

Abstract PhD thesis

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We study the notion of *pre-Cartier category* and the corresponding algebraic version of *pre-Cartier quasitriangular bialgebra*, introduced and investigated in [1]. The latter is a quasitriangular bialgebra equipped with an infinitesimal \mathcal{R} -matrix. The motivating example is provided by the first order degree of a quasitriangular structure on a topological bialgebra. Infinitesimal \mathcal{R} -matrices are Hochschild 2-cocycles which satisfy an infinitesimal quantum Yang–Baxter equation and are 2-coboundaries under some additional assumptions. We classify pre-Cartier structures for some well-known Hopf algebras, such as the Sweedler Hopf algebra and some of its generalizations, namely the Radford Hopf algebras $H_{(r,n,q)}$ and the Hopf algebras $E(n)$, as done in [2]. In the dual setting of pre-Cartier coquasitriangular bialgebras, i.e. coquasitriangular bialgebras equipped with an infinitesimal \mathcal{R} -form, we give examples on $M_q(2)$, $GL_q(2)$, and we provide an infinitesimal FRT construction and a Tannaka–Krein reconstruction theorem.

The axioms of an infinitesimal \mathcal{R} -form involve a pair of actions that satisfy some compatibility conditions, called a *matched pair of actions*. The category of matched pairs of actions on a Hopf algebra is shown to be isomorphic to the category of *Yetter–Drinfeld braces*, novel structures introduced in [3] that generalize cocommutative Hopf braces. In fact, the main examples of Yetter–Drinfeld braces are provided by coquasitriangular Hopf algebras, involving Majid’s transmutation. Just as cocommutative Hopf braces are equivalent to cocommutative post-Hopf algebras, Yetter–Drinfeld braces are equivalent to *Yetter–Drinfeld post-Hopf algebras*, introduced in [5]. We also provide an adjunction between Yetter–Drinfeld post-Hopf algebras and *Yetter–Drinfeld relative Rota–Baxter operators* that becomes an equivalence considering bijective operators. Just as matched pairs of actions produce solutions to the quantum Yang–Baxter equation, we show that the same result can be achieved by using matched pairs of coactions.

Finally, we explore some categorical properties of the category of cocommutative Hopf braces (equivalently, cocommutative Yetter–Drinfeld braces), following [4]. We show that this category is semi-abelian and strongly protomodular, and it shares many properties with the semi-abelian category of cocommutative Hopf algebras. We provide a torsion theory for the category of cocommutative Hopf braces, where the torsion subcategory is given by universal enveloping algebras of post-Lie algebras and the torsion-free subcategory by skew braces. Moreover, the latter form a localization of cocommutative Hopf braces. As an application of the semi-abelianness of the category, we describe central extensions and commutators of cocommutative Hopf braces.

The results presented in this thesis are mainly based on the papers reported below. Some other original results are added.

References

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- [4] M. Gran, A. Sciandra, *Hopf braces and semi-abelian categories*. J. Algebra 690 (2026), 266-303.
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