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Differential Geometry and its Applications 103 (2026) 102347

Contents lists available at ScienceDirect

Differential Geometry and its Applications

journal homepage: www.elsevier.com/locate/difgeo

Toric Kähler-Einstein manifolds immersed in complex projective spaces ☆

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A R T I C L E I N F O

Article history:
Received 27 June 2025
Received in revised form 16 January 2026
Accepted 30 January 2026
Available online xxxx
Communicated by D.V. Alekseevsky

MSC:
53C55
32Q20
32Q40
35J96
35C11

Keywords:
Kähler-Einstein manifolds
Kähler immersions
Toric Kähler manifolds
Complex projective spaces
Calabi's diastasis function

A B S T R A C T

We give a complete list, for $n \leq 6$, of mutually non-isometric $\mathbb{T}^n$-invariant Kähler-Einstein manifolds, where $\mathbb{T}^n$ is the real n-dimensional torus, immersed in a finite dimensional complex projective space endowed with the Fubini-Study metric. This class includes, in particular, all toric Kähler manifolds. This solves, in the aforementioned case, a classical and long-staying problem addressed among others by Calabi and Chern.

1. Introduction

The problem of classifying *Kähler immersions* (i.e. holomorphic isometric immersions) into complex space forms (i.e. Kähler manifolds with constant holomorphic sectional curvature) goes back to the classical works of Calabi and Bochner [9,10]. In the case of Kähler-Einstein metrics, the above problem has been solved for Kähler immersions into either the Euclidean or the hyperbolic spaces (see [37]), remaining largely open for

☆ The first author gratefully acknowledges support by the projects "Finanziamento alla Ricerca" under the contract numbers 53_RBA17MANGIO, 53_RBA21MANGIO, and by the PRIN project 2022 "Real and Complex Manifolds: Geometry and Holomorphic Dynamics" (code 2022AP8HZ9). The second author is supported by the "Starting Grant" under the contact number 53_RSG22SALFIL. Both authors are members of the GNSAGA of the INdAM.

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https://doi.org/10.1016/j.difgeo.2026.102347

the so-called *projectively induced* metrics, namely Kähler metrics arising as the pullback of the Fubini-Study metric via a Kähler immersion into some complex projective space; see, for instance, [15,17,34,11,36,25]. Similar issues arise, for example, in the broader context of extremal Kähler metrics and for notable classes such as Kähler–Ricci solitons, where the existence of projectively induced representatives is far from being fully understood (see, e.g., [3,21–24] and references therein).

In this paper we investigate the problem of classifying projectively induced Kähler-Einstein metrics within the class of connected Kähler manifolds admitting an effective, holomorphic and Hamiltonian action of the real torus $\mathbb{T}^n$ with at least one fixed point. We refer to such manifolds as $\mathbb{T}^n$*-invariant*. This class includes, in particular, all toric Kähler manifolds [2,13,14]. Even in the presence of such symmetry, the above classification problem shows significant analytical and geometric challenges.

Our main result, contained in the theorem below, proves that, for $n \leq 6$, every projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein metric arises from a product of projective spaces endowed with explicit multiples of the Fubini–Study metric.

Theorem 1.1. *For $n \leq 6$, the only projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein manifolds are open subsets of*

$$\mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_k}, \qquad n_1 + \cdots + n_k = n,$$

endowed with the Kähler metric

$$q\big(c_1 g_{FS} \oplus \cdots \oplus c_k g_{FS}\big),$$

where g_{FS} stands for the Fubini-Study metric, $q \in \mathbb{Z}_{>0}$,

$$c_i = \frac{1}{G^{k-1}} \prod_{j \neq i} (n_j + 1), \qquad G = \gcd(n_1 + 1, \ldots, n_k + 1)$$

and gcd *denotes the greatest common divisor.*

This extends the work of [5], who treated the case $n \leq 4$ in the toric setting. Here we reach dimension 6 and work in the more general framework of $\mathbb{T}^n$-invariant manifolds.

1.1. Description of the paper

The starting point of our analysis is the well-known fact that, by a simple integration, the Kähler-Einstein condition $\mathrm{Ric} = \lambda g$ for a Kähler metric g can be expressed in terms of the following complex Monge-Ampère equation for its potentials Φ:

$$\det(\partial\bar{\partial}\Phi) = e^{-\frac{\lambda}{2}(\Phi+\varphi+\bar{\varphi})}, \tag{1}$$

where φ denotes an arbitrary holomorphic function.

In Section 2.1, we study (1) in the $\mathbb{T}^n$-invariant context. Indeed, by exploiting the existence of a distinguished $\mathbb{T}^n$-invariant potential, the Calabi's diastasis function, and by considering a suitable normalization of the Einstein constant, we may assume, without loss of generality, that the equation (1) takes, in suitable real coordinates $x = (x_1, \ldots, x_n)$, exactly the form

$$\det D^2 u = e^{-u}, \qquad u = u(x). \tag{2}$$

A key point is that, since we are assuming that the Kähler-Einstein g is projectively induced, the solutions of (2) have a very rigid structure. More precisely, we prove (cfr. Proposition 2.6) that they need to be written as

$$u(x) = \log\left(\sum_{I\in\mathcal{I}} a_I\, e^{I\cdot x}\right) - \sum_{i=1}^{n} x_i, \tag{3}$$

where $\mathcal{I} \subset \mathbb{N}^n$ is finite, the coefficients satisfy $a_I \geq 0$ for all I and $a_I = 1$ for $|I| \leq 1$.

In Section 2.2 we study the gradient map Du of solutions of (2) of type (3). We show that the closure of its image is a convex polytope which, by results contained in [12,28], must be a *smooth reflexive Delzant polytope* with barycenter at the origin. Conversely, by [8, Theorem 1.1], any such polytope arises as the gradient image of a convex solution of (2) (although not necessarily of type (3)). Hence, taking all this into account, the problem of classifying $\mathbb{T}^n$-invariant Kähler-Einstein projectively induced metrics reduces to identifying which reflexive Delzant polytopes are actually produced by solutions of type (3). We underline that the majority of the smooth reflexive Delzant polytopes, with barycenter at the origin, that are not the closure of the gradient map of any solutions of type (3).

In Section 2.3, we illustrate a fundamental geometric ingredient that is able to select the shape of polytopes associated with the aforementioned particular solutions. This criterion follows from [27, Lemma 2.8].

In Section 2.4, the aforementioned geometrical ingredient is suitably applied to the list of smooth reflexive Delzant polytopes, with barycenter at the origin, given in [31,32] up to dimension 6. As there exist numerous of such polytopes beyond products of simplices, the geometric condition we use becomes essential in excluding the more intricate cases, finally leading to Theorem 1.1.

Notation and conventions

If $I = (I_1, \ldots, I_n) \in \mathbb{N}^n$ is a multi-index, its length is $|I| := \sum_{\alpha=1}^{n} I_\alpha$. For $w = (w_1, \ldots, w_n)$ we denote by w^I the monomial $\prod_{\alpha=1}^{n} w_\alpha^{I_\alpha}$.

2. Proof of Theorem 1.1

2.1. Calabi's diastasis and polynomial solutions of an n-dimensional Monge-Ampère equation

Since Kähler potentials of the Fubini-Study metric g_{FS} are real analytic functions, a Kähler potential Φ of a Kähler metric defined as the holomorphic pullback of g_{FS} is itself real analytic. Thus, in a holomorphic coordinate system on an open set $U \subseteq \mathbb{C}^n$

$$z = (z_1, \ldots, z_n),$$

the function Φ coincides with its power expansion around the origin:

$$\Phi(z) = \sum_{I,J\in\mathbb{N}^n} a_{IJ} z^I \bar{z}^J. \tag{4}$$

The series (4) extends complex analytically to a function $\tilde{\Phi}$ on a neighborhood of the diagonal in $U \times \overline{U}$, where $\overline{U}$ denotes the conjugate of U. This defines the *diastasis function* $\mathrm{D}_0 : U \to \mathbb{R}$ associated with g:

$$\mathrm{D}_0(z) = \tilde{\Phi}(z, \bar{z}) - \tilde{\Phi}(z, 0) - \tilde{\Phi}(0, \bar{z}) + \tilde{\Phi}(0, 0).$$

Moreover, for any Kähler manifold with real analytic metric, there exists a coordinate system, still denoted $z = (z_1, \ldots, z_n)$, in a neighborhood of each point such that

$$\mathrm{D}_0(z) = \sum_{\alpha=1}^{n} |z_\alpha|^2 + \psi, \tag{5}$$

where ψ is a power series of degree ≥ 2 in both z and $\bar{z}$.

Definition 2.1. A coordinate system $(z_1, \ldots, z_n)$ for which (5) holds is called a system of *Bochner coordinates* for the metric g.

Bochner coordinates are uniquely determined up to unitary transformations (see [9,10,16,17,35]).

Lemma 2.2. *Let (M, g) be a projectively induced $\mathbb{T}^n$-invariant Kähler manifold. Let $z = (z_1, \ldots, z_n)$ be Bochner coordinates for g centered at a fixed point p of the $\mathbb{T}^n$-action. Then the diastasis function $\mathrm{D}_0(z)$ can be written as*

$$\mathrm{D}_0(z) = \log\big(P(z)\big), \tag{6}$$

where

$$P(z) = \sum_{I \in \mathcal{I}} a_I |z^I|^2, \tag{7}$$

with $a_I > 0$ and $a_I = 1$ for all $|I| \leq 1$. Here $\mathcal{I}$ is a finite subset of $\mathbb{N}^n$.

Proof. Let $Z_0, \ldots, Z_N$ be homogeneous coordinates on $\mathbb{CP}^N$, $N \geq n$, and let $\zeta_j = Z_j/Z_0$ be affine coordinates around the point $[1, 0, \ldots, 0]$ on $U_0 = \{Z_0 \neq 0\}$. Let $f\colon M \to \mathbb{CP}^N$ be a Kähler immersion. Up to a unitary transformation of $\mathbb{CP}^N$, and possibly shrinking the domain of f, we may assume that $f(p) = [1, 0, \ldots, 0]$ and $f(V) \subset U_0$ for a neighborhood V of p.

By [10, Theorem 7], one can choose Bochner coordinates on U_0 so that, in Bochner coordinates $z = (z_1, \ldots, z_n)$ for g centered at p, the map f is expressed in coordinates as the graph of a holomorphic function:

$$z = (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, f_{n+1}(z), \ldots, f_N(z)),$$

where

$$f_j(z) = \sum_{I \in \mathbb{N}^n} \alpha_{jI} z^I, \qquad j = n+1, \ldots, N.$$

The affine coordinates ζ_j on U_0 are Bochner coordinates for the Fubini-Study metric g_{FS}. Moreover, the diastasis is hereditary (see [10, Prop. 6]), meaning that the diastasis of g is the composition of the diastasis of the ambient space,

$$\log\left(1 + \sum_{j=1}^{N} |\zeta_j|^2\right),$$

with the immersion f. Hence

$$\mathrm{D}_0(z) = \log\left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} |f_j(z)|^2\right).$$

Since the diastasis at a fixed point of a $\mathbb{T}^n$-action depends only on the moduli $|z_\alpha|$ (see e.g. [4]), each f_j must be a monomial, and (6) follows. □

Lemma 2.3. *The Einstein constant of a projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein manifold is a positive rational number.*

Proof. Let $z = (z_1, \ldots, z_n)$ be arbitrary holomorphic coordinates. A Kähler metric g with diastasis D_0 is Einstein if and only if

$$\lambda \frac{i}{2} \partial\bar{\partial} \mathrm{D}_0 = -i\partial\bar{\partial} \log\det(g_{\alpha\bar{\beta}})$$

for some $\lambda \in \mathbb{R}$. By the $\partial\bar{\partial}$-lemma, there exists a holomorphic function φ satisfying

$$\det(g_{\alpha\bar{\beta}}) = e^{-\frac{\lambda}{2}(\mathrm{D}_0+\varphi+\bar{\varphi})}. \tag{8}$$

Passing to Bochner coordinates (still denoted by z), comparison of the series expansions of both sides of (8) shows that $\varphi + \bar{\varphi} \equiv 0$ (see e.g. [17,33]).

Since D_0 is the diastasis of a projectively induced $\mathbb{T}^n$-invariant Kähler metric, Lemma 2.2 implies that (8) may be written as an identity between polynomials in the real variables $x_\alpha := |z_\alpha|^2$:

$$\frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{\alpha,\beta=1}^n}{P^{n-1}} = P^{-\frac{\lambda}{2}+n+1}. \tag{9}$$

Comparing degrees on both sides yields $\lambda \in \mathbb{Q}$ and $\lambda > 0$. □

Remark 2.4. In view of Lemma 2.3, we may write $\lambda = 2s/q \in \mathbb{Q}^+$ with $\gcd(s, q) = 1$. Let $P(|z_1|^2, \ldots, |z_n|^2)$ be a polynomial solution of type (7) to (9). Since $\gcd(2nq, s) = 1$, P must be the q-th power of a polynomial, i.e.

$$P(x_1, \ldots, x_n) = \sum_{I \in \mathcal{I}} a_I x^I, \qquad a_I = 1 \text{ if } |I| \leq 1, \tag{10}$$

is necessarily of the form

$$P(x) = R(x/q)^q,$$

for some polynomial R of the same type. One checks that R solves

$$\frac{\det\left[\left(R\frac{\partial^2 R}{\partial x_\alpha \partial x_\beta} - \frac{\partial R}{\partial x_\alpha}\frac{\partial R}{\partial x_\beta}\right)x_\alpha + R\frac{\partial R}{\partial x_\alpha}\delta_{\alpha\beta}\right]}{R^{n-1}} = R^{n+1-s}. \tag{11}$$

Conversely, if R solves (11), then

$$P(x) = R(x/s)^s$$

solves

$$\frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]}{P^{n-1}} = P^n. \tag{12}$$

Lemma 2.5. *Without loss of generality, in the sense of Remark 2.4, we may assume $\lambda/2 = 1$ in (9).*

We now summarize the results obtained so far in this section in order to state Proposition 2.6, which plays a central role in the proof of Theorem 1.1.

Every Kähler metric admits infinitely many local potentials, but in the real analytic case there is a unique one whose power series contains no terms in z or $\bar{z}$: the diastasis $\mathrm{D}_0(z)$. If (M, g) is a projectively induced $\mathbb{T}^n$-invariant Kähler manifold and p is a fixed point of the $\mathbb{T}^n$-action, then in a holomorphic coordinate system $z = (z_1, \ldots, z_n)$ centered at p, Lemma 2.2 shows that D_0 must be of the form (6). If, moreover, (M, g) is Kähler-Einstein, then D_0 arises from a solution P of (12) as described in Remark 2.4.

Proposition 2.6. *There exists a bijective correspondence between projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein metrics defined near a fixed point and the solutions of the real n-dimensional Monge-Ampère equation (2) of type (3).*

Proof. As explained above, projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein metrics correspond exactly to solutions of type (10) of (12). The substitution $x_i \mapsto e^{x_i}$ shows that (10) solves (12) if and only if (3) solves (2). □

Example 2.7. In the one-dimensional case, (2) reduces to

$$u'' = e^{-u},$$

and the unique solution of type (3) is

$$\log\left(1 + \frac{e^x}{2}\right)^2 - x. \tag{13}$$

Substituting $x \mapsto |z|^2$ in (13) yields a local Kähler potential in the affine coordinate z for the metric $2g_{FS}$ on $\mathbb{CP}^1$.

2.2. Gradient maps of the solutions to the Monge-Ampère equation (2): momentum maps and Delzant polytopes

Lemma 2.8. *Let u be a function of the form (3). Then the closure $\mathcal{P}$ of the image $Du(\mathbb{R}^n)$ of the gradient map Du is the convex hull of $\mathcal{I}$ translated by $-\mathbf{1} = (-1, \ldots, -1)$ and, if $d = \max_{I\in\mathcal{I}} |I|$,*

$$\mathcal{P} \subseteq \left\{ (u_1, \ldots, u_n) \in \mathbb{R}^n \,\middle|\, u_1 \geq -1, \ldots, u_n \geq -1, \sum_{i=1}^n u_i \leq d - n \right\}. \tag{14}$$

In particular, $\mathcal{P}$ is a lattice polytope, namely a polytope whose vertices all have integer coordinates.

Proof. A straightforward computation shows that the values $Du(x)$ of the gradient are (up to translations) convex combinations of the elements of $\mathcal{I}$:

$$Du(x) = \frac{1}{\sum_{I\in\mathcal{I}} a_I e^{I\cdot x}} \sum_{I\in\mathcal{I}} a_I e^{I\cdot x} I - \mathbf{1}. \tag{15}$$

Thus $\mathcal{P} + \mathbf{1}$ is the convex hull of $\mathcal{I}$. Moreover, since $I \in \mathcal{I} \subset \mathbb{N}^n$ and $|I| \leq d$ for all $I \in \mathcal{I}$, we obtain $\sum_{i=1}^n u_i \leq d$ for any $(u_1, \ldots, u_n) \in \mathcal{P} + \mathbf{1}$. □

Remark 2.9. If u is of type (3), then

$$\{(-1,\ldots,-1),(0,-1,\ldots,-1),(-1,0,-1,\ldots,-1),\ldots,(-1,\ldots,-1,0)\}\subset\mathcal{P}.$$

When u is a solution of (2), additional geometric properties of the associated polytope $\mathcal{P}$ can be obtained by using the theory of toric manifolds. In order to better introduce such properties we recall some notions from the context of convex polytopes.

Definition 2.10. Let $u:\mathbb{R}^n\to\mathbb{R}$ be a solution of type (3) to the Monge-Ampère equation (2). The (convex) polytope $\mathcal{P}$ defined as the closure of $Du(\mathbb{R}^n)$ is called the *polytope associated to u.*

Definition 2.11. An n-dimensional convex polytope is called a *Delzant* polytope if and only if all the following properties are fulfilled:

simplicity: at each vertex p, exactly n edges l_i meet;
rationality: $l_i=p+tv_i$, where $t\in\mathbb{R}^+$ and $(v_1,\ldots,v_n)\in\mathbb{Z}^n$;
smoothness: $(v_1,\ldots,v_n)$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Definition 2.12. An n-dimensional lattice polytope $\mathcal{P}$ containing $\mathbf{0}=(0,\ldots,0)$ as an interior point is called *reflexive* if and only if

$$\mathcal{P}=\{y\in\mathbb{R}^n\mid Ay\le\mathbf{1}\},\tag{16}$$

where $A\in\mathbb{Z}^{m,n}$ and $\mathbf{1}$ is the column vector of length m with all entries equal to 1.

Remark 2.13. A reflexive polytope possesses a unique interior point with integer coordinates, which is forced to be the origin in view of Definition 2.12.

In order to prove that polytopes associated to solutions of type (3) to the Monge-Ampère equation (2) are in particular Delzant (see Proposition 2.16 below), we need the following two lemmas.

Lemma 2.14. *A projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein manifold is an open subset of a compact, simply connected and complete manifold M.*

Proof. In [16], it has been proved that every Kähler-Einstein manifold Kähler immersed into a complex projective space can be extended to a complete Kähler-Einstein manifold, still Kähler immersed into the same complex projective space. Since the Einstein constant λ is positive by Lemma 2.3, the Bonnet–Myers theorem implies that M is compact. Moreover, every compact Kähler manifold with positive definite Ricci tensor is simply connected by a well-known result due to Kobayashi [19]. □

Lemma 2.15. *A projectively induced $\mathbb{T}^n$-invariant Kähler-Einstein manifold M is an open subset of a toric Kähler manifold.*

Proof. In view of Lemma 2.14, we may assume without loss of generality that M itself is compact, simply connected and complete. Let $z=(z_1,\ldots,z_n)$ be Bochner coordinates centered at a fixed point of the $\mathbb{T}^n$-action and let $\mathrm{D}_0(z)$ be the diastasis. Define $u:U\cap(\mathbb{C}\setminus\{0\})^n\to\mathbb{R}$ by

$$u(\log|z_1|^2,\ldots,\log|z_n|^2)=\mathrm{D}_0\left(|z_1|^2,\ldots,|z_n|^2\right).\tag{17}$$

Choosing any branch of the complex logarithm, we set holomorphic coordinates $w_i = \log z_i$. Hence, the Kähler form reads locally as

$$\omega = \frac{i}{2}\partial\bar{\partial}u = \frac{i}{2}\sum_{k,j}\frac{\partial^2 u}{\partial w_k \partial \bar{w}_j}\, dw_k \wedge d\bar{w}_j = \sum_{k,j}\frac{\partial^2 u}{\partial r_k \partial r_j}\, dr_k \wedge d\theta_j, \tag{18}$$

where $w_k = r_k + i\theta_k$.

Since M is simply connected and real analytic, each Killing vector field ∂_{θ_k} extends to a unique global Killing vector field on M (see [30], Theorems 1 and 2). Let X_k be the global extension of ∂_{θ_k}. Every Killing vector field on a compact Kähler manifold is real holomorphic (see e.g. [29], Prop. 9.5), and since M is complete we obtain a holomorphic and isometric action of $\mathbb{R}$ on M by means of the flows of each Killing vector field X_k.

Furthermore, for any $1 \leq k, j \leq \dim M$, the commutator $[X_k, X_j]$ is a Killing vector field vanishing on U. Since a nontrivial Killing vector field cannot vanish on a totally geodesic submanifold of real codimension at least 2 (see [20]), the commutator $[X_k, X_j]$ must vanish everywhere on M. Therefore we obtain an effective holomorphic and isometric action $\mathcal{G}$ of $\mathbb{R}^n$ on M.

Let $V \subset M$ be the subset where at least one Killing vector field X_k vanishes. Since $M \setminus V$ consists precisely of maximal dimensional orbits of the action $\mathcal{G}$, the action restricts to $M \setminus V$. It is easy to see that $\mathbb{Z}^n$ is the stabilizer of $\mathcal{G}$ in $(M \setminus V) \cap U$. Hence we obtain a holomorphic and isometric action of the real torus $\mathbb{R}^n/\mathbb{Z}^n$ on $M \setminus V$. Since the stabilizers of $\mathcal{G}$ at points in V contain $\mathbb{Z}^n$, this torus action extends to the whole manifold M.

Every Killing vector field on a compact and simply connected Kähler manifold M is Hamiltonian: indeed, as observed above, such vector fields are real holomorphic and therefore symplectic, i.e. $i_{X_k}\omega$ is closed. Since M is simply connected, $H^1_{\mathrm{dR}}(M) = 0$, and so $i_{X_k}\omega$ is exact.

The existence of such an effective Hamiltonian action shows that M is a toric Kähler manifold. □

Proposition 2.16. *If a function u of type (3) is a solution of the Monge-Ampère equation (2), then its associated polytope $\mathcal{P}$ is Delzant and reflexive.*

Proof. By Lemma 2.15, u can be viewed as a local Kähler potential defined on an open dense subset of an n-dimensional toric Kähler-Einstein manifold M (see (17)). Let ω be the Kähler form of M. From (18) we obtain

$$i_{\partial_{\theta_j}}\omega = -\,d\left(\frac{\partial u}{\partial r_j}\right),$$

so a momentum map

$$\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^n$$

is given by the gradient of u (here $\mathfrak{t}^*$ denotes the dual of the Lie algebra of $\mathbb{T}^n$).

By the results of T. Delzant (see e.g. [12]), $\mu(M) \subset \mathbb{R}^n$ has the properties listed in Definition 2.11. Furthermore, since u is a solution of the Monge-Ampère equation (2), the Ricci form ρ of M equals 2ω (see also the proof of Lemma 2.3 and Remark 2.4). Thus the first Chern class $c_1(M) = \frac{1}{2\pi}[\rho]$ is equal to $\frac{1}{\pi}[\omega]$. Taking into account also that $\mathcal{P} = \mu(M)$ is a lattice polytope (see Lemma 2.8), it follows from McDuff's result [28] that $\mathcal{P}$ contains only one interior point with integer coordinates. By (14), this interior point must be $\mathbf{0}$. Moreover, in [28] it is shown that the condition $c_1(M) = \frac{1}{\pi}[\omega]$ implies that the polytope $\mathcal{P}$ is reflexive, by proving that the *affine distance* of the integer interior point (in our case $\mathbf{0}$) from any facet is equal to 1. □

Remark 2.17. Although our approach starts from the diastasis (see (17)) and the associated real Monge-Ampère equation (2), the symplectic toric picture is naturally recovered in our setting. Indeed, as shown in Proposition 2.16, the gradient map Du coincides with the moment map of the Hamiltonian $\mathbb{T}^n$-action with respect to the Kähler form ω. In particular, the Delzant polytope associated to u is precisely the moment polytope in the sense of symplectic toric geometry (see e.g. [1]).

Even if it is widely known that the existence of Kähler-Einstein metrics on toric manifolds is related to the position of the barycenter of the image of the momentum map (see e.g. [26,8,38]), we include the following (more general) proposition for the sake of completeness.

Proposition 2.18 *([8]). If $\mathcal{P}$ is a convex body containing $\mathbf{0}$ in its interior, then there exists a smooth convex function ϕ solving the Monge-Ampère equation (2) and such that the closure of the image of the gradient map $D\phi$ is $\mathcal{P}$ if and only if $\mathbf{0}$ is the barycenter of $\mathcal{P}$. The solution ϕ is uniquely determined up to the action of the additive group $\mathbb{R}^n$ by translations.*

It is natural to ask about the relationship between separability of solutions to (2), i.e. solutions of the form $u(x_1, \dots, x_k) + v(x_{k+1}, \dots, x_{k+h})$, and decomposability of the associated polytopes, i.e. polytopes which are Cartesian products of lower-dimensional ones. This aspect is clarified by the following propositions.

Proposition 2.19. *Let $u(x_1, \dots, x_k)$ and $v(x_{k+1}, \dots, x_{k+h})$ be solutions of type (3) to the k- and h-dimensional Monge-Ampère equations (2), with associated polytopes $\mathcal{P}$ and $\mathcal{Q}$. Then $u + v$ is a solution of type (3) to the $(k + h)$-dimensional Monge-Ampère equation (2), whose associated polytope is $\mathcal{P} \times \mathcal{Q}$.*

Proof. This follows directly from the definition of the gradient map and the form (3). □

Conversely, we have the following.

Proposition 2.20. *If u is a solution of type (3) to the Monge-Ampère equation (2) whose associated polytope decomposes as a Cartesian product of a k-dimensional polytope $\mathcal{P}$ and an h-dimensional polytope $\mathcal{Q}$, then the k- and h-dimensional Monge-Ampère equations (2) admit solutions of type (3) whose associated polytopes are respectively $\mathcal{P}$ and $\mathcal{Q}$.*

Proof. Since u is a solution of type (3) to the $(k+h)$-dimensional Monge-Ampère equation (2), the polytope $Du(\mathbb{R}^{k+h}) = \mathcal{P} \times \mathcal{Q}$ is reflexive (see Proposition 2.16). It is then easy to see that both $\mathcal{P}$ and $\mathcal{Q}$ must be reflexive as well. Moreover, since the origin of $\mathbb{R}^{k+h}$ is the barycenter of $\mathcal{P} \times \mathcal{Q}$ (see Proposition 2.18), the barycenters of $\mathcal{P}$ and $\mathcal{Q}$ are forced to be their unique interior points with integer coordinates.

Therefore, in view of Proposition 2.18, there exist convex solutions f_1 and f_2 to the k- and h-dimensional Monge-Ampère equations (2), whose associated polytopes are respectively $\mathcal{P}$ and $\mathcal{Q}$. By applying Proposition 2.18 once again, we find that

$$u(x_1, \dots, x_{k+h}) = f_1(x_1 + c_1, \dots, x_k + c_k) + f_2(x_{k+1} + c_{k+1}, \dots, x_{k+h} + c_{k+h})$$

for some $(c_1, \dots, c_{k+h}) \in \mathbb{R}^{k+h}$. Then $f_1(x_1 + c_1, \dots, x_k + c_k)$ and $f_2(x_{k+1} + c_{k+1}, \dots, x_{k+h} + c_{k+h})$ are functions of type (3). □

2.3. Some technical results

For practical reasons, in the following lemma we refer to the Monge-Ampère equation (12) instead of (2), keeping in mind the equivalence established in Proposition 2.6. The next lemma follows from Lemma 2.8 in

[27] and will be useful for determining the shape of the polytopes associated with solutions of type (3) to the Monge-Ampère equation (2).

Lemma 2.21. *Let P be a solution of type (10) to the Monge-Ampère equation (12). Then the restriction $P(0,\dots,0,t,0,\dots,0)$ of P to the i-axis is*

$$P(0,\dots,0,t,0,\dots,0)=\left(1+\frac{t}{k_i}\right)^{k_i}, \tag{19}$$

and the restriction of $\frac{\partial P}{\partial y_j}$ to the i-axis, for $j\neq i$, is

$$\frac{\partial P}{\partial y_j}(0,\dots,0,t,0,\dots,0)=\left(1+\frac{t}{k_i}\right)^{h_{ij}}, \tag{20}$$

for some $k_i\in\mathbb{Z}^+$ and $h_{ij}\in\mathbb{N}$. Moreover,

$$\sum_{\alpha\neq i} h_{i\alpha}=k_i(n-2)+2 \tag{21}$$

and

$$\frac{h_{ij}}{k_i}=\frac{h_{ji}}{k_j}, \tag{22}$$

for any i and j.

Proof. Formula (19) is formula (13) in [27] with $s=1$, where s denotes the constant $\lambda/2$, with λ the Einstein constant. Therefore s may be assumed equal to 1 in view of Lemma 2.4.

Concerning (20), formula (15) of [27] with $s=1$ gives

$$\prod_{j\neq i}\frac{\partial P}{\partial y_j}(0,\dots,0,t,0,\dots,0)=\prod_{\alpha=1}^{R}\left(1+\frac{t}{r_i}\right)^{k_i(n-2)+2}$$

for some $R\in\mathbb{Z}^+$ and $r_i\in\mathbb{R}^+$. In the same Lemma 2.8 of [27] it is proved that the only possibility is $R=1$ and $r_1=k_i$. Hence

$$\prod_{j\neq i}\frac{\partial P}{\partial y_j}(0,\dots,0,t,0,\dots,0)=\left(1+\frac{t}{k_i}\right)^{k_i(n-2)+2}.$$

Formulas (20) and (21) then follow directly.

Finally, (22) follows from the Cauchy-Schwarz Lemma by evaluating the first derivative of (20) at the origin. □

2.3.1. A geometric interpretation of the constants k_i and h_{ij}

Let u be a function of type (3), namely a function reading as

$$u(x)=\log\sum_{I\in\mathcal{I}} a_I e^{I\cdot x}-\sum_{\alpha} x_\alpha,\quad a_I=1 \text{ if } |I|\leq 1, \tag{23}$$

whose gradient image is equal to the interior of a given polytope $\mathcal{P}$.

Considering (15), a direct computation shows that the limit of $\frac{\partial u}{\partial x_j}$ for every x_α different from x_i tending to $-\infty$, is

$$\begin{cases} L_i(x_i) = \frac{1}{\sum_{I\in\hat{\mathcal{I}}} a_I e^{I\cdot x}} \sum_{I\in\hat{\mathcal{I}}} a_I e^{I\cdot x} I_i - 1, & \text{if } j=i \\ -1, & \text{otherwise} \end{cases}$$

where $\hat{\mathcal{I}} = \{I = (I_1,\ldots,I_n) \in \mathcal{I} \,|\, I_\alpha = 0 \ \forall \alpha \neq i\}$. Therefore the limit of the gradient of u, for every x_α different from x_i tending to $-\infty$, provides a parametrization for the interior of the edge l_i of $\mathcal{P}$ starting from $-\mathbf{1}$ and parallel to the i-axis. Furthermore, we have that

$$\lim_{x_i\to+\infty} L_i(x_i) = \max_{I\in\hat{\mathcal{I}}} |I| - 1.$$

Indeed, if $\hat{I} \in \hat{\mathcal{I}}$ is such that $|\hat{I}| = \max_{I\in\hat{\mathcal{I}}} |I|$, then the coefficient $a_{\hat{I}}$ cannot be 0 in view of the bijective correspondence between $\mathcal{I}$ and integer points of $\mathcal{P}$ expressed by Lemma 2.8. By working with the polynomial

$$P(x) = \sum_{I\in\mathcal{I}} a_I x^I, \quad a_I = 1 \text{ if } |I| \leq 1,$$

instead of the function u (23) (this choice is justified by Proposition 2.6) we have that the degree of the restriction to i-axis of P is $\max_{I\in\hat{\mathcal{I}}} |I|$, which is in turn equal to the length of the edge l_i of $\mathcal{P}$.

Moreover, by means of very similar considerations as above that we skip for the sake of brevity, we get that the degree of restriction to the i-axis of the derivative of P with respect to the j-th variable, is an integer value between 0 and the length of the intersection of $\mathcal{P} \cap \ell_{ij}$, where ℓ_{ij} denotes the straight line parallel to the i-axis and passing through the point having all its coordinates equal to -1 except for the j-th one, which is equal to 0.

2.4. Classification of smooth reflexive polytopes: final steps of the proof of Theorem 1.1

In [31], it has been developed an algorithm that has been used to completely classify smooth reflexive polytopes up to dimension 7. Indeed, until then a classification only up to size 5 was known ([6,39,7,18]). However, we are going to consider only polytopes up to dimension 6 because only in this case is present a description [32] in terms of the matrix A, see (16).

Let

$$\mathbf{k} = (k_1,\ldots,k_n)^T \tag{24}$$

and

$$\mathbf{H}_{ij} = \begin{cases} h_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \tag{25}$$

where k_i and h_{ij} are those defined in Lemma 2.21.

By Propositions 2.16, 2.18 and 2.19, we can consider only Delzant reflexive polytopes with barycenter at $\mathbf{0}$ that cannot be decomposed as a cartesian product of lower dimensional polytopes.

In the subsequent subsections we will use some tables containing the matrix A, defined by (16), and the vector $\mathbf{k}$ and the matrix $\mathbf{H}$, defined respectively by (24) and (25), that we have computed by considering what seen in Section 2.3.1. Notice that we consider only the case where each entry of $\mathbf{H}$ attains the maximum value predicted in the aforementioned section, because we are going to realize that only in this case the condition (21) is satisfied.

Table 1
n-dimensional simplex with barycenter at the origin.

A	$\mathbf{k}$	$\mathbf{H}$
$\begin{pmatrix} -\mathrm{Id}_{n\times n} \\ \mathbf{1} \end{pmatrix}$	$\begin{pmatrix} n+1 \\ \vdots \\ n+1 \end{pmatrix}$	$\mathbf{H}_{ij} = n - n\delta_{ij}$

2.4.1. Simplex

The n-simplex with the following $n+1$ vertices

$$(-1,\dots,-1),\ (n,-1,\dots,-1),\ (-1,n,-1\dots),\ \dots,\ (-1,\dots,-1,n)$$

is Delzant and reflexive and, as such, there exists a unique convex solution of the Monge-Ampère (2) associated to it. The aforementioned simplex is described by Table 1.

In Table 1, $\mathrm{Id}_{n\times n}$ is the n-dimensional identity matrix and $\mathbf{1}$ is the row whose entries are 1. We note that the values contained in the table do not contradict (21) and (22). Indeed, in this case we have the unique solution

$$u(x_1,\dots,x_n) = \log\left(1+\sum_{i=1}^{n}\frac{e^{x_i}}{n+1}\right)^{n+1} - \sum_{i=1}^{n} x_i, \tag{26}$$

that, in view of Proposition 2.6, is the solution associated to

$$(\mathbb{CP}^n, (n+1)g_{FS})\,. \tag{27}$$

2.4.2. 1-dimensional case

As we have already seen in the very end of Section 2.1, the only solution in this case is (13), that leads to (27) for $n=1$. In particular, we notice that the associated polytope is the segment from -1 to 1, namely a 1-dimensional simplex, according to Section 2.4.1.

2.4.3. 2-dimensional case

As we have already seen in Section 2.4.1, in this case we have the solution associated to the 2-simplex, namely (26) for $n=2$. Furthermore, in view of the Proposition 2.19 and Section 2.4.2, we have also the solution

$$\log\left(1+\frac{x_1}{2}\right)^2 + \log\left(1+\frac{x_2}{2}\right)^2 - x_1 - x_2,$$

whose associated polytope is the square with vertices $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$, namely the cartesian product of the segment $\{(t,-1)\mid -1\le t\le 1\}$ and the segment $\{(-1,t)\mid -1\le t\le 1\}$. This is the only reflexive Delzant polytope that can be decomposed as a cartesian product of 1-dimensional ones. In view of Proposition 2.6, this solution leads to

$$(\mathbb{CP}^1\times\mathbb{CP}^1,\ 2g_{FS}\oplus 2g_{FS}).$$

There exists another reflexive Delzant polytope with barycenter at the origin, namely the one given by Table 2.

The polytope described by Table 2 is the hexagon $\mathcal{E}$ with vertices $(-1,-1)$, $(-1,0)$, $(0,-1)$, $(0,1)$, $(1,0)$, $(1,1)$. It is easy to realize that such polytope satisfies conditions (21) and (22). The most general function u of type (3) such that the closure of $Du(\mathbb{R}^2)$ is equal to $\mathcal{E}$ reads as

Table 2
Undecomposable 2-dimensional smooth reflexive polytopes with barycenter at the origin (2-simplex excluded).

A	$\mathbf{k}$	$\mathbf{H}$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

$$\log\left(1 + e^{x_1} + e^{x_2} + a_{(1,1)}e^{x_1+x_2} + a_{(2,0)}e^{2x_1+x_2} + a_{(0,2)}e^{x_1+2x_2} + a_{(2,2)}e^{2x_1+2x_2}\right) - x_1 - x_2.$$

If such u is a solution of the Monge-Ampère (2), then it needs to satisfy (19) and (20). Hence

$$u(x_1, x_2) = \log\left(1 + e^{x_1} + e^{x_2} + 2e^{x_1+x_2} + e^{2x_1+x_2} + e^{x_1+2x_2} + a_{(2,2)}e^{2x_1+2x_2}\right) - x_1 - x_2.$$

We can easily compute that

$$\lim_{x_1\to-\infty}\lim_{x_2\to-\infty}\frac{\partial^2}{\partial x_1\partial x_2}\left(e^u \det D^2 u\right) \neq 0$$

independently of $a_{(2,2)}$. Therefore there is no $a_{(2,2)} \in \mathbb{R}$ for which u is a solution of (2).

2.4.4. 3-dimensional case

As said in Section 2.4.1, we have the solution associated to the 3-simplex, namely (26) for $n = 3$. Furthermore, in view of the Proposition 2.19 and Section 2.4.2, we have also the solutions

$$\log\left(1 + \frac{x_1}{2}\right)^2 + \log\left(1 + \frac{x_2}{2}\right)^2 + \left(1 + \frac{x_3}{2}\right)^2 - x_1 - x_2 - x_3$$

and, up to variables renaming,

$$\log\left(1 + \frac{x_1 + x_2}{3}\right)^3 + \log\left(1 + \frac{x_3}{2}\right)^2 - x_1 - x_2 - x_3,$$

which, in view of Proposition 2.6, lead respectively to

$$(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1,\ 2g_{FS} \oplus 2g_{FS} \oplus 2g_{FS})$$

and

$$(\mathbb{CP}^2 \times \mathbb{CP}^1,\ 3g_{FS} \oplus 2g_{FS}).$$

We can easily see that the associated polytopes are respectively, the cube with vertices

$$(-1,-1,-1), (1,-1,-1), (-1,1,-1), (-1,-1,1), (1,1,-1), (1,-1,1), (-1,1,1), (1,1,1),$$

namely the cartesian product of the three segments $\{(-1,-1,t) \mid -1 \leq t \leq 1\}$, $\{(-1,t,-1) \mid -1 \leq t \leq 1\}$ and $\{(t,-1,-1) \mid -1 \leq t \leq 1\}$, and the prism with vertices

$$(-1,-1,-1), (-1,1,-1), (1,-1,-1), (-1,-1,1), (-1,1,1), (1,-1,1)$$

namely the cartesian product of the 2-simplex whose vertices are $(-1,-1,-1), (-1,1,-1), (1,-1,-1)$ and the segment $\{(-1,-1,t) \mid -1 \leq t \leq 1\}$. By taking into account Proposition 2.20, we have no more

Table 3
Undecomposable 3-dimensional smooth reflexive polytopes with barycenter at the origin (3-simplex excluded).

A	$\mathbf{k}$	$\mathbf{H}$
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$

decomposable polytopes to take into account. Indeed, even if there exists another decomposable reflexive Delzant polytope with barycenter at the origin, namely the prism with vertices

$$\begin{gathered}(-1,-1,-1),(-1,0,-1),(0,-1,-1),(0,1,-1),(1,0,-1),(1,1,-1),\\(-1,-1,1),(-1,0,1),(0,-1,1),(0,1,1),(1,0,1),(1,1,1),\end{gathered}$$

we can see directly that it is a cartesian product of an hexagon and a segment. In view of Proposition 2.20, there are no solutions of type (3) associated to such polytope, since there are no 2-dimensional solutions of type (3) associated to the hexagon.

Finally, there is another undecomposable reflexive Delzant polytope with barycenter at the origin, namely the one given by Table 3.

Note that the values contained in Table 3 do not satisfy the condition (22). Therefore we cannot have solutions of type (3) related to such polytope.

2.4.5. 4-dimensional case

As said in Section 2.4.1, we have the solution associated to the 4-simplex, namely (26) for $n = 4$. Moreover, in view of the Proposition 2.20, Proposition 2.19 and taking into account the results of Sections 2.4.2–2.4.4, the only solutions whose associated polytope can be decomposed as a cartesian product of lower dimensional polytopes are

$$\begin{gathered}\log\left(1+\frac{x_1}{2}\right)^2+\log\left(1+\frac{x_2}{2}\right)^2+\left(1+\frac{x_3}{2}\right)^2+\log\left(1+\frac{x_4}{2}\right)^2-x_1-x_2-x_3-x_4,\\ \log\left(1+\frac{x_1+x_2}{3}\right)^3+\left(1+\frac{x_3}{2}\right)^2+\log\left(1+\frac{x_4}{2}\right)^2-x_1-x_2-x_3-x_4,\\ \log\left(1+\frac{x_1+x_2}{3}\right)^3+\log\left(1+\frac{x_3+x_4}{3}\right)^3-x_1-x_2-x_3-x_4,\\ \log\left(1+\frac{x_1+x_2+x_3}{4}\right)^4+\log\left(1+\frac{x_4}{2}\right)^2-x_1-x_2-x_3-x_4.\end{gathered}$$

In view of Proposition 2.6, these solutions respectively lead to

$$\begin{gathered}(\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1,\ 2g_{FS}\oplus 2g_{FS}\oplus 2g_{FS}\oplus 2g_{FS}),\\ (\mathbb{CP}^2\times\mathbb{CP}^1\times\mathbb{CP}^1,\ 3g_{FS}\oplus 2g_{FS}\oplus 2g_{FS}),\\ (\mathbb{CP}^2\times\mathbb{CP}^2,\ 3g_{FS}\oplus 3g_{FS}),\\ (\mathbb{CP}^3\times\mathbb{CP}^1,\ 4g_{FS}\oplus 2g_{FS}).\end{gathered}$$

Beside the 4-simplex, there exists also three further undecomposable reflexive Delzant polytope with barycenter at the origin, namely the ones given by Table 4. Only the first polytope in such table satisfies both conditions (21) and (22). Nevertheless, if we assume the existence of a solution of type (3) to the

Table 4
Undecomposable 4-dimensional smooth reflexive polytopes with barycenter at the origin (4-symplex excluded).

A	$\mathbf{k}$	$\mathbf{H}$
$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$

Table 5
Undecomposable 5-dimensional smooth reflexive polytopes with barycenter at the origin (5-symplex excluded).

A	$\mathbf{k}$	$\mathbf{H}$
$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 5 \\ 5 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 5 & 5 & 0 & 4 & 3 \\ 5 & 5 & 4 & 0 & 3 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 4 & 4 & 0 & 3 & 3 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 3 & 0 & 2 & 1 \\ 3 & 2 & 3 & 0 & 3 \\ 1 & 2 & 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 & 2 \\ 3 & 3 & 3 & 0 & 2 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 4 \\ 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 4 & 4 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 2 \\ 3 & 3 & 3 & 2 & 0 \end{pmatrix}$
$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \\ 4 \\ 4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 2 & 2 & 3 \\ 4 & 4 & 0 & 3 & 3 \\ 4 & 4 & 3 & 0 & 3 \\ 2 & 2 & 2 & 2 & 0 \end{pmatrix}$

Monge-Ampère equation (2) associated to such polytope, we get a contradiction. Indeed, by taking into account (19) and (20), we obtain after long computations that

$$\lim_{x_1 \to -\infty} \lim_{x_2 \to -\infty} \lim_{x_3 \to -\infty} \lim_{x_4 \to -\infty} \frac{\partial^2}{\partial x_1 \partial x_3} \left(e^u \det D^2 u \right) > 0.$$

2.4.6. 5-dimensional case

As already seen in Section 2.4.1, we have the solution associated to the 5-simplex, namely (26) for $n = 5$. Moreover, in view of Proposition 2.19 and 2.20, we can obtain all the solutions associated to decomposable polytopes (as a cartesian product of lower dimensional ones) by taking into account the results of Sections 2.4.2–2.4.5. Beside the 5-simplex, there are also seven further undecomposable reflexive Delzant 5-polytope with barycenter at the origin (see Table 5). Nevertheless, none of them satisfies the condition (22).

Table 6
Undecomposable 6-dimensional smooth reflexive polytopes with barycenter at the origin (6-symplex excluded).

A	$\mathbf{k}$	$\mathbf{H}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&1&-1&-1&-1\\ -1&-1&-1&1&1&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&0&2&2&2\\ 0&0&0&2&2&2\\ 0&0&0&2&2&2\\ 2&2&2&0&0&0\\ 2&2&2&0&0&0\\ 2&2&2&0&0&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&1&-1&-1&-1\\ 0&-1&-1&1&1&0\\ 0&1&1&0&-1&0\\ -1&0&-1&1&1&1\\ 1&0&1&0&0&0\\ 0&0&0&0&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\1\\2\end{pmatrix}$	$\begin{pmatrix} 0&0&0&2&2&2\\ 0&0&0&2&2&2\\ 0&0&0&2&2&2\\ 1&2&2&0&0&1\\ 1&2&2&0&0&1\\ 3&2&3&1&1&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&1&-1&-1&-1\\ 0&0&-1&1&0&0\\ 0&-1&0&0&1&0\\ -1&0&0&0&0&1\\ 1&1&1&0&0&0\\ 0&0&0&1&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&0&2&2&2\\ 0&0&0&2&2&2\\ 0&0&0&2&2&2\\ 1&1&2&0&1&1\\ 1&2&1&1&0&1\\ 2&1&1&1&1&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&0&0&0&-2\\ 0&0&1&0&0&-1\\ 0&0&-1&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&1&2 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\5\\5\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&1&3\\ 0&0&1&1&1&3\\ 1&1&0&1&1&2\\ 5&5&5&0&4&3\\ 5&5&5&4&0&3\\ 1&1&2&1&1&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&0&0&-1&-1\\ 0&-1&1&-1&1&0\\ -1&0&-1&1&0&1\\ 1&0&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&0&0&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&2&2\\ 0&0&1&1&2&2\\ 1&2&0&2&0&1\\ 2&1&2&0&1&0\\ 1&2&0&2&0&1\\ 2&1&2&0&1&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&0&0&-1&-1\\ 0&0&1&0&0&-1\\ -1&-1&0&0&1&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\3\\1\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&2&2\\ 0&0&1&1&2&2\\ 1&1&0&1&1&2\\ 3&3&3&0&3&2\\ 2&2&1&1&0&0\\ 2&2&1&1&0&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&0&0&-1&-1\\ -1&-1&0&0&1&1\\ 1&1&1&0&0&0\\ 0&0&0&1&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\4\\4\\1\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&2&2\\ 0&0&1&1&2&2\\ 3&3&0&4&4&4\\ 4&4&4&0&3&3\\ 2&2&1&1&0&0\\ 2&2&1&1&0&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&0&0&-1&-1\\ 0&0&0&0&1&-1\\ 0&0&0&0&-1&1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&1&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\5\\5\\1\\1\end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&2&2\\ 0&0&1&1&2&2\\ 5&5&0&4&4&4\\ 5&5&4&0&4&4\\ 1&1&1&1&0&2\\ 1&1&1&1&2&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&1&0&0&-1&-1\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&1&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\5\\5\\2\\2\end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&2&2\\ 0&0&1&1&2&2\\ 5&5&0&4&4&4\\ 5&5&4&0&4&4\\ 2&2&2&2&0&2\\ 2&2&2&2&2&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&0&0&0&0&-1\\ 0&1&0&0&0&-1\\ 0&0&1&0&0&-1\\ 0&0&-1&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\3\\3\\1\end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&2\\ 1&0&1&1&1&2\\ 1&1&0&1&1&2\\ 3&3&3&0&3&2\\ 3&3&3&3&0&2\\ 1&1&2&1&1&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&0&0&0&0&-1\\ 0&1&0&0&0&-1\\ 0&-1&1&0&0&0\\ 0&-1&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&1&0&0\\ 0&0&0&0&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\3\\3\\1\end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&2\\ 1&0&1&1&1&2\\ 1&2&0&1&1&1\\ 3&2&3&0&3&3\\ 3&3&3&3&0&2\\ 1&2&1&1&1&0 \end{pmatrix}$
$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1\\ 1&0&0&0&0&-1\\ -1&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&1&0&0&-1\\ 0&0&0&0&0&1\\ 0&0&0&1&1&1 \end{pmatrix}$	$\begin{pmatrix}1\\2\\2\\4\\4\\1\end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&2\\ 2&0&1&2&2&3\\ 2&1&0&2&2&3\\ 4&4&4&0&3&3\\ 4&4&4&3&0&3\\ 2&1&1&1&1&0 \end{pmatrix}$

2.4.7. 6-dimensional case

As already seen in Section 2.4.1, we have the solution associated to the 6-simplex, namely (26) for $n = 6$. Moreover, in view of Proposition 2.19 and 2.20, we can obtain all the solutions associated to decomposable polytopes (as a cartesian product of lower dimensional ones) by taking into account the results of Sections 2.4.2–2.4.6. Beside the 6-simplex, there are also twelve further undecomposable reflexive Delzant 6-polytope with barycenter at the origin, but only one (the first polytope in the Table 6) satisfies both the condition (21) and (22). Nevertheless, if we assume the existence of a solution of type (3) to the Monge-Ampère equation (2) associated to such polytope, we get a contradiction. Indeed, by taking into account (19) and (20), we can consider a linear system

$$\lim_{x_1\to-\infty}\lim_{x_2\to-\infty}\lim_{x_3\to-\infty}\lim_{x_4\to-\infty}\lim_{x_5\to-\infty}\lim_{x_6\to-\infty}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(e^u \det D^2 u\right) = 0$$

where $\alpha, \beta = 1, \ldots, 6$, having the third degree coefficients of the polynomial $P = e^{u+\sum_{i=1}^{6} y_i}|_{y_i=\log x_i}$ as variables. By considering that any coefficient of P cannot be negative, we obtain that such system admits a unique solution. Thus, this result puts us in the position to get, after long computations,

$$\lim_{x_1 \to -\infty} \lim_{x_2 \to -\infty} \lim_{x_3 \to -\infty} \lim_{x_4 \to -\infty} \lim_{x_5 \to -\infty} \lim_{x_6 \to -\infty} \frac{\partial^3}{\partial x_1^2 \partial x_2} \left(e^u \det D^2 u \right) > 0,$$

that clearly contradicts (2).

Data availability

No data was used for the research described in the article.

References

[1] M. Abreu, Kähler geometry of toric manifolds in symplectic coordinates, in: Symplectic and Contact Topology: Interactions and Perspectives, Toronto, 2001, in: Fields Inst. Commun., vol. 35, 2003, pp. 1–24.
[2] M. Atiyah, Convexity and commuting Hamiltonians, Bull. Lond. Math. Soc. 14 (1982) 1–15.
[3] C. Arezzo, C. Li, A. Loi, Gromov-Hausdorff limits and holomorphic isometries, Math. Res. Lett. 32 (3) (2025) 707–738.
[4] C. Arezzo, A. Loi, F. Zuddas, Some remarks on the symplectic and Kähler geometry of toric varieties, Ann. Mat. Pura Appl. 195 (2016) 1287–1304.
[5] C. Arezzo, A. Loi, F. Zuddas, On homothetic balanced metrics, Ann. Glob. Anal. Geom. 41 (2012) 473–491.
[6] V.V. Batyrev, Toroidal Fano 3-folds, Math. USSR, Izv. 19 (1982) 13–25.
[7] V.V. Batyrev, On the classification of toric Fano 4-folds, J. Math. Sci. (N.Y.) 94 (1999) 1021–1050.
[8] R.J. Berman, B. Berndtsson, Real Monge-Ampère equations and Kaehler-Ricci solitons on toric log Fano varieties, Ann. Fac. Sci. Toulouse Math. 6 22 (4) (2013) 649–711.
[9] S. Bochner, Curvature in Hermitian metric, Bull. Am. Math. Soc. 53 (2) (1947) 179–195.
[10] E. Calabi, Isometric imbedding of complex manifolds, Ann. Math. 58 (1) (1953) 1–23.
[11] S.S. Chern, Einstein hypersurfaces in a Kähler manifold of constant sectional curvature, J. Differ. Geom. 1 (1–2) (1967) 21–31.
[12] T. Delzant, Hamiltoniens periodiques et image convex de I'application moment, Bull. Soc. Math. Fr. 116 (1988) 315–339.
[13] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, Invent. Math. 67 (1982) 491–513.
[14] V. Guillemin, Kaehler structures on toric varieties, J. Differ. Geom. 40 (1994) 285–309.
[15] J. Hano, Einstein complete intersections in complex projective space, Math. Ann. 216 (3) (1975) 197–208.
[16] D. Hulin, Sous-variétés complexes d'Einstein de l'espace projectif, Bull. Soc. Math. Fr. 124 (2) (1996) 277–298.
[17] D. Hulin, Kähler–Einstein metrics and projective embeddings, J. Geom. Anal. 10 (3) (2000) 525–528.
[18] M. Kreuzer, B. Nill, Classification of toric Fano 5-folds, Adv. Geom. 9 (1) (2009) 85–97.
[19] S. Kobayashi, Compact Kaehler manifolds with positive Ricci tensor, Bull. Am. Math. Soc. 67 (4) (1961) 412–413.
[20] S. Kobayashi, Fixed points of isometries, Nagoya Math. J. 13 (1958) 63–68.
[21] C. Li, Extensions of extremal Kähler submanifolds of complex projective spaces, Ann. Mat. Pura Appl. (4) 203 (6) (2024) 2825–2845.
[22] A. Loi, F. Salis, F. Zuddas, Extremal Kähler metrics induced by finite or infinite complex space forms, J. Geom. Anal. 31 (8) (2021) 7842–7865.
[23] A. Loi, F. Salis, F. Zuddas, Kähler-Ricci solitons induced by infinite dimensional complex space forms, Pac. J. Math. 316 (2022) 183–205.
[24] A. Loi, F. Salis, F. Zuddas, Two conjectures on Ricci-flat Kähler metrics, Math. Z. 290 (2018) 599–613.
[25] A. Loi, M. Zedda, Kähler Immersions of Kähler Manifolds into Complex Space Forms, Lecture Notes of the Unione Matematica Italiana, vol. 23, Springer, 2018.
[26] T. Mabuchi, Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties, Osaka J. Math. 24 (4) (1987) 705–737.
[27] G. Manno, F. Salis, 2-dimensional Kähler-Einstein metrics induced by finite dimensional complex projective spaces, N.Y. J. Math. 28 (2022) 420–432.
[28] D. McDuff, Displacing Lagrangian toric fibers via probes, Proc. Symp. Pure Math. 82 (2011) 131–160.
[29] A. Moroianu, Lectures on Kähler Geometry, London Mathematical Society Student Texts, vol. 69, 2007.
[30] K. Nomizu, On local and global existence of Killing vector fields, Ann. Math. (2) 72 (1960) 105–120.
[31] M. Øbro, An algorithm for the classification of smooth Fano polytopes, preprint, arXiv:0704.0049.
[32] Dual smooth Fano polytopes, School of Mathematical Sciences at Nottingham University, http://www.grdb.co.uk/.
[33] F. Salis, Projectively induced rotation invariant Kähler metrics, Arch. Math. 109 (2017) 285–292.
[34] B. Smyth, Differential geometry of complex hypersurfaces, Ann. Math. (2) 85 (2) (1967) 246–266.
[35] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differ. Geom. 32 (1) (1990) 99–130.
[36] K. Tsukada, Einstein-Kähler submanifolds with codimension two in a complex space form, Math. Ann. 274 (1986) 503–516.
[37] M. Umehara, Einstein-Kähler submanifolds of complex linear or hyperbolic space, Tohoku Math. J. 39 (3) (1987) 385–389.

[38] X.-J. Wang, X. Zhang, Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. 188 (1) (2004) 87–103.
[39] K. Watanabe, M. Watanabe, The classification of Fano 3-folds with torus embeddings, Tokyo Math. J. 5 (1982) 37–48.