varepsilon(t)\|^2_{H^d} \\ &= \int_0^t \int_\Omega (\theta_{0\varepsilon} + \chi_{0\varepsilon} + \hat{f}_\varepsilon(s))\theta_\varepsilon(s)\,\mathrm{d}s - \int_0^t \int_\Omega \chi_\varepsilon(s)\theta_\varepsilon(s)\,\mathrm{d}s \\ &\le 2(1+K)(\|\theta_{0\varepsilon}\|_H^2 + \|\chi_{0\varepsilon}\|_H^2 + \|\hat{f}_\varepsilon\|^2_{L^2(0,t;H)}) + \frac{1}{4}\|\theta_\varepsilon\|^2_{L^2(0,t;H)}. \end{aligned} \tag{4.14}$$

Now let $r_0 \in D(\beta)$. From (2.15) we have that there exists $\xi_\varepsilon \in L^2(Q)$ with $\xi_\varepsilon \in \beta(\chi_\varepsilon)$ such that $\varepsilon\chi'_\varepsilon + \xi_\varepsilon - \beta^o(r_0) = \gamma(\theta_\varepsilon + u_\varepsilon) - \beta^o(r_0)$ a.e. in Q, thus by multiplying this last equation by $\chi_\varepsilon - r_0$ and observing that $(\xi_\varepsilon(x,t) - \beta^o(r_0))(\chi_\varepsilon(x,t) - r_0) \ge 0$ due to the

monotonicity of β, we infer the existence of a constant $C_1 > 0$ independent of ε, depending only on C_γ, $\beta^o(r_0)$, ε_0, such that

$$\begin{aligned}\frac{\varepsilon}{2}\|\chi_\varepsilon(t)\|_H^2 &\leq \int_0^t\int_\Omega [\gamma(\theta_\varepsilon(s)+u_\varepsilon(s))-\beta^o(r_0)](\chi_\varepsilon(s)-r_0)\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega r_0\chi_\varepsilon'(s)\,\mathrm{d}s\\ &\leq \int_0^t\int_\Omega [C_\gamma(1+|\theta_\varepsilon(s)|+|u_\varepsilon(s)|)+|\beta^o(r_0)|]|\chi_\varepsilon(t)-r_0| + \varepsilon r_0\int_\Omega(\chi_\varepsilon(t)-\chi_{0\varepsilon})\\ &\leq \frac{1}{4}\|\theta_\varepsilon\|^2_{L^2(0,t;H)} + C_1(1+\|\chi_{0\varepsilon}\|_H^2+\|u_\varepsilon\|^2_{L^2(Q)}+K). \qquad (4.15)\end{aligned}$$

Thus adding the two previous inequalities (4.14) and (4.15), we get a constant $C > 0$, independent of ε, such that (4.13) holds. □

4.3. **L^1-estimates.** In this section we perform the necessary L^1-estimates needed in our limit procedure.

Lemma 4.3. *Let* (A1)-(A3) *of Assumptions 2.1 be satisfied. For every $\varepsilon > 0$ let us assume that*

$$f_\varepsilon \in L^2(0,T;H)\cap BV([0,T]\,;L^1(\Omega)), \qquad (4.16)$$

$$u_\varepsilon \in C([0,T]\,;H^1(\Omega)),\quad u_\varepsilon', \Delta u_\varepsilon \in BV([0,T]\,;L^1(\Omega)), \qquad (4.17)$$

$$u_\varepsilon(t)|_\Gamma = u_\varepsilon(0)|_\Gamma \qquad \text{for a.e. } t\in[0,T], \qquad (4.18)$$

$$\theta_{0\varepsilon}\in V,\quad \Delta\theta_{0\varepsilon}\in L^1(\Omega),\quad \chi_{0\varepsilon}\in L^2(\Omega), \qquad (4.19)$$

and

$$\gamma(\theta_{0\varepsilon}(x)+u_\varepsilon(x,0))-\beta(\chi_{0\varepsilon}(x))\subseteq[-\varepsilon,\varepsilon] \qquad \text{for a.e. } x\in\Omega. \qquad (4.20)$$

If $(\theta_\varepsilon,\chi_\varepsilon)$ is such that (2.11)-(2.16) *hold, then $A\theta_\varepsilon\in L^\infty(0,T;L^1(\Omega))$ and there exists a constant $C>0$ independent of ε (depending only on T and $|\Omega|$) such that*

$$\begin{aligned}&\|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}+\|\chi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}\\ &\leq \|\theta_{0\varepsilon}\|_{L^1(\Omega)}+\|\chi_{0\varepsilon}\|_{L^1(\Omega)}+\|u_\varepsilon(0)\|_{L^1(\Omega)}+\|\Delta\theta_{0\varepsilon}\|_{L^1(\Omega)}+\|\Delta u_\varepsilon(0)\|_{L^1(\Omega)}\\ &\quad + C(1+\|f_\varepsilon\|_{L^1(Q)}+\|u_\varepsilon'\|_{L^1(0,T;L^1(\Omega)(\Omega))}+\|\Delta u_\varepsilon\|_{L^1(0,T;L^1(\Omega))}), \qquad (4.21)\end{aligned}$$

and

$$\begin{aligned}&\|\theta_\varepsilon'\|_{L^\infty(0,T;L^1(\Omega))}+\|\chi_\varepsilon'\|_{L^\infty(0,T;L^1(\Omega))}+\|A\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}\\ &\leq C+\|\Delta\theta_{0\varepsilon}\|_{L^1(\Omega)}+\|\Delta u_\varepsilon(0)\|_{L^1(\Omega)}\\ &\quad +\|f_\varepsilon\|_{BV([0,T];L^1(\Omega))}+\|u_\varepsilon'\|_{BV([0,T];L^1(\Omega))}+\|\Delta u_\varepsilon\|_{BV([0,T];L^1(\Omega))}, \qquad (4.22)\end{aligned}$$

where for every $h\in BV([0,T]\,;L^1(\Omega))$ we adopt the convention that $h(0):=\lim_{t\searrow 0}h(t)$.

Proof. Let us observe that by Lemma 3.1 we have that $A\theta_\varepsilon = -\Delta\theta_\varepsilon\in L^2(0,T;H)$ and $\theta_\varepsilon'+\chi_\varepsilon'-\Delta\theta_\varepsilon = f_\varepsilon$ a.e. in Q, hence if we set

$$\overline{\theta}_\varepsilon := \theta_\varepsilon + u_\varepsilon, \qquad \overline{\theta}_{0\varepsilon} := \theta_{0\varepsilon}+u_\varepsilon(0) \qquad (4.23)$$

$$\overline{f}_\varepsilon := f_\varepsilon + u_\varepsilon' - \Delta u_\varepsilon, \qquad (4.24)$$

we have that

$$\overline{\theta}_\varepsilon'(t)+\chi_\varepsilon'(t)-\Delta\overline{\theta}_\varepsilon(t) = \overline{f}_\varepsilon(t) \qquad \text{a.e. in } \Omega, \qquad (4.25)$$

$$\overline{\theta}_\varepsilon(0) = \overline{\theta}_{0\varepsilon} \qquad \text{a.e. in } \Omega, \qquad (4.26)$$

for a.e. $t \in]0,T[$. Moreover let $\xi_\varepsilon \in L^2(Q)$ be such that

$$\varepsilon\chi'_\varepsilon(x,t) + \xi_\varepsilon(x,t) = \gamma\big(\theta_\varepsilon(x,t) + u_\varepsilon(x,t)\big) \qquad \text{for a.e. } x \in \Omega, \tag{4.27}$$

$$\xi_\varepsilon(x,t) \in \beta(\chi_\varepsilon(x,t)) \qquad \text{for a.e. } x \in \Omega, \tag{4.28}$$

for a.e. $t \in]0,T[$. Now we fix $t \in]0,T[$ such that (4.25), (4.27) hold and such that $(\partial|\overline{\theta}_\varepsilon|/\partial t)(x,t)$ and $(\partial|\chi_\varepsilon|/\partial t)(x,t)$ exist for a.e. $x \in \Omega$: the set of such t's has full measure in $]0,T[$ by virtue of [26, Theorem 2.1.11, p. 48]. Now we adapt the argument in [24, p. 1477] to the multivalued case and to the presence of the datum u_ε. Recalling that $\mathrm{sign}_\mu(r)$ is defined by (3.5) for $\mu > 0$, by (4.18) we have that $\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}(t) \in V$ and $\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}(t)) \in V$, hence

$$\begin{aligned}
&-\int_\Omega \Delta\overline{\theta}_\varepsilon(t)\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) \\
&= -\int_\Omega \Delta(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon})\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) - \int_\Omega \Delta\overline{\theta}_{0\varepsilon}\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) \\
&= \int_\Omega |\nabla(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon})|^2\mathrm{sign}'_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) - \int_\Omega \Delta\overline{\theta}_{0\varepsilon}\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}), \\
&\geq -\int_\Omega \Delta\overline{\theta}_{0\varepsilon}\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) \geq -\int_\Omega |\Delta\overline{\theta}_{0\varepsilon}|.
\end{aligned} \tag{4.29}$$

Thus, multiplying (4.25) by $\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon})$, we infer that

$$\int_\Omega (\overline{\theta}'_\varepsilon(t) + \chi'_\varepsilon(t))\mathrm{sign}_\mu(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) \leq \int_\Omega |\Delta\overline{\theta}_{0\varepsilon}| + \int_\Omega |\overline{f}_\varepsilon(t)|, \tag{4.30}$$

and, taking the limit as $\mu \searrow 0$,

$$\int_\Omega (\overline{\theta}'_\varepsilon(t) + \chi'_\varepsilon(t))\mathrm{sign}_0(\overline{\theta}_\varepsilon(t) - \overline{\theta}_{0\varepsilon}) \leq \int_\Omega |\Delta\overline{\theta}_{0\varepsilon}| + \int_\Omega |\overline{f}_\varepsilon(t)|, \tag{4.31}$$

where sign_0 is defined by (3.6). Now fix $x \in \Omega$ such that $(\partial\chi_\varepsilon/\partial t)(x,t)$ and $(\partial|\chi_\varepsilon - \chi_{0\varepsilon}|/\partial t)(x,t)$ exist and define $\xi^t_{0\varepsilon} : \Omega \longrightarrow \mathbb{R}$ in such a way that

$$\xi^t_{0\varepsilon}(x) \in \beta(\chi_{0\varepsilon}(x)) \quad \text{and} \quad \begin{cases} \xi_\varepsilon(x,t) \geq \xi^t_{0\varepsilon}(x) & \text{if } \chi_\varepsilon(x,t) \geq \chi_{0\varepsilon}(x), \\ \xi_\varepsilon(x,t) \leq \xi^t_{0\varepsilon}(x) & \text{if } \chi_\varepsilon(x,t) < \chi_{0\varepsilon}(x). \end{cases}$$

By (4.27), by the monotonicity of γ and β, and by (4.20), we find

$$\begin{aligned}
&\chi'_\varepsilon(x,t)\mathrm{sign}_0(\overline{\theta}_\varepsilon(x,t)-\overline{\theta}_{0\varepsilon}(x))\\
&=\frac{1}{\varepsilon}(\gamma(\overline{\theta}_\varepsilon(x,t))-\xi_\varepsilon(x,t))\mathrm{sign}_0(\overline{\theta}_\varepsilon(x,t)-\overline{\theta}_{0\varepsilon}(x))\\
&=\frac{1}{\varepsilon}[(\gamma(\overline{\theta}_\varepsilon(x,t))-\gamma(\overline{\theta}_{0\varepsilon}(x)))-(\xi_\varepsilon(x,t)-\xi^t_{0\varepsilon}(x))]\mathrm{sign}_0(\overline{\theta}_\varepsilon(x,t)-\overline{\theta}_{0\varepsilon}(x))\\
&\quad+\frac{1}{\varepsilon}(\gamma(\overline{\theta}_{0\varepsilon}(x,t))-\xi^t_{0\varepsilon}(x))\mathrm{sign}_0(\overline{\theta}_\varepsilon(x,t)-\overline{\theta}_{0\varepsilon}(x))\\
&\geq\frac{1}{\varepsilon}[(\gamma(\overline{\theta}_\varepsilon(x,t))-\gamma(\overline{\theta}_{0\varepsilon}(x)))-(\xi_\varepsilon(x,t)-\xi^t_{0\varepsilon}(x))]\mathrm{sign}_0(\chi_\varepsilon(x,t)-\chi_{0\varepsilon}(x))\\
&\quad+\frac{1}{\varepsilon}(\gamma(\overline{\theta}_{0\varepsilon}(x,t))-\xi^t_{0\varepsilon}(x))\mathrm{sign}_0(\overline{\theta}_\varepsilon(x,t)-\overline{\theta}_{0\varepsilon}(x))\\
&=\chi'_\varepsilon\mathrm{sign}_0(\chi_\varepsilon(x,t)-\chi_{0\varepsilon}(x))\\
&\quad+\frac{1}{\varepsilon}(\gamma(\overline{\theta}_{0\varepsilon}(x))-\xi^t_{0\varepsilon}(x))[\mathrm{sign}_0(\overline{\theta}_\varepsilon(x,t)-\overline{\theta}_{0\varepsilon}(x))-\mathrm{sign}_0(\chi_\varepsilon(x,t)-\chi_{0\varepsilon}(x))]\\
&\geq\frac{\partial|\chi_\varepsilon-\chi_{0\varepsilon}|}{\partial t}(x,t)-2\qquad\text{a.e. in } Q.
\end{aligned}\tag{4.32}$$

Hence we have proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\overline{\theta}_\varepsilon(t)-\overline{\theta}_{0\varepsilon}|+|\chi_\varepsilon(t)-\chi_{0\varepsilon}|)\leq\int_\Omega(|\Delta\overline{\theta}_{0\varepsilon}|+|\overline{f}_\varepsilon(t)|+2)\quad\text{for a.e. } t\in\,]0,T[,$$

and integrating in time we obtain

$$\begin{aligned}
&\|\overline{\theta}_\varepsilon(t)-\overline{\theta}_{0\varepsilon}\|_{L^1(\Omega)}+\|\chi_\varepsilon(t)-\chi_{0\varepsilon}\|_{L^1(\Omega)}\\
&\leq t(\|\Delta\overline{\theta}_{0\varepsilon}\|_{L^1(\Omega)}+2|\Omega|)+\int_0^t\|\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s\quad\forall t\in[0,T]\,.
\end{aligned}\tag{4.33}$$

Since $|\|\theta_\varepsilon(t)-\theta_{0\varepsilon}\|_{L^1(\Omega)}-\|u_\varepsilon(t)-u_\varepsilon(0)\|_{L^1(\Omega)}|\leq\|\overline{\theta}_\varepsilon(t)-\overline{\theta}_{0\varepsilon}\|_{L^1(\Omega)}$, from (4.33) we infer that

$$\begin{aligned}
&\|\theta_\varepsilon(t)-\theta_{0\varepsilon}\|_{L^1(\Omega)}+\|\chi_\varepsilon(t)-\chi_{0\varepsilon}\|_{L^1(\Omega)}\\
&\leq\|u_\varepsilon(t)-u_\varepsilon(0)\|_{L^1(\Omega)}+t(\|\Delta\overline{\theta}_{0\varepsilon}\|_{L^1(\Omega)}+2|\Omega|)+\int_0^t\|\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s\qquad\forall t\in[0,T]\,,
\end{aligned}$$

which in particular also implies (4.21). Now for any $h\in\,]0,T[$ and for any $v:[0,T]\longrightarrow V'$ we define $\delta_h v:\,]0,T-h[\longrightarrow V'$ by

$$\delta_h v(t)=v(t+h)-v(t),\qquad t\in\,]0,T-h[\,.\tag{4.34}$$

By (4.25) we get

$$(\delta_h\overline{\theta}_\varepsilon)'(t)+(\delta_h\chi_\varepsilon)'(t)-\Delta(\delta_h\overline{\theta}_\varepsilon(t))=\delta_h\overline{f}_\varepsilon(t)\qquad\text{a.e. in }\Omega\tag{4.35}$$

for a.e. $t\in\,]0,T-h[$. We fix $t\in\,]0,T-h[$ such that (4.35) and (4.27) hold, and such that $(\partial|\delta_h\overline{\theta}_\varepsilon|/\partial t)(x,t)$ and $(\partial|\delta_h\chi_\varepsilon|/\partial t)(x,t)$ exist for a.e. $x\in\Omega$: the set of such t's

has full measure in $]0,T[$ by virtue of [26, Theorem 2.1.11, p. 48]. We test (4.35) by $\mathrm{sign}_\mu(\delta_h\overline{\theta}_\varepsilon) \in V$ and, arguing exactly as in (4.29)–(4.31) we get

$$\int_\Omega \big((\delta_h\overline{\theta}_\varepsilon)'(t) + (\delta_h\chi_\varepsilon)'(t)\big)\,\mathrm{sign}_0(\delta_h\overline{\theta}_\varepsilon(t)) \le \int_\Omega |\delta_h\overline{f}_\varepsilon(t)| \qquad \text{for a.e. } t \in]0,T-h[. \tag{4.36}$$

If we fix $x \in \Omega$ such that $(\partial\delta_h\chi_\varepsilon/\partial t)(x,t)$ and $(\partial|\delta_h\chi_\varepsilon|/\partial t)(x,t)$ exist, by the phase relaxation (4.27) we have that

$$\begin{aligned}&(\delta_h\chi_\varepsilon)'(x,t)\mathrm{sign}_0(\delta_h\overline{\theta}_\varepsilon(x,t))\\ &= \frac{1}{\varepsilon}(\delta_h\gamma(\overline{\theta}_\varepsilon(x,t)) - \delta_h\xi_\varepsilon(x,t))\mathrm{sign}_0(\delta_h\overline{\theta}_\varepsilon(x,t)),\end{aligned}$$

and the same monotonicity argument used in (3.14)-(3.16) yields

$$(\delta_h\chi_\varepsilon)'(x,t)\mathrm{sign}_0(\delta_h\overline{\theta}_\varepsilon(x,t)) \ge \frac{\partial}{\partial t}|\delta_h\chi_\varepsilon|(x,t). \tag{4.37}$$

Thus from (4.36) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\delta_h\overline{\theta}_\varepsilon(t)| + |\delta_h\chi_\varepsilon(t)|) \le \int_\Omega |\delta_h\overline{f}_\varepsilon(t)|, \qquad \text{for a.e. } t \in]0,T-h[,$$

hence integrating in time and exploiting (4.33) we obtain

$$\begin{aligned}&\|\delta_h\overline{\theta}_\varepsilon(t)\|_{L^1(\Omega)} + \|\delta_h\chi_\varepsilon(t)\|_{L^1(\Omega)}\\ &\le \|\delta_h\overline{\theta}_\varepsilon(0)\|_{L^1(\Omega)} + \|\delta_h\chi_\varepsilon(0)\|_{L^1(\Omega)} + \int_0^t \|\delta_h\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s\\ &\le h(\|\Delta\overline{\theta}_{0\varepsilon}\|_{L^1(\Omega)} + 2|\Omega|) + \int_0^h \|\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s + \int_0^t \|\delta_h\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s.\end{aligned} \tag{4.38}$$

Since $|\|\delta_h\theta_\varepsilon(t)\|_{L^1(\Omega)} - \|\delta_h u_\varepsilon(t)\|_{L^1(\Omega)}| \le \|\delta_h\overline{\theta}_\varepsilon(t)\|_{L^1(\Omega)}$, from (4.38) we get

$$\begin{aligned}&\|\delta_h\theta_\varepsilon(t)\|_{L^1(\Omega)} + \|\delta_h\chi_\varepsilon(t)\|_{L^1(\Omega)}\\ &\le \|\delta_h u_\varepsilon(t)\|_{L^1(\Omega)} + h(\|\Delta\overline{\theta}_{0\varepsilon}\|_{L^1(\Omega)} + 2|\Omega|)\\ &\quad + \int_0^h \|\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s + \int_0^t \|\delta_h\overline{f}_\varepsilon(s)\|_{L^1(\Omega)}\,\mathrm{d}s\end{aligned}$$

for every $t \in]0,T-h[$. Hence taking the limit as $h \searrow 0$, using (4.16),(4.17) and [3, Lemma A.1], and by comparison in (2.14), we infer (4.22). □

Lemma 4.4. *Let the assumptions of Lemma 4.3 be satisfied. For every $j \in \mathbb{N}$ set*

$$\Omega_j := \{x \in \Omega \ : \ d(x, \mathbb{R}^d \setminus \Omega) > 1/j\}. \tag{4.39}$$

For $v : Q \longrightarrow \mathbb{R}$ and $k \in \mathbb{R}^d$ define $\tau_k v : \Omega_j \times]0,T[\longrightarrow \mathbb{R}$ by

$$\tau_k v(x,t) := v(x+k,t) - v(x,t), \qquad x \in \Omega_j,\ |k| < 1/j. \tag{4.40}$$

If $(\theta_\varepsilon, \chi_\varepsilon)$ is such that (2.11)-(2.16) hold, then

$$\begin{aligned}\|\tau_k\chi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega_j))} \le{}& \|\tau_k\theta_{0\varepsilon}\|_{L^1(\Omega_j)} + \|\tau_k\chi_{0\varepsilon}\|_{L^1(\Omega_j)} + \|\tau_k u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega_j))}\\ &+ \big\|\tau_k\big(f_\varepsilon + u_\varepsilon' - \Delta u_\varepsilon\big)\big\|_{L^1(\Omega_j\times]0,T[)}.\end{aligned} \tag{4.41}$$

Proof. Observe that (Ω_j) is an increasing sequence of open subsets invading Ω. If we define by $\overline{\theta}_\varepsilon$, $\overline{\theta}_{0\varepsilon}$ and $\overline{f}_\varepsilon$ by (4.23), (4.24), then from Lemma 3.1 we get

$$(\tau_k\overline{\theta}_\varepsilon)'(t)+(\tau_k\chi_\varepsilon)'(t)-\Delta(\tau_k\overline{\theta}_\varepsilon(t))=\tau_k\overline{f}_\varepsilon(t)\quad\text{a.e. in }\Omega_j,\ |k|<1/j \tag{4.42}$$

for a.e. $t\in\,]0,T[$. We fix $t\in\,]0,T[$ such that (4.42) and (4.27) hold, and such that $(\partial|\tau_k\overline{\theta}_\varepsilon|/\partial t)(x,t)$ and $(\partial|\tau_k\chi_\varepsilon|/\partial t)(x,t)$ exist for a.e. $x\in\Omega$: the set of such t's has full measure in $]0,T[$ by virtue of [26, Theorem 2.1.11, p. 48]. We multiply (4.42) by $\mathrm{sign}_\mu(\tau_k\overline{\theta}_\varepsilon)(t)\in V$, and arguing exactly as in (4.29)–(4.31) we get

$$\int_{\Omega_j}((\tau_k\overline{\theta}_\varepsilon)'(t)+(\tau_k\chi_\varepsilon)'(t))\mathrm{sign}_0(\tau_k\overline{\theta}_\varepsilon(t))\le\int_{\Omega_j}|\tau_k\overline{f}_\varepsilon(t)|\qquad\text{for a.e. } t\in\,]0,T[.$$

If we fix $x\in\Omega$ such that $(\partial\tau_k\chi_\varepsilon/\partial t)(x,t)$ and $(\partial|\tau_k\chi_\varepsilon|/\partial t)(x,t)$ exist, by the phase relaxation (4.27) we have that

$$\begin{aligned}&(\tau_k\chi_\varepsilon)'(x,t)\mathrm{sign}_0(\tau_k\overline{\theta}_\varepsilon(x,t))\\&=\frac{1}{\varepsilon}(\tau_k\gamma(\overline{\theta}_\varepsilon)(x,t)-\tau_k\xi_\varepsilon(x,t))\mathrm{sign}_0(\tau_k\overline{\theta}_\varepsilon(x,t)),\end{aligned}$$

where $\xi_\varepsilon\in L^2(Q)$ is such that (4.27)-(4.28) hold. Thus the monotonicity argument used in (3.14)-(3.16) yields

$$(\tau_k\chi_\varepsilon)'(x,t)\mathrm{sign}_0(\tau_k\overline{\theta}_\varepsilon(x,t))\ge\frac{\partial}{\partial t}|\tau_k\chi_\varepsilon|(x,t). \tag{4.43}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_j}(|\tau_k\overline{\theta}_\varepsilon(t)|+|\tau_k\chi_\varepsilon(t)|)\le\int_{\Omega_j}|\tau_k\overline{f}_\varepsilon(t)|,\qquad\text{for a.e. } t\in\,]0,T[,$$

and integrating in time we obtain

$$\begin{aligned}&\|\tau_k\overline{\theta}_\varepsilon(t)\|_{L^1(\Omega_j)}+\|\tau_k\chi_\varepsilon(t)\|_{L^1(\Omega_j)}\\&\le\|\tau_k\overline{\theta}_\varepsilon(0)\|_{L^1(\Omega_j)}+\|\tau_k\chi_\varepsilon(0)\|_{L^1(\Omega_j)}+\int_0^t\|\tau_k\overline{f}_\varepsilon(s)\|_{L^1(\Omega_j)}\,\mathrm{d}s.\end{aligned} \tag{4.44}$$

Since $|\tau_k\theta_\varepsilon(t)|\le|\tau_k\overline{\theta}_\varepsilon(t)|+|\tau_k u_\varepsilon(t)|$, from (4.44) we infer that

$$\begin{aligned}&\|\tau_k\theta_\varepsilon(t)\|_{L^1(\Omega_j)}+\|\tau_k\chi_\varepsilon(t)\|_{L^1(\Omega_j)}\\&\le\|\tau_k\theta_{0\varepsilon}\|_{L^1(\Omega_j)}+\|\tau_k\chi_{0\varepsilon}\|_{L^1(\Omega_j)}+\|\tau_k u_\varepsilon(0)\|_{L^1(\Omega_j)}+\|\tau_k u_\varepsilon(t)\|_{L^1(\Omega_j)}\\&\quad+\int_0^T\|\tau_k\overline{f}_\varepsilon(s)\|_{L^1(\Omega_j)}\,\mathrm{d}s\qquad\forall t\in(0,T],\end{aligned} \tag{4.45}$$

which yields (4.41), due to the definition of $\overline{f_\varepsilon}$ (see (4.24)). □

5. Asymptotic limit to the Stefan problem

Before proceeding with the limit procedure, we need the following uniqueness result for a suitable weak formulation of the Stefan problem.

Lemma 5.1. *Let* (A1) *and* (A4) *of Assumptions 2.1 be satisfied. If* $f \in L^2(0,T;V') + L^1(0,T;H)$, χ_0, $\theta_0 \in L^2(\Omega)$, *then there exists at most one pair* $(\theta,\chi) : Q \longrightarrow \mathbb{R}^2$ *satisfying the following conditions:*

$$\theta \in L^2(0,T;H), \tag{5.1}$$

$$\widehat{\theta} \in L^\infty(0,T;V) \cap H^1(0,T;V'), \tag{5.2}$$

$$\chi \in L^\infty(Q), \tag{5.3}$$

$$\theta(t) + \chi(t) + A\widehat{\theta}(t) = \widehat{f}(t) + \theta_0 + \chi_0 \qquad \text{in } V', \text{ for a.e. } t \in]0,T[, \tag{5.4}$$

$$\chi(x,t) \in \alpha\big(\theta(x,t) + u(x,t)\big) \qquad \text{for a.e. } (x,t) \in Q, \tag{5.5}$$

where $\widehat{\theta}$ *and* $\widehat{f}$ *are defined according to* (4.11). *A pair* (θ,χ) *satisfying* (5.2)-(5.5) *is also called a solution of the Stefan problem in the sense of Baiocchi-Duvaut-Frémond.*

Proof. Let (θ_i,χ_i), $i = 1,2$, be two solutions of (5.2)–(5.5), and set

$$\Theta := \theta_1 - \theta_2, \qquad \mathcal{X} := \chi_1 - \chi_2. \tag{5.6}$$

Taking the difference of the equations (5.4) written for (θ_1,χ_1) and (θ_2,χ_2), we find

$$\widehat{\Theta} \in L^\infty(0,T;V) \cap H^1(0,T;V'), \tag{5.7}$$

$$\mathcal{X} \in L^\infty(Q), \tag{5.8}$$

$$\Theta + \mathcal{X} + A\widehat{\Theta} = 0 \quad \text{in } V', \ \text{ in } \ ]0,T[\,. \tag{5.9}$$

By a comparison in the last equation, we see that $A\widehat{\Theta} \in L^1(0,T;H)$, therefore multiplying (5.9) by Θ, and integrating over $(0,t)\times\Omega$, we get

$$\|\Theta\|^2_{L^2(0,t;H)} + \int_0^t \int_\Omega \mathcal{X}(x,s)\Theta(x,s)\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\int_\Omega |\nabla\widehat{\Theta}(x,t)|^2\,\mathrm{d}x = 0. \tag{5.10}$$

Therefore, since $\mathcal{X}\Theta \geq 0$ a.e. in Q by (5.5) and by the monotonicity of α, from (5.10) we infer that $\Theta = 0$ a.e. in Q, and therefore $A\widehat{\Theta} = 0$ a.e. in Q. By a comparison in (5.9), also $\mathcal{X} = 0$ a.e. in Q, and the uniqueness of problem (5.2)-(5.5) is proved. □

We are now ready to prove our main convergence theorem.

Proof of Theorem 2.3. We can assume that $\varepsilon \leq 1$. Integrating in time the energy balance equation (2.14) we get

$$\theta_\varepsilon + \chi_\varepsilon + A\widehat{\theta}_\varepsilon = \theta_{0\varepsilon} + \chi_{0\varepsilon} + \widehat{f}_\varepsilon \qquad \text{in } V', \text{ a.e. in }]0,T[, \tag{5.11}$$

where $\widehat{\theta}_\varepsilon$ and $\widehat{f}_\varepsilon$ are defined defined according to (4.11). Let us observe that thanks to (A4) of Assumptions 2.1, and to (2.31), we can apply Lemma 4.2, therefore we have that there exist $\theta \in L^2(0,T;H)$, $\chi \in L^\infty(Q)$ such that $\widehat{\theta} \in L^\infty(0,T;V)$ and

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{in } L^2(0,T;H), \tag{5.12}$$

$$\widehat{\theta}_\varepsilon \overset{*}{\rightharpoonup} \widehat{\theta}_\varepsilon \quad \text{in } L^\infty(0,T;V) \cap H^1(0,T;H), \tag{5.13}$$

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(Q), \tag{5.14}$$

therefore we can take the limit in (5.11) and obtain

$$\theta + \chi + A\widehat{\theta} = \theta_0 + \chi_0 + \widehat{f} \qquad \text{in } V', \text{ a.e. in }]0,T[. \tag{5.15}$$

Let $\xi_\varepsilon \in L^2(Q)$ be such that

$$\xi_\varepsilon(x,t) \in \beta(\chi_\varepsilon(x,t)) \qquad \text{for a.e. } (x,t) \in Q, \tag{5.16}$$

$$\varepsilon\chi'_\varepsilon(x,t) + \xi_\varepsilon(x,t) = \gamma(\theta_\varepsilon(x,t) + u_\varepsilon(x,t)) \qquad \text{for a.e. } (x,t) \in Q. \tag{5.17}$$

From (4.21) of Lemma 4.3 we deduce that

$$\|\theta_\varepsilon\|_{L^1(Q)} + \|\chi_\varepsilon\|_{L^1(Q)} \leq C,$$

for a suitable constant $C > 0$ independent of ε. Moreover, recalling notations (4.39), (4.40), from Lemma 4.4 we infer that for every j and every k we have

$$\begin{aligned}
&\|\tau_k\theta_\varepsilon\|_{L^1(\Omega_j\times]0,T[)} + \|\tau_k\chi_\varepsilon\|_{L^1(\Omega_j\times]0,T[)} \\
&= \|\tau_k\theta_\varepsilon\|_{L^1(]0,T[;L^1(\Omega_j))} + \|\tau_k\chi_\varepsilon\|_{L^1(]0,T[;L^1(\Omega_j))} \\
&\leq T(\|(\tau_k\theta_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega_j))} + \|(\tau_k\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega_j))}) \\
&\leq T(\|\tau_k\theta_{0\varepsilon}\|_{L^1(\Omega_j)} + \|\tau_k\chi_{0\varepsilon}\|_{L^1(\Omega_j)} + \|\tau_k u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega_j))}) \\
&\quad + T\|\tau_k(f_\varepsilon + u'_\varepsilon - \Delta u_\varepsilon)\|_{L^1(\Omega_j\times]0,T[)},
\end{aligned} \tag{5.18}$$

therefore from (2.27)-(2.30) and from a suitable versione of the Riesz-Frechet-Kolmogorov theorem (cf. [1, Theorem 2.22]) we infer that, at least for a subsequence which we do not relabel, the following strong convergences hold:

$$\theta_\varepsilon \to \theta \quad \text{in } L^1(Q), \tag{5.19}$$

$$\chi_\varepsilon \to \chi \quad \text{in } L^1(Q). \tag{5.20}$$

Estimate (4.22) yields the strong convergence

$$\varepsilon\chi'_\varepsilon \to 0 \quad \text{in } L^1(Q). \tag{5.21}$$

Therefore, taking possibly a further subsequence, we find a subset $\tilde{Q}$ of full measure in Q such that

$$\theta_\varepsilon(x,t) \to \theta(x,t), \quad u_\varepsilon(x,t) \to u(x,t) \quad \text{in } \mathbb{R},\ \forall (x,t) \in \tilde{Q} \tag{5.22}$$

$$\chi_\varepsilon(x,t) \to \chi(x,t), \qquad \text{in } \mathbb{R},\ \forall (x,t) \in \tilde{Q}, \tag{5.23}$$

$$\varepsilon\chi'_\varepsilon(x,t) \to 0 \qquad \text{in } \mathbb{R},\ \forall (x,t) \in \tilde{Q}. \tag{5.24}$$

Hence by the continuity of γ and by a comparison in (5.17) we have that for every $(x,t) \in \tilde{Q}$ there exists $\xi(x,t) \in \mathbb{R}$ such that

$$\gamma(\theta_\varepsilon(x,t) + u_\varepsilon(x,t)) \to \gamma(\theta(x,t) + u(x,t)) \quad \text{in } \mathbb{R},\ \forall (x,t) \in \tilde{Q} \tag{5.25}$$

$$\xi_\varepsilon(x,t) \to \xi(x,t), \quad \varepsilon\chi'_\varepsilon(x,t) \to 0 \qquad \text{in } \mathbb{R},\ \forall (x,t) \in \tilde{Q}. \tag{5.26}$$

Therefore, taking the limit as $\varepsilon \searrow 0$ in (5.17) we obtain

$$\xi(x,t) = \gamma(\theta(x,t) + u(x,t)).$$

Let us also observe that from (5.16), (5.24) we obtain (see, e.g., [3, Proposition 2.5, p. 27] applied for monotone graphs in $\mathbb{R}$) that

$$\xi(x,t) \in \beta(\chi(x,t)) \quad \forall (x,t) \in \tilde{Q},$$

so that

$$\gamma(\theta(x,t) + u(x,t)) \in \beta(\chi(x,t)),$$

and from the compatibility condition (A5) in Assumptions 2.1 we get $\chi(x,t) \in \alpha(\theta(x,t) + u(x,t))$. Hence we have proved that (θ, χ) solves (5.1)-(5.5), which has at most one solution by virtue of Lemma 5.1. This uniqueness property, together with the fact that the formulation (2.18)-(2.23) is stronger than the formulation of problem (5.2)-(5.5), let us infer that (θ, χ) is indeed the solution of (2.18)-(2.23). By this uniqueness property, we also have that the whole sequence $(\theta_\varepsilon, \chi_\varepsilon)$ converges to (θ, χ).

□

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