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A note on very ample Terracini loci

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Abstract
In this short note we show that, for any ample embedding of a variety of dimension at least two in a projective space, all high enough degree Veronese re-embeddings have non-empty Terracini loci.

Keywords Secant variety · Terracini locus

Mathematics Subject Classification 14N05

1 Introduction

Terracini loci were introduced by the first author and Chiantini in [2]. Their emptiness implies non-defectivity of secant varieties due to the celebrated Terracini's lemma, whereas the converse is not true: there exist non-empty Terracini loci even in the presence of non-defective secants. This triggered the interest for this geometric notion, leading to the results in the aforementioned article. The Terracini locus has been the subject of recent investigations [3, 4], especially for Segre and Veronese varieties, that are crucial in the context of tensors. We start off by defining set-theoretically these loci.

Definition Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety of dimension $n \geq 1$ over an algebraically closed field $\mathbb{K}$. Let $S \subset X_{\mathrm{reg}}$ be a finite subset of smooth points of X whose cardinality is k. Let $(2S, X)$ be the union of the corresponding 2-fat points $(2p, X)$ supported at the points $p \in S$. Then S is in the kth *Terracini locus* $\mathbb{T}_k(X)$ if and only if $h^0(\mathcal{I}_{(2S,X)}(1)) > 0$ and $h^1(\mathcal{I}_{(2S,X)}(1)) > 0$. Equivalently, S is in $\mathbb{T}_k(X)$ whenever the n-dimensional tangent spaces T_pX, for $p \in S$, are linearly dependent and their projective linear span is not the ambient space $\mathbb{P}^N$.

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A consequence of a deep result of Alexander and Hirschowitz [1, Theorem 1.1 and Corollary 1.2] (where in their notation one chooses $m = 2$) states that for any projective variety X there exists a very ample embedding such that all the secant varieties of X under this embedding are non-defective. The aim of this note is to point out that, even in this very ample regime, the emptiness of the corresponding Terracini locus *does not* generally hold. Thus we answer in the negative the question whether a statement similar to the one by Alexander and Hirschowitz works for Terracini loci.

2 Very ample regime

Let $\mathbb{K}$ be an algebraically closed field and let X be a projective variety of dimension n over $\mathbb{K}$. We say that an embeddeding $X \subset \mathbb{P}^r$ of X is not secant defective if for each positive integer k the k-secant variety of X has dimension $\min\{r, k(n+1)-1\}$. For a very ample line bundle L on X, let $\nu_L : X \to |L|^\vee$ denote the associated embedding. The kth secant variety and the kth Terracini locus of $\nu_L(X)$ are denoted $\sigma_k(\nu_L(X))$ and $\mathbb{T}_k(\nu_L(X))$, respectively. We say that $\nu_L(X)$ is *secant non-defective* if $\sigma_k(\nu_L(X))$ is non-defective for *every* $k \geq 1$.

Theorem 1 *Let $n \geq 2$ and X be as above. Let $F, L \in \mathrm{Pic}(X)$, where L is an ample line bundle. Then there exists an integer m_0 (depending only on X, F, L) such that for all $m \geq m_0$ the line bundle $F + mL$ is very ample, $\nu_{F+mL}(X)$ is secant non-defective, and there exists $k > 0$ such that $\sigma_k(\nu_{F+mL}(X)) \neq |F + mL|^\vee$ and $\mathbb{T}_k(\nu_{F+mL}(X)) \neq \emptyset$.*

Proof Let $L = \mathcal{L}(D)$ and define $\alpha = D \cdots D > 0$, the n times self-intersection of the Cartier divisor D. Fix an integral curve $Y \subset X$ such that $Y \cap X_{\mathrm{reg}} \neq \emptyset$, where Y is possibly singular. Let $\beta = Y \cdot D \cdots D$, the intersection of Y with $n-1$ copies of D, i.e. $\beta = \deg(L_{|Y})$ and $\beta > 0$ because L is ample. Fix a real number ε such that $\alpha > \varepsilon > 0$. By the result of Alexander and Hirschowitz [1, Theorem 1.1], by the asymptotic Riemann-Roch and by the ampleness of L, we find an integer m_1 such that for all $m \geq m_1$ we have that: $F + mL$ is very ample, $\nu_{F+mL}(X)$ is secant non-defective, and $h^0(F + mL) \geq \frac{\alpha-\varepsilon}{n!} m^n$.

Thus, for $1 \leq k < \left\lfloor \frac{\alpha-\varepsilon}{(n+1)!} m^n \right\rfloor$, we have $\sigma_k(\nu_{F+mL}(X)) \subsetneq |F+mL|^\vee$. By the asymptotic Riemann-Roch, $h^0(Y, (F + mL)|_Y)$ grows like a linear function of the form βm. Therefore there exists $m_0 \geq m_1$ such that for all $m \geq m_0$ one has $1 \leq h^0(Y, (F + mL)|_Y)/2 < \left\lfloor \frac{\alpha-\varepsilon}{(n+1)!} m^n \right\rfloor$.

Define $k - 1 = \lceil h^0(Y, (F + mL)|_Y)/2 \rceil$. Note that the projective linear span of the curve Y has dimension $\dim\langle Y\rangle \leq 2k-3$. Fix a set $S \subset Y \cap X_{\mathrm{reg}}$ with cardinality k. The zero-dimensional scheme $(2S, X) \cap Y \subset Y$ has degree at least $2k$. Hence, if $(2S, X) \cap Y \subset Y$ was linearly independent, then its projective linear span would be at least $(2k-1)$-dimensional. Therefore $(2S, X) \cap Y$ is linearly dependent, i.e. $h^1(\mathcal{I}_{(2S,X)\cap Y}(1)) > 0$. Moreover, since $k < \left\lfloor \frac{\alpha-\varepsilon}{(n+1)!} m^n \right\rfloor$ and $\deg((2S, X)) = k(n + 1)$, the projective linear span of this scheme cannot fill the ambient space, i.e. one has $h^0(\mathcal{I}_{(2S,X)}(1)) > 0$.

Now, let $Z \subset W$ be two zero-dimensional schemes. Then one has the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_W(1) \longrightarrow \mathcal{I}_Z(1) \longrightarrow \mathcal{I}_Z(1)/\mathcal{I}_W(1) \longrightarrow 0.$$

Here the cokernel sheaf is either zero or supported on a zero-dimensional scheme. Taking the long exact sequence in cohomology, we then find a surjective map in cohomology

$H^1(\mathcal{I}_W(1)) \twoheadrightarrow H^1(\mathcal{I}_Z(1))$. The zero-dimensional scheme $(2S, X)\cap Y$ is a closed subscheme of $(2S, X)$ and so we likewise have a surjection

$$H^1(\mathcal{I}_{(2S,X)}(1)) \twoheadrightarrow H^1(\mathcal{I}_{(2S,X)\cap Y}(1)).$$

Therefore $h^1(\mathcal{I}_{(2S,X)}(1)) > 0$ too. So any collection of k smooth points of $Y \cap X_{\mathrm{reg}}$ are in the kth Terracini locus of $\nu_{F+mL}(X)$. □

Remark 2 Let $X \subset \mathbb{P}^N$ be a projective variety with $\dim X = n \geq 2$ and consider $\nu_d(X)$. For any integer $k > 0$, the set $S^k\nu_d(X_{\mathrm{reg}})$ of all subsets of $\nu_d(X_{\mathrm{reg}})$ with cardinality k is a variety of dimension kn. For $d \gg 0$, the families of $S \in \mathbb{T}_k(\nu_d(X))$ we found in the proof of Theorem 1 on a fixed curve Y have codimension k in $S^k\nu_d(X_{\mathrm{reg}})$. Varying Y, we do not decrease significantly the codimension of $\mathbb{T}_k(\nu_d(X))$ in $S^k\nu_d(X_{\mathrm{reg}})$: the magnitude of this is $O(k)$. We do not have examples for which, when k is increasing with d, $\mathbb{T}_k(\nu_d(X))$ has codimension 1 in $S^k\nu_d(X_{\mathrm{reg}})$, which is the least codimension allowed in view of the secant non-defectivity result in [1].

Proposition 3 *Let $N \geq 1$ and let $C \subset \mathbb{P}^N$ be a smooth and non-degenerate rational curve of degree d. For all $d' \geq d+1-N$, the curve $\nu_{d'}(C) \subset \langle \nu_{d'}(C)\rangle$ has empty Terracini loci.*

Proof Suppose $N = d = 1$ so that $C = \mathbb{P}^1$. For $d' \geq 1$, consider the rational normal curve $\nu_{d'}(\mathbb{P}^1)$. Its kth Terracini locus consists of those subsets $S \subset \mathbb{P}^1$ such that $(2\,S, \nu_{d'}(\mathbb{P}^1))$ does not span $\langle \nu_{d'}(C)\rangle$, i.e. $h^0(\mathcal{I}_{2S}(d')) > 0$, and such that $h^1(\mathcal{I}_{2S}(d')) > 0$. Since $C = \mathbb{P}^1$, for any zero-dimensional scheme $Z \subset C$ either $h^0(\mathcal{I}_Z(d')) = 0$ or $h^1(\mathcal{I}_Z(d')) = 0$. Hence any Terracini locus of the rational normal curve $\nu_{d'}(C)$ is empty.

For the general case, let $d \geq 2$ and $d' \geq d+1-N$. One has $h^1(\mathcal{I}_C(d')) = 0$ [5, Theorem p. 492]. Hence $\nu_{d'}(C)$ is an embedding of $\mathbb{P}^1$ by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(d\cdot d')|$. So this has empty Terracini loci by the first part. □

The case of curves with positive arithmetic genus is treated in the following proposition. Here different behaviours appear according to the parity of the degree.

Proposition 4 *Let C be an integral projective curve over $\mathbb{K}$, with $\mathrm{char}(\mathbb{K}) \neq 2$, whose arithmetic genus is $g > 0$. Let F and L be line bundles on C, where L is ample, of degrees $\alpha = \deg(L)$ and $\beta = \deg(F)$. For each integer $m > 0$, consider the complete linear system $|F+mL|$. Assume that $\beta + m\alpha \geq 4g+2$ and assume that $\beta+m\alpha$ is even. Then $\nu_{F+mL}(C)$ has a non-empty Terracini locus.*

Proof Recall that a line bundle E on C is very ample if $\deg(E) \geq 2\,g+1$ [6, Corollary 3.2, Chapter IV]. Since the Picard group $\mathrm{Pic}^0(C)$ is a quasi-projective irreducible group and $\mathrm{char}(\mathbb{K}) \neq 2$, the kernel of the multiplication morphism $\otimes 2 : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)$ is finite. So this morphism is surjective. Since $\deg(F+mL)$ is even and $\otimes 2$ is surjective, there is a line bundle R_m such that $R_m^{\otimes 2} \cong F+mL$. Thus $\deg(R_m) = (\beta+m\alpha)/2$. Since $\beta + m\alpha \geq 4\,g+2$, the line bundle R_m is very ample. Thus $|R_m| \neq \emptyset$ and a general $S \in |R_m|$ consists of k distinct reduced points and $S \subset C_{\mathrm{reg}}$. Note that $2\,S \in |F+mL|$ and hence $\langle 2\nu_{F+mL}(S)\rangle \subsetneq |F+mL|^{\vee}$ is a hyperplane. Since $\deg(F+mL) > 2g-1$, one has $h^0(F+mL) = \deg(F+mL)+1-g = \deg(2\,S)+1-g$. Since $g > 0$, $2\,S$ does not give $\deg(2\,S)$ independent conditions to $|F+mL|$. Then, by definition, $\nu_{F+mL}(S)$ is in the kth Terracini locus of $\nu_{F+mL}(C)$. □

Corollary 5 *Let $C \subset \mathbb{P}^N$ be an integral and non-degenerate projective curve with arithmetic genus $g = 1$ of degree d over $\mathbb{K}$, with $\mathrm{char}(\mathbb{K}) \neq 2$. If $d' \geq d+1-N$ and $d\cdot d'$ is even, then $\mathbb{T}_{d\cdot d'/2}(\nu_d(C)) \neq \emptyset$. If $d\cdot d'$ is odd, then all Terracini loci of $\nu_{d'}(C)$ are empty.*

Proof Since $d' \geq d + 1 - N$, we have $h^1(\mathcal{I}_C(d')) = 0$ [5, Theorem p. 492]. Hence $\nu_{d'}(C)$ is an embedding of C by a complete linear system. By Proposition 4, if $d \cdot d'$ is even, then $\mathbb{T}_{d \cdot d'/2}(\nu_{d'}(C)) \neq \emptyset$.

Suppose a line bundle L on C has $\deg(L) = 2m + 1$; let $S \subset C_{\mathrm{reg}}$ have cardinality k. Then $\deg(L(-2\,S)) = 2(m - k) + 1 \neq 0$. If $\deg(L(-2\,S)) < 0$, then $h^0(L(-2\,S)) = 0$. If $\deg(L(-2S)) > 0$, by Serre duality, we find $h^1(L(-2S)) = 0$. Therefore any Terracini locus is empty. □

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Declarations

Conflict of Interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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