# An Application of Angular Network Equations: Exact Solution of the PEC Wedge in Biaxial Media

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***Abstract*— The Generalized Wiener Hopf Equations are deduced for a PEC wedge immersed in an anisotropic medium starting from angular network equations. In the case of biaxial medium the Wiener-Hopf model is reduced to Classical Wiener-Hopf Equations that can be solved resorting to exact factorization. The spectral solution allows asymptotic estimates of far-field components in the case of plane wave illumination.**

## I. Introduction

This work presents our recent studies on a novel formulation of the scattering problem constituted by angular regions immersed/made by anisotropic media [1] with a novel application: the scattering by a perfect electrically conducting (PEC) wedge in a biaxial medium illuminated by plane waves, see Fig.1. In [1] we present spectral equations that relates continuous field components defined on the faces of an angular region that can be interpreted as angular network equations, extending the network paradigm presented in [2].

The importance of these equations is the possibility to introduce a network modelling that conceptually simplifies the study of angular regions, also in presence of stratified angular regions. In this work the angular network equations are introduced to formulate the proposed problem through Generalized Wiener Hopf Equations (GWHEs) by applying constitutive relations of biaxial medium (1) and PEC boundary conditions.

$$\underline{\underline{\varepsilon}}_b = \begin{vmatrix} \varepsilon_z & 0 & 0 \\ 0 & \varepsilon_x & 0 \\ 0 & 0 & \varepsilon_y \end{vmatrix}, \underline{\underline{\mu}}_b = \begin{vmatrix} \mu_z & 0 & 0 \\ 0 & \mu_x & 0 \\ 0 & 0 & \mu_y \end{vmatrix} \qquad (1)$$

Two suitable mappings reduce the GWHEs to Classical Wiener-Hopf Equations (CWHEs) that can be exactly factorized. The solution of the GWHEs provide the spectral electromagnetic field that is amenable of asymptotic estimation.

From here on, we assume time harmonic fields $e^{j\omega t}$, plane wave sources with normal incidence and, being $\tilde{\psi}(\rho,\varphi)$ the field, the spectral electromagnetic field is defined by the Laplace Transform:

$$\psi(\eta,\varphi) = \int_0^\infty e^{j\eta\rho}\tilde{\psi}(\rho,\varphi)d\rho \qquad (2)$$

Finally, we report that the literature on scattering by wedges in presence of anisotropic media shows only few solutions for simplified problems [3] with respect to the proposed one.

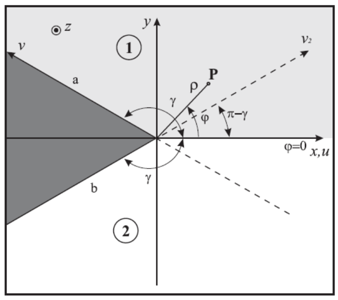


Figure 1: Geometry of the problem: regions 1 and 2 are made by the same biaxial medium and the dark gray region is made by PEC.

## II. Spectral Formulation

### A. Angular Network Equations

The considered problem is constituted of an angular stratification of three angular regions, see Fig.1: the PEC wedge, the angular region 1 (0<φ<γ) and the angular region 2 (−γ<φ<0).

Extending Bresler and Marcuvitz's transversalization method for rectangular/layered regions in arbitrary linear media [4] to angular regions considering oblique Cartesian coordinates *(u,v,z)* we obtain from Maxwell's equations a matrix differential equation in terms of $\psi(\eta,\varphi)$ that can be solved using an extended version of Green's function procedure [1]. Projecting the solution into reciprocal eigenvectors of the matrix differential operator and evaluating the resultant equations at φ=0 we obtain a set of functional equations [1] that relates Laplace transforms of field components defined at the interfaces of the angular region φ=0,γ,−γ. As a result, in regions 1 and 2, we have

$$\begin{aligned} &\frac{\mathrm{H}_{\mathrm{xo}}(\eta)}{2} - \frac{\mathrm{E}_{zo}(\eta)\xi_1}{2\omega\sqrt{\mu_x\mu_y}} = \frac{\mathrm{H}_{\rho\mathrm{a}}(-m_{a1})}{2} + \frac{1}{2}\left(\frac{\xi_1\cos(\gamma)}{\omega\sqrt{\mu_x\mu_y}} - \frac{\eta\sin(\gamma)}{\omega\mu_y}\right)\mathrm{E}_{\mathrm{za}}(-m_{a1}) \\ &\frac{\mathrm{H}_{\mathrm{zo}}(\eta)}{2} + \frac{\omega\sqrt{\varepsilon_x\varepsilon_y}\,\mathrm{E}_{\mathrm{xo}}(\eta)}{2\xi_2} = \left(\frac{\cos(\gamma)}{2} + \frac{\eta\sin(\gamma)}{2\omega\xi_2}\sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\right)\mathrm{H}_{\mathrm{za}}(-m_{a2}) + \frac{\omega\sqrt{\varepsilon_x\varepsilon_y}\,\mathrm{E}_{\rho\mathrm{a}}(-m_{a2})}{2\xi_2} \end{aligned} \qquad (3)$$

$$\begin{aligned} &\frac{\mathrm{H}_{\mathrm{xo}}(\eta)}{2} + \frac{\mathrm{E}_{zo}(\eta)\xi_1}{2\omega\sqrt{\mu_x\mu_y}} = -\frac{\mathrm{H}_{\rho\mathrm{b}}(-m_{a1})}{2} - \frac{1}{2}\left(\frac{\xi_1\cos(\gamma)}{\omega\sqrt{\mu_x\mu_y}} - \frac{\eta\sin(\gamma)}{\omega\mu_y}\right)\mathrm{E}_{\mathrm{zb}}(-m_{a1}) \\ &\frac{\mathrm{H}_{\mathrm{zo}}(\eta)}{2} - \frac{\omega\sqrt{\varepsilon_x\varepsilon_y}\,\mathrm{E}_{\mathrm{xo}}(\eta)}{2\xi_2} = \left(\frac{\cos(\gamma)}{2} + \frac{\eta\sin(\gamma)}{2\omega\xi_2}\sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\right)\mathrm{H}_{\mathrm{zb}}(-m_{a2}) + \frac{\omega\sqrt{\varepsilon_x\varepsilon_y}\,\mathrm{E}_{\rho\mathrm{b}}(-m_{a2})}{2\xi_2} \end{aligned} \qquad (4)$$

with propagation constants $\xi_1 = \sqrt{k_1^{\,2} - \eta^2}$, $k_1 = \omega\sqrt{\mu_{\mathrm{y}}\varepsilon_z}$, $\xi_2 = \sqrt{k_2^{\,2} - \eta^2}$, $k_2 = \omega\sqrt{\mu_{\mathrm{z}}\varepsilon_y}$ (typical of biaxial medium) and where

$$m_{a1} = -\eta\cos(\gamma) + \sqrt{\frac{\mu_x}{\mu_y}}\xi_1\sin(\gamma),\ m_{a2} = -\eta\cos(\gamma) + \sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\xi_2\sin(\gamma).$$

Eqs. (3) and (4) can be interpreted as the angular network equations (two port network) that relate continuous field components in spectral domain at φ=0 (subscript o) , φ=γ (subscript a), φ=–γ (subscript b).

### B. GWHEs of the problem

By imposing the PEC boundary conditions at φ=±γ

$$E_{za} = E_{zb} = E_{\rho a} = E_{\rho b} = 0 \tag{5}$$

we get the system of GWHEs of the problem

$$\frac{H_{xo}(\eta)}{2} - \frac{E_{zo}(\eta)\xi_1}{2\omega\sqrt{\mu_x\mu_y}} = \frac{H_{\rho a}(-m_{a1})}{2} \tag{6}$$

$$\frac{H_{zo}(\eta)}{2} + \frac{\omega\sqrt{\varepsilon_x\varepsilon_y}E_{xo}(\eta)}{2\xi_2} = \left(\frac{\cos(\gamma)}{2} + \frac{\eta\sin(\gamma)}{2\omega\xi_2}\sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\right)H_{za}(-m_{a2})$$

$$\frac{H_{xo}(\eta)}{2} + \frac{E_{zo}(\eta)\xi_1}{2\omega\sqrt{\mu_x\mu_y}} = -\frac{H_{\rho b}(-m_{a1})}{2} \tag{7}$$

$$\frac{H_{zo}(\eta)}{2} - \frac{\omega\sqrt{\varepsilon_x\varepsilon_y}E_{xo}(\eta)}{2\xi_2} = \left(\frac{\cos(\gamma)}{2} + \frac{\eta\sin(\gamma)}{2\omega\xi_2}\sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\right)H_{zb}(-m_{a2})$$

Summing and subtracting the first and the second equations of (6) and (7), it yields the four uncoupled GWHEs

$$H_{xo}(\eta) = \frac{H_{\rho a}(-m_{a1}) - H_{\rho b}(-m_{a1})}{2} \tag{8}$$

$$-\frac{E_{zo}(\eta)\xi_1}{\omega\sqrt{\mu_x\mu_y}} = \frac{H_{\rho a}(-m_{a1}) + H_{\rho b}(-m_{a1})}{2}$$

$$H_{zo}(\eta) = \left(\frac{\cos(\gamma)}{2} + \frac{\eta\sin(\gamma)}{2\omega\xi_2}\sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\right)\left(H_{za}(-m_{a2}) + H_{zb}(-m_{a2})\right)$$

$$\frac{\omega\sqrt{\varepsilon_x\varepsilon_y}E_{xo}(\eta)}{\xi_2} = \left(\frac{\cos(\gamma)}{2} + \frac{\eta\sin(\gamma)}{2\omega\xi_2}\sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\right)\left(H_{za}(-m_{a2}) - H_{zb}(-m_{a2})\right)$$

### C. CWHEs of the problem

Each of the uncoupled equations reported in (8) presents the form

$$G_i(\eta)F_{i+}(\eta) = X_{i+}(-m_{ai}) \qquad \text{i=1,2} \tag{9}$$

where the unknowns $F_{i+}(\eta)$ and $X_{i+}(-m_{ai})$ are plus and minus functions respectively in the planes $\eta$ and $m_{ai}$.

Algebraic manipulations reduce $m_{ai}$ to the following form:

$$m_{ai} = -p_i\eta\cos\gamma_i + p_i\xi_i\sin\gamma_i, \quad \text{i=1,2} \tag{10}$$

with $\tan\gamma_1 = \sqrt{\frac{\mu_x}{\mu_y}}\tan\gamma, \tan\gamma_2 = \sqrt{\frac{\varepsilon_x}{\varepsilon_y}}\tan\gamma, p_i = \frac{\cos\gamma}{\cos\gamma_i}$ i=1,2.

After the introduction of the mappings

$$\eta = -k_i\cos\left[\frac{\gamma_i}{\pi}arc\cos(-\frac{\alpha_i}{k_i})\right], \quad \text{i=1,2} \tag{11}$$

we get $G_i(\eta) = \bar{G}_i(\alpha_i), F_{i+}(\eta) = \bar{F}_{i+}(\alpha_i), X_{i+}(-m_{ai}) = \bar{X}_{i-}(\alpha_i)$.

These mapped unknows $\bar{F}_{i+}(\alpha_i), \bar{X}_{i-}(\alpha_i)$ are respectively plus and minus functions in the $\alpha_i$-plane, after checking their regularity properties; consequently, the GWHEs (8) reduce to uncoupled CWHEs with simple kernels of the form

$$\bar{G}_i(\alpha_i)\bar{F}_{i+}(\alpha_i) = \bar{X}_{i-}(\alpha_i) \qquad \text{i=1,2} \tag{12}$$

## III. Solution

Eqs. (8) in the form (12) show decoupled problems for $E_z$ and $H_z$ polarization with different propagation constants (see first two equations of (8) with respect to the last two).

Illumination of the structure immersed in the biaxial medium $-\gamma < \varphi < \gamma$ by incident planar waves with direction $\varphi_i$ is of two types: 1) $E_z^i = e^{jk_1\rho\cos(\varphi-\varphi_{i1})}$ and 2) $H_z^i = e^{jk_2\rho\cos(\varphi-\varphi_{i2})}$.

Focusing the attention on type 1, we have that only $E_z, H_\rho$ components are present where

$$H_x = \frac{j}{\omega\mu_x}\frac{\partial E_z}{\partial y}, H_y = -\frac{j}{\omega\mu_y}\frac{\partial E_z}{\partial x}, H_\rho = H_x\cos\varphi + H_y\sin\varphi \tag{12}$$

The solution of the CWHEs for $E_z$ polarization is obtained as indicated in [5] with standard procedures and we get:

$$\bar{E}_{zo+}(\alpha_1) = E_{zo}(\eta) = \csc w_1 \hat{E}_{zod}(w_1) = -j\frac{\pi}{\gamma_1}\frac{\sqrt{k_1+\alpha_1}\sin(\frac{\pi}{\gamma_1}\varphi_{i1})\csc\varphi_{i1}}{\sqrt{2}\xi_1(\alpha_1)(\alpha_1-\alpha_{1o})} \tag{13}$$

$$\bar{H}_{xo+}(\alpha_1) = \hat{H}_{xo+}(w_1) = j\frac{k_1}{\omega\sqrt{\mu_x\mu_y}}\frac{\pi}{\gamma_1}\frac{\sin(\frac{\pi}{\gamma_1}\varphi_{i1})}{(\alpha_1-\alpha_{1o})},$$

with $\alpha_1 = -k_1\cos\frac{\pi}{\gamma_1}w_1, \alpha_{1o} = -k_1\cos\frac{\pi}{\gamma_1}\varphi_{i1}, \eta = -k_1\cos w_1$.

From the solution in terms of axial spectra (13), i.e. for φ=0, we obtain the spectra in any direction φ using network representation (3)-(4) and sec. 3.10 of [5] changing the angle ±γ to φ on the RHS of the equations.

Finally, asymptotics is straightforwardly applied for far field computation in terms GO, GTD components.


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