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Forward Backward SDEs Systems for Utility Maximization in Jump Diffusion Models

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Abstract

We consider the classical problem of maximizing the expected utility of terminal net wealth with a final random liability in a simple jump-diffusion model. In the spirit of Horst et al. (Stoch Process Appl 124(5):1813–1848, 2014) and Santacroce and Trivellato (SIAM J Control Optim 52(6):3517–3537, 2014), under suitable conditions the optimal strategy is expressed in implicit form in terms of a forward backward system of equations. Some explicit results are presented for the pure jump model and for exponential utilities.

Keywords Forward backward stochastic differential systems · Jump-diffusions · Utility maximization problem

Mathematics Subject Classification 60H10 · 91G80 · 60G07

1 Introduction

Portfolio's optimization and hedging problems are two classical problems which have been deeply investigated since the beginning of the seventies. Their mathematical formulation in continuous time was pioneered by Merton. In [19] he provides the strategy which maximizes the expected utility of a small investor in closed form by

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exploiting the Markovianity of the model for standard utilities. An alternative approach for not necessarily Markovian models was suggested by [2] and uses convex duality. This methodology has been developed in its full potential in [3, 7, 11, 25] and, in its modern form, in [12] and many others work thereafter. It is well known that convex duality and martingale methods lead to establish the existence and uniqueness of an optimal strategy in general market models and for non classical utility functions, but they do not provide a constructive characterization. A constructive form for the optimal portfolio has been determined for classical utilities in quite general market models by using dynamic programming techniques [9, 14, 15]. Nevertheless, there exist just very few results for non classical utilities. A first work in this direction is given by [16] (see also [17]). In the non-Markovian framework, the problem is solved by dynamic programming and the optimal strategy is given in terms of the value function related to the problem and its derivatives. In turn, the value function is characterized as the solution of a backward stochastic partial differential equation under some assumptions imposed on the value function. Another subsequent work with a constructive characterization is [8], where the hedging problem is studied for general utilities in a Brownian model without resorting to dynamic programming. In this article the optimal strategy is described by means of the utility function with its derivatives and the solution of a fully coupled forward backward system. In [28] the same approach has been generalized to a continuous semimartingale setting.

Further developments can be found in more recent works in Brownian settings (see [24] for a large investor problem with endogenous permanent market impacts and [29] for a stochastic maximum principle algorithm for constrained utilities maximization).

In this paper, we consider a jump diffusion model driven by a Brownian motion and an independent simple Poisson process and study a classical problem of maximization for a general utility using the techniques introduced in [8, 28]. To our present knowledge, it is the first paper dealing with this approach in a model with jumps.

We start by restating Proposition 2.1 in [28], which gives the derivative of the expectation of the terminal net wealth computed in an admissible strategy. The proof follows the same lines as in [28], but we add here an extra condition which was missing. The derivative is proved to be null for the optimal strategy and this represents our necessary condition for optimality.

In our main result we use this key condition to express the optimal strategy in terms of a fully coupled forward–backward stochastic differential system.

Besides, the vice versa of this result states that from a suitable solution of the system, the strategy which satisfies a certain optimality equation is admissible and optimal.

As a result, in the jump diffusion model the optimal strategy is found to be implicitly defined by an equation written in terms of the solution of the forward backward system. Such optimality equation can not be explicitly solved even for the exponential utility, where the system decouples (see e.g., [20–22]). However, the optimal strategy admits an explicit expression in the pure jump model. In this setting, for the pure investment problem we also provide sufficient conditions in order to find a solution to the forward backward system, which is not in general an easy task. Moreover, the related optimal strategy turns out to have a more pleasant expression. Finally, we prove that these sufficient conditions are satisfied for exponential utilities, where an easy solution to the forward backward system and the corresponding optimal strategy are explicitly written.

The paper is organized as follows. In Sect. 2, we introduce the model with the assumptions and state the preliminary proposition, containing the necessary condition for the optimality. In Sect. 3, we consider the jump diffusion model and give the characterization of the optimal strategy in terms of a fully-coupled forward backward system. In Sect. 4, we deal with the pure jump model. The forward backward characterization of the optimal strategy is followed by the application to the pure investment problem. Finally, the results of the previous sections are specified to exponential utilities in Sect. 5.

2 Model Settings and Preliminary Results

We consider the classical problem of maximizing the expected utility of terminal net wealth, with a finite time horizon, when the asset price process is modeled by a jump-diffusion. Specifically, the financial model consists of a bank account which pays no interests and of one risky asset whose (discounted) price S is given by the stochastic exponential of a jump diffusion.

On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we consider two independent processes W and N, both defined on $[0, T]$, where $T < +\infty$ is the fixed time horizon: W is a standard (one-dimensional) Brownian motion and N a homogeneous Poisson process with intensity $\nu > 0$.

The probability space is equipped with the natural filtration generated by W and N (completed by the $\mathbb{P}$-null sets of $\mathscr{F}$). As a consequence, every (local) martingale admits the classical representation as the sum of two stochastic integrals with respect to W and the compensated Poisson martingale $n_t = N_t - \nu t$, respectively (see [10]). This will be extensively used in the sequel, e.g. in (3.9).

For the sake of simplicity, we denote the filtration by $\mathcal{F} = (\mathcal{F}_t, t \in [0, T])$ and we suppose $\mathcal{F}_T = \mathscr{F}$.

We assume the price process S is defined on the filtered probability space $(\Omega, \mathscr{F}, \mathcal{F}, \mathbb{P})$ with a dynamics given by

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \eta_t dn_t), \quad S_0 > 0, \tag{2.1}$$

where the coefficients μ, σ, η are uniformly bounded predictable processes and, to ensure that S is almost surely positive, $\eta > -1$.

Under these assumptions on the coefficients of the model there is no arbitrage in the market.

We investigate the model in two main different cases: in the first one we consider $\sigma^2 > 0$, whereas the second concerns the pure jump model, i.e. $\sigma^2 \equiv 0$.

We denote by π the dollar amount of risky asset in the portfolio and consider the wealth process X^π which evolves according to the self-financing strategy π, i.e.

$$X_t^\pi = X_0^\pi + \int_0^t \pi_s \frac{dS_s}{S_{s^-}}. \tag{2.2}$$

Moreover, let H be a bounded $\mathcal{F}_T$-measurable random variable, representing a liability due at time T.
Given a utility function U defined on the real line, we consider the related problem of maximizing the expected utility of the terminal net wealth

$$to\ maximize \quad \mathbb{E}\big[U\big(X_T^\pi + H\big)\big] \quad over\ all \quad \pi \in \Pi_x, \tag{2.3}$$

where $x > 0$ is a given initial capital and Π_x is the set of admissible strategies defined as

$$\Pi_x = \big\{\pi \in \mathbb{H}^2 \text{ s.t. } X_0^\pi = x\big\},$$

with

$$\mathbb{H}^2 = \left\{\vartheta \text{ predictable process s.t. } \mathbb{E}\left(\int_0^T \vartheta_t^2 dt\right) < +\infty\right\}.$$

Taking into account the boundedness of μ, σ and η, this choice ensures the square integrability of the wealth process (2.2).
Throughout the paper, the utility function U is assumed to be strictly increasing, strictly concave and three times continuously differentiable. Moreover, the following additional conditions are used as needed:

(H1) $\exists\, k > 0$ s.t. the absolute risk aversion satisfies $\mathrm{ARA}(x) = -\frac{U''(x)}{U'(x)} \geq k, \forall\, x \in \mathbb{R}$;
(H2) $\mathbb{E}[U'(\xi)^2] < +\infty$, for a suitable random variable ξ;
(H3) $\mathbb{E}[|U(\xi)|] < +\infty$, for a suitable random variable ξ.

From now on, according to (2.2) we consider the process $X^{0,h}$, where $X_0^{0,h} = 0$ and h is a predictable bounded process, i.e.

$$X_t^{0,h} = \int_0^t h_s \frac{dS_s}{S_{s^-}}.$$

Let us observe that, under our assumptions on the model and due to the boundedness of h, the process $X^{0,h}$ is square integrable.
The following proposition represents the key starting point for the derivation of the main result and is a restatement of Proposition 2.1 in [28]. For this reason, we omit the proof. We just remark that we add here an extra condition which was missing in [28].

Proposition 2.1 *For $\pi^* \in \Pi_x$, let (H2) hold with $\xi = X_T^{\pi^*} + H$. Moreover, suppose that for any predictable bounded process h, there exists $\varepsilon > 0$ such that (H3) is satisfied by $\xi = X_T^{\pi^*+\varepsilon h} + H$.*
Then

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}\big[U\big(X_T^{\pi^*+\varepsilon h} + H\big) - U\big(X_T^{\pi^*} + H\big)\big]}{\varepsilon} = \mathbb{E}\left[U'\big(X_T^{\pi^*} + H\big)\, X_T^{0,h}\right]. \tag{2.4}$$

If, in addition, π^* is optimal for problem (2.3), then

$$\mathbb{E}\left[U'\big(X_T^{\pi^*} + H\big)\, X_T^{0,h}\right] = 0. \tag{2.5}$$

Remark 2.1 Note that the extra integrability condition (H3) required in Proposition 2.1 can be weakened by requiring only the integrability of the negative part of $U(\xi)$.

3 Jump-Diffusion Model

In this section, we consider the dynamics of the price process (2.1) with $\sigma^2 > 0$ and we require σ^{-1} to be bounded.
The next two theorems are the main results of the paper. The first one characterizes the optimal strategy in terms of the solution of a forward–backward system of SDEs. The second theorem represents the vice versa and establishes the existence of an optimal strategy.

Theorem 3.1 *Let $\pi^* \in \Pi_x$ be optimal for problem (2.3) and suppose (H1) holds. Under the assumptions of Proposition 2.1, there exists a smooth function G such that the following forward–backward system admits a solution (X, Y, Z, Ψ):*

$$X_t = x + \int_0^t G(X_{s^-}, Y_{s^-}, Z_s, \Psi_s, \Upsilon_s)(\mu_s ds + \sigma_s dW_s + \eta_s dn_s) \tag{3.6}$$

$$\begin{aligned} Y_t =& H + \int_t^T \left(\frac{U'(\Psi_s + G(X_{s^-}, Y_{s^-}, Z_s, \Psi_s, \Upsilon_s)\eta_s + X_{s^-} + Y_{s^-}) - U'(X_{s^-} + Y_{s^-})}{U''(X_{s^-} + Y_{s^-})} - \Psi_s\right)\nu ds \\ &+ \int_t^T \frac{1}{2}\frac{U'''(X_{s^-} + Y_{s^-})}{U''(X_{s^-} + Y_{s^-})}(Z_s + G(X_{s^-}, Y_{s^-}, Z_s, \Psi_s, \Upsilon_s)\sigma_s)^2 ds, \\ &+ \int_t^T G(X_{s^-}, Y_{s^-}, Z_s, \Psi_s, \Upsilon_s)\,(\mu_s - \eta_s \nu)\, ds - \int_t^T (Z_s dW_s + \Psi_s dn_s), \end{aligned} \tag{3.7}$$

where $\Upsilon_t = (\eta_t, \mu_t, \sigma_t)$.
Moreover,

$$\pi^* = G(X_-, Y_-, Z, \Psi, \Upsilon) \tag{3.8}$$

and the related optimal wealth process is equal to X.

Proof The arguments used in this proof are similar to those in [8, 28]. We start by defining

$$\alpha_t = \mathbb{E}\big[U'\big(X_T^{\pi^*} + H\big)|\mathcal{F}_t\big].$$

The process α is a square integrable martingale since, due to Proposition 2.1, $(H2)$ is satisfied by $\xi = X_T^{\pi^*} + H$. Therefore, α satisfies the backward stochastic differential equation (BSDE)

$$\alpha_t = U'\big(X_T^{\pi^*} + H\big) - \int_t^T (\beta_s dW_s + \gamma_s dn_s), \tag{3.9}$$

where β and γ are respectively the predictable integrand appearing in the martingale representation of α, with respect to W and n. We consider the process Y, where

$$Y_t = (U')^{-1}(\alpha_t) - X_t^{\pi^*},$$

with final value $Y_T = H$. By Itô's formula we can write

$$\begin{aligned} dY_t =& \frac{1}{U''(U'^{-1}(\alpha_{t-}))} d\alpha_t - \frac{1}{2}\frac{U'''(U'^{-1}(\alpha_{t-}))}{\big(U''(U'^{-1}(\alpha_{t-}))\big)^3} d[\alpha]_t^c + (U'^{-1}(\alpha_t) - U'^{-1}(\alpha_{t-})) \\ & - \frac{1}{U''(U'^{-1}(\alpha_{t-}))}\Delta\alpha_t - dX_t^{\pi^*}. \end{aligned} \tag{3.10}$$

Observing that

$$\begin{aligned} U'^{-1}(\alpha_t) - U'^{-1}(\alpha_{t-}) &= U'^{-1}(\gamma_t \Delta N_t + \alpha_{t-}) - U'^{-1}(\alpha_{t-}) \\ &= \Big(U'^{-1}(\gamma_t + \alpha_{t-}) - U'^{-1}(\alpha_{t-})\Big)\,\Delta N_t \end{aligned}$$

(3.10) can be immediately rewritten as

$$\begin{aligned} dY_t =& \frac{1}{U''(U'^{-1}(\alpha_{t-}))}(\beta_t dW_t - \gamma_t \nu dt) - \frac{1}{2}\frac{U'''(U'^{-1}(\alpha_{t-}))}{\big(U''(U'^{-1}(\alpha_{t-}))\big)^3}\beta_t^2 dt \\ & + \Big(U'^{-1}(\gamma_t + \alpha_{t-}) - U'^{-1}(\alpha_{t-})\Big) dN_t - \pi_t^*(\mu_t dt + \sigma_t dW_t + \eta_t dn_t). \end{aligned}$$

After rearranging the dt, dW_t and dn_t terms together and replacing $U'^{-1}(\alpha_t) = X_t^{\pi^*} + Y_t$, we denote the integrands in the martingale terms respectively by Z_t and Ψ_t, i.e. we define the processes

$$\begin{aligned} Z &= \frac{1}{U''(X_-^{\pi^*} + Y_-)}\beta - \pi^*\sigma, \\ \Psi &= -\pi^*\eta + U'^{-1}\Big(\gamma + U'(X_-^{\pi^*} + Y_-)\Big) - \Big(X_-^{\pi^*} + Y_-\Big). \end{aligned} \tag{3.11}$$

Therefore, Y solves the following BSDE

$$dY_t = Z_t dW_t + \Psi_t dn_t$$

$$+\left[U'^{-1}(\gamma_t+U'(X_{t-}^{\pi^*}+Y_{t-}))-(X_{t-}^{\pi^*}+Y_{t-})-\frac{1}{U''(X_{t-}^{\pi^*}+Y_{t-})}\gamma_t\right]\nu dt$$
$$-\left[\frac{1}{2}\frac{U'''(X_{t-}^{\pi^*}+Y_{t-})}{\big(U''(X_{t-}^{\pi^*}+Y_{t-})\big)^3}\beta_t^2+\pi_t^*\mu_t\right]dt, \quad Y_T=H.$$

If in the previous equation we replace β and Ψ using the expressions in (3.11), we obtain

$$\begin{aligned} dY_t =& -\left[\frac{1}{2}\frac{U'''(X_{t-}^{\pi^*}+Y_{t-})}{U''(X_{t-}^{\pi^*}+Y_{t-})}(Z_t+\pi_t^*\sigma_t)^2\right.\\ &\left.-\big(\Psi_t+\pi_t^*\eta_t\big)\,\nu+\frac{1}{U''(X_{t-}^{\pi^*}+Y_{t-})}\gamma_t\nu+\pi_t^*\mu_t\right]dt\\ &+Z_t dW_t+\Psi_t dn_t, \quad Y_T=H. \end{aligned} \tag{3.12}$$

In order to obtain the FBSDE (3.6, 3.7) we will prove that π^* admits an (implicit) representation like (3.8). To get a characterization of the optimal strategy π^*, we use (2.5) of Proposition 2.1. Starting by the application of the integration by parts formula to $U'(X_T^{\pi^*}+H)\,X_T^{0,h}$, for an arbitrarily fixed bounded predictable process h, we have

$$\begin{aligned} U'(X_T^{\pi^*}+H)X_T^{0,h} =& \int_0^T U'(X_{t-}^{\pi^*}+Y_{t-})h_t\big(\mu_t dt+\sigma_t dW_t+\eta_t dn_t\big)\\ &+\int_0^T X_{t-}^{0,h}\,(\beta_t dW_t+\gamma_t dn_t)\\ &+\int_0^T \frac{h_t}{S_{t-}}\,(\beta_t d[S,W]_t+\gamma_t d[S,n]_t)\\ =& \int_0^T \alpha_{t-}h_t\big(\sigma_t dW_t+\eta_t dn_t\big)\\ &+\int_0^T X_{t-}^{0,h}\,(\beta_t dW_t+\gamma_t dn_t)+\int_0^T h_t\eta_t\gamma_t dn_t\\ &+\int_0^T h_t\left(U'(X_{t-}^{\pi^*}+Y_{t-})\mu_t+\beta_t\sigma_t+\gamma_t\eta_t\nu\right)dt. \end{aligned} \tag{3.13}$$

We now check that the first three integrals in (3.13) are martingales.

For the first integral we have

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t \alpha_{s-}h_s\big(\sigma_s dW_s+\eta_s dn_s\big)\right|\right)\le C\,\mathbb{E}\left(\left(\int_0^T \alpha_{t-}^2h_t^2\big(\sigma_t^2dt+\eta_t^2dN_t\big)\right)^{\frac{1}{2}}\right)$$
$$\le C\,\mathbb{E}\left(\sup_{0\le t\le T}|\alpha_{t-}|\left(\int_0^T h_t^2\big(\sigma_t^2dt+\eta_t^2dN_t\big)\right)^{\frac{1}{2}}\right)$$

$$\leq C\left(\mathbb{E}\left(\sup_{0\leq t\leq T}|\alpha_{t-}|\right)^2\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T h_t^2\big(\sigma_t^2 dt+\eta_t^2 dN_t\big)\right)\right)^{\frac{1}{2}}$$
$$\leq C\left(\mathbb{E}\left(\alpha_{T-}^2\right)\right)^{\frac{1}{2}}<\infty,$$

where the constant C can differ from line to line. The first is Burkholder–Davis–Gundy inequality for $p=1$, the third is Cauchy–Schwarz inequality, while the last is a consequence of Doob inequality and the boundedness of h, η and σ. Then the conclusion follows by the integrability assumptions on α.
Exploiting the same chain of inequalities, for the second integral we can write

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t X_{s^-}^{0,h}\left(\beta_s dW_s+\gamma_s dn_s\right)\right|\right)$$
$$\leq C\left(\mathbb{E}\left(X_{T^-}^{0,h}\right)^2\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T\left(\beta_s^2 ds+\gamma_s^2 dN_s\right)\right)\right)^{\frac{1}{2}}<\infty,$$

where the finiteness of the last expression is due to the square integrability of $X^{0,h}$ and α.
We are left to prove that $\int_0^{\cdot} h_s\eta_s\gamma_s dn_s$ is a martingale. A sufficient condition is to show $\mathbb{E}\left(\int_0^T|h_t\eta_t\gamma_t|\nu\, dt\right)<\infty$ for the predictable process $h\eta\gamma$. This is true since h is bounded and by Cauchy–Schwarz

$$\mathbb{E}\left(\int_0^T|\eta_t\gamma_t|\, dt\right)\leq C\left(\mathbb{E}\left(\int_0^T\eta_t^2 dt\right)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T\gamma_t^2 dt\right)\right)^{\frac{1}{2}}<\infty.$$

Taking the expectation in (3.13) and recalling (2.5) we find

$$\mathbb{E}\left(U'(X_T^{\pi^*}+H)\,X_T^{0,h}\right)=\mathbb{E}\left(\int_0^T h_t\left(U'(X_{t^-}^{\pi^*}+Y_{t^-})\mu_t+\beta_t\sigma_t+\gamma_t\eta_t\nu\right)dt\right)$$
$$=\mathbb{E}\left(\int_0^T h_t\left(U'(X_{t^-}^{\pi^*}+Y_{t^-})\mu_t+U''(X_{t^-}^{\pi^*}+Y_{t^-})(Z_t+\pi_t^*\sigma_t)\sigma_t+\gamma_t\eta_t\nu\right)dt\right)=0, \tag{3.14}$$

where in the last equality we replaced $\beta=U''(X_-^{\pi^*}+Y_-)(Z+\pi^*\sigma)$.
Choosing in (3.14) the integrand $h=\mathbb{1}_{\{U'(X_-^{\pi^*}+Y_-)\mu+U''(X_-^{\pi^*}+Y_-)(Z+\pi^*\sigma)\sigma+\gamma\eta\nu>0\}}$ and then $h=\mathbb{1}_{\{U'(X_-^{\pi^*}+Y_-)\mu+U''(X_-^{\pi^*}+Y_-)(Z+\pi^*\sigma)\sigma+\gamma\eta\nu<0\}}$, we deduce

$$U'(X_-^{\pi^*}+Y_-)\mu+U''(X_-^{\pi^*}+Y_-)(Z+\pi^*\sigma)\sigma+\gamma\eta\nu=0$$
$$d\mathbb{P}\otimes dt-\text{a.e. on }[0,T]. \tag{3.15}$$

From (3.11),

$$\gamma = U'(\Psi + \pi^* \eta + X_-^{\pi^*} + Y_-) - U'(X_-^{\pi^*} + Y_-)$$

which plugged into (3.15) gives the equation

$$U'(X_-^{\pi^*} + Y_-)\mu + U''(X_-^{\pi^*} + Y_-)(Z + \pi^* \sigma)\sigma \\ + \Big(U'(\Psi + \pi^* \eta + X_-^{\pi^*} + Y_-) - U'(X_-^{\pi^*} + Y_-)\Big) \eta\nu = 0. \tag{3.16}$$

Note that (3.16) includes the case $\eta = 0$ for which the optimal strategy can be explicitly written,

$$\pi^* = \frac{1}{\sigma}\left(-\frac{U'(X_-^{\pi^*} + Y_-)}{U''(X_-^{\pi^*} + Y_-)}\frac{\mu}{\sigma} - Z\right),$$

as in [8].
For any $\omega \in \Omega$ and $t \in [0, T]$, (3.16) can be seen as $F = 0$, where the function $F : D \times \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$F(w, \pi) = U'(x + y)\mu + U''(x + y)(z\sigma + \pi\sigma^2) \\ + \big(U'(\psi + \pi\eta + x + y) - U'(x + y)\big) \eta\nu, \tag{3.17}$$

for every w in the open set $D = \{w = (x, y, z, \psi, \eta, \mu, \sigma) \in \mathbb{R}^7 : \sigma^2 > 0\}$, and $\pi \in \mathbb{R}$, is continuously differentiable on its domain, because of the smoothness hypotheses on U. Moreover, U being strictly concave, we get $\forall (w, \pi) \in D \times \mathbb{R}$

$$\frac{\partial}{\partial\pi} F(w, \pi) = U''(x + y)\sigma^2 + U''(\psi + \pi\eta + x + y)\eta^2\nu \le U''(x + y)\sigma^2 \\ < \frac{1}{2}U''(x + y)\sigma^2 < 0,$$

so that a global implicit function theorem can be applied (see Theorem 6.1 (2) in the Appendix). Consequently, for any $w \in D$ the equation $F(w, \pi) = 0$ admits a unique solution $\pi \in \mathbb{R}$ and there exists a function G continuously differentiable on D and such that $\{(w, \pi) \in D \times \mathbb{R} : F(w, \pi) = 0\} = \{(w, \pi) \in D \times \mathbb{R} : \pi = G(w)\}$.

Thus, the optimal strategy is

$$\pi^* = G(X_-, Y_-, Z, \Psi, \eta, \sigma, \mu),$$

which is expressed in terms of the solution (Y, X, Z, Ψ) of the forward backward system (3.6, 3.7), where the backward equation (3.7) is obtained replacing (3.8) in (3.12). Besides, we note that (3.6) represents the optimal wealth process. □

Theorem 3.2 *Suppose a solution of system* (3.6, 3.7) *exists with* $Z \in \mathbb{H}^2$ *and* G *a smooth function such that* $F(w, G(w)) = 0$*, where* F *is defined by* (3.17). *Assume*

(H1) and (H2), (H3) with $\xi = X_T + H$, and $U'(X+Y)$ to be a positive martingale such that the stochastic integral $\frac{1}{U'(X_- + Y_-)} \cdot U'(X+Y)$ is a square integrable martingale. Then the optimal strategy π^* exists in the class Π_x and is given by (3.8).

Proof We first check that the strategy defined in (3.8) and satisfying (3.15) is in Π_x. We consider (X^{π^*}, Y, Z, Ψ) satisfying the system (3.6, 3.7) and we write Itô's formula for $U'(X^{\pi^*} + Y)$, i.e.

$$dU'(X_t^{\pi^*} + Y_t) = U'(X_{t-}^{\pi^*} + Y_{t-}) \times \left(\frac{U''(X_{t-}^{\pi^*} + Y_{t-})}{U'(X_{t-}^{\pi^*} + Y_{t-})} (\pi_t^* \sigma_t + Z_t) dW_t + \frac{\gamma_t}{U'(X_{t-}^{\pi^*} + Y_{t-})} dn_t \right)$$

which, using (3.15), can be rewritten as

$$dU'(X_t^{\pi^*} + Y_t) = U'(X_{t-}^{\pi^*} + Y_{t-}) \times \left(-\Big(\frac{\mu_t}{\sigma_t} + \frac{\eta_t \nu}{\sigma_t} \frac{\gamma_t}{U'(X_{t-}^{\pi^*} + Y_{t-})} \Big) dW_t + \frac{\gamma_t}{U'(X_{t-}^{\pi^*} + Y_{t-})} dn_t \right). \quad (3.18)$$

Since we have assumed that $\frac{1}{U'(X_-^{\pi^*} + Y_-)} \cdot U'(X^{\pi^*} + Y)$ is a square integrable martingale, by the martingale representation theorem and the hypotheses on boundedness of η, μ and σ^{-1} we deduce that $\frac{\gamma}{U'(X_-^{\pi^*} + Y_-)} \in \mathbb{H}^2$. Using this fact, together with the hypotheses (H1) on ARA and $Z \in \mathbb{H}^2$, since (3.15) can be rewritten as

$$\pi^* = -\frac{1}{\sigma} \frac{U'(X_-^{\pi^*} + Y_-)}{U''(X_-^{\pi^*} + Y_-)} \left(\frac{\mu}{\sigma} + \frac{\eta \nu}{\sigma} \frac{\gamma}{U'(X_-^{\pi^*} + Y_-)} \right) - \frac{Z}{\sigma},$$

we get that $\pi^* \in \Pi_x$.
Now we are are left to show that π^* is optimal. From (3.18), we observe that the positive martingale $U'(X^{\pi^*} + Y)$ can be written as the Doleans exponential of the martingale

$$M = -\Big(\frac{\mu}{\sigma} + \frac{\eta \nu}{\sigma} \frac{\gamma}{U'(X_-^{\pi^*} + Y_-)} \Big) \cdot W + \frac{\gamma}{U'(X_-^{\pi^*} + Y_-)} \cdot n,$$

where $\frac{\gamma}{U'(X_-^{\pi^*} + Y_-)} > -1$ by the definition of γ and since $U'(X^{\pi^*} + Y)$ is positive.
Since U is a concave function, for any $\pi \in \Pi_x$ we have that

$$U(X_T^{\pi} + H) - U(X_T^{\pi^*} + H) \leq U'(X_T^{\pi^*} + H) \left(X_T^{\pi} - X_T^{\pi^*} \right). \quad (3.19)$$

Thus we define the measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'(X_T^{\pi^*} + H)}{\mathbb{E}\big(U'(X_T^{\pi^*} + H)\big)},$$

and taking the expectation in (3.19) we write

$$\mathbb{E}\left(\frac{U(X_T^{\pi}+H)-U(X_T^{\pi^*}+H)}{\mathbb{E}\big(U'(X_T^{\pi^*}+H)\big)}\right)\le \mathbb{E}^{\mathbb{Q}}\left(X_T^{\pi}-X_T^{\pi^*}\right)$$
$$=\mathbb{E}^{\mathbb{Q}}\left(\int_0^T(\pi_t-\pi_t^*)\frac{dS_t}{S_{t^-}}\right)=0, \tag{3.20}$$

where the last equality is due to the fact that, by the Girsanov theorem and the admissibility of the strategies π and π^*, the stochastic integral $\int_0^\cdot(\pi_t-\pi_t^*)\frac{dS_t}{S_{t^-}}$ is a $\mathbb{Q}$-martingale. In fact, $\frac{1}{S_-}\cdot S$ is a $\mathbb{Q}$-martingale since its predictable quadratic covariation with M is

$$\left\langle\frac{1}{S_-}\cdot S, M\right\rangle=\left\langle\sigma\cdot W+\eta\cdot n, -\left(\frac{\mu}{\sigma}+\frac{\eta\nu}{\sigma}\frac{\gamma}{U'(X_-^{\pi^*}+Y_-)}\right)\cdot W+\frac{\gamma}{U'(X_-^{\pi^*}+Y_-)}\cdot n\right\rangle$$
$$=-\int_0^\cdot\mu_s ds.$$

Moreover, since π and π^* are in Π_x, we get that

$$\int_0^\cdot(\pi_t-\pi_t^*)\frac{dS_t}{S_{t^-}}=\int_0^\cdot(\pi_t-\pi_t^*)(\mu_t dt+\sigma_t dW_t+\eta_t dn_t)$$

is a $\mathbb{Q}$-local martingale. Finally, by the following chain of inequalities we prove that it is a true $\mathbb{Q}$-martingale:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left(\sup_{0\le t\le T}\left|\int_0^t(\pi_s-\pi_s^*)\frac{dS_s}{S_{s^-}}\right|\right)\\
&\le C\,\mathbb{E}^{\mathbb{Q}}\left(\left(\int_0^T(\pi_t-\pi_t^*)^2(\sigma_t^2+\eta_t^2\nu)dt\right)^{\frac{1}{2}}\right)\\
&=C\,\mathbb{E}\left(\frac{U'(X_T^{\pi^*}+H)}{\mathbb{E}\big(U'(X_T^{\pi^*}+H)\big)}\left(\int_0^T(\pi_t-\pi_t^*)^2dt\right)^{\frac{1}{2}}\right)\\
&\le C\frac{\Big(\mathbb{E}\big((U'(X_T^{\pi^*}+H))^2\big)\Big)^{\frac{1}{2}}}{\mathbb{E}\big(U'(X_T^{\pi^*}+H)\big)}\left(\mathbb{E}\left(\int_0^T(\pi_t-\pi_t^*)^2dt\right)\right)^{\frac{1}{2}}<+\infty.
\end{aligned}$$

From (3.20), we have

$$\mathbb{E}\left(U(X_T^{\pi}+H)-U(X_T^{\pi^*}+H)\right)\le 0,\qquad \text{for any } \pi\in\Pi_x$$

which means that π^* is optimal. The proof is complete. □

Remark 3.1 It is worth noting that assumption (H2) in Theorem 3.2 which requires $\mathbb{E}[(U'(X_T^{\pi^*} + H))^2] < +\infty$ is only used to show that, for any $\pi \in \Pi_x$, $\frac{\pi}{S_-} \cdot S$ is a $\mathbb{Q}$-martingale. Therefore, it can be replaced by any other condition which ensures this requirement.

4 Pure Jump Model

In this section we study the case where the asset price is a pure jump process and show that for this model π^* can be written in an explicit form in terms of the forward backward SDE system solution.

We consider (2.1) with $\sigma^2 \equiv 0$, that is the price S is modeled by a pure jump process whose dynamics is

$$dS_t = S_{t^-}(\mu_t dt + \eta_t dn_t), \quad S_0 > 0.$$

We recall that the filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0, T])$ is generated by a standard Brownian motion W and a simple Poisson process N with intensity $\nu > 0$, which implies that the market is incomplete. Moreover, the coefficients μ and η are predictable and bounded with $\eta > -1$ and with the additional assumption $c_1 \le \frac{\mu}{\eta} \le c_2 < \nu$ which guarantees that $\frac{\mu}{\eta\nu}$ is bounded with the upper bound smaller than 1. Note that this assumption on the coefficients will be exploited to define π^* in (4.23) and to guarantee the martingality of M in (4.27). Moreover, since $\mathbb{E}\left(e^{C\langle -\frac{\mu}{\eta\nu} \cdot n\rangle_T}\right) < \infty$ for some $C \ge 1$, one can check that the Doleans exponential of $-\frac{\mu}{\eta\nu} \cdot n$ is a uniformly integrable martingale (see Protter [25]) providing the existence of a martingale measure for S.

Theorem 4.1 *Let $\pi^* \in \Pi_x$ be optimal for problem (2.3) and suppose (H1) holds. Under the assumptions of Proposition 2.1, there exists a solution (X, Y, Z, Ψ) of the following forward–backward system*

$$\begin{aligned} X_t =& x + \int_0^t \left(U'^{-1}\left(U'(X_{s^-} + Y_{s^-})\left(1 - \frac{\mu_s}{\eta_s \nu}\right)\right) - (X_{s^-} + Y_{s^-} + \Psi_s)\right) \\ &\times \left(\frac{\mu_s}{\eta_s} ds + dn_s\right) \end{aligned} \tag{4.21}$$

$$\begin{aligned} Y_t =& H - \int_t^T \left[\left(U'^{-1}\left(U'(X_{s^-} + Y_{s^-})\left(1 - \frac{\mu_s}{\eta_s \nu}\right)\right) - (X_{s^-} + Y_{s^-})\right)\right] \\ &\times \left(1 - \frac{\mu_s}{\eta_s \nu}\right) \nu ds \\ -& \int_t^T \left[\left(\Psi_s + \frac{U'(X_{s^-} + Y_{s^-})}{U''(X_{s^-} + Y_{s^-})}\right) \frac{\mu_s}{\eta_s} - \frac{1}{2} \frac{U'''(X_{s^-} + Y_{s^-})}{U''(X_{s^-} + Y_{s^-})} Z_s^2\right] ds \\ &- \int_t^T (Z_s dW_s + \Psi_s dn_s). \end{aligned} \tag{4.22}$$

Moreover, π^* takes on the form

$$\pi^* = \frac{1}{\eta}\left(U'^{-1}\left(U'(X_-^{\pi^*} + Y_-)\left(1 - \frac{\mu}{\eta\nu}\right)\right) - (\Psi + X_-^{\pi^*} + Y_-)\right), \tag{4.23}$$

and the optimal wealth process is equal to X.

Vice versa, suppose a solution of the system (4.21, 4.22) exists. Assume (H1) and (H2), (H3) with $\xi = X_T + H$, and $U'(X+Y)$ to be a positive martingale. If the strategy (4.23) is in the class Π_x then it is optimal.

Proof Using the same arguments as those for the jump-diffusion case, we can write

$$dY_t = -\left[\frac{1}{2}\frac{U'''(X_{t-}^{\pi^*} + Y_{t-})}{U''(X_{t-}^{\pi^*} + Y_{t-})}Z_t^2 - \left(\Psi_t + \pi_t^*\eta_t\right)\nu + \frac{1}{U''(X_{t-}^{\pi^*} + Y_{t-})}\gamma_t\nu + \pi_t^*\mu_t\right]dt$$
$$+ Z_t dW_t + \Psi_t dn_t, \quad Y_T = H, \tag{4.24}$$

which corresponds to (3.12) when $\sigma = 0$. Condition (3.15) now becomes

$$U'(X_-^{\pi^*} + Y_-)\mu + \gamma\eta\nu = 0 \qquad d\mathbb{P}\otimes dt - \text{a.e. on } [0, T], \tag{4.25}$$

where

$$\gamma = U'(\Psi + \pi^*\eta + X_-^{\pi^*} + Y_-) - U'(X_-^{\pi^*} + Y_-). \tag{4.26}$$

From (4.25) and (4.26), we deduce (4.23). Finally, plugging γ obtained from (4.25) and (4.23) into (4.24), we deduce (4.22).

For the converse, let us observe that the positive martingale $U'(X^{\pi^*} + Y)$ can be written as the Doleans exponential of the martingale

$$M = \frac{U''(X_-^{\pi^*} + Y_-)}{U'(X_-^{\pi^*} + Y_-)}Z \cdot W - \frac{\mu}{\eta\nu}\cdot n, \tag{4.27}$$

whose predictable quadratic covariation with $\frac{1}{S_-}\cdot S$ is $\langle \frac{1}{S_-}\cdot S, M\rangle = -\int_0^\cdot \mu_s ds$. Thus, by Girsanov Theorem, $\frac{1}{S_-}\cdot S$ is a martingale with respect to the $\mathbb{P}$-equivalent measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'(X_T^{\pi^*} + H)}{\mathbb{E}\big(U'(X_T^{\pi^*} + H)\big)}.$$

The conclusion for the optimality of π^* follows as in Theorem 3.2. □

As previously said, it is not easy to find a solution of the coupled system. In the next proposition, we consider the pure investment problem, i.e. $H = 0$, and find a sufficient condition for obtaining an explicit solution to the system (4.21, 4.22) with $Z = 0$. At the end of the next section, we show an example where this condition is satisfied.

Proposition 4.1 *Let $H = 0$ and suppose (H1) holds. Consider the system*

$$X_t = x + \int_0^t \frac{1}{ARA(X_{s^-})} \frac{a_s - \mu_s/\eta_s}{\mu_s - \eta_s \nu} \left(\mu_s ds + \eta_s dn_s\right), \tag{4.28}$$

$$Y_t = U'^{-1}\left(U'(X_t)e^{A_t}\right) - X_t, \tag{4.29}$$

where $a \in \mathbb{H}^2$ and $A_t = -\int_t^T a_s\, ds$, with A_0 deterministic. If $U'(X)e^A$ is a positive martingale, then (X, Y) gives a solution to the forward backward system (4.21, 4.22) with $Z = 0$. Moreover, under (H3) with $\xi = X_T$ the strategy

$$\pi^* = \frac{1}{ARA(X_-)} \frac{a - \mu/\eta}{\mu - \eta\nu} \tag{4.30}$$

is in Π_x and it is optimal.

Proof Applying Itô's formula to $U'(X_t)e^{A_t}$, and taking into account the definition of A_t and $n_t = N_t - \nu t$, we get

$$\begin{aligned}
d\left(U'(X_t)e^{A_t}\right) &= U'(X_{t^-})e^{A_t} a_t dt + U''(X_{t^-})e^{A_t} \frac{1}{ARA(X_{t^-})}\left(a_t - \frac{\mu_t}{\eta_t}\right) dt \\
&\quad + U'(X_t)e^{A_t} - U'(X_{t^-})e^{A_t} \\
&= U'(X_{t^-})e^{A_t}\left(\left(\frac{\mu_t}{\eta_t} + \frac{\nu\gamma_t}{e^{A_t}U'(X_{t^-})}\right) dt + \frac{\gamma_t}{e^{A_t}U'(X_{t^-})} dn_t\right)
\end{aligned} \tag{4.31}$$

where $\gamma_t \Delta N_t = e^{A_t}\left(U'(X_t) - U'(X_{t^-})\right)$.

Since we assumed that $U'(X + Y) = U'(X)e^A$ is a martingale, we find $\gamma = -\frac{\mu}{\eta\nu} e^A U'(X_-)$ and therefore

$$d\left(U'(X_t)e^{A_t}\right) = -U'(X_{t^-})e^{A_t} \frac{\mu_t}{\eta_t \nu} dn_t. \tag{4.32}$$

From (4.29) we get $Y_T = 0$ and, by Itô's formula, using (4.31) it follows

$$\begin{aligned}
dY_t =& \frac{U'(X_{t^-})e^{A_t}}{U''\left(U'^{-1}\left(U'(X_{t^-})e^{A_t}\right)\right)} \frac{\mu_t}{\eta_t} dt + \left[U'^{-1}\left(U'(X_t)e^{A_t}\right) - U'^{-1}\left(U'(X_{t^-})e^{A_t}\right)\right] \\
& - \frac{1}{ARA(X_{t^-})} \frac{a_t - \mu_t/\eta_t}{\mu_t - \eta_t \nu} \left(\mu_t dt + \eta_t dn_t\right) \\
=& \frac{U'(X_{t^-})e^{A_t}}{U''\left(U'^{-1}\left(U'(X_{t^-})e^{A_t}\right)\right)} \frac{\mu_t}{\eta_t} dt \\
& + \left[U'^{-1}\left(U'(X_{t^-})e^{A_t}\left(1 - \frac{\mu_t}{\eta_t \nu}\Delta N_t\right)\right) - U'^{-1}\left(U'(X_{t^-})e^{A_t}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{ARA(X_{t-})}\frac{\mu_t-\eta_t a_t}{\mu_t-\eta_t\nu}\left(\frac{\mu_t}{\eta_t}dt+dn_t\right)\\
=&\frac{U'(X_{t-})e^{A_t}}{U''\left(U'^{-1}\left(U'(X_{t-})e^{A_t}\right)\right)}\frac{\mu_t}{\eta_t}dt\\
&+\left[U'^{-1}\left(U'(X_{t-})e^{A_t}\left(1-\frac{\mu_t}{\eta_t\nu}\right)\right)-U'^{-1}\left(U'(X_{t-})e^{A_t}\right)\right]\Delta N_t\\
&+\frac{1}{ARA(X_{t-})}\frac{\mu_t-\eta_t a_t}{\mu_t-\eta_t\nu}\left(\frac{\mu_t}{\eta_t}dt+dn_t\right)\\
=&\left(\frac{U'(X_{t-})e^{A_t}}{U''\left(U'^{-1}\left(U'(X_{t-})e^{A_t}\right)\right)}+\frac{1}{ARA(X_{t-})}\frac{\mu_t-\eta_t a_t}{\mu_t-\eta_t\nu}\right)\frac{\mu_t}{\eta_t}dt\\
&+\left[U'^{-1}\left(U'(X_{t-})e^{A_t}\left(1-\frac{\mu_t}{\eta_t\nu}\right)\right)-U'^{-1}\left(U'(X_{t-})e^{A_t}\right)\right]\left(1-\frac{\mu_t}{\eta_t\nu}\right)\nu dt\\
&+\left[U'^{-1}\left(U'(X_{t-})e^{A_t}\left(1-\frac{\mu_t}{\eta_t\nu}\right)\right)-U'^{-1}\left(U'(X_{t-})e^{A_t}\right)\right]\frac{\mu_t}{\eta_t}dt\\
&+\left[\frac{1}{ARA(X_{t-})}\frac{\mu_t-\eta_t a_t}{\mu_t-\eta_t\nu}+U'^{-1}\left(U'(X_{t-})e^{A_t}\left(1-\frac{\mu_t}{\eta_t\nu}\right)\right)\right.\\
&\left.-U'^{-1}\left(U'(X_{t-})e^{A_t}\right)\right]dn_t,
\end{aligned}
$$

which corresponds to

$$
\begin{aligned}
dY_t=&\left[\left(U'^{-1}\left(U'(X_{t-}+Y_{t-})\left(1-\frac{\mu_t}{\eta_t\nu}\right)\right)-(X_{t-}+Y_{t-})\right)\right]\left(1-\frac{\mu_t}{\eta_t\nu}\right)\nu dt\\
&+\left(\Psi_t+\frac{U'(X_{t-}+Y_{t-})}{U''(X_{t-}+Y_{t-})}\right)\frac{\mu_t}{\eta_t}dt+\Psi_t dn_t
\end{aligned}
$$

with

$$
\Psi=\frac{1}{ARA(X_-)}\frac{\mu-\eta a}{\mu-\eta\nu}+U'^{-1}\left(U'(X_-)e^{A}\left(1-\frac{\mu}{\eta\nu}\right)\right)-U'^{-1}\left(U'(X_-)e^{A}\right).
$$

Therefore, Y satisfies (4.22) with $Z=0$.
The admissibility of π^* follows from (H1), the assumptions on a_t and on the model coefficients. In order to check the optimality of the strategy, by the converse part of Theorem 4.1, we are left to prove that $U'(X)e^A$ is a positive martingale and $\mathbb{E}\left[\left(U'(X_T)\right)^2\right]<+\infty$. But this is true since, by (4.32), $U'(X)e^A$ is the Doleans exponential of the martingale $M=-\frac{\mu}{\eta\nu}\cdot n$ whose predictable quadratic variation $\langle M\rangle=\int_0^{\cdot}\frac{{\mu_s}^2}{\eta_s^2\nu}ds$ is bounded. In fact, using Novikov condition, $\mathcal{E}(2M)$ is a uniformly integrable martingale and, thus, $[\mathcal{E}_T(M)]^2=\mathcal{E}_T(2\,M)e^{\langle M\rangle_T}$ has finite expectation. □

5 Exponential Utility

In this last section we tailor the results obtained previously to the case of the exponential utility, which has been extensively studied in the literature (e.g., [4, 5, 13, 23, 27]). It can be easily checked that with this particular choice of utility function, the forward-backward system decouples and the problem reduces to the study of a backward stochastic differential equation.

For results for continuous price models, see, e.g., [9, 26] in Brownian setting, and [14, 15, 20] for more general continuous semimartingale models. For models allowing jumps, we quote among others [1, 21, 22] and [18] for the pure jump model. We consider the exponential utility $U(x) = -e^{-\delta x}$ with risk aversion parameter $\delta \in (0, +\infty)$ and a bounded random liability H. In this case the evolution for

$$Y = -\frac{1}{\delta}\log\frac{\alpha}{\delta} - X^{\pi^*}$$

does not depend on the wealth process X^{π^*} and is given by the following backward equation

$$\begin{aligned} dY_t = &\left((e^{-\delta(\Psi_t+\pi_t^*\eta_t)} - 1)\frac{\nu}{\delta} + \frac{1}{2}\delta(Z_t + \pi_t^*\sigma_t)^2 + (\Psi_t + \pi_t^*\eta_t)\nu - \pi_t^*\mu_t\right)dt \\ &+ Z_t dW_t + \Psi_t dn_t, \end{aligned}$$

with final condition $Y_T = H$.
Moreover, Eq. (3.16) in the proof of Theorem 3.1 for the optimal strategy π^* can be rewritten as

$$\mu_t - \delta(Z_t + \pi_t^*\sigma_t)\sigma_t + \left(e^{-\delta(\Psi_t+\pi_t^*\eta_t)} - 1\right)\eta_t\nu = 0 \qquad d\mathbb{P}\otimes dt - \text{a.e. on } [0, T], \tag{5.33}$$

from which we can also deduce that the optimal strategy is independent on the wealth. Let us point out that, employing a notation similar to [21], the driver of the BSDE can be rewritten in the form

$$f_t(z, \psi) = \sup_{\pi\in\mathbb{R}}\left\{-\frac{\delta}{2}\left(\pi\sigma_t - (-z + \frac{\mu_t}{\delta\sigma_t})\right)^2 - [-\psi - \pi\eta_t]_\delta\right\} - \frac{\mu_t}{\sigma_t}z + \frac{\mu_t^2}{2\delta\sigma_t^2},$$

where, for $\psi \in \mathbb{R}$, $[\psi]_\delta = \frac{\nu}{\delta}\left(e^{\delta\psi} - 1 - \delta\psi\right)$. Then, using the results in [20], we deduce that there exists a solution to the BSDE with Y bounded and $Z, \Psi \in \mathbb{H}^2$.
In the pure jump case, the backward evolution of Y reduces to

$$dY_t = \left(\frac{1}{2}\delta Z_t^2 - \frac{\nu}{\delta}\left(1 - \frac{\mu_t}{\eta_t\nu}\right)\ln\left(1 - \frac{\mu_t}{\eta_t\nu}\right) + \frac{\mu_t}{\eta_t}\left(\Psi_t - \frac{1}{\delta}\right)\right)dt + Z_t dW_t + \Psi_t dn_t.$$

Moreover, the optimal strategy π^* can be written in an explicit form, using (5.33) with $\sigma = 0$, as stated in the next proposition.

Proposition 5.1 *A solution of the backward equation*

$$
\begin{aligned}
Y_t = H - \int_t^T \left(\frac{1}{2}\delta Z_s^2 - \frac{\nu}{\delta}\left(1-\frac{\mu_s}{\eta_s \nu}\right)\ln\left(1-\frac{\mu_s}{\eta_s \nu}\right) + \frac{\mu_s}{\eta_s}\left(\Psi_s - \frac{1}{\delta}\right)\right) ds \\
- \int_t^T (Z_s dW_s + \Psi_s dn_s) \qquad (5.34)
\end{aligned}
$$

exists with Y *bounded and* $Z, \Psi \in \mathbb{H}^2$ *and the strategy* π^*

$$
\pi^* = -\frac{1}{\eta}\left(\frac{1}{\delta}\ln\left(1-\frac{\mu_t}{\eta_t \nu}\right) + \Psi\right) \qquad (5.35)
$$

is in Π_x. *In addition, if* $e^{-\delta(X^{\pi^*}+Y)}$ *is a positive martingale and either* Z *or* Ψ *is bounded, then* (5.35) *is optimal.*

Proof The existence of a solution to (5.34) with Y bounded and $Z, \Psi \in \mathbb{H}^2$ is proved in Theorem 1 of Antonelli and Mancini [1]. This theorem requires two main assumptions on the driver f of (5.34), which we can rewrite as

A1) f is measurable and predictable and satisfies $d\mathbb{P} \otimes dt$—a.e.

$$
-\lambda_t - \frac{\delta}{2}z^2 - [-\psi]_\delta \le f_t(z,\psi) \le \lambda_t + \frac{\delta}{2}z^2 + [\psi]_\delta \qquad (5.36)
$$

for a strictly positive predictable process λ such that $\operatorname{essup}_\omega \int_0^T \lambda_t \, dt < +\infty$;

A2) f verifies

$$
f_t(z,\psi) - f_t(z,\psi') \le \zeta_t^{z,\psi,\psi'}(\psi - \psi')\nu \qquad (5.37)
$$

where the process $\zeta^{z,\psi,\psi'}$ is such that $D_1 \le \zeta_t^{z,\psi,\psi'} \le D_2$, with $-1 < D_1 \le 0$ and $D_2 \ge 0$.

In order to check these assumptions we rewrite the driver f in the form

$$
f_t(z,\psi) = \sup_{\pi\in\mathbb{R}} \{\pi\mu_t - [-\psi - \pi\eta_t]_\delta\} - \frac{1}{2}\delta z^2. \qquad (5.38)
$$

Since $\pi = 0 \in \mathbb{R}$, from (5.38) we deduce the validity of the left inequality in (5.36). To prove the right inequality, we consider the function $F(\psi) = f_t(z,\psi) - \frac{\delta}{2}z^2 - [\psi]_\delta$. Using the explicit form of the driver in (5.34), we can rewrite $F(\psi)$ as

$$
F(\psi) = \frac{\nu}{\delta}\left(\left(\ln\left(1-\frac{\mu}{\eta\nu}\right) + 1 + \delta\psi\right) - e^{\delta\psi}\right),
$$

whose maximum is attained at $\psi^* = \frac{1}{\delta}\ln\left(1-\frac{\mu}{\eta\nu}\right)$ and holds

$$F(\psi^*) = 2\,\frac{\nu}{\delta}\left(1-\frac{\mu}{\eta\nu}\right)\ln\left(1-\frac{\mu}{\eta\nu}\right).$$

We deduce that in (5.36) we can take $\lambda = 2\,\frac{\nu}{\delta}\left(1-\frac{\mu}{\eta\nu}\right)\left|\ln\left(1-\frac{\mu}{\eta\nu}\right)\right|$, which is a bounded predictable and strictly positive process, thus the assumption (A1) is verified. Using again the explicit form of the driver, (5.37) can be rewritten in the form

$$(\psi-\psi')\left(\zeta_t^{z,\psi,\psi'}+\frac{\mu}{\eta\nu}\right)\geq 0,$$

and (A2) is easily verified choosing $\zeta^{z,\psi,\psi'} = -\frac{\mu}{\eta\nu}$, $D_1 = -1$ and a suitable constant D_2, which can be found since $\frac{\mu}{\eta}$ is bounded.
From the existence of the solution with Y bounded and $Z, \Psi \in \mathbb{H}^2$ and from (5.35), thanks to the standing assumptions on the model, we deduce that $\pi^* \in \Pi_x$.

Finally, from the vice versa of Theorem 4.1, the optimality of π^* is guaranteed if we assume that either Ψ or Z is bounded. In fact, (H1) and (H3) trivially hold true, whereas (H2) reduces to check that $\mathbb{E}\left[e^{-2\delta(X_T^{\pi^*}+H)}\right]$ is finite.

First we suppose Ψ bounded, so is π^*. Taking into account also the boundedness of H, we get

$$\mathbb{E}\left[e^{-2\delta(X_T^{\pi^*}+H)}\right]\leq C_2\mathbb{E}\left[e^{C_1 N_T}\right]<+\infty,$$

where C_1, C_2 are suitable constants whose specific values are irrelevant.
Assume now Z bounded. Since the positive martingale $e^{-\delta(X^{\pi^*}+Y)}$ is the Doleans exponential of the martingale M defined in (4.27), we get $[\mathcal{E}_T(M)]^2 = \mathcal{E}_T(2\,M)e^{\langle M\rangle_T}$. Moreover, if Z is bounded then

$$\langle M\rangle_T = \int_0^T \delta^2 Z_s^2\,ds + \int_0^T \frac{{\mu_s}^2}{\eta_s^2\nu}\,ds$$

is bounded. Using Novikov condition, $\mathcal{E}(2M)$ is a uniformly integrable martingale and, therefore,

$$\mathbb{E}\left[e^{-2\delta(X_T^{\pi^*}+H)}\right] = \mathbb{E}\left[[\mathcal{E}_T(M)]^2\right]<+\infty.$$

□

Remark 5.1 We notice that, in the case that Z is bounded, the martingale assumption on $e^{-\delta(X^{\pi^*}+Y)}$ can be omitted since it is automatically satisfied.

Remark 5.2 In [18], the exponential utility maximization problem is studied within a BSDE framework and in a Lévy-driven pure jump asset model. Existence of optimal strategies are proved using BMO arguments and assumptions on the solutions of the BSDEs involved which imply the boundedness of the strategies. Although our market model is simpler, Proposition 5.1 represents an attempt to establish conditions for the existence of optimal strategies possibly not bounded.

We conclude the section with an example for the pure investment problem, where the hypotheses of Proposition 4.1 are automatically satisfied.

Proposition 5.2 *If $H = 0$ and $\frac{\mu}{\eta}$ is deterministic then the strategy*

$$\pi^* = -\frac{1}{\delta\eta} \ln\left(1 - \frac{\mu}{\eta\nu}\right)$$

is in Π_x and it is optimal.

Proof We choose in Proposition 4.1,

$$a = \left(\frac{\mu}{\eta\nu} + \left(1 - \frac{\mu}{\eta\nu}\right) \ln\left(1 - \frac{\mu}{\eta\nu}\right)\right)\nu, \tag{5.39}$$

which is a positive bounded process by the assumptions on the model. Then, the processes X and Y become

$$\begin{aligned} X_t =& x + \int_0^t \frac{1}{ARA(X_{s^-})} \frac{a_s - \mu_s/\eta_s}{\mu_s - \eta_s \nu} (\mu_s ds + \eta_s dn_s) \\ =& x + \int_0^t \left(-\frac{1}{\delta\eta_s}\right) \ln\left(1 - \frac{\mu_s}{\eta_s \nu}\right) (\mu_s ds + \eta_s dn_s), \\ Y_t =& U'^{-1}\left(U'(X_t)e^{A_t}\right) - X_t = -\frac{1}{\delta} A_t = \frac{1}{\delta} \int_t^T a_s ds. \end{aligned}$$

On the other hand, we can observe that (5.39) represents the unique choice for a in Proposition 4.1 which makes $U'(X)e^A$ a martingale. In fact, $U'(X)e^A$ is a martingale if and only if

$$e^A \left(U'(X) - U'(X_-)\right) = -\frac{\mu}{\eta\nu} e^A U'(X_-) \Delta N,$$

i.e.

$$\delta e^{-\delta X_- + A} \left(e^{-\delta\left(\frac{1}{\delta} \frac{a - \mu/\eta}{\mu - \eta\nu} \eta\right)} - 1\right) = -\frac{\mu}{\eta\nu} \delta e^{-\delta X_- + A},$$

from which we deduce $\frac{\mu - \eta a}{\mu - \eta\nu} = \ln\left(1 - \frac{\mu}{\eta\nu}\right)$ and thus (5.39). The positivity of $U'(X)e^A$ immediately follows from the conditions on the model.

Since all the assumptions of Proposition 4.1 are satisfied, the processes X and Y give a solution to the forward backward system (4.21, 4.22) with $Z = 0$ and

$$\begin{aligned}\Psi &= \frac{1}{ARA(X_-)}\frac{\mu-\eta a}{\mu-\eta\nu} + U'^{-1}\left(U'(X_-)e^A\left(1-\frac{\mu}{\eta\nu}\right)\right) - U'^{-1}\left(U'(X_-)e^A\right)\\ &= \frac{1}{\delta}\ln\left(1-\frac{\mu}{\eta\nu}\right) - \frac{1}{\delta}\ln\left(e^{-\delta X_-+A}\left(1-\frac{\mu}{\eta\nu}\right)\right) + \frac{1}{\delta}\ln\left(e^{-\delta X_-+A}\right) = 0.\end{aligned}$$

Finally, the strategy is

$$\pi^* = \frac{1}{ARA(X_-)}\frac{a-\mu/\eta}{\mu-\eta\nu} = -\frac{1}{\delta\eta}\ln\left(1-\frac{\mu}{\eta\nu}\right).$$

□

Remark 5.3 We notice that in the proposition above the hypothesis on $\frac{\mu}{\eta}$ can be weakened by assuming deterministic $\int_0^T a_s ds$, with the choice of a in (5.39).

Remark 5.4 In this paper, the theory of [8, 28] has been developed for a market model with Poissonian jumps focusing on some simple but analytically tractable examples, which exhibit a possible interpretation of the solution in terms of the model parameters. Future research directions may include theoretical generalizations of the market price model by considering a general jump measure in the dynamics. Moreover, utilities defined on the positive half line, such as the power and the logarithmic, can be investigated.

6 Appendix

This result is a generalization of Lemma 1 in [6].

Theorem 6.1 *Let A be an open subset of $\mathbb{R}^n$.*
Let us consider a function $F \in C^1(A \times \mathbb{R}; \mathbb{R})$ and suppose there exists a function $g : A \to \mathbb{R}$ such that one of the following conditions holds:

1. $\frac{\partial}{\partial y}F(\mathbf{x}, y) > g(\mathbf{x}) > 0$, $\forall(\mathbf{x}, y) \in A \times \mathbb{R}$,
2. $\frac{\partial}{\partial y}F(\mathbf{x}, y) < g(\mathbf{x}) < 0$ $\forall(\mathbf{x}, y) \in A \times \mathbb{R}$.

Then there exists a function $G \in C^1(A; \mathbb{R})$ such that $F(\mathbf{x}, G(\mathbf{x})) = 0$, $\forall \mathbf{x} \in A$.

Proof We start by proving that for any $\mathbf{x} \in A$ there exists $y(\mathbf{x}) \in \mathbb{R}$ such that $F(\mathbf{x}, y(\mathbf{x})) = 0$.

In order to do this, let us suppose that condition 1 holds. We can proceed in the same way if condition 2 is assumed.

Let us fix $\mathbf{x} \in A$. If $F(\mathbf{x}, 0) = 0$, there is nothing to check. Otherwise, by Lagrange mean value theorem, we have

$$F\left(\mathbf{x}, \frac{|F(\mathbf{x},0)|}{g(\mathbf{x})}\right) = F(\mathbf{x},0) + \frac{\partial}{\partial y}F(\mathbf{x},y)\Big|_{y=y_1(\mathbf{x})}\frac{|F(\mathbf{x},0)|}{g(\mathbf{x})} > F(\mathbf{x},0) + |F(\mathbf{x},0)| \geq 0$$

and

$$F\left(\mathbf{x}, -\frac{|F(\mathbf{x},0)|}{g(\mathbf{x})}\right) = F(\mathbf{x},0) - \frac{\partial}{\partial y}F(\mathbf{x},y)\Big|_{y=y_2(\mathbf{x})}\frac{|F(\mathbf{x},0)|}{g(\mathbf{x})} < F(\mathbf{x},0) + |F(\mathbf{x},0)| \leq 0,$$

where $0 < y_1(\mathbf{x}) < \frac{|F(\mathbf{x},0)|}{g(\mathbf{x})}$ and $-\frac{|F(\mathbf{x},0)|}{g(\mathbf{x})} < y_2(\mathbf{x}) < 0$.

Then, by the intermediate value theorem, there exists $-\frac{|F(\mathbf{x},0)|}{g(\mathbf{x})} < y(\mathbf{x}) < \frac{|F(\mathbf{x},0)|}{g(\mathbf{x})}$ such that $F(\mathbf{x}, y(\mathbf{x})) = 0$.

Let us observe that, since $F(\mathbf{x}, \cdot)$ is strictly monotone, $y(\mathbf{x})$ is univocally determined.

We are left with the task of proving that the function G defined by $G(\mathbf{x}) = y(\mathbf{x})$ is continuously differentiable on A. Let us fix $\mathbf{x}_0 \in A$. Since $F(\mathbf{x}_0, G(\mathbf{x}_0)) = 0$, by the implicit function theorem we can find a neighborhood $U_{\mathbf{x}_0}$ of $\mathbf{x}_0$ and a function $\varphi_{\mathbf{x}_0} \in C^1(U_{\mathbf{x}_0}; \mathbb{R})$ such that $F(\mathbf{x}, \varphi_{\mathbf{x}_0}(\mathbf{x})) = 0$, for any $\mathbf{x} \in U_{\mathbf{x}_0}$. Considering that also F$(\mathbf{x}, G(\mathbf{x})) = 0$ and F is strictly monotone, we deduce that $G = \varphi_{\mathbf{x}_0}$, therefore continuously differentiable, on $U_{\mathbf{x}_0}$. Due to the arbitrariness of $\mathbf{x}_0$ we can then conclude that $G \in C^1(A; \mathbb{R})$ and get the thesis. $\square$

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Declarations

Conflict of interest There is no conflicts of interest to declare.

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