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Excess-Continuous Prox-Regular Sweeping Processes

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Abstract: We consider the Moreau's sweeping processes driven by a time dependent prox-regular set $C(t)$ which is continuous in time with respect to the asymmetric distance e called *the excess*, defined by $e(\mathcal{A},\mathcal{B}) := \sup_{x\in\mathcal{A}} d(x,\mathcal{B})$ for every pair of sets $\mathcal{A}$, $\mathcal{B}$ in a Hilbert space. As observed by J.J. Moreau in his pioneering works, the excess provides the natural topological framework for sweeping processes. Assuming a uniform interior cone condition for $C(t)$, we prove that the associated sweeping process has a unique solution, thereby improving the existing result on continuous prox-regular sweeping processes in two directions: indeed, in the previous literature $C(t)$ was supposed to be continuous in time with respect to the symmetric Hausdorff distance instead of the excess and also its boundary $\partial C(t)$ was required to be continuous in time, an assumption which we completely drop. Therefore our result allows to consider a much wider class of continuously moving constraints.

Keywords: Evolution variational inequalities, sweeping processes, prox-regular sets, Hausdorff distance, excess, functions of bounded variation.
2020 Mathematics Subject Classification: 34G25, 34A60, 47J20, 74C05.

1. Introduction

1.1. Sweeping processes

Sweeping processes are a class of evolution problems with unilateral constraints which were originally introduced in [48, 49] by J. J. Moreau, motivated by problems in elastoplasticity and nonsmooth mechanics (cf., e.g., [52, 54, 56], and the monograph [47]). Later sweeping processes have then found applications in several diverse disciplines: in economic theory (for instance in [24, 30, 31]), in electrical circuits (see, e.g., [1, 2, 3, 9]), in crowd motion modeling (cf., e.g., [11, 25, 42, 43, 44, 45, 46]), and in other fields (see, e.g., papers [32, 39, 57, 67] and their references).

Sweeping processes can be described as follows. Fix a final time $T > 0$, let $\mathcal{H}$ be a real Hilbert space, and let $C(t)$ be a given nonempty, closed subset of $\mathcal{H}$ depending on time $t \in [0,T]$.

One has to find a function $y : [0,T] \longrightarrow \mathcal{H}$ such that

$$y(t) \in C(t) \qquad \forall t \in [0,T]\,, \tag{1}$$

$$y'(t) \in -N_{C(t)}(y(t)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T], \tag{2}$$

$$y(0) = y_0, \tag{3}$$

y_0 being a prescribed point in $C(0)$. Here $y'(t)$ is the derivative of y at t, $\mathcal{L}^1$ is the Lebesgue measure, and $N_{C(t)}(y(t))$ is the (proximal) exterior normal cone to $C(t)$ at $y(t)$ (all the precise definitions needed in the paper will be provided in the next section). In [48] Moreau found a unique Lipschitz continuous function y solving (1)–(3), by assuming that the sets $C(t)$ are convex, and the mapping $t \longmapsto C(t)$ is Lipschitz continuous in time, when the family of closed subsets of $\mathcal{H}$ is endowed with the Hausdorff metric

$$d_{\mathcal{H}}(\mathcal{A},\mathcal{B}) := \max\left\{\sup_{x\in\mathcal{A}} d(x,\mathcal{B}), \sup_{y\in\mathcal{B}} d(x,\mathcal{A})\right\},$$

with $\mathcal{A}$, $\mathcal{B}$ nonempty closed sets in $\mathcal{H}$.

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In the celebrated paper [55], the formulation (1)–(3) was then extended to the case when the mapping $t \longmapsto C(t)$ is of bounded variation and right continuous with respect to $d_{\mathcal{H}}$: it is proved that there is a unique right continuous $y \in BV([0,T];\mathcal{H})$, the space of right continuous $\mathcal{H}$-valued functions with bounded variation, such that there exist a positive measure μ and a μ-integrable function $v : [0,\infty[\longrightarrow \mathcal{H}$ satisfying (1) and (3) together with

$$\mathrm{D}y = v\mu, \tag{4}$$

$$v(t) \in -N_{C(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [0,T], \tag{5}$$

where $\mathrm{D}y$ denotes distributional derivative of y, which is a measure since y is of bounded variation.

Actually in [55] the moving set C is assumed to be only with *bounded retraction*, rather than with bounded variation, i.e. the Hausdorff distance is replaced by the asymmetric distance

$$e(\mathcal{A},\mathcal{B}) := \sup_{x\in\mathcal{A}} d(x,\mathcal{B}),$$

called *excess of $\mathcal{A}$ over $\mathcal{B}$*, with $\mathcal{A}$, $\mathcal{B}$ nonempty closed sets in $\mathcal{H}$. Indeed, as Moreau observed, "some regularity for the motion of the set $C(t)$ is needed only when this set retracts, effectively sweeping the point $y(t)$", so that it is natural to assume only "some unilateral rating for the displacement of sets". This reflects the fact that the dynamics is intrinsically directional in time (and irreversible), hence it is wasteful to impose a control on the movement of the sets which is symmetric in time, such as the continuity with respect to the Hausdorff distance or the bounded variation arising from it.

After the pioneering work of Moreau, convex sweeping processes in the BV framework have received a great deal of attention, but most of the papers addressed theorems involving the Hausdorff distance $d_{\mathcal{H}}$ rather than the excess e, largely due to the fact that the asymmetric excess introduces certain technical difficulties (see [50, 53]). Several developments can be found, e.g., in the monograph [47], in papers [10, 12, 13, 14, 15, 25, 33, 40, 61, 62, 67], and in the references therein. These researches can be generalized in two directions. On one hand one can deal with driving sets which are non-convex, and on the other hand one can relax the assumption that $C(t)$ has bounded variation.

The analysis of non-convex sweeping processes started in [70] and, since then, has called the attention of many other authors, e. g., [5, 8, 17, 21]. Concerning existence and uniqueness results in the non-convex setting, we mention, for instance, [4, 6, 7, 18, 28, 34, 37, 38, 39, 57, 63, 65, 66]. More precisely these papers consider the family of *uniformly prox-regular* subsets of $\mathcal{H}$, which can be described as those closed subsets $\mathcal{K}$ such that there exists $r > 0$ for which the function

$$x \longmapsto d_{\mathcal{K}}(x) := \inf\{\|z - x\| \ : \ z \in \mathcal{K}\}$$

is differentiable on the open set $U_r(\mathcal{K}) := \{x \in \mathcal{H} \ : \ 0 < d_{\mathcal{K}}(x) < r\}$. A fundamental property of uniform prox-regular sets is that any point in $U_r(\mathcal{K})$ admits a unique projection in $\mathcal{K}$, therefore, such sets can be regarded as a natural generalization of convex sets. Uniform prox-regular sets were introduced in [29] under the name *positively reached sets* in the finite dimensional case. Further properties of these sets were investigated in [71] (therein called *weak convex sets*), and the notion of prox-regularity was later extended to infinite dimensional spaces in [20, 22, 58].

The second direction of research about processes driven by convex sets with unbounded variation was instead addressed independently in [14] and in Section 19 (mostly written by A. Vladimirov) of [35]. In such works the moving constraint is required to have non-empty interior: this assures that the solution has bounded variation and that formulation (5) makes sense. For other contributions see, e.g., [16, 36, 47] and their references.

The first paper where the two directions are addressed simultaneously, namely when the moving non-convex constraint has unbounded variation, was [22] where it is assumed that $C(t)$ is continuous in time with respect to the Hausdorff distance $d_{\mathcal{H}}$. In [22] another geometric condition is also required, namely $C(t)$ has uniform interior cone condition, which essentially means that cusps are not admitted on the boundary, thereby assuring the bounded variation of the solution.

A further step into this research can be found in the recent paper [38], where $C(t)$ has uniform interior cone condition, but with special form $C(t) = u(t) - \mathcal{Z}(w(t))$ for given time-dependent $u(t) \in \mathcal{H}$ and parameter $w(t)$ which are assumed to be regulated right-continuous functions, i.e., they admit the one-sided limits

$$u(t+) = u(t),\ u(t-),\ w(t+) = w(t),\ w(t-)$$

at every point $t \in [0,T]$.

As we mentioned above, most papers about prox-regular sweeping processes address theorems where the rating of $C(t)$ is prescribed with respect to the Hausdorff distance, but not to the excess. To the best of our knowledge, in the prox-regular setting, the excess is addresses only in [67] and in [61] where $C(t)$ is assumed to be of bounded retraction: in [67] the uniqueness of solution is proved, while the existence is achieved in [61].

1.2. Aim of this article

The aim of the present paper is to study the case when $C(t)$ moves continuously with respect to the excess. With regard to the results of [22], our improvements are twofold.

First, we consider a general $C(t)$ having uniform interior cone condition, but we assume its continuity in time only with respect to the excess, i.e. we assume that the following excess-continuity modulus

$$\omega(\delta) := \sup\{e(C(s), C(t)) \ : \ 0 \le t - s < \delta\}$$

converges to zero as $\delta \searrow 0$. The motivation for doing this is that the excess continuity captures precisely the time-directionality which is relevant for the evolution, allowing in particular for variations of the constraint sets that are invisible to the dynamics but excluded by Hausdorff-based frameworks. In Example 3.5 we will illustrate this distinction, with a sweeping process which cannot be treated with existing literature.[1] As already said, this can be considered a "return to the origin", in the sense that we reestablish the excess introduced by Moreau as the natural framework for sweeping processes, although it has been abandoned for decades by other authors mostly because of technical reasons.

Second, we remove the assumption made in [22] that $t \longmapsto \partial C(t)$ is continuous w.r.t. the Hausdorff distance: we do not require any kind continuity of $\partial C(t)$, thereby amply enlarging the class of admissible sets. This strong condition appeared as a technical assumption in [22, Lemma 2.3], but it is way too restrictive and superfluous.

Let us point out that relaxing the movement to be excess-continuous is not merely a technical innovation, but has useful analytical consequences as in particular it unlocks the possibility to study general *regulated* prox-regular sweeping processes, i.e. driven by a family of sets $C(t)$ which admits limits $C(t-)$ and $C(t+)$ at every time t. Indeed, the regulated setting can be reduced to the continuous one, provided we are able to construct a time reparametrization in which discontinuities of the driving sets are filled by geodesic interpolations that preserve prox-regularity, an operation that appears to be unavailable under Hausdorff continuity but is instead simple in the excess topology. This extension to the regulated class is a work in progress by the first author and some collaborators.

The paper is organized as follows. In the next section we present some preliminaries and in Section 3 we state the main results of the paper. In Section 4 we perform all the necessary proofs. Section 4 is divided in several subsections corresponding to the various stages of the proof.

2. Preliminaries and notations

In this section we recall the main definitions and tools needed in the paper. The set of integers greater than or equal to 1 will be denoted by $\mathbb{N}$. For every real number $t \in \mathbb{R}$, we define $\lceil t \rceil := \min\{k \in \mathbb{Z} \ : \ t \le k\}$.

2.1. Prox-regular sets

We recall here some notions of non-smooth analysis. We refer the reader to the monographs [19, 64, 68, 69]. Throughout this paper we assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } (x,y) \longmapsto \langle x, y\rangle, \\ \|x\| := \langle x, x\rangle^{1/2}, \quad x \in \mathcal{H}, \end{cases} \tag{6}$$

and we endow $\mathcal{H}$ with the natural metric defined by $(x,y) \longmapsto \|x-y\|$, $x, y \in \mathcal{H}$. If $\rho > 0$ and $x \in \mathcal{H}$ we set

$$B_\rho(x) := \{y \in \mathcal{H} : \|y - x\| < \rho\} \ \text{ and } \ \overline{B}_\rho(x) := \{y \in \mathcal{H} : \|y - x\| \le \rho\}.$$

If $\mathcal{Z} \subseteq \mathcal{H}$, the closure, the boundary and the interior of $\mathcal{Z}$ will be respectively denoted by $\overline{\mathcal{Z}}$, $\partial\mathcal{Z}$, and $\mathrm{int}(\mathcal{Z})$. If $x \in \mathcal{H}$ we also set $d_{\mathcal{Z}}(x) := d(x, \mathcal{Z}) := \inf_{s\in\mathcal{Z}} \|x - s\|$, defining in this way a function $d_{\mathcal{Z}} : \mathcal{H} \longrightarrow \mathbb{R}$.

[1] A simpler example is given by $C(t) := B(0, 2 - t + \lfloor t \rfloor)$. In the Hausdorff topology, it retracts continuously and enlarges with jumps. The instantaneous enlargements do not affect the trajectories, and indeed they are invisible in the excess topology (hence the movement is excess-continuous).

Definition 2.1. If $\mathcal{Z} \subseteq \mathcal{H}$, $\mathcal{Z} \neq \emptyset$, and $y \in \mathcal{H}$ then we set

$$\mathrm{Proj}_{\mathcal{Z}}(y) := \left\{ x \in \mathcal{Z} \ : \ \|x - y\| = \inf_{z \in \mathcal{Z}} \|z - y\| \right\}. \qquad \square \tag{7}$$

Proposition 2.3 below can be found, e.g., in [19, Proposition 1.3]. In fact, given $y \in \mathcal{H}$ and $x \in \mathcal{Z}$, we see that $\|x - y\|^2 \leq \|z - y\|^2$ (resp. $\|x - y\|^2 < \|z - y\|^2$) for all $z \in \mathcal{Z} \setminus \{x\}$ if and only if $x \in \mathrm{Proj}_{\mathcal{Z}}(y)$ (resp. $\{x\} = \mathrm{Proj}_{\mathcal{Z}}(y)$), and writing

$$\|z - y\|^2 = \|z - x\|^2 - 2\langle y - x, z - x\rangle + \|x - y\|^2$$

we then have the following

Proposition 2.2. *If $\mathcal{Z} \subseteq \mathcal{H}$, $\mathcal{Z} \neq \emptyset$, and $y \in \mathcal{H}$, then*

$$x \in \mathrm{Proj}_{\mathcal{Z}}(y) \iff \left[\, x \in \mathcal{Z}, \ \langle y - x, z - x\rangle \leq \tfrac{1}{2}\|z - x\|^2 \ \ \forall z \in \mathcal{Z} \, \right] \tag{8}$$

and

$$\{x\} = \mathrm{Proj}_{\mathcal{Z}}(y) \iff \left[\, x \in \mathcal{Z}, \ \langle y - x, z - x\rangle < \tfrac{1}{2}\|z - x\|^2 \ \ \forall z \in \mathcal{Z} \setminus \{x\} \, \right] \tag{9}$$

If $\mathcal{K}$ is a nonempty closed subset of $\mathcal{H}$ and $x \in \mathcal{K}$, then $N_{\mathcal{K}}(x)$ denotes the *(exterior proximal) normal cone of $\mathcal{K}$ at x* which is defined by setting

$$N_{\mathcal{K}}(x) := \{\lambda(y - x) \ : \ x \in \mathrm{Proj}_{\mathcal{K}}(y), \ y \in \mathcal{H}, \ \lambda \geq 0\}. \tag{10}$$

Let us also recall the following proposition (see, e.g., [19, Proposition 1.5]).

Proposition 2.3. *Assume that $\mathcal{K}$ is a closed subset of $\mathcal{H}$ and that $x \in \mathcal{K}$. We have that $u \in N_{\mathcal{K}}(x)$ if and only if there exists $\sigma \geq 0$ such that*

$$\langle u, z - x\rangle \leq \sigma\|z - x\|^2 \qquad \forall z \in \mathcal{K}. \tag{11}$$

Now we recall the notion of *uniformly prox-regular set*, introduced in [20, Section 4, Theorem 4.1(d)]) under the name of *proximal smooth set*, and generalized in [58, Definition 1.1, Definition 2.4, Theorem 4.1].

Definition 2.4. If $\mathcal{K}$ is a nonempty closed subset of $\mathcal{H}$ and if $r \in \,]0, \infty[$, then we say that $\mathcal{K}$ is *r-prox-regular* if for every y the following implication holds true:

$$0 < d_{\mathcal{K}}(y) < r \implies \begin{cases} \mathrm{Proj}_{\mathcal{K}}(y) \neq \emptyset, \\ x \in \mathrm{Proj}_{\mathcal{K}}\left(x + r\dfrac{y - x}{\|y - x\|}\right) \qquad \forall x \in \mathrm{Proj}_{\mathcal{K}}(y). \end{cases} \tag{12}$$

The family of r-prox-regular subsets of $\mathcal{H}$ will be denoted by $\mathrm{Conv}_r(\mathcal{H})$. We will also indicate by $\mathrm{Conv}_0(\mathcal{H})$ the family of nonempty closed subsets of $\mathcal{H}$, and by $\mathrm{Conv}_\infty(\mathcal{H})$ the family of nonempty closed convex subsets of $\mathcal{H}$. $\square$

It is useful to take into account the following easy property.

Proposition 2.5. *Let $\mathcal{K}$ be a closed subset of $\mathcal{H}$. If $y \in \mathcal{H} \setminus \mathcal{K}$ and $x \in \mathrm{Proj}_{\mathcal{K}}(y)$ then $\mathrm{Proj}_{\mathcal{K}}(x + t(y - x)) = \{x\}$ for every $t \in \,]0, 1[$. Hence if $\mathcal{K}$ is also r-prox-regular for some $r > 0$ it follows that $\mathrm{Proj}_{\mathcal{K}}(u)$ is a singleton for every $u \in \mathcal{H}$ such that $d_{\mathcal{K}}(u) < r$.*

Proof. If $t \in \,]0, 1[$ then, thanks to (8), for every $z \in \mathcal{K} \setminus \{x\}$ we have that

$$\langle x + t(y - x) - x, z - x\rangle = t\langle y - x, z - x\rangle \leq \tfrac{t}{2}\|z - x\|^2 < \tfrac{1}{2}\|z - x\|^2,$$

therefore $\mathrm{Proj}_{\mathcal{K}}(x + t(y - x)) = \{x\}$ by virtue of (9) and the first statement is proved. If $\mathcal{K}$ is also r-prox-regular and $d_{\mathcal{K}}(u) < r$, pick $x \in \mathrm{Proj}_{\mathcal{K}}(u)$ and let $y = x + r(u - x)/\|u - x\|$. Then $x \in \mathrm{Proj}_{K}(y)$ and $u = x + t_u(y - x)$ with $t_u = r/\|u - x\| \in \,]0, 1[$, so that $\{x\} = \mathrm{Proj}_{\mathcal{K}}(u)$. $\square$

Definition 2.6. If $r \in]0,\infty[$ and if $\mathcal{K} \subseteq \mathcal{H}$ is r-prox-regular, then we can define the function $\mathrm{P}_{\mathcal{K}} : \{y \in \mathcal{H} : d_{\mathcal{K}}(y) < r\} \longrightarrow \mathcal{K}$ by setting $\mathrm{P}_{\mathcal{K}}(y) := x$ where $\{x\} = \mathrm{Proj}_{\mathcal{K}}(y)$ for every $y \in \mathcal{H}$ such that $d_{\mathcal{K}}(y) < r$. □

The following characterizations of prox-regularity are very useful. The proofs can be found in [58, Theorem 4.1] and in [23, Theorem 16].

Theorem 2.7. *Let $\mathcal{K}$ be a nonempty closed subset of $\mathcal{H}$ and let $r \in]0,\infty[$. The following statements are equivalent.*

(i) *$\mathcal{K}$ is r-prox-regular.*

(ii) *$d_{\mathcal{K}}$ is differentiable in $\{y \in \mathcal{H} : 0 < d_{\mathcal{K}}(y) < r\}$.*

(iii) *For every $x \in \mathcal{K}$ and $n \in N_{\mathcal{K}}(x)$ we have*

$$\langle n, z - x\rangle \le \frac{\|n\|}{2r}\|z - x\|^2 \qquad \forall z \in \mathcal{K}.$$

2.2. Functions of bounded variation and differential measures

Let I be an interval of $\mathbb{R}$. The set of $\mathcal{H}$-valued continuous functions defined on I is denoted by $C(I;\mathcal{H})$. If $f \in C(I;\mathcal{H})$ then $\|f\|_\infty := \sup\{\|f(t)\| : t \in I\}$ denotes it supremum norm.

Definition 2.8. Given an interval $I \subseteq \mathbb{R}$, a function $f : I \longrightarrow \mathcal{H}$, and a subinterval $J \subseteq I$, the *variation of f on J* is defined by

$$\mathrm{V}(f, J) := \sup\Big\{\sum_{j=1}^{m} \|(f(t_{j-1}) - f(t_j)\| : m \in \mathbb{N},\ t_j \in J\ \forall j,\ t_0 < \cdots < t_m\Big\}.$$

If $\mathrm{V}(f, I) < \infty$ we say that *f is of bounded variation on I* and we set

$$BV(I;\mathcal{H}) := \{f : I \longrightarrow \mathcal{H} \ :\ \mathrm{V}(f, I) < \infty\}.$$ □

It is well known that every $f \in BV(I;\mathcal{H})$ admits one-sided limits $f(t-)$, $f(t+)$ at every point $t \in I$, with the convention that $f(\inf I-) := f(\inf I)$ if $\inf I \in I$, and $f(\sup I+) := f(\sup I)$ if $\sup I \in I$. The family of Borel sets in I is denoted by $\mathscr{B}(I)$. If $\mu : \mathscr{B}(I) \longrightarrow [0,\infty]$ is a measure, then the space of $\mathcal{H}$-valued functions which are integrable with respect to μ will be denoted by $L^1(I,\mu;\mathcal{H})$ or simply by $L^1(\mu;\mathcal{H})$.

For the theory of integration of vector valued functions we refer, e.g., to [41, Chapter VI]. When $\mu = \mathcal{L}^1$, where $\mathcal{L}^1$ is the one dimensional Lebesgue measure, we write $L^1(I;\mathcal{H}) := L^1(I,\mu;\mathcal{H})$.

We recall that a *$\mathcal{H}$-valued measure on I* is a map $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that $\nu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \nu(B_n)$ for every sequence (B_n) of mutually disjoint sets in $\mathscr{B}(I)$. The *total variation of ν* is the positive measure $|\nu| : \mathscr{B}(I) \longrightarrow [0,\infty]$ defined by

$$|\nu|(B) := \sup\Big\{\sum_{n=1}^{\infty} \|\nu(B_n)\| : B = \bigcup_{n=1}^{\infty} B_n,\ B_n \in \mathscr{B}(I),\ B_h \cap B_k = \varnothing \text{ if } h \ne k\Big\}.$$

The vector measure ν is said to be *with bounded variation* if $|\nu|(I) < \infty$. In this case the equality $\|\nu\| := |\nu|(I)$ defines a complete norm on the space of measures with bounded variation (see, e.g. [27, Chapter I, Section 3]).

If $\mu : \mathscr{B}(I) \longrightarrow [0,\infty]$ is a positive bounded Borel measure and if $g \in L^1(I,\mu;\mathcal{H})$, then $g\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ denotes the vector measure defined by

$$g\mu(B) := \int_B g \,\mathrm{d}\mu, \qquad B \in \mathscr{B}(I).$$

Assume that $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation and $f : I \longrightarrow \mathcal{H}$ and $\phi : I \longrightarrow \mathbb{R}$ are two *step maps with respect to ν*, i.e. there exist $f_1, \dots, f_m \in \mathcal{H}$, $\phi_1, \dots, \phi_m \in \mathcal{H}$ and $A_1, \dots, A_m \in \mathscr{B}(I)$ mutually disjoint such that $|\nu|(A_j) < \infty$ for every j and $f = \sum_{j=1}^{m} \mathbb{1}_{A_j} f_j$, $\phi = \sum_{j=1}^{m} \mathbb{1}_{A_j} \phi_j$. Here $\mathbb{1}_S$ is the characteristic function of a set S, i.e. $\mathbb{1}_S(x) := 1$ if $x \in S$ and $\mathbb{1}_S(x) := 0$ if $x \notin S$. For such step maps we define $\int_I \langle f, \mathrm{d}\nu\rangle := \sum_{j=1}^{m} \langle f_j, \nu(A_j)\rangle \in \mathbb{R}$ and $\int_I \phi \,\mathrm{d}\nu := \sum_{j=1}^{m} \phi_j \nu(A_j) \in \mathcal{H}$.

If $St(|\nu|;\mathcal{H})$ (resp. $St(|\nu|)$) is the set of $\mathcal{H}$-valued (resp. real valued) step maps with respect to ν, then the maps

$$St(|\nu|;\mathcal{H}) \longrightarrow \mathcal{H} : f \longmapsto \textstyle\int_I \langle f, \mathrm{d}\nu\rangle \ \text{ and } \ St(|\nu|) \longrightarrow \mathcal{H} : \phi \longmapsto \textstyle\int_I \phi \,\mathrm{d}\nu$$

are linear and continuous when $St(|\nu|;\mathcal{H})$ and $St(|\nu|)$ are endowed with the L^1-seminorms $\|f\|_{L^1(|\nu|;\mathcal{H})} := \int_I \|f\| \,\mathrm{d}|\nu|$ and $\|\phi\|_{L^1(|\nu|)} := \int_I |\phi| \,\mathrm{d}|\nu|$.

Therefore they admit unique continuous extensions $\mathsf{I}_v : L^1(|v|;\mathcal{H}) \longrightarrow \mathbb{R}$ and $\mathsf{J}_v : L^1(|v|) \longrightarrow \mathcal{H}$, and we set

$$\int_I \langle f, \mathrm{d}v\rangle := \mathsf{I}_v(f), \quad \int_I \phi \ \mathrm{d}v := \mathsf{J}_v(\phi), \qquad f \in L^1(|v|;\mathcal{H}), \quad \phi \in L^1(|v|).$$

Now using step functions it is easy to check that $|\int_I \langle f, \mathrm{d}v\rangle| \le \int_I \|f(s)\| \, \mathrm{d}|v|(s)$ for $f \in L^1(|v|;\mathcal{H})$. The following results (cf., e.g., [27, Section III.17.2-3, 358–362]) provide the connection between functions with bounded variation and vector measures which will be implicitly used in this paper.

Theorem 2.9. *For every $f \in BV(I;\mathcal{H})$ there exists a unique vector measure of bounded variation $v_f : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that*

$$v_f(]c,d[) = f(d-) - f(c+), \qquad v_f([c,d]) = f(d+) - f(c-),$$
$$v_f([c,d[) = f(d-) - f(c-), \qquad v_f(]c,d]) = f(d+) - f(c+).$$

whenever $\inf I \le c < d \le \sup I$ and the left hand side of each equality makes sense.

Conversely, if $v : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation, and if $f_v : I \longrightarrow \mathcal{H}$ is defined by $f_v(t) := v([\inf I, t[\cap I)$, then $f_v \in BV(I;\mathcal{H})$ and $v_{f_v} = v$.

Proposition 2.10. *Let $f \in BV(I;\mathcal{H})$, let $g : I \longrightarrow \mathcal{H}$ be defined by $g(t) := f(t-)$, for $t \in \mathrm{int}(I)$, and by $g(t) := f(t)$, if $t \in \partial I$, and let $V_g : I \longrightarrow \mathbb{R}$ be defined by $V_g(t) := \mathrm{V}(g, [\inf I, t] \cap I)$. Then $v_g = v_f$ and $\left|v_f\right|(I) = v_{V_g}(I) = \mathrm{V}(g, I)$.*

The measure v_f is called the *Lebesgue-Stieltjes measure* or *differential measure* of f. Let us see the connection between the differential measure and the distributional derivative. If $f \in BV(I;\mathcal{H})$ and if $\overline{f} : \mathbb{R} \longrightarrow \mathcal{H}$ is defined by

$$\overline{f}(t) := \begin{cases} f(t) & \text{if } t \in I, \\ f(\inf I) & \text{if } \inf I \in \mathbb{R},\ t \notin I,\ t \le \inf I, \\ f(\sup I) & \text{if } \sup I \in \mathbb{R},\ t \notin I,\ t \ge \sup I, \end{cases} \tag{13}$$

then, as in the scalar case, it turns out (cf. [59, Section 2]) that $v_f(B) = \mathrm{D}\overline{f}(B)$ for every $B \in \mathscr{B}(\mathbb{R})$, where $\mathrm{D}\overline{f}$ is the distributional derivative of $\overline{f}$, i.e.

$$-\int_{\mathbb{R}} \varphi'(t)\overline{f}(t) \, \mathrm{d}t = \int_{\mathbb{R}} \varphi \, \mathrm{d}\mathrm{D}\overline{f}, \qquad \forall \varphi \in C^1_c(\mathbb{R};\mathbb{R}),$$

where $C^1_c(\mathbb{R};\mathbb{R})$ is the space of continuously differentiable functions on $\mathbb{R}$ with compact support. Observe that $\mathrm{D}\overline{f}$ is concentrated on I: $\mathrm{D}\overline{f}(B) = v_f(B \cap I)$ for every $B \in \mathscr{B}(I)$, hence in the remainder of the paper, if $f \in BV(I,\mathcal{H})$ then we will simply write

$$\mathrm{D}f := \mathrm{D}\overline{f} = v_f, \tag{14}$$

and from the previous discussion it follows that

$$\|\mathrm{D}f\| = \left|\mathrm{D}f\right|(I) = \|v_f\| \tag{15}$$

and

$$\left|\mathrm{D}f\right|(I) = \mathrm{V}(f, I) \quad \text{if } f(t) \in \{(1-\lambda)f(t-) + \lambda f(t+) : \lambda \in [0,1]\} \ \forall t \in I. \tag{16}$$

3. Main results

In this section we state the main results of the paper. Let us start by providing the rigorous formulation of the problem we are going to solve.

Problem 3.1. *Assume that (6) holds. Given $T > 0$, C, and $y_0 \in C(0)$, one has to find a function $y \in BV([0,T];\mathcal{H})$ such that there exist a measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ and a function $v \in L^1(\mu;\mathcal{H})$ satisfying*

$$y(t) \in C(t) \qquad \forall t \in [0,T], \tag{17}$$
$$\mathrm{D}y = v\mu, \tag{18}$$
$$v(t) \in -N_{C(t)}(y(t)) \qquad \text{for } \mu\text{-a.e. } t \in [0,T], \tag{19}$$
$$y(0) = y_0. \tag{20}$$

In order to state our results we recall the precise notion of excess:

Definition 3.2. If $\mathscr{P}(\mathcal{H})$ denotes the family of all subsets of $\mathcal{H}$, then the *excess* is the function $e : \mathscr{P}(\mathcal{H}) \times \mathscr{P}(\mathcal{H}) \longrightarrow [0,\infty]$ defined by

$$e(\mathcal{Z}_1,\mathcal{Z}_2) := \sup_{z_1\in\mathcal{Z}_1} d(z_1,\mathcal{Z}_2) = \inf\{\rho \geq 0 \ : \ \mathcal{Z}_1 \subseteq \mathcal{Z}_2 + \overline{B}_\rho(0)\}, \qquad \mathcal{Z}_1,\mathcal{Z}_2 \in \mathscr{P}(\mathcal{H}).$$

The extended number $e(\mathcal{Z}_1,\mathcal{Z}_2)$ is called *excess of* $\mathcal{Z}_1$ *over* $\mathcal{Z}_2$. □

Observe that we have $e(\varnothing,\mathcal{Z}) = 0$ for every $\mathcal{Z} \in \mathscr{P}(\mathcal{H})$ and $e(\mathcal{Z},\varnothing) = \infty$ for every $\mathcal{Z} \in \mathscr{P}(\mathcal{H})\setminus\{\varnothing\}$. The excess e is an asymmetric semidistance, in particular the following properties hold (see [53, Section 2a]):

$$\begin{aligned}
&e(\mathcal{Z}_1,\mathcal{Z}_2) = 0 \iff \mathcal{Z}_1 \subseteq \overline{\mathcal{Z}_2} && \forall \mathcal{Z}_1,\mathcal{Z}_2 \in \mathscr{P}(\mathcal{H}),\\
&e(\mathcal{Z}_1,\mathcal{Z}_2) \leq e(\mathcal{Z}_1,\mathcal{Z}_3) + e(\mathcal{Z}_3,\mathcal{Z}_2) && \forall \mathcal{Z}_1,\mathcal{Z}_2,\mathcal{Z}_3 \in \mathscr{P}(\mathcal{H}),\\
&e(\mathcal{Z}_1,\mathcal{Z}_2) = e(\overline{\mathcal{Z}_1},\mathcal{Z}_2) = e(\mathcal{Z}_1,\overline{\mathcal{Z}_2}) = e(\overline{\mathcal{Z}_1},\overline{\mathcal{Z}_2}) && \forall \mathcal{Z}_1,\mathcal{Z}_2 \in \mathscr{P}(\mathcal{H}).
\end{aligned}$$

The main existence theorem is the following. It improves the existing result [22, Theorem 4.2] in two ways: first, the assumption of continuity of the multivalued mapping $t \longmapsto \partial C(t)$ is completely dropped; second, the continuity of the driving set $t \longmapsto C(t)$ is assumed with respect to the asymmetric distance e, the excess, and not with respect to the Hausdorff distance, thereby emphasizing the unilateral character of the rating of the data, as observed by Moreau in its celebrated paper [55] in the convex framework. As a consequence we are able to provide an existence/uniqueness result for a much wider class of moving constraints $C(t)$.

Theorem 3.3. *Let $C : [0,T] \longrightarrow \mathrm{Conv}_r(\mathcal{H})$, a function with values in the family of r-prox-regular subsets of $\mathcal{H}$, satisfying the condition*

$$\forall \varepsilon > 0\ \exists \delta > 0 \ : \ [0 \leq t - s < \delta \implies e(C(s),C(t)) < \varepsilon]. \tag{21}$$

If there exist $R > 0$ and $d > 0$ such that for every $t \in [0,T]$

$$\forall x \in \partial C(t)\ \exists z \in C(t) \cap B_d(x) \ : \ (1-\lambda)x + \lambda B_R(z) \subseteq C(t),\ \forall \lambda \in [0,1]\,, \tag{22}$$

then for every $y_0 \in C(0)$ there exists a unique solution $y \in BV([0,T];\mathcal{H})$ of Problem 3.1. Moreover $y \in C([0,T];\mathcal{H})$ and $\mathrm{V}(y,[0,T]) \leq C$ for a constant $C > 0$ depending only on r, $R > 0$, $d > 0$, and on the continuity modulus of C.

Condition (22) can be referred to as a "uniform inner cone condition", which is a regularity assumption on the boundary of $C(t)$. This regularity assumption on the boundary can be dropped if one assumes that there exists a fixed ball contained on all the sets $C(t)$, but this ball should be big enough (cf. the compatibility condition (25)). This result in contained in the following theorem which generalizes [22, Theorem 4.1] by assuming the continuity of $C(t)$ with respect to the excess only.

Theorem 3.4. *Let $C : [0,T] \longrightarrow \mathrm{Conv}_r(\mathcal{H})$, a function with values in the family of r-prox-regular subsets of $\mathcal{H}$, satisfying the condition*

$$\forall \varepsilon > 0\ \exists \delta > 0 \ : \ [0 \leq t - s < \delta \implies e(C(s),C(t)) < \varepsilon]. \tag{23}$$

Assume that $y_0 \in C(0)$. If

$$\exists w \in \mathcal{H},\ \exists \rho > 0 \ : \ \big[\, B_\rho(w) \subseteq C(t)\ \forall t \in [0,T]\ \big], \tag{24}$$

and

$$(\|y_0 - w\| + \rho)^2 < 2r\rho, \tag{25}$$

then there exists a unique solution $y \in BV([0,T];\mathcal{H})$ of Problem 3.1. Moreover $y \in C([0,T];\mathcal{H})$ and $\mathrm{V}(y,[0,T]) \leq C$ for a constant $C > 0$ depending only on r, w, ρ, and on the continuity modulus of C.

The proofs of the Theorems 3.3 and 3.4 are contained in the following section 4.

We now provide an explicit example showcasing the difference between the Hausdorff continuity and the excess continuity.

Example 3.5. Let $K(t)$ be a given curve of r-prox-regular sets which are continuous with respect to the Hausdorff distance, let $\{t_k\}_{k\in\mathbb{N}}$ be an increasing sequence of times, and define $C(t) = K(t)+B\big(0,\rho(t)\big)$, where $\rho : [0,T] \to [0,r)$ is a function whose discontinuity set is $\{t_k\}_{k\in\mathbb{N}}$, $\rho(t_k) > \rho(t_k-)$ and $\sum_{k\in\mathbb{N}}[\rho(t_k) - \rho(t_k-)] = \infty$.

There are many choices of $K(t)$, for instance a constant set, such that $C(t)$ is excess-continuous, but is neither Hausdorff-continuous nor BV.

Indeed, it is simple to verify that the family $B\big(0,\rho(t)\big)$ is Hausdorff-continuous if and only if ρ is continuous, whereas it is excess-continuous in the sense of (23) if and only if $e\big(B\big(0,\rho(s)\big), B\big(0,\rho(t)\big)\big) = \big(\rho(s) - \rho(t)\big)_+ \to 0$ whenever $t - s \to 0$ with $s < t$.

Particular instances of this construction provide the geodesics mentioned in the introduction used to fill in the jumps to reparametrize the regulated case. □

4. Proofs

4.1. Catching-up algorithm

In the following lemma we describe the Euler implicit discretization of (19). This discretization scheme ((32)–(34)) is usually called "catching-up algorithm" (cf. [55, p. 353]), due to its evident geometrical meaning in the case of sweeping processes (see (35) below).

Lemma 4.1. *For every $t \in [0,T]$ let $C(t)$ be an r-prox-regular subset of $\mathcal{H}$, and assume that (21) holds. If $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that*

$$0 < \varepsilon_{n+1} < \varepsilon_n < r \quad \forall n \in \mathbb{N}, \qquad \lim_{n\to\infty} \varepsilon_n = 0, \tag{26}$$

then there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, such that

$$0 < \delta_{n+1} < \delta_n \quad \forall n \in \mathbb{N}, \qquad \lim_{k\to\infty} \delta_k = 0, \tag{27}$$

$$\sup\{e\,(C(s),C(t)) \ : \ 0 \le t - s \le \delta_n\} < \varepsilon_n < r \quad \forall n \in \mathbb{N}. \tag{28}$$

Moreover for every $n \in \mathbb{N}$ let $\mathfrak{S}_n$ be a subdivision of $[0,T]$ satisfying the conditions

$$\mathfrak{S}_n = \{t_j^n \in [0,T] \ : \ j = 0,\dots,J^n\} \text{ for some } J_n \in \mathbb{N}, \tag{29}$$

$$t_0^n = 0,\ t_{J_n}^n = T, \quad t_{j-1}^n < t_j^n \ \forall j \in \{0,\dots,J^n\}, \quad \max_{j=1,\dots,J^n}(t_j^n - t_{j-1}^n) \le \delta_n, \tag{30}$$

$$\mathfrak{S}_n \subseteq \mathfrak{S}_{n+1}. \tag{31}$$

Then for every $y_0 \in C(0)$ and every $n \in \mathbb{N}$ there exists a family $(y_j^n)_{j=0}^{J^n}$ in $\mathcal{H}$ such that

$$y_0^n := y_0, \tag{32}$$

$$y_j^n \in C(t_j^n) \qquad \forall j \in \mathbb{N},\ 0 \le j \le J^n, \tag{33}$$

$$\frac{y_j^n - y_{j-1}^n}{t_j^n - t_{j-1}^n} \in -N_{C(t_j^n)}(y_j^n) \qquad \forall j \in \mathbb{N},\ 1 \le j \le J^n. \tag{34}$$

Under the above assumptions, condition (34) is equivalent to

$$y_j^n = \mathrm{P}_{C(t_j^n)}(y_{j-1}^n) \qquad \forall j \in \mathbb{N},\ 1 \le j \le J^n. \tag{35}$$

Proof. The existence of the sequence δ_n satisfying (27) and (28) is a straightforward consequence of (21). Now for every $n \in \mathbb{N}$ set

$$y_0^n := y_0 \in C(0) = C(t_0^n);$$

thus from (28) we get $d(y_0^n, C(t_1^n)) \le e\,(C(a),C(t_1^n)) < r$, and by the r-prox-regularity of $C(t_1^n)$ we infer that there exists $y_1^n := \mathrm{P}_{C(t_1^n)}(y_0^n) \in C(t_1^n)$, so that $y_0^n - y_1^n \in N_{C(t_1^n)}(t_1^n)$. For any integer j such that $2 \le j \le J_n$, let us assume that y_{j-1}^n satisfies the inclusions $y_{j-1}^n \in C(t_{j-1}^n)$ and $y_{j-2}^n - y_{j-1}^n \in N_{C(t_{j-1}^n)}(y_{j-1}^n)$. Hence by virtue of (28) we have $d(y_{j-1}^n, C(t_j^n)) \le e\,(C(t_{j-1}^n), C(t_j^n)) < r$, so that there exists

$$y_j^n := \mathrm{P}_{C(t_j^n)}(y_{j-1}^n) \qquad \forall j \in \mathbb{N}, \tag{36}$$

and $y^n_{j-1} - y^n_j \in N_{C(t^n_j)}(y^n_j)$. Therefore, recalling that $N_{C(t)}(z)$ is a cone for every $z \in C(t)$ and for every $t \in [0,T]$, we have recursively defined a family $(y^n_j)_{j=0}^{J^n}$ in $\mathcal{H}$ satisfying (32)–(34). □

Lemma 4.2. *Under the assumptions of Lemma 4.1, let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ such that (26) and (27)–(28) hold, let $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$ satisfying (29)–(31), and let $(y^n_j)_{j=1}^{J^n}$ be the family in $\mathcal{H}$ satisfying (32)–(34).*

For every $n \in \mathbb{N}$ let $y_n : [0,T] \longrightarrow \mathcal{H}$ be the right continuous step function defined by

$$y_n(0) := y^n_0 = y_0, \tag{37}$$

$$y_n(t) := y^n_{j-1} \qquad \text{if } t \in \left[t^n_{j-1}, t^n_j\right[, \quad j \in \mathbb{N},\ 1 \le j \le J^n, \tag{38}$$

which we will call discrete solution of Problem 3.1, *and let $x_n : [0,T] \longrightarrow \mathcal{H}$ be the piecewise affine interpolant of $(y^n_j)_{j=0}^{J^n}$, i.e.*

$$x_n(t) := y^n_{j-1} + \frac{t - t^n_{j-1}}{t^n_j - t^n_{j-1}}(y^n_j - y^n_{j-1}) \quad \text{if } t \in \left[t^n_{j-1}, t^n_j\right[,\ j \in \mathbb{N},\ 1 \le j \le J^n. \tag{39}$$

Then

$$\lim_{n\to\infty} \|y_n - x_n\|_\infty = 0, \tag{40}$$

and y_n solves Problem 3.1 with $C(t)$ replaced by $C^n(t)$ defined by

$$C^n(t) := C(t^n_{j-1}) \qquad \text{if } t \in \left[t^n_{j-1}, t^n_j\right[,$$

i.e. if $\mu_n : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ and $v_n : [0,T] \longrightarrow \mathcal{H}$ are defined by

$$\mu_n := \sum_{j=1}^{J^n} \delta_{t^n_j}, \qquad v_n(t) := \sum_{j=1}^{J^n} (y^n_j - y^n_{j-1}) \mathbb{1}_{\{t^n_j\}}(t) = \mathrm{D}y_n(t), \qquad t \in [0,T],$$

then $v_n \in L^1(\mu_n; \mathcal{H})$ and

$$\mathrm{D}y_n(t) = v_n(t)\mu_n, \quad v_n(t) \in -N_{C^n(t)}(y_n(t)) \qquad \forall t \in [0,T].$$

Proof. Let us observe that by (33), (30), and (28), we have that

$$\|y^n_j - y^n_{j-1}\| = d_{C(t^n_j)}(y^n_{j-1}) \le e(C(t^n_{j-1}), C(t^n_j)) < \varepsilon_n \qquad \forall j \in \mathbb{N},\ 1 \le j \le J^n,$$

thus (40) follows from the equality

$$\|y_n(t) - x_n(t)\| = \left|\frac{t - t^n_{j-1}}{t^n_j - t^n_{j-1}}\right| \|y^n_j - y^n_{j-1}\| \qquad \forall t \in \left]t^n_{j-1}, t^n_j\right[.$$

If $0 \le j \le J^n$, from (34) it follows that

$$\mathrm{D}y_n(\{t^n_j\}) = y^n_j - y^n_{j-1} \in -N_{C(t^n_j)}(y^n_j) = N_{C(t^n_j)}(y_n(t^n_j)),$$

and we also trivially have that

$$\mathrm{D}y_n\left(\left]t^n_{j-1}, t^n_j\right[\right) = 0 \in N_{C(t)}(y_n(s)) \qquad \forall s \in \left]t^n_{j-1}, t^n_j\right[.$$

□

4.2. Uniform convergence of discrete approximations

In the following theorem we discuss the uniform convergence of the approximated solutions. It is worth noting that one of the main innovations is the way we obtain the Cauchy estimate, due to the fact that the C is assumed to be continuous with respect to the excess, and not with respect to the Hausdorff distance.

Theorem 4.3. *For every $t \in [0,T]$ let $C(t)$ be an r-prox-regular subset of $\mathcal{H}$, and assume that (21) holds. Fix $y_0 \in C(0)$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that*

$$0 < \varepsilon_{n+1} < \varepsilon_n < r \quad \forall n \in \mathbb{N}, \qquad \sum_{n=0}^{\infty} \varepsilon_n < \infty. \tag{41}$$

Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ satisfy (27)–(28), let $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$ satisfying (29)–(31), and let $(y_j^n)_j^{J^n}$ be the family in $\mathcal{H}$ satisfying (32)–(34), all of them existing by virtue of Lemma 4.1. For every $n \in \mathbb{N}$, let $y_n : [0,T] \longrightarrow \mathcal{H}$ be the discrete solution of Problem 3.1, i.e. the right continuous step function defined by (37)–(38) and let $x_n : [0,T] \longrightarrow \mathcal{H}$ be the piecewise affine interpolant of $(y_j^n)_{j=0}^{J^n}$ defined by (39). If

$$\sup_{n\in\mathbb{N}} \mathrm{V}(y_n, [0,T]) < \infty,$$

then the sequences $(y_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are uniformly convergent to a function $y \in BV([a,b];\mathcal{H}) \cap C([0,T];\mathcal{H})$ with $y(0) = y_0$ and $y(t) \in C(t)$ for every $t \in [0,T]$.

Proof. We will assume without loss of generality that $\varepsilon_n \le 1$ for all $n \in \mathbb{N}$. Let us recall that observe that by (33), (30), and (28), we have that

$$\|y_j^n - y_{j-1}^n\| = d_{C(t_j^n)}(y_{j-1}^n) \le e\left(C(t_{j-1}^n), C(t_j^n)\right) < \varepsilon_n \qquad \forall j \in \mathbb{N},\ 1 \le j \le J^n. \tag{42}$$

Let us also define the *time anticipating* function $\theta_n : [0,T] \longrightarrow [0,T]$ by

$$\theta_n(0) := 0, \qquad \theta_n(t) := t_j^n \quad \text{if } t \in \left]t_{j-1}^n, t_j^n\right],\ j \in \mathbb{N},\ 1 \le j \le J^n, \tag{43}$$

so that

$$0 \le \theta_n(t) - t \le \delta_n \qquad \forall t \in [0,T], \tag{44}$$

and

$$x_n(\theta_n(t)) \in C(\theta_n(t)) \qquad \forall t \in [0,T]. \tag{45}$$

By (39),(43) and (33), if $t \in \left]t_{j-1}^n, t_j^n\right]$ we have $\|x_n(\theta_n(t)) - x_n(t)\| \le \|y_j^n - y_{j-1}^n\|$, thus, thanks to (42),

$$\|x_n(\theta_n(t)) - x_n(t)\| \le \varepsilon_n \qquad \forall t \in [0,T]. \tag{46}$$

We have, by (34), that

$$x_n'(t) = -\frac{y_{j-1}^n - y_j^n}{t_j^n - t_{j-1}^n} \in N_{C(t_j^n)}(y_j^n) = N_{C(\theta_n(t))}(x(\theta_n(t))) \ \forall t \in \left]t_{j-1}^n, t_j^n\right[,\ \forall j \in \mathbb{N}. \tag{47}$$

Now let us fix $n \in \mathbb{N}$ and $t \in [0,T]$ such that we have $t \ne t_j^n$ and $t \ne t_j^{n+1}$ for all $j \in \{0,\dots,J^n\}$. We obtain

$$\begin{aligned}
2\frac{\mathrm{d}}{\mathrm{d}t}\|x_{n+1}(t) - x_n(t)\|^2 &= \left\langle x_{n+1}'(t) - x_n'(t), x_{n+1}(t) - x_n(t)\right\rangle \\
&= \left\langle x_{n+1}'(t) - x_n'(t), x_{n+1}(t) - x_{n+1}(\theta_{n+1}(t))\right\rangle \\
&\quad + \left\langle x_{n+1}'(t) - x_n'(t), x_{n+1}(\theta_{n+1}(t)) - x_n(\theta_n(t))\right\rangle \\
&\quad + \left\langle x_{n+1}'(t) - x_n'(t), x_n(\theta_n(t)) - x_n(t)\right\rangle \\
&\le \|x_{n+1}'(t) - x_n'(t)\|\|x_{n+1}(t) - x_{n+1}(\theta_{n+1}(t))\| \\
&\quad + \left\langle x_{n+1}'(t) - x_n'(t), x_{n+1}(\theta_{n+1}(t)) - x_n(\theta_n(t))\right\rangle \\
&\quad + \|x_{n+1}'(t) - x_n'(t)\|\|x_n(\theta_n(t)) - x_n(t)\| \\
&\le (\|x_{n+1}'(t)\| + \|x_n'(t)\|)(\varepsilon_{n+1} + \varepsilon_n) \\
&\quad + \left\langle x_{n+1}'(t) - x_n'(t), x_{n+1}(\theta_{n+1}(t)) - x_n(\theta_n(t))\right\rangle,
\end{aligned}$$

therefore

$$
\begin{aligned}
&2\|x_{n+1}(t)-x_n(t)\|^2\\
&=\int_0^t \left\langle x'_{n+1}(s)-x'_n(s), x_{n+1}(s)-x_n(s)\right\rangle \mathrm{d}s\\
&\le (\varepsilon_{n+1}+\varepsilon_n)\int_0^t (\|x'_{n+1}(s)\|+\|x'_n(s)\|)\,\mathrm{d}s\\
&\quad+\int_0^t \left\langle x'_{n+1}(s)-x'_n(s), x_{n+1}(\theta_{n+1}(s))-x_n(\theta_n(s))\right\rangle \mathrm{d}s\\
&\le 4V\varepsilon_n+\int_0^t \left\langle x'_{n+1}(s)-x'_n(s), x_{n+1}(\theta_{n+1}(s))-x_n(\theta_n(s))\right\rangle \mathrm{d}s. \qquad (48)
\end{aligned}
$$

Before proceeding with the estimate of the right-hand side of (48), let us preliminarily observe that from (46) we obtain

$$
\begin{aligned}
&\|x_{n+1}(\theta_{n+1}(t))-x_n(\theta_n(t))\|^2\\
&\le 3\|x_{n+1}(\theta_{n+1}(t))-x_{n+1}(t)\|^2+3\|x_{n+1}(t)-x_n(t)\|^2+3\|x_n(t)-x_n(\theta_n(t))\|^2\\
&\le 3\varepsilon_{n+1}^2+3\|x_{n+1}(t)-x_n(t)\|^2+3\varepsilon_n^2\le 6\varepsilon_n^2+3\|x_{n+1}(t)-x_n(t)\|^2. \qquad (49)
\end{aligned}
$$

Now let $j,k\in\mathbb{N}$ be such that $j\in\{1,\dots,J^n\}$, $k\in\{1,\dots,J^{n+1}\}$, and

$$
t_{j-1}^n\le t_{k-1}^{n+1}<t<t_k^{n+1}\le t_j^n, \qquad (50)
$$

thus $\theta_{n+1}(t)=t_k^{n+1}$ and $x_n(t_{j-1}^n)=y_{j-1}^n\in C(t_{j-1}^n)$, so that thanks to (30) and (28),

$$
d(x_n(t_{j-1}^n),C(\theta_{n+1}(t)))\le e\,(C(t_{j-1}^n),C(t_k^{n+1}))\le\varepsilon_n<r. \qquad (51)
$$

Hence, there exists $\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))$ and, as $-x'_{n+1}(t)\in N_{C(\theta_{n+1}(t))}(x_{n+1}(\theta_{n+1}(t)))$ by (47), using the r-prox-regularity of $C(\theta_{n+1}(t))$, we can write

$$
\begin{aligned}
&\left\langle x'_{n+1}(t), x_{n+1}(\theta_{n+1}(t))-x_n(\theta_n(t))\right\rangle\\
&=\left\langle x'_{n+1}(t), x_{n+1}(\theta_{n+1}(t))-\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))\right\rangle\\
&\quad+\left\langle x'_{n+1}(t), \mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))-x_n(\theta_n(t))\right\rangle\\
&\le\frac{1}{2r}\left\|x_{n+1}(\theta_{n+1}(t))-\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))\right\|^2\\
&\quad+\|x'_{n+1}(t)\|\left\|\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n)))-x_n(\theta_n(t))\right\|. \qquad (52)
\end{aligned}
$$

In the following two computations we estimate the right-hand side of (52). First we observe that from (39),(43), (42), (49), and (51), we infer that

$$
\begin{aligned}
&\left\|x_{n+1}(\theta_{n+1}(t))-\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))\right\|^2\\
&\le 3\,\|x_{n+1}(\theta_{n+1}(t))-x_n(\theta_n(t))\|^2\\
&\quad+3\left\|x_n(\theta_n(t))-x_n(t_{j-1}^n)\right\|^2+3\left\|x_n(t_{j-1}^n)-\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))\right\|^2\\
&=3\,\|x_{n+1}(\theta_{n+1}(t))-x_n(\theta_n(t))\|^2\\
&\quad+3\left\|y_j^n-y_{j-1}^n\right\|^2+3\left\|x_n(t_{j-1}^n)-\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))\right\|^2\\
&\le 3\,\|x_{n+1}(\theta_{n+1}(t))-x_n(\theta_n(t))\|^2+3\varepsilon_n^2+3d(x_n(t_{j-1}^n),C(\theta_{n+1}(t)))^2\\
&\le 18\varepsilon_n^2+9\|x_{n+1}(t)-x_n(t)\|^2+3\varepsilon_n^2+3\varepsilon_n^2, \qquad (53)
\end{aligned}
$$

and that

$$\begin{aligned}
&\left\|\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))) - x_n(\theta_n(t))\right\| \\
&\le \left\|\mathrm{P}_{C(\theta_{n+1}(t))}(x_n(t_{j-1}^n))) - x_n(t_{j-1}^n)\right\| + \left\|x_n(t_{j-1}^n) - x_n(\theta_n(t))\right\| \\
&= d(x_n(t_{j-1}^n), C(\theta_{n+1}(t))) + \left\|y_{j-1}^n - y_j^n\right\| \\
&\le \varepsilon_n + \varepsilon_n = 2\varepsilon_n,
\end{aligned} \tag{54}$$

therefore, from (52), (53), and (54) we obtain that there exists a constant $K_1 > 0$, independent of n, such that

$$\begin{aligned}
&\int_0^t \left\langle x'_{n+1}(t), x_{n+1}(\theta_{n+1}(t)) - x_n(\theta_n(t))\right\rangle \mathrm{d}s \\
&\le \frac{9}{2r}\int_0^t \|x_{n+1}(s) - x_n(s)\|^2 \,\mathrm{d}s + \int_0^t K_1(1 + \|x'_{n+1}(s)\|)\varepsilon_n \,\mathrm{d}s \\
&\le \frac{9}{2r}\int_0^t \|x_{n+1}(s) - x_n(s)\|^2 \,\mathrm{d}s + K_1(T+V)\varepsilon_n.
\end{aligned} \tag{55}$$

Now, recalling (50), observe that $\theta_n(t) = t_j^n$ and $x_{n+1}(\theta_{n+1}(t)) = y_{k-1}^{n+1} \in C(t_{k-1}^{n+1})$, thus, thanks to (30) and (28),

$$d(x_{n+1}(\theta_{n+1}(t)), C(\theta_n(t))) \le e\,(C(t_{k-1}^{n+1}), C(t_j^n)) < \varepsilon_n. \tag{56}$$

Therefore there exists $\mathrm{P}_{C(\theta(t))}(x_{n+1}(\theta_{n+1}(t)))$ and, as $-x'_n(t) \in N_{C(\theta_n(t))}(x_n(\theta_n(t)))$ by (47), using the r-prox-regularity of $C(\theta_n(t))$, we can write

$$\begin{aligned}
&\langle -x'_n(t), x_{n+1}(\theta_{n+1}(t)) - x_n(\theta_n(t))\rangle \\
&= \left\langle -x'_n(t), x_{n+1}(\theta_{n+1}(t)) - \mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t)))\right\rangle \\
&\quad + \left\langle -x'_n(t), \mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t))) - x_n(\theta_n(t))\right\rangle \\
&\le \|x'_n(t)\| \left\|x_{n+1}(\theta_{n+1}(t)) - \mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t)))\right\| \\
&\quad + \frac{1}{2r}\left\|\mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t))) - x_n(\theta_n(t))\right\|^2.
\end{aligned} \tag{57}$$

In order to estimate the right-hand side of (57) let us observe that, by (56), we have

$$\left\|x_{n+1}(\theta_{n+1}(t)) - \mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t)))\right\| = d(x_{n+1}(\theta_{n+1}(t))), C(\theta_n(t))) \le \varepsilon_n, \tag{58}$$

and that, using also (49),

$$\begin{aligned}
&\left\|\mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t))) - x_n(\theta_n(t))\right\|^2 \\
&\le 2\left\|\mathrm{P}_{C(\theta_n(t))}(x_{n+1}(\theta_{n+1}(t))) - x_{n+1}(\theta_{n+1}(t))\right\|^2 + 2\left\|x_{n+1}(\theta_{n+1}(t)) - x_n(\theta_n(t))\right\|^2 \\
&\le 2\varepsilon_n + 12\varepsilon_n^2 + 6\|x_{n+1}(t) - x_n(t)\|^2,
\end{aligned} \tag{59}$$

therefore, from (57), (58), and (59) we obtain that there exists a constant $K_2 > 0$, independent of n, such that

$$\begin{aligned}
&\int_0^t \langle -x'_n(s), x_{n+1}(\theta_{n+1}(s)) - x_n(\theta_n(s))\rangle \,\mathrm{d}s \\
&\le \frac{3}{r}\int_0^t \|x_{n+1}(s) - x_n(s)\|^2 \,\mathrm{d}s + \int_0^t K_2(1 + \|x'_n(s)\|)\varepsilon_n \,\mathrm{d}s \\
&\le \frac{3}{r}\int_0^t \|x_{n+1}(s) - x_n(s)\|^2 \,\mathrm{d}s + K_2(T+V)\varepsilon_n.
\end{aligned} \tag{60}$$

Collecting together (48), (55) and (60) we infer that there exists a constant $K_3 > 0$, independent of n (depending only on T and V), such that

$$\|x_{n+1}(t) - x_n(t)\|^2 \le K_3\varepsilon_n + \frac{9}{2r}\|x_{n+1}(t) - x_n(t)\|^2. \tag{61}$$

By (61) and by the Grönwall lemma we infer that there is a constant $K(T,V,r) > 0$ independent of n (depending only on r, T and V), such that

$$\|x_{n+1}(t) - x_n(t)\|^2 \le K(T,V,r)\varepsilon_n \qquad \forall t \in [0,T]\,, \quad \forall n \in \mathbb{N},$$

so that

$$\sum_{n=1}^{\infty} \sup_{t\in[0,T]} \|x_{n+1}(t) - x_n(t)\| < \infty$$

and the sequence x_n is uniformly Cauchy in $C([0,T];\mathcal{H})$. It follows that there exists a continuous function $y : [0,\infty[\longrightarrow \mathcal{H}$ such that

$$x_n \to y \quad \text{uniformly on } [0,T]\,, \qquad \forall T > 0, \tag{62}$$

thus $y(0) = y_0$. From (62) and from (40) of Lemma 4.2 we also deduce that

$$y_n \to y \quad \text{uniformly on } [0,T]. \tag{63}$$

The assumption that $\sup_{n\in\mathbb{N}} \mathrm{V}(y_n,[0,T]) < \infty$ and the lower semicontinuity of the variation w.r.t. the pointwise convergence yield $y \in BV([0,T];\mathcal{H})$. Finally, if we have $t \in [0,T]$ and $t \in \left[t^n_{j-1}, t^n_j\right[$ for some j, we obtain

$$d(y_n(t), C(t)) = d(y^n_{j-1}, C(t)) \le e\,(C(t^n_{j-1}), C(t)) \le \varepsilon_n. \tag{64}$$

Therefore by (63) and by the closedness of $C(t)$ we infer that $y(t) \in C(t)$ for every $t \in [0,T]$. □

4.3. Uniform estimate of the variation

This subsection is devoted to the uniform estimate of the variation of the approximated solutions. We start by recalling the following lemma (cf. [22, Lemma 2.2]).

Lemma 4.4. *Assume that $m \in \mathbb{N}$, $r > 0$, and $C_1, \dots, C_m$ are r-prox-regular subsets of $\mathcal{H}$ satisfying the inequalities*

$$e\,(C_k, C_{k+1}) < r/2 \quad \forall k \in \{1,\dots,m-1\}, \tag{65}$$

Moreover assume that there exist $w \in \mathcal{H}$ and $\rho > 0$ such that $B_\rho(w) \subseteq C_k$ for all k. Let $y_0 \in \mathcal{H}$ be such that $d_{C_1}(y_0) < r/2$, and define

$$y_k := \mathrm{P}_{C_k}(y_{k-1}) \qquad k = 1,\dots,m,$$
$$\alpha := \max\left\{d_{C_1}(y_0), e\,(C_1,C_2), \dots, e\,(C_{m-1},C_m)\right\} + \|y_0 - w\| + \rho. \tag{66}$$

If

$$\alpha^2 < 2r\rho, \tag{67}$$

then

$$\sum_{k=1}^{m} \|y_k - y_{k-1}\| \le \max\left\{r\frac{\|y_0 - w\|^2 - \rho^2}{2r\rho - \alpha^2}, 0\right\}. \tag{68}$$

Then, following exactly the argument in [22, p. 57], we can estimate the variation of the approximated solutions in the case when the moving set $C(t)$ contains a fixed ball at every time t.

Lemma 4.5. *For every $t \in [0,T]$ let $C(t)$ be an r-prox-regular subset of $\mathcal{H}$, and assume that (21) holds. Moreover assume that*

$$\exists w \in \mathcal{H},\ \exists \rho > 0\ :\ \left[\, B_\rho(w) \subseteq C(t)\ \forall t \in [0,T]\ \right], \tag{69}$$

and

$$(\|y_0 - w\| + \rho)^2 < 2r\rho. \tag{70}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$, and $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$, satisfying (26), (27)–(28), and (29)–(31), and let $(y^n_j)_{j=0}^{I^n}$ is the family in $\mathcal{H}$ defined in (32)–(34) by virtue of Lemma 4.1. If for every $n \in \mathbb{N}$, we let $y_n : [0,T] \longrightarrow \mathcal{H}$ be the discrete solution of Problem 3.1, i.e. the right continuous step function defined by (37)–(38), then

$$\sup_{n\in\mathbb{N}} \mathrm{V}(y_n,[0,T]) < \infty, \tag{71}$$

and the supremum in the previous formula is depending on r, y_0, w, ρ, and on the continuity modulus of C.

Proof. Thanks to (70) we can find $n_r \in \mathbb{N}$ such that

$$(\|y_0 - w\| + \rho + \varepsilon_n)^2 < 2r\rho \qquad \forall n \in \mathbb{N},\ n \geq n_r. \tag{72}$$

$$\varepsilon_n < r/2 \qquad \forall n \in \mathbb{N},\ n \geq n_r. \tag{73}$$

Therefore we conclude by observing that $\mathrm{V}(y_n, [0,T]) = \sum_{j=1}^{J^n} \|y_j^n - y_{j-1}^n\|$ and by applying Lemma 4.4 with $m = J^n$, $C_j = C(t_j^n)$ for $1 \leq j \leq J^n$, that

$$\alpha = \|y_0 - w\| + \rho + \sup_{n \geq n_r} \varepsilon_n. \qquad \square$$

The next lemma is a variant of [22, Lemma 2.3] which will allows us to get rid of the assumption (3) of [22, Theorem 4.2]: namely, thanks to next lemma, in our main theorems we will not need any continuity in time of ∂C, neither w.r.t. the Hausdorff distance nor w.r.t. the excess.

Lemma 4.6. *Let $T > 0$ and $t_0 \in [0,T)$. For every $t \in [0,T]$ let $C(t)$ be an r-prox-regular subset of $\mathcal{H}$ such that*

(i) *$C(t)$ is r-prox-regular for every $t \in [0,T]$;*

(ii) *$e\big(C(t_0), C(t)\big) \leq \omega(t - t_0)$ for every $t \in [t_0, T]$, with ω non-decreasing and such that $\lim_{t \to 0^+} \omega(t) = 0$;*

(iii) *there exist $x \in \mathcal{H}$ and $\rho_0 > 0$ such that $B_{\rho_0}(x) \subseteq C(t_0)$.*

Then for every $\rho \in (0, \rho_0)$ there exists $\tau = \tau(\omega, r, \rho_0, \rho) \in (0,T]$, depending only on ω, r, ρ_0, ρ and independent of x and t_0, such that $B_\rho(x) \subseteq C(t)$ for every $t \in [t_0, \min\{t_0 + \tau, T\}]$.

Proof. Given $\rho \in (0, \rho_0)$, define $\eta := \eta(r, \rho_0, \rho) := \min\{\rho_0 - \rho, r\}$ and let $\tau = \tau(\omega, r, \rho_0, \rho) > 0$ be such that $\omega(\tau) < \min\{\eta, \rho\}$. Observe that τ is independent of x and t_0.

We start by proving that $d\big(x, \partial C(t)\big) \geq \rho$ for every $t \in [t_0, \min\{t_0 + \tau, T\}]$. If it were not the case, we would have some $t \in [t_0, \min\{t_0 + \tau, T\}]$ and $y \in \partial C(t)$ with $\|y - x\| < \rho$. Since $C(t)$ is r-prox-regular, and y belongs to its boundary, we can find $z \in \mathcal{H} \setminus C(t)$ such that $B_\eta(z) \subseteq \mathcal{H} \setminus C(t)$ and $\|z - y\| = \eta < r$.

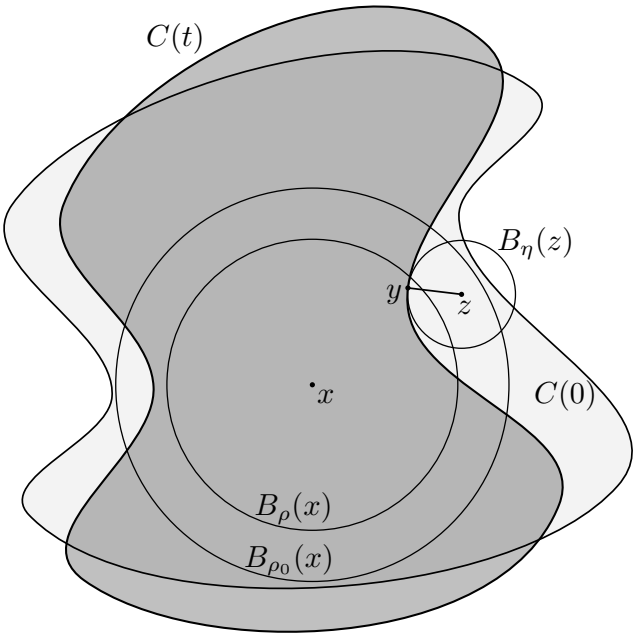


Figure 1. *Visual aid for the proof of Lemma 4.6.*

Figure 1 illustrates this situation. From

$$\|z - x\| \leq \|z - y\| + \|y - x\| < \eta + \rho \leq \rho_0 - \rho + \rho = \rho_0$$

and from iii we deduce $z \in B_{\rho_0}(x) \subseteq C(t_0)$; hence by ii, the definition of τ, and $C(t) \subseteq \mathcal{H} \setminus B_\eta(z)$ we have

$$\eta > \omega(\tau) \geq e\big(C(t_0), C(t)\big) \geq d\big(z, C(t)\big) \geq d\big(z, \mathcal{H} \setminus B_\eta(z)\big) \geq \eta,$$

which is impossible. This concludes the proof of the claim.

Next we show that we have $x \in C(t)$ for every $t \in [t_0, \min\{t_0+\tau, T\}]$. Indeed, if it were not the case, we would have some $t \in [t_0, \min\{t_0+\tau, T\}]$ such that $x \notin C(t)$. From ii, the definition or τ, and recalling that $x \in C(t_0)$ by iii we get

$$d\big(x, C(t)\big) \le e\big(C(t_0), C(t)\big) \le \omega(t) \le \omega(\tau) < \rho,$$

therefore $d\big(x, \partial C(t)\big) = d\big(x, C(t)\big) < \rho$, which is impossible because it violates the first claim.

Finally, from $x \in C(t)$ and $d\big(x, \partial C(t)\big) \ge \rho$ for every $t \in [t_0, \min\{t_0+\tau, T\}]$ we deduce that $B_\rho(x) \subseteq C(t)$ for every $t \in [t_0, \min\{t_0+\tau, T\}]$, and this finishes the proof. □

We are now ready to establish uniform estimates for the variation in the case when $C(t)$ enjoys the uniform inner cone condition.

Theorem 4.7. *For every $t \in [0,T]$ let $C(t)$ be an r-prox-regular subset of $\mathcal{H}$, and assume that (21) holds. Moreover assume that there exist $R > 0$ and $d > 0$ such that for every $t \in [0,T]$*

$$\forall x \in \partial C(t)\ \exists z \in C(t) \cap B_d(x)\ :\ (1-\lambda)x + \lambda B_R(z) \subseteq C(t),\ \forall \lambda \in [0,1]\,, \tag{74}$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}_+$ satisfying (26)–(28), let $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$ satisfying (29)–(31), let $(y_j^n)_{j=0}^{J^n}$ be the family of points in $\mathcal{H}$ defined by (32)–(34) by virtue of Lemma 4.1, and let $y_n : [0,T] \longrightarrow \mathcal{H}$ be the discrete solution of Problem 3.1, i.e. the right continuous step function defined by (37)–(38) in Lemma 4.2. Then

$$\sup_{n\in\mathbb{N}} \mathrm{V}(y_n, [0,T]) < \infty. \tag{75}$$

Proof. Let us first choose $\lambda \in\,]0,1[$ such that $(\lambda d + \lambda R/2)^2 < 2r(\lambda R/2)$, i.e. $\lambda(d + R/2)^2 < rR$. Now let $\tau = \tau(\omega, r, \lambda R, \lambda R/2)$ be the positive number obtained by applying Lemma 4.6 with $\omega(\delta) = \sup\{e(C(s), C(t)) : 0 \le t - s \le \delta\}$ being the continuity modulus of C, $\rho_0 = \lambda R$ and $\rho = \lambda R/2$. Let $\bar n \in \mathbb{N}$ be such that $\varepsilon_{\bar n} < r/2$, $\delta_{\bar n} < \tau/2$ and $(\lambda d + \lambda R/2 + \varepsilon_{\bar n})^2 < 2r(\lambda R/2)$.

Fix $n \in \mathbb{N}$ with $n \ge \bar n$. Given $i \in \{0, \dots, J^n - 1\}$, consider the point $y_i^n \in C(t_i^n)$; there are two cases: either $y_i^n \in \partial C(t_i^n)$, or $y_i^n \in \mathrm{int}(C(t_i^n))$.

Case 1. Assume $y_i^n \in \partial C(t_i^n)$. Since the steps of the partition are shorter than $\delta_{\bar n} < \tau/2$, we can find an index $j \in \{i+1, \dots, J^n\}$ such that $t_j^n \in \big[\min\{t_i^n + \tau/2, T\}, t_i^n + \tau\big]$. Assumption (74) implies that there is some $z_i^n \in C(t_i^n) \cap B_d(y_i^n)$ such that, letting $w_i^n = (1-\lambda)y_i^n + \lambda z_i^n$, we have $B_{\lambda R}(w_i^n) \subseteq C(t_i^n)$, hence from condition (21), Lemma 4.6, and our choice of τ, we deduce that $B_{\lambda R/2}(w_i^n) \subseteq C(t)$ for every $t \in \Big[t_i^n, t_j^n\Big]$. Applying Lemma 4.4 with $C_k = C(t_k^n)$ for $k = i, \dots, j$, $y_0 = y_i^n$, $w = w_i^n$, $\rho = \lambda R/2$, and $\alpha = \lambda d + \lambda R/2 + \varepsilon_{\bar n}$, we deduce

$$\mathrm{V}\Big(y_n, \Big[t_i^n, t_j^n\Big]\Big) \le r\frac{d^2 - (\lambda R/2)^2}{\lambda r R - \alpha^2}.$$

Observe that the numerator is positive because

$$\lambda R/2 = \rho \le \|w_i^n - y_i^n\| \le \|z_i^n - y_i^n\| \le d$$

and the denominator is positive too because by the choice of λ and $\bar n$ we have $\alpha^2 < 2r(\lambda R/2) = \lambda r R$; this is the reason why in the previous estimate of the variation we discarded the max resulting from (68).

Case 2. If $y_i^n \in \mathrm{int}(C(t_i^n))$, then there are two possible situations: either we have $y_j^n = y_i^n$ for every $j \in \{i, \dots, J^n\}$, in which case $\mathrm{V}(y_n, \big[t_i^n, T\big]) = 0$, or there exists $\bar\imath \in \{i+1, \dots, J^n\}$ such that $y_{\bar\imath}^n \ne y_i^n$ and $y_k^n = y_i^n$ for every $k \in \{i, \dots, \bar\imath - 1\}$.

In this latter situation we have $\|y_{\bar\imath}^n - y_{\bar\imath-1}^n\| \le e\big(C(t_{\bar\imath-1}^n), C(t_{\bar\imath}^n)\big) \le \varepsilon_n \le \varepsilon_{\bar n}$, from which $\mathrm{V}(y_n, \big[t_i^n, t_{\bar\imath}^n\big]) \le \varepsilon_{\bar n}$. The point $y_{\bar\imath-1}^n$ cannot belong to $C(t_{\bar\imath}^n)$, otherwise we would have $y_{\bar\imath}^n = \mathrm{P}_{C(t_{\bar\imath}^n)}(y_{\bar\imath-1}^n) = y_{\bar\imath-1}^n$. From this we deduce that $y_{\bar\imath}^n \in \partial C(t_{\bar\imath}^n)$, because if it were in the interior it could not be the point minimizing the distance from $y_{\bar\imath-1}^n$.

We can now replicate the argument of the former case and deduce that there exists an index $j \in \{\bar\imath + 1, \dots, J^n\}$ such that $t_j^n \in \big[\min\{t_{\bar\imath}^n + \tau/2, T\}, t_{\bar\imath}^n + \tau\big]$ and

$$\mathrm{V}\Big(y_n, \Big[t_{\bar\imath}^n, t_j^n\Big]\Big) \le r\frac{d^2 - (\lambda R/2)^2}{\lambda r R - \alpha^2}.$$

In both cases we were able to find $t_j^n \geq \min\{t_i^n + \tau/2, T\}$ such that

$$\mathrm{V}\big(y_n, \left[t_i^n, t_j^n\right]\big) \leq r\frac{d^2 - (\lambda R/2)^2}{\lambda r R - \alpha^2} + \varepsilon_{\bar{n}},$$

therefore it takes at most $\lceil T/(\tau/2)\rceil$ steps to cover the entire interval $[0,T]$ and we find

$$\mathrm{V}(y_n, [0,T]) \leq \lceil 2T/\tau\rceil \left(r\frac{d^2 - (\lambda R/2)^2}{\lambda r R - \alpha^2} + \varepsilon_{\bar{n}}\right). \qquad \square$$

4.4. Limit procedure

In this final subsection we perform the limit procedure needed to recover a solution of Problem 3.1. We start with the following convergence lemma on differential measures.

Lemma 4.8. *If $g_n, g, h \in BV([0,T];\mathcal{H})$ for every $n \in \mathbb{N}$, h is right-continuous, $g_n \to g$ uniformly on $[0,T]$, and $\sup_{n\in\mathbb{N}} \mathrm{V}(g_n,[0,T]) < \infty$, then*

$$\lim_{n\to\infty} \int_{[0,T]} \langle h(t), \mathrm{d}\,\mathrm{D}g_n(t)\rangle = \int_{[0,T]} \langle h(t), \mathrm{d}\,\mathrm{D}g(t)\rangle. \tag{76}$$

Proof. Let us observe that we can assume without loss of generality that $g = 0$ and that h and every g_n are continuous at T. Set $V := \sup_{n\in\mathbb{N}} \mathrm{V}(g_n,[0,T]) < \infty$, and let $\varepsilon > 0$ be arbitrarily fixed. Let $h_\varepsilon = \sum_{j=1}^{m_\varepsilon} h_\varepsilon^j \mathbb{1}_{[t_{j-1},t_j[}$ be a step map with respect to the algebra of subintervals of $[0,T]$ such that $\|h_\varepsilon - h\|_\infty < \varepsilon(2V)^{-1}$, and let $n_\varepsilon \in \mathbb{N}$ be such that $\|g_n\|_\infty < \varepsilon(4m_\varepsilon\|h_\varepsilon\|_\infty)^{-1}$ for every $n \geq n_\varepsilon$. Then for every $n \geq n_\varepsilon$ we have that

$$\begin{aligned}
\left|\int_{[0,T]} h\,\mathrm{d}\,\mathrm{D}g_n\right| &\leq \int_{[0,T]} |h - h_\varepsilon|\,\mathrm{d}\,|\mathrm{D}g_n| + \left|\int_{[0,T]} h_\varepsilon\,\mathrm{d}\,\mathrm{D}g_n\right| \\
&\leq \|h_\varepsilon - h\|_\infty \sup_{n\in\mathbb{N}_+} \mathrm{V}(g_n,[0,T]) + \sum_{j=1}^{m_\varepsilon} \|h_\varepsilon^j\|\|g_n(t_j) - g_n(t_{j-1}-)\| \\
&\leq \varepsilon + 2m_\varepsilon\|h_\varepsilon\|_\infty\|g_n\|_\infty < \varepsilon. \qquad \square
\end{aligned}$$

We can show the limit procedure.

Theorem 4.9. *For every $t \in [0,T]$ let $C(t)$ be an r-prox-regular subset of $\mathcal{H}$, and assume that (21) holds. Fix $y_0 \in C(0)$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ satisfying (41). Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ be a sequence of subdivisions of $[0,T]$, satisfying (27)–(28), and (29)–(31), and let $(y_j^n)_{j=0}^{J^n}$ is the family in $\mathcal{H}$ defined in (32)–(34) by virtue of Lemma 4.1. For every $n \in \mathbb{N}$, we let $y_n : [0,T] \longrightarrow \mathcal{H}$ be the discrete solution of Problem 3.1, i.e. the right continuous step function defined by (37)–(38), assume that $\sup_{n\in\mathbb{N}} \mathrm{V}(y_m,[0,T]) < \infty$, and let $y \in BV([0,T];\mathcal{H}) \cap C([0,T];\mathcal{H})$ be the uniform limit of y_n, which exists by Theorem 4.3. Then y is a solution of Problem 3.1.*

Proof. Let (y_n) be the sequence defined in (37)–(38) of Theorem 4.3 and assume that $y \in BV([0,T];\mathcal{H}) \cap C([0,T];\mathcal{H})$ is its uniform limit. From (34) it follows that at every step n we have that

$$-\mathrm{D}y_n(\{t_j^n\}) = y_j^n - y_{j-1}^n \in N_{C(t_j^n)}(y_j^n) = N_{C(t_j^n)}(y_n(t_j^n)), \tag{77}$$

and we also trivially have that

$$\mathrm{D}y_n\Big(\Big]t_{j-1}^n, t_j^n\Big[\Big) = 0 \in N_{C(t)}(y_n(s)) \qquad \forall s \in \Big]t_{j-1}^n, t_j^n\Big[\tag{78}$$

for every j. Let also consider the continuous interpolation x_n defined in (39). Now fix $\tau \in [0,T[$ and $z_\tau \in C(\tau)$ arbitrarily. By applying Theorem 4.3 with y_0 and $C(t)$ replaced respectively by z_τ and $C(\tau+(T-\tau)t/T)$, we infer the existence of a continuous function $\widetilde{z} : [0,T] \longrightarrow \mathcal{H}$ such that $\widetilde{z}(0) = z_\tau$ and $\widetilde{z}(t) \in C(t)$ for all $t \in [0,T]$. Therefore if we define $z : [\tau,T] \longrightarrow \mathcal{H}$ by

$$z(s) := \widetilde{z}\left(\frac{T}{T-t}(s-\tau)\right) \qquad s \in [\tau,T],$$

then z is continuous, $z(\tau) = z_\tau$, and $z(t) \in C(t)$ for every $t \in [t, T]$. Therefore from (77), (78) we infer that

$$\langle -\mathrm{D}y_n(\{s\}), z_s - y_n(s)\rangle \le \frac{1}{2r}\|\mathrm{D}y_n(s)\|\|z_s - y_n(s)\|^2 \qquad \forall z_s \in C(s),\ \forall s \in [0,T]\,. \tag{79}$$

It follows that for every $h > 0$ small enough we have

$$\int_{[t,t+h]} \langle y_n(s) - z(s), \mathrm{d}\,\mathrm{D}y_n(s)\rangle \le \frac{1}{2r}\int_{[t,t+h]} \|z(s) - y_n(s)\|^2\,\mathrm{d}\left|\mathrm{D}y_n\right|(s) \tag{80}$$

By the boundedness of $V(y_n, [0,T])$ we have that $\left|\mathrm{D}y_n\right|([0,T])$ is bounded, hence there exists a positive Borel measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ such that $\left|\mathrm{D}y_n\right| \overset{*}{\rightharpoonup} \mu$ at least for a subsequence which we dot relabel. Therefore taking the limit in (80) as $n \to \infty$, by the continuity of the integrands, and using the fact that $x_n - y_n \to 0$ uniformly, we infer that

$$\int_{[t,t+h]} \langle y(s) - z(s), \mathrm{d}\,\mathrm{D}y(s)\rangle \le \frac{1}{2r}\int_{[t,t+h]} \|z(s) - y(s)\|^2\,\mathrm{d}\mu(s) \tag{81}$$

For every $k \in \mathbb{N}$ set $I_k := \{t \in I : d(t, \partial I) > 1/k\}$ and let $\phi_k \in C([0,T];\mathbb{R})$ be such that $\mathbb{1}_{I_k} \le \phi_k \le \mathbb{1}_I$. Then by the lower semicontinuity of the pointwise variation w.r.t. the pointwise convergence we have

$$\begin{aligned}\left|\mathrm{D}y\right|(I_k) = \mathrm{V}(y, I_k) &\le \liminf_{n\to\infty} \mathrm{V}(y_k, I_k) = \liminf_{n\to\infty} \left|\mathrm{D}y_n\right|(I_k)\\ &\le \liminf_{n\to\infty} \int_{[0,T]} \phi_k\,\mathrm{d}\left|\mathrm{D}y_n\right| = \int_{[0,T]} \phi_k\,\mathrm{d}\mu\\ &\le \int_{[0,T]} \mathbb{1}_I\,\mathrm{d}\mu = \mu(I),\end{aligned}$$

therefore taking the limit as $k \to \infty$ we infer that $\left|\mathrm{D}y\right|(I) \le \mu(I)$. Since every open set $A \subseteq [0,T]$ is a disjoint union of open intervals, we conclude that $\left|\mathrm{D}y\right|(A) \le \mu(A)$. This inequality and the outer regularity of the two measures finally yield $\left|\mathrm{D}y\right|(B) \le \mu(B)$ for every Borel set B, hence $\mathrm{D}y$ is absolutely continuous with respect to μ.

Thus there exists $v \in L^1(\mu;\mathcal{H})$ such that $\mathrm{D}y = v\mu$, hence

$$\int_{[t,t+h]} \langle y(s) - z(s), v(s)\rangle\,\mathrm{d}\mu(s) \le \frac{1}{2r}\int_{[t,t+h]} \|z(s) - y(s)\|^2\,\mathrm{d}\mu(s). \tag{82}$$

If τ is a μ-Lebesgue point of $s \longmapsto \langle y(s) - z(s), v(s)\rangle$, dividing both sides by $\mu([\tau, \tau + h])$ and taking the limit as $h \searrow 0$ we infer that

$$v(\tau) \in -N_{C(\tau)}(y(\tau)).$$

The conclusion follows from Proposition 2.3 and from the fact that the set of Lebesgue points of $s \longmapsto \langle y(s) - z(s), v(s)\rangle$ has full μ-measure in $[0,T]$. Indeed if L is the set of μ-Lebesgue points for $\tau \longmapsto v(\tau)$, then L has full measure, thus fix $t \in L$, and choose $\zeta_t \in C(t)$ arbitrarily. Since $y \in C([0,T];\mathcal{H})$ we have that t is a μ-Lebesgue points of y, therefore

$$\begin{aligned}&\frac{1}{\mu([t,t+h])}\int_{[t,t+h]} \left|\langle y(\tau) - z(\tau), v(\tau)\rangle - \langle y(t) - z(t), v(t)\rangle\right|\mathrm{d}\tau\\ &\le \frac{1}{\mu([t,t+h])}\int_{[t,t+h]} \left(\|y - z\|_\infty\|v(\tau) - v(t)\| + \|v(t)\|\|y(\tau) - y(t)\|\right)\mathrm{d}\tau,\end{aligned} \tag{83}$$

thus

$$\lim_{h\searrow 0} \frac{1}{\mu([t,t+h])}\int_{[t,t+h]} \langle y(\tau) - z(\tau), v(\tau)\rangle\,\mathrm{d}\mu(\tau) = \langle y(t) - z(t), v(t)\rangle. \qquad \square$$

4.5. Conclusion

In this subsection we prove the main Theorems 3.3 and 3.4 of the paper by collecting together the results in the previous subsections. Indeed let $(y_j^n)_{j=0}^{J^n}$ be the sequence of discrete solution obtained by the catching-up algorithm in Lemma 4.1. Moreover let (y_n) be the sequence of discrete solutions of Problem 3.1, i.e. the sequence of piecewise constant approximated solutions defined by (37)–(38).

Under the assumptions of Theorem 3.4, we have that $\sup_{n\in\mathbb{N}} \mathrm{V}(y_n, [0,T])$ is finite by virtue of Lemma 4.5. Instead under the assumptions of Theorem 3.3, the boundedness of $\mathrm{V}(y_n, [0,T])$ is deduced by exploiting Theorem 4.7.

In both situation we can therefore apply Theorem 4.3 and Theorem 4.9, and infer the existence of a solution of Problem 3.1. Uniqueness is classic and follows for instance from [67, Proposition 3.6].

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