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ON CYCLE SLIPPING IN INFINITE-DIMENSIONAL CONTROL SYSTEMS WITH PERIODIC NONLINEARITIES

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Abstract

In this paper we consider control systems with periodic nonlinearities characterized by countable sets of equilibria, both Lyapunov stable and unstable. The simplest example of such a system is the mathematical pendulum; therefore, these systems are often called "pendulum-like" systems. In pendulum-like systems, the very concept of stability differs from that in systems with a unique equilibrium. Stability is defined as the convergence of any solution to a certain equilibrium. For stable pendulum-like systems, the problem of cycle slipping arises. In the case of a mathematical pendulum, the number of slipped cycles corresponds to the number of rotations of the pendulum around its suspension point. In general, it represents the distance between the initial value of the input and its limit value. In this paper, we obtain frequency-domain estimates for the number of slipped cycles in infinite-dimensional systems using the Popov method of a priori integral indices. These estimates are tighter than those established in previous works. The paper presents an expanded version of the talk delivered at the International Conference on Physics and Control (PhysCon 2024).

Key words

Pendulum-like system, frequency domain estimates, cycle-slipping.

1 Introduction

This paper extends a series of studies on the asymptotic behavior of infinite-dimensional control systems with periodic nonlinearities [Smirnova and Proskurnikov, 2019], [Smirnova et al., 2022], [Elsakov et al., 2023]. These systems are modeled by an integral-differential delay equation. Systems under consideration are often referred to as pendulum-like systems [Leonov et al., 1992] or synchronization systems [Lindsey, 1972], [Hoppensteadt, 1983], [Leonov et al., 1996]. Indeed, they embrace both various kinds of pendulums (i.e. objects without synchronization effect) and phase-locked loops which are aimed to ensure synchronization of two oscillators. To integral-differential equation can also be reduced mathematical description of a system of two vibro-exciters (rotors) installed on a common oscillating platform [Blekhman, 1988], [Smirnova and Proskurnikov, 2021].

These systems are characterized by a countable set of equilibrium points, which can be either Lyapunov stable or unstable. This feature makes the stability analysis of such systems distinct from systems with a single equilibrium point, as the concept of stability itself is modified. In the context of pendulum-like systems, global stability is defined as the convergence of any solution to an equilibrium.

Most of the classical methods of nonlinear systems analysis and design prove to be ineffective for systems with multiple equilibria. Several new procedures have been developed within the framework of classical methods to establish sufficient conditions for the global stability of synchronization systems [Gelig et al., 2004; Leonov et al., 1996]. For infinite-dimensional systems, these procedures have been combined with the Popov method of a priori integral indices [Popov, 1961; Rasvan, 2006]. This integration has led to frequency-domain criteria defined by a set of varying parameters.

Although different procedures yield similar, though not identical, estimates of stability domains, the most effective procedure was identified in [Elsakov et al., 2023]. In this paper, we aim to leverage the capabilities of this procedure specifically to address an additional problem related to asymptotic behavior-namely, the problem of estimating the cycle-slipping number.

The term "cycle slipping" originated from the rotations of a mathematical pendulum, where the number of rotations of the pendulum around its suspension point was defined as the number of slipped cycles [Stoker, 1950]. Generally, for a pendulum-like system, the number of slipped cycles characterizes the "distance" between the initial phase value and its subsequent values. The most interesting scenario, possible only for systems with multiple equilibria, occurs when a solution, beginning in the vicinity of one equilibrium phase, departs from this vicinity due to a non-zero initial frequency and subsequently converges to another equilibrium phase. The distance to this new equilibrium phase is measured by the number of slipped cycles. By employing the Popov method along with auxiliary procedures, it has become possible to obtain frequency-domain estimates for the number of slipped cycles [Perkin et al., 2013; Perkin et al., 2014a; Perkin et al., 2014b].

In this paper, we build on the ideas presented in [Perkin et al., 2014b] using an additional parameter optimization procedure. We also simplify and clarify the cycle-slipping estimation scheme, making it more transparent. Additionally, we introduce an algorithm for selecting the optimal values of the varying parameters.

2 Problem Setup and Preliminaries

Consider a class of control systems with distributed parameters, represented as a feedback interconnection of a time-invariant linear block and a periodic nonlinearity. These systems are described by integral-differential Volterra equations as follows

$$\frac{d\sigma}{dt} = b(t) + \rho\varphi(\sigma(t-h)) - \\ - \int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\, d\tau \quad (t>0). \tag{1}$$

Here $\rho \in \mathbb{R}, h \geq 0; b, \gamma : [0, +\infty) \to \mathbb{R};\ \ \varphi : \mathbb{R} \to \mathbb{R}$. The function φ is Δ– periodic:

$$\varphi(\zeta + \Delta) = \varphi(\zeta) \quad (\Delta > 0). \tag{2}$$

The solution is defined by the initial condition

$$\sigma(t)\Big|_{t\in[-h,0]} = \sigma^o(t),\ \sigma(0+0) = \sigma^o(0), \tag{3}$$

where $\sigma^o(t)$ is supposed to be continuous.

Main Assumptions

The following assumptions A1-A4 are henceforth supposed to hold.

A1. The function $b(t)$ is continuous, the function $\gamma(t)$ is piece-wise continuous.

A2. Two constants $\varkappa_i, M_i > 0$ exist such that

$$|b(t)| \leq M_1 e^{-\varkappa_1 t}, |\gamma(t)| \leq M_2 e^{-\varkappa_2 t}. \tag{4}$$

A3. The nonlinear function $\varphi(\sigma)$ is non-constant, $\mathbb{C}^1$–smooth and has at least two roots over the period. It can be shown that the latter condition is equivalent to the existence of $0 \leq \sigma_1 < \sigma_2 < \Delta$ such that

$$\varphi'(\sigma_1)\cdot\varphi'(\sigma_2) < 0. \tag{5}$$

A4. The inequality holds as follows

$$\int_0^\infty \gamma(t)\, dt \neq \rho, \tag{6}$$

in other words, the steady (DC) gain of the linear part's transfer function

$$K(p) = -\rho e^{-ph} + \int_0^\infty \gamma(t) e^{-pt}\, dt \quad (p \in \mathbb{C}), \tag{7}$$

is non-vanishing: $K(0) \neq 0$.

Assumption **A4** implies that, neglecting the exponentially decaying term $b(t)$, the equilibria of the system are roots of the equation $\varphi(\sigma) = 0$; in view of **A3**, there exists an infinite sequence of such equilibria. In dealing with such systems, the global stability of the system is usually defined as the convergence of any solution to one of the equilibria. This asymptotic property is also called gradient-like behavior.

A Gradient-Like Behavior Criterion

In [Elsakov et al., 2023], the most effective frequency-domain condition for gradient-like behavior is presented. It is formulated in terms of the transfer function of the linear part (7) and has the form of a frequency-domain inequality.

In order to formulate this condition, we introduce the constants

$$A_1 \triangleq \inf_{\sigma\in[0,\Delta)} \varphi'(\sigma) \quad A_2 \triangleq \sup_{\sigma\in[0,\Delta)} \varphi'(\sigma) \tag{8}$$

(evidently $A_1 \cdot A_2 < 0$). We also define the function

$$\Phi(\zeta;\alpha) \triangleq \sqrt{(1-\alpha_1^{-1}\varphi'(\zeta))(1-\alpha_2^{-1}\varphi'(\zeta))} \tag{9}$$

with $\alpha_1 \leq A_1,\ \alpha_2 \geq A_2,$ and $\alpha_i^{-1} = 0$ if $\alpha_i = \pm\infty\ \ (i = 1, 2)$. Let $\alpha \triangleq (\alpha_1, \alpha_2)$.

Finally, we will use the constant

$$\nu(\varepsilon,\tau,\alpha) \triangleq \frac{\int_0^\Delta \varphi(\zeta)\,d\zeta}{\int_0^\Delta |\varphi(\zeta)|\sqrt{\varepsilon+\tau\Phi^2(\sigma;\alpha)}\,d\zeta}, \tag{10}$$

and the function

$$\begin{aligned}\Pi(\omega;\,\varepsilon,\tau,\alpha) \triangleq\; & ReK(\imath\omega)-(\varepsilon+\tau)|K(\imath\omega)|^2+\\ &+\tau(\alpha_1^{-1}+\alpha_2^{-1})\omega ImK(\imath\omega)+\\ &+\tau|\alpha_1^{-1}\alpha_2^{-1}|\omega^2 \qquad (\imath^2=-1).\end{aligned} \tag{11}$$

The stability condition is given by the following.

Theorem 1. [Elsakov et al., 2023]. Suppose there exist $\varepsilon > 0$, $\tau > 0$, $\alpha_1 \le A_1$, and $\alpha_2 \ge A_2$ such that

$$\inf_{\omega\in\mathbb{R}} \Pi(\omega;\varepsilon,\tau,\alpha) > \frac{1}{4}\nu^2(\varepsilon,\tau,\alpha). \tag{12}$$

Then system (1) is gradient-like, that is,

$$\dot\sigma(t)\to 0 \text{ as } t\to+\infty, \tag{13}$$

$$\sigma(t)\to q \text{ as } t\to+\infty, \tag{14}$$

$$\dot\sigma(t)\in L_2[0,+\infty), \tag{15}$$

where the solution's limit is an equilibrium: $\varphi(q) = 0$.

Cycle Slipping Problem

While Theorem 1 specifies the steady-state behavior, it provides no information about the transient behavior of the system. In particular, it remains unclear whether the phase variable converges to the nearest root of φ. For a pendulum, this uncertainty implies that several rotations around the suspension point may occur before the pendulum settles at a stable equilibrium. The number of such rotations representing the maximal deviation of the phase variable from its initial value, $\sigma(0)$–is a key characteristic of the transient regime.

Definition 1. It is said that a solution $\sigma(t)$ of (1) has slipped k cycles (periods of φ) if there exists an instant t_0 such that

$$|\sigma(0)-\sigma(t_0)| = k\Delta, \tag{16}$$

however

$$|\sigma(0)-\sigma(t)| < (k+1)\Delta, \quad \forall t\ge 0. \tag{17}$$

The goal of this paper is to find frequency-domain estimates for the number of slipped cycles.

3 Frequency-Domain Estimates for the Number of Slipped Cycles

In this section we confine ourselves to the systems with

$$-A_1 = A_2. \tag{18}$$

and introduce

$$A \triangleq (A_1, A_2)^T.$$

Then

$$\Phi(\zeta) \triangleq \Phi(\zeta, A) = \sqrt{(1-A^{-2}(\varphi'(\zeta))^2}. \tag{19}$$

Let

$$\sup_{\zeta\in[0,\Delta)} |\varphi(\zeta)| \triangleq m. \tag{20}$$

Consider the function

$$\chi(t) \triangleq \rho\varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\varphi(\tau)\,d\tau, \tag{21}$$

where $\sigma(t)$ is a solution of equation (1).

Notice that

$$\dot\sigma(t) = b(t) + \chi(t). \tag{22}$$

It is evident that

$$|\chi(t)| \le \rho m + \frac{m}{\varkappa_2}M_2(1-e^{-\varkappa_2 t}) \le m(\rho+\frac{M_2}{\varkappa_2}) \triangleq M. \tag{23}$$

Let us consider a solution of (1), (3) with the property

$$\varphi(\sigma(0)) = 0. \tag{24}$$

Introduce the set

$$\Sigma \triangleq \{T : T>0, \sigma(T)=\sigma(0)+N\Delta, N\in\mathbb{Z}\} \tag{25}$$

Proposition 1. Suppose there exist $\varepsilon > 0$, $\tau > 0$, such that for all $\omega \in \mathbb{R}$ the frequency-domain inequality

$$\Pi(\omega;\varepsilon,\tau,A) \ge \delta \tag{26}$$

is valid . Then for a solution of (1), (3) with the property (24) the functionals

$$\begin{aligned}I_T(\varepsilon,\tau,\delta) \triangleq \int_0^T \{&\dot\sigma(t)\varphi(\sigma(t)) + (\tau+\varepsilon)\dot\sigma^2(t)+\\ &+\delta\varphi^2(\sigma(t)) - \tau A^{-2}\dot\varphi^2(\sigma(t))\}\,dt,\end{aligned} \tag{27}$$

where $T \in \Sigma$, are uniformly bounded along the solution:

$$I_T(\varepsilon,\tau,\delta) \le Q(\varepsilon,\tau) \quad (T\in\Sigma), \tag{28}$$

where Q does not depend on T.

Proof. We traditionally use the Popov method of a priory integral indices [Rasvan, 2006]. Let $\sigma(t)$ be a solution of (1), (3) with the property (24) and $T \in \Sigma$. We shall denote

$$\eta(t) \triangleq \varphi(\sigma(t)). \tag{29}$$

According to Popov method determine continuous auxiliary functions:

$$\eta_T(t) \triangleq \begin{cases} 0, \ t < 0, \\ \eta(t), \ 0 < t \leq T, \quad (T \in \Sigma), \\ 0, \ t > T \end{cases} \tag{30}$$

$$b_1(t) \triangleq \begin{cases} b(t) + \rho\varphi(\sigma^o(t-h)), \quad t \in [0,h], \\ b(t), \quad t \geq h, \end{cases} \tag{31}$$

and

$$\sigma_T(t) \triangleq \rho\eta_T(t-h) - \int_0^t \gamma(t-\tau)\eta_T(\tau)\,d\tau. \tag{32}$$

For $t \in [0,T]$ we have

$$\dot{\sigma}(t) = b_1(t) + \sigma_T(t) \quad (t \geq 0). \tag{33}$$

Notice that $\eta_T(t)$ has a piecewise continuous derivative. Notice also that in virtue of (4) and (31)

$$\sigma_T, \eta_T, \dot{\eta}_T \in L_1[0,+\infty) \ \cap \ L_2[0,+\infty). \tag{34}$$

So we can consider the functionals

$$\rho_T \triangleq \int_0^\infty \{\sigma_T(t)\eta_T(t) + \varepsilon\sigma_T^2(t) + \delta\eta_T^2(t) + \\ +\tau(\sigma_T^2(t) - A^{-2}\dot{\eta}_T^2(t))\}\,dt \quad (T \in \Sigma). \tag{35}$$

In virtue of Plancherel theorem we have

$$\rho_T = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \{\mathcal{F}^*[\sigma_T](\imath\omega)\mathcal{F}[\eta_T](\imath\omega) + \\ +(\tau+\varepsilon)|\mathcal{F}[\sigma_T](\imath\omega)|^2 + \delta|\mathcal{F}[\eta_T](\imath\omega)|^2 - \\ -\tau A^{-2}|\mathcal{F}[\dot{\eta}_T](\imath\omega)|^2\}\,d\omega, \tag{36}$$

where by $\mathcal{F}[f](\imath\omega)$ we have denoted the Fourier transform of function $f(t)$ and the symbol $(*)$ stands for the complex conjugation. Since

$$\mathcal{F}[\sigma_T](\imath\omega) = -K(\imath\omega)\mathcal{F}[\eta_T(\imath\omega)] \tag{37}$$

and

$$\mathcal{F}[\dot{\eta}_T](\imath\omega) = \imath\omega\mathcal{F}[\eta_T](\imath\omega) \tag{38}$$

the formula (36) implies that

$$\rho_T = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} (\Pi(\imath\omega;\varepsilon,\tau,A) - \delta)|\mathcal{F}[\eta_T](\imath\omega)|^2\,d\omega. \tag{39}$$

Then it follows from (26) that

$$\rho_T \leq 0 \quad (T \in \Sigma). \tag{40}$$

On the other hand with the help of (30), (33) we obtain

$$\rho_T = I_T + I_{0T} + (\varepsilon+\tau)\int_T^{+\infty} \dot{\sigma}_T^2\,dt, \tag{41}$$

where

$$I_{0T} = \int_0^T \{-\eta(t)b_1(t) + (\tau+\varepsilon)b_1^2(t) - \\ -2(\tau+\varepsilon)\dot{\sigma}(t)b_1(t)\}dt. \tag{42}$$

Notice that in virtue of (22) and (31)

$$\dot{\sigma}(t)b_1(t) = (b(t) + \chi(t))b_1(t). \tag{43}$$

So

$$I_{0T} = \int_0^T \{-\eta(t)b(t) - (\tau+\varepsilon)b^2(t) - \\ -2(\varepsilon+\tau)b(t)\chi(t)\}\,dt + \\ + \int_0^h \{-\rho\eta(t)\varphi(\sigma^0(t-h)) - \\ -(\tau+\varepsilon)\rho^2\varphi^2(\sigma^0(t-h)) - \\ -2\rho(\tau+\varepsilon)\chi(t)\varphi(\sigma^0(t-h))\}\,dt. \tag{44}$$

It follows from (40) and (41) that

$$I_T \leq |I_{0T}| \tag{45}$$

The value of I_{0T} is bounded by a constant which does not depend on T. For example:

$$|I_{0T}| \leq \int_0^T \{m|b(t)| + (\tau+\varepsilon)b^2(t) + \\ +2(\varepsilon+\tau)|\chi(t)|b(t)\}\,dt + \int_0^h \{\rho m^2 + (\tau+\varepsilon)\rho^2 m^2 + \\ +2\rho(\tau+\varepsilon)|\chi(t)|m\}\,dt \tag{46}$$

whence in virtue of (23), (4) we have

$$I_T \leq (m + (\tau+\varepsilon)\rho m + 2(\tau+\varepsilon)M)\rho m h + \\ +\frac{M_1}{\varkappa_1}(m + (\tau+\varepsilon)\frac{M_1}{2} + 2(\tau+\varepsilon)M) \triangleq Q(\varepsilon,\tau). \tag{47}$$

The proposition is proved.

Remark 1. The constant Q may vary, depending on the accuraey of intermediate estimates for the integrals.

The estimate for the number of slipped cycles depends on the value of Q. We shall need the two functions

$$r_j(x;k,\varepsilon,\tau) \triangleq \frac{\int\limits_0^{\Delta} \varphi(\zeta)\, d\zeta + (-1)^j \frac{x}{k}}{\int\limits_0^{\Delta} |\varphi(\zeta)|\sqrt{\varepsilon + \tau\Phi^2(\zeta)}\, d\zeta}, \qquad (48)$$

where $\varepsilon, \tau, x \in \mathbb{R}; k \in \mathbb{N}; j = 1, 2$.

Theorem 2. Suppose there exist positive ε, τ and $l \in \mathbb{N}$ such that the inequalities

$$\inf_{\omega \in \mathbb{R}} \Pi(\omega; \varepsilon, \tau, A) > \frac{1}{4} r_j^2(Q(\varepsilon,\tau); l, \varepsilon, \tau) \quad (j = 1, 2), \qquad (49)$$

where Q is borrowed from (28), are valid. Then any solution of (1), (3) with the property (24) slips less that l cycles:

$$|\sigma(t) - \sigma(0)| < l\Delta, \quad \forall t > 0. \qquad (50)$$

Remark 2. It follows from Theorem 1 that if the inequalities (49) are true for certain $\varepsilon, \tau > 0$, then equation (1) is gradient-like. Indeed depending on the sign of $\int\limits_0^{\Delta} \varphi(\zeta)\ d\zeta$, for any $l \in \mathbb{N}$ and $Q > 0$ we can see that either

$$|r_1(Q; l, \varepsilon, \tau)| > |\nu(\varepsilon, \tau, A)| \qquad (51)$$

or

$$|r_2(Q; l, \varepsilon, \tau)| > |\nu(\varepsilon, \tau, A)|. \qquad (52)$$

Proof. Let $\varepsilon_0 > 0$ be so small that the inequalities

$$\Pi(\omega; \varepsilon, \tau, A) > \frac{1}{4} r_j^2(Q + \varepsilon_0; l, \varepsilon, \tau) \quad (j = 1, 2). \qquad (53)$$

are valid. We use the denotation

$$P(\zeta; \varepsilon, \tau) \triangleq \sqrt{\varepsilon + \tau\Phi^2(\zeta)} \qquad (54)$$

and define the functions

$$Y_j(\zeta) = \varphi(\zeta) - r_j \cdot |\varphi(\zeta)| P(\zeta; \varepsilon, \tau) \quad (j = 1, 2), \qquad (55)$$

where

$$r_j \triangleq r_j(Q + \varepsilon_0; l, \varepsilon, \tau). \qquad (56)$$

Notice that Y_j is Δ-periodic and

$$\int\limits_0^{\Delta} Y_j(\zeta)\, d\zeta = (-1)^{j+1} \frac{Q + \varepsilon_0}{l}. \qquad (57)$$

For $T \in \Sigma$ consider the functionals $I_T(\varepsilon, \tau, \frac{r_j^2}{4})$ and transform them as follows

$$I_T(\varepsilon, \tau, \frac{r_j^2}{4}) = \int\limits_{\sigma(0)}^{\sigma(T)} Y_j(\zeta)\, d\zeta +$$
$$+ \int\limits_0^T \{\dot\sigma(t)\varphi(\sigma(t)) + \varepsilon\dot\sigma^2(t) + \qquad (58)$$

Then

$$I_T(\varepsilon, \tau, \frac{r_j^2}{4}) = \int\limits_{\sigma(0)}^{\sigma(T)} Y_j(\zeta)\, d\zeta +$$
$$+ \int\limits_0^T (\frac{r_j}{2}|\varphi(\sigma(t))| + P(\zeta; \varepsilon, \tau)\dot\sigma(t))^2\, dt \qquad (59)$$

whence

$$I_T(\varepsilon, \tau, \frac{r_j^2}{4}) \geq \int\limits_{\sigma(0)}^{\sigma(T)} Y_j(\zeta)\, d\zeta. \qquad (60)$$

Assume now that for some $\bar t > 0$

$$\sigma(\bar t) = \sigma(0) + l\Delta. \qquad (61)$$

Then

$$\int\limits_{\sigma(0)}^{\sigma(\bar t)} Y_1(\zeta)\, d\zeta =$$
$$= l \int\limits_0^{\Delta} Y_1(\zeta)\, d\zeta = Q + \varepsilon_0 > Q, \qquad (62)$$

Notice that $\bar t \in \Sigma$, so it follows from (60), (62) that

$$I_{\bar t}(\varepsilon, \tau, \frac{r_j^2}{4}) > Q, \qquad (63)$$

which contradicts (28). Similarly, if

$$\sigma(\hat t) = \sigma(0) - l\Delta \quad (\hat t > 0) \qquad (64)$$

we have

$$\int\limits_{\sigma(0)}^{\sigma(\hat t)} Y_2(\zeta)\, d\zeta =$$
$$= -l \int\limits_0^{\Delta} Y_2(\zeta)\, d\zeta = Q + \varepsilon_0, \qquad (65)$$

Then

$$I_{\hat{t}}(\varepsilon, \tau, \frac{r_j^2}{4}) > Q, \tag{66}$$

which contradicts (28). So we have demonstrated that

$$\sigma(0) - l\Delta < \sigma(t) < \sigma(0) + l\Delta \quad (\forall t > 0). \tag{67}$$

Thus Theorem 2 is proved.

Theorem 3. Suppose that all the conditions of Theorem 2 hold. Then any solution of (1), (3) slips less than $(l+1)$ cycles:

$$|\sigma(t) - \sigma(0)| < (l+1)\Delta \quad (\forall t > 0). \tag{68}$$

Proof. Suppose

$$\sigma_0 - \Delta < \sigma(0) < \sigma_0, \tag{69}$$

where

$$\varphi(\sigma_0) = 0. \tag{70}$$

If

$$\sigma_0 - \Delta < \sigma(t) < \sigma_0 \quad (\forall t > 0), \tag{71}$$

then the solution slips 0 cycles.

Let

$$\sigma(0) < \sigma(t) < \sigma_0 \quad \text{for } 0 < t < t_1, \tag{72}$$

and

$$\sigma(t_1) = \sigma_0. \tag{73}$$

Then since

$$\sigma(t) - \sigma(0) = (\sigma(t) - \sigma(t_1)) + (\sigma_0 - \sigma(0)),$$

we affirm that

$$\sigma(t) - \sigma(0) < l\Delta + \Delta = (l+1)\Delta. \tag{74}$$

Analogously suppose

$$\sigma_0 - \Delta < \sigma(t) < \sigma_0 \quad \text{for } t \in (0, t_2), \tag{75}$$

and

$$\sigma(t_2) = \sigma_0 - \Delta. \tag{76}$$

Then

$$\sigma(t) - \sigma(0) = (\sigma(t) - \sigma(t_2)) + (\sigma_0 - \Delta - \sigma(0)), \tag{77}$$

and

$$\sigma(t) - \sigma(0) > -l\Delta - \Delta = -(l+1)\Delta. \tag{78}$$

Theorem 3 is proved.

Theorem 3 gives the opportunity to estimate the genuine value of the number of slipped cycles.

Let

$$\Pi_0(\varepsilon, \tau) \triangleq \inf_{\omega \in \mathbb{R}} \Pi(\omega; \varepsilon, \tau). \tag{79}$$

Introduce the denotations

$$L(\varepsilon, \tau) \triangleq \int_0^\Delta \sqrt{\varepsilon + \tau \Phi^2(\zeta)} |\varphi(\zeta)| \, d\zeta, \tag{80}$$

and

$$W(\varepsilon, \tau) \triangleq 2L(\varepsilon, \tau)\sqrt{\Pi_0(\varepsilon, \tau)} - \left| \int_0^\Delta \varphi(\zeta) \, d\zeta \right|. \tag{81}$$

Since Q and l are positive , the system of two inequalities (49) is equivalent to inequality

$$2\sqrt{\Pi_0(\varepsilon, \tau)} > \frac{\left| \int_0^\Delta \varphi(\zeta) \, d\zeta \right| + \frac{Q(\varepsilon, \tau)}{l}}{L(\varepsilon, \tau)}. \tag{82}$$

So if for certain $\varepsilon > 0, \tau > 0, l \in \mathbb{N}$

$$l > \frac{Q(\varepsilon, \tau)}{W(\varepsilon, \tau)} \tag{83}$$

then any solution of (1), (3) slips less than $(l+1)$ cycles. It follows from (83) that genuine number of slipped cycles k satisfies the estimate

$$k \leq 1 + \left\lfloor \left(\frac{Q(\varepsilon, \tau)}{W(\varepsilon, \tau)} \right) \right\rfloor, \tag{84}$$

where $\lfloor y \rfloor$ stands for the integer floor of y.

But the estimate (84) is not optimal as different pairs (ε, τ) may give different value of $Q(\varepsilon, \tau)$ and $W(\varepsilon, \tau)$. Let E be the set of pairs of positive values ε and τ such that inequality (12) is true , i.e.

$$E \triangleq \{(\varepsilon, \tau) : \varepsilon > 0, \tau > 0, \Pi_0(\varepsilon, \tau) > \frac{1}{4}\nu^2(\varepsilon, \tau, A)\}. \tag{85}$$

Then the optimal estimate for k takes the form

$$k \leq \inf_{(\varepsilon, \tau) \in E} \left\lfloor \left(\frac{Q(\varepsilon, \tau)}{W(\varepsilon, \tau)} + 1 \right) \right\rfloor. \tag{86}$$

4 Example (mathematical pendulum)

The simplest example of pendulum-like system is the mathematical pendulum in the absence of rotating force. Its equation is as follows:

$$\ddot{\sigma} + a\dot{\sigma} + \sin\sigma = 0 \quad (a > 0). \tag{87}$$

For mathematical pendulum $h = 0$,

$$\begin{aligned} Q(\varepsilon,\tau) = (a^{-1} + a^{-2}(\varepsilon+\tau))|\dot{\sigma}(0)| + \\ +0.5a^{-1}(\varepsilon+\tau)\dot{\sigma}^2(0), \end{aligned} \tag{88}$$

$$\begin{aligned} \Pi(\omega;\varepsilon,\tau) &= \frac{a-\varepsilon-\tau}{a^2+\omega^2} + \tau\omega^2, \\ L(\varepsilon,\tau) &= \int_0^{2\pi} (\sin\sigma)\sqrt{\varepsilon + \tau\sin^2\sigma}\, d\sigma, \\ W(\varepsilon,\tau) &= 2L(\varepsilon,\tau)\sqrt{\Pi_0(\varepsilon,\tau)}. \end{aligned} \tag{89}$$

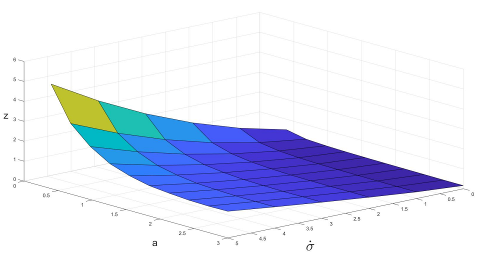

Figure 1. Estimate of the number of slipped cycles

In Fig.1 the spacial plot of function

$$z \triangleq \inf_{(\varepsilon,\tau)\in E} Z(\varepsilon,\tau), \tag{90}$$

where

$$Z(\varepsilon,\tau) = 1 + \lfloor \frac{Q(\varepsilon,\tau)}{W(\varepsilon,\tau)} \rfloor, \tag{91}$$

is shown.

5 Conclusion

In this paper, we address the problem of cycle slipping, which is relevant for control systems with periodic nonlinearities and infinite sets of equilibria (known also as “pendulum-like systems”). For infinite-dimensional pendulum-like systems, new frequency-domain estimates for the number of slipped cycles are obtained.

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