APPROXIMATION OF VISCOUS TRANSPORT AND CONSERVATIVE EQUATIONS WITH ONE SIDED LIPSCHITZ VELOCITY FIELDS*

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Abstract. The aim of this work is to investigate semi-Lagrangian approximation schemes on unstructured grids for viscous transport and conservative equations with measurable coefficients that satisfy a one-sided Lipschitz condition. To establish the convergence of the schemes, we exploit the characterization of the solution for these equations expressed in terms of measurable time-dependent viscosity solution and, respectively, duality solution. We supplement our theoretical analysis with various numerical examples to illustrate the features of the schemes.

Key words. one-sided Lipschitz condition, viscous transport equation, measure-valued solution, semi-Lagrangian schemes.

MSC codes. 35K20, 35D30, 49L25, 65M12.

1. Introduction.

We investigate semi-Lagrangian approximation schemes for the viscous transport equation

$$-\frac{\partial u}{\partial t} - \mathrm{tr}[a(t,x)D^2u] + b(t,x)\cdot Du = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \quad \text{with } u(T,\cdot) = u_T, \tag{1.1}$$

and the viscous conservative equation

$$\frac{\partial f}{\partial t} - D^2\cdot[a(t,x)f] - \mathrm{div}(b(t,x)f) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \quad \text{with } f(0,\cdot) = f_0. \tag{1.2}$$

The well-posedness of these problems has been extensively studied in the literature. For the first-order case where $a(t,x) \equiv 0$, relevant results can be found in [1, 15]. The second-order case has been investigated in [21, 17], and a more comprehensive list of references is available in [23]. These works generally assume that the divergence of b is absolutely continuous with respect to the Lebesgue measure.

In this work, we consider $b : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ as a measurable function satisfying the one-sided Lipschitz condition ((OSL) condition in short)

$$(b(t,x) - b(t,y))\cdot(x-y) \geq -C(t)|x-y|^2, \quad \forall t\in[0,T],\ x,y\in\mathbb{R}^d, \tag{1.3}$$

for some $C \in L^1(0,T)$ with $C \geq 0$. Additionally, a is assumed to be a regular, degenerate matrix. Since the (OSL) framework does not rule out the possibility that the measure $\mathrm{div}\, b$ may include a singular component, this limitation prevents the application of the aforementioned theories to the present situation.

For the first-order case satisfying the (OSL) condition, the problems (1.1) and (1.2) have been addressed for example in [5, 6, 10, 28] (see Remark 2.11 concerning the sign of b). In [23], a general theory has been recently developed both in the first order case and in the second order one. In this latter work, various characterizations of the "good" solutions, i.e. solutions obtained as limits of equations with regular approximations of b, are provided. Specifically, the "good" solutions are characterized as measurable time-dependent viscosity solutions for the transport equation, and duality solutions for the conservative equation.

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Regarding the approximation of these equations in the regular case, a vast body of literature exists due to their relevance in diverse fields such as fluid dynamics, financial mathematics, optimal transport, and Mean-Field Games. For first-order equations satisfying the (OSL) condition, we refer to [13, 14, 20, 7]. The analysis of upwind-type finite volume schemes with rough coefficients in the DiPerna–Lions setting has been carried out also in [26, 27], where both explicit and implicit formulations are interpreted probabilistically in terms of stochastic flows, while the second-order case has been recently studied in [25]. Our approach shares this stochastic interpretation but focuses on the (OSL) framework, which has, at our knowledge, not treated elsewhere and it extends beyond the DiPerna–Lions regularity assumptions.

Here, we examine semi-Lagrangian approximation schemes for (1.1) and (1.2) under the (OSL) condition. For an introduction to semi-Lagrangian scheme for the first-order case, we refer to [16], while the second-order case has been studied, for instance, in [4, 8, 11, 12]. Semi-Lagrangian methods offer a distinct advantage over traditional approaches like finite differences and finite volumes. Instead of directly approximating the partial differential equations themselves, they focus on approximating the underlying dynamics. This allows to employ the assumption (OSL) to derive properties of the discrete trajectories, which subsequently inform the approximation scheme. These properties enable us to establish the convergence of the semi-Lagrangian approximation scheme for the viscous transport equation by exploiting the characterization of the solution as time-measurable viscosity solution. The proof utilizes a classical stability result by Barles and Souganidis [3], adapted to this specific context.

For the conservation equation, we adopt the scheme proposed in [11, 12], which is designed to handle linear and nonlinear Fokker-Planck equations with continuous coefficients on structured grids. For linear equations, our work generalizes their results to encompass cases involving irregular coefficients and general unstructured grids. Moreover, we precisely characterize the scheme's limit as a duality solution of (1.2). These generalizations present new challenges, particularly in establishing the regularity properties of the discrete solution, which are critical for achieving convergence in the limit. At the discrete level, we derive a duality formula analogous to the one in the continuous setting and, exploiting this formula and the convergence result for (1.1), we prove that the scheme converges to the duality solution of (1.2) in the sense of weak convergence of measures.

While the results presented in this paper apply to the first-order problem, it is important to note that, in this case, error estimates for a finite difference scheme are derived in terms of the $L^\infty([0,T], L^1(\mathbb{R}^d))$ norm for the transport equation (see [13]) and the p-Wasserstein distance for conservative equation (see [14]).

In the numerical section, we show in practice the performances of the numerical techniques proposed here. We consider an example in a one-dimensional case, where it is possible to compare the approximation with the analytical solution of the problem and two examples in the two dimensional case, the first one with a continuous vector field, and the second one with a discontinuous vector field verifying the (OSL) condition.

Notations. Let $\mathcal{B}(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$ represent the set of bounded and bounded continuous functions respectively, and $C^{0,1}(\mathbb{R}^d)$ the set of the Lipschitz continuous function in $\mathbb{R}^d$. We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of non-negative measures on $\mathbb{R}^d$, defined with respect to the Borel σ-algebra $\Sigma_B(\mathbb{R}^d)$. This space is equipped with the weak topology $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$. Next, we consider the space $C([0,T];\mathcal{M}(\mathbb{R}^d))$, which represents flows of measures. This space is endowed with the topology of uniform convergence over $[0,T]$.

Let $\mathcal{P}(\mathbb{R}^d)$ be the subset of $\mathcal{M}(\mathbb{R}^d)$ consisting of probability measures defined on $(\mathbb{R}^d, \Sigma_B(\mathbb{R}^d))$. Furthermore, we define $\mathcal{P}_p(\mathbb{R}^d)$ as the space of probability measures that have a finite p-th order moment for $p \geq 1$, given by:

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \, \mu(dx) < \infty \right\}.$$

This space is equipped with the standard Wasserstein distance $\mathcal{W}_p$ (see [30]). For $p = 1$, by the Kantorovich–Rubinstein duality formula, we have for $\mu, \mu' \in \mathcal{P}_1(\mathbb{R}^d)$

$$\mathcal{W}_1(\mu, \mu') = \sup \left\{ \int_{\mathbb{R}^d} \phi(x) d(\mu - \mu') : \phi : \mathbb{R}^d \longrightarrow \mathbb{R} \text{ is 1-Lipschitz continuous} \right\}.$$

For a locally bounded function $g : \mathbb{R}^d \to \mathbb{R}$, we define

$$\liminf_{z \to x} {}_* \, g(z) = \lim_{r \to 0} \inf\{g(z) : z \in B(0, r)\},$$
$$\limsup_{z \to x} {}^* \, g(z) = \lim_{r \to 0} \sup\{g(z) : z \in B(0, r)\}.$$

2. The model problem. The results of this section are taken from [23], whose notation we will adhere to. We assume that $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a measurable function satisfying: for all $t \in [0, T]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \frac{|b(t, x)|}{1 + |x|} \leq C_0(t), \\ &(b(t, x) - b(t, y)) \cdot (x - y) \geq -C_1(t)|x - y|^2, \end{aligned} \tag{2.1}$$

for some non negative functions $C_0, C_1 \in L^1(0, T)$. Moreover $a = \frac{1}{2}\sigma\sigma^T$ for $\sigma : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times r}$ satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{|\sigma(t, x)|}{1 + |x|} + \sup_{y, z \in \mathbb{R}^d} \frac{|\sigma(t, y) - \sigma(t, z)|}{|y - z|} \leq C_2(t) \tag{2.2}$$

for a non negative function $C_2 \in L^2(0, T)$. We consider the SDE

$$\begin{cases} d_s \Phi_{s,t}(x) = -b(s, \Phi_{s,t}(x))ds + \sigma(s, \Phi_{s,t}(x))dW_s & s \in [t, T], \\ \Phi_{t,t}(x) = x \in \mathbb{R}^d, \end{cases} \tag{2.3}$$

where $W : \Omega \times [0, T] \longrightarrow \mathbb{R}^r$ is a standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

PROPOSITION 2.1. *For every $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mathbb{P}$-almost surely, there exists a unique strong solution $\Phi_{s,t}(x)$ of (2.3) defined on $[t, T] \times \mathbb{R}^d$. For all $p \in [2, \infty)$, there exists a constant $C = C_p > 0$ depending only on the assumptions (2.1) and (2.2) such that*

$$\mathbb{E}|\Phi_{s,t}(x) - \Phi_{s,t}(y)|^p \leq C|x - y|^p \quad \textit{for all } 0 \leq t \leq s \leq T \textit{ and } x, y \in \mathbb{R}^d, \tag{2.4}$$

$$\mathbb{E}|\Phi_{s,t}(x)|^p \leq C(|x|^p + 1) \quad \textit{for all } 0 \leq t \leq s \leq T \textit{ and } x \in \mathbb{R}^d, \tag{2.5}$$

and

$$\mathbb{E}|\Phi_{s_1,t}(x) - \Phi_{s_2,t}(x)|^p \leq C(1 + |x|)|s_1 - s_2|^{p/2}$$

for all $t \in [0, T]$, $s_1, s_2 \in [t, T]$, and $x \in \mathbb{R}^d$. Moreover, for all $0 \leq r \leq s \leq t \leq T$, we have $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$ with probability one.

If $(b^\varepsilon)_{\varepsilon > 0}$ is a regularization of b satisfying

$$\begin{cases} (b^\varepsilon)_{\varepsilon > 0} \subset L^1([0, T], C^{0,1}(\mathbb{R}^d)), \quad \lim_{\varepsilon \to 0} b^\varepsilon = b \textit{ a.e. in } [0, T] \times \mathbb{R}^d, \textit{ and} \\ b^\varepsilon \textit{ satisfies } (2.1) \textit{ uniformly in } \varepsilon > 0, \end{cases} \tag{2.6}$$

then the corresponding stochastic flows Φ^ε converge locally uniformly to Φ as $\varepsilon \to 0$ with probability one.

2.1. The viscous transport equation.

THEOREM 2.2. *Let $u_T \in C(\mathbb{R}^d)$ and define $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ by*

$$u(t,x) = \mathbb{E}[u_T(\Phi_{T,t}(x))]. \tag{2.7}$$

If $(b^\varepsilon)_{\varepsilon>0}$ satisfy (2.6) and u^ε is the solution of (1.1) with velocity b^ε, then, as $\varepsilon \to 0$, u^ε converges locally uniformly to u. Moreover, if $u_T \in C^{0,1}(\mathbb{R}^d)$, then

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(t,x)-u(t,y)|}{|x-y|} + \sup_{(r,s,x)\in[0,T]\times\mathbb{R}^d} \frac{|u(r,x)-u(s,x)|}{|r-s|^{1/2}(1+|x|)} < \infty,$$

and u is a distributional solution of (1.1).

Although (2.7) represents a distributional solution that arises uniquely through the regularization of b, distributional solutions are generally not unique. In [23], it is proved that the function u given by (2.7) can be uniquely identified as the solution of (1.1) in time measurable viscosity sense. We define, for $(t,x,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\underline{b}(t,x,p) = \liminf_{z\to x}{}_* b(t,z)\cdot p \quad \text{and} \quad \overline{b}(t,x,p) = \limsup_{z\to x}{}^* b(t,z)\cdot p. \tag{2.8}$$

For fixed $(t,x) \in [0,T] \times \mathbb{R}^d$, $\underline{b}(t,x,\cdot)$ and $\overline{b}(t,x,\cdot)$ are Lipschitz continuous on $\mathbb{R}^d$, and, for fixed $(t,p) \in [0,T]$, $\underline{b}(t,\cdot,p)$ and $\overline{b}(t,\cdot,p)$ are respectively lower semicontinous (l.s.c.) and upper semicontinuous (u.s.c.).

DEFINITION 2.3. *An u.s.c. (resp., l.s.c.) function u is called a viscosity subsolution (resp. supersolution) of (1.1) if $u(\cdot,T) \le u_T$ (resp., $\ge$) and for all $\psi \in C^2(\mathbb{R}^d)$*

$$-\frac{d}{dt}\max_{x\in\mathbb{R}^d}\{u(t,x)-\psi(x)\} \\ \le \inf\left\{\mathrm{tr}[a(t,y)D^2\psi(y)] - \underline{b}(t,y,D\psi(y)) : y \in \arg\max\{u(t,\cdot)-\psi(\cdot)\}\right\}$$

$$\Big(\textit{resp.}\quad -\frac{d}{dt}\min_{x\in\mathbb{R}^d}\{u(t,x)-\psi(x)\} \\ \ge \sup\left\{\mathrm{tr}[a(t,y)D^2\psi(y)] - \overline{b}(t,y,D\psi(y)) : y \in \arg\min\{u(t,\cdot)-\psi(\cdot)\}\right\}\Big)$$

in distributional sense. If $u \in C([0,T]\times\mathbb{R}^d)$ is both a subsolution and supersolution, we say that u is a viscosity solution.

We have the following Comparison Principle:

THEOREM 2.4. *If u and v are respectively a sub and supersolution of (1.1) such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{u(t,x)}{1+|x|} + \sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\frac{-v(s,y)}{1+|y|} < \infty,$$

then $t \mapsto \sup_{x\in\mathbb{R}^d}\{u(t,x)-v(t,x)\}$ is nondecreasing.

As a consequence of the previous result, the "good" distributional solution of (1.1) can be uniquely characterized.

COROLLARY 2.5. *Assume $u_T \in C(\mathbb{R}^d)$ is uniformly continuous and $u_T\cdot(1+|x|)^{-1} \in L^\infty(\mathbb{R}^d)$. Then (2.7) is the unique viscosity solution of (1.1).*

REMARK 2.6. *As noted in [23, Subsection 3.1.3], there generally exist distributional solutions to (1.1) that are not viscosity solutions to the problem. The uniqueness of distributional solutions can be ensured by requiring that $b \in C([0,T]\times\mathbb{R}^d)$ satisfies the (OSL) condition in (2.1) and that $u_T \in C^{0,1}(\mathbb{R}^d)$.*

Remark 2.7 (Filippov interpretation.). *Throughout the paper, the characteristics, i.e. the solution of* (2.3)*, can be interpreted in the sense of Filippov differential inclusions. The map* $x \mapsto \Phi_{s,t}(x)$ *is therefore the unique Filippov flow associated with the discontinuous vector field* b*, which exists and is stable under the OSL condition (see [23]). This framework allows us to treat merely measurable drifts for which classical flows are not available.*

Remark 2.8 (On the one-sided Lipschitz (OSL) condition.). *The one-sided Lipschitz (OSL) assumption plays a fundamental role in the well-posedness of both the transport and conservative equations. In the first-order case, it allows the characteristics associated with the vector field* b *to merge, while preventing them from diverging exponentially. Hence, trajectories starting from distinct points may intersect, but the flow remains uniquely defined in the Filippov sense and satisfies an exponential stability estimate. In contrast, reversing the sign in the OSL inequality would lead to a loss of uniqueness for the backward problem and to an instability of the flow. This case is examined in [23]. In the present work, we instead restrict our attention to the setting where the (OSL) condition holds, as it guarantees uniqueness and stability of the solution, properties essential for constructing a convergent numerical scheme.*

2.2. The conservative equation. For the conservative equation (1.2), the tendency of the backward flow to concentrate on sets of zero Lebesgue measure suggests that, even if f_0 is absolutely continuous with respect to the Lebesgue measure, the solution $f(t,\cdot)$ may develop a singular component for t positive. Hence, it is natural the introduce a notion of solution in the sense of measure.

Definition 2.9. *A map* $f \in C([0,T],\mathcal{M})$ *(where* $\mathcal{M}$ *is the space of non-negative measures on* $\mathbb{R}^d$*) is called a solution of* (1.2) *if, for all* $t \in [0,T]$ *and* $g \in C_b(\mathbb{R}^d)$*,*

$$\int_{\mathbb{R}^d} g(x) f(t,dx) = \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{t,0}(x))] f_0(dx),$$

where $\Phi_{t,0}$ *is the stochastic flow satisfying* (2.3).

Solutions as in the previous definition are called duality solutions because $\mathbb{E}[g(\Phi_{t,0}(x))]$ is the solution of (1.1) with terminal value g at time t (cfr. [9]). If f_0 is a probability measure, then $f(t,\cdot)$ is the law of the stochastic process $\Phi_{t,0}(X_0)$, where X_0 is a random variable with law f_0.

Theorem 2.10. *Assume that* $f_0 \in \mathcal{M}$*. Then, there exists a unique duality solution* f *of* (1.2)*. If, for* $\varepsilon > 0$*,* f^ε *is the solution corresponding to* b^ε *as in* (2.6)*, then, as* $\varepsilon \to 0$*,* f^ε *converges weakly in the sense of measures to* f*. If* $1 \le p \le \infty$*,* $f_0, g_0 \in \mathcal{P}_p$*, and* f *and* g *are the corresponding duality solutions, then, for some* $C > 0$ *depending on* p *and the constants in* (2.1)*,* $\mathcal{W}_p(f(t), g(t)) \le C\mathcal{W}_p(f_0, g_0)$*.*

Remark 2.11. *Note that, in the conservative first-order case, the results presented in this section apply to the equation*

$$\frac{\partial f}{\partial t} + \operatorname{div}(-b(t,x)f) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \quad \text{with } f(0,\cdot) = f_0. \tag{2.9}$$

therefore, consistently with the corresponding second-order problem (1.2) *which requires the use of the forward flow defined by* (2.3)*, the velocity field, also in this first-order case is* $-b$*. This notation choice has been made to keep all the study consistent with [23].*

3. The numerical scheme. In this section, we consider semi-Lagrangian schemes for (1.1) and (1.2) defined on an unstructured mesh. Given $N \in \mathbb{N}$ we set $h := T/N$ and $t_k := kh$ for $k = 0,\dots,N$. Moreover, given $\Delta x > 0$, we consider a triangular mesh $\mathcal{T}^{\Delta x} = (T_\tau)_{\tau \in \mathcal{K}^{\Delta x}}$ such that $\bigcup_{\tau \in \mathcal{K}^{\Delta x}} T_\tau = \mathbb{R}^d$, $\operatorname{int}(T_\tau) \cap \operatorname{int}(T_\lambda) = \emptyset$ if $\tau \neq \lambda$. We assume that $\sup_\tau[\operatorname{diam}(T_\tau)] \le \Delta x$ and $\inf_\tau[\operatorname{height}(T_\tau)] > 0$. We set $\Delta = (h, \Delta x)$.
Given the set of the vertices of the triangulation $\mathcal{V}^{\Delta x} = \{x_i\}_{i \in \mathcal{I}^{\Delta x}}$, we consider a $\mathbb{P}_1$ basis $(\beta_i)_{i \in \mathcal{I}^{\Delta x}}$, where $\beta_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a polynomial of degree less than or equal to 1 and satisfies $\beta_i(x_j) = 1$ if

$i = j$ and $\beta_i(x_j) = 0$, otherwise. Moreover, the support $\mathrm{supp}(\beta_i)$ of β_i is compact and

$$0 \le \beta_i \le 1 \quad \forall\, i \in \mathcal{I}^{\Delta x}, \qquad \sum_{i \in \mathcal{I}^{\Delta x}} \beta_i(x) = 1 \quad \forall\, x \in \mathbb{R}^d. \tag{3.1}$$

Let $B(\mathcal{V}^{\Delta x})$ be the space of bounded functions on $\mathcal{V}^{\Delta x}$ and, for $\phi \in B(\mathcal{V}^{\Delta x})$, let $\phi(i)$ be its value at x_i. We consider the following linear interpolation operator

$$I[\phi](\cdot) := \sum_{i \in \mathcal{I}^{\Delta x}} \beta_i(\cdot)\, \phi(i) \qquad \text{for } \phi \in B(\mathcal{V}^{\Delta x}).$$

Given $\phi \in C_b(\mathbb{R}^d)$, let us define $\hat{\phi} \in B(\mathcal{V}^{\Delta x})$ by $\hat{\phi}(i) := \phi(x_i)$ for all $i \in \mathcal{I}^{\Delta x}$. If $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ is Lipschitz with constant L, then

$$\sup_{x \in \mathbb{R}^d} |I[\hat{\phi}](x) - \phi(x)| \le L \Delta x. \tag{3.2}$$

Moreover, if $\phi \in C^2(\mathbb{R}^d)$ with bounded second order derivatives, then

$$\sup_{x \in \mathbb{R}^d} |I[\hat{\phi}](x) - \phi(x)| \le C \Delta x^2 \tag{3.3}$$

with C independent of Δx.

We consider a standard mollifier $\rho \in C^\infty(\mathbb{R}^d)$, $\rho \ge 0$, $\mathrm{supp}(\rho) \subset B(0,1)$, and $\int_{\mathbb{R}^d} \rho dx = 1$. For $h > 0$, we set

$$\rho_{\sqrt{h}}(x) = h^{-\frac{d}{2}} \rho\left(\frac{x}{\sqrt{h}}\right),$$

where h is the time step.

We define

$$\begin{aligned} b_h^k(x) &= \frac{1}{h} \int_{t_k}^{t_{k+1}} b_h(s,x) ds \quad \text{where} \quad b_h(s,x) = (b(s,\cdot) * \rho_{\sqrt{h}})(x), \\ \sigma_{h,\ell}^k(x) &= \frac{1}{h} \int_{t_k}^{t_{k+1}} \sigma_\ell(s,x) ds, \quad \ell = 1, \dots, r. \end{aligned} \tag{3.4}$$

We approximate the stochastic flow (2.3) by a discrete-time and countably-state space Markov chain $\{\Phi_{k,m}^\Delta(x_s)\ ;\ k = m, \dots, N\}$ with state space $\mathcal{V}^{\Delta x}$ and transition probabilities

$$\begin{aligned} p_{ji}^k &:= \mathbb{P}\left(\Phi_{k+1,m}^\Delta(x_s) = x_i \,\middle|\, \Phi_{k,m}^\Delta(x_s) = x_j\right) \\ &= \frac{1}{2r} \sum_{\ell=1}^r \left[\beta_i(\mathcal{X}_{\ell,+}^k(x_j)) + \beta_i(\mathcal{X}_{\ell,-}^k(x_j))\right], \quad \forall\, i, j \in \mathcal{I}^{\Delta x}, \\ \mathbb{P}&\left(\Phi_{m,m}^\Delta(x_s) = x_j\right) = \delta_{s,j}, \end{aligned} \tag{3.5}$$

where, for $x \in \mathbb{R}^d$, $k = 0, \dots, N-1$ and $\ell = 1, \dots, r$

$$\begin{aligned} \mathcal{X}_{\ell,+}^k(x) &:= x - h b_h^k(x) + \sqrt{rh}\, \sigma_{h,\ell}^k(x), \\ \mathcal{X}_{\ell,-}^k(x) &:= x - h b_h^k(x) - \sqrt{rh}\, \sigma_{h,\ell}^k(x). \end{aligned} \tag{3.6}$$

Remark 3.1. *The previous approximation is motivated by the following construction. Given the equation (2.3), we approximate*

$$\Phi_{t+h,t}(x) = \Phi_{t,t}(x) - \int_t^{t+h} b(s, \Phi_{s,t}(x)) ds + \int_t^{t+h} \sigma(s, \Phi_{s,t}(x)) dW_s$$

by the r-dimensional random walk with N time steps

$$\tilde{\Phi}^h_{t_{k+1},t_k}(x_i) = x_i - h\, b^k_h(x_i) + \sqrt{rh}\, \sigma^k_h(x_i)\, Z_k, \tag{3.7}$$

where b^k_h, σ^k_h defined as in (3.4) and $Z_k = (Z^k_1, \dots, Z^k_r)$ is a r-valued mean-zero unit-variance random variable, i.e. for all $\ell = 1, \dots, r$,

$$\mathbb{P}(Z^k_\ell = 1) = \mathbb{P}(Z^k_\ell = -1) = \frac{1}{2r} \qquad \text{and} \qquad \mathbb{P}\left(\bigcup_{1 \le \ell_1 < \ell_2 \le r} \{Z^k_{\ell_1} \neq 0\} \cap \{Z^k_{\ell_2} \neq 0\} \right) = 0. \tag{3.8}$$

If $\tilde{\Phi}^h_{t_{k+1},t_k}(x_i)$ is not a point of the grid, we interpolate the value by means of the basis $(\beta_i)_{i \in \mathcal{I}^{\Delta x}}$ to get the Markov chain (3.5). For more details, we refer to [8, 12].

Remark 3.2 (Motivation for the spatial regularization). *The spatial smoothing of b with a mollifier of width $\sqrt{h}$ is not a purely technical device, but an essential part of the scheme. Choosing the same scale $\sqrt{h}$ as the time step guarantees consistency between the discrete stochastic flow and the continuous dynamics. This regularization also allows us to exploit the discrete counterpart of the one-sided Lipschitz stability estimate, mirroring the behavior of the continuous Filippov flow.*

A classical choice of mollifier in $\mathbb{R}^d$ is

$$\rho(x) = \begin{cases} C_d \exp\left(-\dfrac{1}{1 - |x|^2} \right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \ge 1, \end{cases}$$

where the constant $C_d > 0$ is chosen so that $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. For any $h > 0$, we define the scaled mollifier

$$\rho_{\sqrt{h}}(x) = h^{-d/2} \rho\left(\frac{x}{\sqrt{h}} \right),$$

which satisfies $\int_{\mathbb{R}^d} \rho_{\sqrt{h}}(x)\, dx = 1$ and $\operatorname{supp} \rho_{\sqrt{h}} = \overline{B(0, \sqrt{h})}$.

The regularization procedure adopted here is reminiscent of the technique proposed by Carlini and Silva [12], where the drift field is mollified to recover Peano-type solutions of the conservative equation. Our framework, however, differs in that b is merely measurable and the convergence analysis relies on the duality between the transport and conservative formulations, together with the stability of viscosity solutions.

It is worth also to notice that since b is only measurable, one must ensure that the formulation of (1.1) and (1.2) is independent of the particular representative of b. Replacing b by another measurable vector field that coincides with it almost everywhere does not change the associated flow or the PDE solutions, because the regularized fields b_ε converge to the same limit almost everywhere. Hence, the notion of viscosity and duality solutions, both given in a distributional sense, as suggested in [23], is well defined modulo changes on null sets.

3.1. The numerical scheme for the viscous transport equation. For a function $u : \{0, \dots, N\} \times B(\mathcal{V}^{\Delta x}) \longrightarrow \mathbb{R}$, $u_k(i)$ denotes its value at time step k and grid point x_i. We consider the following backward explicit scheme for (1.1)

$$\begin{cases} u^\Delta_k(i) = S^\Delta(u^\Delta_{k+1}, i, k) & \text{for all } i \in \mathcal{I}^\Delta,\ k = 0, \dots, N-1, \\ u^\Delta_N(i) = u_T(x_i), & \text{for all } i \in \mathcal{I}^\Delta, \end{cases} \tag{3.9}$$

where $S^\Delta : B(\mathcal{V}^{\Delta x}) \times \mathcal{I}^{\Delta x} \times \{0, \dots, N-1\} \longrightarrow \mathbb{R}$ is defined as

$$\begin{aligned} S^\Delta(w, i, k) :=\ & \tfrac{1}{2r} \textstyle\sum_{\ell=1}^r \Big(I[w](\mathcal{X}^k_{\ell,+}(x_i)) + I[w](\mathcal{X}^k_{\ell,-}(x_i)) \Big) \\ =\ & \textstyle\sum_{j \in \mathcal{I}^{\Delta x}} \tfrac{1}{2r} \sum_{\ell=1}^r \Big(\beta_j(\mathcal{X}^k_{\ell,+}(x_i)) + \beta_j(\mathcal{X}^k_{\ell,-}(x_i)) \Big) w_j. \end{aligned} \tag{3.10}$$

and $\mathcal{X}^k_{\ell,\pm}(x_i)$ defined in (3.6). Recalling (3.5), the scheme (3.9) can be rewritten as

$$u_k^\Delta(i) = \sum_{j\in I^\Delta} p_{ij}^k u_{k+1}^\Delta(j). \tag{3.11}$$

3.2. The numerical scheme for the conservative equation. Now we describe the derivation of the numerical scheme for (1.2). Given the initial measure f_0, to each simplex $T_\tau \in \mathcal{T}^{\Delta x}$, we associate the barycenter B_τ^Δ and we define $f_\tau^B = \int_{T_\tau} f_0(dx)$. We define

$$f_0^\Delta = \sum_i f_0^\Delta(i)\delta_{x_i} \quad \text{where} \quad f_0^\Delta(i) = \sum_{\tau\in\mathcal{K}^{\Delta x}} \beta_i(B_\tau^\Delta) f_\tau^B, \tag{3.12}$$

i.e., the value $f_0^\Delta(i)$ is calculated as the sum of the weights f_τ^B, each scaled by the barycentric coordinate $\beta_i(B_\tau^\Delta)$, for all triangles in which x_i is a vertex. We have by (3.1)

$$\sum_{i\in\mathcal{I}^{\Delta x}} f_0^\Delta(i) = \sum_{i\in\mathcal{I}^{\Delta x}} \sum_{\tau\in\mathcal{K}^{\Delta x}} \beta_i(B_\tau^\Delta) f_\tau^B = \sum_{\tau\in\mathcal{K}^{\Delta x}} f_\tau^B = \int_{\mathbb{R}^d} f_0(dx). \tag{3.13}$$

Given $f_0^\Delta(i)$, we consider the following explicit scheme

$$f_{k+1}^\Delta(i) = \frac{1}{2r}\sum_{\ell=1}^r \sum_{j\in\mathcal{I}^{\Delta x}} \left[\beta_i(\mathcal{X}^k_{\ell,+}(x_j)) + \beta_i(\mathcal{X}^k_{\ell,-}(x_j))\right] f_k^\Delta(j) \quad \forall\, i\in\mathcal{I}^{\Delta x},\ \ k=0,\dots,N-1. \tag{3.14}$$

The measure in $\mathcal{P}_1(\mathbb{R}^d)$ defined by

$$f_k^\Delta = \sum_{i\in\mathcal{I}^{\Delta x}} f_k^\Delta(i)\delta_{x_i} \qquad k=0,\dots,N, \tag{3.15}$$

gives an approximation of the measure $f(kh,dx)$, solution to the equation (1.2). To explain the scheme (3.14), let us assume that, at a given step k, we have computed the measure $f_k^{\Delta x}$ on the discrete set $\mathcal{V}^{\Delta x}$. At the next time step, for each vertex x_i, we follow the characteristic path using the Euler scheme (3.6). The mass $f_k(i)$ is then distributed linearly among the vertices of the triangles T_τ that are reached by the trajectories $\mathcal{X}_k^{\ell,\pm}(x_i)$. The updated measure $f_{k+1}^{\Delta x}$ at each vertex x_j is computed by collecting the mass transported along all characteristic trajectories whose terminal points lie in triangles sharing x_j as a vertex. The contribution of each trajectory is weighted according to the barycentric coordinates of its endpoint within the corresponding triangle. By (3.5), the scheme (3.14) can be also rewritten as

$$f_{k+1}^\Delta(i) = \sum_{j\in I^\Delta} p_{ji}^k f_k^\Delta(j). \tag{3.16}$$

4. Convergence of the scheme for the viscous transport equation. We aim to prove the convergence of the scheme (3.9) to the "good" solution of equation (1.1). In the following, we replace the sublinear growth conditions in (2.1) and (2.2) with the stronger assumptions

$$\sup_{x\in\mathbb{R}^d} |b(t,x)| \le C_0(t), \tag{4.1}$$

$$\sup_{x\in\mathbb{R}^d} |\sigma(t,x)| \le C_2(t), \tag{4.2}$$

with $C_0, C_2 \in L^\infty(0,T)$, i.e. $b,\sigma\in L^\infty([0,T],L^\infty(\mathbb{R}^d))$. Note that a similar assumption concerning b is also made in [14, 13]. We also assume that C_1 in the (OSL) condition is bounded.

To prove the convergence of the scheme, we use an intermediate, semi-discrete in time scheme. On one hand, for the solution of this scheme, it is possible to demonstrate certain properties similar to those satisfied by the continuous problem, and thus prove its convergence through a stability argument; in this part, the property (2.1) plays a key role. On the other hand, the regularity of the solution to the semi-discrete scheme allows us to estimate, in the L^∞-norm, its distance from the solution of (3.9).

We consider the semi-discrete scheme

$$
\begin{cases} u_k^h(x) = S^h(u_{k+1}^h(\cdot), x, k) & \text{for all } x \in \mathbb{R}^d,\ k = 0, \dots, N-1, \\ u_N^h(x) = u_T(x), & \text{for all } x \in \mathbb{R}^d, \end{cases} \tag{4.3}
$$

where $S^h : \mathcal{B}(\mathbb{R}^d) \times \mathbb{R}^d \times \{0, \dots, N-1\} \longrightarrow \mathbb{R}$ is defined as

$$
\begin{aligned} S^h(w, x, k) &:= \tfrac{1}{2r} \textstyle\sum_{\ell=1}^r \Big(w(\mathcal{X}_{\ell,+}^k(x)) + w(\mathcal{X}_{\ell,-}^k(x)) \Big) \\ &= \tfrac{1}{2r} \textstyle\sum_{\ell=1}^r \big[w(x - h b_h^k(x) + \sqrt{rh}\sigma_{h,\ell}^k(x)) + w(x - h b_h^k(x) - \sqrt{rh}\sigma_{h,\ell}^k(x)) \big]. \end{aligned} \tag{4.4}
$$

Define a Markov chain by

$$
\begin{cases} \Phi_{m+1,k}^h(x) = \Phi_{m,k}^h(x) - h b_h^m(\Phi_{m,k}^h(x)) + \sqrt{rh}\sigma_h^m(\Phi_{m,k}^h(x)) Z^m & m = k+1, \dots, N-1 \\ \Phi_{k,k}^h(x) = x, \end{cases} \tag{4.5}
$$

where Z^m is a sequence of r-valued random variables independent of $\Phi_{m,k}^h(x)$ and satisfying (3.8), while b_h^m, σ_h^m are defined as in (3.4) with m in place of k. Note that $\Phi_{k+1,k}^h(x) = \tilde{\Phi}_{t_{k+1},t_k}^h(x)$ where $\tilde{\Phi}^h$ is defined in (3.7).

Proposition 4.1. *Assume that $u_T \in C_b(\mathbb{R}^d)$. For $h > 0$, let $u^h = \{u_k^h\}_{k=0}^N$ be the solution of (4.3). Then*

$$
u_k^h(x) = \mathbb{E}[u_T(\Phi_{N,k}^h(x))], \quad x \in \mathbb{R}^d,\ k = 0, \dots, N \tag{4.6}
$$

and $u_k^h(x)$ is bounded and continuous in $\mathbb{R}^d$ for any $k = 0, \dots, N$. Moreover, if u_T is Lipschitz continuous, then u^h is also Lipschitz continuous in x with a constant L independent of h.

Proof. We first prove (4.6). The identity is obvious for $k = N$ since $\Phi_{N,N}^h(x) = x$. For $k = N-1$, we have

$$
\begin{aligned} &\mathbb{E}[u_T(\Phi_{N,N-1}^h(x))] \\ &= \mathbb{E}[u_T(\Phi_{N-1,N-1}^h(x) - h b_h^{N-1}(\Phi_{N-1,N-1}^h(x)) + \sqrt{rh}\sigma_h^{N-1}(\Phi_{N-1,N-1}^h(x)) Z^{N-1})] \\ &= \mathbb{E}[u_T(x - h b_h^{N-1}(x) + \sqrt{rh}\sigma_h^{N-1}(x) Z^{N-1})] = \frac{1}{2r} \sum_{\ell=1}^r \big(u_T(\mathcal{X}_{\ell,+}^k(x)) + u_T(\mathcal{X}_{\ell,-}^k(x)) \big) \\ &= S^h(u_N, x, N-1) = u_{N-1}^h(x) \end{aligned}
$$

and therefore (4.6) for $k = N-1$. For $k = N-2$, observing that $\Phi_{N,N-2}^h = \Phi_{N,N-1}^h \circ \Phi_{N-1,N-2}^h$ and since (4.6) holds for $k = N-1$, we have

$$
\begin{aligned} &\mathbb{E}[u_T(\Phi_{N,N-2}^h(x))] = \mathbb{E}[u_T(\Phi_{N,N-1}^h(\Phi_{N-1,N-2}^h(x)))] \\ &= \mathbb{E}[u_{N-1}^h(\Phi_{N-1,N-2}^h(x))] = \mathbb{E}[u_{N-1}^h(x - h b_h^{N-2}(x) + \sqrt{rh}\sigma_h^{N-2}(x) Z^{N-2})] \\ &= \frac{1}{2r} \sum_{\ell=1}^r \big[u_{N-1}^h(x - h b_h^{N-2}(x) + \sqrt{rh}\sigma_{h,\ell}^{N-2}(x)) + u_{N-1}^h(x - h b_h^{N-2}(x) - \sqrt{rh}\sigma_{h,\ell}^{N-2}(x)) \big] \\ &= S^h(u_{N-1}, x, N-2) = u_{N-2}^h(x), \end{aligned}
$$

which gives (4.6) for $k = N-2$. Iterating, we get (4.6) for $k = 0, \dots, N-1$.
By the assumption on u_T and (4.6), it follows immediately that $\sup_{k=0,\dots,N} \|u_k^h\|_{L^\infty} \le \|u_T\|_{L^\infty}$. To prove the continuity of u_k^h, $k = 0, \dots, N$, we first give a stability estimate for the discrete

trajectories (4.5). Given $x, y \in \mathbb{R}^d$, we denote by $\Phi^h_{m,k}$, $\Psi^h_{m,k}$ the discrete trajectories such that $\Phi^h_{k,k} = x$, $\Psi^h_{k,k} = y$. We have

$$\begin{aligned}
\mathbb{E}|\Phi^h_{m+1,k} - \Psi^h_{m+1,k}|^2 &= \mathbb{E}|\Phi^h_{m,k} - \Psi^h_{m,k}|^2 + h^2\mathbb{E}|b^m_h(\Phi^h_{m,k}) - b^m_h(\Psi^h_{m,k})|^2 \\
&+ hr\mathbb{E}|(\sigma^m_h(\Phi^h_{m,k}) - \sigma^m_h(\Psi^h_{m,k}))Z^m|^2 - 2h\mathbb{E}\big[(\Phi^h_{m,k} - \Psi^h_{m,k})(b^m_h(\Phi^h_{m,k}) - b^m_h(\Psi^h_{m,k}))\big] \\
&+ 2\sqrt{hr}\,\mathbb{E}\big[(\Phi^h_{m,k} - \Psi^h_{m,k})(\sigma^m_h(\Phi^h_{m,k}) - \sigma^m_h(\Psi^h_{m,k}))Z^m\big] \\
&+ 2\sqrt{h^3 r}\,\mathbb{E}\big[(b^m_h(\Phi^h_{m,k}) - b^m_h(\Psi^h_{m,k}))(\sigma^m_h(\Phi^h_{m,k}) - \sigma^m_h(\Psi^h_{m,k}))Z^m\big].
\end{aligned} \tag{4.7}$$

By (4.1), we estimate

$$\begin{aligned}
h^2\mathbb{E}\left|b^m_h(\Phi^h_{m,k}) - b^m_h(\Psi^h_{m,k})\right|^2 &= h^2\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}}\int_{\mathbb{R}^d} b(s,y)(\rho_{\sqrt{h}}(\Phi^h_{m,k} - y) - \rho_{\sqrt{h}}(\Psi^h_{m,k} - y))dyds\right|^2 \\
&\le h^2\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}} |b(s,y)|\left(\int_{B(0,\sqrt{h})}\|D\rho_{\sqrt{h}}\|_\infty dy\right)|\Phi^h_{m,k} - \Psi^h_{m,k}|ds\right|^2 \\
&\le h^2\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}} C_0(s)h^{-\frac{1}{2}}|\Phi^h_{m,k} - \Psi^h_{m,k}|ds\right|^2 \\
&\le h^2\left(h^{-\frac{1}{2}}\right)^2\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}} C_0(s)|\Phi^h_{m,k} - \Psi^h_{m,k}|^2 ds\right|^2 \\
&\le h\mathbb{E}|\Phi^h_{m,k} - \Psi^h_{m,k}|^2\left(\frac{1}{h}\int_{t_m}^{t_{m+1}} C_0(s)ds\right)^2.
\end{aligned}$$

By (2.2) and (4.2), since Z^m independent of $\Phi^h_{m,k} - \Psi^h_{m,k}$, we estimate

$$\begin{aligned}
hr\mathbb{E}|(\sigma^m(\Phi^h_{m,k}) - \sigma^m(\Psi^h_{m,k}))Z^m|^2 &= hr\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}}(\sigma(s,\Phi^h_{m,k}) - \sigma(s,\Psi^h_{m,k}))Z^m ds\right|^2 \\
&\le hr\mathbb{E}|\Phi^h_{m,k} - \Psi^h_{m,k}|^2\left|\frac{1}{h}\int_{t_m}^{t_{m+1}} C_2(s)ds\right|^2\mathbb{E}[|Z^m|^2].
\end{aligned}$$

By (2.1), it follows that

$$-2h\mathbb{E}(\Phi^h_{m,k} - \Psi^h_{m,k})(b^m_h(\Phi^h_{m,k}) - b^m_h(\Psi^h_{m,k})) \le 2h\mathbb{E}|\Phi^h_{m,k} - \Psi^h_{m,k}|^2\left(\frac{1}{h}\int_{t_m}^{t_{m+1}} C_1(s)ds\right).$$

Since Z^m is independent of $\Phi^h_{m,k} - \Psi^h_{m,k}$ and $\mathbb{E}[Z^m] = 0$, the last two terms on the right-hand side of the identity (4.7) vanish. Hence, replacing the previous estimates in (4.7), we finally get the stability estimate

$$\mathbb{E}|\Phi^h_{m+1,k} - \Psi^h_{m+1,k}|^2 \le (1 + Ch)\mathbb{E}|\Phi^h_{m,k} - \Psi^h_{m,k}|^2$$

and, iterating on m,

$$\mathbb{E}|\Phi^h_{m+1,k} - \Psi^h_{m+1,k}|^2 \le (1 + Ch)^{m+1-k}|x - y|^2, \qquad m = k-1, \dots, N-1 \tag{4.8}$$

for C depending only on the constants in assumptions (2.1), (2.2), (4.1) and (4.2).

We also estimate the dependence of $\Phi^h_{m,k}$ with respect to m. Given $m \in \{k, \dots, N\}$, we have

$$\begin{aligned}
&\mathbb{E}|\Phi^h_{m+1,k}(x) - \Phi^h_{m,k}(x)|^2 = h^2\mathbb{E}|b^m_h(\Phi^h_{m,k})|^2 + hr\mathbb{E}|\sigma^m(\Phi^h_{m,k})Z^m|^2 \\
&+ 2h\sqrt{hr}\mathbb{E}[b^m_h(\Phi^h_{m,k})\sigma^m(\Phi^h_{m,k})Z^m] = h^2\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}}\int_{\mathbb{R}^d} b(s,y)\rho_{\sqrt{h}}(\Phi^h_{m,k} - y)dyds\right|^2
\end{aligned}$$

$$+ hr\mathbb{E}\left|\frac{1}{h}\int_{t_m}^{t_{m+1}} \sigma(s,\Phi^h_{m,k})Z^m ds\right|^2 \le h^2\left|\frac{1}{h}\int_{t_m}^{t_{m+1}} C_0(s)ds\right|^2 + hr\left[\frac{1}{h}\int_{t_m}^{t_{m+1}} |C_2(s)|^2 ds\right] \le Ch$$

with C independent of h. In similar way, we estimate

$$\mathbb{E}|\Phi^h_{m+\bar{m},k}(x) - \Phi^h_{m,k}(x)|^2 \le C\bar{m}h, \qquad m = k,\dots,N, \quad 0 \le \bar{m} \le N-m. \tag{4.9}$$

The estimate (4.8) and the representation formula (4.6) give immediately the continuity of u^h_k for any $k = 0,\dots,N$. Moreover, if u_T is Lipschitz continuous with constant L_T, then, given $x, y \in \mathbb{R}^d$, we have by (4.8)

$$\begin{aligned} |u^h_k(x) - u^h_k(y)| &\le \mathbb{E}|u_T(\Phi^h_{N,k}(x)) - u_T(\Phi^h_{N,k}(y))| \le L_T[\mathbb{E}|\Phi^h_{N,k}(x) - \Phi^h_{N,k}(y)|^2]^{\frac{1}{2}} \\ &\le L_T e^{C(N-k)h}|x-y| \le L|x-y| \end{aligned} \tag{4.10}$$

with L independent of h. Hence the uniform Lipschitz property for u^h_k, $k = 0,\dots,N$ follows. ∎

Remark 4.2 (Lipschitz regularity of u). *If the terminal condition u_T is Lipschitz continuous, the discrete approximations u^h remain uniformly Lipschitz in space with a constant independent of h. Passing to the limit in the viscosity sense, and using the contraction property of the stochastic flow associated with the OSL field, we conclude that the limiting solution u is also Lipschitz in x. This is consistent with what already stated in Theorem 2.2.*

We now estimate the distance between the solutions of (3.9) and (4.3).

Proposition 4.3. *Assume that u_T is Lipschitz continuous. Let $\bar{u}^\Delta$ be the linear interpolation of the solution of (3.9) on the triangulation $\mathcal{T}^{\Delta x}$, i.e. $\bar{u}^\Delta_k(x) = I[u^\Delta_k](x)$, and u^h the solution of (4.3). Then*

$$\|u^h_k - \bar{u}^\Delta_k\|_{L^\infty(\mathbb{R}^d)} \le L(N-k+1)\Delta x, \qquad k = 0,\dots,N, \tag{4.11}$$

where L is as in Prop. 4.1.

Proof. At step N, the estimate follows from (3.2). At step $N-1$, we have for $x \in \mathbb{R}^d$

$$\begin{aligned} &|u^h_{N-1}(x) - \bar{u}^\Delta_{N-1}(x)| = |u^h_{N-1}(x) - \sum_{i\in\mathcal{I}^{\Delta x}} \beta_i(x)\bar{u}^\Delta_{N-1}(x_i)| \\ &\le \sum_{i\in\mathcal{I}^{\Delta x}} \beta_i(x)|u^h_{N-1}(x) - u^h_{N-1}(x_i)| + \sum_{i\in\mathcal{I}^{\Delta x}} \beta_i(x)|u^h_{N-1}(x_i) - \bar{u}^\Delta_{N-1}(x_i)|. \end{aligned} \tag{4.12}$$

Since $\beta_i(x) \neq 0$ if and only if x_i is a vertex of the triangle containing x, by (4.10) we have

$$|u^h_{N-1}(x) - u^h_{N-1}(x_i)| \le L\Delta x. \tag{4.13}$$

Moreover, since $\bar{u}^\Delta_{N-1}(x_i) = u^\Delta_{N-1}(x_i)$, we have

$$u^h_{N-1}(x_i) = \frac{1}{2r}\sum_{\ell=1}^{r}\left(u_T(\mathcal{X}^k_{\ell,+}(x_i)) + u_T(\mathcal{X}^k_{\ell,-}(x_i))\right),$$

$$\bar{u}^\Delta_{N-1}(x_i) = \frac{1}{2r}\sum_{\ell=1}^{r}\left(I[\hat{u}_T](\mathcal{X}^k_{\ell,+}(x_i)) + I[\hat{u}_T](\mathcal{X}^k_{\ell,-}(x_i))\right).$$

Subtracting the previous equation and using (3.2), we get

$$|u^h_{N-1}(x_i) - u^\Delta_{N-1}(x_i)| \le L_T\Delta x. \tag{4.14}$$

Plugging (4.13) and (4.14) into (4.12), we get (4.11) for $k = N-1$. Following a similar argument for $k = N-2, N-3, \dots$ and exploiting the estimate obtained at the previous step to bound $u^h_k(x_i) - \bar{u}^\Delta_k(x_i)$, we obtain (4.11) for any $k = 0,\dots,N$. ∎

We now discuss the convergence of $\bar{u}^\Delta$ to u. We first extend u^h and $\bar{u}^\Delta$ to $[0,T]\times\mathbb{R}^d$ by setting, for $\phi_k = u_k^h,\ \bar{u}_k^\Delta$,

$$\phi(t,x) = \frac{(k+1)h-t}{h}\phi_k(x) + \frac{t-kh}{h}\phi_{k+1}(x), \quad t\in[kh,(k+1)h), \quad k=0,\dots,N-1.$$

We use a definition of L^1-viscosity solution equivalent to Def. 2.3 which seems more suitable for stability properties (see [22] for the equivalence of the two definitions and [2, 24] for related stability properties). We set

$$H(t,x,p,X) = \mathrm{tr}[a(t,x)X] - b(t,x)p \tag{4.15}$$

and

$$\overline{H}(t,x,p,X) = \mathrm{tr}[a(t,x)X] - \underline{b}(t,x,p),$$
$$\underline{H}(t,x,p,X) = \mathrm{tr}[a(t,x)X] - \bar{b}(t,x,p),$$

where $\underline{b}$, $\bar{b}$ are defined in (2.8).

Definition 4.4. *An u.s.c. (respectively, l.s.c.) function $u:\mathbb{R}^d\times[0,T]\longrightarrow\mathbb{R}$ is called a subsolution (respectively, supersolution) of (1.1) if $u(\cdot,T)\le u_T(x)$ (resp., $\ge$) and*

(i) for any $\phi\in C^{1,2}((0,T)\times\mathbb{R}^d)$ and any $\gamma\in L^1(0,T)$ such that the function $u(t,x)-\phi(t,x)-\int_t^T\gamma(s)ds$ has a local maximum (resp., minimum) at $(t_0,x_0)\in\mathbb{R}^d\times(0,T)$;

(ii) for any continuous function $G:(0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times S^d\longrightarrow\mathbb{R}$ such that

$$\overline{H}(t,x,p,X)-\gamma(t)\le G(t,x,p,X)$$
$$\Big(\textit{resp.}\quad \underline{H}(t,x,p,X)-\gamma(t)\ge G(t,x,p,X)\Big)$$

for all (x,p,X) in a neighborhood of $(x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0))$ and almost all t in a neighborhood of t_0, we have

$$-\frac{\partial\phi}{\partial t}(t_0,x_0)\le G(t_0,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0))$$
$$\left(\textit{resp.}\quad -\frac{\partial\phi}{\partial t}(t_0,x_0)\ge G(t_0,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0))\right).$$

If $u\in C([0,T]\times\mathbb{R}^d)$ is both a sub and supersolution, we say u is a solution of (1.1).

Remark 4.5 (Equivalence of the viscosity formulations.). *Definitions 2.3 and 4.3 provide two equivalent formulations of viscosity solutions for equation (1.1). The first one is stated in the measurable viscosity sense, which is convenient when the drift b is only measurable in time, while the second one is written in the classical pointwise framework used for stability and convergence arguments. Under the present assumptions on a and b, these notions coincide. In particular, if u satisfies Definition 2.3, then it also satisfies Definition 4.3, and conversely; this equivalence follows from the localization argument and the comparison principles and it has been discussed and proven in [22].*

Theorem 4.6. *Assume that u_T is Lipschitz continuous. Then, for $|\Delta|=|(h,\Delta x)|\to 0$ with $\Delta x/h\to 0$, the function $\bar{u}^\Delta$, given by the linear interpolation of the solution of (3.9) on $[0,T]\times\mathbb{R}^d$, converges to the solution u of (1.1), locally uniformly in $[0,T]\times\mathbb{R}^d$.*

Proof. By (4.11), we have

$$\sup_{t\in[0,T]}\|u^h(t)-u^\Delta(t)\|_{L^\infty(\mathbb{R}^d)}\le LT\frac{\Delta x}{h}$$

with L independent of h and Δx. Hence, it is sufficient to show that u^h converges, locally uniformly in $[0,T]\times\mathbb{R}^d$, to u and, for this, we use the stability argument in [3] adapted to Def. 4.4 (see [24] for a related result). We set

$$\underline{u}(t,x) := \liminf_{\substack{h\to 0\\(s,y)\to(t,x)}} u^h(s,y), \qquad \overline{u}(t,x) := \limsup_{\substack{h\to 0\\(s,y)\to(t,x)}} u^h(s,y).$$

We verify that $\overline{u}$ is a subsolution of (1.1). Notice that, due to (4.8) and (4.9), we have that u^h is uniformly continuous in x and t. Hence, recalling that $u^h(T,x) = u_T(x)$, we have $\overline{u}(T,x) = u_T(x)$.

Let $\phi \in C^2((0,T)\times\mathbb{R}^d)$ and $\gamma \in L^1(0,T)$ be such that $\bar{u}(t,x) - \phi(t,x) - \int_0^t \gamma(s)ds$ has a local maximum point at $(t_0,x_0) \in (0,T)\times\mathbb{R}^d$ and let G be a continuous function such that

$$\overline{H}(t,x,p,X) - \gamma(t) \le G(t,x,p,X) \tag{4.16}$$

in a neighborhood of $(x_0, D\phi(t_0,x_0), D^2\phi(t_0,x_0))$ and for almost all t in a neighborhood of t_0. We have to prove that

$$\frac{\partial\varphi}{\partial t}(t_0,x_0) \le G(t_0,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0)). \tag{4.17}$$

It is not restrictive to assume that (t_0,x_0) is a strict global maximum point and that, additionally, $\sup_{t\in(0,T)} \|\phi(\cdot,t)\|_{C^2(\mathbb{R}^d)}$ is finite. We set

$$H^h(w,x,k) = \frac{S^h(w,x,k) - w(x)}{h}, \tag{4.18}$$

where S^h is defined as in (4.4). We have for $\psi(\cdot) = \phi(t_0,\cdot)$

$$\lim_{h\to 0} \sum_{k=[t/h]}^{N-1} hH^h(\psi,x_0,k) = \int_t^T H(s,x_0,D\psi(x_0),D^2\psi(x_0))ds, \quad \forall t\in[0,T], \tag{4.19}$$

locally uniformly in x (see Lemma 4.1 at the end of the proof). For $t \in [0,T]$, we define

$$\mathcal{B}^h(t,x) = u^h(t,x) - \phi(t,x) - \int_t^T \gamma(s)ds - \mathcal{E}_h(t),$$

where

$$\mathcal{E}_h(t) = \sum_{k=[t/h]}^{N-1} hH^h(\phi(t_0,\cdot),x_0,k) - \int_t^T H(s,x_0,D\phi(t_0,x_0),D^2\phi(t_0,x_0))ds.$$

By (4.19), since $\limsup^*_{h\to 0} \mathcal{B}^h(t,x) = u^*(t,x) - \phi(t,x) - \int_t^T \gamma(s)ds$, there exists a sequence (t^h,x^h), with $t^h = k^h h$, of global maximum points for $\mathcal{B}^h - \phi$ converging to (t_0,x_0) for $h \to 0^+$. We set $C^h = \mathcal{B}^h(t^h,x^h)$ and we observe that

$$\begin{aligned}
&u^h(t,x) \le \phi(t,x) + \int_t^T \gamma(s)ds + \mathcal{E}_h(t) + C^h, \qquad \forall (t,x)\in\mathbb{R}^d\times(0,T),\\
&u^h(t^h,x^h) = \phi(t^h,x^h) + \int_{t^h}^T \gamma(s)ds + \mathcal{E}_h(t^h) + C^h.
\end{aligned}$$

Hence, by (4.3)

$$\begin{aligned}
\phi(t^h,x^h) + \int_{t^h}^T \gamma(s)ds + \mathcal{E}_h(t^h) + C^h &= u^h(t^h,x^h) = u^h_{k^h}(x^h) = S^h(u^h_{k^h+1}(\cdot),x^h,k^h)\\
&\le S^h(\phi(t^h+h,\cdot),x^h,k^h) + \int_{t^h+h}^T \gamma(s)ds + \mathcal{E}_h(t^h+h) + C^h
\end{aligned}$$

and, recalling (4.18), we have

$$-\frac{\phi(t^h+h,x^h)-\phi(t^h,x^h)}{h}\leq H^h(\phi(t^h+h,\cdot),x^h,k^h)-\frac{1}{h}\int_{t^h}^{t^h+h}\gamma(s)ds$$
$$+\frac{1}{h}\left(\mathcal{E}_h(t^h+h)-\mathcal{E}_h(t^h)\right).$$

By the definition of $\mathcal{E}_h$, we get

$$\begin{aligned}
&-\frac{\phi(t^h+h,x^h)-\phi(t^h,x^h)}{h}\leq H^h(\phi(t^h+h,\cdot),x^h,k^h)-\frac{1}{h}\int_{t^h}^{t^h+h}\gamma(s)ds\\
&+H^h(\phi(t_0,\cdot),x_0,k^h)+\frac{1}{h}\int_{t^h}^{t^h+h}H(x_0,s,D\phi(t_0,x_0),D^2\phi(t_0,x_0))ds\\
&=H^h(\phi(\cdot,t^h+h),x^h,k^h)-H^h(\phi(t_0,\cdot),x_0,k^h)\\
&+\frac{1}{h}\int_{t^h}^{t^h+h}\left(H(x_0,s,D\phi(t_0,x_0),D^2\phi(t_0,x_0))-\gamma(s)\right)ds.
\end{aligned}$$

Passing to the limit for $h\to 0$, recalling (4.16) and since

$$H^h(\phi(t^h+h,\cdot),x^h,k^h)-H^h(\phi(t_0,\cdot),x_0,k^h)\longrightarrow 0\quad\text{for } h\to 0,$$

we get (4.17).

By a similar argument, it is possible to show that $\underline{u}$ is a supersolution of (1.1). Hence by Theorem 2.4, we get $u=\overline{u}=\underline{u}$ and the local uniform convergence of u^h to u. □

LEMMA 4.1. *Let $\psi\in C^2(\mathbb{R}^d)$. Then*

$$\lim_{h\to 0}\sum_{k=[t/h]}^{N-1}hH^h(\psi,x,k)=\int_t^T H(s,x,D\psi(x),D^2\psi(x))ds,\qquad\forall t\in[0,T),\tag{4.20}$$

locally uniformly in $x\in\mathbb{R}^d$.

Proof. We first observe that

$$\begin{aligned}
H^h(\psi,x,k)&=\frac{1}{2r}\sum_{\ell=1}^{r}\frac{1}{h}\left[\psi(x-hb_h^k(x)+\sqrt{rh}\sigma_{h,\ell}^k(x))+\psi(x-hb_h^k(x)-\sqrt{rh}\sigma_{h,\ell}^k(x))-2\psi(x)\right]\\
&=\operatorname{tr}\left[\frac{1}{2}\sigma_k^h(\sigma_k^h)^TD^2\psi(x)\right]-b_k^h(x)D\psi(x)+o(h)
\end{aligned}$$

where $o(h)\longrightarrow 0$ for $h\to 0$. Recalling that $T=Nh$ and (3.4), we have

$$\begin{aligned}
\sum_{k=[t/h]}^{N-1}hH^h(\psi,x,k)&=\sum_{k=[t/h]}^{N-1}h\left[\frac{1}{2}\operatorname{tr}\left(\frac{1}{h}\int_{kh}^{(k+1)h}\sigma(s,x)ds\frac{1}{h}\int_{kh}^{(k+1)h}\sigma^T(s,x)dsD^2\psi(x)\right)\right.\\
&\left.-\left(\frac{1}{h}\int_{kh}^{(k+1)h}\int_{\mathbb{R}^d}\rho_{\sqrt{h}}(x-y)b(s,y)dyds\right)D\psi(x)\right]+o(h)\\
&=\frac{1}{2}\operatorname{tr}\left(\int_{[t/h]h}^{T}\sigma(s,x)\sigma^T(s,x)dsD^2\psi(x)\right)\\
&-\left(\int_{[t/h]h}^{T}\int_{\mathbb{R}^d}\rho_{\sqrt{h}}(x-y)b(s,y)dyds\right)D\psi(x)\\
&+\sum_{k=[t/h]}^{N-1}h\frac{1}{2}\operatorname{tr}\left(\frac{1}{h}\int_{kh}^{(k+1)h}\sigma(s,x)ds\frac{1}{h}\int_{kh}^{(k+1)h}\sigma^T(s,x)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{h}\int_{kh}^{(k+1)h}\sigma(s,x)\sigma^T(s,x)ds\Big)D^2\psi(x)+o(h)\\
&=\int_t^T H(x,s,D\psi(x),D^2\psi(x))ds-\int_{[t/h]h}^{t}H(x,s,D\psi(x),D^2\psi(x))ds\\
&+\int_{[t/h]h}^{T}\int_{\mathbb{R}^d}\rho_{\sqrt{h}}(x-y)(b(s,y)-b(s,x))dyds\\
&+\sum_{k=[t/h]}^{N-1}h\frac{1}{2}\operatorname{tr}\left(\frac{1}{h}\int_{kh}^{(k+1)h}\sigma(s,x)ds\frac{1}{h}\int_{kh}^{(k+1)h}\sigma^T(s,x)ds\right.\\
&\left.-\frac{1}{h}\int_{kh}^{(k+1)h}\sigma(s,x)\sigma^T(s,x)ds\right)D^2\psi(x)+o(h).
\end{aligned}
$$

Passing to the limit for $h\to 0$, since $[t/h]h\to t$, $b_h\longrightarrow b$ a.e. and σ is regular, by dominated convergence we get (4.20). ∎

5. Convergence of the scheme for the conservative equation. The aim of this section is to show that the scheme (3.14) converges to the unique duality solution, in the sense of Definition 2.9, of the conservative equation (1.2). In this section, besides (2.1), (4.1), (4.2), we assume that

$$
f_0\in\mathcal{P}_2(\mathbb{R}^d)\quad\text{and}\quad \operatorname{supp}[f_0]\subset B(0,R)\quad\text{for some } R>0 \tag{5.1}
$$

where $\operatorname{supp}[f_0]$ denotes the support of the measure f_0.

LEMMA 5.1. *Let f_0^Δ be defined as in (3.12). Then $f_0^\Delta\in\mathcal{P}_2(\mathbb{R}^d)$ and $\mathcal{W}_1(f_0^\Delta,f_0)\longrightarrow 0$ as $\Delta\to 0$.*

Proof. Recall that, by (3.12), $f_0^\Delta=\sum_i f_0^\Delta(i)\delta_{x_i}$ where $f_0^\Delta(i)=\sum_{\tau\in\mathcal{K}^{\Delta x}}\beta_i(B_\tau^\Delta)f_\tau^B$ and $f_\tau^B=\int_{T_\tau}f_0(dx)$. By (3.15), we have $\int_{\mathbb{R}^d}f_0^\Delta(dx)=\int_{\mathbb{R}^d}f_0(dx)=1$. Moreover

$$
\begin{aligned}
\int_{\mathbb{R}^d}|x|^2 f_0^\Delta(dx)&=\sum_{i\in\mathcal{I}^{\Delta x}}|x_i|^2 f_0^\Delta(i)=\sum_{i\in\mathcal{I}^{\Delta x}}|x_i|^2\sum_{\tau\in\mathcal{K}^{\Delta x}}\beta_i(B_\tau^\Delta)\int_{T_\tau}f_0(dx)\\
&=\sum_{\tau\in\mathcal{K}^{\Delta x}}\int_{T_\tau}\sum_{i\in\mathcal{I}^{\Delta x}}\beta_i(B_\tau^\Delta)|(x_i-x)+x|^2 f_0(dx).
\end{aligned}
$$

Since $|x_i-x|\le\Delta x$ for $x,x_i\in T_\tau$, $\beta_i(B_\tau^\Delta)=0$ if $x_i\notin T_\tau$ and $\sum_{i\in\mathcal{I}^{\Delta x}}\beta_i(B_\tau^\Delta)=1$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^d}|x|^2 f_0^\Delta(dx)&\le\Delta x^2\sum_{\tau\in\mathcal{K}^{\Delta x}}\int_{T_\tau}\sum_{i\in\mathcal{I}^{\Delta x}}\beta_i(B_\tau^\Delta)f_0(dx)+\sum_{\tau\in\mathcal{K}^{\Delta x}}\int_{T_\tau}|x|^2\sum_{i\in\mathcal{I}^{\Delta x}}\beta_i(B_\tau^\Delta)f_0(dx)\\
&\le\Delta x^2+\int_{\mathbb{R}^d}|x|^2 f_0(dx).
\end{aligned}
$$

We claim that $\mathcal{W}_1(f_0^\Delta,f^\Delta)\longrightarrow 0$ for $\Delta\to 0$. For ϕ 1-Lipschitz, we have

$$
\begin{aligned}
\int_{\mathbb{R}^d}\phi(x)(f_0^\Delta(dx)-f_0(dx))&=\int_{\mathbb{R}^d}\phi(x)\left(\sum_{i\in\mathcal{I}^{\Delta x}}f_0^\Delta(i)\delta_{x_i}-f_0(dx)\right)\\
&=\sum_{\tau\in\mathcal{K}^\Delta}\left(\sum_{i\in\mathcal{I}^\Delta}\beta_i(B_\tau^\Delta)f_\tau^B\phi(x_i)-\int_{T_\tau}\phi(x)f_0(dx)\right)\\
&=\sum_{\tau\in\mathcal{K}^\Delta}\left[\int_{T_\tau}\left(\sum_{i\in\mathcal{I}^\Delta}\beta_i(B_\tau^\Delta)\phi(x_i)-\phi(x)\right)f_0(dx)\right].\\
&\le\sup_\tau\{\operatorname{diam}T_\tau\}\sum_{\tau\in\mathcal{K}^\Delta}\int_{T_\tau}f_0(dx)\le\Delta x\int_{\mathbb{R}^d}f_0(dx)
\end{aligned}
$$

and therefore the claim follows. ∎

Lemma 5.2. *Let u^Δ be the solution of* (3.9). *Then*

$$u_k^\Delta(i) = \mathbb{E}[u_T(\Phi_{N,k}^\Delta(x_i))], \qquad i \in \mathcal{I}^\Delta,\ k = 0, \ldots, N, \tag{5.2}$$

where $\Phi_{N,k}^\Delta(x_i)$ is defined as in (3.5).

Proof. We prove that $u_k^\Delta(i)$ defined as in (5.2) gives a solution of (3.11). For $k = N$, since $\mathbb{P}(\Phi_{N,N}^\Delta(x_i) = x_j) = \delta_{i,j}$, we get $u_N^\Delta(i) = u_T(x_i)$. For $k = N-1$, by (3.5) and (5.2) we have

$$\begin{aligned}
&\mathbb{E}[u_T(\Phi_{N,N-1}^\Delta(x_i))] = \sum_{j\in\mathcal{I}^\Delta} \mathbb{P}(\Phi_{N,N-1}^\Delta(x_i) = x_j) u_T(x_j) \\
&= \sum_{j\in\mathcal{I}^\Delta} \sum_{k\in\mathcal{I}^\Delta} \mathbb{P}(\Phi_{N,N-1}^\Delta(x_i) = x_j | \Phi_{N-1,N-1}^\Delta(x_i) = x_k) \mathbb{P}(\Phi_{N-1,N-1}^\Delta(x_i) = x_k) u_T(x_j) \\
&= \sum_{j\in\mathcal{I}^\Delta} \mathbb{P}(\Phi_{N,N-1}^\Delta(x_i) = x_j | \Phi_{N-1,N-1}^\Delta(x_i) = x_i) u_T(x_j) = \sum_{j\in\mathcal{I}^\Delta} p_{ij}^{(N-1)} u_N^\Delta(j) = u_{N-1}^\Delta(i)
\end{aligned}$$

and hence we get (5.2) for $k = N-1$. Following a similar argument for $k = N-2, N-3, \ldots$, we obtain that the solution of (3.11) is given by the representation formula (5.2). ∎

Lemma 5.3. *Let f_0^Δ be given by* (3.12) *and f^Δ by* (3.15). *Then, for $g \in C_b(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} g(x) f_k^\Delta(dx) = \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{k,0}^\Delta(x))] f_0^\Delta(dx). \tag{5.3}$$

Proof. For $k = 0$, (5.3) is obvious since $\Phi_{0,0}^\Delta(x_i) = x_i$. For $k = 1$, by (3.5) we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{1,0}^\Delta(x))] f_0^\Delta(dx) = \sum_{j\in\mathcal{I}^\Delta} f_0^\Delta(j) \mathbb{E}[g(\Phi_{1,0}^\Delta(x_j))] \\
&= \sum_{j\in\mathcal{I}^\Delta} f_0^\Delta(j) \Big(\sum_{i\in\mathcal{I}^\Delta} g(x_i) \mathbb{P}(\Phi_{1,0}^\Delta(x_j) = x_i) \Big) = \sum_{j\in\mathcal{I}^\Delta} f_0^\Delta(j) \Big(\sum_{i\in\mathcal{I}^\Delta} g(x_i) p_{ji}^{(0)} \Big) \\
&= \sum_{i\in\mathcal{I}^\Delta} g(x_i) \Big(\sum_{j\in\mathcal{I}^\Delta} p_{ji}^{(0)} f_0^\Delta(j) \Big) = \sum_{i\in\mathcal{I}^\Delta} g(x_i) f_1^\Delta(i) = \int_{\mathbb{R}^d} g(x) f_1^\Delta(dx),
\end{aligned}$$

and (5.3) follows. Repeating a similar argument for $k = 2, 3, \ldots, N$, we obtain the statement. ∎

Lemma 5.4. *The scheme* (3.15) *is conservative. Moreover, if $\Delta x^2/h$ is uniformly bounded for $\Delta \to 0$, then there exists a constant C, independent of Δ, such that for k, $k' \in \{0, \ldots, N\}$*

$$\int_{\mathbb{R}^d} |x|^2 f_k^\Delta(dx) \le C, \tag{5.4}$$

$$\mathcal{W}_1(f_{k'}^\Delta, f_k^\Delta) \le C(h|k'-k|)^{\frac{1}{2}}. \tag{5.5}$$

Proof. We first prove that the scheme is conservative. Indeed, for all $k = 0, \ldots, N-1$,

$$\begin{aligned}
\sum_{i\in\mathcal{I}^{\Delta x}} f_{k+1}^\Delta(i) &= \sum_{i\in\mathcal{I}^{\Delta x}} \frac{1}{2r} \sum_{\ell=1}^{r} \sum_{j\in\mathcal{I}^{\Delta x}} f_k^\Delta(j) \left[\beta_i(\mathcal{X}_{\ell,+}^k(x_j)) + \beta_i(\mathcal{X}_{\ell,-}^k(x_j)) \right] \\
&= \frac{1}{2r} \sum_{j\in\mathcal{I}^{\Delta x}} f_k^\Delta(j) \sum_{\ell=1}^{r} \sum_{i\in\mathcal{I}^{\Delta x}} \left[\beta_i(\mathcal{X}_{\ell,+}^k(x_j)) + \beta_i(\mathcal{X}_{\ell,-}^k(x_j)) \right] = \sum_{j\in\mathcal{I}^{\Delta x}} f_k^\Delta(j).
\end{aligned}$$

Moreover, by (3.13), we have $\sum_{i\in\mathcal{I}^{\Delta x}} f_k^\Delta(i) = \sum_{i\in\mathcal{I}^{\Delta x}} f_0^\Delta(i) = \int_{\mathbb{R}^d} df_0 = 1$.

We now prove (5.4). With a computation similar to the previous one and taking into account (3.3), we have

$$\int_{\mathbb{R}^d} |x|^2 f_{k+1}^\Delta(dx) = \sum_{i\in\mathcal{I}^{\Delta x}} |x_i|^2 f_{k+1}^\Delta(i)$$

$$
\begin{aligned}
&= \frac{1}{2r} \sum_{j \in \mathcal{I}^{\Delta x}} f_k^\Delta(j) \sum_{\ell=1}^{r} \sum_{i \in \mathcal{I}^{\Delta x}} |x_i|^2 \left[\beta_i(\mathcal{X}_{\ell,+}^k(x_j)) + \beta_i(\mathcal{X}_{\ell,-}^k(x_j))\right] \\
&= \sum_{j \in \mathcal{I}^{\Delta x}} f_k^\Delta(j) \frac{1}{2r} \sum_{\ell=1}^{r} \left(I[|\cdot|^2](\mathcal{X}_{\ell,+}^k(x_j)) + I[|\cdot|^2](\mathcal{X}_{\ell,-}^k(x_j))\right) \\
&\le \sum_{j \in \mathcal{I}^{\Delta x}} f_k^\Delta(j) \frac{1}{2r} \sum_{\ell=1}^{r} \left(|\mathcal{X}_{\ell,+}^k(x_j)|^2 + |\mathcal{X}_{\ell,-}^k(x_j)|^2 + C\Delta x^2\right) \\
&= \sum_{j \in \mathcal{I}^{\Delta x}} f_k^\Delta(j) \frac{1}{2r} \sum_{\ell=1}^{r} \Big(|x_j|^2 + h^2 |b_k^h(x_j)|^2 + 2hr|\sigma_k^{\ell,h}(x_j)|^2 - 2hx_j b_k^h(x_j)\Big) + C\Delta x^2 \\
&= \int_{\mathbb{R}^d} |x|^2 f_k^\Delta(dx)(1 + Ch) + Ch + C\Delta x^2,
\end{aligned}
$$

where we used condition the (OSL) assumption in (2.1) to estimate $-2hx_j b_k^h(x_j)$. Indeed

$$
-hx_j b_k^h(x_j) = -hx_j \frac{1}{h} \int_{kh}^{(k+1)h} b_h(x_j, s) ds \le |x_j|^2 \int_{kh}^{(k+1)h} C_1(s) ds + x_j \int_{kh}^{(k+1)h} b_h(s, 0) ds.
$$

Iterating, we get

$$
\begin{aligned}
\int_{\mathbb{R}^d} |x|^2 f_{k+1}^\Delta(dx) &\le (1 + Ch)^{k+1} \int_{\mathbb{R}^d} |x|^2 f_0^\Delta(dx) + (k+1)(Ch + \Delta x^2) \\
&\le e^{(k+1)h} \int_{\mathbb{R}^d} |x|^2 f_0^\Delta(dx) + (k+1)h \left(C + \frac{\Delta x^2}{h}\right).
\end{aligned}
$$

To prove (5.5), given a 1-Lipschitz function ϕ, by (5.3) and (4.9) we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} g(x)(f_k^\Delta(dx) - f_{k'}^\Delta(dx)) = \int \left[\mathbb{E}[g(\Phi_{k,0}^\Delta(x))] - \mathbb{E}[g(\Phi_{k',0}^\Delta(x))]\right] f_0^\Delta(dx) \\
&\le \sum_j \mathbb{E}|\Phi_{k,0}^\Delta(x_j) - \Phi_{k',0}^\Delta(x_j)| f_0^\Delta(dx) \le \sum_j (\mathbb{E}|\Phi_{k,0}^\Delta(x_j) - \Phi_{k',0}^\Delta(x_j)|^2)^{\frac{1}{2}} f_0^\Delta(dx) \\
&\le C(h(k-k'))^{\frac{1}{2}}.
\end{aligned}
$$

∎

We extend f^Δ given by (3.15) to an element of $C([0,T], \mathcal{P}_1(\mathbb{R}^d))$ by setting

$$
f^\Delta(t) = \frac{(k+1)h - t}{h} f_k^\Delta + \frac{t - kh}{h} f_{k+1}^\Delta(x), \quad t \in [kh, (k+1)h), \ k = 0, \dots, N-1. \tag{5.6}
$$

It is immediate that f^Δ satisfies the properties corresponding to (5.4) and (5.5) for any $t \in [0,T]$. By [12, Lemma 2.1] the sequence f^Δ converges, up to a subsequence, in the space $C([0,T); \mathcal{P}_1(\mathbb{R}^d))$. In the next result, we identify the limit as the duality solution of (1.2).

Theorem 5.1. *For $\Delta = (h, \Delta x) \to 0$ with $\Delta x^2/h$ uniformly bounded, the measure f^Δ defined in (5.6) converges to the duality solution f of (1.2) in $C([0,T); \mathcal{P}_1(\mathbb{R}^d))$.*

Proof. Consider a converging subsequence of f^Δ (still indexed by Δ) and assume that $k_h h \to t$ for $h \to 0$. Then, given a 1-Lipschitz function g, by Def. 2.9 and (5.3), we have that

$$
\begin{aligned}
\int_{\mathbb{R}^d} g(x) f(t, dx) &= \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{t,0}(x))] f_0(dx), \\
\int_{\mathbb{R}^d} g(x) df_{k_h}^\Delta(x) &= \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{k_h,0}^\Delta(x))] f_0^\Delta(dx).
\end{aligned}
$$

Moreover $\mathbb{E}[g(\Phi_{t,0}(x))]$ is the solution at time 0 of (1.1) with final datum $g(x)$ at time t and $\mathbb{E}[g(\Phi_{k,0}^\Delta(x))]$ is the solution of the scheme (3.9) at step 0 with final datum $\hat{g}(i) = g(x_i)$ at step k.

Hence, for $\Phi^h_{k_h,0}$ defined as in (4.5), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} g(x)(f^\Delta_{k_h}(dx) - f(t,dx)) = \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi^\Delta_{k_h,0}(x))] f^\Delta_0(dx) - \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{t,0}(x))] f_0(dx)\\
&= \int_{\mathbb{R}^d} \left(\mathbb{E}[g(\Phi^\Delta_{k_h,0}(x)) - \mathbb{E}[g(\Phi^h_{k_h,0}(x))]\right) f^\Delta_0(dx)\\
&+ \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi^h_{k_h,0}(x))] f^\Delta_0(dx) - \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{t,0}(x))] f_0(dx),
\end{aligned}$$

By (3.2) and Prop. 4.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\mathbb{E}[g(\Phi^\Delta_{k_h,0}(x))] - \mathbb{E}[g(\Phi^h_{k_h,0}(x))]\right) f^\Delta_0(dx)\\
&= \int_{\mathbb{R}^d} \left(\mathbb{E}[g(\Phi^\Delta_{k_h,0}(x))] - \mathbb{E}[I[g(\Phi^\Delta_{k_h,0}(\cdot))](x)]\right) f^\Delta_0(dx)\\
&+ \int_{\mathbb{R}^d} \left(\mathbb{E}[I[g(\Phi^\Delta_{k_h,0}(\cdot))](x)] - \mathbb{E}[g(\Phi^h_{k_h,0}(x))]\right) f^\Delta_0(dx)\\
&\le C\left(\Delta x + \frac{\Delta x}{h}\right) \int_{\mathbb{R}^d} f^\Delta_0(dx).
\end{aligned} \tag{5.7}$$

Arguing as in (4.10), we have that the function $\mathbb{E}[g(\Phi^h_{k_h,0}(x))]$ is Lipschitz continuous in x with constant L, uniformly in Δ. Hence

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}[g(\Phi^h_{k_h,0}(x))] f^\Delta_0(dx) - \int_{\mathbb{R}^d} \mathbb{E}[g(\Phi_{t,0}(x))] f_0(dx)\\
&= L\int_{\mathbb{R}^d} \frac{1}{L}\mathbb{E}[g(\Phi^h_{k_h,0}(x))](f^\Delta_0(dx) - f_0(dx))\\
&+ \int_{\mathbb{R}^d} \Big(\mathbb{E}[g(\Phi^h_{k_h,0}(x))] - \mathbb{E}[g(\Phi_{t,0}(x))]\Big) f_0(dx)\\
&\le L\mathcal{W}_1(f^\Delta_0, f_0) + \int_{\mathbb{R}^d} \left(\mathbb{E}[g(\Phi^h_{k_h,0}(x))] - \mathbb{E}[g(\Phi_{t,0}(x))]\right) f_0(dx).
\end{aligned} \tag{5.8}$$

By (5.7) and (5.8), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} g(x)(f^\Delta_{k_h}(dx) - f(t,dx)) \le C\left(\Delta x + \frac{\Delta x}{h}\right) \int_{\mathbb{R}^d} f^\Delta_0(dx)\\
&+ L\mathcal{W}_1(f^\Delta_0, f_0) + \int_{\mathbb{R}^d} \left(\mathbb{E}[g(\Phi^h_{k_h,0}(x))] - \mathbb{E}[g(\Phi_{t,0}(x))]\right) f_0(dx).
\end{aligned} \tag{5.9}$$

By (5.1) and the proof of Theorem 4.6, $\mathbb{E}[g(\Phi^h_{k_h,0}(x))]$ converges to $\mathbb{E}[g(\Phi_{t,0}(x))]$ uniformly in $[0,T] \times \mathrm{supp}[f_0]$. Moreover, by Lemma 5.1, f^Δ_0 converges to f_0 in $\mathcal{W}_1$. Therefore, passing to the limit in (5.9), we get

$$\lim_{\Delta\to 0} \int_{\mathbb{R}^d} g(x)(f^\Delta_{k_h}(dx) - f(t,dx)) = 0 \quad \text{for any } g \text{ 1-Lipschitz}.$$

Hence $\mathcal{W}_1(f^\Delta(t), f(t)) \longrightarrow 0$ for $\Delta \to 0$ for any $t \in [0,T]$. Since any converging subsequence of f^Δ converges to the duality solution f of (1.2), we conclude that all the sequence f^Δ converges fo f. ∎

Remark 5.2. *Equation (5.9) provides a bound that can be viewed as a first–order estimate in the Wasserstein distance $\mathcal{W}_1$, provided the discretization parameters Δx and h are chosen in a balanced manner (for example, with $\Delta x^2/h$ uniformly bounded). Although the convergence of the last term in Equation (5.9) is ensured, to the best of our knowledge no convergence bounds are currently available for the transport equation in this setting. Hence, while this observation indicates a potential direction for establishing quantitative convergence rates, the issue is not straightforward and will be left for future investigation.*

6. Numerical illustration. In this section, we provide some numerical example to illustrate the properties of the scheme.

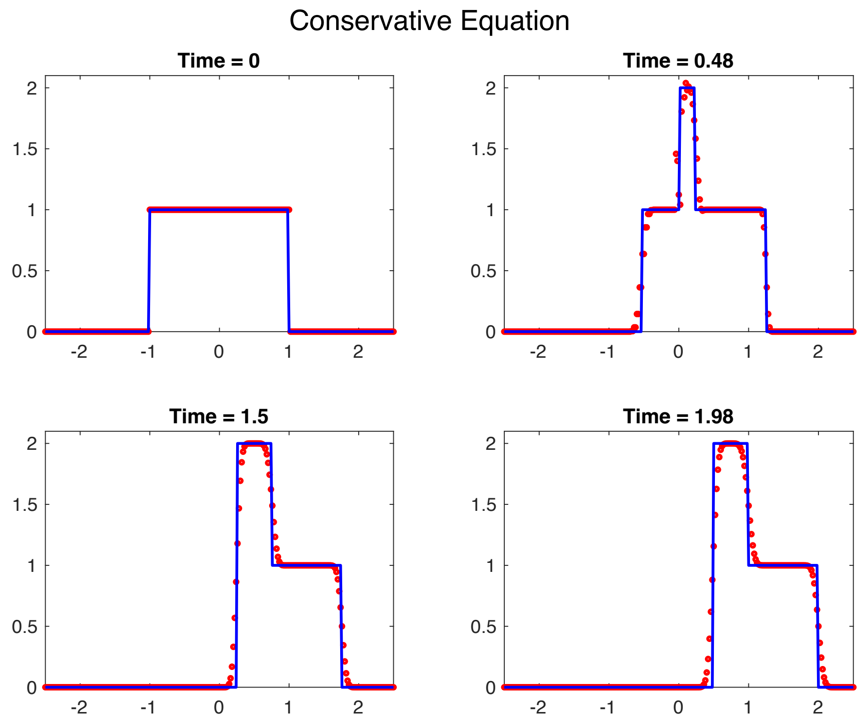

FIG. 6.1. *Test 1. (Forward) Conservative equation. The discretization parameters are* $\Delta = (\Delta x, h) = (0.02, 0.06)$.

Test 1. We consider the case of a first order one-dimensional problem for which we can compute the analytical solutions, allowing direct comparison with the numerical approximations. This example is taken by [14], where a similar analysis for an upwind scheme is carried out. We study equations (1.1) and (2.9) with $a(t,x) \equiv 0$ and

$$-b(t,x) = \begin{cases} 1 & \text{for } x < 0, \\ \frac{1}{2} & \text{for } x \geq 0; \end{cases}$$

since the velocity field is $-b$ in its forward formulation (cfr. Remark 2.11). Note that such jump in the velocity field verify the assumption (OSL). Coefficients of type arise pretty naturally in supply chain models and similar problems involving (discontinuous) congested transportation flows [18, 19]. We choose as initial condition for the conservative equation $f_0(x) = \mathbb{1}_{[-1,1]}(x)$. Then, the analytical solution is given by

$$f(t,x) = \begin{cases} \mathbb{1}_{[-1+t,0)}(x) + 2 \cdot \mathbb{1}_{[0,t/2)}(x) + \mathbb{1}_{[t/2,1+t/2)}(x) & \text{for } t \leq 1, \\ 2 \cdot \mathbb{1}_{[1/2(t-1),t/2)}(x) + \mathbb{1}_{[t/2,1+t/2]}(x) & \text{for } t > 1. \end{cases}$$

We observe that the scheme is highly effective to approximate the compression wave, even with pretty high discretization parameters.

Figure 6.1 shows the numerical results with discretization parameters $(\Delta x, h) = (0.02, 0.06)$. Note that no restrictive condition on the choice of h is required. Since this is the purely inviscid case ($a \equiv 0$), the scheme correctly captures the expected solution, without any oscillation around the discontinuity of b. We highlight that here and in the following, to avoid some boundary effect

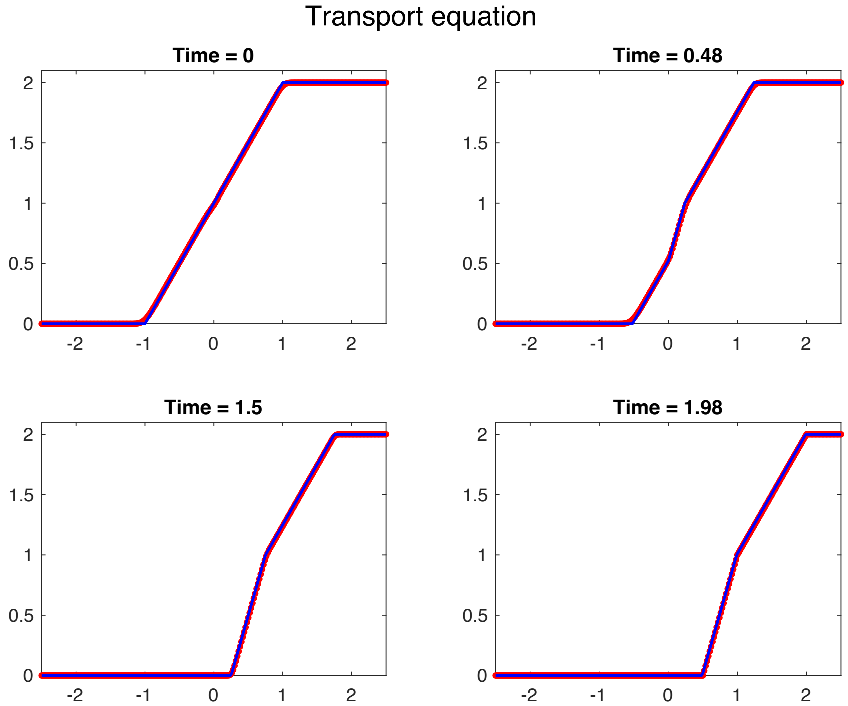

FIG. 6.2. *Test 1. (Backward) Inviscid transport equation. The initial solution is chosen as the cumulative distribution function of the exact solution for the conservative case. Comparison between exact and numerical solutions at various times. Discretization parameters:* $\Delta = (\Delta x, h) = (0.02, 0.06)$.

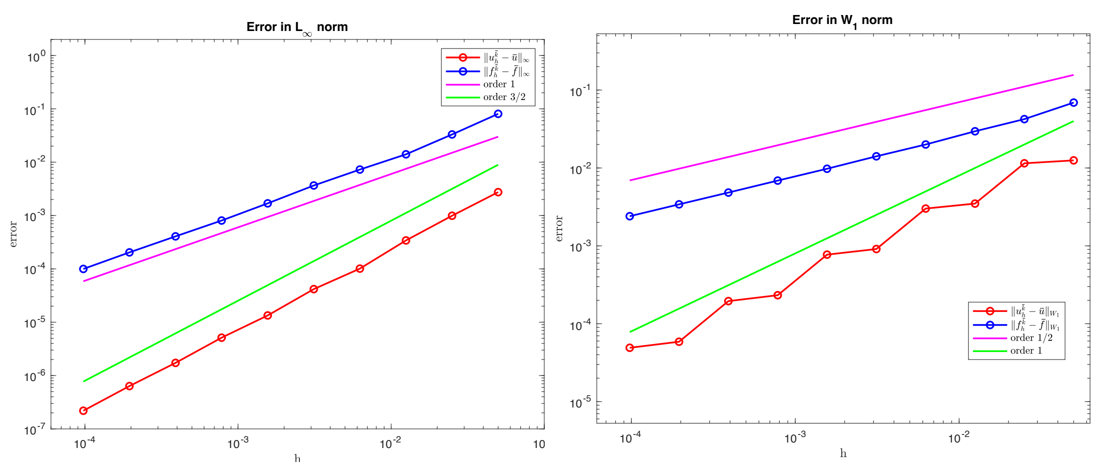

FIG. 6.3. *Test 1. Comparison of convergence rates in* L^∞ *and Wasserstein norms for the two numerical schemes.*

not considered in this study, the approximation is performed in a numerical domain sufficiently big (in this case the interval $[-5, 5]$). We stress also on the role of the mollifier $\rho : \mathbb{R} \to \mathbb{R}$ which, scaled by the time step $\rho_{\sqrt{h}}(x) = h^{-1/2}\rho(x/\sqrt{h})$, provide a regularization of the discontinuous velocity field that vanishes when $h \to 0^+$. In this test we take

$$\rho(x) := \frac{1}{C}\, e^{-\frac{1}{1-x^2}}\, \mathbb{1}_{[-1,1]}, \text{ with } C = \int_{-1}^{1} e^{-\frac{1}{1-x^2}}\, dx.$$

Exploiting the duality between (1.1) and (1.2), we consider as final condition for the transport equation (1.1) the cumulative distribution of $f(2,x)$, obtaining

$$u_T(x) = \begin{cases} 0 & \text{for } x < 1, \\ 2x - \frac{1}{2} & \text{for } 1/2 \le x < 1, \\ x & \text{for } 1 \le x < 2, \\ 2 & \text{for } x \ge 2. \end{cases}$$

The corresponding analytical solution for $t \in [0,2]$ is given by

$$u(t,x) = \begin{cases} (x+1-t)\mathbb{1}_{[-1+t,0)}(x) + (2x+1-t)\cdot\mathbb{1}_{[0,\frac{t}{2})}(x) + (x-\frac{t}{2}+1)\mathbb{1}_{[\frac{t}{2},1+\frac{t}{2})}(x) \\ \qquad\qquad +2\,\mathbb{1}_{[1+\frac{t}{2},+\infty)}(x), & \text{for } t \le 1, \\ (2\,x+1-t)\,\mathbb{1}_{[\frac{1}{2}(t-1),\frac{t}{2})}(x) + (x-\frac{t}{2}+1)\mathbb{1}_{[\frac{t}{2},1+\frac{t}{2}}(x) + 2\,\mathbb{1}_{[1+\frac{t}{2},+\infty)}(x), \\ & \text{for } t > 1. \end{cases}$$

Essentially, we use the backward transport equation to recover the initial state of the system. Thus, we expect the approximate solution to satisfy the relation $u_x(t,x) \approx f(t,x)$, which holds exactly for the analytical solution under some regularity assumption. As shown in Figure 6.2, the numerical approximation matches the analytical solution very closely.

The good performance of the scheme is further confirmed in Figure 6.3, where the convergence rates are reported both in L^∞-norm and in Wasserstein distance

$$\|\phi\|_{L^\infty} = \max_i |\phi_i|, \quad \mathcal{W}_1(\mu,\nu) = \int_{-\infty}^{+\infty} |F_\mu(x) - F_\nu(x)|\,dx,$$

where F_μ and F_ν are the cumulative distribution functions of μ and ν, respectively.

The numerical experiments display first–order convergence for the transport equation and a slightly lower rate for the conservative formulation when the error is measured in the $\mathcal{W}_1$ distance. This behavior can be heuristically explained by the structure of the schemes. In the transport case, the update is purely Lagrangian: each grid value is transported along the approximate flow, so that the global error mainly stems from the time discretization and from the regularization of the velocity field, both of order $O(h)$. In contrast, the conservative scheme involves an additional redistribution of mass among cell barycenters, which introduces a weak numerical diffusion. This diffusive effect smooths the transported density and slightly degrades the observed order of convergence. The phenomenon is consistent with the smoothing estimate given in Lemma 5.4 (see Eq. (5.5)), which quantifies the averaging produced by the stochastic flow in the conservative setting.

Test 2. In the second test, we move to a two-dimensional case. The aim is to study the properties of the scheme with respect to a triangular grid. We consider a problem set in $[-2,2]^2 \subset \mathbb{R}^2$, with a continuous drift

$$-b(t,x) = (2 - \max(|x_1|,|x_2|))\,(x_2,-x_1),$$

a rotating velocity field that degenerates near the origin and along the boundary of $[-2,2]^2$ (see Figure 6.4). For this domain, we employ a triangular mesh generator with $\Delta x = 0.08$ (corresponding to the maximum triangle area). The mesh is generated using the well-known software `Triangle` [29], which produces a quality, conforming Delaunay triangulation of bounded 2D domains.

The other problem parameters are set as follows: $h = 2\Delta x$ and $a(t,x) \equiv \frac{1}{2}\sigma\sigma^T$, with $\sigma = 10^{-3}\,\mathrm{Id}_2$, where Id_2 is the identity matrix of dimension 2. The time interval is $[0,T] = [0,1.5]$. In this case, we take as mollifier

$$\rho(x) := \frac{1}{C}\,e^{-\frac{1}{1-(x_1^2+x_2^2)}}\,\mathbb{1}_{B(0,1)}, \text{ with } C = \int_{B(0,1)} e^{-\frac{1}{1-(x_1^2+x_2^2)}}\,dx.$$

We begin by considering the backward viscous transport equation with final condition

$$u_T(x) = \sqrt{(x_1-1)^2 + x_2^2}.$$

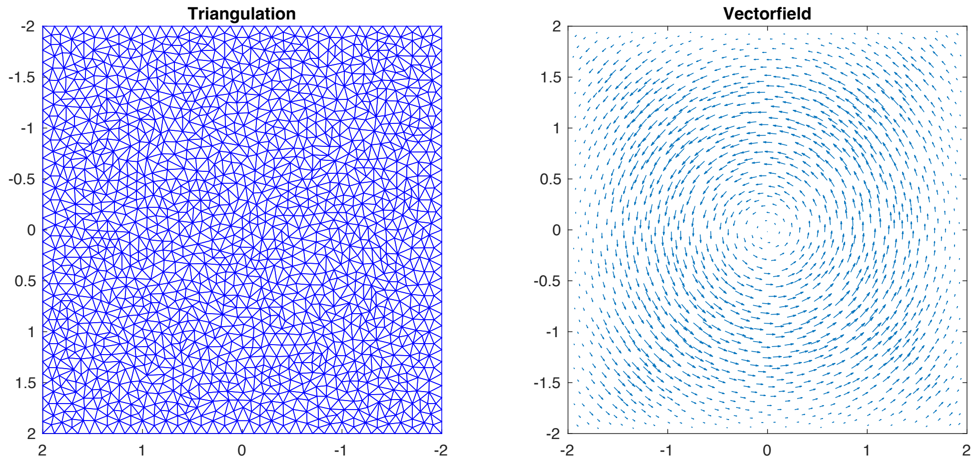

Fig. 6.4. *Test 2. Triangulation used ($\Delta x = 0.08$) and vector field b.*

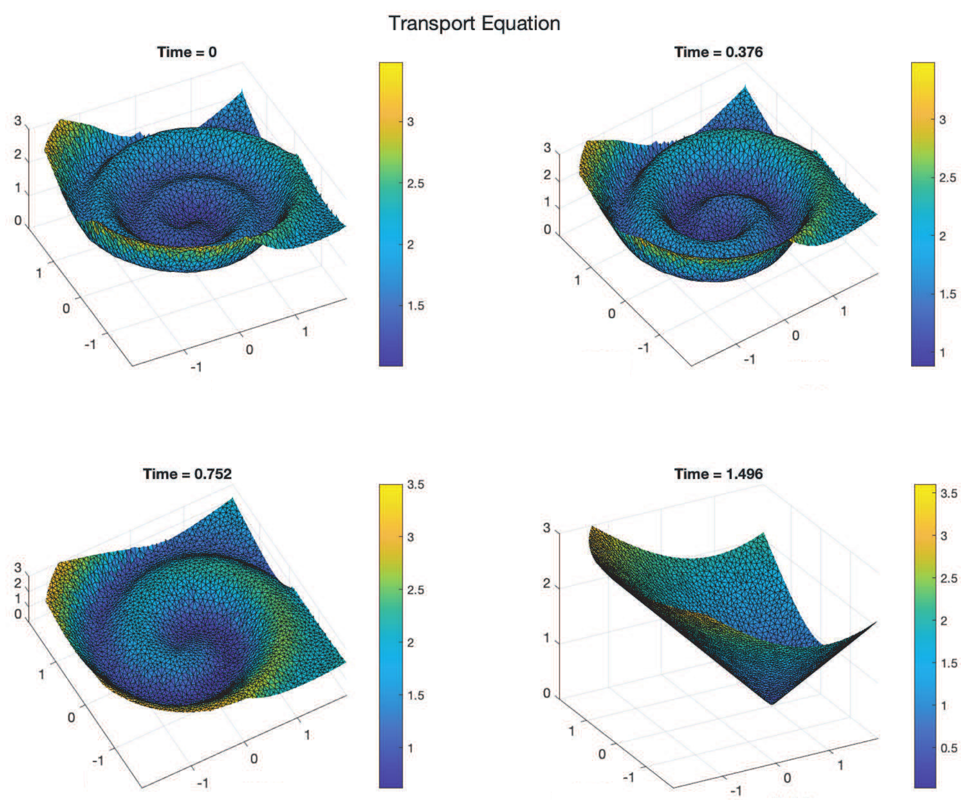

Fig. 6.5. *Test 2. Solution of the transport equation at various moments of its evolution.*

Figure 6.5 shows the approximate solution at different times during its evolution. In this case, the presence of a nondifferentiable point in the final condition (only Lipschitz continuity is required by Theorem 4.6) does not affect the performance of the numerical scheme, which successfully approximates the solution without introducing spurious oscillations.

We now turn to analyzing the same choice of parameters for the viscous conservation equation,

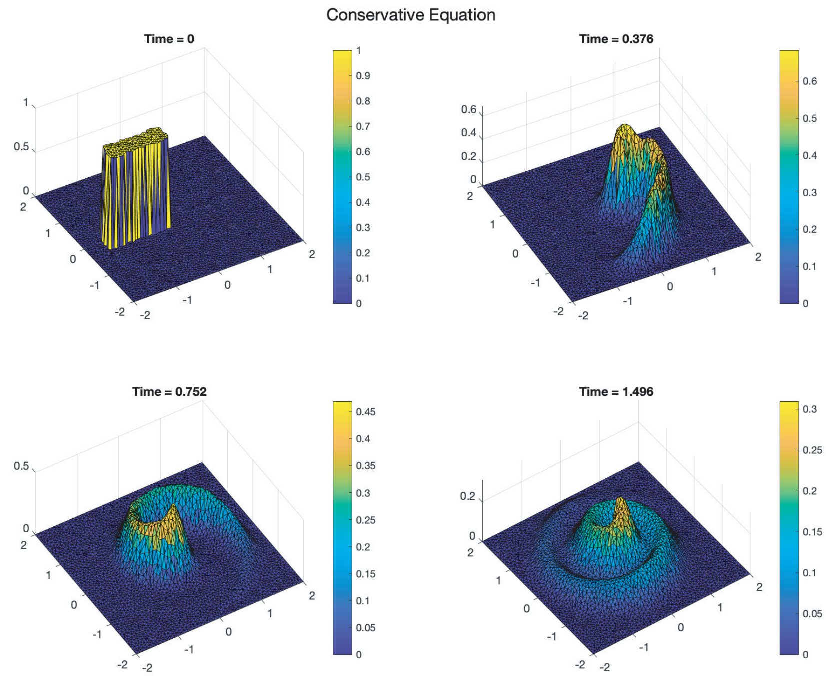

FIG. 6.6. *Test 2. Solution of the conservation equation at various moments of its evolution.*

with initial condition

$$f_0(x) = \mathbb{1}_{[-1.5,-0.1]\times[-0.25,0.25]}(x),$$

where the initial data is clearly discontinuous. Figure 6.6 displays the approximate solution at various times during its evolution. It is possible to observe that a higher density remains concentrated near the center of the domain (where the vector field b is smaller), while the rest of the density moves more rapidly in a non-uniform circular motion.

In Figure 6.7 we observe how various choices of the diffusion coefficient σ influences the solution. We report here the previous test with, respectively, $\sigma = 10^{-2}\mathrm{Id}_2$ (left) and $\sigma = 0.3\,\mathrm{Id}_2$ (right).

Test 3. In our third test we pass to the case of a discontinuous velocity field b, time-varying, that verifies the (OSL) condition. We consider

$$-b(t,x,y) = \begin{cases} (3/2,0) & x < t \\ (1/2,0) & x \geq t \end{cases}$$

where we can notice that the discontinuity moves in the same direction of the motion with unitary speed, while beyond the discontinuity, since the speed is lower, we expect then that a compression wave will appear. In this test the diffusion coefficient is variable, while the scenario remains convection dominant. We set

$$\sigma(t,x) = 0.1\,|\cos(\pi\,x_1)\cos(\pi\,x_2)|\,\mathrm{Id}_2$$

where we observe that such function degenerate to zero in some points of the domain.

In Figure 6.8 we observe the (backward) evolution of the viscous transport equation in the time interval $[0,0.8]$, starting from the final condition

$$u_T(x) = \sqrt{(x_1-0.8)^2 + x_2^2}.$$

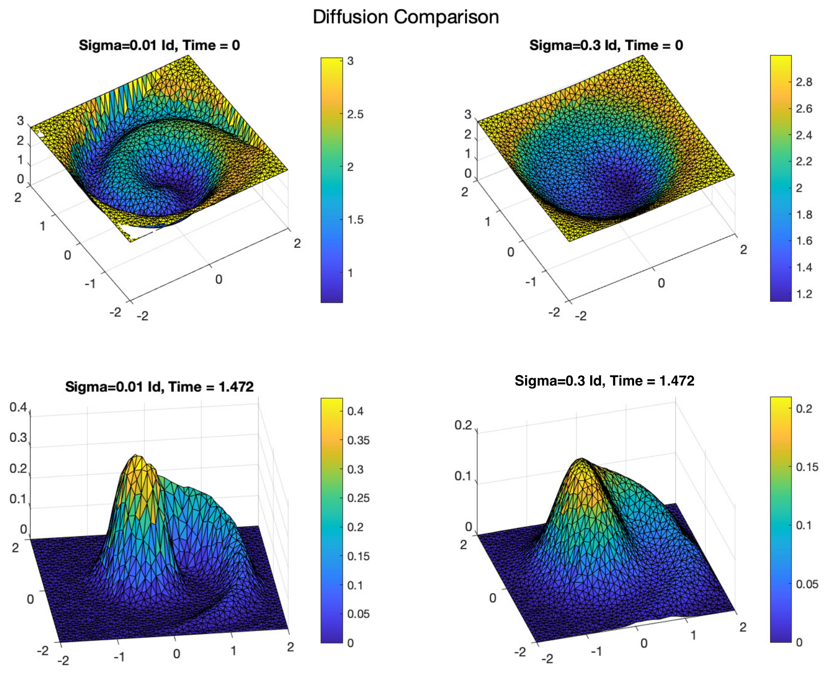

FIG. 6.7. *Test 2. Comparison for various choices of σ: respectively, $\sigma = 10^{-2}\,Id_2$ (left) and $\sigma = 0.3\,Id_2$ (right).*

Here the discretization parameters are set as $(\Delta x, h) = (0.01, 0.02)$. In the figure, the red line marks the position of the discontinuity in the velocity field b. Such discontinuity is visible in the gradient of the approximated solution, while the diffusion tends to smooth up the minimum point of the initial solution. Also in this case, to avoid boundary effects, the problem is computed in a much larger domain (in particular $[-4, 4]^2$).

We now turn to analyzing the same choice of parameters for the viscous conservation equation, with initial condition

$$f_0(x) = \mathbb{1}_{[-1,1]\times[-0.5,0.5]}(x).$$

Figure 6.9 shows the approximate solution at several time points during its evolution. A compression wave can be observed, as indicated by the fact that the peak concentration at the final time exceeds the maximum of the initial condition. This concentration effect is smoothed by diffusion and by the fact that the discontinuity propagates in the same direction as the drift.

7. Conclusions. In this work, we developed and analyzed semi-Lagrangian schemes on unstructured meshes for viscous transport and conservative equations with coefficients satisfying a one-sided Lipschitz condition. By leveraging a probabilistic framework and the dual characterization of solutions as viscosity and duality solutions, we proved convergence under minimal regularity assumptions. The schemes exhibit strong robustness in the presence of irregular data and extend classical semi-Lagrangian methods beyond the Di Perna-Lions setting.

Numerical results confirm convergence to the correct solution and validate the method's accuracy and stability on unstructured grids, making it particularly effective for high-dimensional and geometrically complex domains. The approach is especially relevant for Mean Field Games, where monotone interactions and degenerate dynamics frequently arise. Its compatibility with measure-theoretic formulations and probabilistic interpretations enhances its suitability for modeling socio-economic and collective behavior dynamics.

Future directions include extending the method to fully nonlinear second-order equations and systems with additional couplings or controls.

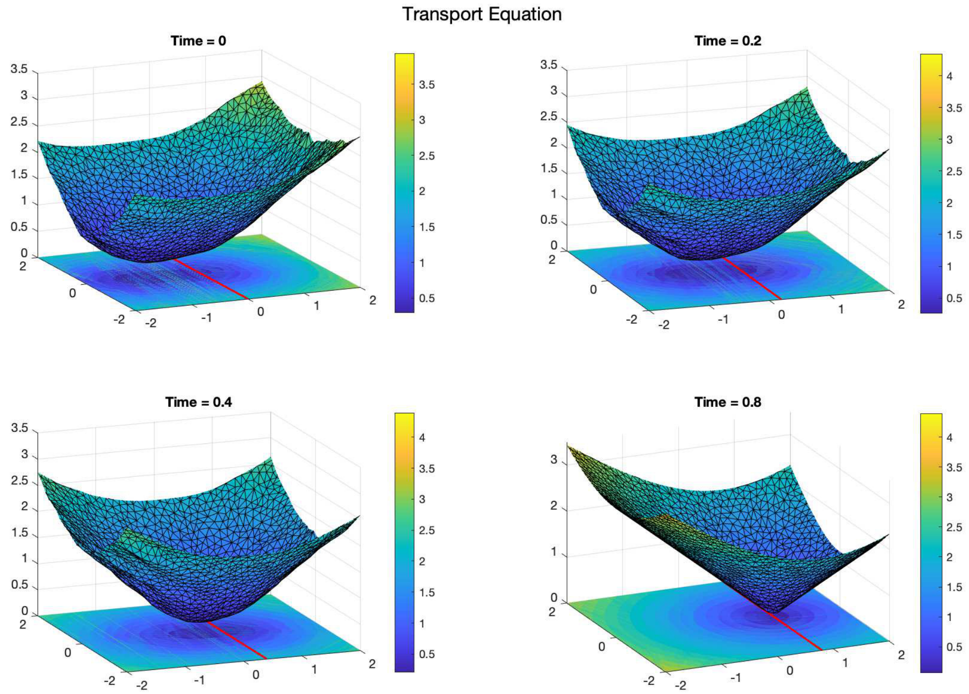

FIG. 6.8. *Test 3. Transport equation. The red line marks the position, varying in time, of the discontinuity in the velocity field. The contour lines are also displayed for a better interpretation of the plot.*

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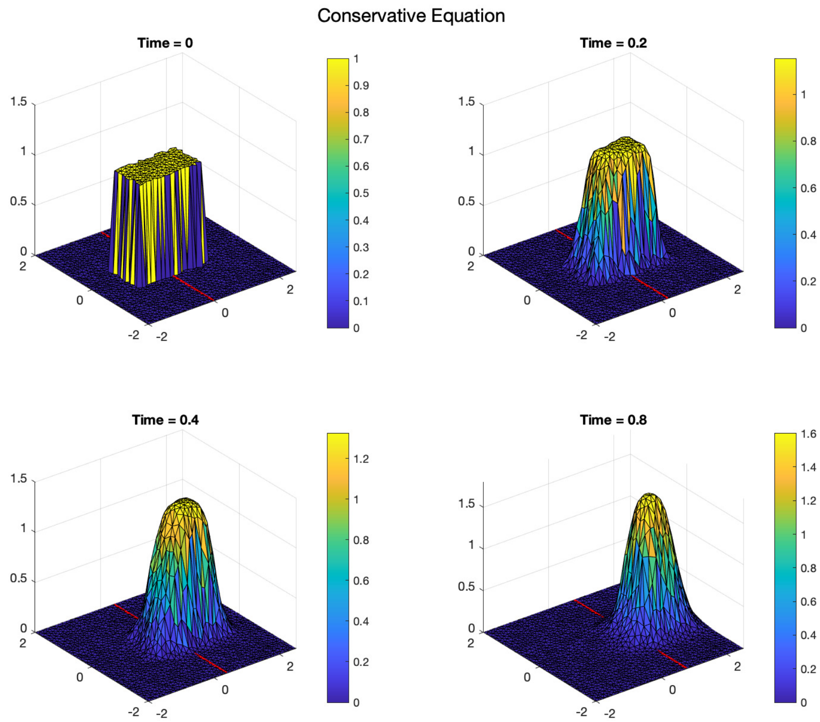

FIG. 6.9. *Test 3. Conservation equation. The red line marks the position, varying in time, of the discontinuity in the velocity field. The initial density concentrates while passing through the moving discontinuity.*

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