The Pyramid Cell and its Associated Scalar and Vector Bases for FEM and MoM Applications

# The Pyramid Cell and its Associated Scalar and Vector Bases for FEM and MoM Applications

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***Abstract*—Curl- and divergence-conforming vector bases for the pyramid have been recently obtained using a multiplicative constructive technique, that is by multiplying the lowest order vector functions by polynomial sets complete up to a given order. So far the technique has been successfully employed to obtain hierarchical bases of arbitrarily high order, which will be illustrated at the conference, and is now used to produce interpolatory vector bases. This presentation briefly summarizes the theoretical results produced up to now and the difficulties that still have to be overcome in order to obtain the interpolatory bases as well. Preliminary results will be presented at the conference.**

## I. Introduction

Three-dimensional electromagnetic codes can model complicated geometries using meshes made up of identically shaped cells, for example using only tetrahedrons or only bricks (i.e. hexahedrons). However, modern codes and solvers should be more flexible and use hybrid meshes formed by cells of different shapes, that is a mixture of tetrahedrons, bricks, prisms and pyramids. This requires the use and possibly the development of very effective and ready-to-use meshers (not always within everyone's reach). Hybrid meshes can often be obtained more simply by refining an extremely loose *"starting"* mesh with cells of different shapes, as it occurs using $h$-adaptive techniques. Before doing this, however, one has to overcome several difficulties encountered both in the definition of the scalar shape functions that describe curved or distorted cells, and in the definition of vector basis functions for the numerical solution of differential and/or integral equations. As a matter of fact, the literature devoted to the use of pyramidal cells is relatively recent and far from vast (cf. [1]–[3]), unlike that concerning the use of tetrahedral, prismatic or brick-shaped cells, of which everything is known by now (see for example [4]). Since *hierarchical* vector bases of high order for the pyramid have been described very recently in [2], [3], in what follows we can concentrate only on the problems related to the development and use of the *interpolatory* shape functions and the *interpolatory* vector basis functions for pyramids, of which nothing is yet available in the literature. The presentation of new results on interpolatory functions for the pyramid is postponed to the oral presentation of this paper at the conference. The main findings related to the pyramid's hierarchical bases will also be presented at the conference.

## II. Interpolatory Shape Functions for Pyramids

All cells of identical shape, for example all the tetrahedrons in a given mesh, are obtained by mapping a single, well-defined parent cell into the various cells of the observer domain, which we call the child domain. Thus, all tetrahedrons derive from the same parent cell, and there is only one (different) parent cell for all prisms, one for all the bricks, and also one parent cell for all pyramids. The interested reader can find in [4] the interpolatory shape functions that define straight or curved tetrahedral, prismatic and brick cells of any order, together with higher order vector basis of the curl- and divergence-conforming kind, both hierarchical and interpolatory.

In particular, the shape functions that we use for the aforementioned cells are scalar polynomials of the parent variables that interpolate a well-defined set of points of the parent cell [4]. It is by assigning the position in the child space of the points of this interpolatory grid that the (straight or distorted) shape of a child cell is determined. Note that in the vast majority of applications it is usually sufficient to use low-order shape functions (say of the second or at most third order) which give less problems because they require a relatively small number of interpolation points. In fact, for higher orders, great care must be taken to ensure that the cells are not *warped*, that is, with a Jacobian of the transformation from the parent space to the child space that changes sign within the cell. In any case, to obtain conforming meshes, the interpolation points on a face of the pyramid in common with that of an adjacent cell (perhaps of different shape) must be the same for both cells, with continuous shape functions on the common face. That is, the shape functions of the pyramid must necessarily simplify into polynomials of the parent variables on the pyramid faces.

Unfortunately, however, the functions for the pyramids are considerably more complicated because its apex is the common point of four (triangular) faces and not of three, as instead it happens for the vertices of the other differently shaped cells. As discussed in [3], the shape functions of the pyramid on its bounding faces are, by continuity, polynomials of the parent variables which however become fractional functions of the same parent variables within the pyramid. To obtain

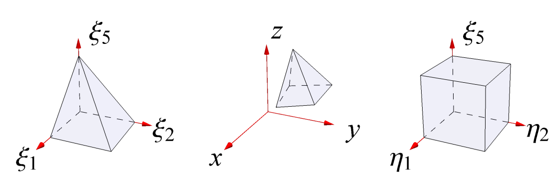


Fig. 1. The shape functions map the parent pyramid on the left to the child pyramid in the center. In the grandparent space ($\boldsymbol{\eta}$, $\xi_5$) the shape functions and the basis functions take polynomial form, while the pyramid is described by the cubic cell shown on the right. Figure taken from [2].

polynomial shape functions we need a variables transformation to work in a new grandparent space, where the pyramid is mapped by a grandparent cube (see Fig. 1). In the grandparent space the shape functions assume polynomial form and, with a few more expedients, the same happens to the vector basis functions [2], [3]. Although we know that in the grandparent space the shape functions are interpolatory scalar polynomials which simplify into polynomials of the parent variables on the pyramid faces, it is far from simple to derive these shape functions in general by simple algorithms. Fortunately, as said, shape functions of higher than third order are almost never necessary, so we deem it will suffice to show the shape functions up to third order at the conference.

## III. Interpolatory Vector Bases for Pyramids

As explained in [2], [3], for the pyramid, as for all the other elements, we now know how to build hierarchical vector bases of arbitrarily high order; so why should we also bother building a pyramid's interpolatory vector bases?

Aside from the fact that this complements the interpolatory families given in [4], the main motivation for this further effort is due to the fact that codes using the higher order interpolatory vector bases of [4] are readily obtained by adding a couple of subroutines and a few loop statements to the codes using the lowest order basis functions, i.e. those of zero order. This is because in [4] we routinely and repeatedly use Silvester's one-dimensional interpolatory polynomials to obtain the higher order vector bases. Of course, for the pyramid things can get more complicated. However, if it were enough to add a few routines to a zero-order code to use higher-order pyramids, we would have obtained in a very simple way a code able to provide useful results to validate more sophisticated codes that use higher order hierarchical bases.

The construction technique to build the vector bases is the same used in [2]–[4], so that we derive the basis functions by groups, i.e., those based on the edges of the pyramid (this is a group that exists only for the curl-conforming bases), those based on the faces of the pyramid (these functions exist both for the curl- and for the divergence-conforming bases) and those based on the volume of the pyramid (that we call bubbles). All groups can be obtained by multiplying the lowest order vector functions by a polynomial set complete to the order $p$, so that the $p$-th order base is obtained eliminating the redundancies. This way of proceeding does not create many problems as long as we are satisfied with deriving the basis functions based on the face and edges of the pyramid, but we were unable to apply it to build the volume-based functions until we got the lowest-order fundamental bubbles reported in [2], [3].

After having obtained the fundamental bubbles, we have applied this constructive technique to obtain the interpolatory vector bases which seemed to us the simplest to obtain, namely the divergence-conforming ones. As we did for the other types of cells of different shapes, we have used Silvester polynomials extensively and, as we intend to show at the conference, it seems we can show that divergence-conforming interpolatory bases of arbitrarily high order for the pyramid can be obtained by implementing moderately simple algorithms, thereby adding few subroutines to a zero-order code.

We are now applying the same constructive technique to obtain the curl-conforming interpolatory bases for the pyramid; hopefully, at the conference, we will be able to show the main results for these latter bases too.

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