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ABSTRACT

Introducing the slope of a curve as the limit of the slope of secant lines is a well-known challenge in mathematics education. As an alternative, three other approaches can be recognized, based on linear approximation, based on multiplicities, or based on transition points. In this study we investigated which of these approaches fits students most by analyzing students' inventions during a lesson scenario revolving around a design problem. The problem is set in a context that is meaningful to students and invites them to invent methods to construct a tangent line to a curve: an implementation of the guided reinvention principle from Realistic Mathematics Education (RME). The teaching scenario is based on the phased lesson structure of the Theory of Didactical Situations (TDS). The scenario was tested with 44 groups of three students in six grade 9 or 10 classrooms. We classified the strategies used by students and, using the emergent models-principle from RME, investigated to which of the four approaches the student strategies connect best. The results show that the groups produced a variety of strategies in each classroom and these strategies contributed to a meaningful institutionalization of the notion of slope of a curve.

1. Introduction

The mathematical notion of slope of a curve is a mathematization of the common sense idea of the steepness of a path. Geometrically it is defined as the slope of the tangent line (if it exists). In practice, students' intuitions of what a tangent line is do not match up nicely with the common definition of a tangent line as a limit of secant lines. For instance, in a test by Orton (1977) 43 out of 110 calculus students had difficulty seeing the tangent line as a limit of secant lines, and similar observations are found in the work of Ferrini-Mundy and Geuther Graham (1991). Vinner (1982) observed that early experiences of the tangent line in circle geometry introduce a belief that the tangent is the same as a bounding line: a line that touches but does not cross the curve. A study among 196 Greek students (grade 12) for their understanding of tangent lines reached similar conclusions (Biza, Christou, & Zachariades, 2008). In the Greek curriculum the first tangent lines students encounter are all bounding lines (as in the case of circles and parabolas). A precise classification of student work showed how students have difficulty with the transition from this geometric point of view to an analytic point of view of tangent lines in terms of the slope of a curve (Biza et al., 2008).

In general, the transition between geometric and analytic/algebraic representations of slope is problematic for students (Orton, 1983). For example, students struggle to approximate the slope from graphs (by computing $\frac{\Delta y}{\Delta x}$ for two points), both for lines and for more general curves. Similarly, the relation between instant rate of change of a function and the slope of the graph of that function is challenging for students (Habre & Abboud, 2006). Once students learn to compute the slope symbolically and algorithmically (computing derivatives by applying rules), they seem to have a preference for this, and the geometric interpretation is lost, if it was

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ever there. The computational focus of students seems to originate in a daily educational practice with a quick shift from the conceptual introduction to calculation procedures (Thompson, 1994). And with this topic in particular this is problematic since the calculation algorithms do not align with any idea of tangent line that might initially be meaningful for students. A way to learn new concepts in a way that endows them with meaning is by guided reinvention, a design principle from the Realistic Mathematics Education (RME) framework (Freudenthal, 1991). Freudenthal had, in our interpretation, a *recursive* idea of what meaningfulness is: something has meaning if it is *based on* meaningful ideas and experiences; the ultimate sources of meaningful experiences are everyday life experiences. Students' everyday experiences of curved slopes, e.g. playground slides, marble tracks or paths in hilly landscapes, give them a embodied sense of (relative) steepness: for a smooth curve they can tell (a) out of two points on the curve where the slope is steeper; (b) where on the curve the slope is steepest; and (c) where the slope is ascending, descending or horizontal. One could describe this as embodied knowledge; it does not take any (mathematical) reflection to consider these issues, instead perhaps imagining sliding or walking along the curve. What is lacking is an absolute quantification of steepness. An issue is whether a quantitative measure of steepness actually *exists* for a specific point on a curve. This too can be addressed using embodied (tactile) knowledge, this time of smoothness. By (imagining) sliding a hand along a curve one can intuitively detect at which points the curve is not smooth. These bodily experiences can be used for teaching slope (Tall, 2013). The aim of this study is to investigate how a teaching scenario based on embodied knowledge of slope can support students' reinvention of the notion of slope of a curve.

2. Theoretical framework

2.1. Reinvention and emergent models

A reinvention task does not need to aim for immediate rigorous mathematical results from the students. The emergent models design principle explains how students' activity at several levels are part of a reinvention process (Doorman & Gravemeijer, 2009; Gravemeijer, 1999). Within this principle *models* are understood as student-generated ways of organizing their activity with observable and mental tools (Zandieh & Rasmussen, 2010). The theoretical four levels of activity relating to emergent models are: *within the task setting*, *referential*, *general* and *formal*. For the 60–90 minutes lesson and task in this study, the goal is for the students to reach a referential level (their solution as a *model of* slope) and for the teacher to push that to a general level (introducing a *model for* slope based on the students activity). Hence, it is the teacher's task to develop the students' models that refer to the task setting into models that "facilitate a focus on interpretations and solutions independent of situation-specific imagery" (Gravemeijer, 1999).

This approach needs problems/tasks that can be solved in a variety of ways to elicit students' levels of understanding of modelling slope. Students' solutions ideally show the various difficulties and opportunities the students have in the process of learning the new concept. This way the solutions of the students anticipate on the whole class learning process: "The cross-sectional view of the class (the different levels of understanding of the students in a class at one particular moment) that is produced in this way can show at the same time a longitudinal section of the intended learning trajectory" (van den Heuvel-Panhuizen, 2005).

The reinvention and emergent models principles have been applied successfully in recent years in upper secondary schools and bachelor university level: in calculus (Doorman, 2005; Doorman & Gravemeijer, 1999; Gilboa, Kidron, & Dreyfus, 2019; Herbert & Pierce, 2008; Oehrtman, Swinyard, & Martin, 2014), in linear algebra (Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012), in abstract algebra (Larsen, 2013), in statistics (Schwartz & Martin, 2004), and for the teaching of bifurcation diagrams (Rasmussen, Dunmyre, Fortune, & Keene, 2019). Calling it "construction" instead of "reinvention" Tall's approach to teaching slope in Tall (1987) is an older successful application of the reinvention principle on the basis of examples and non-examples. Some of the studies aim at the reinvention of a definition (based on examples and non-examples), and others aim at development of the conceptual understanding, without a formal definition being the aim. Indeed, students can develop meaningful insights into the tangent-concept without being able to formulate a formal definition of a tangent (Gilboa et al., 2019). Moreover, in our approach the definitions of slope and tangent that connect to the concepts that emerges from the students' activity need not align with the standard definitions. This is in contrast with for example the work of Larsen (2013), where the abstract concept of group is reinvented, and the description of a group – as a set with certain operations – is a fixed endpoint. More in line is the work of Wawro and colleagues (Wawro et al., 2012), who give students a task to reinvent the notion of linear dependence. These students proceed to produce four different equivalent characterizations of linear dependence, allowing the instructor to bring out connections. In all of these approaches the instructor seems to have an active, intervening role during the reinvention process. In our study the instructor's role is different.

2.2. Unguided inquiry in a-didactical phases

The emergent models principle describes a general trajectory from situational mathematics to formal, but does not give a framework for implementation in classroom practice. For this one might turn to the *guided* reinvention principle from RME, but we claim that every reinvention intervention should contain episodes with just organizational guidance or no guidance at all, for at least two reasons: (1) students are to experience ownership of the reinvention; (2) students should have the freedom to apply the knowledge and methods that are meaningful to them. Neither can be achieved by providing didactical guidance at every step along the way.

It has been criticized under the name of the *constructivist teaching fallacy* (Mayer, 2004) that constructivist learning should require constructivist teaching in the form of (unguided) behavioral activity. Indeed, periods of unguided activity lead to seemingly inefficient processes, leaving the learning goals out of reach: students' struggle (or even stagnation), divergence of applied approaches, and, in the end, suboptimal solutions for the problem. So why not just guide? Constructivist learning might be achievable through teaching without unguided periods, but will this teaching be able to address points (1) and (2)? Nevertheless, any constructivist teaching that includes unguided episodes should address the issues of struggle, divergence and suboptimal solutions.

First, the struggle of students has the potential to be productive in mathematics classrooms (Warshauer, 2015), and failure in tasks (that are demanding) can lead to deeper understanding than direct instruction (Kapur & Bielaczyc, 2012; Kapur, 2010). Struggle may slow things down, but also lead to a deeper engagement with the subject matter.

Secondly, unguided activity may lead to divergent approaches by students, including suboptimal ones. The way to deal with this is to implement episodes of converging activity in the lesson; where convergence is towards the (mathematical) learning goals. For these classroom implementation issues we draw on ideas elaborated in Brousseau's Theory of Didactical Situations (Brousseau, 2002), TDS for short. He makes a distinction between didactical and a-didactical phases in a lesson. In didactical phases the teacher actively intervenes in students learning processes, whereas in a-didactical phases the teacher only observes.

TDS proposes a phased lesson structure. First the problem (*milieu*) is set out (didactical *devolution* phase) and worked on by students (a-didactical *action* phase), three phases followed that aimed at converging to the learning goals: a phase where students present their results; a phase where students discuss the validity of their results, and a phase where the teacher relates the students' models to the target knowledge (the so-called institutionalization phase).

Task design in TDS is oriented on creating a situation that captures the epistemological essence of the mathematical target knowledge (Artigue et al., 2019), in our case the notion of a slope of a curve. This study investigates to what extent our task situation has the potential to confront students with the epistemic obstacle of quantifying the steepness of a curve. Our concern is whether our task offers opportunities for developing and formulating a variety of approaches that is rich and deep enough, capturing the conceptual essences, allowing a meaningful institutionalization of the notion of slope of a curve. This study aims to demonstrate how the emergent models principle from RME contributes to the analysis of student work for such opportunities. This study is not concerned with further issues of classroom implementation of TDS-scenarios, but we do hypothesize that performing an analysis based on emergent models could support teachers in preparing an institutionalization phase and a further learning trajectory.

3. A-priori analysis

In order to genuinely build on students informal ideas the teacher needs to be able to recognize their relation to not just the standard approaches to the tangent line and the slope of a curve, but ideally to any approach. For this reason a comprehensive a-priori analysis of these concepts follows, including informal approaches with, if known to us, a formalization. We are not aware of a previous deliberate juxtaposition of approaches to the tangent line and slope of a curve, though all of these approaches are mentioned, but not together, in a Dutch book by Kindt (Kindt, 2015). The approaches were also found spread out through the book Calculus Unlimited (Marsden & Weinstein, 1981). Since many teachers' first pedagogical example of a computation of slope is in the context of a parabola, we shall add this as an example for each approach.

Approach S (limit of secant lines). Informally, the tangent line to a curve c in a point P can be seen as a limit of secant lines (where a secant line is a line through P and another point Q on the curve), see Fig. 1. The common formal definition of the tangent line makes use of the notion of limit from analysis: Assuming the curve is the graph of a function f, the tangent line at P is the line through $P(x_0, f(x_0))$ with slope $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$, if the limit exists. Geometrically, this entails taking a limit of secant lines through the points $P(x_0, f(x_0))$ and $Q(x_0 + \Delta x, f(x_0 + \Delta x))$ with $\Delta x \neq 0$.

The formal definition of slope of the curve in P is then, of course, $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.

Example. We want to compute the slope of the parabola described by $y = x^2$ in the point (x_0, x_0^2). To separate the algebra steps and the conceptual steps we first translate the parabola so that the point of interest is in the origin (see Fig. 2): $y = -x_0^2 + (x + x_0)^2$, which simplifies to $y = 2x_0 x + x^2$.

Applying the well-known definition to the translated curve in (0,0) we get $\lim_{\Delta x \to 0} \frac{2x_0 \Delta x + (\Delta x)^2}{\Delta x} = 2x_0$.

Approach L (best linear approximation). This approach, advocated by Tall in e.g. (Tall, 2013), is based on the act of zooming in

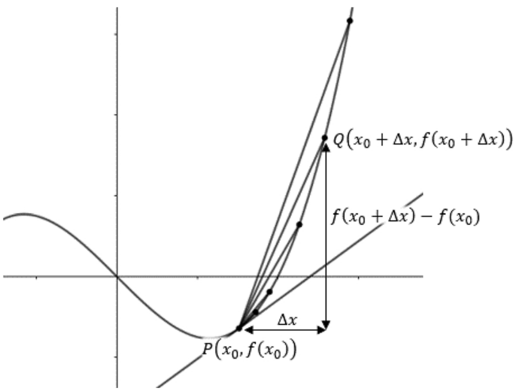

Fig. 1. The tangent line as a limit of secant lines.

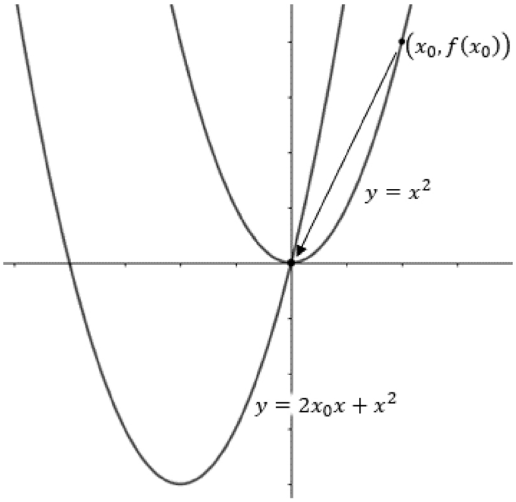

Fig. 2. Translating the parabola so that the point of interest lands in the Origin.

on the curve at a point. Visualizing the curve around P under high enough magnification (and low enough resolution) will show it as linear, if the curve is smooth at P. Informally, one sees the tangent line as the line that is indistinguishable from the curve after enough magnification. One could say that the two are indistinguishable because the tangent line is the best linear approximation of the curve at, and vice versa, see Fig. 3 (left).

We present a formalization of this idea is closely related to the previous approach, as found in (Marsden & Weinstein, 1981). Again, we assume the curve is the graph of a function f. The equation $y = f(x_0) + m(x - x_0)$ describes a line l_m through $P(x_0, f(x_0))$ with slope m. The line l_m is a tangent line at P iff for every $\varepsilon > 0$ there is an open interval $I \ni x_0$ such that for $x \in I$, with $x \neq x_0$ one has $|f(x) - [f(x_0) + m(x - x_0)]| < \varepsilon\, |x - x_0|$, see Fig. 3 (right). This definition has great practical and insightful value in case the function f can be described by a locally (around x_0) convergent (Taylor) series $\sum_{k\geq 0} a_k(x - x_0)^k$ (i.e. if f is analytic), since each higher order term $a_k(x - x_0)^k$ (with $k \geq 2$) can easily be understood to vanish more rapidly than linear when approaching x_0. Sangwin presents a similar approach in his work on limit-free derivatives (Sangwin, 2011), where he defines and computes derivatives by truncating the Taylor series (obtained algebraically) at the linear term, to obtain a linear approximation.

Example. Considering the equation for the translated parabola $y = 2x_0x + x^2$ we see without computation that the linear approximation at (0,0) is $y = 2x_0x$, from which follows that the slope is $2x_0$.

Approach A (multiplicity intersection points). This approach to tangent lines, recently advocated by Michael Range (2018, Michael Range, 2011), goes back to algebraic ideas about tangent lines by Fermat, Descartes and Hudde, so from before Barrow, Newton and Leibniz (Grabiner, 2011). Informally, the point of view is that a line is tangent to the curve in point P, if a small rotation of the line around P leads to one or more new nearby intersection points of the line and the curve. One could consider this is the

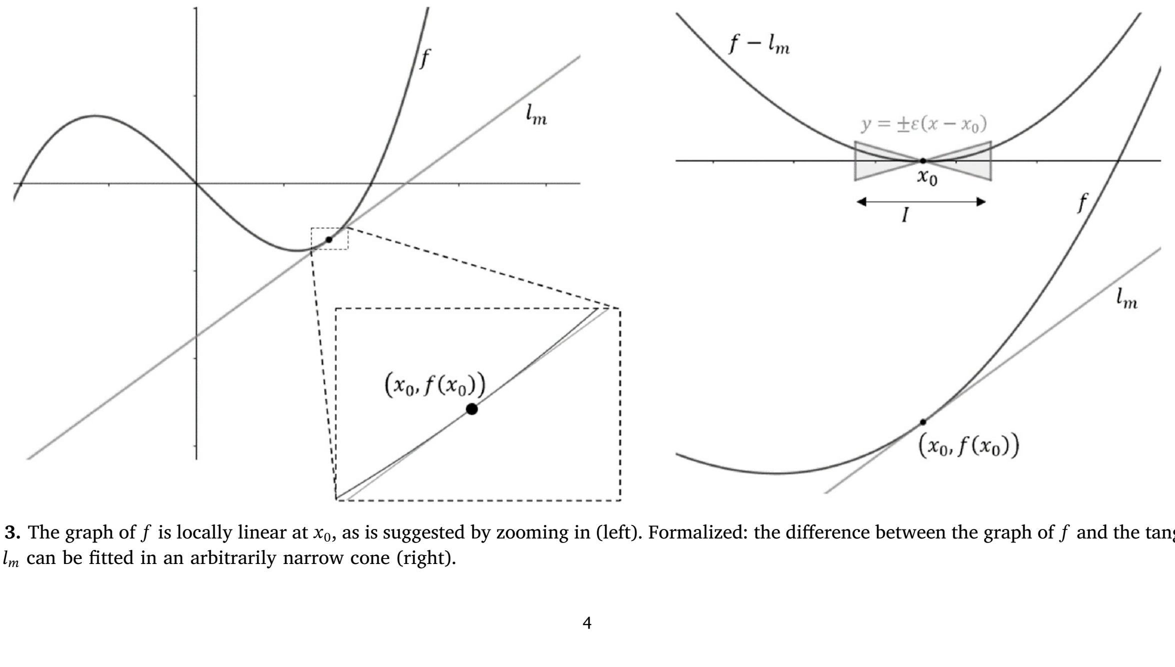

Fig. 3. The graph of f is locally linear at x_0, as is suggested by zooming in (left). Formalized: the difference between the graph of f and the tangent line l_m can be fitted in an arbitrarily narrow cone (right).

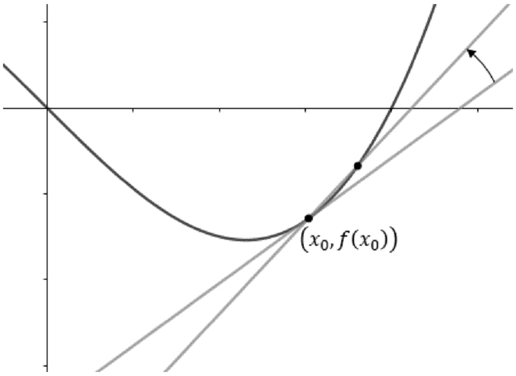

Fig. 4. The tangent line meets the curve in a multiple point; a slight rotation of the tangent line reveals a new nearby intersection point.

inverse of the secant line approach. The idea is that the point of tangency is actually a *multiple* point and the rotation reveals this by separating the multiple point into multiple points, see Fig. 4.

This can be formalized for algebraic functions f: the line l_m is tangent to the graph of f in $P(x_0, f(x_0))$ iff the solution $x = x_0$ of the equation $f(x) = f(x_0) + m(x - x_0)$ has a multiplicity of 2 or higher. This approach is unfortunately limited to algebraic functions; in case of transcendent functions the notion of multiplicity for the solution of the equation is not defined. The informal formulation leads into difficulty in case of, for example, the (famous in the context of differentiability) function $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. The slope in (0,0) equals zero, so the tangent line is $y = 0$. The problem is that $x = 0$ is an accumulation point (on the x -axis) of the (countably infinite) solution set of $x^2 \sin\frac{1}{x} = 0$.

Example. We equate $y = 2x_0 x + x^2$ (parabola) and $y = mx$ (line through the origin) to find $2x_0 x + x^2 = mx$. The solution $x = 0$ has multiplicity 2 only if this equation is equivalent to $x^2 = 0$, that is if $m = 2x_0$.

Approach T (transitions in overtaking). This approach is introduced in the book *Calculus Unlimited* (Marsden & Weinstein, 1981) and can be viewed as a formalization of Euclid's informal definition of a tangent line (in Proposition 16 in the Elements): a tangent line to a curve in a point P is a line $l$through P such that no other line through P fits in the space between the curve and l. Marsden and Weinstein mathematized the meaning of "fits" in a way that uses the idea of rotating the line around P as in the previous approach, but now, instead of focusing on intersection points, the focus is on whether the line overtakes the curve or the curve overtakes the line. Informally the idea is as follows: For a line with a slope smaller than the tangent line's slope, the curve overtakes the line at; if the slope of the line is greater than the tangent line's slope, then the line overtakes the curve. If there is exactly one value of the slope of the line where the first case changes into the second, then that is the slope of the curve, see Fig. 5.

Formally: the line l_m is tangent to the graph of f in $P(x_0, f(x_0))$ iff

- for all $n < m$ the difference $f(x) - [f(x_0) + n(x - x_0)]$ is defined on an open interval around x_0 and changes sign from negative to positive at $x = x_0$, and
- for all $n > m$ the difference $f(x) - [f(x_0) + n(x - x_0)]$ is defined on an open interval around x_0 and changes sign from positive to negative at $x = x_0$.

Example. We investigate $f(x) - [f(0) + nx] = x^2 + 2x_0 x - nx$, where f refers to the translated parabola. This equals

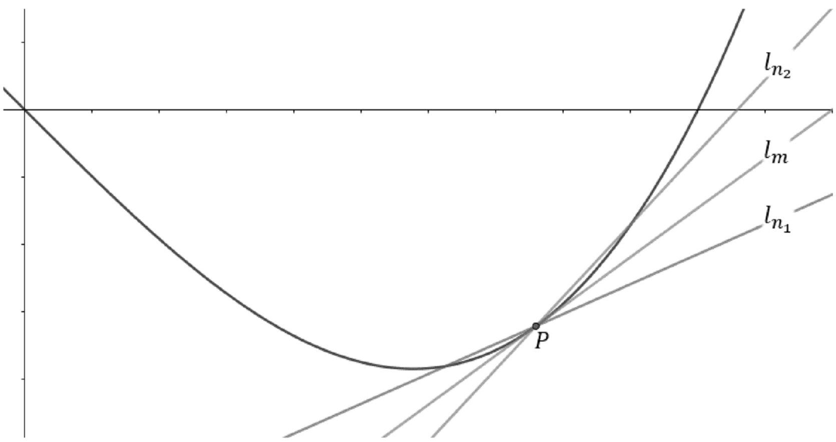

Fig. 5. Line l_{n_2} overtakes the curve at P. The curve overtakes line l_{n_1} at P. The tangent line is the transition case.

$x(x + 2x_0 - n)$. Which changes sign at $x = 0$, except if $n = 2x_0$.

In the example it becomes obvious how close approach T is to approach S. For algebraic functions whether a sign change takes place at a zero can be concluded from the multiplicity of the zero, so the approaches are computationally equal. The strength of approach T is that is "sign change/overtaking" can replace "multiplicity" beyond algebraic functions, and is more directly accessible.

4. Research questions

We designed a task based on a meaningful context, that begs to be mathematized by the target knowledge and activates previous embodied knowledge of slope. Our aim is to better understand how to design tasks that support students' reinvention. At stake are our main choices of design principles, inspired by RME – rich, meaningful contexts and the emergent models principle – and by TDS – working in an a-didactical action phase, followed by phases leading to institutionalization. Our main research question is: to what extent and in what way does a reinvention task based on these principles have the potential to lead to informal student models from which the more formal models representing the target knowledge can emerge? Related to this we have the following sub-questions:

1 What referential activities (design strategies and validation strategies) do the task evoke? What informal models do students construct?
2 How do these activities and models relate to the four formal models we present in our a-priori analysis? Do some models emerge more often than others?

5. Method

5.1. Task description

The slope reinvention task that we used is expected to support students to draw on their embodied knowledge of steepness and smoothness. To this purpose the didactical approach to slope of a curve is through geometry, not analysis: the slope of the curve in a point is seen as the slope of the tangent line. Students are challenged to design within a coordinate system a slide or ski jump consisting of a linear part and a curved part (see Fig. 6).

Each part is to be described by an equation and the point where the curves meet smoothly, without bumps, has to be indicated.

A mathematician's solution to the task consists of (1) a description of a function $f: D \to \mathbb{R}$ on a domain $D \subset \mathbb{R}$, such that the graph of f accounts for the bended bit, (2) a point $x_0 \in D$, and (3) an equation for a line that accounts for the straight bit, e.g. $y = f(x_0) + f'(x_0)(x - x_0)$. For example, $f(x) = x^2$, with chosen point $(-1,1)$, and line equation $y = 1 - 2(x + 1)$. In short: students need to construct and describe a curve, a point, and a tangent line in that point, before they know how to do so by standard mathematical methods. The order "curve – point – line" is not imposed on students, so they will not necessarily conceive the task as constructing a tangent line to a chosen curve in a chosen point. Instead, they may interpret the task by constructing a tangent *curve* to a chosen line and the point comes out last.

Students are supposed to already be familiar with several curves and their algebraic description. In particular, they are supposed to understand the notion of slope of a line, and how it is represented in an equation for a line. The task intends to let students use this pre-knowledge of slope of a line to make sense of the slope of a curve. The task strongly draws attention to the fact that you do have a number to describe the slope of a line, but do not yet have a method to quantify the slope of the curve. Central to the task is finding a way to deal with this issue. In that sense the task answers to the requirements of a Didactical Phenomenology (Freudenthal, 1983): the task situation begs to be mathematized by means of the target knowledge. Or, in terms of TDS, the "winning" strategy coincides with the target knowledge.

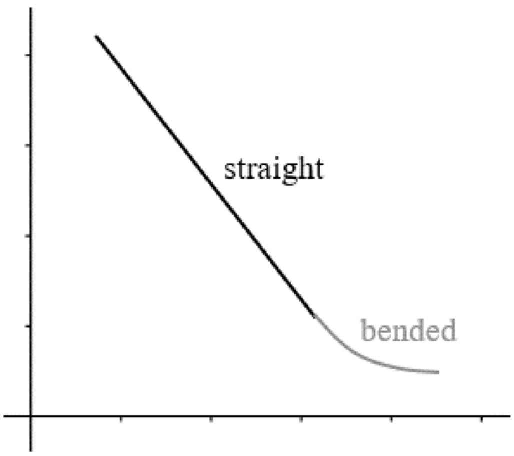

Fig. 6. Task: design a slide or ski jump with one straight part and one bended part.

The slide context of the task is chosen to activate the students' embodied knowledge of steepness and smoothness. The task of teaching students about the steepness or tangency, without referring to real-world experiences and sensations would seem nonsensical. On contrary, the slide-task puts these experiences on the foreground. Students are invited to use every-day language (e.g. “meeting without a bump”, “here it's steeper than there”, “here the line touches”), and pictures describing their experiences with this common object (slide/ski jump) to support their mathematization process.

5.2. Implementation

We invited teachers in our network from the Utrecht region to participate in the study. Five of them were willing to participate (one with two classes), all teaching at upper secondary pre-university schools (the Dutch name for this is: VWO) classes of grade 9 or 10 (Dutch grade 3 and 4). In grade 10 students either have chosen for a science oriented stream or a humanities oriented stream.

The lesson was designed and piloted as part of the European Erasmus + project Meria in several countries (Bos et al., 2019).

For each lesson the students formed groups of three, seated together by joining three tables. The groups were formed spontaneously by the students. Different classes were given different amounts of time in the action phase (ranging from 25 to 50 min), so that we could register the influence of available time on the outcome. If students had a graphical calculator or GeoGebra to their disposal and knew how to use it, they were invited, but not urged, to use it. We did not aim to measure the effect of tool use on the outcome in detail.

For all lessons (except one) we had two or three observers which were allocated to a group in consultation with the teacher. So roughly 30% of the groups had an observer. The goal was to have more detailed information on groups that teachers judged to be different in level and possibly in approach. Observers had an observation form to use to report. Some approaches, that were observed in the pilot, were already presented on the form with a box to tick. Observers were asked to (1) report on the steps that students suggested and/or took, and (2) quote sentences used by students, if those sentences were about specific ideas related to the slope of the curve. Students worked on a work sheet with an extra column on the right. They were requested to use this column to explain what they were drawing or computing at that point on the working sheet. Our data consist of the student worksheets and the observation forms.

The data were analyzed by the first and third author and compared. Another form of triangulation was achieved by comparing observations on the worksheet to reports on the observation forms, for observed groups.

5.3. Method of analysis

In a preliminary investigation of all students' works we discerned seven reoccurring slide design strategies, as described in the Table 1 and Fig. 7.

Then we studied the data again to agree, for each group, on a description of their work and, if possible, a description of the design process in the action phase. Then we classified the strategies using the labels in Table 1.

As a final step we decided for each group whether their result could serve as a *model of* or building block in the institutionalization, and, if so, if one approach, **S**, **L**, **A** or **T** clearly emerged from it as a *model for*. The other two possibilities here were **V**, if there were various options, and **N**, if none of the previous applied. The ascription of this label was based on (1) the design strategy and (2) the validation method. Validating the result means doing something to know your design is correct. The validation can be a-didactical or didactical. So, in case students did not wonder why their design was correct or not, the teacher intervened during group work and drew their attention to this issue, if judged necessary suggesting an approach matching their design strategy. We discerned three ways to validate the designs.

Zooming in on the graphs. If students worked using graphical software, like GeoGebra, or a graphical calculator a validation often used was *zooming in* on the point where the line and curve meet. This way students got a better view to judge (1) whether the line and curve meet smoothly or (2) whether what seemed to be one intersection point were actually two or no intersection points. Obviously, zooming in does not objectively verify a design, but it can falsify a design. If students commented on the locally linear nature of the curve this contributed to a classification as **L**, but otherwise zooming in did not necessarily lead to **L**. Instead, if they

Table 1
Classification labels for student design strategies.

Label	Description
D	**D**raw a line and curve and find the equations from the data (like points, slope of a line, or top of a parabola) taken from the drawing
PS	Choose a line and a curve (equation); then vary the **P**arameter for the **S**lope of the line
PT	Choose a line and a curve (equation); then vary the **P**arameter to **T**ranslate the line
PC	Choose a line and a curve (equation); then vary the **P**arameter(s) of the **C**urve
A	Use **A**lgebraic means to find a good design: e.g. computing intersection points
HS	Use the tangent line perpendicular to the **S**ymmetry axis of a **H**yperbola
C	Use the tangent line perpendicular to the radius of a **C**ircle
R	The **R**est: strategies not mentioned above
O	**O**bscure, untraceable strategies, no data on strategy
N	**N**o serious attempt registered

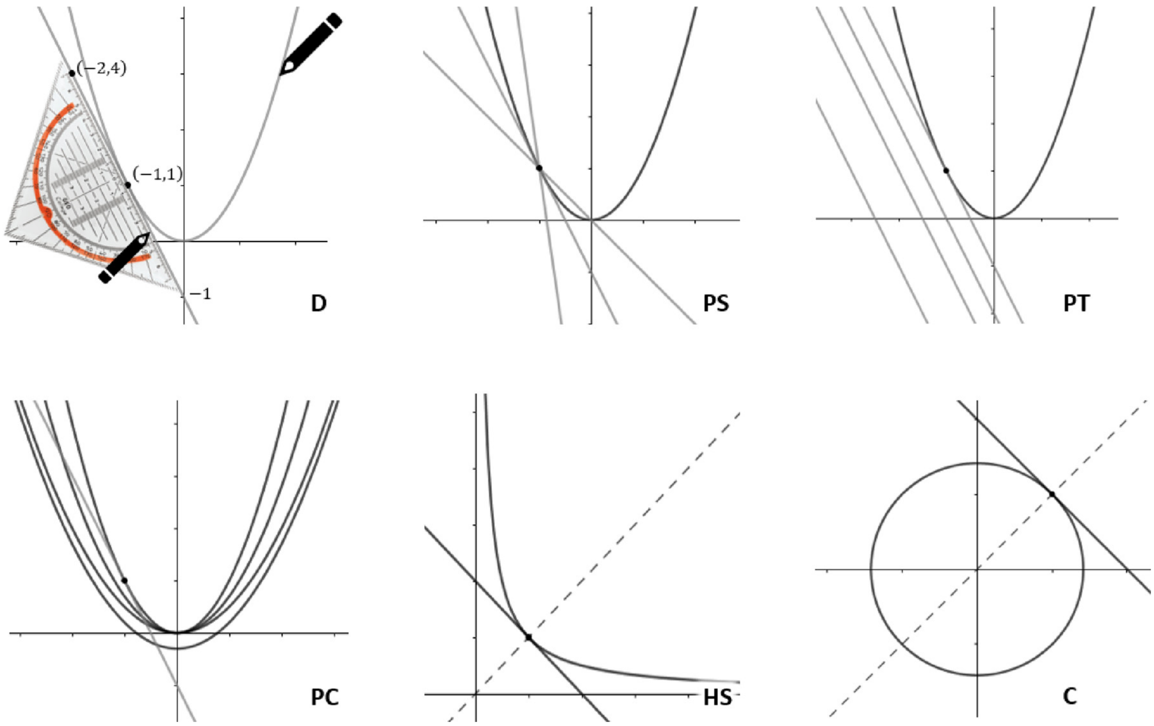

Fig. 7. Students design strategies depicted.

commented on the intersection point(s), this contributed to a classification as **A**.

Computing intersection points. If students computed the intersection points of the line and the curve, they found confirmation about whether the two intersect. Finding two nearby intersection points could suggest the solution is almost correct, and finding just one solution (possibly recognized as a double point) could suggest the design is correct. If students computed the intersection points algebraically or numerically, we interpreted it as a strong cue to classify as **A**.

Geometric arguments and mirror symmetry. In case of a circle strategy (**C**), students knew the solution was correct from the theorem that states that the tangent line is perpendicular to the diameter. In case of a hyperbola (**HS**), where students find the tangent line in a point on the symmetry axis, they could know the solution is correct from the result that this tangent line is perpendicular to the symmetry axis. In fact, the symmetry argument also applies to the circle, where diameters are symmetry axes. In both cases the perpendicular to the curve plays an important role. Imagine putting a mirror on that perpendicular. The curve and it mirror image will meet smoothly, see Fig. 8.

As soon as the mirror rotates away from the perpendicular this will not be the case anymore. So the perpendicular position marks a specific transition point while rotating the mirror around the point on the curve. Therefore we took this validation based on symmetry as a cue to connect to approach **T**.

On top of this, we took the strategy **PS** (varying the slope of the line) as a cue for approach **T**, since varying the slope is a crucial ingredient in this approach. Combinations of **D**, **PT** and/or **PC** generally led to a classification as **V**, unless complimentary observations suggested otherwise. In most of these cases the teacher could suggest a validation method (zooming in, computing intersection points, or otherwise), which, in turn, guided students' thinking towards one approach or another.

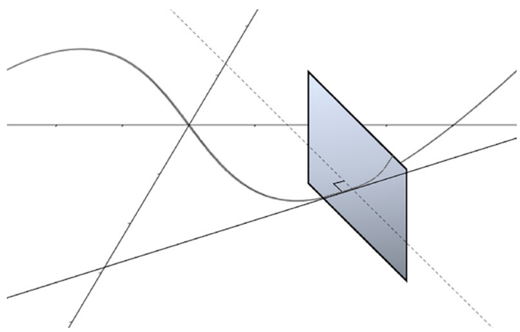

Fig. 8. A curve meets its mirror image smoothly, if the mirror is perpendicular to the tangent line.

Table 2
Data for the six classes where the experimental lesson took place.

Class label	School	Grade	Length of action phase in minutes	Number of groups (n = 44)	% of groups with a (nearly) correct solution
1	I	9	25	8	12,5%
2	I	9	25	7	100%
3	II	9	25	3	66,7%
4	III	10	25	7	42,9%
5	II	10	35	9	55,6%
6	II	10	50	10	80%

6. Results

The experimental lesson was given in six classes in three different pre-university schools I, II and III, see Table 2.

We first present five samples of illustrative cases that are representative for the observed variety in strategies of the students and then we present an overview of all observed strategies and connecting approaches.

Case 1 (Class 4, classified as C/R → L). This group had an observer. The students used GeoGebra to visualize their designs. Their first solution attempt was based on a circle, but then they switched to a parabola. Their equations were $y = 2x + 8$ and $y = \frac{1}{2}x^2 + 2x + 8$. A correct design: the line and curve are tangent in the point (0,8). They wrote on their work sheet

b in the formula of the parabola and a in the formula of the line must be the same.

(they had the formulas $y = ax^2 + bx + c$ and $y = ax + b$ in mind). They specified this to the observer who quoted their remark in her observation form:

the directional coefficients of the line and the curve *must be the same*

So this group tried to relate the directional coefficient of the line to a parameter in the equation for the parabola and succeeded, because in the point $(0, c)$ the parameter b does represent the slope of the parabola. Students noted that the only difference between the two equations is the term $\frac{1}{2}x^2$. For the teacher there is great opportunity to build on this informal idea towards the approach via linear approximation (**L**). This group produced the line algebraically by omitting the quadratic term, as if they truncate a Maclauren (Taylor) series off after the linear term.

Case 2 (Class 5, classified as HS → T). This group had an observer. Their design consisted of a hyperbola, described by the equation $y = \frac{1}{x}$, and a line, described by the equation $y = -x + 2$ (see Fig. 9). This is correct; the line and curve meet smoothly at the point (1,1). The observer noted that students explicitly mention symmetry as a justification of their design.

However, when the teacher joined the students' table and asked how they could find out whether the design was correct, they mentioned that they would like to zoom in on the intersection point. They remarked that to have absolute certainty, they would like to "zoom in forever". Even though we classified this group as **T**, it is worth mentioning that this remark would have also been a natural starting point for a discussion on limits, because the students indicate they need a way to deal with *the infinite* (in the sense of infinitesimal).

Case 3 (Class 5, classified as PC/PS/PT → L. This group used a graphical calculator (GC) to visualize their designs. First the students fixed a seemingly random line and tried to adjust the parameters in the equation for the parabola. Then they changed strategy and fixed the parabola to $y = \left(\frac{3}{10}x\right)^2$ and fixed a point on it (10,9). Next, they wanted to adjust the parameters of the line. In the end, they settled for $y = \frac{15}{10}x - 6$, which is not correct. Graphing it on a GC on a "standard" scale shows a convincing picture, since the second intersection point is quite close to (10,9). When students spontaneously began to validate their design, one students said

> I think that when the line touches the curve, in that small part the equation of the parabola must be the same as the line.

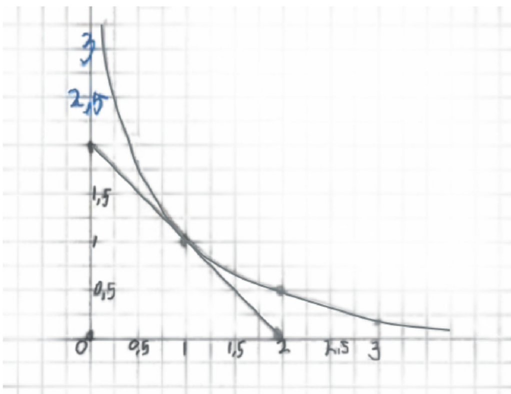

Fig. 9. Students' drawing of a design based on a hyperbola (case 2).

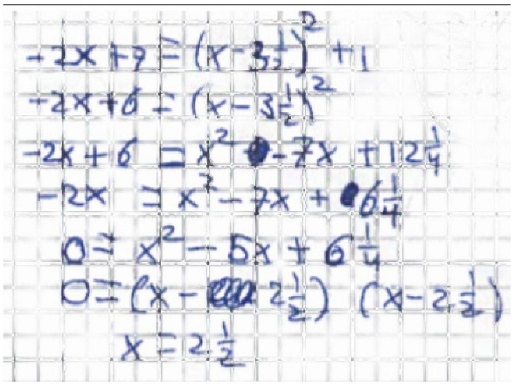

Fig. 10. Students' work from case 4.

The students did not manage to mathematize this statement, but it phrases the idea of local linear approximation in an informal way. So in this case the classification as **L** is a consequence of the validation and not of the design itself.

Case 4 (Class 5, classified as PC → A). This group started by drawing a line and a parabola.

Then the students derived an equation for the line from the graph: $y = -2x + 7$. Next, they found an equation for the parabola: $y = \left(x - \frac{7}{2}\right)^2 + 1$. When the teacher arrived at this table, the students explained that they wanted $\left(\frac{7}{2}, 1\right)$ as the coordinates for the top of the parabola, since the line intersected the x -axis in $\left(\frac{7}{2}, 0\right)$. Even though they could not explain why this was needed, it does give them a correct solution. When the teacher challenged them to explain why they knew their design was correct, they produced the computation in Fig. 10. They explained that, because there was a single intersection point, the design was good. Therefore, this groups' work was classified by **A**.

Case 5 (Class 6, classified as A → A). This group relied on algebraic methods from the beginning of their attempt. If the students had a picture, it was just in mind. They chose for a hyperbola, $y = \frac{6}{x}$, and a line described by $y = -x + 5$.

They computed the intersection points and found two, at $x = 2$ and $x = 3$. At this point they remark (inside the bottom "circle", see Fig. 11): "not good, we need 1 outcome". They realize they can adjust the "5" in the equation, which means translating the line, to force having just one solution. In the next line the "5" is replaced by "$2\sqrt{6}$", which allows a unique solution, with multiplicity 2. So, their new line is $y = -x + 2\sqrt{6}$, a correct solution. Consequently, this was classified as an algebraic approach.

Table 3 shows how we classified the strategies and connecting approaches of these five sample groups and the remaining 39 groups. Fig. 12 presents the absolute frequency of the student strategies. Note that these do not add up to 44, since some groups followed more than one strategy. The cases of untraceable strategies come from groups without observer, often using GeoGebra or a graphical calculator. Those groups would forget to write down most of what they were doing. In group 3 and 6 there are no cases where the strategy is untraceable. In these lesson the teacher prompted and helped students to make their strategies explicit.

Fig. 13 shows the relative frequencies of the approaches to the tangent line and slope of a curve to which the students' work connects. Note that the secant line approach (**S**) is not occurring.

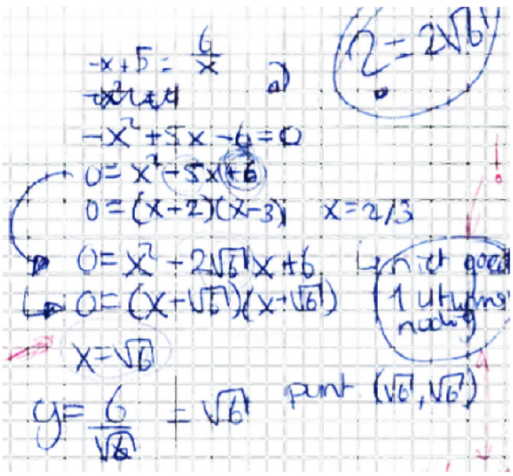

Fig. 11. Students' work on a design using algebraic methods (case 5).

Table 3
Strategies and connecting approach per group.

Class label	School	Strategies per group	Connecting approach
1	A	PC, PC, O, PT, HS, D/PS, O, N	V, V, N, T, N, T, N
2	A	PC/PT, O/PT, O, O, O, PS/PC, O	V, V, V, V, V, V, V
3	B	A, C, C	A, T, N
4	C	PT/PC, D, O, C/R, O, R, N	V, V, V, L, N, L, N
5	B	D, R, PC, O, R, PC/PS/PT, PC, HS, O	V, V, V, N, A, L, A, T, N
6	B	A, C, D, PT/PC/C, D, R, PC/PT/PS/A, C, D, PC	A, N, L, V, N, V, A, V, V, A

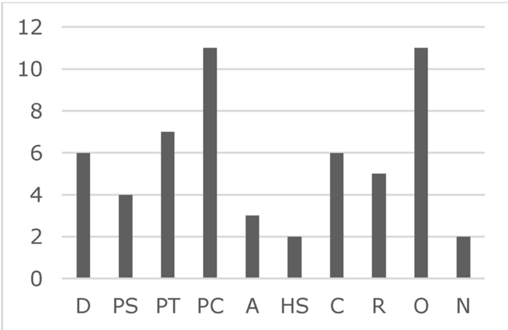

Fig. 12. Frequency of student strategies.

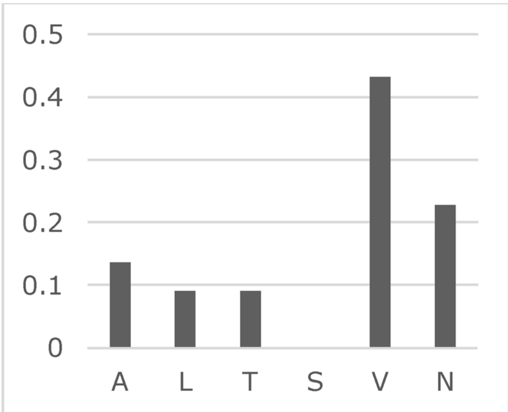

Fig. 13. Relative frequency of approaches.

7. Discussion

We first discuss what can be concluded with respect to our sub-questions, and then address the main research question.

7.1. Sub-question 1: strategies

Tabel 3 and Fig. 12 show that students use a wide range of strategies. The most frequently applied strategy was to fix the line and vary the parameters of the curve (11 occurrences of **PC**). An experienced mathematician would chose to fix the curve and compute the slope in a point. Student groups that follow strategy **PC** or **PT** go the opposite way: they begin by fixing the slope (of the line). In case of **PT** they pursue this by suitably translating the line until it touches the curve; in case of **PC** by suitably transforming the curve until it touches the line. So, varying the slope of a line (**PS**), with the curve and point fixed, is less favored: only four occurrences.

What is appealing about fixing the line, and in particular its slope, before fixing the curve? We see four possible explanations: (1) *Without* a standard technique to apply to the problem, the students might consider the order line-curve or curve-line as unimportant; (2) for students making an equation for a line is easier than making an equation for a curve, so they prefer to begin with this; (3) the problem formulation leads students this way: students may see the straight bit of the slide as most important and the curved bit as a less important ramp; (4) We put the entry for the line equation before the one for the curve on the final answer work sheet. They were supposed to fill it in at the end of the action phase, but they could have looked at it.

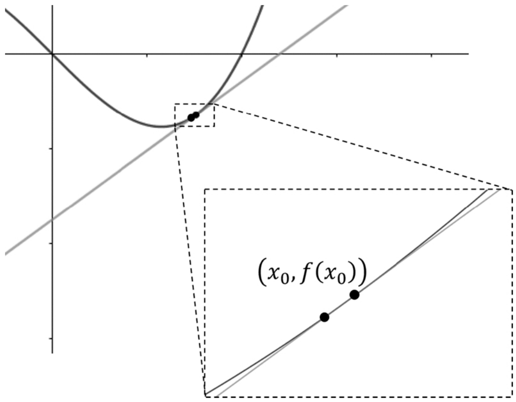

Fig. 14. Design strategy not observed: zooming in to construct the line.

In class 5 and 6 students had more time in the action phase. In those classes some groups used that time to switch strategy.

In class 2 students used GeoGebra and in class 1 they didn't. There is a remarkable difference between those classes in the number of nearly correct solutions (100% versus 12,5%, see Table 1). GeoGebra helped students to experiment: dragging lines and curves across the plane without computation, and, complimentary, changing parameters without drawing the graphs again. In many groups, students knew or found out that GeoGebra can provide the equations of lines and curves. Altogether, GeoGebra facilitated students to arrive at a correct or nearly correct solution with less mathematical demand, but still allowing for conceptual reflections.

7.2. Subquestion 2: emergent models

As is clear from Fig. 14 the secant lines approach (**S**) never emerged from students' work. A condition for this approach – that is met in few cases – is that students first fix the curve. Next, students should fix a point on the curve, which happened in even less cases. On top of that, student should then have the idea to construct a line *using another point on the curve*. This did not occur. This would only be a first step, where the next would be to move the second point closer to the first. We think the reason that students do not choose a second point on the curve is that they intuitively know that, if there is a second intersection point, the design will have a bump. Students want a solution that is *immediately correct*. Deliberately beginning with a bumpy slide and working your way towards a better one is an unfamiliar strategy for students. Moreover, even if they are willing to pursue this strategy, they may realize that it has the serious difficulty – the problem of limits – that the perfect solutions seems out of reach. The conclusion is that from the slide assignment the *tangent line as a limit of secant lines*-model does not emerge from the informal student models for good reasons. The fact that this model does not appear in students work on a situation that begs to be mathematized by tangent lines and the slope of curve confirms the status of epistemic obstacle of the secant line approach.

For 91% of the groups the curve was a conic section, with parabolas most popular (61%), so students had the means to compute the intersection point(s) algebraically. We observed that only science oriented students spontaneously use algebra in either of these phases. This is no surprise, because these are students that usually feel more comfortable with algebra and would apply it in new situations. Algebraic validation was sometimes proposed by the teacher to individual groups and in classroom discussions. Fig. 14 shows that for 14% of the groups the algebraic approach (**A**) emerged naturally, but in many more groups (classified as **V**) the teacher had the opportunity to conduct a group in that direction. Note that for groups classified as **V** the teacher could alternatively connect the student work to the other approaches, in particular **L**. Students using GeoGebra or the GC had the opportunity to compute the intersection points digitally. Remarkably, this use has not been observed.

The locally linear approach (**L**) emerged from 9% of the groups (see Fig. 14), but in many of the groups classified as **V**, the teacher had the option to connect students work to **L** as well. The approach is closely related to the idea of zooming in on the curve, but students did not zoom in *to construct* a line. Zooming until the curve looks more or less like a line, then choosing two points to construct a line with, did not occur.

Rather, after they had a candidate line and curve, they would zoom in *to validate* the design. Probably, the reasons why zooming in is not used as a means of construction are the same as the reasons why students do not use secant lines (students want to construct an immediately correct design, etcetera). Still, zooming in as a validation method alone is not enough to connect the students reasoning to approach **L**. It matters what students are looking for: Are they checking whether they really have one intersection point and not two, then we should classify as **A**; are they checking whether the line and curve are indistinguishable, then we should classify as **L**. In many cases from the data it could not be decided which of the two applied so we classified as **V**, although in some cases, like case 1 and 3 above, we could classify as **L**.

The "transition in overtaking" approach (**T**) is another approach that requires students to first fix the curve and a point. For this reason it did not emerge from many groups (9%, see Fig. 14). Of this 9% a majority is based on students applying a symmetry, whose connection to **T** is not as strong as other connections. Also, in students' discussions about the validation of design the idea of *overtaking lines and curves* was not observed. So approach **T** does not emerge convincingly. This comes as a surprise, since *rotating a ruler at a point until it touches the curve* a-priori seemed a very accessible strategy and the language of *overtaking* seemed readily available.

We conclude that the intersection point approach (**A**) is slightly favoured over the other approaches (see Fig. 14: 14% versus 9% **L**, 9% **T**, 0% **S**), in particular by science oriented students. Given the many occurrences of **V** (which in most cases means **A** or **L** is a possible emergent model), we can state that these together are the favourite ones. An explanation for the popularity of the intersection point approach **A** is that students spend a lot of time on computing intersection points algebraically in lower secondary, so to them it is a readily available and meaningful frame of reference and tool. We conclude that the slide activity succeeded in clarifying what pre-knowledge and tools are meaningful to students to approach the problem of tangent lines. Additionally, the slide activity makes clear what connecting approaches to slope have good potential to become meaningful to the students.

7.3. Main question: supporting reinvention

For 77% of the groups, and 100% of the classes, there were *models of* that could be developed into *models for* (see Fig. 14). In every class multiple connecting approaches were found suitable, so the teacher had a choice, either to single one out or to discuss several approaches and possibly combine them into more complete picture of the target knowledge. The previous section supports the conclusion that in many lessons a reinvention process took place. How? It is important to stress the RME *emergent models*-principle. The students did not fully reinvent the formal definition of slope of a curve. Instead, students solutions and strategies represented informal models of the notion. Together with a teacher these models could be developed during the institutionalization phase and later lessons into more general models based on mostly the intersection points approach (**A**) and locally linear approximation approach (**L**).

Biza studied a lesson design for the meaning of slope of a curve based on the diagnostic teaching methodology (Biza, 2011). This was a *guided* reinvention study focussed both on concept and definition development where the instructor provided well-timed cues with examples and non-examples. She observed that students' personal meaning of the definition of evolved significantly, but also in various ways, in response to the lesson. We like to emphasize that in our study students developed the concepts independently of the teacher in an a-didactical (*non-guided*) action phase, and therefore the sense of ownership was preserved and only techniques that are meaningful to the students were used. As a consequence the students' personal meanings of the slope concept developed in various ways. Andrews-Larson and collaborators (Andrews-Larson et al., 2017) stress the importance to connect to students' thinking in a reinvention task. We see this role situated in the institutionalization phase, a phase that is very demanding for the teacher. Analysing students work for informal models during the action phase, without interfering in their work, is a challenge. In the last experiment, class 6, we improved the scenario by spreading the activity over two lessons. Students handed in their work sheets at the end of the action phase, which coincided with the end of the first lesson. This gave the teacher enough time to prepare for the institutionalization looking for the potential in students' approaches.

It is tempting to conclude from our study that approach **A** and **L** are generally more accessible and meaningful for secondary school students. Their didactic potential has been observed before and deployed for teaching approaches, see (Michael Range, 2018) for **A** and (Tall, 2013) for **L**. The task however has characteristics that evoke these approaches. First of all the task is geometric in nature and appeals to embodied knowledge of students concerning steepness and smoothness; a task about average growth of a quantity – or in kinematic context (average speed to instant speed) – could still provoke the secant line approach, although we are not aware of any unguided reinvention task that achieves this. Secondly, a small adjustment to the task may already lead to different results. If the task would have required to fix the curve equation for the slide before the linear equation, approach **S** and **T** perhaps would have become more likely. This would also make the task more directive, limiting the approaches the students might take in favor of certain approaches to slope. It is important to note that the task was stated in *most open form* for this experiment. The strength of the task it that is manages to activate students to apply meaningful methods from which didactically potent formal approaches (**A** and **L**) to the slope of a curve emerge – and not the standard approach **S**.

We can think of no reason why a teacher implementing our activity in a different school (in a different country) would have a very different experience than described in this paper. It is important to note that the outcomes do depend on the students' mathematics level, and on the pre-knowledge of synthetic/analytic geometry and algebra of the class. If the mathematics level is average or lower, or time is limited, we suggest encouraging students to use graphing software, like GeoGebra, for reasons discussed before.

Is the reinvention principle suitable and feasible at secondary school level? Our cases contributes to the point of view that it is suitable, since (1) there were hardly any groups that were not engaged (2 out of 44 ($\approx$ 4,5%) did no serious attempt, see **N** in Table 3) and (2) as mentioned before in 77% of the groups (in 100% of the classes) produced informal models. So constructivist learning can be achieved through constructivist teaching. The feasibility depends on the effort a teacher is willing to make. The teacher needs to prepare and study the subject more thoroughly than usually to be able to recognize non-standard strategies and approaches in the students' work. Since students' reinvention may be non-standard (as in our case), they may also need to provide a meaningful bridge from the students' reinvention (which maybe based on approach **L** or **A**) to what is required by the curriculum (probably based on approach **S**).

Measuring whether the slide task made the slope-concept more meaningful to students would be a self-fulfilling prophecy within the RME-framework. According to RME the concept is meaningful if it is based on what is already meaningful to the student. So the issue is to what extent task characteristics activate experiences that connect to and support the development of mathematical notions aimed at. This study illustrates how everyday embodied experiences of steepness and smoothness can be activated in a classroom setting and have the potential to make slope of a curve meaningful, and how to institutionalize this notion with all students in a classroom practice. Further research is needed to investigate whether these students really have grasped and retained the meaning of slope, even when calculational procedures enter the learning trajectory.

Declaration of Competing Interest

None.

CRediT authorship contribution statement

Rogier Bos: Conceptualization, Methodology, Formal analysis, Investigation, Writing - original draft, Visualization. **Michiel Doorman:** Conceptualization, Methodology, Writing - review & editing. **Margherita Piroi:** Formal analysis, Investigation.

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