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A Coevolutionary Model for Actions and Opinions in Social Networks

Lorenzo Zino, Mengbin Ye, and Ming Cao

***Abstract*— In complex social networks, the decision-making mechanisms behind human actions and the cognitive processes that shape opinion formation processes are often intertwined, leading to complex and varied collective emergent behavior. In this paper, we propose a mathematical model that captures such a coevolution of actions and opinions. Following a discrete-time process, each individual decides between binary actions, aiming to coordinate with the actions of other members observed on a network of interactions and taking into account their own opinion. At the same time, the opinion of each individual evolves due to the opinions shared by other members, the actions observed on the network, and, possibly, an external influence source. We provide a global convergence result for a special case of the coupled dynamics. Steady state configurations in which all the individuals take the same action are then studied, elucidating the role of the model parameters and the network structure on the collective behavior of the system.**

I. Introduction

The study of mathematical models of social dynamics, having captured the attention of various scientific communities for several decades, has recently become increasingly popular in the systems and controls community. Established concepts and techniques from dynamical systems have allowed researchers to shed light on human behavior, predicting the evolution of a community and elucidating how individuals' dynamics shape the emergence of complex collective behaviors [1].

A key area of such mathematical models focuses on the formation of opinions in social communities, through the lens of opinion dynamics models [2]–[4]. Most of these models assume individuals have opinions, represented by variables taking values on a continuous interval, which evolve through interaction and learning of the opinions of others in a social network. The field has been extensively studied beginning from the 1950s, with a number of classical contributions.

As second area, evolutionary game theory has emerged as a powerful paradigm to represent and study decision-making processes in network systems [5]–[7]. A standard framework assumes that each individual aims to maximise a payoff by dynamically choosing which action to take from a finite set, taking into account the actions of others on a network.

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Clear evidence from social-psychological literature and empirical studies suggest that the two complex social processes of opinion dynamics and decision-making are deeply intertwined, whereby an individual's action may be influenced by their own opinion, and the opinion formation process may be shaped by the observed actions of others [8], [9]. Given this evidence, it is perhaps surprising that there have been few efforts to provide a unified mathematical modeling framework that captures a coupling between actions and opinions in social networks. Two recent works considered opinion dynamics models which assume each individual has coevolving private and expressed opinions [10], [11], but a decision-making process is lacking. Other models posit that each individual's action is a quantized output of an opinion dynamics process, but independent of others' actions [12]–[14], inconsistent with the extensive literature from evolutionary game theory. A decision-making model was proposed in [15], but with fixed private opinions. A general model for the coevolution of actions and opinions is still missing, capable of capturing and predicting complex behavioral phenomena of importance, such as the emergence of unpopular norms in which a majority of individuals select an action they privately reject [15].

In this paper, we propose a novel model that captures the coevolution of actions and opinions. In the model, a population of rational individuals interact on a network, revising their actions and opinions asynchronously according to a discrete-time process. Specifically, individuals decide on a binary action, aiming to coordinate with the actions of others observed on the network, and taking into account their own opinion. Simultaneously, they update their opinion, depending on the opinions shared by others, the actions observed on the network, and, possibly, an external influence. The proposed model lies at the interface between the opinion dynamics and evolutionary game literature; the dynamics of the actions and opinions are coupled seamlessly, while each separate dynamics inherits the fundamental features of their separate grounding frameworks. The effect of the network structure on a simplified version of the model with bounded rationality was investigated in [16], via numerical simulations.

In addition to the formalization of the coevolutionary model, the main theoretical contributions are: i) a discussion on the model motivation, offering an explanation of its intuitive intertwined mechanisms; ii) a rigorous convergence result for a special case of the dynamics; and iii) the extensive analysis of the configurations in which all the individuals choose the same action, named pure configurations.

The rest of the paper is organized as follows. Section II provides mathematical preliminaries. In Section III, we propose the model. Section IV provides the main theoretical

results. Section V concludes the paper.

II. Mathematical preliminaries

The set of real, nonnegative real, strictly positive real, and nonnegative integer numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, and $\mathbb{Z}_+$, respectively. The n-column vector of all ones and zeros is given by $\mathbf{1}$ and $\mathbf{0}$, respectively. A vector $\boldsymbol{x}$ is denoted with bold font, with ith entry x_i. A matrix $\boldsymbol{A}$ is denoted with bold capital letter, with a_{ij} the jth entry of its ith row.

A. Graph Theory

A weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{W})$ is a tuple, where $\mathcal{V} = \{1, \dots, n\}$ is the set of n vertices of $\mathcal{G}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of undirected edges, so that $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$, and $\boldsymbol{W} = \boldsymbol{W}^\top \in \mathbb{R}_+^{n \times n}$ is a symmetric nonnegative stochastic matrix such that $w_{ij} = w_{ji} > 0$ if and only if $(i,j) \in \mathcal{E}$, and $\sum_{j=1}^n w_{ij} = 1$ for all i. Edges $(i,i) \in \mathcal{E}$ are called self-loops. The neighbor set of vertex i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$. Given a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{W})$, its adjacency matrix $\boldsymbol{A} \in \{0,1\}^{n \times n}$ has entries $a_{ij} = 1 \Leftrightarrow w_{ij} > 0$ for all $j \neq i$ and $a_{ij} = 0$ otherwise. That is, $\boldsymbol{A}$ has zero diagonal entries and its offdiagonal entries have the same zero-nonzero pattern as $\boldsymbol{W}$. We define $d_i \triangleq \sum_{j=1}^n a_{ij}$ as the degree of node i. If $\mathcal{G}$ has no self-loops, then $d_i = |\mathcal{N}_i|$. A graph $\mathcal{G}$ is *connected* if and only if there is a path of edges between every pair of nodes.

B. Game Theory

We consider a set of individuals $\mathcal{V} = \{1, \dots, n\}$, each of whom may choose an action from an action set $\mathcal{A}_i$, and has payoff function $\pi_i : \mathcal{A}_i \to \mathbb{R}$. With $x_i \in \mathcal{A}_i$ representing the action of individual i, let us collect the actions of all n individuals into the vector $\boldsymbol{x} = [x_1, \dots, x_n]^\top \in \prod_i \mathcal{A}_i$. The function $\pi_i = \pi_i(s \,|\, \boldsymbol{x}, \boldsymbol{u})$ determines the payoff that individual i receives for playing action $s \in \mathcal{A}_i$, given the action configuration $\boldsymbol{x}$ of the individuals in the game, and, possibly, some external variable vector $\boldsymbol{u}$.

At discrete time instant $k \in \mathbb{Z}_+$, an individual $i \in \mathcal{V}$ may revise their action. A classical concept in game theory is *best-response* updating. Formally, given the set of payoff functions $\pi_i(\cdot \,|\, \boldsymbol{x}, \boldsymbol{u})$ the best-response actions are defined as

$$\mathcal{B}_i\big(\pi_i(\cdot|\, \boldsymbol{x}, \boldsymbol{u})\big) \triangleq \operatorname{argmax}_{s \in \mathcal{A}_i} \pi_i(s \,|\, \boldsymbol{x}, \boldsymbol{u}).$$

Several implementations of best-response dynamics can be defined, depending on how to resolve the case $\mathcal{B}_i$ is not a singleton. We consider the following *best-response dynamics*:

$$x_i(k+1) = \begin{cases} \mathcal{B}_i(\pi_i(\cdot|\, \boldsymbol{x}, \boldsymbol{u})) & \text{if } |\mathcal{B}_i(\pi_i(\cdot|\, \boldsymbol{x}, \boldsymbol{u}))| = 1, \\ x_i(k) & \text{otherwise.} \end{cases} \tag{1}$$

That is, if multiple actions maximize individual i's payoff, then individual i does not change their action.

In our analysis, we will consider functions $F(\boldsymbol{x}, \boldsymbol{u})$ of the actions and the external variable vector. Focusing on some individual i, we will sometimes denote $F(\boldsymbol{x}, \boldsymbol{u}) = F(s, \boldsymbol{x}_{-i}, \boldsymbol{u})$, meaning individual i takes action $x_i = s$ and the action configuration of all the other individuals is given by $\boldsymbol{x}_{-i} = [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]^\top$. Thus, writing $F(s, \boldsymbol{x}_{-i}, \boldsymbol{u})$ and $F(s', \boldsymbol{x}_{-i}, \boldsymbol{u})$ enables us to compare the value of the function F when individual i selects s or s' from $\mathcal{A}_i$; this will be used in subsequent analysis.

III. Model

We consider a population $\mathcal{V} = \{1, \dots, n\}$ of n individuals that interact on a weighted undirected network, represented by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{W})$, where $\boldsymbol{W}$ is a stochastic weight matrix, i.e., $\boldsymbol{W}\mathbf{1} = \boldsymbol{W}^\top \mathbf{1} = \mathbf{1}$.

Each individual $i \in \mathcal{V}$ has a two-dimensional state variable $[x_i, y_i]^\top \in \{-1, +1\} \times [-1, 1]$. The former is the binary action that i can take, the latter is a continuously distributed opinion, which quantifies individual i's *preference* for one of the two actions. Specifically, $y_i = -1$, $y_i = +1$, and $y_i = 0$ represent an individual i who maximally prefers action -1, $+1$, and is neutral, respectively. The actions and opinions of all the individuals are gathered into two n-dimensional vectors $\boldsymbol{x} \in \{-1, +1\}^n$, and $\boldsymbol{y} \in [-1, +1]^n$, respectively.

The state variables evolve according to a discrete-time updating rule. At each discrete time $k \in \mathbb{Z}_+$, a single individual $i \in \mathcal{V}$ is selected to revise their state, while all other individuals do not modify their state. We will make the following minimal assumption of the activation rule.

Assumption 1. *There exists a $T < \infty$ such that in every time-window $[k, k+T)$, $k \in \mathbb{Z}_+$, each individual $i \in \mathcal{V}$ activates at least once.*

Note that many activation rules such as deterministic sequential rules satisfy Assumption 1.

The action and the opinion of the selected individual i are updated simultaneously, as follows.

Action update: individual i's action is updated according to a best-response dynamics. Specifically, when $\boldsymbol{x}(k) = \boldsymbol{x}$ and $\boldsymbol{y}(k) = \boldsymbol{y}$, we define the following payoff function:

$$\begin{aligned} \pi_i(+1|\boldsymbol{x}, \boldsymbol{y}) &= \lambda_i y_i + \frac{(1-\lambda_i)(1+\alpha)}{d_i} \sum_{j \in \mathcal{N}_i} a_{ij} \frac{1+x_j}{2}, \\ \pi_i(-1|\boldsymbol{x}, \boldsymbol{y}) &= -\lambda_i y_i + \frac{1-\lambda_i}{d_i} \sum_{j \in \mathcal{N}_i} a_{ij} \frac{1-x_j}{2}, \end{aligned} \tag{2}$$

where $\alpha \geq 0$ is a parameter that models a possible evolutionary advantage for action $+1$ over action -1 and $\lambda_i \in [0,1]$ is the *commitment* of individual i to their current opinion. The payoff function is comprised of two summands: the former takes into account individual i's preference for the action, and the latter accounts for the tendency to coordinate with the others' actions. Then, we let

$$x_i(k+1) = \mathcal{B}_i(\boldsymbol{x}(k), \boldsymbol{y}(k)). \tag{3}$$

Opinion update: the same individual i's opinion updates as

$$y_i(k+1) = (1-\gamma_i) \sum_{j \in \mathcal{V}} w_{ij} \big(\mu_i x_j(k) + (1-\mu_i) y_j(k)\big) + \gamma_i u_i, \tag{4}$$

where $\mu_i \in [0,1]$, called *susceptibility*, measures the influence of the neighbors' actions on the opinion of individual i and $\gamma_i \in [0,1]$ is the individual's *attachment* to an existing prejudice or external influence source $u_i \in [-1, 1]$.

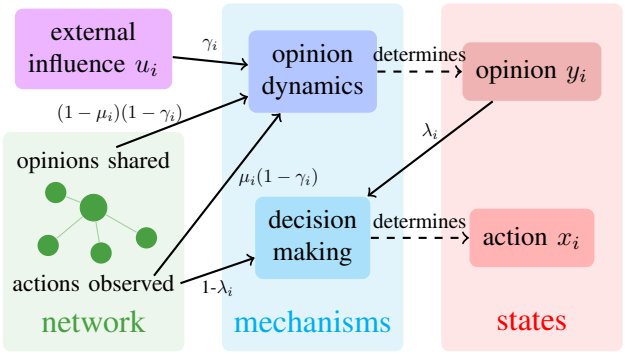

Fig. 1: Schematic of the two coupled dynamics.

A. Model Motivation and Explanation

Decision-Making: The decision-making process is inspired by network coordination games [5]–[7], widely used to model collective decision-making. When $\lambda_i = 0$, the payoff function in Eq. (2) reduces to the normalized sum of the payoff individual i receives in a pairwise coordination game with neighbors $j \in \mathcal{N}_i$ (possibly asymmetric, if $\alpha > 0$).

When $\lambda_i > 0$, individual $i \in \mathcal{V}$ selects $x_i(k+1) = +1$ if and only if

$$\frac{1}{d_i}\sum_{j\in\mathcal{N}_i} a_{ij}x_j(k) > -\frac{1}{2+\alpha}\left(\alpha + 4\frac{\lambda_i}{1-\lambda_i}y_i(k)\right).$$

Thus, the proposed model introduces a *state-dependent* threshold for deciding on the action, which depends on individual i's opinion $y_i(k)$, i.e., the individual's preference for action $+1$ or -1. The proposed Eq. (3) therefore yield an intuitive modification to the standard threshold dynamics of network coordination games, by incorporating an individual's preference directly into the best-response dynamics.

Opinion Dynamics: The model for opinion dynamics is inspired by the seminal Friedkin–Johnsen model [3], which is recovered (in an asynchronous implementation) by setting $\mu_i = 0$. Before presenting further discussion, we prove the following result, which guarantees that the opinions $y_i(k)$, and thus the coevolutionary model, are always well-defined under Eq. (4). The proof is omitted, but follows from the observation that $y_i(k+1)$ is a convex combination of i) the opinions $y_j(k)$ of the individual's neighbors $j \in \mathcal{N}_i$, ii) their actions $x_j(k)$, and iii) individual i's constant prejudice u_i.

Lemma 1. *Consider a system that evolves according to Eqs. (3) and (4). If $y_i(0) \in [-1,1]$ for all $i \in \mathcal{V}$, then $y_i(k) \in [-1,1]$ for all $i \in \mathcal{V}$ and $k \geq 0$.*

Thus, the proposed dynamics extends the existing linear weighted averaging mechanism, commonly adopted to model the integration of learned information [17]. In particular, we extend the mechanism by explicitly assuming that an individual's opinion is shaped by the observed actions of others, consistent with the social-psychological literature [8] and empirical studies [9].

Coupled Evolutionary Dynamics: Our model presents an intuitive extension to two classical class of models for social dynamics, i.e., coordination games on networks [5], [6] and opinion dynamics [4] (specifically, the Friedkin–Johnsen model [3]). In particular, we have coupled the dynamics by i) defining two variables for each individual i to represent an action x_i and an opinion y_i, and ii) including y_i in the decision-making dynamics Eq. (3), and neighboring actions x_j for $j \in \mathcal{N}_i$ in the opinion dynamics Eq. (4).

The two modifications are simple, and ensure the two dynamics remain consistent with the fundamental philosophies of the separate modeling frameworks; individuals select actions that maximize a payoff, and individuals' opinions evolve via weighted averaging of exogenous influences. Moreover, the coupled dynamics can be intuitively understood (see, e.g., Fig. 1), and the impact of y_i in the action payoff function Eq. (2) and of x_j in the opinion dynamics Eq. (4) are immediate and clear to see. Nonetheless, and as the rest of the paper will illustrate, the resulting coevolutionary dynamics allows for more complex collective behavior to be captured, and the dynamical analysis becomes highly nontrivial.

IV. Main Results

Here, we analyze the limiting behavior of the proposed coevolutionary model, and special action configurations.

A. Convergence

It turns out that while the coupling of the coevolutionary model yields intuitive dynamics, the theoretical analysis is rendered challenging. Thus, we establish a general convergence result for a special case of the dynamics in which individuals' susceptibility $\mu_i = 0$ for all $i \in \mathcal{V}$.

Theorem 1. *Consider a given population of $n \geq 2$ individuals interacting on a connected network $\mathcal{G}$, where at each time instant k, and consistent with Assumption 1, a single individual $i \in \mathcal{V}$ is selected to update their state $[x_i(k), y_i(k)]^\top$ according to Eq. (3) and Eq. (4). Suppose further that $\mu_i = 0$ and $\lambda_i \in [0,1)$ for all $i \in \mathcal{V}$. Then, for $k \to \infty$, $\boldsymbol{x}(k) \to \bar{\boldsymbol{x}}$ and $\boldsymbol{y}(k) \to \bar{\boldsymbol{y}}$, where $\bar{\boldsymbol{x}} \in \{-1,+1\}^n$ and $\bar{\boldsymbol{y}} \in [-1,+1]^n$ are constant vectors.*

The proof is in the Appendix, and we provide a brief sketch here. The underlying approach draws inspiration from the theory of potential games [18]. First, asymptotic convergence of the opinions is established using a potential function that increases at each step until it reaches its unique maximum value. Then, we show that when the opinions are sufficiently close to their steady states, a different potential function for the actions always increases over a sufficiently long interval of time, even if it can sometimes decrease for short intervals of time. Thus, the potential function converges to a maximum, which implies convergence of the actions.

Simulations in Fig. 2 suggest that a stronger convergence result can be obtained by generalizing to allow $\mu_i > 0$, for some $i \in \mathcal{V}$, which is a key future direction. However, the analysis in such a general scenario poses some technical issues. To be more precise, each one of the two mechanisms (i.e., action and opinion updates), when considered on its own, is a (weighted) potential game [18]. Even though, the linear combination of two potential games is a weighted potential

game [19, Section 2.4.1.1], the coevolutionary dynamics is not a best-response for the combined game, since the two best-responses with respect to the action and the opinion are performed separately in Eq. (3) and Eq. (4), respectively. Thus, when $\mu_i > 0$, the convergence results for best-response dynamics applicable to weighted potential games cannot be used, and the current proof cannot be easily extended since the potential associated with the opinions may decrease when individual i updates their action.

B. Analysis of pure configurations

The convergence result in Section IV-A provides insight on the long-term collective behavior of the system, guaranteeing convergence to a steady state. In general, multiple steady states can be present, depending on the model parameters and the network structure, as suggested in Fig. 2.

Here, we analyze the two *pure configurations* $\boldsymbol{x} = \pm\mathbf{1}$, that is, full coordination states in which all the individuals take the same action ± 1, respectively. Without imposing that $\mu_i = 0$ for all i, we establish a necessary and sufficient condition on the individuals' opinions that guarantees pure configurations to be steady states of the action dynamics. This general result is used to study the coevolutionary dynamics, determining a sufficient condition for the pure configurations to be steady states, depending on the initial opinions, the model parameters, and the external influence.

Proposition 1. *The configuration $\boldsymbol{x} = -\mathbf{1}$ is a steady state of the action dynamics under Eq. (3) if and only if*

$$\sup_{k\in\mathbb{Z}_+} y_i(k) \leq \frac{1-\lambda_i}{2\lambda_i}, \quad \forall\, i \in \mathcal{V}; \tag{5}$$

while $\boldsymbol{x} = +\mathbf{1}$ is a steady state of Eq. (3) if and only if

$$\inf_{k\in\mathbb{Z}_+} y_i(k) \geq -\frac{(1-\lambda_i)(1+\alpha)}{2\lambda_i}, \quad \forall\, i \in \mathcal{V}. \tag{6}$$

Proof. We consider Eq. (5). Necessity is proved by observing that, if Eq. (5) is not verified, then there exists $k \in \mathbb{Z}_+$ and $i \in \mathcal{V}$ such that $y_i(k) > (1-\lambda_i)/2\lambda_i$. The opinion of node i is not modified until their following activation. When the node i activates at some time $\tilde{k} \geq k$, then they revises their action and opinion. From Eq. (2), we compute

$$\pi_i\left(+1|-\mathbf{1}, \boldsymbol{y}(\tilde{k})\right) = \lambda_i y_i(\tilde{k}) > \frac{1-\lambda_i}{2},$$
$$\pi_i\left(-1|-\mathbf{1}, \boldsymbol{y}(\tilde{k})\right) = -\lambda_i y_i(k) + 1 - \lambda_i < \frac{1-\lambda_i}{2}.$$

Hence, according to Eq. (3), $x_i(\tilde{k}+1) = +1$, implying that $\boldsymbol{x} = -\mathbf{1}$ is not a steady state. Sufficiency is proved by observing that, Eq. (5) implies that $y_i(k) \leq (1-\lambda_i)/2\lambda_i$, for all $k \in \mathbb{Z}_+$ and $i \in \mathcal{V}$. Hence, $\pi_i(+1|-\mathbf{1}, \boldsymbol{y}(k)) \leq (1-\lambda_i)/2$ and $\pi_i(-1|-\mathbf{1}, \boldsymbol{y}(k)) \geq (1-\lambda_i)/2$, $\forall i \in \mathcal{V}$ and $k \in \mathbb{Z}_+$, guaranteeing that action -1 is always the best response, according to Eq. (3). Eq. (6) follows a similar argument. □

Remark 1. *Proposition 1 provides a bound on how much an action can be unpopular, before individuals stop adopting it. If $\boldsymbol{x}(0) = -\mathbf{1}$, individual i will deviate from -1 if and only if their support for $+1$ satisfies $y_i(k) > (1-\lambda_i)/2\lambda_i > 0$, i.e, a positive opinion may not be sufficient. The model parameters are key to determine whether pure configurations are steady states: if $\lambda_i < 1/3$, $\forall\, i \in \mathcal{V}$, then the pure configuration $\boldsymbol{x} = -\mathbf{1}$ is always a steady state, regardless of the individuals' opinions, whereas $\boldsymbol{x} = +\mathbf{1}$ is always a steady state if $\lambda_i < (1+\alpha)/(3+\alpha)$, $\forall\, i \in \mathcal{V}$. Note that $(1+\alpha)/(3+\alpha) > 1/3$ and increases as the evolutionary advantage α grows, favoring the advantageous pure configuration $\boldsymbol{x} = +\mathbf{1}$ being a steady state with respect to the disadvantageous one $\boldsymbol{x} = -\mathbf{1}$.*

Figures 2(c–d) offer a simple example to elucidate the remark. Being $\lambda_i = 0.2$, $\forall i$, the pure configuration $\boldsymbol{x} = -\mathbf{1}$ is a steady state and, even though the majority of the nodes have a positive opinion, none deviate from action -1, which yields the emergence of an unpopular norm.

Proposition 1 is leveraged to establish easy-to-check conditions on the initial opinions, the external influence, and the model parameters that guarantee the pure configurations to be steady states of Eq. (3). Even though these conditions are more conservative that the general result in Proposition 1, being only sufficient, they can be easily checked and provide insight into the role of the model parameters on the collective behavior of the system. Due to space constraints, we focus our analysis on the pure configuration $-\mathbf{1}$; using the same techniques, similar results can be established for $+\mathbf{1}$.

Corollary 1. *The pure configuration $\boldsymbol{x} = -\mathbf{1}$ is a steady state of Eq. (3) if*

$$\max_{i\in\mathcal{V}} q_i \leq \frac{1-\lambda_{max}}{2\lambda_{max}},$$

where $\lambda_{max} := \max_{i\in\mathcal{V}} \lambda_i$ and

$$q_i := \max\left\{ y_i(0), \frac{\gamma_i u_i - \mu_i(1-\gamma_i)}{\gamma_i + \mu_i(1-\gamma_i)} \right\}, \tag{7}$$

where we assume $q_i = y_i(0)$, if $\gamma_i = \mu_i = 0$

Proof. When $\boldsymbol{x}(k) = -\mathbf{1}$, Eq. (4) updates as an (asynchronous) Friedkin–Johnsen model [3], that is,

$$y_i(k+1) = (1-\beta_i)\sum_{j\in\mathcal{V}} w_{ij} y_i(k) + \beta_i b_i,$$

with $\beta_i = \mu_i + \gamma_i - \mu_i\gamma_i$ and $b_i = (\gamma_i u_i - \mu_i(1-\gamma_i))/(\gamma_i + \mu_i(1-\gamma_i))$. From the fact that $\beta_i \in [0,1]$ and $b_i \in [-1,1]$ for all $i \in \mathcal{V}$, it follows that $y_i(k)$ is always bounded from above by the maximum among all the entries of the initial opinion $y_i(0)$ and the entries of the exogenous input b_i for which $\beta_i > 0$. By substituting in Eq. (5) and minimizing with respect to all $i \in \mathcal{V}$, we obtain the claim. □

Besides the individuals' initial opinions and their commitment λ_i, the external influence u_i and the parameters γ_i, μ_i of the nodes on which u_i is exerted are key. In particular, in order to ensure the pure configuration $\boldsymbol{x} = -\mathbf{1}$ is a steady state, we need that all those individuals that have large positive external influence $u_i \gg 0$ to also have large susceptibility $\mu_i \gg 0$, so that the maximum in Eq. (7) is attained by the initial condition $y_i(0)$. The condition in Corollary 1 can be checked in a distributed fashion: each node can independently

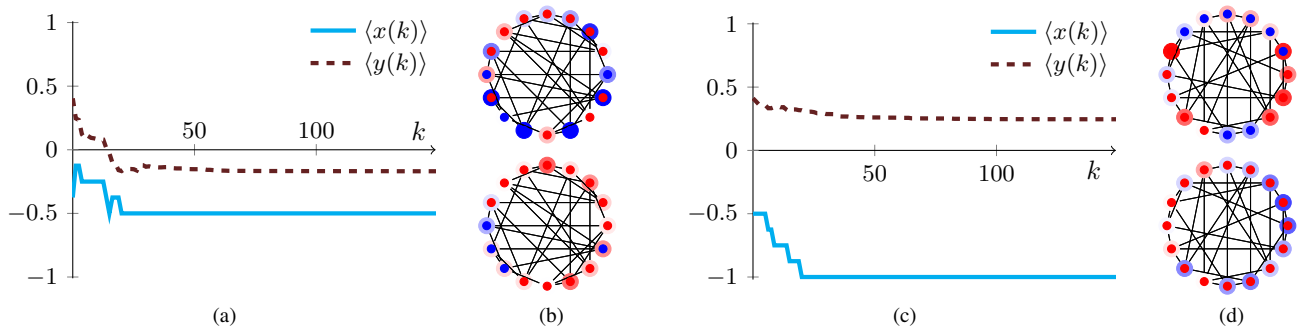

Fig. 2: Two simulations of the coevolutionary dynamics. In (a,c) we illustrate the evolution of average action (solid cyan) and opinion (dashed brown). In (b,d) the initial (above) and steady state (below) configurations of actions (inner circles) and opinions (external circles) are represented through a color code: red (blue) is used for -1 ($+1$) and intensity represents the absolute value of the opinion. Common parameters are $n = 16$, $\alpha = 0.1$, $\mu_i = \gamma_i = 0.2$, $\forall i \in \mathcal{V}$, the network is generated as a random regular graph with degree $d_i = 4$, initial condition and external influence are set uniformly at random in their domain. In (a,b), $\lambda_i = 0.6$, $\forall i \in \mathcal{V}$; in (c,d), $\lambda_i = 0.2$, $\forall i \in \mathcal{V}$.

evaluate their own quantity q_i; the maximum can then be computed in a distributed fashion (and similar for λ_i) [20].

V. Conclusions

In this paper, we proposed a novel mathematical model to capture the complex coevolution of decision-making processes and opinion dynamics in social networks. We established a convergence result for a specific case of the coupled dynamics, and studied pure configurations for the general dynamics.

These promising preliminary results suggest several avenues of future research. First, game-theoretic tools such as the notion of Stackelberg games can be leveraged to study convergence in more general scenarios. Second, a characterization of all the steady states of the dynamics should be performed. Finally, a case study should be analyzed to calibrate the model parameters and test its predictive performance in a real-world scenario.

Appendix

Opinion convergence: Since $\mu_i = 0, \forall\, i \in \mathcal{V}$, Eq. (4) reduces to

$$y_i(k+1) = (1-\gamma_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(k) + \gamma_i u_i, \tag{8}$$

which can be framed as a best response dynamics [21], whereby the payoff of $i \in \mathcal{V}$ for having opinion s is

$$\sigma_i(s|\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{2}\Big[(1-\gamma_i)\sum_{j\in\mathcal{V}} w_{ij}(s-y_j)^2 + \gamma_i(s-u_i)^2\Big]. \tag{9}$$

Consider the following potential function:

$$\Phi(\boldsymbol{y}) = -\frac{1}{2}\sum_{i\in\mathcal{V}}\Big(\sum_{j\in\mathcal{V}}\frac{w_{ij}}{2}(y_i-y_j)^2 + \frac{\gamma_i}{1-\gamma_i}(y_i-u_i)^2\Big).$$

From the fact that $w_{ij} = w_{ji}$ for all $i, j \in \mathcal{V}$, one can obtain that, with a fixed individual $i \in \mathcal{V}$, $\Phi(\boldsymbol{y})$ can be expressed as

$$\Phi(\boldsymbol{y}) = -\frac{1}{2}\sum_{j\in\mathcal{N}_i} w_{ij}(y_i-y_j)^2 - \frac{1}{2}\sum_{k\in\mathcal{V}}\frac{\gamma_k}{1-\gamma_k}(y_k-u_k)^2 - \frac{1}{4}\sum_{k\in\mathcal{V}\setminus\{i\}}\sum_{\ell\in\mathcal{V}\setminus\{i\}} w_{k\ell}(y_k-y_\ell)^2.$$

It can be verified that, for any $s, s' \in [-1,1]$,

$$\sigma_i(s|\boldsymbol{y}) - \sigma_i(s'|\boldsymbol{y}) = (1-\gamma_i)\big(\Phi(s,\boldsymbol{y}_{-i}) - \Phi(s',\boldsymbol{y}_{-i})\big). \tag{10}$$

Verify that Φ can be expressed as

$$\Phi(\boldsymbol{y}) = -\frac{1}{2}\left(\boldsymbol{y}^\top(\boldsymbol{I}_n - \boldsymbol{W} + \boldsymbol{\Gamma})\boldsymbol{y} - 2\boldsymbol{y}^\top\boldsymbol{\Gamma}\boldsymbol{u} + \boldsymbol{u}^\top\boldsymbol{\Gamma}\boldsymbol{u}\right),$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_1/(1-\gamma_1), \ldots, \gamma_n/(1-\gamma_n))$. If $\boldsymbol{\Gamma} = \boldsymbol{0}_{n\times n}$, implying $\gamma_i = 0$ for all $i \in \mathcal{V}$, then $\boldsymbol{I}_n - \boldsymbol{W}$ is known to be positive semidefinite with a single eigenvalue at 0, with associated eigenvector $\boldsymbol{1}$ [22, Chapter 2]. Thus Φ attains its maximum value of $\Phi^* = 0$ at $\boldsymbol{y} = \theta\boldsymbol{1}$, for all scalar θ (note that according to Lemma 1, $\theta \in [-1,1]$). In this instance, $\theta\boldsymbol{1}$ is clearly a steady state of the dynamics Eq. (4). If $\boldsymbol{\Gamma} \neq \boldsymbol{0}_{n\times n}$, then it has at least one strictly positive diagonal entry. According to [22, Theorem 2.3], $\boldsymbol{I}_n - \boldsymbol{W} + \boldsymbol{\Gamma}$ is positive definite, and thus Φ has a unique maximum, Φ^*. By an abuse of notation, let $\Phi(k) = \Phi(\boldsymbol{y}(k))$. Recalling Assumption 1, it follows from Eq. (10) and the fact that Eq. (8) is a best-response to the payoff function Eq. (9) with $\mu_i = 0$, that $\Phi(k)$ is nondecreasing and $\Phi(k+T) - \Phi(k) > 0$, for all $k \geq 0$, and $\Phi(k) \neq \Phi^*$. It follows that $\lim_{k\to\infty}\Phi(k) = \Phi^*$. This in turn implies that the opinions will converge to a steady state, i.e., $\lim_{k\to\infty} y_i(k) = y_i^* \in [-1,1]$ for all i. Hence, for every $\varepsilon > 0$, there exists a $\tau_\varepsilon \in \mathbb{Z}_+$ such that

$$|y_i(k) - y_i^*| \leq \varepsilon\,, \quad \forall\, i \in \mathcal{V} \text{ and } \forall\, k \geq \tau_\varepsilon. \tag{11}$$

Action convergence: consider the potential function

$$\Psi(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{8}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} a_{ij}\Big((1+\alpha)(1+x_j)(1+x_i) + (1-x_j)(1-x_i)\Big) + \sum_{i\in\mathcal{V}}\frac{\lambda_i d_i}{1-\lambda_i} y_i x_i.$$

Since $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$, it follows that

$$\Psi(+1,\boldsymbol{x}_{-i},\boldsymbol{y}) - \Psi(-1,\boldsymbol{x}_{-i},\boldsymbol{y}) = \frac{1}{2}\sum_{j\in\mathcal{V}} a_{ij}(2x_j + \alpha(1+x_j)) + 2\frac{\lambda_i d_i}{1-\lambda_i} y_i. \tag{12}$$

Next, one can use Eq. (2) and that Eq. (12) to obtain

$$\pi_i(+1\,|\,\boldsymbol{x},\boldsymbol{y}) - \pi_i(-1\,|\,\boldsymbol{x},\boldsymbol{y}) = \frac{1-\lambda_i}{d_i}\left(\Psi(+1,\boldsymbol{x}_{-i},\boldsymbol{y}) - \Psi(-1,\boldsymbol{x}_{-i},\boldsymbol{y})\right), \quad (13)$$

with the final equality obtained from Eq. (12). Because $\boldsymbol{x}$ belongs to the set $\{-1,+1\}^n$ of finite size, this implies that, for $s, s' \in \{-1,+1\}$, there exists a $\varphi > 0$ satisfying

$$\varphi \triangleq \min_{i\in\mathcal{V},\boldsymbol{x}\in\{-1,+1\}^n} \frac{d_i}{1-\lambda_i}\left(\pi_i(s\,|\,\boldsymbol{x},\boldsymbol{y}^*) - (s'\,|\,\boldsymbol{x},\boldsymbol{y}^*)\right), \quad (14)$$

subject to $\pi_i(s\,|\,\boldsymbol{x},\boldsymbol{y}^*) - \pi(s'\,|\,\boldsymbol{x},\boldsymbol{y}^*) > 0$. In other words, φ is the minimum scaled payoff increase, among all individuals, for switching action when following the best-response dynamics in Eq. (3), with scaling constant $d_i/(1-\lambda_i)$.

Consider a positive constant $\varepsilon \geq 0$ satisfying $\varepsilon < \frac{\varphi}{2}\min_{i\in\mathcal{V}}\frac{1-\lambda_i}{\lambda_i d_i}$. Without loss of generality, suppose that, for some $k \geq \tau_\varepsilon$, individual i is activated and changes action from $x_i(k) = -1$ to $x_i(k+1) = +1$. By an abuse in notation, let $\Psi(k) = \Psi(\boldsymbol{x}(k),\boldsymbol{y}(k))$. Following similar computations to those in Eq. (12), observe that

$$\Psi(k+1) - \Psi(k) = \frac{d_i}{1-\lambda_i}\big(\pi_i(+1|\boldsymbol{x},\boldsymbol{y}) - \pi_i(-1|\boldsymbol{x},\boldsymbol{y})\big) + \frac{\lambda_i d_i}{1-\lambda_i}\big(y_i(k+1) + y_i(k) - 2y_i^*\big) > \varphi - \frac{2\lambda_i d_i}{1-\lambda_i}\varepsilon > 0, \quad (15)$$

where the second equality is obtained by adding and subtracting $2\lambda_i d_i y_i^*/(1-\lambda_i)$ and substituting in Eq. (13), the first inequality comes from Eq. (11), and the last inequality from Eq. (14) and the definition of ε. In other words, for any $k \geq \tau_\varepsilon$, if the individual active at time k changes action, the potential Ψ necessarily increases from k to $k+1$.

Let $k' \geq \tau_\varepsilon$ be some time instant immediately after i has changed action (if no such k' exists, the actions are already at a steady state), i.e., $x_i(k'-1) \neq x_i(k')$. Let $k'' > k'$ be the next time instant in which an individual's action has changed, i.e. $\boldsymbol{x}(k') = \boldsymbol{x}(k'+1) = \ldots = \boldsymbol{x}(k''-1) \neq \boldsymbol{x}(k'')$. Obviously, if $k'' = \infty$, then the actions have already converged to a steady state: $\lim_{k\to\infty}\boldsymbol{x}(k) = \boldsymbol{x}(k')$.

Suppose that $k'' < \infty$. We are going to show that there exists a sufficiently small $\bar{\varepsilon}$ such that for all $k' \geq \tau_{\bar{\varepsilon}} \geq \tau_\varepsilon$, the potential function increases as $\Psi(k'') > \Psi(k')$. Let $\varepsilon' \leq \varepsilon$ satisfy $k' = \tau_{\varepsilon'}$. From Eq. (11), observe that

$$|\Psi(k''-1) - \Psi(k')| \leq 2\varepsilon'(k''-k'-1)\max_{i\in\mathcal{V}}\frac{\lambda_i d_i}{1-\lambda_i}.$$

In other words, between time instants k' and $k''-1$, the potential Ψ decreases by at most $2\varepsilon'(k''-k'-1)\max_{i\in\mathcal{V}}\lambda_i d_i/(1-\lambda_i)$. However, the analysis leading to Eq. (15) implies that $\Psi(k'') - \Psi(k''-1) > \varphi - 2\varepsilon'\min_{k\in\mathcal{V}}\frac{\lambda_k d_k}{1-\lambda_k}$. It follows that

$$\varphi - 2\varepsilon'\left(\min_{k\in\mathcal{V}}\frac{\lambda_k d_k}{1-\lambda_k} + (k''-k'-1)\max_{i\in\mathcal{V}}\frac{\lambda_i d_i}{1-\lambda_i}\right) > 0, \quad (16)$$

which implies $\Psi(k'') - \Psi(k') > 0$. From the fact that $\varepsilon' \to 0$ as $k' \to \infty$ as recorded above Eq. (11), and $k' < k'' < \infty$, we conclude that there exists a $\bar{k}$ such that Eq. (16) holds for all $k' \geq \bar{k}$. Thus, there exists a $\bar{\varepsilon}$ satisfying $\bar{k} = \tau_{\bar{\varepsilon}}$ such that the potential function Ψ, which is bounded from above, converges to a (possibly local) maximum, implying convergence of the actions to a steady state. □

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