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Research Article

Multivariate conditional aging intensity functions and load-sharing models

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Abstract

The aging intensity functions analyze the aging property quantitatively, in the sense that the larger the aging intensity, the stronger the tendency of aging. They are useful tools to describe reliability properties of distributions. In the literature, the aging intensity functions have been studied in the univariate and bivariate case but without considering the possibility of observing a dynamic history. In this paper, the concept of aging intensity function is extended to the multivariate case by the use of the multivariate conditional hazard rate functions. Some properties of those functions are studied and a focus on the bivariate case is performed. Finally, the multivariate conditional aging intensity functions are studied for the order dependent version of the time-homogeneous load-sharing model and a study on the comparison among surviving components in a system is provided.

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1. Introduction

Let X be a non-negative random variable with probability density function (pdf) f, survival function $\overline{F}$ and hazard rate function r. The Aging Intensity (AI) function of X is defined as

$$L(t) = \frac{r(t)}{\frac{1}{t}\int_0^t r(x)dx} = \frac{-tf(t)}{\overline{F}(t)\log\overline{F}(t)}, \tag{1.1}$$

i.e., $L(t)$ is the ratio of the instantaneous failure rate $r(t)$ to the average failure rate in the interval $(0,t)$. Some useful explanations are presented in Section 2 where some basic references are presented. The survival function and the aging intensity function are strictly related each other, see [18] for details.

In the literature, the aging intensity functions have been mainly studied for univariate distributions, also with reference to inverse distributions [11]. Furthermore, they have been generalized in order to obtain some characterization results [2, 19]. Recently, the concept of aging intensity function has been extended to bivariate, absolutely continuous and discrete, distributions [20] without taking into consideration the possibility of conditioning on a dynamic observed hystory. The study of aging intensity functions for multivariate

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distributions is still an open issue. In this paper, we try to fill this gap by providing definitions of multivariate conditional aging intensity functions and studying some of their properties. As well as the definition of aging intensity function is based on the hazard rate function, the definition of the multivariate conditional aging intensity functions will be based on the multivariate conditional hazard rate functions whose definition is recalled below.

Let $X_1, \ldots, X_n$ be non-negative random variables with an absolutely continuous joint distribution. Such an assumption will be fixed along the paper. For a fixed index $j \in [n] = \{1, \ldots, n\}$ and $i_1, \ldots, i_k \in [n]$ with $j \notin I = \{i_1, \ldots, i_k\}$, and an ordered sequence $0 \leq t_1 \leq \cdots \leq t_k$, the multivariate conditional hazard rate function $\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k)$ is defined as follows:

$$\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}\left(X_j \leq t + \Delta t \,\middle|\, X_{i_1} = t_1, \ldots, X_{i_k} = t_k, \min_{h \notin I} X_h > t \right).$$

Furthermore, we use the notation

$$\begin{aligned} \lambda_j(t|\emptyset) &= \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}\left(X_j \leq t + \Delta t \,\middle|\, \min_{h \in [n]} X_h > t \right) \\ &= \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}\left(X_j \leq t + \Delta t \,|\, X_{1:n} > t \right). \end{aligned}$$

For further details about multivariate conditional hazard rate functions one may refer to [14,15]. See also the reviews presented in the papers [15,16].

Based on the definition of multivariate conditional hazard rate functions, here we introduce the multivariate conditional aging intensity functions. We observe that those functions will depend on the dynamic history, observed up to the calendar time t, for the random vector $(X_1, \ldots, X_n)$. In particular it is defined only for the components surviving at t and it depends on the failure times of the components which have failed before t (see Definition 3.1). The given definition entails that the multivariate conditional aging intensity functions establish the tendency to aging of random variables and allow us to make comparisons among surviving components at a fixed time. Moreover, it is possible to observe that the tendency of aging of a component proceeds continuously with the exception of the failure times of different components when it may undergo a sudden variation due to the stochastic dependence of components. Hence, it is of great interest to study the continuity of multivariate conditional aging intensity functions.

The structure of the paper is described as follows. An overview of the aging intensity functions in the univariate case is presented in Section 2. The definition of the multivariate conditional aging intensity functions is given in Section 3 in which some examples and properties are presented and a study on the continuity is provided. In Section 4, a focus on the bivariate case with formulas based on joint survival and density functions is given and a study related to Gumbel's type I bivariate exponential distribution is performed. The study of aging intensity for a well-known multivariate model, known as time-homogeneous load-sharing one, is presented in Section 5 together with examples of application of multivariate conditional aging intensity functions to make comparisons among random variables. Finally, in Section 6 conclusions are given.

2. Aging intensity functions

The notion of aging intensity (AI) function has been introduced in [6] as the ratio of instantaneous failure rate and a baseline failure rate, as recalled in (1.1). The evaluation of the AI function for some well known models is presented in [6] where it is also introduced the notion of average aging intensity in order to study models characterized by quasi-constant failure rate. Some properties of AI functions are presented in [10] where, in particular, a new stochastic order (aging intensity order) based on the AI functions is defined. More precisely, based on the fact that the larger the aging intensity, the stronger the tendency to aging, a random variable X is said to be smaller than another random variable Y in the AI order, denoted by $X \leq_{AI} Y$, if $L_X(t) \geq L_Y(t)$, for all $t > 0$. As mentioned above, the failure rate function, or the survival function, uniquely determines the AI function but not conversely. In fact, the AI function of a non-negative random variable determines a family of survival functions through the relation

$$\overline{F}(t) = \exp\left[\log k \exp\left(\int_a^t \frac{L(x)}{x} dx\right)\right], \quad t \in (0, +\infty),$$

where $k = \overline{F}(a)$ for some arbitrary chosen $a \in (0, +\infty)$, see [18] for more details and for the proof of this result. There are several families of distributions in which the parameter k reduces to be one of the model. For instance, if X follows the Weibull distribution, $X \sim W2(\alpha, \lambda)$, with survival function $\overline{F}(t) = \exp(-\lambda t^{\alpha})$, $t > 0$, then the AI function is constant and expressed as $L(t) = \alpha$. Then, $L(t) = \alpha$ determines the subfamily of the family of the Weibull distributions with fixed parameter α and varying parameter $\lambda > 0$. Based on these considerations, it is possible to use the shape of an estimated aging intensity function in order to discover the underlying distribution of some data. A survey of characterization results based on AI functions for different types of Weibull distributions is presented in [5].

Recently, the study of AI functions has been extended to other different fields. In fact, there has been a great interest on the study of quantile function. Then, a quantile-based aging intensity function is introduced in [17] in which some of its properties are presented and some stochastic comparisons of random variables are performed by using this measure. Furthermore, the problem of the local linear estimation of the conditional aging intensity function when the variable of interest is subject to random right-censored is analyzed in [8].

The study of aging intensity functions has been extended to some simply systems by preserving the assumption of independence. In [1] the authors have proved that if X is the lifetime of a series system formed by n independent components, then the aging intensity function of X satisfy

$$\min_{1 \leq i \leq n} L_{X_i}(x) \leq L_X(x) \leq \max_{1 \leq i \leq n} L_{X_i}(x),$$

where $L_{X_i}(\cdot)$ is the AI function of the i-th component. About parallel systems, they proved that if X and Y are the lifetimes of parallel systems with n and m independent and identically distributed components, then for $n > m$, $X \leq_{AI} Y$. However, the possibility of considering a mode of dependence among components is not yet foreseen and, in this perspective, the necessity of defining a more general form of aging intensity functions emerges.

3. Multivariate conditional aging intensity functions

Definition 3.1. Let $(X_1, \ldots, X_n)$ be a random vector whose components are non-negative random variables with an absolutely continuous joint distribution. For an ordered sequence $h_1, \ldots, h_j$, $I = \{h_1, \ldots, h_k\} \subset \{1, \ldots, n\}$, $k = |I|$, the Multivariate Conditional Aging Intensity (MCAI) function is defined as

$$L_j(t|h_1, \ldots, h_k; t_1, \ldots, t_k) = \frac{\lambda_j(t|h_1, \ldots, h_k; t_1, \ldots, t_k)}{\frac{1}{t}\sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \lambda_j(x|h_1, \ldots, h_i; t_1, \ldots, t_i) dx}, \tag{3.1}$$

where $0 \equiv t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} \equiv t$, $j \notin I$ and $\min_{l \notin I} X_l > t$. In the case in which $I = \emptyset$, the MCAI fucntion can be expressed as

$$L_j(t|\emptyset) = \frac{\lambda_j(t|\emptyset)}{\frac{1}{t}\int_0^t \lambda_j(x|\emptyset) dx}. \tag{3.2}$$

Remark 3.2. If $X_1, \ldots, X_n$ are independent, then MCAI functions reduce to the classical aging intensity functions since in this case the multivariate conditional hazard rates are equal to the hazard rates independently of I and the order of its elements. In fact,

$$\begin{aligned}\lambda_j(t|i_1,\ldots,i_k;t_1,\ldots,t_k) &= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\mathbb{P}\left(X_j \le t+\Delta t \,\middle|\, X_{i_1}=t_1,\ldots,X_{i_k}=t_k, \min_{h\notin I} X_h > t\right)\\ &= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\mathbb{P}\left(X_j\le t+\Delta t\,|X_j>t\,\right) = r_j(t).\end{aligned}$$

Then, from (3.1), we get

$$L_j(t|i_1,\ldots,i_k;t_1,\ldots,t_k) = \frac{r_j(t)}{\frac{1}{t}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} r_j(x)dx} = \frac{r_j(t)}{\frac{1}{t}\int_0^t r_j(x)dx} = L_j(t).$$

In a similar manner, we get $L_j(t|\emptyset) = L_j(t)$.

Remark 3.3. It is of interest to consider the comparison between two models with proportional multivariate conditional hazard rate functions. In fact, this assumption brings to models which preserve the monotonicity properties of a fixed hazard rate function. This circumstance can be interpreted as a generalization of the classical proportional hazard rate model introduced in [3] about the univariate case and it can be formalized as follows. Let $\mathbf{X} = (X_1,\ldots,X_n)$ and $\mathbf{Y} = (Y_1,\ldots,Y_n)$ be two random vectors. If there exists a constant $a>0$ such that $\lambda_j^{(\mathbf{Y})}(t|i_1,\ldots,i_{|I|};t_1,\ldots,t_{|I|}) = a\lambda_j^{(\mathbf{X})}(t|i_1,\ldots,i_{|I|};t_1,\ldots,t_{|I|})$ for all $0<t_1<\cdots<t_{|I|}<t$, $j\notin I = \{i_1,\ldots,i_{|I|}\}\subset\{1,\ldots,n\}$, then

$$L_j^{(\mathbf{X})}(t|i_1,\ldots,i_{|I|};t_1,\ldots,t_{|I|}) = L_j^{(\mathbf{Y})}(t|i_1,\ldots,i_{|I|};t_1,\ldots,t_{|I|}).$$

In the following, we study the continuity of the MCAI functions associated to a fixed component. The critical points are the ones in which the other components fail. In fact, if we consider a time between two consecutive failures, the expression of the MCAI function is given in (3.1) without changing the parameters and then its continuity is guaranteed by the continuity of multivariate conditional hazard rate functions that is assured under the assumption of absolutely continuous joint distribution. Moreover, the denominator of MCAI function is continuous with respect to t, and a discontinuity of the function can be caused only by a jump in the numerator. In the following proposition, an expression for the size of the jump discontinuity is given.

Proposition 3.4. *Let $(X_1,\ldots,X_n)$ be a random vector with non-negative components and $I=\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}$. Let $t_1,\ldots,t_k$ be the failure times of the components $i_1,\ldots,i_k$, respectively. Then, the size of the jump discontinuity at t_k of the MCAI function of component $j\notin I$, is given by*

$$\begin{aligned}&L_j(t_k|i_1,\ldots,i_k;t_1,\ldots,t_k) - L_j(t_k|i_1,\ldots,i_{k-1};t_1,\ldots,t_{k-1})\\ &\quad= \frac{\lambda_j(t_k|i_1,\ldots,i_k;t_1,\ldots,t_k) - \lambda_j(i_1,\ldots,i_{k-1};t_1,\ldots,t_{k-1})}{\frac{1}{t_k}\sum_{r=0}^{k-1}\int_{t_r}^{t_{r+1}}\lambda_j(x|h_1,\ldots,h_r;t_1,\ldots,t_r)dx}.\end{aligned} \tag{3.3}$$

Proof. From the definition of MCAI functions, we have

$$\begin{aligned}&L_j(t_k|i_1,\ldots,i_k;t_1,\ldots,t_k) - L_j(t_k|i_1,\ldots,i_{k-1};t_1,\ldots,t_{k-1})\\ &\quad= \frac{\lambda_j(t_k|i_1,\ldots,i_k;t_1,\ldots,t_k)}{\frac{1}{t_k}\sum_{r=0}^{k}\int_{t_r}^{t_{r+1}}\lambda_j(x|h_1,\ldots,h_r;t_1,\ldots,t_r)dx}\\ &\quad\quad- \frac{\lambda_j(i_1,\ldots,i_{k-1};t_1,\ldots,t_{k-1})}{\frac{1}{t_k}\sum_{r=0}^{k-1}\int_{t_r}^{t_{r+1}}\lambda_j(x|h_1,\ldots,h_r;t_1,\ldots,t_r)dx},\end{aligned} \tag{3.4}$$

hence, by observing that t_{k+1} is the point of evaluation of the MCAI function and then it is equal to t_k, the ratios in (3.4) have a common denominator and the thesis follows. $\square$

From the above proposition, we can conclude that the jumps of the MCAI functions, i.e., changes in the aging tendency, may occur only at the failure times of other components. The failure of a component may then produce a shock for a different component. However, not necessarily a component is affected by the failure of another one. For instance, if the components are independent the continuity of the MCAI functions is guaranteed also under failures. Hence, by using Proposition 3.4, we have the following corollary about the continuity of the MCAI functions.

Corollary 3.5. *Let* $(X_1, \ldots, X_n)$ *be a random vector with non-negative components and* $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$. *Then, for* $j \notin I$,

$$\lim_{t \to t_k^+} L_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \lim_{t \to t_k^-} L_j(t|i_1, \ldots, i_{k-1}; t_1, \ldots, t_{k-1})$$

if, and only if

$$\lambda_j(t_k|i_1, \ldots, i_k; t_1, \ldots, t_k) = \lambda_j(t_k|i_1, \ldots, i_{k-1}; t_1, \ldots, t_{k-1}).$$

Moreover, from the expression given in (3.3), it follows that the sign of the jump is determined by the difference $\lambda_j(t_k|i_1, \ldots, i_k; t_1, \ldots, t_k) - \lambda_j(i_1, \ldots, i_{k-1}; t_1, \ldots, t_{k-1})$, since the denominator in (3.3) is positive. Hence, the jump is upward if $\lambda_j(t_k|i_1, \ldots, i_k; t_1, \ldots, t_k) > \lambda_j(i_1, \ldots, i_{k-1}; t_1, \ldots, t_{k-1})$, i.e., if the failure of the component i_k at time t_k increases the hazard of component j, while it is downward if $\lambda_j(t_k|i_1, \ldots, i_k; t_1, \ldots, t_k) < \lambda_j(i_1, \ldots, i_{k-1}; t_1, \ldots, t_{k-1})$.

4. The bivariate case

In the applications, there are several situations in which a model can be described by two random variables with a certain mode of dependence. Hence, it is of interest to specialize the concept of MCAI functions for bivariate distributions. In the literature, it has been already presented a definition of the bivariate aging intensity function [20]. We remark that this definition is different from the one considered in this paper since it is based on the failure rates gradient defined in [7]. For a random vector (X_1, X_2) with joint survival function $\overline{F}(\cdot, \cdot)$, the failure rates gradient is defined as $(r_1(t_1, t_2), r_2(t_1, t_2))$ where

$$r_1(t_1, t_2) = -\frac{\partial}{\partial t_1} \log \overline{F}(t_1, t_2), \quad r_2(t_1, t_2) = -\frac{\partial}{\partial t_2} \log \overline{F}(t_1, t_2). \tag{4.1}$$

Hence, the existent bivariate aging intensity functions are defined as

$$\mathcal{L}_1(t_1, t_2) = \frac{r_1(t_1, t_2)}{\frac{1}{t_1} \int_0^{t_1} r_1(x, t_2) dx}, \quad \mathcal{L}_2(t_1, t_2) = \frac{r_2(t_1, t_2)}{\frac{1}{t_2} \int_0^{t_2} r_1(t_1, x) dx}. \tag{4.2}$$

As one can see, the above definition does not take in account the possibility of observing a dynamic history. In the following, we extend the concept of bivariate aging intensity by considering stochastic dependence and the possibility of observing a dynamic history. For a random vector of dimension two, (X_1, X_2), we have to consider four aging intensity functions depending on how many variables and which ones assume a value greater than t. If $X_1 > t$ and $X_2 = t_2 < t$ then we consider

$$L_1(t|2; t_2) = \frac{t\lambda_1(t|2; t_2)}{\int_0^{t_2} \lambda_1(x|\emptyset) dx + \int_{t_2}^t \lambda_1(x|2; t_2) dx},$$

if $X_2 > t$ and $X_1 = t_1 < t$ we have

$$L_2(t|1; t_1) = \frac{t\lambda_2(t|1; t_1)}{\int_0^{t_1} \lambda_2(x|\emptyset) dx + \int_{t_1}^t \lambda_2(x|1; t_1) dx},$$

and if $X_1, X_2 > t$ we consider

$$L_j(t|\emptyset) = \frac{t\lambda_j(t|\emptyset)}{\int_0^t \lambda_j(x|\emptyset) dx}, \quad j = 1, 2.$$

In the case in which $t_1 \geq t_2$, the joint probability density function $f(t_1,t_2)$ can be expressed in terms of the multivariate conditional hazard rate functions as

$$f(t_1,t_2)=\lambda_2(t_2|\emptyset)\lambda_1(t_1|2;t_2)\exp\left[-\int_0^{t_2}(\lambda_1(u|\emptyset)+\lambda_2(u|\emptyset))\,du-\int_{t_2}^{t_1}\lambda_1(u|2;t_2)du\right]. \tag{4.3}$$

From (4.3) we get

$$\log\left(\frac{f(t_1,t_2)}{\lambda_2(t_2|\emptyset)\lambda_1(t_1|2;t_2)}\right)=-\int_0^{t_2}(\lambda_1(u|\emptyset)+\lambda_2(u|\emptyset))\,du-\int_{t_2}^{t_1}\lambda_1(u|2;t_2)du,$$

and then

$$\int_0^{t_2}\lambda_1(u|\emptyset)du+\int_{t_2}^{t_1}\lambda_1(u|2;t_2)du=-\left[\int_0^{t_2}\lambda_2(u|\emptyset)du+\log\left(\frac{f(t_1,t_2)}{\lambda_2(t_2|\emptyset)\lambda_1(t_1|2;t_2)}\right)\right],$$

where we can observe that the LHS is the denominator of $L_1(t|2;t_2)$. Hence, we can express $L_1(t|2;t_2)$ in a different way as

$$L_1(t|2;t_2)=\frac{-t\lambda_1(t|2;t_2)}{\int_0^{t_2}\lambda_2(u|\emptyset)du+\log\left(\frac{f(t,t_2)}{\lambda_2(t_2|\emptyset)\lambda_1(t|2;t_2)}\right)}. \tag{4.4}$$

Then, by taking into account the following relations between the multivariate conditional hazard rate functions and the joint density function and the joint survival function,

$$\begin{aligned}
\lambda_1(t|\emptyset)&=\frac{-\frac{\partial}{\partial t_1}\overline{F}(t_1,t)|_{t_1=t}}{\overline{F}(t,t)},\ t\geq 0,\\
\lambda_2(t|\emptyset)&=\frac{-\frac{\partial}{\partial t_2}\overline{F}(t,t_2)|_{t_2=t}}{\overline{F}(t,t)},\ t\geq 0,\\
\lambda_1(t|2;t_2)&=\frac{f(t,t_2)}{-\frac{\partial}{\partial t_2}\overline{F}(t,t_2)},\ t>t_2\geq 0,\\
\lambda_2(t|1;t_1)&=\frac{f(t_1,t)}{-\frac{\partial}{\partial t_1}\overline{F}(t_1,t)},\ t>t_1\geq 0.
\end{aligned}$$

the MCAI function can be written as

$$L_1(t|2;t_2)=\frac{\frac{tf(t,t_2)}{\frac{\partial}{\partial t_2}\overline{F}(t,t_2)}}{\int_0^{t_2}\frac{-\frac{\partial}{\partial t_2}\overline{F}(u,t_2)\Big|_{t_2=u}}{\overline{F}(u,u)}du+\log\left(\overline{F}(t_2,t_2)\frac{\frac{\partial}{\partial t_2}\overline{F}(t,t_2)}{\frac{\partial}{\partial v}\overline{F}(t_2,v)\big|_{v=t_2}}\right)}. \tag{4.5}$$

About $L_1(t|\emptyset)$ we can get in a similar way the following expression:

$$L_1(t|\emptyset)=\frac{-t\frac{\frac{\partial}{\partial t_1}\overline{F}(t_1,t)\Big|_{t_1=t}}{\overline{F}(t,t)}}{\int_0^{t}\frac{-\frac{\partial}{\partial t_1}\overline{F}(t_1,u)\Big|_{t_1=u}}{\overline{F}(u,u)}du}. \tag{4.6}$$

The expressions in (4.5)–(4.6) are useful in the sense that they allow to obtain expressions for MCAI functions without involving the multivariate conditional hazard rate functions which may be of difficult evaluation and they are based only on joint probability density and survival functions. In the following subsection, the MCAI functions are obtained for a family of bivariate distributions by applying (4.5)–(4.6).

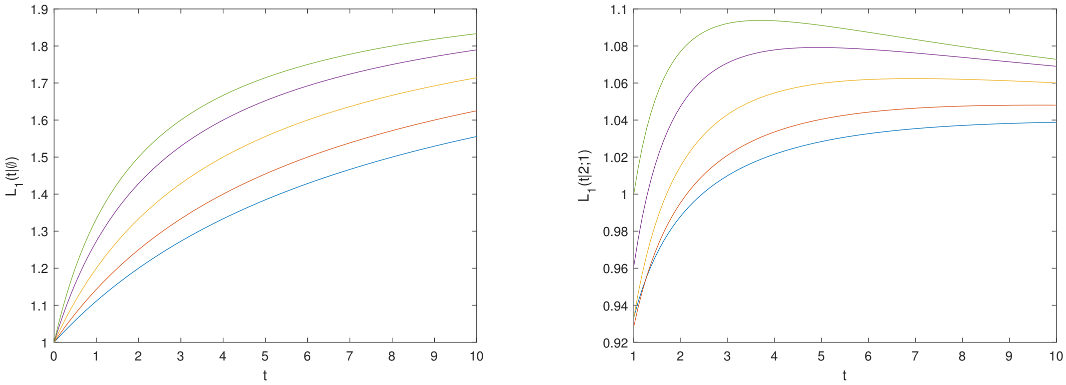


Figure 1. Plot of $L_1(t|\emptyset)$ and $L_1(t|2;1)$ with $\theta = 0.25$ (blue), $1/3$ (red), 0.5 (yellow), 0.75 (violet) and 1 (green).

4.1. Gumbel's type I bivariate exponential distribution

In this subsection, the MCAI functions of a well-known bivariate distribution, the Gumbel's type I bivariate exponential distribution, are obtained. This kind of distribution has attracted the interest of researchers since it has a wide range of applications including competing risks, extreme values, failure times, regional analyses of precipitation, and reliability [9].

Let us consider the Gumbel's type I bivariate exponential distribution with parameter $\theta \in [0,1]$

$$F(x,y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, \quad x, y \geq 0,$$

whose joint density and survival function are respectively expressed as

$$f(x,y) = e^{-(x+y+\theta xy)} \left[(1+\theta x)(1+\theta y) - \theta\right],$$

$$\overline{F}(x,y) = e^{-(x+y+\theta xy)}.$$

About the failure rates gradient (4.1), we have

$$r_1(x,y) = 1 + \theta y, \quad r_2(x,y) = 1 + \theta x,$$

and then from (4.2) we obtain

$$\mathcal{L}_1(x,y) = 1, \quad \mathcal{L}_2(x,y) = 1. \tag{4.7}$$

Now, we aim to compute the bivariate aging intensity functions defined here. We use (4.5) to evaluate the aging intensity function $L_1(t|2;t_2)$ and we get

$$L_1(t|2;t_2) = \frac{-t\left[(1+\theta t)(1+\theta t_2) - \theta\right]}{(1+\theta t)\left(\frac{\theta t_2^2}{2} - t - \theta t t_2 + \log\left(\frac{1+\theta t}{1+\theta t_2}\right)\right)}. \tag{4.8}$$

If $\theta = 0$ we are in the independent case and (4.8) reduces to $L_1(t|2;t_2) = 1$ as the aging intensity function of the exponential distribution is equal to 1.

By using (4.6) we can express $L_1(t|\emptyset)$ as

$$L_1(t|\emptyset) = \frac{2(1+\theta t)}{2+\theta t}. \tag{4.9}$$

In Figure 1 we plot the aging intensity functions related to component 1 for different choices of θ. For $L_1(t|2;t_2)$ we choose the value $t_2 = 1$ and so the function is plotted for $t \geq 1$.

The size of the jump at time t_2 for the MCAI function of component 1 is given by

$$\begin{aligned} L_1(t_2|2;t_2) - L_1(t_2|\emptyset) &= \frac{-t_2\left[(1+\theta t_2)^2-\theta\right]}{(1+\theta t_2)\left(\frac{\theta t_2^2}{2}-t_2-\theta t_2^2\right)} - \frac{2(1+\theta t_2)}{2+\theta t_2} \\ &= \frac{2[(1+\theta t_2)^2-\theta]}{(1+\theta t_2)(2+\theta t_2)} - \frac{2(1+\theta t_2)}{2+\theta t_2} \\ &= \frac{-2\theta}{(1+\theta t_2)(2+\theta t_2)}, \end{aligned}$$

i.e., it is a negative jump with the exception of the case $\theta = 0$ in which there are independence and the continuity of the MCAI function.

5. Aging intensities for time-homogeneous load-sharing models

In this section, we focus attention on a special class of models known as time-homogeneous load-sharing (THLS) models. First, we consider models for which the multivariate conditional hazard rate functions does not depend on the times $t_1, \ldots, t_k$ so that the instantaneous risk of a given unit only depends on the current time and on the set of surviving ones. This class of models is known as load-sharing models. The random vector $(X_1, \ldots, X_n)$ is distributed according to a load-sharing model if, for $I \subset [n]$, $k = |I| \geq 1$ and $j \notin I$, there exist functions $\mu_j(t|I)$ such that, for all $0 \leq t_1 \leq \cdots \leq t_k$,

$$\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \mu_j(t|I). \tag{5.1}$$

Moreover, it is possible to consider models in which the functions $\mu_j(t|I)$ depend on the order of the elements of I. This different version of the load-sharing model was recently studied in [4] and named order dependent load-sharing model (ODLS). We remark that load-sharing models can be seen as a particular case of ODLS models. Furthermore, a load-sharing model is said to be time-homogeneous if the multivariate conditional hazard rate functions do not even depend on the time t. Hence, $(X_1, \ldots, X_n)$ is distributed according to a THLS model if there exist non-negative numbers $\mu_j(I)$ and $\mu_j(\emptyset)$ such that, for all $t \geq 0$,

$$\mu_j(t|I) = \mu_j(I), \quad \lambda_j(t|\emptyset) = \mu_j(\emptyset). \tag{5.2}$$

For further details and related properties of load-sharing and THLS models one may refer to [12, 13]. Of course, it is possible to introduce the ordered version of THLS models. In this case, we have an order dependent time-homogeneous load-sharing model (ODTHLS) and the parameters $\mu_j(I)$ are expressed as $\mu_j(i_1, \ldots, i_k)$.

Let $(X_1, \ldots, X_n)$ be distributed according to an ODTHLS model, then $\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \mu_j(i_1, \ldots, i_k)$ and the MCAI functions can be expressed as

$$\begin{aligned} &L_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) \qquad (5.3)\\ &= \frac{t\mu_j(i_1, \ldots, i_k)}{\sum_{i=0}^k \int_{t_i}^{t_{i+1}} \mu_j(h_1, \ldots, h_i)dx} \\ &= \frac{t\mu_j(i_1, \ldots, i_k)}{\sum_{i=0}^k (t_{i+1}-t_i)\mu_j(h_1, \ldots, h_i)} \\ &= \frac{t\mu_j(i_1, \ldots, i_k)}{t_1\mu_j(\emptyset) + (t_2-t_1)\mu_j(i_1) + \cdots + (t_k - t_{k-1})\mu_j(i_1, \ldots, i_{k-1}) + (t-t_k)\mu_j(i_1, \ldots, i_k)} \end{aligned}$$

In the case in which $X_{1:n} > t$ we have to consider $L_j(t|\emptyset)$ that is given by

$$L_j(t|\emptyset) = \frac{\mu_j(\emptyset)}{\frac{1}{t}t\mu_j(\emptyset)} = 1. \tag{5.4}$$

Proposition 5.1. *$L_j(t|\emptyset) = 1$ for all $t > 0$, $j = 1, \ldots, n$ if, and only if, $\lambda_j(t|\emptyset)$ is constant for all j with respect to t.*

Proof. If $\lambda_j(t|\emptyset)$ is constant for all j then we have shown in (5.4) that the MCAI functions related to the empty set are constant and equal to 1. Conversely, let us suppose $L_j(t|\emptyset) = 1$ for all $t > 0$, $j = 1, \dots, n$. Then, from (3.2) we have

$$\frac{\lambda_j(t|\emptyset)}{\frac{1}{t}\int_0^t \lambda_j(x|\emptyset)dx} = 1,$$

and then

$$\lambda_j(t|\emptyset) = \frac{1}{t}\int_0^t \lambda_j(x|\emptyset)dx.$$

By the mean value theorem for definite integrals we get

$$\lambda_j(t|\emptyset) = \lambda_j(\tilde{t}|\emptyset),$$

where $\tilde{t} \in (0, t)$. Hence, $\lambda_j(t|\emptyset)$ can not be strictly monotone in an arbitrary small interval and then it has to be constant. □

In the following theorem, a characterization of ODTHLS models is given in terms of MCAI functions. In particular, in this case the MCAI functions are constant and equal to 1 or hyperbolas.

Theorem 5.2. *Let $(X_1, \dots, X_n)$ be a random vector with non-negative components. Then, $(X_1, \dots, X_n)$ is distributed according to an ODTHLS model if, and only if, the MCAI functions can be expressed as*

$$\begin{aligned} L_j(t|\emptyset) &= 1, \\ L_j(t|i_1, \dots, i_k; t_1, \dots, t_k) &= \frac{t}{(t - t_k) + C(i_1, \dots, i_k; t_1, \dots, t_k)}, \end{aligned} \tag{5.5}$$

where $C(i_1, \dots, i_k; t_1, \dots, t_k) = \frac{t_1\mu_j(\emptyset) + (t_2 - t_1)\mu_j(i_1) + \dots + (t_k - t_{k-1})\mu_j(i_1, \dots, i_{k-1})}{\mu_j(i_1, \dots, i_k)} > 0$ is constant with respect to t.

Proof. If $(X_1, \dots, X_n)$ is distributed according to an ODTHLS model, then $L_j(t|\emptyset) = 1$ and

$$\begin{aligned} &L_j(t|i_1, \dots, i_{|I|}; t_1, \dots, t_{|I|}) \\ &= \frac{t\mu_j(i_1, \dots, i_k)}{t_1\mu_j(\emptyset) + (t_2 - t_1)\mu_j(i_1) + \dots + (t_k - t_{k-1})\mu_j(i_1, \dots, i_{k-1}) + (t - t_k)\mu_j(i_1, \dots, i_k)} \\ &= \frac{t}{(t - t_k) + \frac{t_1\mu_j(\emptyset) + (t_2 - t_1)\mu_j(i_1) + \dots + (t_k - t_{k-1})\mu_j(i_1, \dots, i_{k-1})}{\mu_j(i_1, \dots, i_k)}}, \end{aligned}$$

and then, by letting $C(i_1, \dots, i_k; t_1, \dots, t_k) = \frac{t_1\mu_j(\emptyset) + (t_2 - t_1)\mu_j(i_1) + \dots + (t_k - t_{k-1})\mu_j(i_1, \dots, i_{k-1})}{\mu_j(i_1, \dots, i_k)}$ we get the result.

Conversely, by Proposition 5.1, if $L_j(t|\emptyset) = 1$ for all $t > 0$, $j = 1, \dots, n$, then $\lambda_j(t|\emptyset) = \mu_j(\emptyset)$ is constant for all j. Let us now consider the case in which $|I| = 1$. We have

$$\begin{aligned} L_j(t|i; t_1) &= \frac{t\lambda_j(t|i; t_1)}{\int_0^{t_1} \lambda_j(x|\emptyset)dx + \int_{t_1}^t \lambda_j(x|i; t_1)dx} \\ &= \frac{t\lambda_j(t|i; t_1)}{t_1\mu_j(\emptyset) + \int_{t_1}^t \lambda_j(x|i; t_1)dx}, \end{aligned}$$

and then by the assumptions

$$\frac{t\lambda_j(t|i; t_1)}{t_1\mu_j(\emptyset) + \int_{t_1}^t \lambda_j(x|i; t_1)dx} = \frac{t}{(t - t_1) + C(i; t_1)}. \tag{5.6}$$

From (5.6), we get

$$(t-t_1)\lambda_j(t|i;t_1)+C(i;t_1)\lambda_j(t|i;t_1)=t_1\mu_j(\emptyset)+\int_{t_1}^{t}\lambda_j(x|i;t_1)dx, \tag{5.7}$$

and, by differentiating both sides of (5.7) with respect to t, we obtain

$$\lambda_j(t|i;t_1)+\lambda_j'(t|i;t_1)+C(i;t_1)\lambda_j'(t|i;t_1)=\lambda_j(t|i;t_1),$$

which is equivalent to

$$(1+C(i;t_1))\lambda_j'(t|i;t_1)=0. \tag{5.8}$$

Since $C(i;t_1)>0$, in order to satisfy (5.8), $\lambda_j(t|i;t_1)$ needs to be constant, $\lambda_j(t|i;t_1) = \mu_j(i)$. Moreover, if in (5.7) we take the limit $t\to t_1^+$ we get

$$C(i;t_1)=\frac{t_1\mu_j(\emptyset)}{\mu_j(i)}.$$

By induction, we can consider the case in which $|I|>1$ and we obtain

$$\frac{t\mu_j(i_1,\dots,i_k)}{t_1\mu_j(\emptyset)+(t_2-t_1)\mu_j(i_1)+\cdots+(t_k-t_{k-1})\mu_j(i_1,\dots,i_{k-1})+\int_{t_k}^{t}\lambda_j(x|i_1,\dots,i_k;t_1,\dots,t_k)dx}$$
$$=\frac{t}{(t-t_k)+C(i_1,\dots,i_k;t_1,\dots,t_k)}.$$

By following the same steps of the case $|I|=1$, we get that $\lambda_j(t|i_1,\dots,i_k;t_1,\dots,t_k)$ needs to be constant,

$$\lambda_j(t|i_1,\dots,i_k;t_1,\dots,t_k)=\mu_j(i_1,\dots,i_k),$$
$$C(i_1,\dots,i_k;t_1,\dots,t_k)=\frac{t_1\mu_j(\emptyset)+(t_2-t_1)\mu_j(i_1)+\cdots+(t_k-t_{k-1})\mu_j(i_1,\dots,i_{k-1})}{\mu_j(i_1,\dots,i_k)}.$$

□

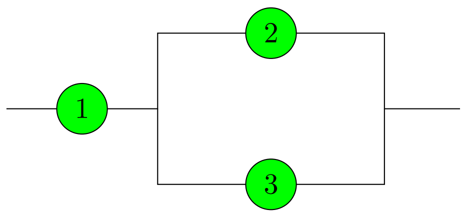
It is of interest to study what happens for the MCAI functions of surviving components in the failure time of other ones. From Proposition 3.4, the sign of the size of the jump for ODTHLS model is determined by the difference $\mu_j(i_1,\dots,i_k)-\mu_j(i_1,\dots,i_{k-1})$, and, in particular, the continuity of the MCAI function is given by the condition $\mu_j(i_1,\dots,i_k)=\mu_j(i_1,\dots,i_{k-1})$.

5.1. Comparisons among surviving components

In this subsection, we show an application of MCAI functions. For the sake of simplicity, we consider coherent systems whose lifetimes are distributed according to ODTHLS models. We use the MCAI functions to make comparisons among surviving components and to discover which component ages faster than the others.

Let us consider a coherent system S formed by three components X_1, X_2, X_3 and whose lifetime T_S is described as

$$T_S=\min\{X_1,\max\{X_2,X_3\}\}.$$



Let us suppose that the component 2 failed at time t_1 and that at time $t > t_1$ the components 1 and 3 are still working, i.e., the system is still working. Moreover, (X_1, X_2, X_3) is distributed according to an ODTHLS model and the parameters of interest are expressed as

$$\mu_1(\emptyset) = 2, \quad \mu_3(\emptyset) = 1,$$
$$\mu_1(2) = 2, \quad \mu_3(2) = 2.$$

The aging intensity functions of components 1 and 3 at time t are expressed as

$$\begin{aligned} L_1(t|2;t_1) &= \frac{2}{\frac{1}{t}[2t_1 + 2(t-t_1)]} = 1, \\ L_3(t|2;t_1) &= \frac{2}{\frac{1}{t}[t_1 + 2(t-t_1)]} = \frac{2t}{2t-t_1}. \end{aligned}$$

Then, we have

$$L_3(t|2;t_1) > L_1(t|2;t_1) \Leftrightarrow \frac{2t}{2t-t_1} > 1 \Leftrightarrow 2t > 2t - t_1 \Leftrightarrow t_1 > 0,$$

and so the component 3 suffers more than 1 the failure of component 2 by aging faster. Moreover, we can observe that the MCAI function of component 1 is constantly equal to 1 and hence continuous also at time t_1, in fact $\mu_1(\emptyset) = \mu_1(2)$. Furthermore, since $\mu_3(2) > \mu_3(\emptyset)$, we expect an upward jump for the MCAI function of component 3 at time t_1, that is

$$L_3(t_1|2;t_1) - L_3(t_1|\emptyset) = \frac{2t_1}{2t_1 - t_1} - 1 = 1.$$

We can do comparisons among surviving components without fixing the values of parameters. In this case, about the aging intensities, we have

$$\begin{aligned} L_1(t|2;t_1) &= \frac{t\mu_1(2)}{\mu_1(\emptyset)t_1 + \mu_1(2)(t-t_1)}, \\ L_3(t|2;t_1) &= \frac{t\mu_3(2)}{\mu_3(\emptyset)t_1 + \mu_3(2)(t-t_1)}. \end{aligned}$$

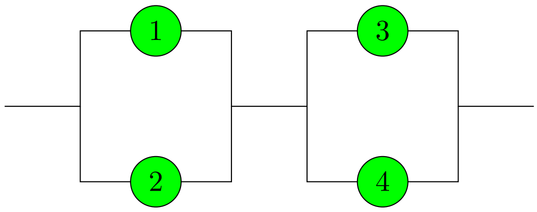
Then, we can compare the aging intensities as

$$\begin{aligned} L_3(t|2;t_1) > L_1(t|2;t_1) &\Leftrightarrow \frac{t\mu_3(2)}{\mu_3(\emptyset)t_1 + \mu_3(2)(t-t_1)} > \frac{t\mu_1(2)}{\mu_1(\emptyset)t_1 + \mu_1(2)(t-t_1)} \\ &\Leftrightarrow \frac{\mu_3(2)}{\mu_3(\emptyset)t_1 + \mu_3(2)(t-t_1)} > \frac{\mu_1(2)}{\mu_1(\emptyset)t_1 + \mu_1(2)(t-t_1)} \\ &\Leftrightarrow \frac{\mu_3(\emptyset)t_1 + \mu_3(2)(t-t_1)}{\mu_3(2)} < \frac{\mu_1(\emptyset)t_1 + \mu_1(2)(t-t_1)}{\mu_1(2)} \\ &\Leftrightarrow \frac{\mu_3(\emptyset)t_1}{\mu_3(2)} + (t-t_1) < \frac{\mu_1(\emptyset)t_1}{\mu_1(2)} + (t-t_1) \\ &\Leftrightarrow \frac{\mu_3(\emptyset)}{\mu_3(2)} < \frac{\mu_1(\emptyset)}{\mu_1(2)}. \end{aligned}$$

We can observe that the comparison is not dependent on t_1 and t. We remark that, when we compare the aging intensities of ODTHLS components the dependence of t is always lost whereas if the number of failed components is greater than one, the times of failure will be involved in the comparisons. In fact, from Equation (5.5), we can easily deduce that the comparison is based on the value of $C(i_1, \ldots, i_k; t_1, \ldots, t_k)$ which is a constant with respect to t.

Let us consider a coherent system S formed by four components X_1, X_2, X_3, X_4 and whose lifetime T_S is described as

$$T_S = \min\{\max\{X_1, X_2\}, \max\{X_3, X_4\}\}.$$



Let us suppose that the component 2 failed at time t_1, the component 4 failed at time $t_2 > t_1$ and that at time $t > t_2$ the components 1 and 3 are still working, i.e., $T_S > t$. Moreover, (X_1, X_2, X_3, X_4) is distributed according to an ODTHLS model. The aging intensity functions of components 1 and 3 at time t are expressed as

$$\begin{aligned} L_1(t|2,4;t_1,t_2) &= \frac{t\mu_1(2,4)}{\mu_1(\emptyset)t_1 + \mu_1(2)(t_2-t_1) + \mu_1(2,4)(t-t_2)}, \\ L_3(t|2,4;t_1,t_2) &= \frac{t\mu_3(2,4)}{\mu_3(\emptyset)t_1 + \mu_3(2)(t_2-t_1) + \mu_3(2,4)(t-t_2)}. \end{aligned}$$

Then, we can compare the aging intensities as

$$L_3(t|2,4;t_1,t_2) > L_1(t|2,4;t_1,t_2)$$
$$\Leftrightarrow \frac{\mu_3(2,4)}{\mu_3(\emptyset)t_1 + \mu_3(2)(t_2-t_1) + \mu_3(2,4)(t-t_2)} > \frac{\mu_1(2,4)}{\mu_1(\emptyset)t_1 + \mu_1(2)(t_2-t_1) + \mu_1(2,4)(t-t_2)}$$
$$\Leftrightarrow t_1\frac{\mu_3(\emptyset)}{\mu_3(2,4)} + (t_2-t_1)\frac{\mu_3(2)}{\mu_3(2,4)} < t_1\frac{\mu_1(\emptyset)}{\mu_1(2,4)} + (t_2-t_1)\frac{\mu_1(2)}{\mu_1(2,4)}.$$

6. Conclusion

In this paper, the notion of multivariate conditional aging intensity (MCAI) function is introduced by using the multivariate conditional hazard rate functions. Some properties of MCAI functions are presented and an application to the load-sharing model is presented. This new kind of aging intensity takes into account the dependency among random variables and then it will be useful in concrete applications which will be the subject of future studies.

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References

[1] S. Bhattacharjee, A.K. Nanda and S.K. Misra, *Reliability analysis using ageing intensity function*, Statist. Probab. Lett. **83** (5), 1364–1371, 2013.

[2] F. Buono, M. Longobardi and M. Szymkowiak, *On generalized reversed aging intensity functions*, Ric. Mat. **71**, 85–108, 2022.

[3] D.R. Cox, *Regression models and life-tables*, J. R. Stat. Soc. Ser. B. Stat. Methodol. **34** (2), 187–220, 1972.

[4] R. Foschi, G. Nappo and F.L. Spizzichino, *Diagonal sections of copulas, multivariate conditional hazard rates and distributions of order statistics for minimally stable lifetimes*, Depend. Model. **9**, 394–423, 2021.

[5] R.L. Giri, A.K. Nanda, M. Dasgupta, S.K. Misra and S. Bhattacharjee, *On ageing intensity function of some Weibull models*, Comm. Statist. Theory Methods, Doi:10.1080/03610926.2021.1910845, 2021.

[6] R. Jiang, P. Ji and X. Xiao, *Aging property of unimodal failure rate models*, Reliab. Eng. Syst. Saf. **79** (1), 113–116, 2003.

[7] N.L. Johnson and S. Kotz, *A vector multivariate hazard rate*, J. Multivariate Anal. **5**, 53–66, 1975.

[8] S. Khardani and A. Benkhaled, *A nonparametric estimation of the conditional ageing intensity function in censored data: A local linear approach*, Math. Slovaca **71** (2), 429–438, 2021.

[9] S. Nadarajah and S. Kotz, *Reliability for some bivariate exponential distributions*, Math. Probl. Eng., Doi:10.1155/MPE/2006/41652, 2006.

[10] A.K. Nanda, S. Bhattacharjee and S.S Alam, *Properties of aging intensity function*, Statist. Probab. Lett. **77** (4), 365–373, 2007.

[11] M. Rezaei and V.A. Khalef, *On the reversed average intensity order*, JSRI **11**, 25–39, 2014.

[12] S.M. Ross, *A model in which component failure rates depend on the working set*, Nav. Res. Logist. Q. **31** (2), 297–300, 1984.

[13] Z. Schechner, *A load-sharing model: the linear breakdown rule*, Nav. Res. Logist. Q. **31**, 137–144, 1984.

[14] M. Shaked, and J.G. Shanthikumar, *Multivariate stochastic orderings and positive dependence in reliability theory*, Math. Oper. Res. **15** (3), 545–552, 1990.

[15] M. Shaked and J.G. Shanthikumar, *Multivariate conditional hazard rate functions - an overview*, Appl. Stoch. Models Bus. Ind. **31** (3), 285–296, 2015.

[16] F.L. Spizzichino, *Reliability, signature, and relative quality functions of systems under time-homogeneous load-sharing models*, Appl. Stoch. Models Bus. Ind. **35** (2), 158–176, 2018.

[17] S.M. Sunoj and R.S. Rasin, *A quantile-based study on ageing intensity function*, Comm. Statist. Theory Methods **47** (22), 5474–5484, 2018.

[18] M. Szymkowiak, *Characterizations of distributions through aging intensity*, IEEE Trans. Rel. **67** (2), 446–458, 2018.

[19] M. Szymkowiak, *Generalized aging intensity functions*, Reliab. Eng. Syst. Saf. **178**, 198–208, 2018.

[20] M. Szymkowiak, *Lifetime Analysis by Aging Intensity Functions*, Springer, 2020.