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The Wiener-Hopf Technique in Computational Electromagnetics - Part 1: Fundamentals

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Abstract: The Wiener-Hopf technique provides an essential analytical and semi-analytical methodology in applied mathematics and mathematical physics for solving boundary-value problems, formulated by means of partial differential equations and/or integral equations. By exploiting the properties of the formulation in spectral domain after the application of integral transformation, rigorous solutions can be obtained for computational physics problems. In electromagnetics the method allows effective analysis of problems involving diffraction phenomena, guided propagation with discontinuities, antenna radiation and modern applications. Recent advances have extended the capabilities of the method to the analysis of complex electromagnetic (and not only) problems involving different geometries and arbitrary media rather than rectangular objects in isotropic media. One of the best benefits of the method is that intrinsically allows physical interpretation by means of the spectral domain. Part 1 is devoted to fundamental mathematical concepts.

Index Terms: Wiener-Hopf method, applied mathematics, mathematical physics, computational physics, computational electromagnetics, electromagnetism, integral equations, Green's function, spectral domain, network representations, regularization, Fredholm factorization, wave motion, diffraction, discontinuities, antenna radiation, complex materials.

1. Introduction

The Wiener-Hopf (WH) technique in electromagnetics has been successfully applied to wave propagation in waveguide devices and junctions, open waveguides, scattering and diffraction by sharp structures, slot antennas and other interesting applications. Beyond canonical geometries, modern applications include propagation in complex media, metasurface modeling, novel antennas, electromagnetic compatibility, and hybrid numerical-analytical schemes of computation. Nowadays, WH method produces closed-form or semi-analytic solutions, that provide fundamental wave phenomena understanding, high-frequency asymptotic estimations, and benchmarks for full-numerical simulations across different areas of physics and in other disciplines.

In 1931 Wiener and Hopf invented this powerful analytical technique for solving boundary-value problems governed by a special type of singular integral equations, i.e. the semi-convolutional integral equations in physical domain that take origin from problems with semi-infinite geometries/supports [1]. This methodology can be applied to any branch of engineering as well as to many other fields where applied mathematics and mathematical physics are involved, demonstrating its efficacy, powerfulness and versatility as mathematical tool in countless applications. The first major monograph devoted entirely to WH theory was proposed by Noble in 1958 [2] with a systematic exposition of the method that uses spectral domain with Laplace transform, the concept of the factorization method and the contour integral analysis for the solution of classical and modified WH equations. The book recognizes the importance of Jones's method proposed in the early 50s [3-6] for direct application to partial differential equations formulations. Other fundamental and methodological monographies on WH method are the ones by Gohberg [7-8] and Weinstein [9] which report the background of the method with operator theory, function-theoretic properties and the physical application in spectral domain with the factorization procedure. We note that this report of treatises does not pretend to be omni-comprehensive but a starting point with a fundamental theoretical background, fundamental mathematics and general applications that can be extended starting from the references of [10] and expanded with the more recent monograph [11] by Daniele and Zich(2014). In the field of electromagnetic applications, we also acknowledge the books by Mittra and Lee [12] in 1971, by Hashimoto, Idemen, and Tretyakov

[13] in 1993 with contributions of Kobayashi, Buyukaksoy, Serbest and Luneburg beyond the already cited [6],[9],[11]. In 1993 [13] and in 2013 [14] Kobayashi also classified the modified WH geometries proposing dedicated and effective solution procedures.

Since 1978, Daniele expanded the capability of WH method by introducing WH explicit/exact factorization techniques for particular classes of matrices of arbitrary order [15-16]. In [11] he presented the state of the art on analytical procedures for the solution of matrix and modified WH equations. In 2001 Daniele developed the theory for the analysis of structure of angular shapes (wedges) with the definition of Generalized Wiener-Hopf equations (GWHEs) [17] and a technique to reduce them to Classical Wiener-Hopf equations (CWHEs) by a spectral transformation.

Inspired by Vekua [18], the Fredholm factorization method was introduced to reduce the factorization problem to the solution of a Fredholm integral equation of second kind (FIE). It is an approximate, versatile and effective factorization and solution method of matrix WH equations introduced in early 2000s. It was described by Daniele and Lombardi in scientific papers of 2005 and 2007 [19-20] and applied in practical scattering wedge problems in the framework of GWHEs starting from [21] in 2006. The two volume monographs [22-23] by Daniele and Lombardi in 2020 report all recent and past findings on the application of the Generalized Wiener-Hopf technique (GWHT) to impenetrable and penetrable wedge scattering problems with the most complete analytical and approximate factorization techniques and with practical engineering and physical insights. The introduction of Fredholm factorization has been an important progress in approximate factorization of WH equations and their solutions, that supersedes the ability of other not-directly-related-to-physics techniques, for example the one relying on Pade' approximation [24]. In fact, Fredholm factorization preserves the spectral physical properties of the problem like exact analytical spectral solutions when available. Moreover, Fredholm factorization method does not require specialized-in-problem techniques such as in classical/full-analytical WH solutions [2],[11] or such as in Jones-like methods with pre-factorization [2-6], [11]. It is a general procedure that transforms the factorization and the solution problem into explicit Fredholm integral equations of second kind whose convergence properties allow high precision spectral solutions with simple quadrature schemes [25] from which spectral inversion and asymptotic estimates provide the evaluation of physical quantities.

Spectral methods in wedge scattering problems have gathered a fundamental role to investigate the physical phenomenon of diffraction. The state of the art provides several effective and consolidated spectral methods to handle in particular the scattering by wedges where only a propagation constant is present. Among them, we identify the Sommerfeld-Malyuzhinets (SM) method [26-31], the Kontorovich-Lebedev (KL) transform method [32-34]. In presence of isotropic media these methods are capable to effectively handle wedge problems not solvable in closed form by reducing them to FIEs. For these problems, also the GWHT applies and the GWHEs can be reduced to FIEs by means of a spectral mapping that transforms them into CWHEs [17–23].

Recently, Daniele and Lombardi have extended the WH methodology, supported by the Fredholm factorization method, to more general problems with subdomains made by arbitrarily linear media where the GWHEs are defined into more than two complex planes, in particular for angular region problems, where no transformation allows the reduction to CWHEs [35-36]. New developments are also introduced to deal with semi-infinite and finite penetrable sub-domains where incomplete WH equations hold and entire spectral unknowns with exponential phase factors are defined [37-38]. Following the philosophy proposed in [11],[39], for such complex problems, both WH formulations and FIEs modeling can be interpreted through equivalent networks to avoid redundancy for a practical and systematic application of the proposed methods.

Finally, we note that the proposed semi-analytical approach for the solution of exact WH formulations allows the study of new and intricate scattering, propagation and radiation problems, preserving the robust features of analytical solutions in spectral methods, including the asymptotic estimation of fields and the physical interpretation of fields through spectral singularities.

The first part of this review is focused on the fundamentals with sections organized as follows: WH background in wave motion, fundamental definitions in spectral domain, CWHEs, exact factorization, classifications of WH equations, Fredholm factorization, new tools for solution, network formalism. The second part will be devoted to recent advances for novel applications.

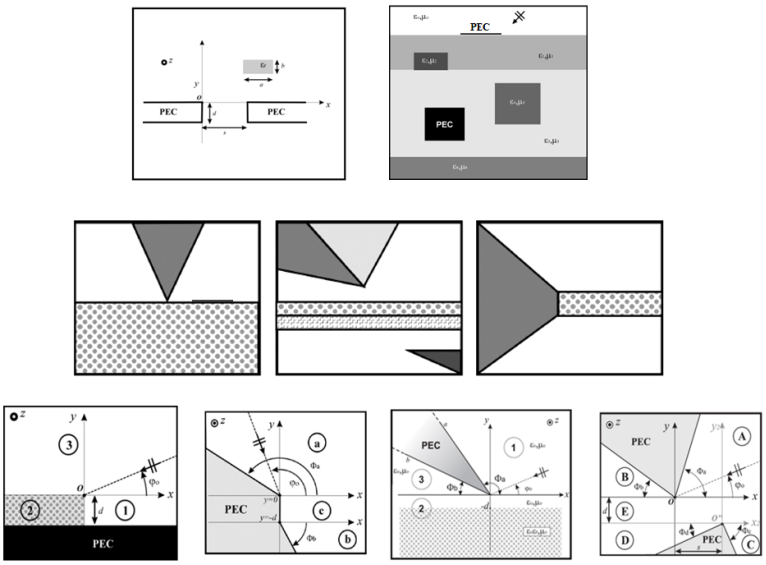


Fig. 1. Orthogonal Cartesian WH geometries and oblique Cartesian WH geometries, with infinite, semi-infinite and finite extension.

2. Wiener-Hopf Background in Wave Motion

The accurate analysis of the scattering of time-harmonic waves by objects of arbitrary shape and materials is a challenging mathematical problem, involving integral equations (IEs) and/or partial differential equations (PDEs) whose solutions are unique with the statement of boundary conditions. A fundamental approach for studying physical properties of complex problems is to decompose it into canonical sub-structures with simple geometries for which exact formulations and solutions are available. The study of solutions of canonical problems is not only useful for providing a deep description of the physical phenomena but also for their integration into more complex problems and for validating alternative computational electromagnetic (CEM) methods.

Canonical Wiener-Hopf regions are defined as geometries that allow the scattering problem to be reduced to WH formulation through equations in spectral domain. The vast literature on the WH technique, partially presented in the Introduction, shows that canonical WH geometries are, in practice, those composed of homogeneous sub-regions where separation of variables in Maxwell's equations is applicable. By denoting with u_i $(i=1,2,3)$ the coordinates that allow the separation of variables, we classify the WH canonical geometries into cartesian, cylindrical, spherical and so on according to the used coordinate systems with infinite, semi-infinite and finite extension. In this review we consider time harmonic fields in regions that are invariant with respect to one coordinate (say $u_3=z$ for cartesian and cylindrical shapes) and filled by media which are strictly passive. It produces the so called 2.5 dimensional problems where the fields are in practice functions of u_1 and u_2, times a space harmonic variation in u_3. In particular we focus the attention on isotropic media for which comparison and synergy with other spectral techniques are possible, such the popular SM method [26-31] and the KL transform method [32-34]. However, with respect to SM and KL, an extended version of the WH methodology remains applicable even in the presence of arbitrary linear media, although the involved mathematics becomes substantially more complex because of the presence of multiple propagation constants. Fig. 1 presents several structures that can be classified into two types of WH geometries: orthogonal Cartesian geometries described by the orthogonal coordinates $u_1=x$, $u_2=y(u_3=z)$ and oblique Cartesian geometries described by straight not orthogonal coordinates $u_1=u$, $u_2=v(u_3=z)$, separated by an oblique azimuthal angle γ.

By applying the Fourier/Laplace transform to the field with respect to one coordinate (say u_1), the 2.5-dimensional problems can be described by WH equations in terms of one spectral variable with u_1 removed by the transformation and u_3 removed by the invariance.

In general, the classical application of the WH technique requires three fundamental steps:

(1) the deduction of functional equations in the spectral domain of all subregions that constitute the overall geometry, followed by the enforcement of boundary conditions to obtain the WH equations,

(2) the solution of the resulting system of WH equations using exact or semi-analytical/approximate factorization techniques,
(3) the elaboration of the WH solutions for useful physical and engineering insights.
In this first part of the review, we focus the attention on the three steps reported above at a fundamental level. Furthermore, we introduce the application of WH technique to complex scattering problems that go beyond rectangular, semi-infinite, isotropic structures, i.e. those traditionally solvable in closed form through the classical WH procedure involving kernel factorization, decomposition, and the application of Liouville's theorem [2].

3. Wiener-Hopf Unknowns in Spectral Domain

The representation of physical quantities in the space-time domain can be spectrally accomplished using Laplace or Fourier transforms with bilateral, unilateral or finite support depending on the geometrical decomposition and the media involved in the problem under investigation. Beyond simplifying the evaluation of the unknown physical quantities, their spectral representations have a fundamental physical significance (see for example their role in Quantum Mechanics and in Signal theory and applications). Even if they have similar definitions, the Laplace and Fourier transforms are conceptually different. In particular, the Laplace transform yields functions of a complex variable, whereas the Fourier transform generally produces tempered distributions. Making reference to time variations, the Fourier transforms are often preferred in bilateral form for spectral signal analysis as harmonic composition, while Laplace are preferred in unilateral form for complete analysis due to discontinuities. The Fourier spectra is interpreted as a continuous composition of harmonics, while the Laplace spectra is characterized by singularities that represent the components of the physical quantities. While investigating time-harmonic electromagnetic field problem, Fourier transforms are preferable for variations in the time while Laplace transforms are considered more appropriate for variations in space in particular for the structures with discontinuities as considered in this work.
The application of spectral transformations allows the formulation of the PDE/IE problems in spectral domain in terms of functional equations. The fundamental unknowns in classical WH technique for electromagnetic problems in spectral domain are unilateral Fourier Transforms or Laplace Transforms of tangential field components defined on suitable semi-infinite interfaces that separate two different subdomains/objects, while bilateral and finite transforms can be considered extending classical WH technique. In the fundamental case of unilateral transformation, we define the plus and minus Laplace transforms (1) of a tangential field *f(u)*

$$F_+(\eta)=\int_0^{\infty} f(u)e^{j\eta u}du, \quad F_-(\eta)=\int_{-\infty}^{0} f(u)e^{j\eta u}du \qquad \text{with } \eta = js \tag{1}$$

$$f_+(u)=\frac{1}{2\pi j}\int_{B_+} F_+(\eta)e^{-j\eta u}d\eta, \qquad f_-(u)=\frac{1}{2\pi j}\int_{B_-} F_-(\eta)e^{-j\eta u}d\eta \tag{2}$$

where $u=u_1$ is the spatial coordinate of the support of the considered interface (by omitting the dependence of f(u_1) on the other spatial coordinates u_2, u_3) and η is a spectral variable defined in the complex plane (the standard definition of Laplace transform in time domain is with $\eta = js$). The integrals that defines $F_+(\eta)$ and $F_-(\eta)$ do exist and converge for values of η located respectively in an upper and lower half-plane $\mathrm{Im}[\eta]>a_+$ and $\mathrm{Im}[\eta]<a_-$. Thus, they constitute analytical elements that, by analytical continuation, define the plus and minus Laplace transforms of *f(u)* in the whole η complex plane, whose composition provides the so called bilateral Laplace transform $F(\eta)=F_+(\eta)+F_-(\eta)$. While singularities of $F_+(\eta)$ and $F_-(\eta)$ can occur in complementary half-plane, the upper half-plane $\mathrm{Im}[\eta]>a_+$ and the lower half-plane $\mathrm{Im}[\eta]<a_-$ are respectively regularity portions of the complex plane for $F_+(\eta)$ and $F_-(\eta)$. In their respective regularity half-plane, the Laplace transforms are vanishing at infinity. We call non-standard plus (minus) Laplace transform if its singularities lie in the half plane $\mathrm{Im}[\eta]>0$ ($\mathrm{Im}[\eta]<0$) yielding $a_+>0$ ($a_-<0$). Once known the plus and the minus Laplace transforms of a physical quantity *f(u), f(u)* is obtained by the inverse Laplace

transforms (2) where $f_+(u)=f(u)H(u)$ and $f_-(u)=f(u)H(-u)$ (*H(x)* is the Heaviside step function, $f(u)=f_+(u)+f_-(u)$) and, B_+ and B_- are called Bromwich contour lines respectively an horizontal line above the singularities of $F_+(\eta)$ and an horizontal line below the singularities of $F_-(\eta)$, that allow the application of Cauchy theorem to (2) with a closure of the contours. To limit the presence of singularities on the real axis of the complex plane η in wave motion problems, we assume small vanishing losses in the media, as commonly used in WH analysis. In the application of WH method, the following generalizations of Cauchy decomposition formula are useful to separate standard parts from non-standard parts of arbitrary plus and minus Laplace transforms

$$F_+(\eta)=\frac{1}{2\pi j}\int_{\gamma_1}\frac{F_+(\eta')}{\eta'-\eta}d\eta'+F_+^{n.s.}(\eta)\quad \eta\in\mathbb{R},\quad F_-(\eta)=-\frac{1}{2\pi j}\int_{\gamma_2}\frac{F_-(\eta')}{\eta'-\eta}d\eta'+F_-^{n.s.}(\eta)\quad \eta\in\mathbb{R}$$
$$F_+^{n.s.}(\eta)=-\frac{1}{2\pi j}\int_{\gamma_2}\frac{F_+(\eta')}{\eta'-\eta}d\eta'\quad \eta\in\mathbb{R},\quad F_-^{n.s.}(\eta)=\frac{1}{2\pi j}\int_{\gamma_1}\frac{F_-(\eta')}{\eta'-\eta}d\eta'\quad \eta\in\mathbb{R} \tag{3}$$

where the integration lines are the “smile” contour γ_1 and the “frown” contour γ_2 shown in Fig. 2. In case of plus (minus) functions the standard part $F_+^{s.}(\eta)$ ($F_-^{s.}(\eta)$) is analytical for $\mathrm{Im}[\eta]\geq 0$ ($\mathrm{Im}[\eta]\leq 0$) while the non-standard part $F_+^{n.s.}(\eta)$ ($F_-^{n.s.}(\eta)$) present singularities for $\mathrm{Im}[\eta]>0$ ($\mathrm{Im}[\eta]<0$) but it is regular for $\mathrm{Im}[\eta]>a_+$ ($\mathrm{Im}[\eta]<a_-$), with $a_+>0$ ($a_-<0$). The representations (3) maintain validity outside of the real axis by analytical continuation.

The plus and minus Laplace transforms are called in the WH method plus and minus WH unknowns. Changing the sign in the argument η, the plus WH unknowns become minus WH unknowns and vice versa.

When the support of the considered physical interface is a segment, the Laplace transform is defined in a finite interval (for instance *0-s*):

$$F_o(\eta)=\int_0^s f(u)e^{j\eta u}du \tag{4}$$

where $F_o(\eta)$ and $F_o(-\eta)$ are entire functions respectively of plus and minus type because, at infinity, they exponentially diverge (vanish) respectively in the half-plane $\mathrm{Im}[\eta]<0$ ($\mathrm{Im}[\eta]>0$) and $\mathrm{Im}[\eta]>0$ ($\mathrm{Im}[\eta]<0$). For finite Laplace transform, the Cauchy formula (3) hold but do not present non-standard parts.

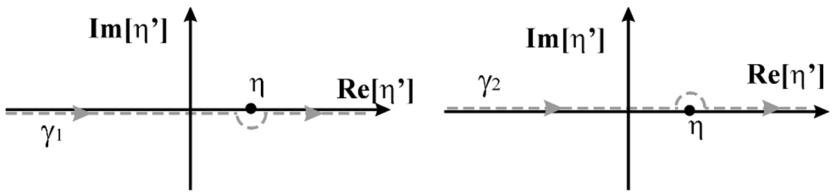


Fig. 2. The “smile” contour γ_1 and the “frown” γ_2 integration lines.

We recall that the standard plus and minus parts of a complex function $F(\eta)=F_+(\eta)+F_-(\eta)$ vanishing at infinity can be obtained using standard Cauchy decomposition formula (5), a special case of Plemelj’s formulae, whose validity can be extended through analytic continuation.

$$F_+(\eta)=\frac{1}{2\pi j}\int_{\gamma_1}\frac{F(\eta')}{\eta'-\eta}d\eta'=\frac{1}{2}F(\eta)+\frac{1}{2\pi j}P.V.\int_{-\infty}^{\infty}\frac{F(\eta')}{\eta'-\eta}d\eta',\quad \mathrm{Im}[\eta]=0$$
$$F_-(\eta)=-\frac{1}{2\pi j}\int_{\gamma_2}\frac{F(\eta')}{\eta'-\eta}d\eta'=\frac{1}{2}F(\eta)-\frac{1}{2\pi j}P.V.\int_{-\infty}^{\infty}\frac{F(\eta')}{\eta'-\eta}d\eta',\quad \mathrm{Im}[\eta]=0 \tag{5}$$

4. Classical Wiener-Hopf Equation and Solution Procedure

In general, the WH formulations in spectral domain yield a system of *n* equations with *n* plus WH unknowns and *n* minus WH unknowns. The normal form of the WH equations can be written as:

$$G(\eta)F_+(\eta)=F_-(\eta)+F_s(\eta) \tag{6}$$

where: 1) $G(\eta)$ is a known matrix of order/dimension *n* depending on the properties of the problem, i.e. in electromagnetics depending on the geometries of the scattering objects and their media,
2) the bilateral Fourier/Laplace transform $F_s(\eta)$ is a known (vector) function depending on the forcing terms, i.e. in electromagnetics depending on the physical exciting sources, like plane waves or concentrated sources,
3) $F_-(\eta)$ and $F_+(\eta)$ are the WH (vector) unknowns having complementary regularity half-planes and defined in semi-infinite physical support as reported in the previous section, i.e. in electromagnetics they are physical quantities defined at truncated interfaces
4) the normal form means that the minus (vector) unknown appears isolated.
According to the above properties we classify (6) as *classical* WH equations of order/dimension *n* in *normal form* and it is called *homogeneous* for vanishing source term $F_s(\eta)$, as for example in EM problems with plane waves illumination (12). The singularity present in $G(\eta)$ and $[G(\eta)]^{-1}$ are called *structural* as they are related to the problem geometries and media and not to the sources. In electromagnetics the role of the structural singularities is important for their physical meaning as they are related to field contributions generated by the structures of the problems like scattered waveguide modes, surface waves, leaky waves, lateral waves and so on, while the other singularities in the WH formulation are directly related to the sources as incident and related fields. The initial assumption of lossy media enforces that no singularity (pole, branch point) is present on the real axis of the complex plane η.

Even though the formulation involves two unknowns (plus and minus of the same dimension), the *classical* WH equations constitute a mathematically well-posed, i.e. *closed*, problem [8]. In contrast, we consider *unclosed* WH equations to be those in which the number of plus and minus WH unknowns are unbalanced. The classical solution procedure for classical WH equations is based on kernel factorization, decomposition, and the application of Liouville's theorem [2]. The factorization of the kernel is the multiplicative decomposition

$$G(\eta)=G_-(\eta)G_+(\eta) \tag{7}$$

where the factorized matrices $G_-(\eta)$, $G_+(\eta)$ and their inverses are regular in the standard half-planes $\mathrm{Im}[\eta]\le 0$ and $\mathrm{Im}[\eta]\ge 0$ and $\|G_{-,+}(\eta)\|/\eta\to 0,\ \left\|\left[G_{-,+}(\eta)\right]^{-1}\right\|/\eta\to 0$ as $\eta\to\infty$. From (6) and (7) we have

$$G_+(\eta)F_+(\eta)=G_-^{-1}(\eta)F_-(\eta)+G_-^{-1}(\eta)F_s(\eta) \tag{8}$$

With standard $F_+(\eta)$ and $F_-(\eta)$ in (8), as for problems with sources located in finite regions, $G_-^{-1}(\eta)F_s(\eta)$ can be decomposed in standard plus and minus components resorting to (5)

$$\left[G_-^{-1}(\eta)F_s(\eta)\right]_+=\frac{1}{2\pi j}\int_{\gamma_1}\frac{G_-^{-1}(\eta')F_s(\eta')}{\eta'-\eta}d\eta',\quad \left[G_-^{-1}(\eta)F_s(\eta)\right]_-=-\frac{1}{2\pi j}\int_{\gamma_2}\frac{G_-^{-1}(\eta')F_s(\eta')}{\eta'-\eta}d\eta',\quad \eta\in\mathbb{R} \tag{9}$$

whose validity can be extended through analytic continuation. It yields:

$$G_+(\eta)F_+(\eta)-\left[G_-^{-1}(\eta)F_s(\eta)\right]_+=G_-^{-1}(\eta)F_-(\eta)+\left[G_-^{-1}(\eta)F_s(\eta)\right]_-=w(\eta) \tag{10}$$

We observe that $w(\eta)$ is an entire function because it is regular both in $\mathrm{Im}[\eta]\geq 0$ (first member) and in $\mathrm{Im}[\eta]\leq 0$ (second member). Since it is vanishing at infinity according to Liouville's theorem is vanishing in the whole complex plane *η,* thus the solution of the WH equation (6) is

$$F_+(\eta)=\left[G_+(\eta)\right]^{-1}\left[G_-^{-1}(\eta)F_s(\eta)\right]_+,\quad F_-(\eta)=-G_-(\eta)\left[G_-^{-1}(\eta)F_s(\eta)\right]_- \tag{11}$$

Alternatively, if the problem under investigation presents non-standard $F_+(\eta)$ and $F_-(\eta)$, as for sources located at infinity (representable through plane waves), the above procedure requires a slight modification because $F_s(\eta)=0$. In fact, the plus and the minus Laplace transforms of a plane wave $f_s(u)=e^{-j\eta_o u}$ in lossy media are respectively:

$$F_{s+}(\eta)=\int_0^{\infty}e^{-j\eta_o u}e^{j\eta u}du=\frac{j}{\eta-\eta_o}\quad,\quad F_{s-}(\eta)=\int_{-\infty}^{0}e^{-j\eta_o u}e^{j\eta u}du=-\frac{j}{\eta-\eta_o} \tag{12}$$

that result non-standard correspondingly for $\mathrm{Im}[\eta_o]>0, \mathrm{Im}[\eta_o]<0$. We observe that the composition of the analytical continuation of the two integrals (12) yields a bilateral Laplace transform that vanishes $F_s(\eta)=F_{s-}(\eta)+F_{s+}(\eta)=0$, thus (6) becomes homogenous, i.e. $G(\eta)F_+(\eta)=F_-(\eta)$, although non-standard components are present in $F_+(\eta)$ and/or $F_-(\eta)$ which are related to plane wave components interpretable by Geometrical Optics (GO). This property might be considered weird and a question arises: *where is the plane wave source if the spectra is null?* To rebuild the plane wave, we need to split the inverse bilateral Laplace transform into two parts using unilateral inversion formula (2) in $F_s(\eta)=F_{s-}(\eta)+F_{s+}(\eta)$

$$f_{s+}(u)=\frac{1}{2\pi j}\int_{B_+}F_{s+}(\eta)e^{-j\eta u}d\eta=e^{-j\eta_o u}H(u),\qquad f_{s-}(u)=\frac{1}{2\pi j}\int_{B_-}F_{s-}(\eta)e^{-j\eta u}d\eta=e^{-j\eta_o u}H(-u) \tag{13}$$

yielding $f_s(u)=f_{s+}(u)+f_{s-}(u)=e^{-j\eta_o u}$. The major benefit of applying the Laplace transform is to avoid the use of tempered distributions (in contrast to the Fourier transform). For example, the bilateral Fourier transform of a plane wave $f_s(u)=e^{-j\eta_o u}$ is the distribution $2\pi\delta(\eta-\eta_o)$.
After the factorization of the kernel in homogenous WH equations, we have

$$G_+(\eta)F_+(\eta)=G_-^{-1}(\eta)F_-(\eta)_-=w_o(\eta) \tag{14}$$

where offending (non-standard) singularities like poles are present in $F_+(\eta)$ and/or $F_-(\eta)$ yielding poles in $w_o(\eta)$. This prevents an immediate application of Liouville's theorem. For the sake of simplicity, considering $F_+(\eta)$ and $F_-(\eta)$ respectively with an offending pole in $-\beta_o$ ($\mathrm{Im}[\beta_o]<0$) and $+\alpha_o$ ($\mathrm{Im}[\alpha_o]<0$), we modify (14) into (15) where *T* and *R* are residues of the non-standard parts of $F_+(\eta)$ and $F_-(\eta)$, also obtainable one from each other from $G(\eta)F_+(\eta)=F_-(\eta)$.

$$G_+(\eta)F_+(\eta)-G_+(-\beta_o)\frac{T}{\eta+\beta_o}-\left[G_-(\alpha_o)\right]^{-1}\frac{R}{\eta-\alpha_o}=G_-^{-1}(\eta)F_-(\eta)-G_+(-\beta_o)\frac{T}{\eta+\beta_o}-\left[G_-(\alpha_o)\right]^{-1}\frac{R}{\eta-\alpha_o}=w(\eta) \tag{15}$$

In (15) the $w(\eta)$ is an entire function, regular in the entire complex plane, that allows application of Liouville's theorem, proving the solution:

$$F_+(\eta)=\left[G_+(\eta)\right]^{-1}G_+(-\beta_o)\frac{T}{\eta+\beta_o}+\left[G_+(\eta)\right]^{-1}\left[G_-(\alpha_o)\right]^{-1}\frac{R}{\eta-\alpha_o}=\left[G(\eta)\right]^{-1}F_-(\eta)$$
$$F_-(\eta)=G_-(\eta)G_+(-\beta_o)\frac{T}{\eta+\beta_o}+G_-(\eta)\left[G_-(\alpha_o)\right]^{-1}\frac{R}{\eta-\alpha_o}=\left[G(\eta)\right]F_+(\eta) \quad (16)$$

We remark that the residues *T* and *R* of non-standard part of $F_+(\eta)$ and $F_-(\eta)$ are known, being the residues of their non-standard GO components: $T=\mathrm{Res}[F_+^{GO}(\eta)]_{-\beta_o}$, $R=\mathrm{Res}[F_-^{GO}(\eta)]_{\alpha_o}$.

5. State of the Art on the Exact Factorization of Matrices

Scalar kernels can generally be factorized explicitly by using logarithmic decomposition [2],[11]:

$$G(\alpha)=e^{\log[G(\alpha)]}=e^{\psi(\alpha)},\quad \log[G(\alpha)]=\psi_-(\alpha)+\psi_+(\alpha)$$
$$G_-(\alpha)=\exp\left[\psi_-(\alpha)\right],\quad G_+(\alpha)=\exp\left[\psi_+(\alpha)\right] \quad (17)$$

The above factorization formula applies also to matrix kernel provided that the matrices $\psi_-(\alpha),\psi_+(\alpha)$ commute [11],[15],[16], but this property occurs only in special cases. Explicit matrix factorizations are available in the literature for triangular matrices, rational matrices, and matrices that commute with rational matrices (a notable special case is the Daniele-Khrapkov factorization [11],[15],[16]). Although important in diffraction problems, the class of matrices that commute with rational matrices is limited. Moreover, determining whether a given matrix does commute with a polynomial matrix is often nontrivial. Exact factorization in nontrivial problems typically leads to transcendental multivalued functions defined through integrals that are very difficult to be analyzed and estimated. Up to now, literature has not provided a general technique to construct the exact factorization of arbitrary matrices. This lack of methodology has historically discouraged many scholars from the study and the application of the WH technique. Nowadays, to overcome this limitation, the authors of this review have introduced an approximate/semi-analytical but general methodology that reduces the factorization problem to the solution of a FIE (Fredholm factorization) [19-20]. Moreover, often it is preferable to use the Fredholm factorization method described in the following sections, even when an exact factorization is, in principle, available.

6. Wiener-Hopf Equations Classification

In this section, we classify the WH equations, going beyond the *classical* WH equations presented in the previous section. In general, we consider *closed* WH equations the ones with same number of plus and minus unknowns among which the classical WH equations are considered. Other closed WH equations are the one containing also plus and minus entire/finite unknowns but again with same number. While dealing with rectangular shapes, the WH equations can always be expressed in terms of a single complex/spectral variable *η* for isotropic regions, or more in general for problems involving a single propagation constant. *Unclosed* WH equations present the plus and minus unknowns in different numbers and also entire/finite Laplace unknowns can be included in the formulation. Among the unclosed WH equations we have the *longitudinally modified* WH equations [11] and the *transversally modified* WH equations [11] that take origin from problems with the occurrence of finite/segmented geometries or changes in the boundary conditions, respectively along the direction and in the perpendicular direction of the spectral transformation.
An example of the first type is the finite strip problem while of the second type is the step problem. Below the two types of equations are reported in homogenous form (hiding source terms)

$$G(\eta)F_o(\eta)+e^{j\eta s}F_+(\eta)=F_-(\eta) \quad (18)$$
$$G(\eta)F_+(\eta)+H(\eta)F_+(-\eta)=F_-(\eta) \quad (19)$$

where $G(\eta),H(\eta)\,(s>0)$ are known kernel terms and $F_+(\eta),F_-(\eta),F_o(\eta)$ are WH unknowns respectively of plus, minus and entire plus (finite) types. Both (18) and (19) present three WH

unknowns but of different type (respectively $F_+(\eta), F_-(\eta), F_o(\eta)$ and $F_+(\eta), F_+(-\eta), F_-(\eta)$). Although (18) and (19) are unclosed WH equations not written in normal form, mathematical manipulations allow to rewrite them in terms of matrix closed WH equations in normal form.
In presence of rectangular shapes, we also define the *incomplete* WH equations (first raw of Fig.1):

$$G(\eta)F_+(\eta) = F_-(\eta) + D(\eta) \otimes f(\eta) \tag{20}$$

This type of equations involve plus and minus WH unknowns $F_\pm(\eta)$, a WH type kernel $G(\eta)$ and at least one additional non-WH unknown *f(η)* which appears under a linear operator $D(\eta)\otimes$ of algebraic or integral form along a physical coordinate.
Among the non-rectangular shapes filled/immersed by/in isotropic media, we present angular structures that yield *Generalized* WH equations (GWHEs) [17], [19], [21-23], [40]. GWHEs are characterized by the plus and minus WH unknowns that are defined into different complex planes but related together. In this case we use oblique Cartesian coordinates related to cylindrical coordinate system with axial symmetry z, and we define the plus radial Laplace transforms of the tangential field components at a specific azimuthal angle *φ* via

$$F_+(\eta) = \int_0^\infty f(\rho) e^{j\eta\rho} d\rho \tag{21}$$

The GWHEs present the following homogenous form

$$G(\eta)F_+(\eta) = X_+(-m(\eta,\gamma)) \tag{22}$$

where $F_+(\eta)$ are plus Laplace spectral tangential field components along a specific direction (say *φ=0*) defined in the complex plane *η*, while $X_+(-m(\eta,\gamma))$ are plus Laplace spectral tangential field components along another specific direction (say *φ=γ*) which is defined in the *-m(η,γ)* complex plane, i.e. it is minus functions in *m*. In (22) $m = m(\eta,\gamma) = -\eta\cos\gamma + \xi(\eta)\sin\gamma$ is a known conformal transformation specified by the propagation properties in the medium (the propagation constant *k*) and by the aperture angle *γ* of the region with $\xi(\eta) = \sqrt{k^2 - \eta^2}$.

7. Approximate/Semi-Analytical Factorization: The Fredholm Factorization of WH Equations

Inspired by Russian books devoted to the solution of the Hilbert problem [18], the Fredholm factorization method was introduced to reduce the factorization problem to the solution of a FIE, see Section 1 and [19-20]. Given the effective solutions achieved in numerous WH problems over the past twenty-five years, these authors claim that the Fredholm factorization method provides an efficient approximate/semi-analytical methodology for essentially all WH equations preserving the spectral physical properties like analytical spectral solutions when available: the physical properties are maintained in the analytical expressions of the WH kernel inside the kernel of the FIE and in the singularities of the sources.
In this section, first, we present the Fredholm factorization applied to *classical* WH equations (6), that can be immediately extended to other classes of equations, as for example to the modified equations because they can be rewritten in terms of matrix closed (classical) WH equations in normal form (see Section 6). The procedure starts from the application of the additive decomposition formula of an arbitrary analytic function into plus and minus components (5) and the generalized Cauchy decomposition formula (3). The Fredholm factorization directly applies to the WH equations and it eliminates the minus unknown providing integral equations (our preference is to keep the plus unknowns, but a very similar procedure can be applied to keep minus ones). Let us consider the homogeneous WH equation in normal form ($G(\eta)F_+(\eta) = F_-(\eta)$), through (3) we get

$$\frac{1}{2\pi j}\int_{\gamma_1}\frac{G(\eta')F_+(\eta')}{\eta'-\eta}d\eta'=\frac{1}{2\pi j}\int_{\gamma_1}\frac{(G(\eta')-G(\eta))F_+(\eta')}{\eta'-\eta}d\eta'+\frac{1}{2\pi j}G(\eta)\int_{\gamma_1}\frac{F_+(\eta')}{\eta'-\eta}d\eta'= \tag{23}$$
$$\frac{1}{2\pi j}\int_{\gamma_1}\frac{(G(\eta')-G(\eta))F_+(\eta')}{\eta'-\eta}d\eta'+G(\eta)F_+(\eta)-G(\eta)F_+^{ns}(\eta)=\frac{1}{2\pi j}\int_{\gamma_1}\frac{F_-(\eta')}{\eta'-\eta}d\eta'=F_-^{ns}(\eta)$$

that yields considering the absence of singularity for $\eta'=\eta$

$$G(\eta)F_+(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{(G(\eta')-G(\eta))F_+(\eta')}{\eta'-\eta}d\eta'=F_-^{ns}(\eta)+G(\eta)F_+^{ns}(\eta),\quad \eta\in\mathbb{R} \tag{24}$$

Eq. (24) is a FIE in terms of plus unknowns and with compact kernel by construction [18],[20], where the minus unknowns are eliminated. The source in (24) is constituted by non-standard parts of $F_+(\eta)$ and $F_-(\eta)$ that in case of EM problems with plane wave excitation are non standard GO components. According to our experience, the FIEs (24) are with fast convergent kernel, thus they are amenable of semi-analytical solution with sample and hold discretization and reconstruction formula (25) in a limited interval where A and h are respectively the truncation and the step parameters such that A/h is an integer number. The Fredholm factorization with this discretization process yields a semi-analytical solution (25) that preserves exactly the spectral properties of the physical problems in terms of non-standard pole contributions of the sources and the singularities contained exactly in the analytic WH kernel $G(\eta)$.

$$F_+(\eta)=-\frac{[G(\eta)]^{-1}}{2\pi j}h\sum_{i=-A/h}^{A/h}\frac{(G(hi)-G(\eta))F_+(hi)}{\eta'-\eta}+[G(\eta)]^{-1}F_-^{ns}(\eta)+F_+^{ns}(\eta),\quad \eta\in\mathbb{R} \tag{25}$$

Starting from the estimation of $F_+(\eta)$ on the real axis and yielding $F_-(\eta)=G(\eta)F_+(\eta)$, analytical continuation allows their evaluation in the entire complex plane η, enabling effective asymptotic estimation of quantities in natural domain as for closed form analytical spectral solutions.
The above Fredholm factorization method can be straightforward applied to classical equations in non-normal form, for instance let us consider (26) where we apply the “smile” Cauchy decomposition formula (3).

$$A(\eta)F_{1+}(\eta)+F_-(\eta)+B(\eta)\exp(j\eta s)F_{2+}(-\eta)=0 \tag{26}$$

From the first two terms we get (27) and from the third term (28) where we have replaced η' with -η' under the integral sign, maintaining only plus unknowns. The resultant equation is a FIE.

$$A(\eta)F_{1+}(\eta)+\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{A(\eta')-A(\eta)}{\eta'-\eta}F_{1+}(\eta')d\eta'+A(\eta)F_{1+}^{n.s.}(\eta)+F_-^{n.s.}(\eta),\quad \eta\in\mathbb{R} \tag{27}$$
$$\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{-B(-\eta')\exp(-j\eta' s)+B(\eta)\exp(j\eta s)}{\eta'+\eta}F_{2+}(\eta')d\eta'+B(\eta)\exp(j\eta s))F_{2+}^{n.s.}(-\eta),\ \eta\in\mathbb{R} \tag{28}$$

We will discuss in the next section and in part 2 the application of the Fredholm factorization to other classes of WH equations.

8. Tools for Managing non-classical Wiener-Hopf Equations

Classical WH literature reports specialized-in-problem techniques to get full-analytical closed form WH solutions [2],[11] or to get approximate solutions as in Jones like method with specialized pre-factorization [2-6], [11]. These categories of methods strongly limit the set of solvable problems.
Alternatively, an extended version of Fredholm factorization method can be a general tool to get effective high-precision approximate factorization and/or solution for any class of WH equations. From the classification given in Section 6, we first consider the unclosed *longitudinally modified*

WH equation (18), where we introduce $F_-(\eta)=F_{\pi+}(-\eta)$ and we have additional equations to make the system closed: we add one equation replacing η with $-\eta$ and multiplying by $e^{j\eta s}$, plus two equations derived from (18) and the previous one where we define the entire/finite plus function $F_{\pi o}(\eta)=e^{j\eta s}F_o(-\eta)$. We get the following system of closed WH equations (29) with plus and minus functions $F_o(\eta),F_{\pi o}(\eta),F_+(\eta),F_{\pi+}(\eta)$ and $F_o(-\eta),F_{\pi o}(-\eta),F_+(-\eta),F_{\pi+}(-\eta)$.

$$\begin{aligned}
&G(\eta)F_o(\eta)+e^{j\eta s}F_+(\eta)=F_{\pi+}(-\eta)\\
&e^{j\eta s}G(-\eta)F_o(-\eta)+F_+(-\eta)=e^{j\eta s}F_{\pi+}(\eta)\\
&e^{j\eta s}G(\eta)F_{\pi o}(-\eta)+e^{j\eta s}F_+(\eta)=F_{\pi+}(-\eta)\\
&G(-\eta)F_{\pi o}(\eta)+F_+(-\eta)=e^{j\eta s}F_{\pi+}(\eta)
\end{aligned} \tag{29}$$

Although the system (29) now can be written in normal form, even in simple cases, no exact factorization of the matrix kernel is available in literature. In [4] Jones proposed an efficient ad hoc methodology to (18) that is based on specialized indispensable pre-factorization of kernel and transformation of the equation into integral equation representation, see [11, sect.6.1.2], [2, sect.5.5]. On the contrary, the proposed approximate methodology, Fredholm factorization, is directly implementable to (29). For what concerns the unclosed *transversally modified* WH equation (19) we obtain a closed system of WH equations by adding one equation replacing η with $-\eta$

$$\begin{aligned}
&G(\eta)F_+(\eta)+H(\eta)F_+(-\eta)=F_-(\eta)\\
&G(-\eta)F_+(-\eta)+H(-\eta)F_+(\eta)=F_-(-\eta)
\end{aligned} \tag{30}$$

that can be easily normalized into

$$\begin{aligned}
&\bar{G}(\eta)\bar{F}_+(\eta)=\bar{F}_-(\eta)\\
&\bar{G}(\eta)=\frac{1}{G(-\eta)}\begin{vmatrix} H(-\eta) & 1\\ G(\eta)G(-\eta)-H(\eta)H(-\eta) & -H(\eta)\end{vmatrix},\quad \bar{F}_+(\eta)=\begin{vmatrix}F_+(\eta)\\ F_-(-\eta)\end{vmatrix},\quad \bar{F}_-(\eta)=\begin{vmatrix}F_+(-\eta)\\ F_-(\eta)\end{vmatrix}
\end{aligned} \tag{31}$$

Again, also in simple cases, no exact matrix factorization of the kernel is available, however a slight modification of the previously cited Jones's ad hoc method is still available and very accurate [11, sect.6.1.3], [2, sect.5.3]. Alternatively, Fredholm factorization is always directly applicable to this set of equations (30) without requesting specialized ad-hoc pre-factorization.
The classical methodology to handle incomplete WH equations as (20) is to resort to Mittag-Leffler representation of $D(\eta)\otimes f(\eta)$ in terms of an infinite number of samples of the WH unknowns using analytic properties of the equations. Solutions can be obtained by applying specialized Jones's method and imposing a system of infinite equations in terms of the infinite samples (to be truncated), see for instance [41]. Alternatively, a modified Fredholm factorization method is again of great help for the general solution of incomplete WH equations. In this case we resort to Cauchy representation (3) of the samples, i.e. (32), of the unknowns after Mittag-Leffler representation, that allows a direct formulation of the solution procedure in terms of a FIE whose integral kernel is augmented because of the application of the Cauchy formula (see for instance [37]).

$$F_+(-\eta_n)=\frac{1}{2\pi j}\int_{-\infty}^{+\infty}\frac{F_+(\eta')}{\eta'+\eta_n}d\eta'+F_+^{n.s.}(-\eta_n)\quad \mathrm{Im}[\eta_n]<0,\ F_-(\eta_n)=-\frac{1}{2\pi j}\int_{-\infty}^{+\infty}\frac{F_+(\eta')}{\eta'-\eta_n}d\eta'+F_-^{n.s.}(\eta_n)\quad \mathrm{Im}[\eta_n]<0 \tag{32}$$

The systems of GWHEs (22) arising in angular regions filled/immersed by/in isotropic media can be transformed into CWHEs in a new complex plane α by means of a spectral mapping of the form

$$\eta(\alpha)=-k\cos\left(\frac{\gamma}{\pi}\cos\left(-\frac{\alpha}{k}\right)\right) \tag{33}$$

The mapped unknowns $\bar{F}_+(\alpha)$ and $\bar{X}_+(-\alpha)$ in the α plane corresponds to $F_+(\eta)$ and $X_{i+}(-m(\eta,\gamma))$ and are respectively plus and minus functions in this unique new complex plane α. The CWHEs take the normal closed homogenous form

$$\bar{G}(\alpha)\bar{F}_+(\alpha) = \bar{X}_+(-\alpha) \tag{34}$$

Again, except in special cases with particular boundary conditions and aperture angles, the factorization of the matrix kernel in (34) is not available in closed form, while Fredholm factorization method can be always applied with no restrictions in α domain [21-23,40].
However, in complex problems, where planar and angular regions are connected together (Fig. 1), spectral mapping makes the analytical continuation cumbersome or not possible [39,41-43]. Alternatively, to get the physical fields, we can resort to Kirchhoff representations in spectral domain that require a duplication of GWHEs where the roles of generalized minus and plus unknowns are interchanged [36], interchanging in the mapped plane α with -α if (33) is applied.
While considering arbitrary linear media problems with multiple propagation constants, GWHEs present WH unknowns defined in more than two complex planes, although related together. This property prevents the possibility to define a mapping to transform GWHEs into CWHEs. In this context, recently we have introduced the Cauchy representations to obtain integral equation representations in a unique complex plane [36], amenable of direct Fredholm factorization. The procedure requires the duplication of the GWHEs that can be obtained resorting to the general methodology to deduct the GWHEs in arbitrary angular regions [35]-[36]. The spectral solution of FIEs allows computation of physical fields through Kirchhoff representations. This extension of the methodology allows one to study WH geometries of Fig. 1 in the presence of arbitrary linear media.

9. Deduction of WH Equations thought Network Representations

In the application of the WH technique, the most difficult step is to obtain the WH equations without resorting to specialized ad-hoc procedure for each problem.
In this section, we introduce systematic and rigorous treatment of electromagnetic field representations in complex structures through WH equations that have equivalent network modelling. The idea of using network modelling originates from transversalization of Maxwell's equations used for studying stratified media [44-45], where transmission line modelling and lumped equivalent models are derived in spectral domain. We extend the same philosophy in the context of WH method to more general structures as the ones listed in this review and shown in Fig. 1.
WH problems can be partitioned into canonical subregions for which, starting from fundamental differential equations (Maxwell's equations in electromagnetics) and applying spectral domain, we obtain spectral functional equations in terms of WH unknowns. The imposition of boundary conditions yields the system of WH equations of the problem. Functional equations, WH equations and FIEs obtained via Fredholm factorization are all interpretable for each subdomain as a sub-network. This property allows the systematic composition of complex problems whose subdomains are of canonical shapes (Fig. 1) described through orthogonal and/or oblique Cartesian coordinates with infinite, semi-infinite and finite extension. The connection of the sub-networks of the subdomains creates the entire network that provides the mathematical formulation of the overall problem. Network analysis yields the system of WH equations and related FIEs in terms of WH unknowns, i.e. spectral physical quantities. We emphasize the fact that the use of domain decomposition and sub-networks make the procedure systematic and general avoiding ad hoc procedures as in classical WH literature. We here list the properties and the procedure in using network modelling:
1) each sub-region of WH canonical shape is constituted by a polygonal cross-section whose sides can be identified and modeled as ports of a multiport,
2) the voltage V and the current I on a port are related to the transforms of the two components of respectively the electrical field and the magnetic field that are tangent to the side,
3) the network equations of the sub-region are related to the voltage and the current of the sides,
4) the parameters of network equations of a sub-region do not depend on the other sub-regions and consequently are evaluated once and for all for each kind of canonical shape,

5) the interaction among different sub-regions is represented by a network of multi-ports where the current and voltage continuity on the ports enforces the boundary conditions which are present on the side in common to two adjacent sub-regions,
6) in case of impenetrable discontinuities, such as strips, the voltage and the current can be discontinuous on the two sides of the interface,
7) the WH unknowns are suitable combinations of voltage and/or currents on the ports,
8) starting from the global connected network, circuit analysis systematically provides the WH equations of the problem avoiding redundancy.
To implement the procedure, starting from the transversalization of Maxwell's equations, either for rectangular geometries or for angular geometries in a homogeneous electromagnetic medium, we get in spectral domain an ordinary vector first-order differential equation with a spectral 4x4 matrix $M(\eta)$. The eigenstructure of this matrix provides the spectral dyadic Green's functions both in Cartesian coordinates for rectangular shapes [46] and oblique Cartesian coordinates for angular shapes [47]. The characteristic Green's function procedure allows to write the solution of the ordinary differential equations in spectral domain that can be interpreted as spectral functional equations connecting the WH unknowns, i.e. the spectral transformations of tangent field components of the different faces of the sub-regions. As classified previously, WH unknowns can be interpreted as network quantities. The circuital equations of sub-regions having forms of half space, slabs, semi slab, angular regions in isotropic media are reported in [11], [22-23], [37-39], [41-43], [48-51], while extension to more complex media is investigated in [36]. In part 2 we present some of these structures, with circuit modelling either from spectral functional equations and after Fredholm factorization, together with numerical results and independent validations.

10. Conclusions of Part 1

The WH technique is a powerful and versatile tool for the rigorous investigation of boundary-value problems in computational electromagnetics with a solid background in applied mathematics and mathematical physics. Part 1 is devoted to establishing the fundamental mathematical concepts and formulation principles that are at the base of the WH method, to allow a full comprehension of the up-to-date solution methodologies with recent findings and to allow the introduction to novel applications in complex electromagnetic problems that will be proposed in part 2.

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