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Survey paper

Bounded confidence opinion dynamics: A survey[☆]

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A R T I C L E I N F O

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A B S T R A C T

At the beginning of this century, Hegselmann and Krause proposed a dynamical model for opinion formation that is referred to as the Bounded Confidence Opinion Dynamics (BCOD) model and that has since attracted a wide interest from different research communities. The model can be viewed as a dynamic network, in which each agent is endowed with a state variable representing an *opinion* and two agents interact if the distance between their opinions does not exceed a constant confidence bound. This relation of instantaneous proximity between the opinions naturally induces a dynamic interaction graph. At each stage of the opinion iteration, all agents synchronously update their opinion to the average of all opinions that belong to the neighbors in the interaction graph.

BCOD models exhibit a broad variety of phenomena that cannot be studied by traditional methods, and their analysis has enriched the systems and control field with a number of novel mathematical tools. This fact, together with the existence of an extensive literature on the topic scattered across different fields, calls for a systematic presentation of the existing results on this class of dynamic networks. The aim of this survey is to provide an overview of BCOD models with time-synchronous interactions, with possibly asymmetric and heterogeneous confidence bounds. Conditions on the different classes of BCOD which ensure the convergence (in finite time or asymptotically) of the opinions are discussed, and the possible structures of the terminal opinions are described. The numerous phenomena highlighted in the literature from numerical studies, e.g., the characterization of steady state behaviors and the sensitivity to confidence thresholds, are also reviewed. Finally, some recent modifications and applications of BCOD models are discussed, and suggestions of directions for future research are provided.

1. Introduction

Models of opinion formation or opinion dynamics (Anderson, Dabbene, Proskurnikov, Ravazzi, & Ye, 2020; Castellano, Fortunato, & Loreto, 2009; Friedkin, 2015; Mastroeni, Vellucci, & Naldi, 2019; Noorazar, Vixie, Talebanpour, & Hu, 2020; Proskurnikov & Tempo, 2017; Ravazzi, Dabbene, Lagoa, & Proskurnikov, 2021) have been proposed in mathematical sociology and sociophysics in an endeavor to portray the rich and diverse behaviors of social groups, capturing the most important effects of the temporal evolution of behaviors and attitudes of the social agents caused by their interactions. The mathematical theory of such models naturally complements the classical social network analysis (Freeman, 2004) that is primarily focused on structural properties of social networks and the computational social science (Lazer et al., 2009) that aims to understand collective human behavior through the analysis of data sets related to social networks. Opinion dynamics modeling has been empowered by recent progress in control of multi-agent systems and large-scale dynamical networks.

Opinion dynamics is now a topic of growing relevance for the systems and control community. The search query "opinion dynamics" on ScienceDirect platform returns more than 300 papers published in Automatica; the same search in the database IEEExplore returns more than 200 works. Wiener's prediction (Wiener, 1954) that cybernetics and control as its indispensable part would play a key role in social sciences is thus coming true.

1.1. Summary

Bounded Confidence Opinion Dynamics (BCOD) models form one of the widely investigated classes of opinion dynamics models. They capture the *homophily* behavior, i.e., the tendency of individuals to be attracted by other individuals having similar opinions.

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The dynamics of interest in this survey is induced in the model by updating the opinions of all agents according to the average opinion of the agent's neighbors, i.e., those agents who have opinions belonging to its confidence interval.

The survey provides an overview of BCOD with time-synchronous interactions between the agents (for brevity, we refer them to as *synchronous* BCOD). After presenting the interpretations of the concept of opinion and giving a brief introduction to the field of opinion dynamics models, properties and conditions on the different classes of synchronous BCOD models which ensure the convergence (in finite time or asymptotically) of the opinions are provided, and the structures of the terminal opinion clusters are described. The technical proofs of the theoretical results are reported in Appendix B. Moreover, the numerous phenomena investigated in the literature through numerical studies, e.g., the characterization of steady state behaviors, are also reviewed. The most "mature" modifications of classical synchronous BCOD models and their principal properties are briefly discussed. Finally, a few applications of synchronous BCOD models are presented.

This introduction continues by framing the topic of the survey within the literature on opinion dynamics and by discussing how graphs and opinion formations constitute the basic ingredients for our analysis. Readers who are familiar with these arguments may move directly to Section 1.4, which focuses on bounded confidence models.

1.2. Agents, opinions, interaction graphs

In this survey, we are primarily interested in microscopic or *agents-based* models that portray the evolution of a social group by specifying the behavior of its individual members, which are called *agents* or *social actors*. At each time instant, the agents are characterized by their *opinions*. The term "opinion" admits various interpretations in human and technical sciences. In its broadest sense, this term stands for a numerical characteristic of a social agent, which is altered due to interactions with other agents. In social psychology, an opinion can be interpreted as a cognitive orientation (attitude) towards a particular object, action, event or issue (Friedkin, 2015). However, many other interpretations are available. Opinions can, e.g., stand for sets of cultural traits characterizing an individual (Axelrod, 1997), measure an agent's skill in a particular field (Xie et al., 2016) or characterize an agent's knowledge set, i.e., the aggregate of information and skills received from the others (Yokomatsu & Kotani, 2020).

The agents are represented by nodes of a graph (referred to as *social network*) whose edges (generally, directed) represent some relations or influence among the agents. The graph can be static, time-varying or even depend on the opinions. The dependence on the opinion values for the neighbors' selection allows one to model the *homophily* behavior, that is, the tendency of individuals to establish relations with similar people (Granovetter, 1973; Lobel & Sadler, 2016; McPherson, Smith-Lovin, & Cook, 2001; Rivera, Soderstrom, & Uzzi, 2010).

1.3. Models of opinion formation

Opinion formation models studied in the engineering literature primarily deal with real-valued opinions whose dynamics are governed by differential or difference equations. Such dynamic models can be examined by Lyapunov analysis and other control-theoretic methods and may be divided into two major classes of linear and nonlinear dynamical systems.

1.3.1. Linear models

Several classes of linear models of opinion formation have been thoroughly studied in the literature. One such class originates from the classical French–DeGroot model of iterative averaging (DeGroot, 1974; French, 1956; Harary, 1959) and its modification proposed by Friedkin and Johnsen (1990, 1999). At each stage of the opinion iteration, the agents update their opinions by taking convex combinations of their own and the others' opinions; the weights of this convex combination determine a weighted directed graph (generally, time-varying). Properties of linear averaging-based models are closely related to the dynamics of Markov chains and are well documented in the literature (Bolouki & Malhame, 2015; Bullo, 2022; Friedkin, 2015; Martin & Hendrickx, 2016; Parsegov, Proskurnikov, Tempo, & Friedkin, 2017; Proskurnikov & Tempo, 2017; Tian & Wang, 2018). It is known that the usual French–DeGroot system usually converges to consensus, whereas full or partial "stubbornness" of the agents (i.e., their anchorage at their initial opinions) leads to opinion cleavage. The iterative averaging admits a game-theoretic interpretation (Bauso & Cannon, 2018; Ghaderi & Srikant, 2014) and is related to some problems of multi-agent control, e.g., constrained consensus and containment control (Proskurnikov, Calafiore, & Cao, 2020; Proskurnikov & Cao, 2017).

Another broad class of linear models has been inspired by the theory of structural balance (Altafini & Lini, 2015; Facchetti, Iacono, & Altafini, 2011; Heider, 1946). Models of this sort describe opinion dynamics over signed graphs, where negatively weighted arcs stand for antagonistic or competitive interactions (repulsion between the opinions), whereas positive weights correspond to trust or cooperation between individuals (whose opinions attract). We refer the reader to the recent survey by Shi, Altafini, and Baras (2019) for a detailed review of these models. One of their principal properties is the emergence of bimodal polarization or "bipartite consensus" (Altafini, 2013; Angeli & Manfredi, 2019; He, Liu, Hu, & Fang, 2021; Liu, Chen, Basar, & Belabbas, 2017; Proskurnikov & Tempo, 2018) in the case where the signed graph is structurally balanced.

It should be noted that the interest in opinion formation modeling was inspired, to a great extent, by the problem of *community cleavage* (Friedkin, 2015), known also as the Abelson's *diversity puzzle* (Kurahashi-Nakamura, Mäs, & Lorenz, 2016): to build a dynamic model capable to explain both *consensus* among the agents and persistent *disagreement* between their opinions, in particular, the splitting of the opinions into several large *clusters*. The existing linear models can portray consensus and disagreement behaviors; however, in the case of disagreement, the clusters are usually determined by the weighted *interaction graph* rather than the initial opinion profile. In some models (e.g., the Friedkin–Johnsen model), the terminal opinions are pairwise different, that is, each cluster consists of only one opinion. These limitations stimulate the development of more advanced nonlinear models able to capture more complex socio-psychological phenomena.

1.3.2. Nonlinear models

It was Abelson (1964, 1967) (see also Hunter, Danes, and Cohen (1984)) who suggested that the actual mechanisms of the opinion ("attitude") formation could be nonlinear. However, rigorous examination of such models started much later and was enabled by the progress in nonlinear systems (in particular, Lyapunov-based methods). In recent decades, a number of nonlinear models have been proposed in the literature; some of them exhibit very rich and complex opinion-forming behaviors such as, e.g., multistable agreement and disagreement manifolds (Bizyaeva, Franci, & Leonard, 2022). Whereas linear opinion formation models have been thoroughly studied, nonlinear models (even relatively simple) still present challenges.

The vast majority of nonlinear models of opinion formation inherit the averaging-based mechanism of the French–DeGroot model (DeGroot, 1974) or its continuous-time counterpart (Abelson, 1964). This structure was initially proposed by Abelson

(1964) and was lacking experimental confirmations. However, recent experiments with small-size groups (Friedkin, Proskurnikov, & Bullo, 2021) and large-scale social media (Kozitsin, 2022) demonstrate that the convex combination mechanism of opinion formation is more than just a mathematical abstraction, even though the structure of averaging weights is not easy to find. In particular, the steady state opinions of a social group tend to fall into the convex hull spanned the initial opinions. A peculiar feature of the nonlinear opinion dynamics models is the dependence of the influence weights on the opinions. The coupling between the influence weights and the opinions implies that the influence graph evolves along with the opinions.

Opinion-dependent weights can portray various effects observed in real societies. In "polar opinion dynamics" models (Amelkin, Bullo, & Singh, 2017), such weights are caused by the opinion-dependent *susceptibility* function (e.g., people with extreme opinions can be more reluctant in changing their opinions than people with neutral opinions). In models of *biased assimilation* (Dandekar, Goel, & Lee, 2013; Xia, Ye, Liu, Cao, & Sun, 2020), the agents are biased towards their own opinions, which increases the importance of self-opinions and scales down the influence of the neighbors. In this way, individuals draw undue support to their initial positions (Dandekar et al., 2013). Even more sophisticated are appraisal-based models, where the influence weights are altered by a dynamic (opinion-driven) appraisal mechanism (Anderson et al., 2020; Kang & Li, 2022; Tian et al., 2022). In the bounded confidence models we consider below, the weights are nonlinear discontinuous functions of the opinions.

1.4. Bounded confidence opinion dynamics

Among the nonlinear opinion formation mechanisms, the class of BCOD models is perhaps the one that has been investigated the most. BCOD models arose from the pioneering works by Deffuant, Neau, Amblard, and Wiesbuch (2000), Hegselmann and Krause (2002) and Krause (2000). Even though these two models exhibit similar steady state behaviors, e.g., clustering and consensus, their formal analyses require theoretical frameworks which are very different (Castellano et al., 2009). The reason is that in the model proposed by Deffuant and Weisbuch at each time-step only a pair of agents is randomly chosen and their opinions are updated according to the classical BCOD rule, which motivates the attribute of asynchronous BCOD for this model. Instead, this survey deals with *synchronous* BCOD models, in the sequel for simplicity just indicated with BCOD, which means that the opinions of all agents are updated simultaneously at each period.

BCOD has attracted visible attention of systems and control theorists, as witnessed, e.g., by the recent works Altafini and Ceragioli (2018), Blondel, Hendrickx, and Tsitsiklis (2009), Ceragioli and Frasca (2012), Chazelle and Wang (2017), Chen, Su, Mei, and Bullo (2020), Etesami (2019), Etesami and Başar (2015), Frasca, Tarbouriech, and Zaccarian (2019), Kang and Li (2022), Kolarijani, Proskurnikov, and Esfahani (2021), Su, Chen and Hong (2017), Vasca, Bernardo, and Iervolino (2021) and Yang, Dimarogonas, and Hu (2014) in leading control journals. The goal of this survey is to overview the recent progress in the analysis of BCOD and the mathematical tools developed for their examination.

In BCOD, the graph topology is dynamic: at each time point the neighbors of each agent are individuals with similar opinions or, mathematically, the individuals whose opinions belong to the agent's *confidence interval*. Similar to the conventional French–DeGroot linear model, an agent updates its opinion at each time point by the average of the opinions of its neighbors (Hegselmann & Krause, 2002). BCOD models can thus be considered as a particular class of (state-dependent) piecewise affine systems (Iervolino, Tangredi, & Vasca, 2016) or mixed logical dynamical systems (Bernardo & Vasca, 2020).

From a sociological viewpoint, the BCOD mechanism prominently manifests the phenomenon of *homophily* (McPherson et al., 2001): the agents readily assimilate opinions of like-minded individuals and disregard dissimilar opinions. The importance of BCOD, however, is not confined to social sciences. Indeed, similar dynamical networks arise, for instance, in algorithms of flocking and swarming with distance-based (or "nearest neighbor") graphs (Cucker & Smale, 2007; Motsch & Tadmor, 2014; Reynolds, 1987; Tanner, Jadbabaie, & Pappas, 2007; Vicsek, Czirók, Ben-Jacob, Cohen, & Shochet, 1995). Hence, an overview of the theoretical studies on BCOD and a systematic presentation of the different model substructures provides an important insight for the examination of more general classes of systems.

The characteristics of the confidence thresholds allow one to divide BCOD models into several groups. A model is said to be *homogeneous* when the same confidence thresholds characterize all agents, and *heterogeneous* otherwise; the model is said to be *symmetric* if the same confidence thresholds are used to select the neighbors with lower and upper opinions, and *asymmetric* otherwise (Hegselmann & Krause, 2002). Both asymmetry and heterogeneity lead to behaviors that are not found in symmetric models (Bernardo & Vasca, 2020; Hegselmann & Krause, 2002). On the other hand, the properties of homogeneous BCOD cannot be easily derived by particularizing the theoretical results available for the heterogeneous case. This survey provides a systematic presentation of the available theoretical results and of their applicability to the different categories of BCOD. Whereas this survey is primarily dedicated to discrete-time models (stemming from the Hegselmann–Krause system), BCOD can also be considered in continuous time. The analysis of continuous-time BCOD models requires additional tools; the key problem is the potential absence of classical solutions, whose existence can be proved only for almost all initial conditions (Blondel, Hendrickx, & Tsitsiklis, 2010; Yang et al., 2014). However, this problem can be avoided by "smoothing" discontinuous nonlinearities (Ceragioli & Frasca, 2012; Yang et al., 2014). Confining oneself to *classical* solutions, the analysis of continuous-time BCOD models is similar in spirit to discrete-time BCOD (Jabin & Motsch, 2014; Motsch & Tadmor, 2014) (such models are briefly mentioned in Section 9). Generalized (non-classical) Filippov and Krasovskii solutions are beyond the scope of this survey because their theory relies on essentially different mathematical tools (Ceragioli & Frasca, 2012; Piccoli & Rossi, 2021; Stamoulas & Rathinam, 2018).

Starting from classical BCOD, many modifications have been proposed in the literature, which differ by assumptions on the confidence thresholds, communication graph and exogenous signals. Section 10 briefly reviews the major peculiarities of these models.

1.5. Problems in question

Despite its conceptual simplicity, a rigorous analysis of BCOD is a nontrivial task due to the presence of model discontinuities that become especially sophisticated for heterogeneous models (Hegselmann, 2004).

A key problem, when dealing with BCOD models, is their asymptotic behavior. Unlike classical control systems, BCOD exhibits infinitely many possible equilibria, depending on the initial conditions. However, the typical behavior of a BCOD is that some of the equilibria correspond to consensus (all opinions are equal) and others correspond to opinion profiles split into two or more clusters, inside which the opinions become identical. A basic question arising in BCOD is whether each solution converges to one of the two types of equilibria mentioned above (consensus and clustering) or if more complex asymptotic behaviors can exist and, if yes, under what conditions. More specific questions are: What conditions ensure convergence to a unanimous value for all opinions at the steady state? And to clustering? Is it possible to characterize the form of the asymptotic behavior? Is the convergence to the steady state occurring in finite time? Is it possible to estimate the convergence time? Existing results and open issues related to these questions are discussed in this paper.

1.6. Organization of the survey

The rest of this survey paper is organized as follows. Section 2 introduces notation and basic concepts of opinion modeling. Section 3 presents discrete-time BCOD and some definitions. In Section 4, the typical steady state behaviors shown by BCOD are introduced. Section 5 describes the properties which are satisfied by more general classes of BCOD, i.e., in the presence of heterogeneity and asymmetric confidence intervals. Section 6 is dedicated to homogeneous BCOD, and some convergence properties are provided for this special case, while in Section 7 the heterogeneous case is discussed. In Section 8, BCOD is analyzed in presence of stubborn agents. Section 9 describes the main features of continuous-time BCOD. Section 10 includes a concise review of some recent modifications of classical BCOD. In Section 11, the applications of BCOD are briefly presented. Section 12 concludes the paper by tracing directions for future research. Basic definitions of graph theory are given in Appendix A. The proofs of the theoretical results presented in the paper are reported in Appendix B.

2. Preliminaries and notation

The following notation is adopted throughout the paper: $\mathbb{N}$ ($\mathbb{N}_0$) is the set of positive (non-negative) integers, $\mathbb{R}$ ($\mathbb{R}_0^+$) is the set of (non-negative) real numbers. Symbol N stands for the number of agents in a network, agents that are numbered by indices from set $\mathcal{I} = \{1, \dots, N\} \subset \mathbb{N}$.

For a number $\alpha \in \mathbb{R}$, $\lceil \alpha \rceil$ denotes the smallest integer larger than or equal to α, $\lfloor \alpha \rfloor$ denotes the largest integer smaller than or equal to α. For a sequence of scalars or vectors $s = \{s(k)\}_{k\in\mathbb{N}_0}$, we use s^+ to denote the shifted sequence: $s^+(k) = s(k+1)$.

Given a set $\mathcal{S} \subseteq \mathbb{R}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. A partition of $\mathcal{S}$ is a finite family $\{\mathcal{S}_\mu\}_{\mu=1}^M$ of its *disjoint* subsets $\mathcal{S}_\mu \subseteq \mathcal{S}$ such that $\bigcup_{\mu=1}^M \mathcal{S}_\mu = \mathcal{S}$.

A matrix $A \in R^{N\times N}$ is row-stochastic if it is non-negative and the sum of the elements of each row of A is equal to 1. Given a row-stochastic matrix A, A^k converges to a rank-one (equal-row) matrix as $k \to \infty$ if and only if there exists a $h \in \mathbb{N}$ such that A^h has a positive column. The trace of the square matrix A, denoted $\mathrm{tr}(A)$, is the sum of the entries on the main diagonal of A (or, equivalently, the sum of its eigenvalues).

In the following, $\mathcal{G} = \{\mathcal{I}, \mathcal{E}\}$ stands for a (directed) graph[1] with $\mathcal{I}$ the node set and $\mathcal{E}$ the edge set. The neighbor set of a node i is defined as $\mathcal{N}_i = \{j \in \mathcal{I} \mid (j, i) \in \mathcal{E}\}$. A graph is *weighted* if a non-negative weight is associated to each edge. A finite sequence of graphs $\mathcal{G}(1), \mathcal{G}(2), \dots, \mathcal{G}(M)$, $M \in \mathbb{N}$, $M \geq 2$, with the same node set is *union* (*composition*) *jointly rooted* if their union (composition) is rooted. An infinite sequence of graphs with the same node set is *repeatedly union* (*composition*) *jointly rooted* if there exist contiguous, nonempty, bounded, finite time-intervals $[k_i, k_{i+1})$, $i \geq 0$, starting at $k_0 \in \mathbb{N}_0$, such that each finite sequence $\mathcal{G}(k_i), \mathcal{G}(k_i+1), \dots, \mathcal{G}(k_{i+1})$ is jointly union (composition) rooted.

[1] For the reader's convenience, basic concepts of graph theory are summarized in Appendix A.

2.1. Opinion formation via iterative averaging

In this section, we introduce a sufficiently general class of discrete-time opinion formation models that are referred to as models of *iterative averaging*. At each time-step $k \in \mathbb{N}_0$, a social network consists of $N \in \mathbb{N}$ agents, whose opinions are represented through scalar state variables $x_i \in \mathbb{R}$, $i \in \mathcal{I}$, and a graph $\mathcal{G}(k) = (\mathcal{I}, \mathcal{E}(k))$ whose edge (i, j) represents *influence* of the agent i on the agent j. The graph $\mathcal{G}(k)$ determines the agents' neighbor sets $\mathcal{N}_i(k)$, and, vice versa, the knowledge of all neighbor sets $\mathcal{N}_i(k)$ allows to determine $\mathcal{G}(k)$. In the models considered below, we most typically deal with the situation that the dynamic graph is determined by the opinions. Choosing specific rules for the selection of the neighbors, e.g., the tendency to connect with similar agents represents the so-called *homophily* principle (McPherson et al., 2001). Some works also consider agents that prefer to connect with agents with dissimilar opinions (heterophilic agents) (Yokomatsu & Kotani, 2020).

2.1.1. The French-type model

The simplest iterative averaging procedure that was proposed (in the case of a static graph) by French (1956) is described by the difference equations

$$x_i^+ = \frac{1}{|\mathcal{N}_i|} \sum_{j\in\mathcal{N}_i} x_j \tag{1}$$

for all $i \in \mathcal{I}$, with initial conditions $x_i(0) \in \mathbb{R}$, $i \in \mathcal{I}$. For the sake of brevity, time is omitted here: $x_i := x_i(k)$, $x_i^+ := x_i(k+1)$, $x_i^{+h} := x_i(k+h)$, $\mathcal{N}_i := \mathcal{N}_i(k)$. The neighbor set $\mathcal{N}_i$ of the agent i in the models considered below includes i; hence, $|\mathcal{N}_i| \geq 1$ for all $i \in \mathcal{I}$, and the model (1) can thus be rewritten as

$$x_i^+ = x_i + \frac{1}{|\mathcal{N}_i|} \sum_{j\in\mathcal{N}_i} (x_j - x_i). \tag{2}$$

The structure (2) highlights that the evolution of each agent's opinion is driven by the deviations between its own and the neighbors' opinions (Tangredi, Iervolino, & Vasca, 2016). In particular, by considering pairwise interactions on the right-hand side of (2), one could say that each neighbor agent $j \in \mathcal{N}_i$ tends to attract the opinion of the agent i towards its own opinion, in the sense that it contributes to increase (decrease) x_i if $x_j > x_i$ ($x_j < x_i$).

2.1.2. A compact matrix form: DeGroot-type model

It is convenient to rewrite the system (1) in a compact matrix form by introducing the opinion vector $\mathbf{x} \in \mathbb{R}^N$

$$\mathbf{x} := \begin{pmatrix} x_1 & x_2 & \dots & x_N \end{pmatrix}^\top, \tag{3}$$

and defining the (time-varying) adjacency matrix $A := A(k)$ whose entries are

$$a_{ij} := \begin{cases} \frac{1}{|\mathcal{N}_i|}, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Obviously, the matrix defined in such a way is row-stochastic. The system (1) shapes into the equation

$$\mathbf{x}^+ = A\mathbf{x}. \tag{5}$$

It should be noticed that even if the graph $\mathcal{G}$ is undirected, the matrix defined in (4) is usually not symmetric, except for special situations (for instance, A is symmetric if $\mathcal{G}$ is undirected and regular, that is, all nodes have the same number of neighbors). Moreover, A need not be column-stochastic, and hence the dynamics (1) do not preserve the sum (or, equivalently, the average) of the opinions (Iervolino, Vasca, & Tangredi, 2018).

Some properties of the French model (1) remain valid for an *arbitrary* choice of row-stochastic weight matrices A. The system (5) with a general row-stochastic matrix is known as the DeGroot model (DeGroot, 1974). Whereas most BCOD models considered in this survey are special cases of the French model, some advanced models discussed in Section 10 are special cases of the DeGroot model (5) in which $A = A(k)$ differs from (4).

2.1.3. The range of opinions

It is easy to verify that in iterative averaging the opinions remain bounded between the minimum and maximum initial opinions for all time-steps.

Proposition 1. *Consider the opinion dynamics model (5). The opinions satisfy the inequalities*

$$x_i^+ \le \max_{j\in\mathcal{N}_i} x_j, \tag{6a}$$

$$x_i^+ \ge \min_{j\in\mathcal{N}_i} x_j \tag{6b}$$

for all $i \in \mathcal{I}$. Hence, the opinions remains bounded: $x_i \in [\min_{i\in\mathcal{I}} x_i(0), \max_{i\in\mathcal{I}} x_i(0)]$ for all $i \in \mathcal{I}$ and $k \in \mathbb{N}_0$.

In view of Proposition 1, the diameter of the convex hull spanned by the opinions (the *range of opinions* defined below) is nonincreasing and may serve for stability analysis. Also, the maximum and the minimum of the opinions converge to finite limits as $k \to \infty$.

Definition 2 (*Range of Opinions*)**.** The *range* of opinions constituting a vector $\mathbf{x} \in \mathbb{R}^N$ from (3) is

$$\upsilon(\mathbf{x}) = \max_{i\in\mathcal{I}} x_i - \min_{i\in\mathcal{I}} x_i = \max_{i,j\in\mathcal{I}} (x_i - x_j). \tag{7}$$

The inequality $\upsilon(\mathbf{x}^+) \le \upsilon(\mathbf{x})$ following from Proposition 1 can be further tightened using the relation coming from Markov chains theory and involving the so-called *ergodicity coefficient* $\tau_1(A)$ (Seneta, 1980, Theorem 3.1)

$$\tau_1(A) := 1 - \min_{i,j\in\mathcal{I}} \sum_{w\in\mathcal{I}} \min\{a_{iw}, a_{jw}\}. \tag{8}$$

Proposition 3. *Consider the opinion dynamics model (5) where A is row-stochastic. The range of opinions satisfies the inequality*

$$\upsilon(\mathbf{x}^+) \le \tau_1(A)\upsilon(\mathbf{x}) \tag{9}$$

for all $k \in \mathbb{N}_0$.

2.1.4. Asymptotic consensus

In the systems and control literature, the system (5) is often referred to as the (*first-order* discrete-time) consensus protocol (Cao, Morse, & Anderson, 2008; DeGroot, 1974; Ren & Beard, 2005). Reaching an asymptotic consensus as a result of the iterative opinion averaging is one of the important questions (although, as will be discussed later, BCOD often fails to reach consensus and converges to a clustering equilibrium). The following sufficient consensus criterion is broadly known in the literature.

Lemma 4. *If the time-varying neighbor sets $\mathcal{N}_i(k)$ for all $i \in \mathcal{I}$ determine a repeatedly composition jointly rooted sequence of graphs $\mathcal{G}(k)$, then each solution of the dynamical system (1) enjoys the* asymptotic consensus *of the opinions, i.e., there exists $c \in \mathbb{R}$ such that*

$$\lim_{k\to\infty} x_i(k) = c \tag{10}$$

for all $i \in \mathcal{I}$, where the convergence is exponentially fast.

Lemma 4 has been proved by Cao et al. (2008) (Theorem 3). As discussed in Cao et al. (2008) (pp. 596–597), the presence of a self-loop at each node allows one to replace the words "repeatedly composition jointly rooted" by "repeatedly union jointly rooted" (which is easier to be validated in practice). In this form, Lemma 4 in fact extends to a more general class of DeGroot models (5) with time-varying matrices $A = A(k)$. We refer the reader to the surveys Proskurnikov et al. (2020), Proskurnikov and Tempo (2018) for details.

Remark 5. Obviously, if an asymptotic consensus (10) is reached, then $\upsilon(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$. The inverse implication is also valid due to Proposition 1. Indeed, the limits $\bar{M} = \lim_{k\to\infty} \max_{i\in\mathcal{I}} x_i(k)$ and $\bar{m} = \lim_{k\to\infty} \min_{i\in\mathcal{I}} x_i(k)$ exist. It is obvious that $\upsilon(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} \bar{M} - \bar{m}$. Hence, if $\upsilon(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$, then $\bar{M} = \bar{m}$, which entails (10) (where $c = \bar{M} = \bar{m}$).

2.1.5. Convergence under reciprocal interactions

As has been discussed, the Abelson's diversity puzzle has motivated the development of models that portray not only consensus but also different opinions at steady state. If such a model inherits the structure of the DeGroot model (5), then the sufficient condition from Lemma 4, obviously, should fail to hold for some solutions. A natural question then arises on whether the opinions converge. More formally, do the limits in (10) always exist or can some solutions fail to have a limit and persistently oscillate[2]?

Although the condition of the asymptotic consensus is stronger than the convergence of each opinion, consensus conditions are, paradoxically, studied much better than general convergence criteria that are known only in special situations. One of these situations is the case of *reciprocal* interactions.

Lemma 6. *Suppose that the stochastic matrices $A(k)$ in (5) are type-symmetric, that is, there exists $K \ge 1$ such that*

$$K^{-1} a_{ji}(k) \le a_{ij}(k) \le K a_{ji}(k) \tag{11}$$

for all $i, j \in \mathcal{I}$ and $k \in \mathbb{N}_0$. Suppose also that the diagonal entries of all matrices are uniformly positive, i.e., $a_{ii}(k) \ge \delta > 0$ for all $i \in \mathcal{I}$ and $k \in \mathbb{N}_0$. Then, the sequence $\mathbf{x}(k)$ has a limit $\mathbf{x}^\infty = \lim_{k\to\infty} \mathbf{x}(k)$. Furthermore, $\mathbf{x}_i^\infty = \mathbf{x}_j^\infty$ for any pair of agents such that

$$\sum_{k=0}^{\infty} a_{ij}(k) = \infty. \tag{12}$$

Lemma 6 is a special case of a more general convergence criteria established in Bolouki and Malhame (2015) (Theorem 1) whose alternative proof is in Proskurnikov et al. (2020) (Theorem 5). The type-symmetry condition (11) means that if i influences j at some period k, then j also influences i and the intensity of these mutual influences are commensurate. In reality, this condition can be substantially relaxed and replaced by a so-called "cut balance" condition stating that if a group of agents $\mathcal{I}_1 \subset \mathcal{I}$ influences the remaining agents from the group $\mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1$, then $\mathcal{I}_2$ also influences $\mathcal{I}_1$ and the mutual influences of the two groups are commensurate. The convergence criterion expressed by Lemma 6 has been generalized by Xia, Shi, Meng, Cao, and Johansson (2019), where the type-symmetry and cut balance conditions are replaced by their non-instantaneous relaxations. The convergence criterion from Lemma 6 can be restated as follows.

[2] Recall that opinions remain bounded due to Proposition 1, thus $-\infty < \liminf_{k\to\infty} x_i(k) \le \limsup_{k\to\infty} x_i(k) < \infty$. If the middle inequality is strict, the sequence $x_i(k)$ oscillates between its lower and its upper limits as $k \to \infty$.

Corollary 7. *If the assumptions of Lemma 6 hold, then the backward matrix products converge, i.e., there exists* $A^\infty = \lim_{k\to\infty}(A(k)A(k-1)\dots A(0))$. *Furthermore, if the condition in (12) holds, then the i-th and j-th rows of* A^∞ *are identical.*

Notice that in BCOD models (and other models of opinion formation) the matrices $A(k)$ usually satisfy an additional restriction: their nonzero entries are *uniformly* positive, i.e., a constant $\eta > 0$ exists such that

$$a_{ij}(k) \in \{0\} \cup [\eta, 1] \tag{13}$$

for all $i, j \in \mathcal{I}$ and $k \in \mathbb{N}_0$. Under this extra assumption, Lemma 6 and Corollary 7 have been first established by Lorenz (2005); alternative proofs can be found in Proskurnikov and Tempo (2018) (Lemma 1), Blondel, Hendrickx, Olshevsky, and Tsitsiklis (2005) (Theorem 5) and Frasca and Fagnani (2018) (Theorem 3.2). If (11) and (13) hold, then the condition (12) admits a simple interpretation: the arcs (j, i) and (i, j) appear in the sequence of graphs $\{\mathcal{G}(k)\}_{k=0}^\infty$ infinitely often. A pair of agents satisfying (12) is thus said to interact *persistently*; connecting such pairs by edges, one obtains a *persistent graph*.

Definition 8 (*Persistent Graph*)**.** The graph $\mathcal{G}^\infty = \{\mathcal{I}, \mathcal{E}^\infty\}$, where $\mathcal{E}^\infty = \{(j, i) \mid (12) \text{ holds}\}$, is called the persistent graph of the sequence $\{A(k)\}_{k\in\mathbb{N}_0}$.

Trivially, the persistent graph is undirected when (11) holds. Lemma 6 entails the following simple corollary.

Corollary 9. *If the assumptions of Lemma 6 hold, then the steady opinions of two agents i and j that belong to the same connected component of* $\mathcal{G}^\infty$ *coincide, i.e.,* $x_i^\infty = x_j^\infty$.

3. Bounded confidence opinion dynamics

In this section, we formally introduce BCOD and some related definitions. Like in the previous section, we confine ourselves with *scalar* opinions; a brief discussion on the multidimensional case is provided in Section 10. As we will see, scalar-valued BCOD enjoys a number of properties that fail to hold in the multidimensional case and are proved by using special mathematical tools.

3.1. Definitions

BCOD models manifest the homophily principle and stipulate that at each time-step each agent selects as neighbors those agents who have an opinion belonging to its *confidence* interval (Hegselmann & Krause, 2002). In particular, the set of the indices of the agents who are neighbors of the i-th agent is given by

$$\mathcal{N}_i = \mathcal{N}_i(\mathbf{x}) := \{j \in \mathcal{I} \mid -\ell_i \le x_j - x_i \le u_i\}, \tag{14}$$

with $\ell_i \in \mathbb{R}_0^+$ and $u_i \in \mathbb{R}_0^+$ called lower and upper confidence thresholds of the agent i, respectively.

Definition 10 (*BCOD*)**.** BCOD (with scalar opinions) is the procedure of iterative averaging (1), where the neighbor set of each agent i is defined by (14).

The interval $[x_i - \ell_i, x_i + u_i]$ is called *confidence interval* of the agent i, $i \in \mathcal{I}$. An agent such that $\ell_i = 0$ ($u_i = 0$) is said to be *one-sided confident*. An agent i is said to be *stubborn*[3] if $\ell_i = u_i = 0$.

[3] Stubborn agents are sometimes called *radicals* (Hegselmann & Krause, 2015) or *zealots* (Verma, Swami, & Chan, 2014).

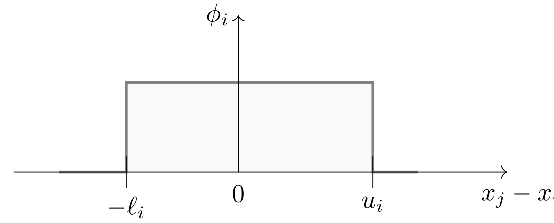

Fig. 1. An example of influence function $\phi_i(x_i, x_j)$.

For BCOD, the cardinality of the set $\mathcal{N}_i$ is given by

$$|\mathcal{N}_i(\mathbf{x})| = \sum_{j\in\mathcal{I}} \phi_i(x_i, x_j), \tag{15}$$

where $\phi_i : \mathbb{R}^2 \to \{0, 1\}$, $i, j \in \mathcal{I}$, is the *influence function* (see Fig. 1) defined as

$$\phi_i(x_i, x_j) = \begin{cases} 1, & \text{if } -\ell_i \le x_j - x_i \le u_i, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Substituting (15) and (16) into (1), BCOD is written as

$$x_i^+ = \frac{1}{\sum_{j\in\mathcal{I}} \phi_i(x_i, x_j)} \sum_{j\in\mathcal{I}} \phi_i(x_i, x_j) x_j \tag{17}$$

for all $i \in \mathcal{I}$, with $\phi_i(x_i, x_j)$, $i, j \in \mathcal{I}$, defined in (16). The compact matrix form (5) of BCOD is as follows

$$\mathbf{x}^+ = A(\mathbf{x})\mathbf{x}, \tag{18}$$

where the adjacency matrix $A(\mathbf{x})$ is given by

$$A(\mathbf{x}) := \Theta(\mathbf{x})^{-1}\Phi(\mathbf{x}). \tag{19}$$

Here, $\Theta(\mathbf{x})$ is the diagonal matrix whose main diagonal entries are $\Theta(\mathbf{x})_{ii} = |\mathcal{N}_i(\mathbf{x})|$, i.e., the number of neighbors of the agent i, and $\Phi(\mathbf{x})$ is the $N \times N$ matrix whose entries are $\Phi(\mathbf{x})_{ij} = \phi_i(x_i, x_j)$. The model (17) can be also rewritten in a mixed logical dynamical form (Bernardo & Vasca, 2020) and as a system with a piecewise affine structure (Iervolino et al., 2016).

Henceforth, the symbol $\mathcal{G}(\mathbf{x})$ denotes the graph determined by the neighbor sets (14) whose definition is as follows.

Definition 11 (*Confidence Graph*)**.** The confidence graph of the opinion vector $\mathbf{x}$ is $\mathcal{G}(\mathbf{x}) = (\mathcal{I}, \mathcal{E}(\mathbf{x}))$ such that $(j, i) \in \mathcal{E}(\mathbf{x})$ if $\phi_i(x_i, x_j) = 1$, and $(j, i) \notin \mathcal{E}(\mathbf{x})$ otherwise, $i, j \in \mathcal{I}$.

Definition 11, together with (16), implies that any node of a confidence graph has a self-loop.

3.2. Classification of BCOD

The properties of BCOD substantially depend on the structure of confidence thresholds ℓ_i and u_i, $i \in \mathcal{I}$; the following classes of BCOD (17) are distinguished.

Definition 12 (*Classes of BCOD*)**.** BCOD (17) is

- *symmetric* if the confidence interval of each agent is centered at the agent's opinion, i.e., $\ell_i = u_i$ for all $i \in \mathcal{I}$ (possibly, $\ell_i \ne \ell_j$ for some $i \ne j$), and *asymmetric* otherwise;
- *homogeneous* if all agents have the same confidence interval, i.e., $\ell_i = \ell$ and $u_i = u$ for all $i \in \mathcal{I}$ (possibly, $\ell \ne u$), and *heterogeneous* otherwise.

An asymmetric heterogeneous BCOD with $\ell_i = \ell$ or, alternatively, $u_i = u$ for all $i \in \mathcal{I}$ is called *one-sided heterogeneous*. An asymmetric BCOD where all agents are one-sided confident, i.e., $\ell_i = 0$ ($u_i = 0$) for all $i \in \mathcal{I}$, is called *one-sided confident* (such BCOD can be homogeneous or heterogeneous).

For symmetric homogeneous BCOD, the adjacency matrix $A(\mathbf{x})$ in (18) is *type-symmetric*, i.e., the conditions in (11) hold, and, in particular, Lemma 6 and Corollary 7 are applicable. This case has been most studied in the literature (Bhattacharyya, Braverman, Chazelle, & Nguyen, 2013; Blondel et al., 2009; Dittmer, 2001; Krause, 2000; Lorenz, 2005; Sun, 2010).

3.3. Technical definitions

In this subsection, we collect several definitions that will be used in the subsequent analysis of BCOD.

During the opinion evolution, an agent's opinion can increase or decrease depending on the neighbors' opinions. It appears, however, that some BCOD models preserve the *order* of opinions, so an agent's opinion cannot "jump over" the opinions of its neighbors. We give a formal definition.

Definition 13 (*Order-preservation*)**.** The opinion dynamics model (1) is said to satisfy the order-preservation property if $(x_i - x_j)(x_i^+ - x_j^+) \geq 0$ for any $i, j \in \mathcal{I}$ and $k \in \mathbb{N}_0$.

As it will be discussed later on, this property, substantially simplifying the system's analysis, is not satisfied by some types of BCOD. In general, validation of this property is non-trivial; however, one class of order-preserving BCOD, generalizing the classical Hegselmann–Krause model, has been considered by Hendrickx (2008).

Silent agents are agents who are in consensus with all their neighbors (and thus do not change the opinion at the next time-step) but are not in consensus with some other agents. A silent agent, for instance, can have no neighbors at all, and in this case we call it isolated. The formal definition is as follows.

Definition 14 (*Silent and Isolated Agents*)**.** An agent $i \in \mathcal{I}$ is said to be silent at the time-step k if $x_j = x_i$ for all $j \in \mathcal{N}_i$ and there exists a $q \in \mathcal{I} \setminus \mathcal{N}_i$ with $x_q \neq x_i$. In particular, a silent agent $i \in \mathcal{I}$ is said to be isolated at the time-step k if $\mathcal{N}_i = \{i\}$.

The definitions of crack and d-chain allow one to relate the values of the opinions to the connectivity of the confidence graph. If two agents are connected at some time-step (i.e., at least one of them influences the other) and become disconnected at the next time-step, we say that a crack has occurred between these two agents. More formally, one can provide the following definition.

Definition 15 (*Crack*)**.** Consider the opinion dynamics model (1) and any two agents who are connected at the time-step k, i.e., $i, j \in \mathcal{I}$ with $i \neq j$ such that $i \in \mathcal{N}_j$ and/or $j \in \mathcal{N}_i$. If at the next time-step the two agents are disconnected, i.e., $i \notin \mathcal{N}_j^+$ and $j \notin \mathcal{N}_i^+$, a *crack* has occurred between the agents i and j at the time-step $k+1$.

By using Definitions 13 and 15, one can show that if the order-preservation property holds, then the possible increase in the number of components of the confidence graph at each step is equal to the number of cracks occurring at that time-step.

Definition 16 (*d-chain*)**.** Consider the opinion dynamics model (1). Say $x_1 \leq x_2 \leq \cdots \leq x_N$. The opinion vector (3) is a d-chain at the time-step k if $x_{i+1} - x_i \leq d$ for all $i \in \mathcal{I} \setminus \{N\}$, i.e., the distance between adjacent opinions is less than or equal to d.

The concept of d-chain in Definition 16 can be related to the connectivity of the confidence graph by using the maximum and minimum confidence thresholds:

$$d_M = \max_{i \in \mathcal{I}} \max\{\ell_i, u_i\}, \tag{20a}$$

$$d_m = \min_{i \in \mathcal{I}} \min\{\ell_i, u_i\}. \tag{20b}$$

In the case of homogeneous BCOD, the opinion vector is a d_m-chain if and only if the confidence graph is strongly connected. Moreover, the opinion vector is a d_M-chain if and only if the confidence graph is weakly connected.

For heterogeneous BCOD, the situation is more involved. If the opinion vector is a d_m-chain, then the confidence graph is strongly connected, but the opposite implication does not hold. If the opinion vector is a d_M-chain, then the confidence graph might be either (weakly or strongly) connected or disconnected. Conversely, assuming the connectivity of the graph, one can only say that if the confidence graph is weakly connected then the opinion vector is a d-chain for some $d \leq d_M$. Nothing more can be said, in general, under strong connectivity.

4. Steady state behaviors

In this section, we are interested in the analysis of steady state behaviors of BCOD.

One typical situation is when all opinions converge to the same value, which is called *a consensus opinion*, and can be formally defined as follows.

Definition 17 (*Consensus*)**.** A consensus state is a vector $\mathbf{x}$ of BCOD (17) at which all opinions are coincident, i.e., $x_i = \bar{c}$ for all $i \in \mathcal{I}$ with some $\bar{c} \in \mathbb{R}$. The value $\bar{c}$ is said to be the consensus opinion of the agents.

Consensus states possess several important properties that are summarized in the following remark.

Remark 18. According to (14) and (17), each consensus state is an equilibrium of BCOD. From (7), it directly follows that consensus states are opinion vectors of zero range, i.e., $\upsilon(\mathbf{x}) = 0$. Furthermore, a solution of BCOD (17) converges to a consensus if and only if $\upsilon(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$ (see Remark 5).

Clearly, at a consensus state every pair of agents is connected, i.e., the graph $\mathcal{G}(\mathbf{x})$ is complete, and then the adjacency matrix $A(\mathbf{x})$ from (19) has rank one. On the other hand, if the graph $\mathcal{G}(\mathbf{x})$ is complete at some time-step k, then $\mathbf{x}^+ = \mathbf{x}(k+1)$ is a consensus state in view of (17). Therefore, an equilibrium $\mathbf{x} = \mathbf{x}^+$ is a consensus if and only if its graph is complete.

Another situation of interest is when a subgroup of agents is disconnected from the other agents and the subgraph of $\mathcal{G}(\mathbf{x})$ consisting of the agents of each subgroup is a weakly connected component of $\mathcal{G}(\mathbf{x})$. Note that this situation applies to every opinion vector, which need not be a steady state of the model (17). On the other hand, a typical steady state behavior exhibited by BCOD is the coincidence of the opinions in each disconnected subgroup of agents. This situation is called *clustering*[4] and requires considering not only the values of the opinions but also the confidence graph.

Definition 19 (*Clustering*)**.** A clustering is an opinion vector $\mathbf{x}$ of BCOD (17) for which there exists a partition $\{\mathcal{C}_\mu\}$ of $\mathcal{I}$ and a sequence of pairwise different constants $\{\bar{c}_\mu\}$ (indexed by $\mu = 1, \ldots, M$, with $M \geq 2$) such that the subgraph consisting of the agents in $\mathcal{C}_\mu$ is a cluster[5] and $x_i = \bar{c}_\mu$ for all $i \in \mathcal{C}_\mu$.

[4] If a solution converges to a clustering equilibrium, it is sometimes said that opinion dynamics exhibits *asymptotic clustering* (Altafini & Ceragioli, 2018) or *fragmentation* (Hegselmann & Krause, 2002; Su, Guo, Chen, Chen, & Li, 2019).

[5] A *cluster* is a complete isolated component of a graph.

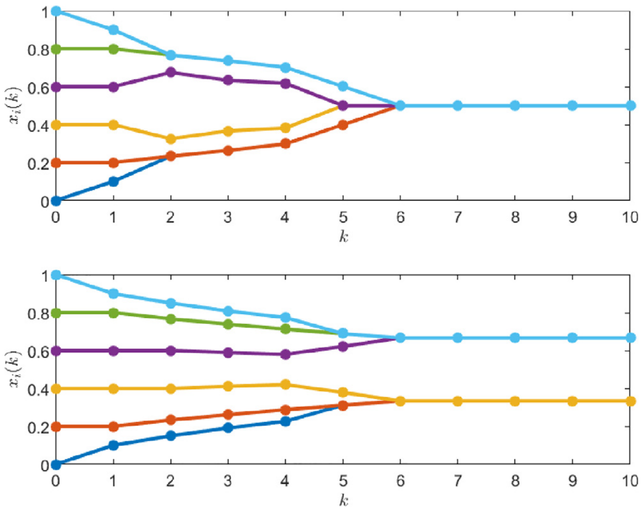

Fig. 2. Opinions of the BCOD in Example 20.

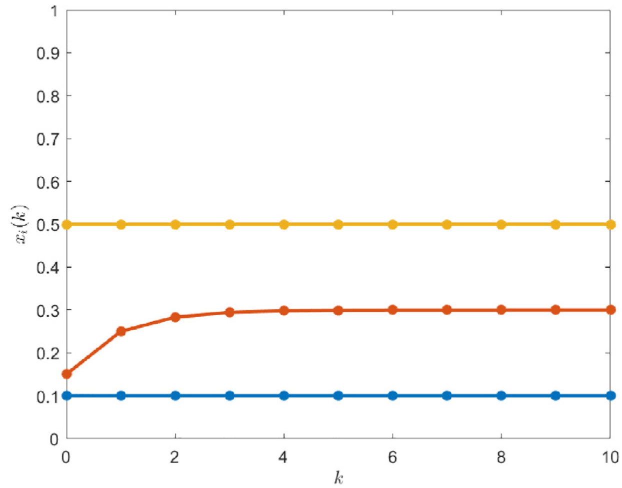

Fig. 3. Opinions of the BCOD in Example 21.

It can be easily shown that clustering is an equilibrium point of BCOD. Indeed, $\phi_i(x_i, x_j) = 1$ for any pair $i, j \in \mathcal{I}$ for which it is $x_i = x_j$. Definition 19 implies that at any clustering it is $\phi_i(x_i, x_j) = 1$ for any $i, j \in \mathcal{C}_\mu$ and $\phi_i(x_i, x_j) = 0$ for any $i \in \mathcal{C}_{\mu_1}$, $j \in \mathcal{C}_{\mu_2}$, $\mu_1 \neq \mu_2$. By definition, a clustering is not a consensus state.

The next example illustrates consensus and clustering.

Example 20. Consider the homogeneous BCOD with $N = 6$ agents, $x_i(0) = 0.2(i-1)$, $i = 1, \ldots, 6$, $\ell_i = u_i = 0.35$ for all $i \in \mathcal{I}$. The corresponding simulation results are shown in the upper plot of Fig. 2. A consensus is reached at $k = 6$ with opinion 0.5, which is the average of the initial conditions. By considering the same initial conditions but $\ell_i = u_i = 0.2$ for all $i \in \mathcal{I}$, the results are those reported in the lower plot of Fig. 2. In this second case, a clustering is reached at $k = 6$, and the agents are partitioned into two clusters with opinion values 0.334 and 0.666.

In the sequel, we will show that *homogeneous* BCOD cannot have equilibria different from consensus and clustering. In general, BCOD may have other types of equilibria, as shown by the following example.

Example 21. Consider the heterogeneous BCOD with $N = 3$ agents and an opinion vector $\mathbf{x}$ such that $x_1 < x_3$, $x_2 = (x_1+x_3)/2$, $x_3 - x_2 > \ell_3$, $x_2 - x_1 > u_1$, $x_2 - x_1 \leq \ell_2$ and $x_3 - x_2 \leq u_2$. It is easy to verify that this is an equilibrium, and clearly it is not a consensus state (the opinions are pairwise different), but it is also not a clustering because the agents 1 and 3 are isolated and the agent 2 is connected with the other two (edges incident to 2); hence, the graph $\mathcal{G}(\mathbf{x})$ is weakly connected and not rooted. Fig. 3 shows a transient with $x_1(0) = 0.1$, $x_2(0) = 0.15$, $x_3(0) = 0.5$, $\ell_1 = u_1 = \ell_3 = u_3 = 0$, $\ell_2 = u_2 = 0.4$; the opinions converge asymptotically to the above equilibrium point. Clearly, by starting with the same initial condition, the simulation results will not change if one selects $u_1 < 0.05$, $\ell_3 < 0.2$ and arbitrary ℓ_1 and u_3.

An intuitive conjecture arises that a general BCOD cannot exhibit a non-constant steady state behavior, that is, oscillate persistently; to the best of the authors' knowledge, a formal proof of this result is still missing for heterogeneous BCOD. On the other hand, for some classes of BCOD it is possible to provide sufficient conditions for the convergence of the opinion vector to a consensus or a clustering state in finite time. In these situations, one can define the convergence time[6] as follows.

Definition 22 (*Convergence Time*)**.** Consider a BCOD (17) with given initial opinions $\mathbf{x}(0)$. If the system reaches a constant steady state in finite time, then the convergence time is the minimum (finite) time-step $\bar{k} \in \mathbb{N}_0$ such that $x_i^+ = x_i$ for all $i \in \mathcal{I}$ and $k \geq \bar{k}$.

On the other hand, there are examples of BCOD for which the convergence to a constant steady state is reached only asymptotically. In these situations, one may look for conditions of convergence in finite time to regions around a consensus or a clustering state. To this aim, we introduce the concepts of *practical consensus* and *practical clustering* proposed by Bernardo, Vasca and Iervolino (2021) and Vasca et al. (2021). According to the following definitions, the agents in a practical consensus or a subgroup of agents of a practical clustering are connected and share a small interval of opinion values rather than a unique one.

Definition 23 (*ϵ-practical Consensus*)**.** Given a sufficiently small $\epsilon > 0$, a solution $\mathbf{x}(k)$ of BCOD (17) is said to have reached an ϵ-practical consensus if there exists a finite $\bar{k} \in \mathbb{N}_0$ such that $v(\mathbf{x}) \leq \epsilon$ and each graph $\mathcal{G}(\mathbf{x}(k))$ is weakly connected for any $k \geq \bar{k}$.

Definition 23 directly implies that if the opinions belong to an ϵ-practical consensus, then the inequalities

$$|x_i - x_j| \leq \epsilon \tag{21}$$

hold for all $i, j \in \mathcal{I}$. It is evident from (21) that the range of opinions at a practical consensus is such that $v(\mathbf{x}) \leq \epsilon$. Since $v(\mathbf{x})$ is nonincreasing, the set of opinion vectors corresponding to ϵ-practical consensus is forward invariant under BCOD dynamics. For homogeneous BCOD, if $\epsilon \leq d_M$ then the condition (21) implies that the confidence graph is weakly connected (complete for the symmetric case) and remains weakly connected (respectively, complete) for all future time-steps. In order to have this implication in the case of heterogeneous BCOD, one should consider $\epsilon \leq d_m$.

A consensus (Definition 17) is a special case of an ϵ-practical consensus. However, differently from the consensus, Definition 23 does not imply that the vector $\mathbf{x}$ is an equilibrium point of BCOD.

[6] The convergence time is sometimes called *dynamics termination time* (Etesami & Başar, 2015; Touri & Nedić, 2011).

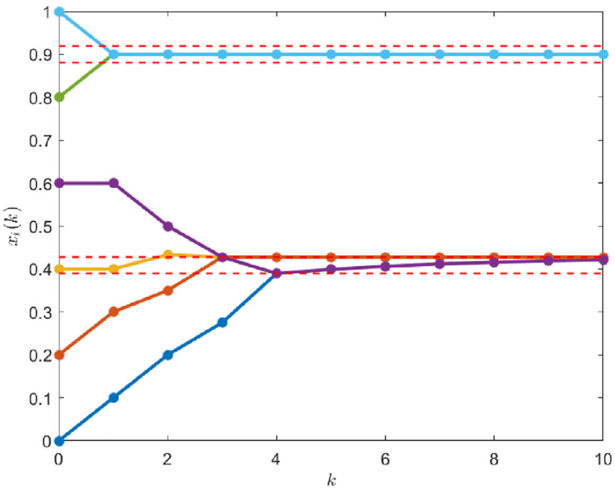

Fig. 4. Opinions of the BCOD in Example 25. The dashed lines correspond to intervals of amplitude $\epsilon = 0.0382$.

If the opinion vector reaches an ϵ-practical consensus, then it cannot reach any clustering in future time-steps because the corresponding graph would be disconnected. On the other hand, the following definition can be given.

Definition 24 (*ϵ-practical Clustering*)**.** Given a sufficiently small $\epsilon > 0$, a solution $\mathbf{x}(k)$ of BCOD (17) is said to have reached an ϵ-practical clustering if there exist a finite $\bar{k} \in \mathbb{N}_0$ and a partition $\{\mathcal{C}_\mu\}$ of $\mathcal{I}$ (indexed by $\mu = 1, \dots, M$, with $M \geq 2$) such that $v(\mathbf{x}_\mu) \leq \epsilon$, where $\mathbf{x}_\mu \subset \mathbf{x}$ is the opinion vector of the agents belonging to $\mathcal{C}_\mu$, and the subgraph consisting of the agents in $\mathcal{C}_\mu$ is a weakly connected component of $\mathcal{G}(\mathbf{x}(k))$ for any $k \geq \bar{k}$.

The ϵ-practical clustering is a sort of generalization of the clustering concept in the sense of Definition 19. The following example shows a situation in which clustering is reached only asymptotically, but a practical clustering is reached in finite time.

Example 25. Consider the BCOD with $N = 6$ agents, $x_i(0) = 0.2(i-1)$, $i = 1, \dots, 6$, $\ell_i = u_i = 0.2$ for all $i \in \mathcal{I}$ except for $\ell_2 = \ell_5 = 0$. The result of a numerical simulation is shown in Fig. 4. At $k = 0$, the graph is connected. The BCOD converges asymptotically to an equilibrium with two clusters: the agents 5 and 6 reach consensus at opinion 0.9, while the remaining agents converge to the terminal opinion 0.4278. However, an ϵ-practical clustering with $\epsilon = 0.0382$ is reached at $k = 4$.

5. Common properties

The propositions recalled in this section express properties which are valid for all classes of BCOD.

The following proposition shows that if a crack occurs (or has occurred) at some time-step between two adjacent agents with a relative distance larger than d_M, then the two agents will not influence each other in the future.

Proposition 26. *Consider BCOD (17). If there exist a time-step $k \in \mathbb{N}_0$ and $i, j \in \mathcal{I}$ such that $x_j - x_i > d_M$ and there does not exist any agent $q \in \mathcal{I}$ such that $x_q \in (x_i, x_j)$, then $x_j^+ - x_i^+ > d_M$.*

The proof of Proposition 26 is reported in Bernardo, Altafini, and Vasca (2022) (Lemma 13) for the homogeneous case and can be easily extended to the heterogeneous case by using similar arguments. A direct consequence of Proposition 26 is that if a crack with a relative distance larger than d_M occurs, no consensus can be reached.

A classical criterion for the asymptotic convergence to a consensus is based on the connectivity of the graph, as shown by the following result which is implied by a consensus criterion from Cao et al. (2008).

Theorem 27. *Consider BCOD (17). An asymptotic consensus (10) is reached in BCOD by a given solution $\mathbf{x}(k)$ if and only if the infinite sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ is repeatedly composition jointly rooted. Moreover, if a solution to BCOD (17) reaches asymptotic consensus (10), then $\mathcal{G}(\mathbf{x}(k))$ is rooted for k large enough.*

Recall that the strong connectivity of each graph along a solution of (17) implies that any sequence of these graphs is repeatedly union (and composition) jointly rooted. The following example shows that the condition of *repeatedly* jointly rooted must be carefully considered.

Example 28. Consider the heterogeneous BCOD with $N = 3$ agents, $x_1(0) = 0.1$, $x_2(0) = 0.3$, $x_3(0) = 0.5$, $\ell_1 = \ell_2 = u_2 = \ell_3 = u_3 = 0.05$ and $u_1 = 0.3$. The opinions converge to a clustering at $k = 6$ with values 0.316 (for the agent 2) and 0.485 (for the agents 1 and 3). The sequence of confidence graphs is characterized by the following sets of edges:

$$\mathcal{E}(0) = \{(2, 1)\} \cup SL,$$
$$\mathcal{E}(1) = \{(2, 1), (3, 1)\} \cup SL,$$
$$\mathcal{E}(2) = \{(1, 2), (2, 1), (3, 1)\} \cup SL,$$
$$\mathcal{E}(3) = \mathcal{E}(4) = \{(3, 1)\} \cup SL,$$
$$\mathcal{E}(k) = \{(1, 3), (3, 1)\} \cup SL, \quad k \geq 5,$$

where the self-loops have been indicated with the set $SL = \{(1, 1), (2, 2), (3, 3)\}$. It is easy to verify that the union of the confidence graphs is rooted at the node 3 for any $k \geq 1$ and it is also strongly connected for any $k \geq 5$. By considering any interval $[0, \bar{k})$ with $\bar{k} \geq 5$, the corresponding union (and composition) of confidence graphs is jointly rooted. On the other hand, this condition is not satisfied for any sequence of confidence graphs starting after any $k \geq \bar{k}$, and then the repeatedly jointly rooted condition required for consensus in Theorem 27 does not hold.

The assumptions of Theorem 27 do not exclude the possibility that the confidence graph is disconnected at some time-step, as shown by the following example.

Example 29. Consider the heterogeneous BCOD with $N = 3$ agents, $x_1(0) = 0.1$, $x_2(0) = 0.3$, $x_3(0) = 0.5$, $\ell_1 = \ell_2 = u_2 = u_3 = 0.1$, $u_1 = 0.3$ and $\ell_3 = 0$. The results of a simulation are shown in Fig. 5. The solution exhibits an asymptotic consensus, but, as shown in Fig. 6, the agent 3 was initially isolated, i.e., the graph corresponding to the initial opinions was not connected. It is easy to verify that for this example the repeatedly jointly rooted condition holds.

The situation occurring in Example 29, i.e., having a confidence graph disconnected and then reaching a consensus later on, is not necessarily associated to the initial condition in the sense that the confidence graph may disconnect also during the system evolution.

Theorem 27 establishes sufficient conditions for asymptotic consensus. A natural question arising from this result is whether one can also prove finite-time convergence. Let us first consider the simple situation when all opinions are sufficiently close to each other. The following proposition shows that if the range of opinions is such that $v(\mathbf{x}) \leq d_m$, then a consensus is reached in at most one time-step (Bernardo et al., 2022, Lemma 11).

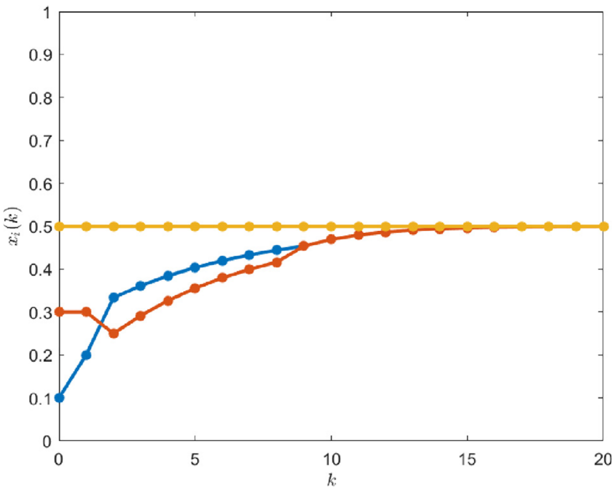

Fig. 5. Opinions of the BCOD in Example 29.

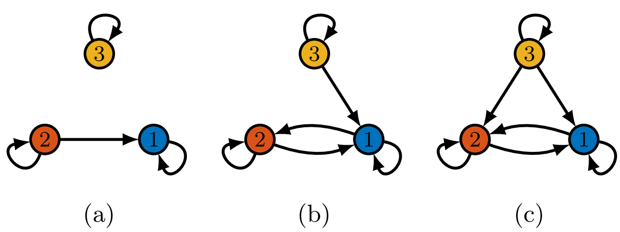

Fig. 6. Confidence graphs for the opinions in Fig. 5: at $k = 0$ (a), for $k = 1, \dots, 7$ (b), for any $k \geq 8$ (c).

Proposition 30. *Consider BCOD (17). If at some time-step $k \in \mathbb{N}_0$ it is $v(\mathbf{x}) \leq d_m$, then $x_i^+ = \bar{c}$ with $\bar{c} = \frac{1}{N}\sum_{i=1}^{N} x_i$, i.e., at the next time-step consensus corresponding to the average of the opinions is reached.*

The proof of Proposition 30 directly follows by considering that if $v(\mathbf{x}) \leq d_m$ in (17) it is $\phi_i(x_i, x_j) = 1$ for all $i, j \in \mathcal{I}$, i.e., $\mathcal{G}(\mathbf{x})$ is complete, and then the right-hand side of (17) is the same for all $i \in \mathcal{I}$. In particular, note that in Proposition 30 for $d_m = 0$ it is $v(\mathbf{x}) = 0$, which already corresponds to a consensus. Moreover, the condition $v(\mathbf{x}) \leq d_m$ is not necessary for reaching consensus in one step, e.g., consider the case $N = 3$, $x_3 - x_2 = x_2 - x_1 = d_m$, $\ell_2 = u_2 = d_m$ and $\ell_3 = u_1 \geq 2d_m$ which implies $x_i^+ = x_2$ for all $i \in \{1, 2, 3\}$. It should be noticed that in Example 29 the convergence is asymptotic; this is possible because $d_m = 0$, and then the condition $v(\mathbf{x}) \leq d_m$ (in this case is equivalent to $v(\mathbf{x}) = 0$) is never satisfied in finite time.

The following theorem shows that if all confidence thresholds are strictly positive, i.e., $d_m > 0$, then the sequence $v(\mathbf{x}(k))$ terminates in finite time and, therefore, the sequences $\max_{i\in\mathcal{I}} x_i(k)$ and $\min_{i\in\mathcal{I}} x_i(k)$ also terminate in finite time.

Theorem 31. *Consider BCOD (17) and assume that $d_m > 0$. There exists a finite time-step $\bar{k} \in \mathbb{N}_0$ such that $v(\mathbf{x}^+) = v(\mathbf{x})$ for all $k \geq \bar{k}$. In particular, if $v(\mathbf{x}) \leq 2d_m$ any solution $\mathbf{x}(k)$ converges to 2 clusters at most.*

A special case of Theorem 31 dealing with symmetric heterogeneous models has been proved in Su, Gu, Wang and Yu (2017) (see Theorem 3.2, Remark 3.5 and Corollary 3.6 therein). Theorem 31 shows that the evolution of the opinions will be confined in finite time to some level set of the range function $\{\mathbf{x} : v(\mathbf{x}) = const\}$. However, the solutions inside this set may not be constant.

The results in Theorems 27 and 31 allow one to provide conditions for the convergence to a consensus in finite time, as the following theorem states.

Theorem 32. *Consider BCOD (17) and assume that $d_m > 0$. A consensus is reached in finite time in BCOD by a given solution $\mathbf{x}(k)$ if and only if the sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ is repeatedly composition jointly rooted.*

From Theorem 32, the following result directly descends.

Corollary 33. *Consider BCOD (17) and assume that $d_m > 0$. A consensus is reached in finite time in BCOD by a given solution $\mathbf{x}(k)$ if each graph of the sequence $\mathcal{G}(\mathbf{x}(k))$ is strongly connected.*

This result, for the specific class of homogeneous BCOD, follows from Theorem 7.3 in Motsch and Tadmor (2014).

For symmetric homogeneous BCOD, if an agent becomes silent or isolated (Definition 14) during the time evolution, then it remains in the same condition for all future time-steps. This is not the case for the other classes of BCOD, e.g., see Example 29. The following theorem shows that the convergence to a consensus is guaranteed in finite time if no agent remains silent for any time-step (Etesami & Başar, 2015, Theorem 7).

Theorem 34. *Consider BCOD (17) and assume that $d_m > 0$. A consensus is reached in finite time in BCOD by a given solution $\mathbf{x}(k)$ if and only if there exists a finite time interval $\Delta \in \mathbb{N}$ such that no agent is silent for more than Δ consecutive time-steps.*

The theoretical results presented above can be summarized in the following implications:

$$\mathcal{G}(\mathbf{x}(k))^{\text{rcjr}} \overset{\text{Theorem 27}}{\Longleftrightarrow} \text{asymptotic consensus}$$
$$d_m > 0 \overset{\text{Theorem 31}}{\Longrightarrow} v(\mathbf{x}) \text{ constant in finite time}$$
$$d_m > 0;\ \mathcal{G}(\mathbf{x}(k))^{\text{rcjr}} \overset{\text{Theorem 32}}{\Longleftrightarrow} \text{consensus in finite time}$$
$$d_m > 0;\ \text{no permanent silent agents} \overset{\text{Theorem 34}}{\Longleftrightarrow} \text{consensus in finite time}$$

where $\mathcal{G}(\mathbf{x}(k))^{\text{rcjr}}$ stands for the sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ being repeatedly composition jointly rooted. By comparing the last two lines in the list above, it follows that, under the assumption $d_m > 0$, the condition $\mathcal{G}(\mathbf{x}(k))^{\text{rcjr}}$ is equivalent to the absence of permanently silent agents. Unfortunately, finding apriori conditions for ensuring the absence of permanently silent agents or that the graphs are repeatedly jointly rooted over the entire time evolution is anything but easy. On the other hand, by restricting the analysis to specific classes of BCOD one can provide sufficient conditions for the convergence in finite time to a consensus or a clustering, as shown in the sequel.

6. Homogeneous model

According to Definition 12, BCOD is said to be homogeneous if the agents have equal confidence intervals with the same thresholds, i.e., $\ell_i = \ell$ and $u_i = u$ for all $i \in \mathcal{I}$, with $\ell, u \in \mathbb{R}_0^+$. The influence functions (16) thus equal

$$\phi_i(x_i, x_j) = \phi(x_i, x_j) = \begin{cases} 1, & -\ell \leq x_j - x_i \leq u, \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

The homogeneity of the model allows one to prove that the order of the opinions is preserved.

Proposition 35. *BCOD (17) with influence function (22) enjoys the order-preservation property (see Definition 13).*

Proposition 35 derives from Lemma 2 in Krause (2000). A formal proof has been provided in Blondel et al. (2009) (Proposition 1) in the case where $\ell = u$.

The order-preservation property in Proposition 35 allows one to consider, without loss of generality, that in homogeneous BCOD the agents can be ordered such that $x_1 \le x_2 \le \cdots \le x_N$ for any time-step $k \in \mathbb{N}_0$.

6.1. Symmetric model

In this subsection, we consider symmetric homogeneous BCOD, i.e., opinion dynamics (17) with (22) and $\ell = u$.

6.1.1. Convergences to consensus and clustering

The convergence properties of symmetric BCOD are expressed by the following theorem, which shows that the opinions converge in finite time to a steady state (either a consensus or a clustering).

Theorem 36. *Consider BCOD (17) with influence function (22) and $\ell = u > 0$. A constant steady state which is either a consensus or a clustering is reached in finite time in BCOD by any solution $\mathbf{x}(k)$, and the corresponding convergence time $\bar{k}$ satisfies*

$$\bar{k} \le 2\left(N - 1 + \left\lceil \frac{2N^2}{d} v(\mathbf{x}(0)) \right\rceil\right) \tag{23}$$

with $d = \ell = u$. In particular, a consensus is reached if and only if $\mathbf{x}(k)$ is a d-chain with $d = \ell = u$ for all $k \in \mathbb{N}_0$.

Notice that the first statement of Theorem 36, that is, the finite time convergence of opinions, has been first proved in Dittmer (2001) (Theorem 9). This statement also follows from Lemma 6, because in the case $\ell = u$ the matrices $A(\mathbf{x})$ are type-symmetric; this alternative way of proving the finite-time convergence has been employed by Lorenz (2005). Both approaches, however, do not provide an upper bound on the convergence time.

The final part of Theorem 36 (consensus condition) has been first proved in Krause (2000) (Theorem 2). This statement is a direct consequence of Theorem 32 because for the symmetric homogeneous model the opinion vector is a d-chain if and only if the confidence graph is connected. A sufficient condition for the connectivity of the confidence graph for symmetric homogeneous BCOD is provided in Sun (2010) (Proposition 5.1), where it is proved that if $|\mathcal{N}_i(\mathbf{x}(0))| \ge \frac{N+1}{2}$ for all $i \in \mathcal{I}$, i.e., the number of neighbors of each agent at the initial time-step is larger than half of the number of agents, then the confidence graph remains connected.

The estimate for convergence time is a special case of a more general statement from Theorem 40 below, which appeared first in Bernardo et al. (2022) (Theorem 17). Sorting the opinions in the ascending order, one can assume, without loss of generality, that the distance between adjacent opinions does not exceed d (otherwise, BCOD decomposes into several smaller systems evolving independently). Then, $v(\mathbf{x}(0)) = O(N)$, and thus (23) estimates the worst-case convergence time as $O(N^3)$. On the other hand, if all agents are arbitrarily spaced but belong to a predefined interval, then $v(\mathbf{x}(0))$ is bounded and (23) implies that $\bar{k}$ is $O(N^2)$.

Theorem 36 does not provide any indication on the form of the steady state situation, e.g., the number of clusters that are reached. Clearly, by excluding the trivial case when the agents with the minimum and maximum opinions are isolated, an upper bound on the number of clusters is given by $\lfloor \frac{v(\mathbf{x}(0))}{d} \rfloor$. This bound is strict for arbitrary initial conditions. In fact, this number of clusters can be obtained by considering agents divided into $\lfloor \frac{v(\mathbf{x}(0))}{d} \rfloor$ subgroups and by taking initial opinions such that any two agents in the same subgroup have a relative distance less than d and any two agents belonging to different subgroups have a relative distance larger than d. Many studies have been devoted to the analysis of clustering behaviors by assuming uniform initial conditions. In this case, numerical simulations in Bernardo and Vasca (2020), Blondel et al. (2009), Hegarty and Wedin (2016), Hegselmann and Krause (2002), Kou, Zhao, Peng, and Shi (2012), Lorenz (2006, 2007), Srivastava, Bernardo, Altafini, and Vasca (2023) and Schawe, Fontaine, and Hernández (2021) have shown that, as one would expect, the number of clusters tends to increase by decreasing the confidence threshold d. However, there exist numerical tests which show that an increase in d can increase the number of clusters (Proskurnikov & Tempo, 2018, Fig. 1). Moreover, for a fixed d, the number of clusters tends to increase as the initial range of opinions $v(\mathbf{x}(0))$ increases. Blondel, Hendrickx, and Tsitsiklis (2007) have analyzed the so-called "2d-conjecture", according to which for initial opinions uniformly distributed the opinions converge to $\lceil \frac{v(\mathbf{x}(0))}{2.2d} \rceil$ clusters and the relative distance between adjacent clusters is approximately equal to $2.2d$. However, numerical tests have shown that the latter conjecture may provide only an upper bound on the number of clusters (Proskurnikov & Tempo, 2018). In synthesis, rigorously characterizing the possible forms of the steady states is still an open issue even for the simple case of symmetric homogeneous BCOD.

6.1.2. Energy function

The convergence to a constant steady state of the opinions in symmetric homogeneous BCOD has been analyzed also by using the energy function approach, which is discussed in this section.

The energy function is obtained starting from a quadratic function of the opinions weighted by the Laplacian of the confidence graph. The Laplacian matrix (function) $L(\mathbf{x}) : \mathbb{R}^N \to \{-1, 0, 1, \ldots, N\}^{N \times N}$ of the confidence graph is given by

$$L(\mathbf{x}) = \Theta(\mathbf{x}) - \Phi(\mathbf{x}), \tag{24}$$

where $\Theta(\mathbf{x})$ and $\Phi(\mathbf{x})$ are defined in (19). The matrix (24) is zero-row-sum and is symmetric for symmetric homogeneous BCOD.

From (24), one can write

$$\operatorname{tr}(L(\mathbf{x})) = \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j) - N, \tag{25}$$

which means that the trace of the Laplacian matrix provides the number of edges over the confidence graph excluding the self-loops (Etesami, 2019). Moreover, it is

$$\begin{aligned}
\mathbf{x}^\top L(\mathbf{x})\mathbf{x} &= \mathbf{x}^\top \Theta(\mathbf{x})\mathbf{x} - \mathbf{x}^\top \Phi(\mathbf{x})\mathbf{x} \\
&= \sum_{i \in \mathcal{I}} \theta_i x_i^2 - \mathbf{x}^\top \begin{pmatrix} \sum_{j \in \mathcal{I}} \phi(x_1, x_j) x_j \\ \vdots \\ \sum_{j \in \mathcal{I}} \phi(x_N, x_j) x_j \end{pmatrix} \\
&= \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j) x_i^2 - \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j) x_i x_j \\
&= \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j) x_i (x_i - x_j),
\end{aligned} \tag{26}$$

where $\theta_i = \sum_{j \in \mathcal{I}} \phi(x_i, x_j)$, $i = 1, \ldots, N$, are the diagonal elements of the matrix $\Theta(\mathbf{x})$. For each pair of connected agents, say i and j, in (26) there are two terms that can be grouped in the form

$$x_i(x_j - x_i) + x_j(x_j - x_i) = x_i^2 + x_j^2 - 2x_i x_j = (x_i - x_j)^2,$$

which substituted in (26) allows one to write

$$\mathbf{x}^\top L(\mathbf{x})\mathbf{x} = \sum_{i,j\in\mathcal{I}} \phi(x_i, x_j)x_i(x_i - x_j) = \frac{1}{2}\sum_{i,j\in\mathcal{I}} \phi(x_i, x_j)(x_i - x_j)^2. \quad (27)$$

It should be noted that, in general, the function $\mathbf{x}^\top L(\mathbf{x})\mathbf{x}$ given by (27) is not monotone in time along the system's solutions, also if the confidence graph remains connected. An alternative function, which strictly decreases until the opinion dynamics terminates, is given by the so-called energy function defined as

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^\top L(\mathbf{x})\mathbf{x} - \frac{1}{2}\mathrm{tr}(L(\mathbf{x}))d^2 + \frac{1}{2}N(N-1)d^2, \quad (28)$$

which has been introduced by Roozbehani, Megretski, and Frazzoli (2008). By using (25) and (27), the energy function (28) can be rewritten as

$$\mathcal{E}(\mathbf{x}) = \frac{1}{2}\sum_{i,j\in\mathcal{I}} g(x_i, x_j), \quad (29)$$

where

$$g(x_i, x_j) = \phi(x_i, x_j)(x_i - x_j)^2 + (1 - \phi(x_i, x_j))d^2 = \begin{cases} (x_i - x_j)^2, & \text{if } |x_i - x_j| \le d, \\ d^2, & \text{otherwise.} \end{cases} \quad (30)$$

Clearly, each function $g(x_i, x_j)$, $i, j \in \mathcal{I}$, is Lipschitz continuous and non-negative meaning that the function $\mathcal{E}(\mathbf{x})$ is piecewise-smooth, Lipschitz continuous and non-negative. The energy function $\mathcal{E}(\mathbf{x})$ evaluated along the system's solutions has many local minima corresponding to the different constant steady state solutions at consensus or clustering. In particular, the lowest value of $\mathcal{E}(\mathbf{x})$ is obtained at any consensus for which it is $\mathcal{E}(\mathbf{x}) = 0$, see (30). For any clustering, it is

$$\mathcal{E}(\mathbf{x}^+) = \mathcal{E}(\mathbf{x}) = \sum_{\substack{\mu_1,\mu_2\in\mathcal{I}_c \\ \mu_1\neq\mu_2}} |\mathcal{C}_{\mu_1} \parallel \mathcal{C}_{\mu_2}|d^2 > 0,$$

where $\mathcal{C}_\mu$, $\mu \in \mathcal{I}_c = \{1, \ldots, M\}$, are the sets of agents grouped by clusters.

Although $\mathcal{E}(\mathbf{x})$ is not globally convex, for BCOD (17) it is strictly decreasing as long as some agent moves, as shown by the following proposition proved in Roozbehani et al. (2008) (Theorem 1) and Bhattacharyya et al. (2013) (Theorem 4.3) (also for multi-dimensional systems).

Proposition 37. *Consider BCOD (17) with influence function (22) and $\ell = u$. The energy function (28) satisfies for all $k \in \mathbb{N}_0$ the inequality*

$$\mathcal{E}(\mathbf{x}^+) - \mathcal{E}(\mathbf{x}) \le -2\sum_{i\in\mathcal{I}}(x_i^+ - x_i)^2. \quad (31)$$

A similar result has also been obtained for opinion dynamics with more general influence functions; see Jabin and Motsch (2014), Motsch and Tadmor (2014) and Proskurnikov and Tempo (2018).

Proposition 37 represents an alternative way for showing the fact, here expressed as part of Theorem 36, that the opinion dynamics terminate in finite time reaching a consensus or a clustering. Remarkably, energy-based arguments allow to prove the convergence of BCOD on the circle (Hegarty, Martinsson, & Wedin, 2016), which exhibits properties similar to classical BCOD.

Functions different from (29) have been used for the stability analysis of BCOD. A local piecewise quadratic Lyapunov function has been considered by Tangredi, Iervolino, and Vasca (2017) for finding sufficient conditions for the convergence to a consensus. The convergence in finite time has been proved by Coulson, Steeves, Gharesifard, and Touri (2015) and Mohajer and Touri (2013) by adopting the following function

$$\mathcal{E}(\mathbf{x}) = |\mathcal{C}_{x_1}|(x_N - x_1) + (x_N - x_\nu), \quad (32)$$

where (by assuming without loss of generality an ordered sequence of indices) x_N is the maximum opinion, x_1 is the minimum opinion, $\mathcal{C}_{x_1}$ is the set of all agents having the minimum opinion, i.e., $\mathcal{C}_{x_1} = \{i \in \mathcal{I} \mid x_i = x_1\}$, and x_ν is the smallest opinion strictly larger than the minimum one, i.e., $\nu = \min\{|\mathcal{C}_{x_1}| + 1, N\}$.

6.2. Asymmetric model

In the following, we exclude the trivial case $d_M = 0$ and separately consider the case $d_m = 0$, which corresponds to one-sided confident agents, i.e., agents who connect only with lower ($\ell > 0$ and $u = 0$) or upper ($\ell = 0$ and $u > 0$) neighbors.

It should be noticed that asymmetric BCOD satisfies the properties in Section 5, e.g., the range of opinions becomes constant in finite time and the absence of silent agents allows consensus.

6.2.1. Convergence to a consensus

The analysis of convergence to a consensus can be started by considering the particular situation when the opinions are close enough. The following proposition is proved in Bernardo et al. (2022) (Lemma 12).

Proposition 38. *Consider BCOD (17) with influence function (22) and assume that $d_m > 0$. A consensus is reached in finite time in BCOD by a given solution $\mathbf{x}(k)$ if there exists a time-step $k^* \in \mathbb{N}_0$ such that $v(\mathbf{x}(k^*)) \le d_M$. In particular, the convergence time $\bar{k}$ satisfies*

$$\bar{k} \le k^* + \left\lceil N\frac{d_M - d_m}{d_m} \right\rceil + 1. \quad (33)$$

Analogously to Theorem 36, which deals with the symmetric case, one should expect that also in the asymmetric case the convergence to a consensus can be reached if at each time-step there exists at least one agent able to propagate its opinion to all the others. The presence of at least one root in the confidence graph corresponds to the opinion vector being a d-chain with $d = d_M$; indeed, according to Definition 16, this corresponds to $\phi(x_i, x_{i+1}) = 1$ or $\phi(x_{i+1}, x_i) = 1$ for all $i \in \mathcal{I} \setminus \{N\}$. In particular, the following result holds.

Corollary 39. *Consider BCOD (17) with influence function (22). An asymptotic consensus is reached in BCOD by a given solution $\mathbf{x}(k)$ if and only if the opinion vector $\mathbf{x}(k)$ is a d-chain with $d = d_M$ for all $k \in \mathbb{N}_0$. Hence, a consensus is reached in finite time if $d_m > 0$.*

The result above has been proved in Bernardo et al. (2022) (Theorem 14) and also derives as a corollary of Theorem 27 by considering that in homogeneous BCOD the graph corresponding to a d-chain with $d = d_M$ is rooted and one of its roots is the agent with the minimum (if $\ell > u$) or maximum (if $u > \ell$) opinion.

6.2.2. Convergence to a constant steady state

Conditions for which the confidence graph remains rooted over the system's evolution are not easy to find. On the other hand, it is possible to prove that the opinions always converge to a constant steady state which is a consensus or a clustering, i.e., in homogeneous BCOD it is not possible to have situations like that in Example 21. Moreover, the convergence is in finite time if $d_m > 0$, as shown by the following theorem.

Theorem 40. *Consider BCOD (17) with influence function (22). Any solution $\mathbf{x}(k)$ converges to a constant steady state which is either a consensus or a clustering. In particular, if $d_m > 0$ the steady state is reached in finite time and any convergence time $\bar{k}$ satisfies*

$$\bar{k} \leq 2\left(N - 1 + \left\lceil \frac{N}{\min\left\{d_m, \frac{d_M}{2N}\right\}} v(\mathbf{x}(0)) \right\rceil\right). \tag{34}$$

Theorem 40 has been proved in Bernardo et al. (2022) (Theorem 17) by introducing the set of agents who are not silent, i.e., they have at least one neighbor whose opinion is different from their own:

$$\mathcal{M} = \{i \in \mathcal{I} \mid \exists j \in \mathcal{N}_i \text{ with } x_j \neq x_i\}. \tag{35}$$

Theorem 40 implies that if $d_m > 0$ the set $\mathcal{M}$ becomes empty after a finite number of steps. The following subsets of $\mathcal{M}$ are also used:

$$\mathcal{M}_{\min} = \arg\min_{i \in \mathcal{M}} x_i, \quad \mathcal{M}_{\max} = \arg\max_{i \in \mathcal{M}} x_i, \tag{36}$$

that is, $\mathcal{M}_{\min}$ ($\mathcal{M}_{\max}$) is the set of indices of agents with the minimum (maximum) opinion among those who have at least one neighbor with an opinion different from their own. By definition, any agent with opinion less (larger) than x_q, with $q \in \mathcal{M}_{\min}$ ($\mathcal{M}_{\max}$), will not change its opinion at the next time-step.

By using the sets (35) and (36), the following result can be proved (Bernardo et al., 2022, Lemma 16).

Proposition 41. *Consider BCOD (17) with influence function (22) and $d_m > 0$. For agent q, denote $\mathcal{N}_{=q} = \{i \in \mathcal{I} \mid x_i = x_q\}$ and $\mathcal{N}^+_{=q} = \{i \in \mathcal{I} \mid x_i^+ = x_q^+\}$.*

If $u \geq \ell$, then for any agent $q \in \mathcal{M}_{\min}$ at least one of the conditions C1–C4 holds:

C1. $|\mathcal{N}^+_{=q}| > |\mathcal{N}_{=q}|$, i.e., at the next time-step the number of agents with the same opinion of the agent q increases;
C2. $\mathcal{N}^+_{=q} = \mathcal{N}^+_q$, i.e., at next time-step all neighbors of q have the same opinion;
C3. $x_q^{+2} - x_q > \frac{u}{2N^2}$, i.e., the increase of the opinion of q after two time-steps is at least $u/(2N^2)$;
C4. $x_q^+ - x_q > \frac{\ell}{N}$, i.e., the increase of the opinion of q after one time-step is at least ℓ/N.

If $u \leq \ell$, then for any agent $q \in \mathcal{M}_{\max}$ at least one of the conditions C1, C2, C5, C6 holds:

C5. $x_q^{+2} - x_q < -\frac{\ell}{2N^2}$, i.e., the decrease of the opinion of q after two time-steps is at least $\ell/(2N^2)$,
C6. $x_q^+ - x_q < -\frac{u}{N}$, i.e., the decrease of the opinion of q after one time-step is at least u/N.

It should be noticed that the occurrences of C1 and C2 do not require that x_q^+ is equal to x_q.

Proposition 41 contains the main conditions used in Bernardo et al. (2022) for deriving the upper bound of the convergence time expressed by (34).

Theorem 40 does not provide any indication on the form of the steady state solution, e.g., the value of opinions at a consensus, the number of clusters eventually reached and the number of agents in each cluster. In Bernardo and Vasca (2020), Hegselmann and Krause (2002) and Lorenz (2007), it has been shown via numerical simulations that the number of clusters tends to increase by decreasing the confidence thresholds ℓ and u, but some exceptions may occur (Lorenz, 2006; Proskurnikov & Tempo, 2018).

Example 42. Consider the asymmetric homogeneous BCOD with $N = 10$ agents. We considered a truncation of all real numbers with 8 decimal digits. The choice of a truncation on the decimal numbers, i.e., to fix a desired precision on the numerical results, is important because of the equality conditions which determine the edges of the confidence graphs. For each value of ℓ in the set $\{0.1, 0.2, 0.5, 0.6, 0.8, 0.9\}$, 35 values of u uniformly distributed in the interval $[0.1, 1]$ have been chosen. For each pair $\{\ell, u\}$, 20.000 runs have been done. For each run, the initial opinions have been chosen as pseudorandom numbers with uniform distribution in the interval $[0, 1]$ by using the Matlab command `rand`. The plots in Fig. 7 show the average (red), maximum (blue) and minimum (cyan) convergence time. It should be noticed that in many experiments the convergence time is larger than N. Fig. 8 shows the same statistics for the number of clusters. For each value of ℓ, the maximum and average number of clusters usually (but not always) decreases as u increases.

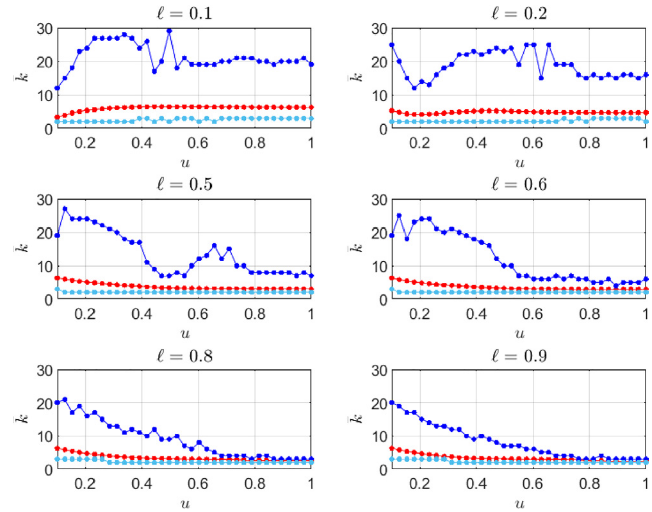

Fig. 7. Convergence time in the tests from Example 42. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

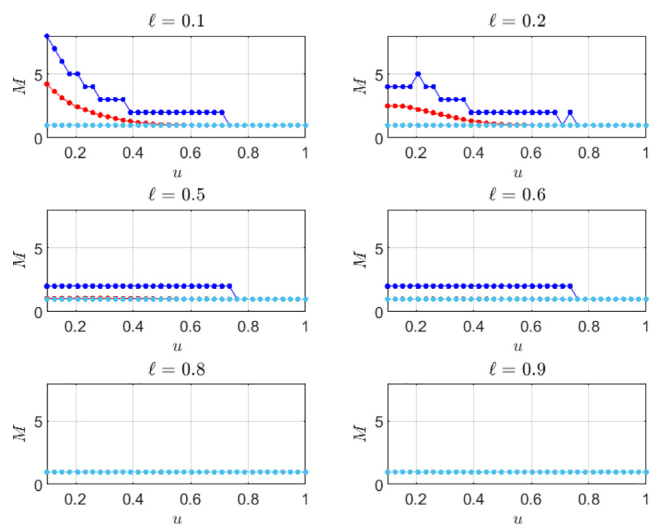

Fig. 8. Number of clusters in the tests from Example 42.

Numerical results in Bernardo and Vasca (2020) show that the average of the opinions at the steady state tends to move in the direction of the asymmetry of the confidence interval, i.e., it increases by increasing the upper threshold u and decreases by increasing the lower threshold ℓ. A formal sensitivity analysis of the steady state behaviors with respect to the model parameters is anything but trivial and represents an open issue.

6.2.3. One-sided confident model

In homogeneous BCOD where all agents are one-sided confident, i.e., all agents have a null lower or upper confidence threshold, it is $d_m = 0$. If $\ell = 0$ ($u = 0$), the agents are not influenced by lower (upper) neighbors and from (17) all opinions are nondecreasing (nonincreasing), i.e., $x_i^+ \geq x_i$ ($x_i^+ \leq x_i$) for all $i \in \mathcal{I}$ and $k \in \mathbb{N}_0$.

The following properties of one-sided confident BCOD can be directly derived as particular cases of $d_m \geq 0$:

- any crack cannot be recovered, i.e., Proposition 26 holds;
- the repeatedly composition jointly rooted property for the infinite sequence of confidence graphs is a criterion for the asymptotic convergence to a consensus, i.e., Theorem 27 holds;
- Proposition 30 is trivial since the condition $v(\mathbf{x}) \leq d_m$ already corresponds to a consensus;
- the opinions preserve their order, i.e., Proposition 35 holds.

The convergence in finite time to a steady state or to a consensus cannot be derived by Corollary 39 and Theorem 40 because the hypothesis $d_m > 0$ does not hold. The following example of a one-sided confident BCOD shows the convergence to an asymptotic consensus.

Example 43. Consider the one-sided confident BCOD with $d_M = u$ and $d_m = \ell = 0$ and an opinion vector $\mathbf{x}$ such that $c - u \leq x_1 < c$ and $x_i = c$ for all $i \in \mathcal{I} \setminus \{1\}$. From (17) with (22), it follows that $x_1^+ = \frac{1}{N}x_1 + \frac{N-1}{N}c$ and $x_i^+ = x_i$ for all $i \in \mathcal{I} \setminus \{1\}$. Then, one can write

$$x_1^{+h} = \frac{1}{N^h}x_1 + c(N-1)\left(\frac{1}{N^h} + \cdots + \frac{1}{N}\right),$$

which converges asymptotically to c, despite the opinion vector is a d_M-chain.

By excluding the trivial cases corresponding to initial conditions already at a consensus or a clustering, the condition $d_m > 0$ expressed by Corollary 39 and Theorem 40 is also necessary for the convergence in finite time to a constant steady state. This is implied by the following proposition (Bernardo et al., 2022, Lemma 18).

Proposition 44. *Consider BCOD (17) with influence function (22) and $d_m = 0$. For any $i, j \in \mathcal{I}$, if $x_i \neq x_j$ at some time-step $k \in \mathbb{N}_0$, then $(x_i - x_j)(x_i^+ - x_j^+) > 0$.*

It is easy to verify that in one-sided confident BCOD with $\ell = 0$ ($u = 0$) for any solution $\mathbf{x}(k)$ that converges to an asymptotic consensus, such consensus will correspond to the maximum (minimum) initial opinion.

In order to analyze the convergence of the opinions, the concepts of ϵ-practical consensus and ϵ-practical clustering in Definitions 23 and 24, respectively, can be considered. In particular, if all opinions have a mutual distance less than or equal to d_M, the system converges in finite time to a practical consensus, as shown by the following proposition.

Proposition 45. *Consider BCOD (17) with influence function (22) and $d_m = 0$. Given a solution $\mathbf{x}(k)$, if there exists a finite time-step $k^* \in \mathbb{N}_0$ such that $v(\mathbf{x}(k^*)) \leq d_M$, then for any $\delta \in (0, d_M]$ there exists a finite $\Delta \in \mathbb{N}_0$ such that $v(\mathbf{x}(k^* + \Delta)) \leq \delta$ with*

$$\Delta \leq \left\lceil N\frac{d_M - \delta}{\delta} \right\rceil. \tag{37}$$

By using arguments similar to those in the proof of Corollary 39 and Theorem 40, the convergence in finite time to an ϵ-practical consensus or an ϵ-practical clustering can be proved. In particular, if $\ell = 0$ ($u = 0$) then at an ϵ-practical consensus it is $x_i \in [x_N(0) - \epsilon, x_N(0)]$ ($x_i \in [x_1(0), x_1(0) + \epsilon]$) for all $i \in \mathcal{I}$.

6.3. Historical remarks: Convergence time estimates

The problems of the convergence in finite time to a constant steady state and the determination of an upper bound on the convergence time have been considered by several authors, as discussed below. In Blondel et al. (2009), it is shown that the worst-case convergence time depends on N.

Several papers investigated possible expressions for upper bounds on the convergence time. The different results are synthesized in Table 1. Only the results proposed by Bernardo et al. (2022) and Shen and Sun (2009) are valid for asymmetric BCOD. The upper bound $N^8 + N$, which is valid also for multidimensional opinions, has been proposed by Etesami and Başar (2015) for symmetric BCOD. An upper bound of the order N^5 has been provided by Martínez et al. (2007). By restricting the result in Shen and Sun (2009) to symmetric models, one obtains that the convergence time $\bar{k}$ must satisfy the inequality

$$\bar{k} \leq \sum_{n=2}^{N}(n-1)\left\lceil \frac{\ln(n-1)}{\ln n - \ln(n-1)} \right\rceil + 1, \tag{38}$$

which becomes very conservative when N increases. The upper bound $32N^4$ for the convergence time has been proved by Touri and Nedić (2011). More specifically, the bound presented in that paper is given by a product of N^2 and a function that depends on the initial opinions and the adjoint dynamics. However, in order to obtain an explicit formula for the upper bound, this function has been bounded by a term proportional to N^2, which leads to a bound $O(N^4)$. By using the energy function (31), the constant 32 has been replaced by 2 in Martinsson (2016) (this estimate remains valid for multidimensional opinions, see Section 10). The less tight upper bound $4N^3 + 2N$ has been found in Bhattacharyya et al. (2013) by improving on the bound $N^{O(N)}$ resulting from a more general theorem in Chazelle (2011). Other bounds of the third order of the number of agents have been proposed in Mohajer and Touri (2013), i.e., $3N^3 + N$, and Coulson et al. (2015), i.e., $6N^3 - 7N^2 - 2N$, by using the energy function (32). For asymmetric homogenous BCOD, Bernardo et al. (2022) have provided the upper on the convergence time expressed in (34). An alternative upper bound, which also depends on the number of agents and the confidence thresholds, has been presented by Shen and Sun (2009).

It is known that the worst-case convergence time (for a specially constructed initial condition) is $\Omega(N^2)$; the relevant example has been provided in Wedin and Hegarty (2015) (Theorem 1). In the configurations examined by Wedin and Hegarty (2015), however, one has $v(\mathbf{x}(0)) = O(N)$, which corresponds, as has been already mentioned, to the upper estimate $O(N^3)$. Hence, a gap between the lower and upper bounds on the convergence time remains. To find an exact value of the convergence time as a function of $\mathbf{x}(0)$ is a difficult open problem that has been solved only for special situations, e.g., the convergence time for equally spaced agents (and d small enough) is known to be $5N/6 + O(1)$ (Hegarty & Wedin, 2016, Remark 1.4).

7. Heterogeneous model

According to Definition 12, BCOD is said to be heterogeneous if the agents have different confidence intervals, i.e., $\ell_i \neq \ell_j$ or $u_i \neq u_j$ for some $i, j \in \mathcal{I}$, with $\ell_i, u_i \in \mathbb{R}_0^+$, $i \in \mathcal{I}$.

Table 1
Upper bound expressions for the convergence time.

Type of model	Upper bound	Reference
Symmetric	N^8+N	Etesami and Başar (2015) (Theorem 2)
Symmetric	$O(N^5)$	Martínez, Bullo, Cortés, and Frazzoli (2007) (Theorem III.1)
Asymmetric	$\sum_{n=2}^{N}(n-1)\left\lceil\frac{\ln(n-1)d_M-\ln(d_m)}{\ln n-\ln(n-1)}\right\rceil+1$	Shen and Sun (2009) (Theorem 3.1)
Symmetric	$N^{O(N)}$	Chazelle (2011) (Theorem 2.1)
Symmetric	$32N^4$	Touri and Nedić (2011) (Theorem 2)
Symmetric	$2N^4$	Martinsson (2016) (Theorem 1.1)
Symmetric	$6N^3-7N^2-2N$	Coulson et al. (2015) (Theorem 3.1)
Symmetric	$4N^3+2N$	Bhattacharyya et al. (2013) (Theorem 3.1)
Symmetric	$3N^3+N$	Mohajer and Touri (2013) (Theorem 1)
Asymmetric	$2\left(N-1+\left\lceil\frac{N}{min\left\{d_m,\frac{d_M}{2N}\right\}}v(x(0))\right\rceil\right)$	Theorem 40 (Bernardo et al., 2022, Theorem 17)

Heterogeneous BCOD allows one to capture realistic behaviors, such as an agent may be influenced by another but not vice versa and two disconnected agents may reconnect. For heterogeneous BCOD, the order-preservation property given in Definition 13 may be violated, as shown by Example 29 (see Fig. 5).

Heterogeneous BCOD can exhibit constant steady states which are neither a consensus nor a clustering, as shown in Example 21. The reasonable conjecture that asymmetric heterogeneous BCOD converges asymptotically to a *constant* steady state has not been formally proved yet. Moreover, in this case one should consider a definition of clustering which is weaker than Definition 19 because Example 21 shows that constant steady states with different opinions but without the corresponding subgraphs being disjoint are possible.

An exhaustive numerical analysis of heterogeneous BCOD is a nontrivial task due to the high number of parameters. However, numerical simulations have shown the tendency that heterogeneity leads to consensus more easily than homogeneity (Liang, Yang, & Wang, 2013; Lorenz, 2010). Moreover, numerical tests have shown that the number of clusters at a clustering usually increases as the fraction of agents with low confidence threshold increases and, keeping the distribution of the confidence bounds fixed, the number of clusters decreases as the number of agents increases (Kou et al., 2012).

Some (not many, indeed) theoretical results have been obtained in the literature for particular classes of heterogeneous BCOD. In symmetric heterogeneous BCOD (Definition 12), the confidence bounds are symmetric, and then the influence function (16) can be rewritten as

$$\phi_i(x_i,x_j)=\begin{cases}1, & \text{if } |x_j-x_i|\le d_i,\\ 0, & \text{otherwise,}\end{cases} \tag{39}$$

with $i,j\in\mathcal{I}$ and $d_i=\ell_i=u_i$. For this particular class of heterogeneous BCOD, Mirtabatabaei and Bullo (2012) have conjectured that any solution $\mathbf{x}(k)$ converges to a constant steady state and found sufficient conditions for the existence of a finite time after which the confidence graph remains constant (see Conjectures 2.1–2.3, Lemma 4.2, Theorem 4.4 and Proposition 4.11 therein). Moreover, Etesami (2020) has shown that the energy function in the form (28) is not a valid Lyapunov function for symmetric heterogeneous BCOD (see Fig. 6 therein).

Another special (and simpler) case of heterogeneous BCOD is that of one-sided heterogeneous BCOD in which the agents have equal upper confidence thresholds and different lower confidence thresholds or vice versa. Let $\eta_i\in[0,\ell]$ be the lower-neighborhood parameter associated to the agent i, according to which its confidence interval is $[x_i-\ell+\eta_i, x_i+u]$, with $\ell,u\in\mathbb{R}_0^+$. Hence, in one-side BCOD with different lower thresholds the influence function (16) can be rewritten as

$$\phi_i(x_i,x_j)=\begin{cases}1, & \text{if } -\ell+\eta_i\le x_j-x_i\le u,\\ 0, & \text{otherwise,}\end{cases} \tag{40}$$

with $i,j\in\mathcal{I}$. Analogously, given the upper-neighborhood parameter $\eta_i\in[0,u]$, then in one-side BCOD with different upper thresholds the influence function (16) can be written as

$$\phi_i(x_i,x_j)=\begin{cases}1, & \text{if } -\ell\le x_j-x_i\le u-\eta_i,\\ 0, & \text{otherwise,}\end{cases} \tag{41}$$

with $i,j\in\mathcal{I}$. Obviously, the case in which the condition $\eta_i=\eta_j$ holds for all $i,j\in\mathcal{I}$ corresponds to homogeneous BCOD.

For some particular cases, a bound on the convergence time for one-sided heterogeneous BCOD can be given (Bhattacharyya et al., 2013; Coulson et al., 2015). To this aim, the following lemma, which is analogous of Proposition 41, can be proved.

Lemma 46. *Consider BCOD (17). Say q an agent and define the sets* $\mathcal{N}_{=q}=\{i\in\mathcal{I}\mid x_i=x_q\}$ *and* $\mathcal{N}_{=q}^{+}=\{i\in\mathcal{I}\mid x_i^+=x_q^+\}$.

If $u\ge\ell$ *and the influence function (40) is considered, then for any agent* $q\in\mathcal{M}_{\min}$ *at least one of the conditions C1–C4 holds:*

- *C1.* $|\mathcal{N}_{=q}^{+}|>|\mathcal{N}_{=q}|$, *i.e., at the next time-step the number of agents with the same opinion of the agent q increases;*
- *C2.* $\mathcal{N}_{=q}^{+}=\mathcal{N}_{q}^{+}$, *i.e., at next time-step all neighbors of q have the same opinion;*
- *C3.* $x_q^{+2}-x_q>\frac{u}{2N^2}$, *i.e., the increase of the opinion of q after two time-steps is at least* $u/(2N^2)$;
- *C4.* $x_q^{+}-x_q>\frac{\ell-\eta_{\max}}{N}$, *where* $\eta_{\max}=\max_{i\in\mathcal{I}}\eta_i$, *i.e., the increase of the opinion of q after one time-step is at least* $(\ell-\eta_{\max})/N$.

If $u\le\ell$ *and the influence function (41) is considered, then for any agent* $q\in\mathcal{M}_{\max}$ *at least one of the conditions C1, C2, C5, C6, holds:*

- *C5.* $x_q^{+2}-x_q<-\frac{\ell}{2N^2}$, *i.e., the decrease of the opinion of q after two time-steps is at least* $\ell/(2N^2)$,
- *C6.* $x_q^{+}-x_q<-\frac{u-\eta_{\max}}{N}$, *i.e., the decrease of the opinion of q after one time-step is at least* $(u-\eta_{\max})/N$.

Lemma 46 can be used to determine an upper bound on the convergence time in some particular cases.

Theorem 47. *Consider BCOD (17) with (40) and assume that* $u\ge\ell>\eta_{\max}$, *with* $\eta_{\max}=\max_{i\in\mathcal{I}}\eta_i$. *Any solution* $\mathbf{x}(k)$ *converges in finite time to a constant steady state which is either a consensus or a clustering. Moreover, any convergence time* $\bar{k}$ *satisfies*

$$\bar{k}\le 2\left(N-1+\left\lceil\frac{N}{\min\left\{\ell-\eta_{\max},\frac{u}{2N}\right\}}v(\mathbf{x}(0))\right\rceil\right). \tag{42}$$

If $\ell\ge u>\eta_{\max}$ a result similar to Theorem 47 can be proved by exchanging the roles of u and ℓ and by considering the influence function (41).

Lemma 46 and Theorem 47 have been proved in Bhattacharyya et al. (2013) (Lemma 3.4 and Theorem 3.3, respectively) for $\ell=u$

in (40) and (41) and can be easily extended by applying similar arguments to those used in Bernardo et al. (2022) for the homogeneous case.

In Coulson et al. (2015) (Theorem 3.1), Lyapunov arguments have been used to prove that one-sided heterogeneous BCOD reaches a constant steady state within a time interval proportional to N^3.

8. Stubbornness

The concept of stubbornness introduces the possibility that some agents, independently from system dynamics, do not change their opinions but can influence those of the other agents. A *stubborn* (or *radical*) agent $\sigma \in \mathcal{I}$ is such that $\ell_\sigma = u_\sigma = 0$. This definition, together with (16) and (17), implies that the opinion of a stubborn agent is constant, i.e., $x_\sigma^+ = x_\sigma$ for all $k \in \mathbb{N}_0$. By considering Definition 14, it follows that a stubborn agent is silent for any time-step. For the sake of simplicity, in the following we consider the presence of a single stubborn agent.

It is easy to verify that the presence of a stubborn agent does not influence the validity of Propositions 1 and 3, i.e., the opinion of any agent at the next time-step is bounded by the minimum and maximum opinions of its neighbors and the range of opinions is non-increasing.

It is also easy to find situations such that during the time evolution the opinion of some agents intersects with that of the stubborn. On the other hand, in the case of homogeneous confidence intervals for the non-stubborn agents, the opinions of these agents preserve their order, i.e., Proposition 35 holds for all $i \in \mathcal{I} \setminus \{\sigma\}$. Clearly, the presence of a stubborn agent does not influence the fact that, in general, the order preservation property is not verified for the heterogeneous model. Also in the presence of a stubborn agent, any crack between two adjacent agents with a relative distance larger than d_M cannot be recovered, i.e., Proposition 26 holds, as shown by the following example.

Example 48. Consider the heterogeneous BCOD with $N = 5$ agents, $\ell_i = 0.1$, $i \in \mathcal{I} \setminus \{3\}$, $u_i = 0.1$, $i \in \mathcal{I} \setminus \{1, 3\}$, $u_1 = 0.4$, $\ell_3 = u_3 = 0$. Then, the agent 3 is stubborn. Fig. 9 shows the simulation results for $\mathbf{x}(0) = [0.2, 0.3, 0.4, 0.6, 0.7]^\top$. The solution converges asymptotically to a clustering corresponding to the opinion 0.6382 (for the agents 1, 4 and 5 who reach this value at $k = 5$) and the stubborn opinion 0.4 which is asymptotically reached by the agent 2. The corresponding confidence graphs are shown in Fig. 10.

The analysis of convergence in finite time for BCOD in presence of a stubborn agent is not relevant. In order to motivate this, we show that Proposition 30 holds only in a very particular case. By assuming that $v(\mathbf{x}) \le d_m$ at the time-step k, all agents except the stubborn at the next time-step have the same opinion. Such a shared opinion is given by

$$\bar{x}^+ = \frac{1}{N}\left(\sum_{i\in\mathcal{I}\setminus\{\sigma\}} x_i + x_\sigma\right). \tag{43}$$

Say $\bar{x}^{++}$ the opinion at the time-step $k+2$, which is shared by all agents except for the stubborn agent. Then, one can write

$$\bar{x}^{++} = \frac{N-1}{N}\bar{x}^+ + \frac{1}{N}x_\sigma, \tag{44}$$

which shows that the only possibility that the convergence to the stubborn opinion is in finite time (in at most one time-step, indeed) is that when the confidence graph excluding the stubborn becomes complete, i.e., $v(\mathbf{x}^-) > d_m$ and $v(\mathbf{x}) \le d_m$ with $\mathbf{x}^- = \mathbf{x}(k-1)$, and the mean of the opinions of all agents except that of the stubborn is equal to the stubborn opinion, i.e., $\sum_{i\in\mathcal{I}\setminus\{\sigma\}} x_i = (N-1)x_\sigma$.

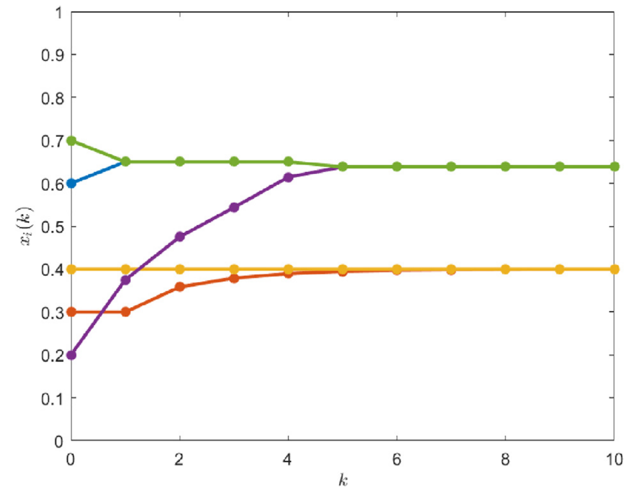

Fig. 9. Opinions of the BCOD in Example 48.

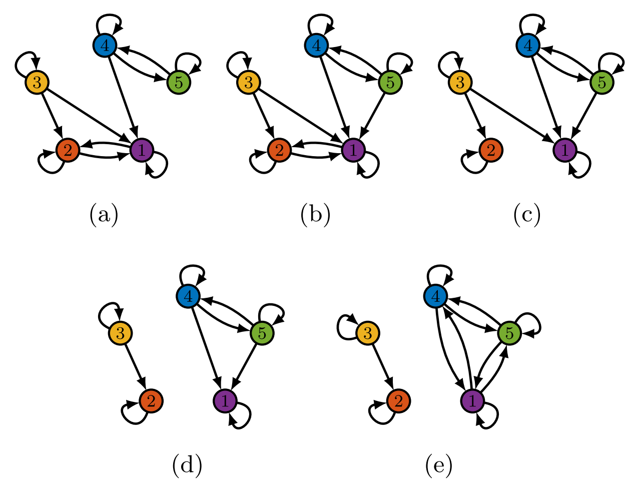

Fig. 10. Confidence graphs for the opinions in Fig. 9: at $k = 0$ (a), at $k = 1$ (b), at $k = 2$ (c), at $k = 3$ (d), for any $k \ge 4$ (e).

If the stubborn opinion diffuses over the network then the opinions of a group of agents usually converge to the stubborn opinion (Krause, 2015). For homogeneous BCOD with a single stubborn agent, if the stubborn does not become isolated then the opinions of a group of agents, not necessarily all of them, converge to the stubborn opinion. The following theorem is proved in Hegselmann and Krause (2015) (Constant Signal Theorem) for symmetric models.

Theorem 49. *Consider BCOD (17) with influence function (22) and $d = \ell_i = u_i$ for all $i \in \mathcal{I} \setminus \{\sigma\}$, where σ is the index of a stubborn agent. The following conditions hold:*

- *C1. there exist a (possibly empty) set $\mathcal{A}$ of agents and a finite time-step $\bar{k}$ such that the opinions of the agents in $\mathcal{A} \cup \{\sigma\}$ are a d-chain for all $k \ge \bar{k}$;*
- *C2. the opinions of the agents in $\mathcal{A}$ will converge to the stubborn opinion;*
- *C3. the opinions of the agents in $\mathcal{I} \setminus \mathcal{A}$ will converge in finite time to a constant steady state with opinions different from that of the stubborn.*

The result above can be easily generalized to the case of asymmetric homogeneous BCOD. Theorem 49 does not provide any indication about the number of agents included in the set $\mathcal{A}$. Simulation results in Hegselmann and Krause (2015) obtained by assuming a stubborn opinion close to the minimum or maximum opinions have shown that the number of agents in $\mathcal{A}$ tends to increase with the confidence bound.

A similar result to Theorem 49 has been proved in Chazelle and Wang (2017) (Theorem 1.2) for the case of multiple stubborn agents. In particular, it has been shown that the opinions converge asymptotically to a fixed point for any number of stubborn agents and also for agents with a multidimensional state. However, it should be noticed that the steady state in the presence of stubborn agents need not be a clustering in the sense of Definition 19, as shown by Example 21.

The model with stubborn agents exhibits also some counterintuitive properties. Paradoxically, an increase in the number of stubborn agents can drastically decrease the number of agents whose opinions coincide with that of a stubborn agent (Hegselmann & Krause, 2015).

A formal analysis for the convergence properties of heterogeneous BCOD in the presence of stubborn agents is still an open issue.

9. Continuous-time model

The theory of discrete-time BCOD can be generalized, to some extent, to continuous-time dynamical systems (Blondel et al., 2010; Yang et al., 2014):

$$\dot{x}_i(t) = \sum_{j\in\mathcal{I}} \phi_i(x_i, x_j)(x_j(t) - x_i(t)), \tag{45}$$

where $\phi_i(x_i, x_j)$ is defined in (16). The principal limitation of the continuous-time theory is the absence of a classical (Carathéodory) solution for some initial conditions (even in the homogeneous symmetric case, the solution existence has been proved for almost all $\mathbf{x}(0)$). A continuous-time counterpart of Lemma 6 (Proskurnikov & Tempo, 2018, Lemma 5) allows one to prove the convergence of each classical solution in the case of homogeneous BCOD (45); the work Yang et al. (2014) has given some sufficient conditions for the asymptotic convergence to a consensus. Classical solutions of (45) enjoy the order-preservation property (Blondel et al., 2010).

The problem of solution existence can be resolved in two ways. On one hand, generalized solutions can be introduced such as Krasovskii or Filippov solutions (Ceragioli & Frasca, 2012; Piccoli & Rossi, 2021). A generalized solution is not uniquely determined by its initial condition and, in general, may exhibit some “pathological” behavior, e.g., a solution starting at an equilibrium point may leave it and converge to another equilibrium (Ceragioli & Frasca, 2012). On the other hand, one can replace the discontinuous indicator function (16) by its “smoothened” version, e.g., $\phi_i(x_i, x_j) = \psi(x_i - x_j)$, where ψ is an even C^1-smooth function supported on $[-d, d]$. Such continuous-time BCOD has been studied in many works (Ceragioli & Frasca, 2012; Jabin & Motsch, 2014; Motsch & Tadmor, 2014; Stamoulas & Rathinam, 2018; Yang et al., 2014); their behaviors appear to be similar to discrete-time symmetric homogeneous BCOD.

An extension of continuous-time BCOD with symmetric and homogeneous confidence thresholds and antagonistic interactions has been analyzed by Altafini and Ceragioli (2018), Ceragioli et al. (2016) and He, Liu, Wu, and Fang (2020). In these models, the sign of each term on the right-hand side of (45) depends on the sign of the product of the two corresponding opinions. It has been shown that the order-preservation property is valid and the asymptotic convergence to a steady state solution holds (Altafini & Ceragioli, 2018, Theorem 2).

10. Generalizations of BCOD

In the last decade, numerous modifications of standard BCOD have been proposed. In such models, the key idea of bounded confidence (that is, the agents do not take into consideration too dissimilar opinions) is preserved, while the dynamical mechanism of opinion formation (17) is altered in different ways in order to capture additional features of the complex social influence mechanism.

10.1. Multidimensional BCOD

Up to now, models with real-valued opinions have been considered. In many situations, however, opinions are naturally represented by vectors, being, e.g., subjective probability distributions (DeGroot, 1974), decisions on how to distribute some resource between several entities (Friedkin, Proskurnikov, Mei, & Bullo, 2019), etc. Vector-valued opinions can be used to characterize belief systems (Friedkin, Proskurnikov, Tempo, & Parsegov, 2016), multidimensional threat appraisal (Friedkin et al., 2021) or sets of cultural traits (Axelrod, 1997).

The definition of BCOD (as well as the more general DeGroot dynamics) can be easily extended to multidimensional opinions, using a convenient matrix notation form (DeGroot, 1974; Friedkin, 2015). Namely, if one replaces the scalar opinions $x_i \in \mathbb{R}$ by the s-dimensional vectors $x_i = (x_{i1}, \dots, x_{is})^\top \in \mathbb{R}^s$, then the opinion vector (3) is naturally replaced by the $N \times s$ matrix

$$\mathbf{x} = \begin{bmatrix} x_1^\top \\ \vdots \\ x_N^\top \end{bmatrix} = \begin{bmatrix} x_{11} & \dots & x_{1s} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{Ns} \end{bmatrix} \in \mathbb{R}^{N\times s}.$$

Using this convention, the DeGroot equation (5) generalizes to the multidimensional opinion case. A multi-dimensional counterpart of Proposition 1 ensures that the convex hull $\mathcal{D}(\mathbf{x}) = \operatorname{conv}\{x_1, \dots, x_N\}$ spanned by the opinions is non-expanding, i.e., $\mathcal{D}(\mathbf{x}^+) \subseteq \mathcal{D}(\mathbf{x})$ for all $k \in \mathbb{N}_0$. Introducing some norm on $\mathbb{R}^s$, one can define the range of opinions $v(\mathbf{x}) = \max_{i,j} \|x_i - x_j\|$, which corresponds to the diameter of the set $\mathcal{D}(\mathbf{x})$, and prove easily that this function does not increase along the solutions (5).

In order to generalize the Eqs. (17)–(19) to vector-valued opinions, the confidence intervals have to be replaced by Minkowski sums $x_i + \mathcal{O}_i \subset \mathbb{R}^s$, and functions $\phi_i(x_i, x_j)$ from (16) have to be defined as

$$\phi_i(x_i, x_j) = \begin{cases} 1, & \text{if } x_j - x_i \in \mathcal{O}_i, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

A most typical choice of the confidence set $\mathcal{O}_i$ (which has to be bounded) is the ball in some norm $\|\cdot\|$ on $\mathbb{R}^s$, i.e.,

$$\mathcal{O}_i = \{z \in \mathbb{R}^s : \|z\| \le d_i\}, \tag{47}$$

being a multidimensional counterpart of the symmetric interval $[-d_i, d_i]$ for all $i \in \mathcal{I}$ in the scalar case[7]. The concepts of consensus, clustering and confidence graph generalize then to multidimensional BCOD.

The model with confidence sets (47) corresponding to the Euclidean norm on $\mathbb{R}^s$ has been introduced in the seminal work by Nedic and Touri (2012) who have proved that, in the homogeneous case, i.e., $d_1 = \dots = d_N$, BCOD terminates in finite time. The original proof in Nedic and Touri (2012) (Proposition 4) is based on a special quadratic Lyapunov function with time-varying coefficients. An alternative proof, applicable to a general

[7] To the best of the authors' knowledge, models with more general sets $\mathcal{O}_i \subset \mathbb{R}^s$, $s \ge 2$, (which, in theory, can be asymmetric and non-convex) have not been studied.

norm, can be obtained by using Lemma 6 (Proskurnikov et al., 2020; Proskurnikov & Tempo, 2018). For the case of the Euclidean norm in (47), polynomial upper bounds for the convergence time have been obtained (Bhattacharyya et al., 2013; Etesami & Başar, 2015; Etesami, Başar, Nedić, & Touri, 2013; Martinsson, 2016). It has been shown (Martinsson, 2016, Theorem 1.1) that the multidimensional homogeneous BCOD terminates in no more than $2N^4$ steps independent of the dimension s; it is also known (Bhattacharyya et al., 2013, Theorem 5.1) that for $s \geq 2$ it is $\bar{k} \geq N^2/28$ for some initial conditions. The upper estimates of the convergence time are based on the energy-type Lyapunov function whose structure is similar to (28). Two principal differences in the analysis of scalar and multidimensional BCOD are, first, the absence of the order-preservation property in the vector-valued case and, second, more intricate dynamics of confidence graphs $\mathcal{G}(\mathbf{x}(k))$ that may not only loose but also acquire connectivity as the opinions evolve. There is, in particular, no counterpart of Proposition 26 in the multidimensional case. However, De Pasquale and Valcher (2022) have provided sufficient conditions for the order-preservation property in homogeneous multidimensional BCOD when ℓ_∞-norm is used in the confidence sets (47) (see Proposition 20 therein).

The convergence properties of heterogeneous multidimensional BCOD, even for the Euclidean norm case, have not been much studied. It can be shown that Theorem 27 remains valid in the multidimensional case. Moreover, Theorem 34 has been proposed for the more general case of heterogeneous multidimensional BCOD (Etesami & Başar, 2015, Theorem 7). Chazelle and Wang (2017) have considered heterogeneous multidimensional models where each pair of agents is characterized by a confidence bound and then the influence graphs are undirected. For this class of BCOD, in Theorem 1.3 they have proved that the system converges asymptotically to a fixed-point attractor and, if any pair of agents have a positive confidence threshold, then the communication network converges to a fixed graph.

10.2. Self-weights and susceptibility to social influence

In classical BCOD, an agent assigns equal influence weights to self and to the others; in particular, the larger number of neighbors $\mathcal{N}_i(\mathbf{x})$ the agent i has, the lower is its self-influence weight $a_{ii}(\mathbf{x})$. The work Urbig, Lorenz, and Herzberg (2008) and the subsequent papers Fu, Zhang, and Li (2015) and Han, Huang, and Yang (2019) relax this assumption by introducing the agent's level of self-confidence, which is a constant value $\alpha_i \in [0, 1]$ that characterizes the agent's susceptibility to social influence; the remaining weight $1-\alpha_i$ is equally allocated between the neighbors in the confidence graph:

$$x_i^+ = \alpha_i x_i + \frac{1-\alpha_i}{\sum_{j\in\mathcal{I}\setminus\{i\}} \phi_i(x_i, x_j)} \phi_i(x_i, x_j) x_j \tag{48}$$

(if the denominator vanishes, it is supposed that $x_i^+ = x_i$). A similar model has been considered by Chazelle and Wang (2017) with the only difference that the summation in (48) is taken over the whole set $\mathcal{I}$ (in this situation, the self-weight admits a lower bound $a_{ii}(\mathbf{x}) \geq \alpha_i$):

$$x_i^+ = \alpha_i x_i + \frac{1-\alpha_i}{\sum_{j\in\mathcal{I}} \phi_i(x_i, x_j)} \phi_i(x_i, x_j) x_j. \tag{49}$$

Notice that $\alpha_i = 1$ is equivalent to the stubbornness of the agent i: in this situation, one has $x_i^+ = x_i$ for all $k \in \mathbb{N}_0$.

Models with static self-weight usually do not enjoy the finite convergence time property. Notice, however, that in the case where $0 < \alpha_i < 1$ and $\ell_i = u_i = d$ for all $i \in \mathcal{I}$ (the confidence bounds are symmetric and homogeneous), Lemma 6 is applicable, which ensures asymptotic convergence. Another convergence result is the aforementioned Chazelle and Wang (2017) (Theorem 1.2), where some of the agents are stubborn, whereas the others have $\alpha_i < 1$ and obey symmetric homogeneous BCOD equations. To the best of the authors' knowledge, the analysis of solutions convergence in (48) and (49) is far from being completed. An elegant result of Chazelle and Wang (2017) (Theorem 1.1) entails that the solutions of (49) obey the inequality

$$\sum_{i\in\mathcal{I}} \sum_{k=0}^{\infty} |x_i(k+1) - x_i(k)|^2 \leq \frac{N^2}{4}. \tag{50}$$

Formally, this condition does not imply the existence of a limit $\lim_{k\to\infty} \mathbf{x}(k)$, showing, however, that the increment of the opinion vector $\mathbf{x}^+ - \mathbf{x}$ vanishes as $k \to \infty$. The condition in (50) has been proved by using the algorithmic approach based on a distributed Lyapunov function.

10.3. BCOD models with "truth-seekers"

The series of works Douven and Hegselmann (2022), Douven and Riegler (2010), Glass and Glass (2021), Hegselmann and Krause (2006) and Kurz and Rambau (2011) have proposed a modification of BCOD that involves a so-called "truth" parameter:

$$x_i^+ = \alpha_i \tau + \frac{1-\alpha_i}{\sum_{j\in\mathcal{I}} \phi_i(x_i, x_j)} \sum_{j\in\mathcal{I}} \phi_i(x_i, x_j) x_j. \tag{51}$$

Although the structure of (51) may seem similar to (48), the coefficient α_i has a principally different interpretation and is not related to an agent's self-influence weight. It is supposed that some of the agents ("truth-seekers") are interested to learn the value of some numerical parameter, which is called the "truth". Although the agents may not know the truth exactly, they may have access to or generate new data (arguments, evidences, test results, etc.) that point in the direction of τ (Hegselmann & Krause, 2006). This leads to the presence of a constant term $\alpha_i\tau$, which Hegselmann and Krause (2006) have called the "objective component". The coefficient $\alpha_i \in [0, 1)$ characterizes the agent's desire to find the true value or the "attractivity" of the truth for the agent. Formally, one may also consider agents with $\alpha_i = 1$ who are aware of the truth variable and have $x_i \equiv \tau$.

Similar to models with static self-influence weights, the model (51) does not converge in finite time. The convergence of this model has been proved only in the case of homogeneous and symmetric confidence bounds. The first proof proposed in Chazelle (2011) (Theorem 2.2) is based on a method of power series (s-energy method). A more elementary proof, based on the theory of averaging inequalities, has been later found in Proskurnikov et al. (2020) (Theorem 6). It has been proved, in particular, that the opinions of the truth-seekers (agents with $\alpha_i > 0$) eventually asymptotically converge to the truth τ, a property which appears to be nontrivial and, in fact, fails to hold for asymmetric BCOD (Kurz & Rambau, 2011, p. 867).

In order to capture the measurement errors on the truth value, the model (51) has been extended by Douven (2010) by adding some noise to the truth τ. A constant steady state solution cannot be reached in this case due to the presence of noise, but for sufficiently large confidence thresholds and strengths of the attraction of truth a non-constant practical consensus around the true value is achieved. Glass and Glass (2021) have proposed a variant of the model (51) where each agent is influenced by the true opinion τ only if τ lies within its confidence interval, i.e., at each time-step k the weight α_i is set equal to 0 if $\phi_i(x_i, \tau) = 0$.

A modified BCOD which includes self-belief and true value has been proposed by Li, Li, Du, Tang and Fan (2022). Accordingly, the model is expressed as

$$x_i^+ = \alpha_i \tau + (1-\alpha_i) \Big(\beta_i x_i + \frac{1-\beta_i}{\sum_{j\in\mathcal{I}} \phi_i(x_i,x_j)} \sum_{j\in\mathcal{I}} \phi_i(x_i,x_j) x_j\Big), \quad (52)$$

where α_i and β_i are the strength of the attraction of truth and the self-belief of the agent i, respectively.

Finally, a sophisticated hierarchical BCOD with truth-seekers has been proposed by Zhao, Zhang, Tang, and Kou (2016). In this model, the agents are divided into two classes: the leaders and the followers. A leader's opinion is influenced by the truth and the other leaders' opinions, but not by the followers' opinions. The followers have no access to the truth, but they are influenced by each other and by the leaders. The resulting dynamics is as follows. Assume that the first N_1 agents are followers, and define the sets $\mathcal{I}_F = \{1,\dots,N_1\}$ and $\mathcal{I}_L = \{N_1+1,\dots,N\}$. The follower i updates its opinion according to

$$x_i^+ = \frac{\alpha_i}{\sum_{j\in\mathcal{I}_L} \phi_i(x_i,x_j)} \sum_{j\in\mathcal{I}_L} \phi_i(x_i,x_j) x_j + \frac{1-\alpha_i}{\sum_{j\in\mathcal{I}_F} \phi_i(x_i,x_j)} \sum_{j\in\mathcal{I}_F} \phi_i(x_i,x_j) x_j, \quad (53)$$

where $\alpha_i \in [0,1]$ is the trust degree of the agent i on the leaders. The leader i updates its opinion according to

$$x_i^+ = w_i \tau + \frac{1-w_i}{\sum_{j\in\mathcal{I}_L} \phi_i(x_i,x_j)} \sum_{j\in\mathcal{I}_L} \phi_i(x_i,x_j) x_j, \quad (54)$$

where $\tau \in \mathbb{R}$ is the truth and $w_i \in [0,1]$ characterizes the leader's desire to find the truth. This model with two leader groups having opposite opinions has been analyzed through numerical simulations by Zhao, Kou, Peng, and Chen (2018). It has been shown that the followers do not converge to the leaders' opinions, i.e., in such a scenario leader groups have no complete influence power on the followers' opinions.

It can be noticed that the constant term in the models (51)–(54) makes them similar to the seminal Friedkin–Johnsen model (Friedkin, 2015), where the constant vector is usually the initial opinion $\mathbf{x}(0)$, but can also be a more general vector of prejudice (Proskurnikov & Tempo, 2017). Sometimes the constant value τ is also interpreted as an agent's prejudice (Su et al., 2019). Unlike the standard Friedkin–Johnsen model, however, the influence weights depend on the opinion vector $\mathbf{x}$.

10.4. Random noises and disturbances

A number of randomized versions of BCOD have been proposed in the literature. The opinion dynamics becomes random in presence of stochastic exogenous signals influencing the opinions, such as communication noises, environmental disturbances and jumps that are used to portray the "free will" effect (Pineda & Buendía, 2015; Pineda, Toral, & Hernandez-Garcia, 2009; Pineda, Toral, & Hernández-García, 2013).

An arbitrarily small additive disturbance $\xi = \xi(k)$ in the iterative averaging dynamics (5), i.e.,

$$\mathbf{x}^+ = A\mathbf{x} + \xi,$$

as can be easily seen, destroys the boundedness of opinions: Proposition 1 fails to be valid, and, generally speaking, the range of opinions can grow unbounded. In many situations, however, the opinions have to stay in a predefined interval, being, e.g., subjective probabilities (DeGroot, 1974) or certainties of belief (Friedkin et al., 2016). Hence, most BCOD models with additive noises introduce a projection map, preventing the updated opinion x_i^+ from leaving the desired interval.

Assuming, without loss of generality, that the opinions are confined to the interval [0, 1], consider the projector

$$\Pi_{[0,1]}(y) = \begin{cases} 1, & \text{if } y > 1, \\ y, & \text{if } y \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \quad (55)$$

A natural extension of BCOD (17) (Chen, Su, Ding, & Hong, 2019; Su, Chen et al., 2017; Su et al., 2019; Volkova, Manita, & Manita, 2019) is then given by

$$x_i^+ = \Pi_{[0,1]}\Big(\frac{1}{\sum_{j\in\mathcal{I}} \phi_i(x_i,x_j)} \sum_{j\in\mathcal{I}} \phi_i(x_i,x_j) x_j + \xi_i\Big), \quad (56)$$

where $\xi_i = \xi_i(k)$ are random environmental disturbances that are typical supposed to be i.i.d., although this assumption can be substantially relaxed (Chen et al., 2019). In particular, one can suppose that the source of the signal ξ_i is not the environment but a noisy communication: the opinions x_j, $j \in \mathcal{N}_i$, used by the agent i are polluted by noises $\zeta_{ij}(k)$ (Chen et al., 2019). In this situation, $\xi_i(k)$ is the average of the noises $\zeta_{ij}(k)$.

The presence of noises, obviously, makes asymptotic consensus (and even) the convergence of the opinions impossible. For symmetric homogeneous BCOD (56) with the confidence range $d = \ell_i = u_i$ for all $i \in \mathcal{I}$, an elegant result in Su, Chen et al. (2017) (Theorem 5) ensures that in presence of bounded disturbances $|\xi_i(k)| \le d/2$ that are i.i.d. (and satisfy some non-restrictive technical assumptions), a d-practical consensus (see Definition 23) is achieved in finite time with probability 1.

An example of non-additive random disturbance has been considered in a model from Pineda et al. (2013). At each time-step, the agents have the opportunity to change their opinions at random values with some probability equal to p or update it according to BCOD with probability $1-p$. Two possible ranges within which the random value is selected have been proposed: (i) the full interval of possible opinions, for instance, $[\min_{i\in\mathcal{I}} x_i(0), \max_{i\in\mathcal{I}} x_i(0)]$; (ii) a random interval around the current opinion, i.e., $[x_i - \xi, x_i + \xi]$, with $\xi \in \mathbb{R}$ and $i \in \mathcal{I}$. A constant steady state cannot be reached due to the possible choice of random opinions. Numerical simulations have shown that the model exhibits practical clustering (Definition 24); practical clusters containing few agents appear between practical clusters with many agents.

10.5. BCOD with general influence functions

The key property of homophily, that is, assigning influence weights only to agents with similar opinions, is preserved if one replaces the influence function (16) by more general nonlinearities. As has been already mentioned, one can consider the general function (46), where $\mathcal{O}_i$ is not a closed interval or a closed ball (47), but some other bounded set. For instance, in Blondel et al. (2009), *open* symmetric confidence intervals $\mathcal{O}_i = (-d, d)$ have been considered. It can be shown that the properties of such BCOD are not principally different from the symmetric homogeneous BCOD discussed in Section 6, although, obviously, the sets of equilibrium points differ.

In a model from Bernardo, Vasca et al. (2021) and Vasca et al. (2021), the sets $\mathcal{O}_i$ are unions of disjoint open intervals:

$$\phi_i(x_i,x_j) = \begin{cases} 1, & \text{if } -\ell_i < x_j - x_i < -\epsilon, \\ 1, & \text{if } \epsilon < x_j - x_i < u_i, \\ 0, & \text{otherwise.} \end{cases} \quad (57)$$

The (small) similarity interval is introduced to reflect the fact that two agents with similar opinions do not interact; this is a special case of heterophilous behavior (Motsch & Tadmor, 2014).

More generally, one can consider the situation where the influence function $\phi_i(x_i, x_j)$ in the definition of BCOD is replaced by a more general function $\phi_{ij}(x_i, x_j)$ that is determined not only by the agent i, but also by the agent j. For instance, the model with "reputation" of agents has been proposed by Blondel et al. (2009), Chen, Glass, and McCartney (2016) and Douven and Riegler (2010) as a variant of symmetric heterogeneous BCOD, where $\phi_{ij}(x_i, x_j) = \phi_i(x_i, x_j)r_j$, with $\phi_i(x_i, x_j)$ is defined in (16) (where $\ell_i = u_i = d$) and $r_j \in \mathbb{R}^+$ is the reputation of the agent j. Numerical tests have shown that, by reducing the larger reputations and by increasing the confidence thresholds, it is easier to achieve a consensus. A BCOD in which the reputation is proportional to the opinion distance has been proposed by Xi, Liu, and Chai (2022).

In Chen, Zhang, Xie and Li (2017) and Xu, Cai, Wu, Ai, and Xu (2020), an extension of BCOD with static self-influence weights (48) is provided by introducing the concept of *media literacy* for which each agent selects its neighbors not only based on their opinions' closeness but also on their social similarities. Then, the opinions are updated according to (48), where $\phi_i(x_i, x_j)$ is replaced by

$$\phi_{ij}(x_i, x_j) = \begin{cases} 1, & \text{if } w(|m_j - m_i| - \bar{m}) \\ & \quad +(1-w)(|x_j - x_i| - d_i) \le 0, \\ 0, & \text{otherwise,} \end{cases} \tag{58}$$

and $\phi_{ii}(x_i, x_i) = 0$ for all $i \in \mathcal{I}$, where $d_i \in \mathbb{R}$ and $m_i \in \mathbb{R}$ are the confidence threshold and the media literacy of the agent i, respectively, $\bar{m} \in \mathbb{R}$ is the lower limit of the media literacy that the agents can accept and $w \in [0, 1]$ is the weight given to the media literacy. Numerical simulations lead to a conjecture that, for $\bar{m}$ being large and w close to 1, a consensus can be reached regardless of the confidence bounds.

A number of models (Dietrich, Martin, & Jungers, 2016; Jabin & Motsch, 2014; Motsch & Tadmor, 2014) introduce even more general coupling functions $\phi_{ij}(x_i, x_j)$ that attain not only binary values, e.g., $\phi_{ij}(x_i, x_j) = f(\|x_i - x_j\|)$, where $f : [0, \infty) \to [0, \infty)$ is some function with a compact support. Properties of such models are studied mainly in the situations where the weight matrix is type-symmetric and the graph $\mathcal{G}(x)$ is undirected; the convergence of every solution can be proved by using Lemma 6 or energy-type Lyapunov functions (Proskurnikov & Tempo, 2018). Moreover, under some assumption on the function $f(\|x_i - x_j\|)$, Dietrich et al. (2016) have analyzed the diameter of a cluster and its distance from the other agents to provide a lower bound on the number of time-steps required for the cluster before merging with the others (see Theorem 1 therein).

In Hendrickx (2008), a similar influence function $\phi_i(x_i, x_j) = f(x_j - x_i)$ has been considered, where $f : \mathbb{R} \to \mathbb{R}^+$ is a general function with a finite support (this function need not be even). It is shown that this BCOD enjoys the order-preservation property if $\log f$ is a concave function (that attains the value $-\infty$ outside the support of f); this condition is almost necessary (modulo some technical assumptions) for the order-preservation property (Hendrickx, 2008, Theorem 2).

Finally, the influence functions can encode additional *communication constraints*. In classical BCOD, it is assumed that the agents are aware of each other's opinions, which, of course, is a simplification: an agent interacts with its inner circle and not with the whole group. There exist a number of models where the agents can be influenced only by the adjacent agents determined by a communication graph, usually undirected (Etesami, 2019; Fortunato, 2005; Fotakis, Palyvos-Giannas, & Skoulakis, 2016; Lanchier & Li, 2022; Li & Zhang, 2010; Parasnis, Franceschetti, & Touri, 2018, 2021; Schawe et al., 2021). Generalizing symmetric homogeneous BCOD, one can replace $\phi_i(x_i, x_j)$ in (16) by

$$\phi_{ij}(x_i, x_j) = \begin{cases} 1, & \text{if } |x_j - x_i| \le d \text{ and } (j, i) \in \mathcal{E}, \\ 0, & \text{otherwise,} \end{cases} \tag{59}$$

where d is the confidence bound and $\mathcal{E}$ is the edge set of a communication graph. It has been shown that the convergence time and the number of clusters at the steady state decrease as the confidence bound increases. Fotakis et al. (2016) have proved the convergence to a steady state of the model (59) by analyzing the temporal graph connectivity (see Theorem 2 therein). A lower bound on its convergence time has been provided in Parasnis et al. (2018) (Proposition 3) by using the concept of conductance (Kannan, Vempala, & Vetta, 2004). Moreover, under the assumption that the influence graph remains connected, an upper bound of the order $O(N^3 \log N)$ on the time-steps required to achieve a consensus has been given in Parasnis et al. (2021) (Proposition 4). Chen, Wu, Wang and Li (2017) have analyzed a BCOD with limited interactions which includes static self-influence weights and reputation. A model with truth-seeking, noises, communication constraints and reputation has been presented by Riegler and Douven (2010).

10.6. Group influence

The effect of group influence (or group pressure) on an agent's opinion has been examined in some works. The public and private opinions (possibly different) of the agents are considered, and each agent can observe only the expressed opinions of the others (Cheng, Luo, & Yu, 2020; Cheng & Yu, 2019, 2022; Hou, Li, & Jiang, 2021). Denote $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ the private and public opinions of the i-th agent at the time-step k, respectively. At each time-step, the agent i first expresses its public opinion y_i, then observes the public opinion of the others and updates its private opinion as

$$x_i^+ = \alpha_i y_i \quad + \frac{1 - \alpha_i}{1 + \sum_{j \in \mathcal{I}} \phi_i(x_i, y_j)} \left(x_i + \sum_{j \in \mathcal{I}} \phi_i(x_i, y_j) y_j \right), \tag{60}$$

where $\alpha_i \in (0, 1]$ is called *self-persuasion* of the agent i and the influence function $\phi_i(x_i, y_j)$ is defined as

$$\phi_i(x_i, y_j) = \begin{cases} 1, & \text{if } |y_j - x_i| \le d_i \text{ and } j \ne i, \\ 0, & \text{otherwise,} \end{cases} \tag{61}$$

with $d_i \in \mathbb{R}$ is the confidence threshold of the agent i. Once the private opinion is updated, the agent i updates its public opinion according to

$$y_i^+ = (1 - p_i) x_i^+ + p_i \frac{1}{N} \sum_{j \in \mathcal{I}} y_j, \tag{62}$$

where $y_i^+ := y_i(k+1)$ and $p_i \in (0, 1]$ is called *group pressure* of the agent i. It has been proved that the gap between public opinions is less than or equal to the gap between private opinions and when a consensus is reached the private and public opinions have the same value (Cheng et al., 2020, Theorem 1). This result has been proved by showing that the sequence of influence graphs is composition jointly rooted.

The averaged group opinion also appears in Chen et al. (2019), which has introduced a generalization of (56) as follows

$$x_i^+ = \Pi_{[0,1]} \left(\alpha_i \bar{x} + \frac{1 - \alpha_i}{\sum_{j \in \mathcal{I}} \phi_i(x_i, x_j)} \sum_{j \in \mathcal{I}} \phi_i(x_i, x_j) x_j \quad + \zeta_{ij} \right), \tag{63}$$

where $\zeta_{ij} \in [-\eta, \eta]$ are noises, $\bar{x} = \frac{1}{N} \sum_{j \in \mathcal{I}} x_j$ is the average of the opinions intended as a background opinion which influences all agents and $\alpha_i \in [0, 1]$ is the weight given to the background opinion.

11. Applications and validation of BCOD

Recently, opinion dynamics models have been tested by using real data on different application fields (Bernardo, Wang et al., 2021; Dong, Zhan, Kou, Ding, & Liang, 2018; Li, Liu and Chai, 2022; Zha et al., 2020). More specifically, some authors have considered practical validations for synchronous bounded confidence mechanisms.

Homogeneous BCOD with self-weights has been proposed by Wan, Ma, and Pan (2018) to describe the dynamic nature of online consumer reviews in the context of e-commerce. According to this mechanism, the viewer's opinion depends on the number of reviews considered, their order, which is determined by the specific e-commerce, and the confidence bounds. The model has been tested using the reviews of 100 products from Amazon.com, by showing that the opinions on a product converge to a consensus or at most two clusters.

A randomized opinion dynamics model in which agents interact according to both the bounded confidence strategy and the geographic distance has been proposed by Haensch, Dragovic, Börgers, and Boghosian (2023) to describe the opinions' evolution about Covid-19 vaccination in the United States. In order to model external influences, such as those from mass media and governors, the agent set includes two stubborn agents having opposite extreme opinions. Simulation results have validated the proposed BCOD by using real data collected in two different time frames.

A symmetric homogeneous BCOD with limited interactions and variable confidence bound has been used for the community detection problem by Morarescu and Girard (2010). According to this approach, the communities are the sets of agents that reach the same opinion. The effectiveness of the proposed approach has been shown through a comparison with the typical community detection approaches on three examples.

Pilyugin and Campi (2019) have proposed a dynamical model of voting process that is based on the principle of bounded confidence and contains the synchronous BCOD model as a special case. Voters have to choose between two alternatives, and real-valued opinions varying in $[-1, 1]$ measure the preferences for these two options. The principle of opinion dynamics reflects the phenomenon of "reinforcement" in social psychology: "an agent's opinion has a tendency to reinforce and drift towards a higher level of belief in the absence of opposite voices" (Pilyugin & Campi, 2019). Convergence of the proposed model and the structure of equilibria have been examined under certain conditions.

In a broader sense, some authors consider as an "application" the study of control strategies for achieving specific steady state behaviors in discrete-time BCOD. For instance, Ding et al. (2016) have proposed an optimization strategy to ensure that a consensus is progressively reached in BCOD for any confidence bound. At each time-step, the solution of an optimization problem determines the minimum number of opinions to adjust, respecting some threshold constraints on the adjustments.

Other approaches consist in changing the confidence intervals in order to achieve some desired behaviors. Models in which the confidence thresholds of some silent agents are increased allow to achieve the complete confidence graph in finite time, i.e., a consensus is reached (Iervolino et al., 2018). Bernardo, Vasca et al. (2021) and Vasca et al. (2021) have proposed other confidence threshold adaptation policies, where it is assumed that the lower (upper) confidence threshold of each agent is increased such that it would connect with at least one neighbor having a lower (upper) opinion, and each agent has a maximum number of interacting agents. These policies, with sufficiently large thresholds, ensure a finite-time practical consensus, as proved in Vasca et al. (2021) (Theorem 9) and Bernardo, Vasca et al. (2021) (Theorem 4).

An optimal opinion control has been proposed by Hegselmann, König, Kurz, Niemann, and Rambau (2015) in order to maximize the number of opinions that reach a desired opinion interval at a given time instant. In this control strategy, there is one agent that freely chooses its opinion at each time-step. Liang et al. (2019) have addressed a problem of optimal adjusting the initial opinions in order to achieve a consensus in BCOD at a given time-step. Kurz (2015) has proposed a control algorithm to minimize the convergence time by determining the opinion values of some strategic agents at each time instant.

12. Conclusions

An overview of synchronous BCOD and, more specifically, of the properties satisfied by different classes of this system has been presented. The study concentrates on discrete-time models with scalar opinions and possibly asymmetric and heterogeneous confidence intervals. It is shown that heterogeneity, asymmetric confidence intervals and stubbornness, even though they introduce local differences in the system evolution, do not destroy the typical asymptotic behavior in BCOD models, which is characterized by convergence to (practical) consensus or clustering. Conditions for having such steady states in the different classes of BCOD are discussed.

Several directions for future research emerge from the analysis proposed in this survey, mainly from the theoretical, numerical and applications points of view. From a formal perspective, there are fundamental conjectures that still require rigorous justifications, such as: the absence of non-vanishing oscillations also for asymmetric heterogeneous models; the sensitivity of the convergence time and the number of clusters on the confidence thresholds; the convergence in finite time to a practical clustering also in the presence of stubborn agents; which properties are still valid in the presence of multiple stubborn agents. Solving these problems is far from trivial in general, but help in this sense could be achieved by assuming some simplifying hypotheses, such as for example specific initial conditions. Furthermore, the concept of practical consensus seems to be an approach that has not yet been fully exploited in the literature. More generally, similar overviews to the one reported in this survey but dedicated to asynchronous and continuous-time BCOD could be useful contributions for the research field.

This survey is mainly focused on the existing rigorously proven results. On the other hand, the literature on numerical analyses of BCOD is huge, but a comprehensive organization of the evidences which have been highlighted in simulations is still missing. The proposed presentation of the theoretical results existing for BCOD models could provide the basis for a structured discussion of numerical analyses available in the literature.

Last but not least, by providing a systematic analysis of the main properties of the various BCOD models, we also aim at highlighting which main features could be of relevance from an application perspective. The applications briefly discussed herein show that the main challenge ahead is still to understand how to use these dynamical behaviors in a constructive way in the various contexts in which they could play a role, which potentially span from engineering to sociology, from psychology to economics.

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Appendix A. Basic concepts of graph theory

A graph $\mathcal{G}$ is a pair of sets $\{\mathcal{I}, \mathcal{E}\}$, where $\mathcal{I}$ is the node set and $\mathcal{E}$ is the set of edges (or arcs) between the nodes. In a *directed* graph (digraph), the arcs are ordered pairs of nodes $\mathcal{E} \subseteq \mathcal{I} \times \mathcal{I}$, and the pair $(i, j) \in \mathcal{E}$ is said to be an edge from i to j. In an *undirected* graph, each edge $\{i, j\}$ is an unordered pair of nodes. In the literature, undirected graphs are often considered as a subclass of digraphs by replacing each undirected edge $\{i, j\}$ by a pair of directed arcs (i, j), (j, i).

The node j is a *neighbor of the node* i if $(j, i) \in \mathcal{E}$, i.e., *there exists an edge from* j *to* i. The neighbor set of a node i is defined as $\mathcal{N}_i = \{j \in \mathcal{I} \mid (j, i) \in \mathcal{E}\}$. A graph is *weighted* if a non-negative weight is associated to each edge. The adjacency matrix A of a weighted graph is a $N \times N$ matrix whose element a_{ij} is the positive weight of the edge from the node j to the node i if $(j, i) \in \mathcal{E}$, and 0 otherwise[8]. The in-degree (out-degree) of the node i of a graph is the number of the edges which are incident to (outgoing from) i. A path is a sequence of edges that joins a sequence of pairwise distinct nodes. An undirected graph is connected if there exists a path between any pair of nodes. A digraph is *strongly connected* if for every pair of distinct nodes i and j there exists a directed path from i to j. A digraph is *weakly connected* if replacing all of its directed edges with undirected edges produces a connected undirected graph. A graph is *complete* if $(i, j) \in \mathcal{E}$ for any $i, j \in \mathcal{I}$.

A subgraph $\mathcal{G}^* = \{\mathcal{I}^*, \mathcal{E}^*\}$ of $\mathcal{G} = \{\mathcal{I}, \mathcal{E}\}$ is a graph such that $\mathcal{I}^* \subseteq \mathcal{I}$ and $\mathcal{E}^* \subseteq \mathcal{E}$, and it inherits all the arcs from the parent graph that connect the nodes from $\mathcal{I}^*$, i.e., $\mathcal{E}^* = \mathcal{E} \cap (\mathcal{I}^* \times \mathcal{I}^*)$. The induced subgraph is thus uniquely determined by its node set $\mathcal{I}^*$. A subgraph $\mathcal{G}^*$ is a *strongly (weakly) connected component* of the graph $\mathcal{G}$ if $\mathcal{G}^*$ is strongly (weakly) connected and any other subgraph of $\mathcal{G}$ strictly containing $\mathcal{G}^*$ is not strongly (weakly) connected. It can be shown that the node sets of strongly (weakly) connected components constitute a partition of the node set $\mathcal{I}$, i.e., each node of a graph belongs to only one strongly (weakly) connected component. Notice that weakly connected components of an undirected graph are strongly connected components; in this case, they are called connected components. A weakly connected component of a graph is *isolated* in the sense that any agent in that component has no edges with agents outside the component; this condition does not hold for strongly connected components.

A node i is a *root*[9] of the graph $\mathcal{G}$ if there exists a path from i to j for all $j \in \mathcal{I}$ (Cao et al., 2008). The graph $\mathcal{G}$ is *rooted at* i if the node i is a root. If a graph is strongly connected then it is rooted at every node; on the other hand, if a graph is rooted then it is weakly connected. Some examples of different types of graphs are shown in Fig. 11.

The *union* of two graphs $\mathcal{G}_1 = \{\mathcal{I}, \mathcal{E}_1\}$ and $\mathcal{G}_2 = \{\mathcal{I}, \mathcal{E}_2\}$ is the graph with the same node set and the edge set $\mathcal{E}_1 \cup \mathcal{E}_2$. The *composition* of two graphs $\mathcal{G}_1 = \{\mathcal{I}, \mathcal{E}_1\}$ and $\mathcal{G}_2 = \{\mathcal{I}, \mathcal{E}_2\}$ having the same node set, denoted by $\mathcal{G}_2 \circ \mathcal{G}_1$, is the graph with the same node set and the edge set such that (i, j) is an edge of the composition if $(i, q) \in \mathcal{E}_1$ and $(q, j) \in \mathcal{E}_2$ for some $q \in \mathcal{I}$. In general, the composition is not commutative, i.e., $\mathcal{G}_2 \circ \mathcal{G}_1 \neq \mathcal{G}_1 \circ \mathcal{G}_2$. If each node of the graphs has a self-loop, then all edges of the union of the graphs are also edges of their composition.

[8] In dynamical networks theory (Bullo, 2022), the opposite convention is sometimes used: $(j, i) \in \mathcal{E}$, i.e., there exists an edge from j to i, if and only if $a_{ji} > 0$. The convention herein is adopted without loss of generality and follows the pioneering work on opinion dynamics (French, 1956) and the tradition of multi-agent control (Moreau, 2005; Ren & Beard, 2005). This convention is important for the consistency of some definitions and statements reported in the paper.

[9] More formally, a node i is the root of an (out-branched) spanning tree of the graph (Chebotarev & Agaev, 2002). Rooted graphs are also referred to as graphs with spanning trees (Ren & Beard, 2005) or quasi strongly connected graphs.

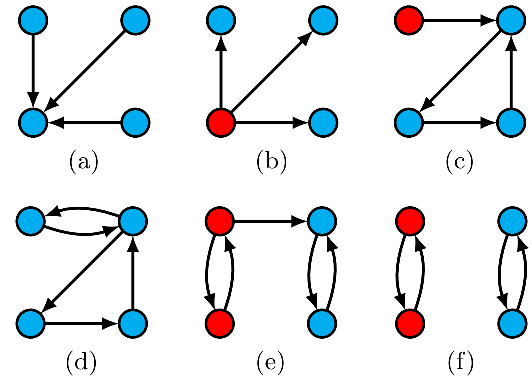

Fig. 11. Examples of graphs: weakly connected but not rooted (a), rooted at the red node but not strongly connected (b), rooted at the red node and the blue nodes constitute a strongly connected component (c), strongly connected but not complete (d), two strongly connected components which are not clusters (e), two clusters (f). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Appendix B. Proofs

Proof of Proposition 1. Since $\mathcal{N}_i$ contains at least the i-th agent and by considering (1) which is the scalar version of (5), it follows

$$x_i^+ = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} x_j \leq \frac{1}{|\mathcal{N}_i|} |\mathcal{N}_i| \max_{j \in \mathcal{N}_i} x_j = \max_{j \in \mathcal{N}_i} x_j$$

for all $i \in \mathcal{I}$, which proves (6a), and

$$x_i^+ = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} x_j \geq \frac{1}{|\mathcal{N}_i|} |\mathcal{N}_i| \min_{j \in \mathcal{N}_i} x_j = \min_{j \in \mathcal{N}_i} x_j$$

for all $i \in \mathcal{I}$, which proves (6b).

The second part of the proof directly follows from the condition $\min_{j \in \mathcal{I}} x_j(0) \leq x_i(0) \leq \max_{j \in \mathcal{I}} x_j(0)$, for all $i \in \mathcal{I}$, and the application of (6).

Proof of Proposition 3. By applying Definition 2 together with (5) one can write

$$\begin{aligned} v(\mathbf{x}^+) &= \max_{i \in \mathcal{I}} x_i^+ - \min_{i \in \mathcal{I}} x_i^+ = \max_{i,j \in \mathcal{I}} (x_i^+ - x_j^+) \\ &= \max_{i,j \in \mathcal{I}} \sum_{w \in \mathcal{I}} (a_{iw} - a_{jw}) x_w. \end{aligned}$$

Moreover, it is

$$\begin{aligned} \sum_{w \in \mathcal{I}} (a_{iw} - a_{jw}) x_w &= \sum_{w \in \mathcal{I}} \left(a_{iw} - \min\{a_{iw}, a_{jw}\}\right) x_w \\ &\quad - \sum_{w \in \mathcal{I}} \left(a_{jw} - \min\{a_{iw}, a_{jw}\}\right) x_w \\ &\leq \sum_{w \in \mathcal{I}} \left(a_{iw} - \min\{a_{iw}, a_{jw}\}\right) \max_{w \in \mathcal{I}} x_w \\ &\quad - \sum_{w \in \mathcal{I}} \left(a_{iw} - \min\{a_{iw}, a_{jw}\}\right) \min_{w \in \mathcal{I}} x_w \\ &= \sum_{w \in \mathcal{I}} a_{iw} \left(\max_{w \in \mathcal{I}} x_w - \min_{w \in \mathcal{I}} x_w\right) \\ &\quad - \sum_{w \in \mathcal{I}} \min\{a_{iw}, a_{jw}\} \left(\max_{w \in \mathcal{I}} x_w - \min_{w \in \mathcal{I}} x_w\right) \\ &= \left(1 - \sum_{w \in \mathcal{I}} \min\{a_{iw}, a_{jw}\}\right) \left(\max_{w \in \mathcal{I}} x_w - \min_{w \in \mathcal{I}} x_w\right) \\ &= \left(1 - \sum_{w \in \mathcal{I}} \min\{a_{iw}, a_{jw}\}\right) v(\mathbf{x}), \end{aligned}$$

where in the second last step we used the condition that the matrix A is row-stochastic.

Then, it is

$$\begin{aligned} v(\mathbf{x}^+) &= \max_{i,j\in\mathcal{I}} \sum_{w\in\mathcal{I}} (a_{iw} - a_{jw}) x_w \\ &\le \max_{i,j\in\mathcal{I}} \left(1 - \sum_{w\in\mathcal{I}} \min\{a_{iw}, a_{jw}\}\right) v(\mathbf{x}) \\ &= \left(1 - \min_{i,j\in\mathcal{I}} \sum_{w\in\mathcal{I}} \min\{a_{iw}, a_{jw}\}\right) v(\mathbf{x}), \end{aligned}$$

which completes the proof.

Proof of Lemma 4. The proof is reported in Cao et al. (2008) (Theorem 3, p. 596).

Proof of Lemma 6. Lemma 6 follows from Proskurnikov et al. (2020) (Theorem 5) by noticing that the type-symmetry (11) is a special case of the cut-balance condition (Proskurnikov et al., 2020, Eq. (14)). Notice also the difference in the notation: the matrices $A(k)$ in Proskurnikov et al. (2020) are denoted by $W(k)$.

The first part of Lemma 6 establishing convergence of each solution (also in the general case of cut-balance graph) was first published as Bolouki and Malhame (2015) (Theorem 1).

Proof of Corollary 7. The proof is straightforward by applying Lemma 6 to the initial condition $x(0) = \mathbf{e}_m$ (the coordinate basis vector whose m-th element is 1 and the others are 0). Denoting $\Pi(k) := A(k-1)\dots A(0)$, $k \ge 1$, Lemma 6 ensures that $x(k) = \Pi(k)\mathbf{e}_m$ converges as $k \to \infty$ for all $m = 1, \dots, N$, that is, each column of $\Pi(k)$ has a limit as $k \to \infty$. This proves the existence of a limit

$$A^\infty = \lim_{k\to\infty} \Pi(k),$$

and the limit of the solution with initial condition $x(0)$ is nothing else than $\lim_{k\to\infty} x(k) = \lim_{k\to\infty} \Pi(k)x(0) = A^\infty x(0)$. Furthermore, the i-th and j-th elements of the latter vector are coincident for each value $x(0)$, which is possible only if the i-th and j-th rows of A^∞ are equal.

Proof of Corollary 9. The proof is straightforward from the second statement of Lemma 6. Indeed, we know that the limits of the opinions x_i^∞ and x_j^∞ are the same for every agents i and j that satisfy (12), that is, for each i and j that are connected by an edge in the persistent graph. Hence, the same equality $x_i^\infty = x_j^\infty$ holds for every two agents i and j that are connected by a *path* of an arbitrary length in the persistent graph (this can be proved, e.g., via the induction on the path's length).

Proof of Proposition 26. By hypothesis there exist $i, j \in \mathcal{I}$ such that at the time-step k it is $x_i < x_j - d_M$ and there does not exist any agent $q \in \mathcal{I}$ such that $x_q \in (x_i, x_j)$. Then, it is $\phi_i(x_i, x_r) = 0$ for any $r \in \mathcal{I}$ such that $x_r > x_i$, which implies $x_i^+ \le x_i$. Similarly, from $x_j > x_i + d_M$ it is $\phi_i(x_i, x_p) = 0$ for any $p \in \mathcal{I}$ such that $x_p < x_j$, and thus $x_j^+ \ge x_j$. Therefore, it is $x_i^+ \le x_i < x_j - d_M \le x_j^+ - d_M$, which implies the thesis.

Proof of Theorem 27. The sufficiency part of the proof is a direct application of Lemma 4 to (17).

For the necessary part, assume that for a given initial condition $\mathbf{x}(0)$ the solution of (17) reaches an asymptotic consensus (10). First consider the case $d_m > 0$, i.e., there are no stubborn or one-sided confident agents. By hypothesis, it is $v(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$ and then for any $\epsilon \in (0, d_m)$ there exists a finite time-step k_ϵ such that $v(\mathbf{x}(k)) \le \epsilon$ which implies that $\mathcal{N}_i(k) = \mathcal{I}$ for all $i \in \mathcal{I}$ and all $k \ge k_\epsilon$. The graph $\mathcal{G}(\mathbf{x}(k_\epsilon))$ is then complete and the composition of the graphs in the interval $[0, k_\epsilon]$ will be rooted. From Proposition 3, any graph $\mathcal{G}(\mathbf{x}(k))$ for $k \ge k_\epsilon$ will be complete and then the infinite sequence of graphs is repeatedly composition jointly rooted.

Consider the case of the presence of one-sided confident agents, all of them with zero lower (upper) bounds, i.e., $\ell_i = 0$ ($u_i = 0$) for all $i \in \mathcal{I}$ such that i is a one-sided confident agent. Say

$$\tilde{d}_m = \min_{i\in\mathcal{I}} \min\{\ell_i, u_i \mid \ell_i > 0, u_i > 0\}. \tag{64}$$

By repeating the arguments above with $\epsilon \in (0, \tilde{d}_m)$ it follows that the graph $\mathcal{G}(\mathbf{x}(k_\epsilon))$ is at least rooted and then, since $v(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$ by hypothesis, the infinite sequence of graphs is repeatedly composition jointly rooted.

Consider now the case of the presence of one-sided confident agents with zero upper or lower thresholds, and consider any two of them i and j such that $\ell_i = 0$ and $u_j = 0$. Since the opinion x_i can only increase and the opinion x_j can only decrease, it must be $x_i \le x_j$ for all k (with the equality holding only at the consensus value) otherwise consensus would never be reached. Analogously, if $u_i = 0$ and $\ell_j = 0$ it must be $x_i \ge x_j$ for all k (with the equality holding only at the consensus value). Then, by repeating the arguments above with $\epsilon \in (0, \tilde{d}_m)$ and $\tilde{d}_m$ defined by (64), it follows that the infinite sequence of graphs is repeatedly composition jointly rooted.

If one or more stubborn agents having the same opinion are present and an asymptotic consensus (10) is reached, then the opinion at a consensus, say c, must be the opinion of all stubborn agents. By hypothesis it is $v(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$ and then for any $\epsilon \in (0, \tilde{d}_m)$ with $\tilde{d}_m$ defined by (64), there exists a finite time-step k_ϵ such that $v(\mathbf{x}(k)) \le \epsilon$ which implies that $\phi_i(x_i, c) = 1$ for all $i \in \mathcal{I}$ and, by using Proposition 3, for all $k \ge k_\epsilon$. The graph $\mathcal{G}(\mathbf{x}(k_\epsilon))$ is then rooted from all stubborn agents, and the composition of the graphs in the interval $[0, k_\epsilon]$ will be rooted. Since any graph $\mathcal{G}(\mathbf{x}(k))$ for $k \ge k_\epsilon$ will be at least rooted from the stubborn agents, the infinite sequence of graphs is repeatedly composition jointly rooted and the necessary part of the proof is complete.

The final statement of the theorem is a direct consequence of the arguments above, i.e., when k increases the graphs $\mathcal{G}(\mathbf{x}(k))$ eventually become rooted.

Proof of Proposition 30. By hypothesis, there exists a finite time-step such that $v(\mathbf{x}) \le d_m$. Then, it is $\phi_i(x_i, x_j) = 1$ for all $i, j \in \mathcal{I}$, i.e., the graph $\mathcal{G}(\mathbf{x})$ is complete, and from Proposition 3 it follows that all future confidence graphs are complete. Therefore the right-hand side of (17) is the same for all $i \in \mathcal{I}$ which implies that $x_i^+ = \frac{1}{N}\sum_{i=1}^N x_i$.

Proof of Theorem 31. Say m and M one of the agents with the minimum and maximum opinion at the time-step k, respectively, i.e., $x_m = \min_{i\in\mathcal{I}} x_i$ and $x_M = \max_{i\in\mathcal{I}} x_i$. Note that the agents m and M could change from one step to the next. Since the minimum opinion is nondecreasing and, according to Proposition 1, is bounded in the range of the initial opinions, there exists a real number $c_m \in [\min_{i\in\mathcal{I}} x_i(0), \max_{i\in\mathcal{I}} x_i(0)]$ such that

$$\lim_{k\to\infty} x_m = c_m. \tag{65}$$

From (65), it follows that for any $\epsilon > 0$ there exists a finite time-step $\hat{k}$ such that $x_m \in (c_m - \epsilon, c_m]$ for all $k \ge \hat{k}$. Suppose that $\epsilon \in (0, d_m/N^2)$, define the set

$$\mathcal{A} = \{i \in \mathcal{I} \mid x_i \in (c_m - \epsilon, c_m + (N-1)\epsilon]\},$$

and denote $\hat{\mathcal{A}} := \mathcal{A}(\mathbf{x}(\hat{k}))$, $\hat{x}_i = x_i(\hat{k})$ and $\hat{x}_i^+ = x_i(\hat{k}+1)$ for all $i \in \mathcal{I}$.

Suppose that $\hat{\mathcal{A}} = \mathcal{I}$, i.e., all agents are in the set $\hat{\mathcal{A}}$. Then, it is

$$x_{i_M} - x_{i_m} < c_m + (N-1)\epsilon - c_m + \epsilon = N\epsilon \leq \frac{d_m}{N} \tag{66}$$

for all $k \geq \hat{k}$, and a consensus is reached at most at the time-step $\hat{k}+1$, according to Proposition 30.

Otherwise, if $|\hat{\mathcal{A}}| < N$, suppose that there exists an agent q such that $\hat{x}_q \in (c_m + (N-1)\epsilon, c_m + d_m - \epsilon]$. Thus, by definition of influence function (16), any agent in $\hat{\mathcal{A}}$ is influenced by q, i.e., $\phi_i(\hat{x}_i, \hat{x}_q) = 1$ for all $i \in \hat{\mathcal{A}}$. By using (17), the following inequalities

$$\begin{aligned}
\hat{x}_i^+ &\geq \frac{1}{|\hat{\mathcal{A}}|+1}\left(\sum_{j\in\hat{\mathcal{A}}} \hat{x}_j + \hat{x}_q\right) \\
&> \frac{1}{|\hat{\mathcal{A}}|+1}(|\hat{\mathcal{A}}|(c_m - \epsilon) + c_m + (N-1)\epsilon) \\
&= c_m - \epsilon + \frac{N}{|\hat{\mathcal{A}}|+1}\epsilon \geq c_m
\end{aligned}$$

hold for all $i \in \hat{\mathcal{A}}$ since $N/(|\hat{\mathcal{A}}|+1) \geq 1$ by hypothesis. The inequalities above contradict the condition in (65). Then, it must be $\hat{x}_q > c_m + d_m - \epsilon$ for all $q \notin \hat{\mathcal{A}}$. By using (17), it is

$$\begin{aligned}
\hat{x}_q^+ &= \frac{1}{|\hat{\mathcal{N}}_q|}\left(\sum_{j\in\hat{\mathcal{A}}\cap\hat{\mathcal{N}}_i} \hat{x}_j + \sum_{j\in\hat{\mathcal{N}}_i\setminus\hat{\mathcal{A}}} \hat{x}_j\right) \\
&\geq \frac{1}{|\hat{\mathcal{N}}_q|}\left((|\hat{\mathcal{N}}_q|-1)\hat{x}_m + \hat{x}_q\right) \\
&> \frac{1}{|\hat{\mathcal{N}}_q|}\left((|\hat{\mathcal{N}}_q|-1)(c_m-\epsilon) + c_m + d_m - \epsilon\right) \\
&= c_m + \frac{d_m}{|\hat{\mathcal{N}}_q|} - \epsilon \geq c_m + \frac{d_m}{|\hat{\mathcal{N}}_q|} - \epsilon \\
&> c_m + (N-1)\epsilon
\end{aligned}$$

for all $q \notin \hat{\mathcal{A}}$, where $\hat{\mathcal{N}}_q := \mathcal{N}_i(\hat{k})$ is the neighbor set of the agent q at the time-step $\hat{k}$. The inequalities above imply that if the agent q is not in the set $\hat{\mathcal{A}}$, it will never be included in this set, i.e., $q \notin \mathcal{A}$ for all $k \geq \hat{k}$. Moreover, if an agent $q \notin \hat{\mathcal{A}}$ influences some agent $r \in \hat{\mathcal{A}}$, by applying (17), it follows

$$\begin{aligned}
\hat{x}_r^+ &\geq \frac{1}{|\hat{\mathcal{A}}|+1}\left(\sum_{j\in\hat{\mathcal{A}}} \hat{x}_j + \hat{x}_q\right) \\
&> \frac{1}{|\hat{\mathcal{A}}|+1}\left(|\hat{\mathcal{A}}|(c_m-\epsilon) + c_m + d_m - \epsilon\right) \\
&\geq c_m + \frac{d_m}{|\hat{\mathcal{A}}|+1} - \epsilon \geq c_m + \frac{d_m}{N} - \epsilon \\
&> c_m + (N-1)\epsilon.
\end{aligned}$$

These inequalities and the definition of the set $\mathcal{A}$ imply that if an agent $r \in \hat{\mathcal{A}}$ is influenced by agents which are not included in the set $\hat{\mathcal{A}}$, then the agent r will leave the set $\hat{\mathcal{A}}$ at the next time-step. Due to the finite number of agents, there exists a finite time-step $\bar{k}_m \in \mathbb{N}_0$ from which the agents in $\mathcal{A}$ have no neighbors outside the set $\mathcal{A}$, i.e.,

$$\mathcal{N}_i \cap (\mathcal{I} \setminus \mathcal{A}) = \emptyset \tag{67}$$

for all $i \in \mathcal{A}$ and $k \geq \bar{k}_m$. By combining (65) and (67), it follows that $x_i = c_m$ for all $i \in \mathcal{A}$ and $k \geq \bar{k}_m$ and thus the minimum opinion remains constant.

By applying similar arguments to the maximum agent, it follows that there exists a finite time-step $\bar{k}_M \in \mathbb{N}_0$ such that the maximum opinion remains constant at the value c_M. Then, the range of opinions is constant for all $k \geq \bar{k}$, with $\bar{k} = \max\{\bar{k}_m, \bar{k}_M\}$.

Suppose now that $\nu(\mathbf{x}) \leq 2d_m$, and denote $\mathcal{F} = \{i \in \mathcal{I} \mid x_i = c_m \text{ or } x_i = c_M\}$ the set of agents having the minimum or the maximum opinion. If $\mathcal{F} = \mathcal{I}$ at $\bar{k}$, then the opinions are at a consensus (if $c_m = c_M$) or at a clustering with $M = 2$. Otherwise, if $\mathcal{I} \setminus \mathcal{F} \neq \emptyset$ at $\bar{k}$, then there exists at least one agent q such that either $x_q - c_m \leq d_m$ or $c_M - x_q \leq d_m$. The latter inequalities contradict the condition (67) and the analogous one for the agents having maximum opinion, respectively. Then, the proof is complete.

Proof of Theorem 32. The sufficiency part starts by considering that if the sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ is repeatedly composition jointly rooted then by applying Theorem 27 it is $\nu(\mathbf{x}(k)) \xrightarrow[k\to\infty]{} 0$. Moreover, from Theorem 31 there exists a finite time-step such that the range of opinions becomes constant, which implies that there exists a finite $\bar{k} \in \mathbb{N}$ such that $\nu(\mathbf{x}(k)) = 0$ for all $k \geq \bar{k}$.

For the necessary part, assume that a consensus is reached in finite time, say $\bar{k}$. Then, the graph $\mathcal{G}(\mathbf{x}(\bar{k}))$ is complete, and the composition of all graphs in $[0, \bar{k}]$ is complete and then rooted too. By considering contiguous, nonempty, bounded time-intervals $[k_i, k_{i+1})$, $i \geq 0$, with $k_0 = 0$, $k_1 = \bar{k}$ and arbitrary k_{i+1} with $i \geq 1$, the composition of the graphs in each subinterval will be rooted, and then the sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ is repeatedly composition jointly rooted.

Proof of Corollary 33. The proof directly follows from Theorem 32 because if each graph of the sequence $\mathcal{G}(\mathbf{x}(k))$ is strongly connected then any subsequence of these graphs is composition jointly rooted.

Proof of Theorem 34. The sufficiency part of the theorem can be proved by induction. For $N = 1$ the theorem is trivial because the initial time corresponds to the convergence time and the agent is never silent. Assume now that the theorem holds for some $\bar{N} \in \mathbb{N}$ with $\bar{N} < N$. We now prove that the theorem is satisfied also for N agents. First, we show that there exists a finite time-step $\bar{k}$ such that the left product of $\bar{k}$ consecutive matrices $A(\mathbf{x})$ defined in (18) will have at least one positive column.

Consider the agent 1, and define the set

$$S = \{i \in \mathcal{I} \mid (A(\mathbf{x}(k))A(\mathbf{x}(k-1))\dots A(\mathbf{x}(0)))_{i1} > 0\} \tag{68}$$

and its complement $S^c = \mathcal{I} \setminus S$. Since $\phi_{ii} = 1$ for all $i \in \mathcal{I}$, if $i \in S$ it is $i \in S^+$ for any $k \in \mathbb{N}_0$. If at the time-step $\bar{k}$ it is $S = \mathcal{I}$, the hypothesis of having a positive column in the left product of $\bar{k}$ consecutive matrices $A(\mathbf{x})$ holds. Otherwise, assume that it exists a time-step k^* since then the set S will not change over time. From the definition (68) any agent in S^c cannot be influenced by those in S. Since the set S contains at least the agent 1, the inequality $|S^c| < N$ holds for all $k \geq k^*$. Then, by using the induction assumption, the agents in S^c will reach a steady state in finite time interval $\bar{\Delta}$. However, under the hypothesis of the theorem, the agents cannot remain silent for more Δ time-steps. Then, after at most $\bar{\Delta} + \Delta$ time-steps at least one agent is added in the set S. Due to the finite number of the agents, $T = (N+1)(\bar{\delta} + \Delta)$ time-steps guarantee the satisfaction of the condition $S = \mathcal{I}$, which implies that the first column of the matrix $A(\mathbf{x}(k))A(\mathbf{x}(k-1))\dots A(\mathbf{x}(0))$ is positive.

Since by definition the non-null entries of $A(\mathbf{x})$ are bounded from below by $1/N$, the minimum positive entry of the left product of T consecutive matrices $A(\mathbf{x})$ is equal to $(1/N)^T$. By using (9) in Proposition 3, after T time-steps the range of opinions decreases by at least $(1/N)^T$. Therefore, there exists a finite time-step such that $v(\mathbf{x}) \leq d_m$ which implies the convergence to a consensus at the next time-step, according to Proposition 30.

The necessary part of the theorem is straightforward by considering the fact that, according to Definition 14, at a consensus all agents share the same opinion and then no silent agents exist.

Proof of Proposition 35. For $N = 1$, the statement is trivial. For $N > 1$, consider the agents q and r with $x_q \leq x_r$ and their sets of neighbors defined as $\mathcal{N}_q = \{i \in \mathcal{I} \mid -\ell \leq x_i - x_q \leq u\}$ and $\mathcal{N}_r = \{i \in \mathcal{I} \mid -\ell \leq x_i - x_r \leq u\}$, respectively. Let $\mathcal{N}_{qr} = \mathcal{N}_q \cap \mathcal{N}_r$, assume that $\mathcal{N}_{qr} \neq \emptyset$, and define $\widehat{\mathcal{N}}_q = \mathcal{N}_q \backslash \mathcal{N}_{qr}$ and $\widehat{\mathcal{N}}_r = \mathcal{N}_r \backslash \mathcal{N}_{qr}$. The means of the opinions in the three sets $\mathcal{N}_{qr}$, $\widehat{\mathcal{N}}_q$ and $\widehat{\mathcal{N}}_r$ are given by

$$\bar{x}_{qr} = \frac{\sum_{i \in \mathcal{N}_{qr}} x_i}{|\mathcal{N}_{qr}|}, \ \bar{x}_q = \frac{\sum_{i \in \widehat{\mathcal{N}}_q} x_i}{|\widehat{\mathcal{N}}_q|}, \ \bar{x}_r = \frac{\sum_{i \in \widehat{\mathcal{N}}_r} x_i}{|\widehat{\mathcal{N}}_r|}, \tag{69}$$

respectively. Clearly, it is $\bar{x}_q \leq \bar{x}_{qr}$ because $x_q \leq x_r$ by hypothesis and $\widehat{\mathcal{N}}_q$ contains at least the agent q. Since the opinion evolution (17) is the mean of the opinions in the neighbor set, it follows

$$x_q^+ = \frac{|\widehat{\mathcal{N}}_q|\bar{x}_q + |\mathcal{N}_{qr}|\bar{x}_{qr}}{|\widehat{\mathcal{N}}_q| + |\mathcal{N}_{qr}|} \leq \bar{x}_{qr}. \tag{70}$$

Similarly, the inequality $x_r^+ \geq \bar{x}_{qr}$ holds for the agent r. From the last two inequalities, it follows $x_q^+ \leq x_r^+$, thus completing the proof.

Proof of Theorem 36. The convergence in finite time of any solution $\mathbf{x}(k)$ to a constant steady state which is either a consensus or a clustering and the upper bound on the convergence time expressed by (23) is a direct application of Theorem 40 to the case of homogeneous *symmetric* BCOD.

Let us prove the second part of the statement, i.e., a consensus is reached if and only if $\mathbf{x}(k)$ is a d-chain with $d = \ell = u$. First note that for symmetric homogeneous BCOD the graphs $\mathcal{G}(\mathbf{x}(k))$ are undirected and then the rooted condition is equivalent to the connected one. From Definition 16, it follows that the opinion vector is a d-chain if and only if the (undirected) confidence graph is connected.

Assume that $\mathbf{x}(k)$ is a d-chain. Then, each graph is connected, and the sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ is repeatedly composition jointly rooted. Therefore, by using Theorem 32, it follows that a consensus is reached in finite time, which completes the sufficiency part.

Assume that a consensus is reached in finite time. Then, from Theorem 32 it follows that the sequence of graphs $\mathcal{G}(\mathbf{x}(k))$ is repeatedly composition jointly rooted (and connected). This implies that any graph of the sequence must be connected, as it can be shown by contradiction. Indeed, if a crack would occur at some time-step, from Proposition 26 with $d_M = d$ it follows that the two agents involved in that crack will never interact anymore in the future and the hypothesis of reaching a consensus would be contradicted. Then, the necessary part is also proved, and the proof is complete.

Proof of Proposition 37. The proof can be obtained by combining the arguments of Theorem 7.3 in Motsch and Tadmor (2014), Theorem 4.3 in Bhattacharyya et al. (2013) and Theorem 2 in Roozbehani et al. (2008), so as detailed below.

Consider (30). It is easy to verify that

$$g(x_i^+, x_j^+) - g(x_i, x_j) \leq \phi(x_i, x_j)[(x_i^+ - x_j^+)^2 - (x_i - x_j)^2]. \tag{71}$$

Indeed, if $\phi(x_i, x_j) = \phi(x_i^+, x_j^+)$ the equality holds in (71). If $\phi(x_i, x_j) = 0$ and $\phi(x_i^+, x_j^+) = 1$, then the left hand side of (71) is equal to $(x_i^+ - x_j^+)^2 - d^2$ that is non-positive because $\phi(x_i^+, x_j^+) = 1$, which is equivalent to $|x_i^+ - x_j^+| \leq d$ and then (71) holds with the right-hand side being null. If $\phi(x_i, x_j) = 1$ and $\phi(x_i^+, x_j^+) = 0$ then (71) becomes

$$d^2 - (x_i - x_j)^2 \leq [(x_i^+ - x_j^+)^2 - (x_i - x_j)^2]$$

that is verified because $\phi(x_i^+, x_j^+) = 0$, which is equivalent to $|x_i^+ - x_j^+| > d$.

By using (71), from (29) and by exploiting the condition $\phi(x_i, x_j) = \phi(x_j, x_i)$, one can write

$$\begin{aligned}
\mathcal{E}(\mathbf{x}^+) - \mathcal{E}(\mathbf{x}) &\leq \frac{1}{2} \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j)[(x_i^+ - x_j^+)^2 - (x_i - x_j)^2] \\
&= \frac{1}{2} \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j)[(x_i^+ - x_i) - (x_j^+ - x_j)] \\
&\quad (x_i^+ - x_j^+ + x_i - x_j) \\
&= \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j)(x_i^+ - x_i)(x_i^+ - x_j^+ + x_i - x_j).
\end{aligned}$$

By defining $x_i^+ = x_i + \Delta_i$ and $x_j^+ = x_j + \Delta_j$, it follows

$$\mathcal{E}(\mathbf{x}^+) - \mathcal{E}(\mathbf{x}) \leq \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j)\Delta_i[2(x_i - x_j) + \Delta_i - \Delta_j].$$

By using the condition

$$\begin{aligned}
\sum_{j \in \mathcal{I}} \phi(x_i, x_j)\Delta_i &= \sum_{j \in \mathcal{I}} \phi(x_i, x_j)x_i^+ - \sum_{j \in \mathcal{I}} \phi(x_i, x_j)x_i \\
&= |\mathcal{N}_i|x_i^+ - \sum_{j \in \mathcal{I}} \phi(x_i, x_j)x_i \\
&= |\mathcal{N}_i| \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{I}} \phi(x_i, x_j)x_j - \sum_{j \in \mathcal{I}} \phi(x_i, x_j)x_i \\
&= \sum_{j \in \mathcal{I}} \phi(x_i, x_j)(x_j - x_i),
\end{aligned}$$

it follows

$$\begin{aligned}
\mathcal{E}(\mathbf{x}^+) - \mathcal{E}(\mathbf{x}) &\leq \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j)\Delta_i(-\Delta_i - \Delta_j) \\
&= -2 \sum_{i,j \in \mathcal{I}} \phi(x_i, x_j)(x_i^+ - x_i)^2 \\
&= -2 \sum_{i \in \mathcal{I}} |\mathcal{N}_i|(x_i^+ - x_i)^2 \\
&\leq -2 \sum_{i \in \mathcal{I}} (x_i^+ - x_i)^2,
\end{aligned}$$

where the symmetry condition $\phi(x_i, x_j) = \phi(x_j, x_i)$ has been used once again.

Proof of Proposition 38. If $v(\mathbf{x}(k^*)) \leq d_m$, Proposition 30 can be applied, and the proof directly follows because consensus is reached at the next time-step at most.

Consider now the case $v(\mathbf{x}(k^*)) \in (d_m, d_M]$. Suppose that $d_m = \ell$ and $d_M = u$, and say m one of the agents with the minimum opinion, i.e., $m \in \{q \in \mathcal{I} \mid x_q = \min_{i \in \mathcal{I}} x_i\}$. By hypothesis, any agent with the minimum opinion is influenced by all the other agents, i.e., $\mathcal{N}_m = \mathcal{I}$ for any $k \geq k^*$. By using (2) with (15) and (22), it is $x_m^+ > x_m + \frac{d_m}{N}$. From Proposition 3, Proposition 35 and the last inequality, after at most $\left\lceil N \frac{d_M - d_m}{d_m} \right\rceil$ time-steps the condition $v(\mathbf{x}) \leq d_m$ holds, and by applying Proposition 30 consensus is reached at the successive time-step at most. Analogously, if $d_m = u$ and $d_M = \ell$, consider M as one of the agents with the maximum opinion, i.e., $M \in \{q \in \mathcal{I} \mid x_q = \max_{i\in\mathcal{I}} x_i\}$. Then, $\mathcal{N}_M = \mathcal{I}$ for any $k \geq k^*$ and $x_M^+ < x_M - \frac{d_m}{N}$ from which the condition $v(\mathbf{x}) \leq d_m$ is verified after at most $\left\lceil N\frac{d_M - d_m}{d_m}\right\rceil$ time-steps, and consensus is reached at the successive time-step at most, which completes the proof.

Proof of Corollary 39. Assume that the opinion vector $\mathbf{x}(k)$ is a d-chain with $d = \ell$ for all $k \in \mathbb{N}_0$. Then, it is straightforward to show that the diagonal and lower off-diagonal entries of any matrix $A(\mathbf{x}(k))$ defined in (18) are not null. It is easy to verify that the entries (i, j) with $i \leq j$ of the product of $N-1$ matrices having this property are positive.

Now, define the product of the first $N-1$ matrices A which appear in (18), i.e., $B_0 = A(\mathbf{x}(N-2))\cdot\ldots\cdot A(\mathbf{x}(0))$. By applying (18) one can define $\tilde{\mathbf{x}}(0) = \mathbf{x}(N-1) = B_0\mathbf{x}(0)$, and by recursively using (18) one can write $\tilde{\mathbf{x}}(k) = \mathbf{x}((N-1)(k+1)) = B_k B_{k-1}\ldots B_0\mathbf{x}(0)$, where $B_k = A(\mathbf{x}((N-1)(k+1)-1))\cdot\ldots\cdot A(\mathbf{x}((N-1)k))$, $k \in \mathbb{N}_0$. From the algebraic property indicated above, the entries (i, j) with $i \leq j$ of B_k, $k \in \mathbb{N}_0$, are positive. Then, according to Proposition 3, it is $v(\tilde{\mathbf{x}}(k+1)) < v(\tilde{\mathbf{x}}(k))$ for all $k \in \mathbb{N}_0$, or equivalently there exists $q \in (0, 1)$ such that $v(B_k\tilde{\mathbf{x}}(k)) \leq qv(\tilde{\mathbf{x}}(k))$ for all $k \in \mathbb{N}_0$. This inequality implies the asymptotic convergence to 0 of the range of opinions $v(\tilde{\mathbf{x}})$. In particular, the sequence $\{v(\tilde{\mathbf{x}})^{+h}\}_{k+h}$, with $v(\tilde{\mathbf{x}})^{+h}$ representing the range of opinions $v(\tilde{\mathbf{x}})$ at the time-step $k+h$, is a converging subsequence of $\{v(\mathbf{x})^{+h}\}_{k+h}$, with $h \in \mathbb{N}_0$. Since the limit point of $\{v(\tilde{\mathbf{x}})^{+h}\}_{k+h}$ is 0, the sequence $\{v(\mathbf{x})^{+h}\}_{k+h}$ converges to 0 too. Then, there exists a finite time-step such that $v(\mathbf{x}) \leq u$, and by using Proposition 30 consensus is reached in at most one time-step.

If $\mathbf{x}(k)$ is a d-chain with $d = u$ for all $k \in \mathbb{N}_0$, similar arguments can be applied by considering that the diagonal and upper off-diagonal entries of any matrix $A(\mathbf{x}(k))$ defined in (18) are not null.

Assume now that a consensus has been reached. As a consequence, any crack cannot be occurred because of Proposition 26. This completes the proof.

Proof of Theorem 40. If a crack never occurs, then by applying Corollary 39 a consensus is reached. When a crack occurs, the agents are divided into independent groups, since Proposition 26 holds. This grouping process can be iterated but must stop in finite time due to the finite number of agents. Hence, there exists a time-step $k \in \mathbb{N}_0$ such that the agents are grouped in independent d-chains, and then each subgroup will reach a "local" consensus, according to Corollary 39.

Assume that $u \geq \ell$. Due to the finite number of agents, the following conditions hold:

i. Since the model is homogeneous, two agents with the same opinion will have the same opinion at any future time-step and then condition C1 in Proposition 41 can occur at most $N-1$ (not necessarily consecutive) time-steps;
ii. If condition C2 in Proposition 41 occurs, then the agent q will not belong anymore to $\mathcal{M}$ for any future time-step; therefore, condition C2 can occur at most $N-1$ (not necessary consecutive) time-steps;
iii. If one of the conditions C3 and C4 in Proposition 41 holds, from $x_q^{+2} \geq x_q^+$ it follows that $x_q^{+2} > x_q + \min\left\{\frac{\ell}{N}, \frac{u}{2N^2}\right\}$. From Proposition 1, it follows that this condition can hold at most a number of time-steps (not necessarily consecutive) such that the entire convex hull of the initial opinions is covered.

By applying Proposition 41 with $u \geq \ell$ and by using i., ii. and iii. above, it follows that the condition $\mathcal{M} = \emptyset$, which corresponds to having reached a constant steady state, holds after at most $\bar{k}_1$ time-steps with

$$\bar{k}_1 \leq 2\left(N - 1 + \left\lceil \frac{N}{\min\left\{\ell, \frac{u}{2N}\right\}} v(\mathbf{x}(0))\right\rceil\right). \tag{72}$$

Assume that $u \leq \ell$. Conditions i. and ii. above hold. Moreover,

iv. If one of the conditions C5 and C6 in Proposition 41 holds, from $x_q^{+2} \leq x_q^+$ it follows that $x_q^{+2} < x_q + \min\left\{\frac{u}{N}, \frac{\ell}{2N^2}\right\}$. From Proposition 1, it follows that this condition can hold at most a number of time-steps (not necessarily consecutive) such that the entire convex hull of the initial opinions is covered.

By applying Proposition 41 with $u \leq \ell$ and by using i., ii. and iv., it follows that the condition $\mathcal{M} = \emptyset$ holds after at most $\bar{k}_2$ time-steps with

$$\bar{k}_2 \leq 2\left(N - 1 + \left\lceil \frac{N}{\min\left\{u, \frac{\ell}{2N}\right\}} v(\mathbf{x}(0))\right\rceil\right). \tag{73}$$

By combining (72), for which $\ell = d_m$ and $u = d_M$ because $u \geq \ell$, together with (73), for which $\ell = d_M$ and $u = d_m$ because $u \leq \ell$, the thesis with (34) follows.

Proof of Proposition 41. Assume that $u \geq \ell$, and consider $q \in \mathcal{M}_{\min}$. From (36) and $u \geq \ell$, the agent q cannot have lower neighbors and has at least one upper neighbor by implying that $x_q^+ > x_q$. From the homogeneity of the model, it follows that $\mathcal{N}_{=q}$ cannot lose elements over time, i.e., $|\mathcal{N}_{=q}^+| \geq |\mathcal{N}_{=q}|$. Let $r = \min\{i \in \mathcal{I} \mid x_i > x_q\}$. By hypothesis, it is $r \in \mathcal{N}_q$ and $r \notin \mathcal{N}_{=q}$. One of the following alternatives must occur:

a. If $\mathcal{N}_q = \mathcal{N}_r$ (and then $q \in \mathcal{N}_r$), from the right-hand side of (17) it follows $x_q^+ = x_r^+$, which implies $r \in \mathcal{N}_{=q}^+$, and then condition C1 holds.
b. Otherwise, if $\mathcal{N}_q \neq \mathcal{N}_r$ one of the following alternatives must occur:
   - b1. If there exists an agent s such that $s \in \mathcal{N}_r$ and $s \notin \mathcal{N}_q$, it is $x_s - x_q > u$, and then $x_q^+ \neq x_r^+$, i.e., $r \notin \mathcal{N}_{=q}^+$, by implying condition C1 does not hold and $\mathcal{N}_{=q}^+ = \mathcal{N}_{=q}$. By using (2) with (15) and (22), it follows

$$\begin{aligned} x_r^+ &\geq x_r + \frac{(x_q - x_r)(|\mathcal{N}_r| - 2) + (x_q + u - x_r)}{|\mathcal{N}_r|} \\ &= x_q + \frac{u + x_r - x_q}{|\mathcal{N}_r|} \geq x_q + \frac{u + x_r - x_q}{N} \\ &> x_q + \frac{u}{N}. \end{aligned} \tag{74}$$

We can now consider two alternative cases:

b1.1. If $x_q^+ - x_q < \frac{u}{2N}$, by using (74) it is $x_r^+ > x_q^+ - \frac{u}{2N} + \frac{u}{N}$ and then $x_r^+ - x_q^+ > \frac{u}{2N}$. Now:

b1.1.1. If $x_r^+ - x_q^+ > u$, then $r \notin \mathcal{N}_q^+$, and by using Proposition 26 condition C2 holds.

b1.1.2. Otherwise, it is $\frac{u}{2N} < x_r^+ - x_q^+ \leq u$, which implies $r \in \mathcal{N}_q^+$. By using (2) with (15) and (22), since the agent q cannot acquire lower neighbors because of Proposition 35, one can write

$$\begin{aligned} x_q^{+2} &> x_q^+ + \frac{u}{2N|\mathcal{N}_q^+|} \geq x_q^+ + \frac{u}{2N^2} \\ &\geq x_q + \frac{u}{2N^2}, \end{aligned} \tag{75}$$

and condition C3 holds.

b1.2. Otherwise, if $x_q^+ - x_q \geq \frac{u}{2N}$ because of (6b) with $i = q$ and Proposition 35, condition C3 holds.

b2. Otherwise, if it does not exist an agent s such that $s \in \mathcal{N}_r$ and $s \notin \mathcal{N}_q$, since there are no agents with opinions in the interval (x_q, x_r) and $\mathcal{N}_r \neq \mathcal{N}_q$, it must be $\mathcal{N}_r = \mathcal{N}_q \setminus \mathcal{M}_{\min}$, i.e., the agent q is not a lower neighbor of r. By using (2) with (15) and (22), one can write

$$x_q^+ \geq x_q + \frac{x_r - x_q}{|\mathcal{N}_q|} \geq x_q + \frac{x_r - x_q}{N} > x_q + \frac{\ell}{N}, \tag{76}$$

and condition C4 holds.

For the case $u \leq \ell$ analogous arguments can be applied. From (36) and $u \leq \ell$, the agent q cannot have upper neighbors and has at least one lower neighbor by implying that $x_q^+ < x_q$. From the homogeneity of the model, it follows that $\mathcal{N}_{=q}$ cannot lose elements over time, i.e., $|\mathcal{N}_{=q}^+| \geq |\mathcal{N}_{=q}|$. Let $r = \max\{i \in \mathcal{I} \mid x_i < x_q\}$. By hypothesis, it is $r \in \mathcal{N}_q$ and $r \notin \mathcal{N}_{=q}$. One of the following alternatives must occur:

a. If $\mathcal{N}_q = \mathcal{N}_r$ (and then $q \in \mathcal{N}_r$), from the right-hand side of (17) it follows $x_q^+ = x_r^+$ which implies $r \in \mathcal{N}_{=q}^+$, and then condition C1 holds.

b. Otherwise, if $\mathcal{N}_q \neq \mathcal{N}_r$ one of the following alternatives must occur:

b1. If there exists an agent s such that $s \in \mathcal{N}_r$ and $s \notin \mathcal{N}_q$, it is $x_s - x_q < -\ell$, and then $x_q^+ \neq x_r^+$, i.e., $r \notin \mathcal{N}_{=q}^+$, by implying condition C1 does not hold and $\mathcal{N}_{=q}^+ = \mathcal{N}_{=q}$. By using (2) with (15) and (22), it follows

$$\begin{aligned} x_r^+ &\leq x_r + \frac{(x_q - x_r)(|\mathcal{N}_r| - 2) + (x_q - \ell - x_r)}{|\mathcal{N}_r|} \\ &= x_q - \frac{\ell + x_q - x_r}{|\mathcal{N}_r|} \leq x_q - \frac{\ell + x_q - x_r}{N} \\ &< x_q - \frac{\ell}{N}. \end{aligned} \tag{77}$$

We can now consider two alternative cases:

b1.1. If $x_q^+ - x_q > -\frac{\ell}{2N}$, by using (77) it is $x_r^+ < x_q^+ + \frac{\ell}{2N} - \frac{\ell}{N}$ and then $x_r^+ - x_q^+ < -\frac{\ell}{2N}$. Now:

b1.1.1. If $x_r^+ - x_q^+ < -\ell$, then $r \notin \mathcal{N}_q^+$, and by using Proposition 26 condition C2 holds.

b1.1.2. Otherwise, it is $-\ell \leq x_r^+ - x_q^+ < -\frac{\ell}{2N}$, which implies $r \in \mathcal{N}_q^+$. By using (2) with (15) and (22), since the agent q cannot acquire upper neighbors because of Proposition 35, one can write

$$\begin{aligned} x_q^{+2} &< x_q^+ - \frac{\ell}{2N|\mathcal{N}_q^+|} \leq x_q^+ - \frac{\ell}{2N^2} \\ &\leq x_q - \frac{\ell}{2N^2}, \end{aligned} \tag{78}$$

and condition C5 holds.

b1.2. Otherwise, if $x_q^+ - x_q \leq -\frac{\ell}{2N}$ because of (6a) with $i = q$ and Proposition 35, condition C5 holds.

b2. Otherwise, if it does not exist an agent s such that $s \in \mathcal{N}_r$ and $s \notin \mathcal{N}_q$, since there are no agents with opinions in the interval (x_r, x_q) and $\mathcal{N}_r \neq \mathcal{N}_q$, it must be $\mathcal{N}_r = \mathcal{N}_q \setminus \mathcal{M}_{\max}$, i.e., the agent q is not an upper neighbor of r. By using (2) with (15) and (22), one can write

$$x_q^+ \leq x_q + \frac{x_r - x_q}{|\mathcal{N}_q|} \leq x_q + \frac{x_r - x_q}{N} < x_q - \frac{u}{N}, \tag{79}$$

and condition C6 holds.

Proof of Proposition 44. Assume that $\ell = 0$ and there exists a pair $q, r \in \mathcal{I}$ such that $x_q < x_r$. If $r \notin \mathcal{N}_q$ the agents q and r are not influenced by each other. By using (6a) with the substitution $i = r$ and by considering that $x_r^+ \geq x_r$ for the definition of one-sided confidence BCOD and that for any $i \in \mathcal{N}_q$ it is $x_i < x_r$ by hypothesis, it will be $x_i^+ > x_j^+$.

Consider now the case $r \in \mathcal{N}_q$. If $\mathcal{N}_r = \{r\}$ it is $x_q^+ = x_q > x_r^+$ because of Proposition 35 and the fact that q is also influenced by itself. In order to consider the case that r has some upper neighbor different from itself, consider the averages of the opinions in (69). For any $i \in \mathcal{N}_{qr}$ it is $x_i \geq x_r$ because $\ell = 0$, then it is $\bar{x}_r \geq \bar{x}_{qr}$ and one obtains

$$x_r^+ = \frac{|\widehat{\mathcal{N}}_r|\bar{x}_r + |\mathcal{N}_{qr}|\bar{x}_{qr}}{|\widehat{\mathcal{N}}_r| + |\mathcal{N}_{qr}|} \geq \bar{x}_{qr}. \tag{80}$$

The set $\widehat{\mathcal{N}}_q$ includes at least the agent q itself, and for any $i \in \widehat{\mathcal{N}}_q$ it is $x_i \in [x_q, x_r)$ which implies $\bar{x}_q < \bar{x}_{qr}$. Then, one can write

$$x_q^+ = \frac{|\widehat{\mathcal{N}}_q|\bar{x}_q + |\mathcal{N}_{qr}|\bar{x}_{qr}}{|\widehat{\mathcal{N}}_q| + |\mathcal{N}_{qr}|} < \bar{x}_{qr}. \tag{81}$$

By combining (80) and (81), it follows $x_q^+ < x_r^+$ which completes the proof for the case $\ell = 0$.

The proof for $u = 0$ can be obtained by applying similar arguments.

Proof of Proposition 45. If $v(\mathbf{x}) \leq \delta$ the statement is trivial. For $\delta < v(\mathbf{x}) \leq d_M$, assume that $d_M = u$ and $d_m = \ell = 0$. By hypothesis, the agent 1 is influenced by all the agent set, i.e., $\mathcal{N}_1 = \mathcal{I}$ for any future time-step. From (2) with (15) and (22), it is

$$x_1^+ > x_1 + \frac{\delta}{N}. \tag{82}$$

By using Proposition 3, Proposition 35 and (82), after at most $\left\lceil N\frac{d_M - \delta}{\delta} \right\rceil$ time-steps the condition $v(\mathbf{x}) = x_N - x_1 \leq \delta$ holds.

Analogously, if $d_m = u = 0$ and $d_M = \ell$, one can consider the N-th agent whose opinion at the next time-step is such that

$$x_N^+ < x_N - \frac{\delta}{N} \tag{83}$$

from which the condition $v(\mathbf{x}) = x_N - x_1 \leq \delta$ is verified after at most $\left\lceil N\frac{d_M - \delta}{\delta} \right\rceil$ time-steps.

Proof of Lemma 46. Assume that $u \geq \ell$ and the influence function in (40) is considered. The proof is the same as for Proposition 41 when $u \geq \ell$ by subsiding the influence function (22) with the one in (40), except the alternative b2 which is as follows.

b2. If it does not exist an agent s such that $s \in \mathcal{N}_r$ and $s \notin \mathcal{N}_q$, since there are no agents with opinions in the interval (x_q, x_r) and $\mathcal{N}_r \neq \mathcal{N}_q$, it must be $\mathcal{N}_r = \mathcal{N}_q \setminus \mathcal{M}_{\min}$, i.e., the agent q is not a lower neighbor of r. By using (2) with (15) and (40), one can write

$$\begin{aligned} x_q^+ &\geq x_q + \frac{x_r - x_q}{|\mathcal{N}_q|} \geq x_q + \frac{x_r - x_q}{N} \\ &> x_q + \frac{\ell - \eta_r}{N} \geq x_q + \frac{\ell - \eta_{\max}}{N}, \end{aligned} \tag{84}$$

and condition C4 holds.

Assume, now, that $u \leq \ell$ and the influence function in (41) is considered. The proof is the same as for Proposition 41 when $u \geq \ell$ by subsiding the influence function (22) with the one in (41), except the alternative b2 which is as follows.

b2. If it does not exist an agent s such that $s \in \mathcal{N}_r$ and $s \notin \mathcal{N}_q$, since there are no agents with opinions in the interval (x_r, x_q) and $\mathcal{N}_r \neq \mathcal{N}_q$, it must be $\mathcal{N}_r = \mathcal{N}_q \setminus \mathcal{M}_{\max}$, i.e., the agent q is not an upper neighbor of r. By using (2) with (15) and (41), one can write

$$x_q^+ \le x_q + \frac{x_r - x_q}{|\mathcal{N}_q|} \le x_q + \frac{x_r - x_q}{N} < x_q - \frac{u - \eta_r}{N} \le x_q - \frac{u - \eta_{\max}}{N}, \quad (85)$$

and condition C6 holds.

Proof of Theorem 47. Assume that $u \ge \ell$ and the influence function in (40) is considered. Due to the finite number of agents, the following conditions hold:

i. By definition of $\mathcal{M}_{\min}$ in (36) the agent q will be always the same until condition C2 in Lemma 46 will be verified because of a swap between the agent q and any upper agent cannot occur due to $u \ge \ell$, (6b) and the homogeneity of the upper thresholds. Since an agent with opinion equal to x_q will have the same opinion of q at any future time-step, then condition C1 in Lemma 46 can occur at most $N-1$ (not necessary consecutive) time-steps;

ii. If condition C2 in Lemma 46 occurs, then the agent q will not belong anymore to $\mathcal{M}$ for any future time-step; therefore, condition C2 can occur at most $N-1$ (not necessary consecutive) time-steps;

iii. If one of the conditions C3 and C4 in Lemma 46 hold, from $x_q^{+2} \ge x_q^+$ it follows that

$$x_q^{+2} > x_q + \min\left\{\frac{\ell - \eta_{\max}}{N}, \frac{u}{2N^2}\right\}. \quad (86)$$

From Proposition 1, it follows that this condition can hold at most a number of time-steps (not necessarily consecutive) such that the entire convex hull of the initial opinions is covered.

By applying Lemma 46 with $u \ge \ell$ and the influence function in (40) and by using i., ii. and iii. above, it follows that after at most $\bar{k}_1$ time-steps with

$$\bar{k}_1 \le 2\left(N - 1 + \left\lceil \frac{N}{\min\left\{\ell - \eta_{\max}, \frac{u}{2N}\right\}} v(\mathbf{x}(0)) \right\rceil\right) \quad (87)$$

the condition $\mathcal{M} = \emptyset$ holds, which corresponds to having reached a constant steady state.

Assume that $u \le \ell$ and the influence function in (41) is considered. Conditions i. and ii. above hold for $q \in \mathcal{M}_{\min}$. Moreover,

iv. If one of the conditions C5 and C6 in Lemma 46 hold, from $x_q^{+2} \le x_q^+$ it follows that

$$x_q^{+2} < x_q + \min\left\{\frac{u - \eta_{\max}}{N}, \frac{\ell}{2N^2}\right\}. \quad (88)$$

From Proposition 1, it follows that this condition can hold at most a number of time-steps (not necessarily consecutive) such that the entire convex hull of the initial opinions is covered.

By applying Lemma 46 with $u \le \ell$ and by using i., ii. and iv., it follows that after at most $\bar{k}_2$ time-steps with

$$\bar{k}_2 \le 2\left(N - 1 + \left\lceil \frac{N}{\min\left\{u - \eta_{\max}, \frac{\ell}{2N}\right\}} v(\mathbf{x}(0)) \right\rceil\right) \quad (89)$$

the condition $\mathcal{M} = \emptyset$ holds.

By combining (87), for which $\ell = d_m$ and $u = d_M$ because $u \ge \ell$, together with (89), for which $\ell = d_M$ and $u = d_m$ because $u \le \ell$, the thesis with (42) follows.

Proof of Theorem 49. For definition of stubborn agent, it is $\phi_{si}(x_\sigma, x_i) = 0$ for all $i \in \mathcal{I} \setminus \{\sigma\}$ such that $x_i \neq x_\sigma$. From Definition 16, it follows that for each agent $i \in \mathcal{A}$ there exists a d-chain $(i_1, i_2, \dots, i_h)$, i.e., it is $\phi_{ii_1}(x_i, x_{i_1}) = 1, \phi_{i_1 i_2}(x_{i_1}, x_{i_2}) = 1, \dots, \phi_{i_h s}(x_{i_h}, x_\sigma) = 1$. Conversely, it is $\phi_{ij}(x_i, x_j) = 0$ for all $i \notin \mathcal{A} \cup \{\sigma\}$ and $j \in \mathcal{A} \cup \{\sigma\}$. Thus, since the order-preservation property given in Definition 13 is satisfied, by assuming that $\sigma = 1$, the matrix $A(\mathbf{x})$ defined in (18) will have the following structure

$$A(\mathbf{x}) = \begin{bmatrix} B(\mathbf{x}_1) & 0 \\ 0 & C(\mathbf{x}_2) \end{bmatrix} \quad (90)$$

with $\mathbf{x}_1 = [x_1 \ \dots \ x_{|\mathcal{A}|+1}]^\top$ and $\mathbf{x}_2 = [x_{|\mathcal{A}|+2} \ \dots \ x_N]^\top$. The above block structure of the matrix $A(\mathbf{x})$ could change over time.

Proposition 26 implies that $\mathcal{A}^+ \subseteq \mathcal{A}$ and, due to the finite number of agents, it must exist a finite time-step $\bar{k}$ such that $\mathcal{A}^+ = \mathcal{A}$ for all $k \ge \bar{k}$. Then, condition C1 is proved.

From the block structure in (90), it follows that the sets $\mathcal{A} \cup \{\sigma\}$ and $\mathcal{I} \setminus \{\mathcal{A} \cup \{\sigma\}\}$ can be analyzed separately for $k \ge \bar{k}$. Consider the first set. According to Corollary 8.5.10 in Krause (2015), the set $\mathcal{A} \cup \{\sigma\}$ will converge to a consensus if the following three conditions are satisfied: (i) the diagonal of the matrix $B(\mathbf{x}_1)$ is positive; (ii) the minimum positive entry is lower bounded by a positive constant; (iii) $B(\mathbf{x}_1)$ is coherent. From the influences functions (22) with $d = \ell = u$ for $i \in \mathcal{A}$ and (39) for $i = 1$, it follows that in (90) $b_{ii} \ge 1/N$ for all $i \in \mathcal{A}$ and $b_{ij} \ge 1/N$ for all $i, j \in \mathcal{A}$ such that $\phi_i(x_i, x_j) = 1$, by implying that conditions (i) and (ii) are valid. Now we show that condition (iii) holds too. The matrix $B(\mathbf{x}_1)$ is coherent if any two of its nonempty saturated sets have a nonempty intersection. The set $\mathcal{A}_1 \subseteq \mathcal{A} \cup \{\sigma\}$ is saturated for $B(\mathbf{x}_1)$ if $b_{ij} > 0$ and $i \in \mathcal{A}_1$ imply $j \in \mathcal{A}_1$. Consider two nonempty sets $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A} \cup \{\sigma\}$, $i \in \mathcal{A}_1$ and $j \in \mathcal{A}_2$. If $i = j = \sigma$, then $\sigma \in \mathcal{A}_1 \cap \mathcal{A}_2$. If $i = \sigma$ and $j \neq \sigma$, for definition of the set $\mathcal{A}$ there exists a d-chain $(i_1, i_2, \dots, i_h)$ such that $b_{ji_1} > 0, b_{i_1 i_2} > 0, \dots, b_{i_h \sigma} > 0$. Since $\mathcal{A}_2$ is saturated, it must be $i_1 \in \mathcal{A}_2, i_2 \in \mathcal{A}_2, \dots, \sigma \in \mathcal{A}_2$. Thus $\sigma \in \mathcal{A}_1 \cap \mathcal{A}_2$. If $i \neq \sigma$ and $j = \sigma$, similar considerations can be applied. Finally, if $i \neq \sigma$ and $j \neq \sigma$ for definition of the set $\mathcal{A}$ there exist a d-chain from i to σ and a d-chain from j to σ. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are saturated, it must be $\sigma \in \mathcal{A}_1 \cap \mathcal{A}_2$. Then, condition (iii) is also satisfied, and the convergence to a consensus is proved. Since the stubborn will not change its opinion over time, the consensus opinion must correspond to the stubborn opinion, which implies condition C2 holds.

Finally, consider the set $\mathcal{I} \setminus \{\mathcal{A} \cup \{\sigma\}\}$. By using Theorem 36 for such set, condition C3 directly follows.

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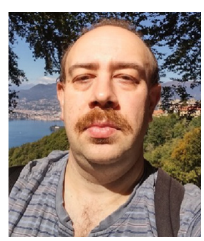

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