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Design, modelling and experimental characterization of complex photonic structure: from microring resonator to hybrid integrated tunable laser

By

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Declaration

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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2026

* This dissertation is presented in partial fulfillment of the requirements for **Ph.D. degree** in the Graduate School of Politecnico di Torino (ScuDo).

I would like to dedicate this thesis to my loving parents and Andrea

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Abstract

Silicon photonics has become a fundamental technology for the development of photonic integrated circuits, driven by the compatibility with CMOS, scalability, and reduced fabrication costs. Among the most relevant building blocks, microring resonators (MRRs) and hybrid integrated tunable lasers play a central role in applications such as optical communications, signal processing, and microwave photonics. However, as integration density and optical power increase, nonlinear optical, free carrier, and thermal effects strongly affect device performance, stability, and dynamical behavior. It is imperative that these effects are modelled with precision in order to guarantee reliable design and optimisation.

The present doctoral thesis addresses the modelling and physical interpretation of nonlinear phenomena in silicon-based integrated photonic devices, with particular emphasis on microring resonators with complex waveguide cross sections and hybrid integrated tunable lasers employing silicon and silicon nitride passive feedback cavities. The primary objective is to develop computationally efficient models that are also physically accurate, with the capability to capture the interplay between optical fields, carrier dynamics, and thermal effects. These models will be validated through comparison with experimental measurements.

A two-dimensional distributed modelling framework based on the finite element method has been developed within the SISCAP platform to describe nonlinear effects in MRR. The proposed model represents a refinement over conventional lumped approaches in that it resolves the spatial distribution of free carriers and temperature. This facilitates accurate analysis of devices with non-standard geometries, including rib or hybrid Si/PolySi waveguides. Numerical results demonstrate that waveguide geometry has a significant

effect on carrier diffusion and heat dissipation, thereby governing nonlinear losses and resonance stability. In the course of the present study, a range of configurations was analysed. Of these, the Si/PolySi three-wing rib geometry was found to provide improved mitigation of nonlinear and thermal effects. This is due to enhanced lateral carrier spreading and reduced carrier lifetime. The model's accuracy has been validated through quantitative congruence with experimental transmission measurements, thereby substantiating its status as a reliable and adaptable design instrument for high Q MRRs.

The thesis then investigates the nonlinear dynamics of hybrid integrated tunable lasers incorporating narrowband passive feedback based on silicon nitride resonators. The experimental and numerical results obtained have demonstrated the emergence of a self-pulsing regime, characterised by the presence of optical spectra resembling combs. A thorough examination has shown that this regime originates from the undamping of relaxation oscillations. These oscillations are driven by asymmetric detuning between the lasing frequency and the reflectivity peak of the passive cavity. Time-delayed simulations demonstrate a high degree of congruence with optical and radio frequency measurements, thereby substantiating the proposed physical interpretation. The pulsing operation is stable and self-contained, and is achieved purely through passive cavity design, without requiring saturable absorbers or external modulation.

Finally, hybrid tunable integrated laser architectures employing microring based feedback cavities are analysed, with a comparison made between silicon nitride and silicon platforms. It has been determined that Kerr nonlinearities in silicon nitride are negligible, thereby ensuring the stability of laser operation. Conversely, silicon-based cavities permit the development of more compact designs; however, this is achieved at the expense of an increased sensitivity to free carrier-induced instabilities. In conclusion, the present thesis provides validated modelling tools and clear physical understanding of nonlinear dynamics in silicon photonic devices, thus supporting the design of compact, stable, and CMOS-compatible integrated photonic sources.

Publications

Papers in journals

- S. Cucco, M. Novarese, S. R. Garcia, J. Bovington and M. Gioannini, "Design of Silicon Photonic Microring Resonators With Complex Waveguide Cross-Sections and Minimal Non-Linearity," in IEEE Photonics Journal, vol. 17, no. 4, pp. 1-10, Aug. 2025, Art no. 3000510.
- Stefania Cucco, Anzal Memon, Cristina Rimoldi, Marco Novarese, Lorenzo L. Columbo, Klaus-Jochen Boller, and Mariangela Gioannini, "Novel type of nonlinear pulsation dynamics in a hybrid III-V/SiN integrated tunable laser with a frequency-selective mirror: experiments and theory" submitted for publication in Optics Letter.

Conference proceedings

- S. Cucco, M. Novarese and M. Gioannini, "Design of Si/polySi microrings with complex waveguide cross-sections and minimal non-linearity," 2023 International Conference on Numerical Simulation of Optoelectronic Devices (NUSOD), Turin, Italy, 2023, pp. 65-66.
- S. Cucco, M. Novarese and M. Gioannini, "2D NL model and design of SISCAP microring resonator with complex waveguide cross-section," 2024 IEEE Silicon Photonics Conference (SiPhotonics), Tokyo Bay, Japan, 2024, pp. 1-2.
- Cucco, S., Novarese, M., Gioannini, M. (2024). Design of Microrings with Complex Waveguide Cross-Sections to Reduce Non-linear Effects of Silicon. In: Witzens, J., Poon, J., Zimmermann, L., Freude, W.

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- S. Cucco, M. Novarese and M. Gioannini, "Design of Microring Resonators with Various Waveguide Cross Sections to Minimize Nonlinear Effects in SISCAP Platform," 2024 Italian Conference on Optics and Photonics (ICOP), Firenze, Italy, 2024, pp. 1-4.
- C. Rimoldi et al., "Self-Generation of Optical Frequency Combs in Hybrid III-V/SiN Integrated Tunable Lasers: Theory and Experiments," 2024 IEEE 29th International Semiconductor Laser Conference (ISLC), Orlando, FL, USA, 2024, pp. 1-2.
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- S. Cucco et al., "Experimental and Theoretical Study of Frequency Combs in Hybrid Lasers with a Narrow-Band Mirror," 2025 International Conference on Numerical Simulation of Optoelectronic Devices (NUSOD), Lodz, Poland, 2025, pp. 113-114.

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Chapter 1

Introduction and motivation

1.1 Silicon photonics Technology overview and state of the art

The continuous evolution of integrated computing systems has gradually shifted performance bottlenecks from processing units to communication channels. Increasing integration density, parallelisation and architectural complexity have made efficient data transfer critical for overall system performance in terms of both bandwidth and energy consumption. Conventional electrical interconnects are increasingly limited by resistive losses, crosstalk, latency and power dissipation, and are difficult to scale to the data rates required by modern architectures. Silicon photonics has emerged as a technological response to these limitations, introducing light as an information carrier within and between integrated circuits. Due to its ability to provide high bandwidth density, low energy per bit and compatibility with CMOS industrial processes, silicon-based integrated photonics is a distinctive platform among emerging technologies. Unlike photonic solutions based on III–V materials, the silicon approach leverages the existing manufacturing infrastructure of the microelectronics industry, making the transition from research to large-scale production realistic. Technological processes, integration strategies and system architectures are key considerations in this field. Recent scientific reviews suggest that silicon photonics is being industrialised according to a well-defined roadmap. There has also been a gradual

transition from traditional pluggable optics to co-packaged optics (CPO), as well as electronic photonic integration at package and wafer levels [1]. Within this framework, concepts such as the co-design of devices and circuits and the joint optimization of performance, power consumption, and area play a central role.

1.1.1 Application domains of silicon photonics

Based on this technology, many different applications have been created. These have made silicon photonics do more than it was originally used for, which was to connect optical devices. Silicon photonics was originally developed to address the need for high-speed, low-cost optical transceivers for communication networks. Since the 1980s, exploiting CMOS-compatible fabrication processes has been key to reducing production costs and facilitating industrial adoption. This phase is now considered mature, with millions of transceivers manufactured and deployed annually in data centre communication infrastructures operating at data rates of hundreds of gigabits per second. As on-chip optical telecommunications have reached technological capability, silicon photonics has progressively expanded towards new application domains. In particular, silicon has emerged as a highly versatile platform for integrated sensing applications. The combination of this technology with the capacity for heterogeneous integration of additional functional materials has resulted in the development of photonic sensors for environmental, industrial and biomedical monitoring, as well as integrated LiDAR systems for imaging and navigation applications[1], [2]. Another emerging domain is that of integrated quantum photonics [1], [3]. Silicon photonics is currently considered one of the most promising platforms for realising compact, scalable quantum photonic circuits, thanks to the ability to integrate a large number of passive and active components with precise control over optical phase and interference. These features make silicon ideal for developing secure quantum communication systems and photonic-based quantum computing architectures. A major driving force behind recent developments in silicon photonics is its application in artificial intelligence [2], [3] and neuromorphic computing [2]. The exponential growth in the volume of data to be processed and transferred has highlighted the need for novel

hardware solutions that can deliver high bandwidth while maintaining low energy consumption. In this context, integrated photonics is being explored as both a technology for high-performance interconnects and a potential platform for neuromorphic computing [1]. Here, optical signals enable highly parallel vector operations with reduced energy expenditure. The wide range of applications described above is enabled by the heterogeneous integration of different materials on the silicon platform. In addition to silicon, silicon nitride is commonly used for visible-wavelength operation and low-loss passive components; III–V materials are used to integrate on-chip laser sources; and emerging materials, such as graphene and lithium niobate, are being investigated to realise ultra-high-speed modulators and detectors. This plan to make better platforms is one of the main things that is making silicon photonics the way it is today.

1.1.2 Cost and manufacturing considerations

Beyond technical performance, fabrication and integration costs represent a decisive factor for the large-scale adoption of silicon photonics, which is why it is so important to consider these factors when making decisions. The cost of a silicon photonic chip is highly dependent on the access mode to fabrication and the production volume. In research and prototyping contexts, Multi-Project Wafer (MPW) services enable the cost of the wafer to be shared among multiple users, with an investment per design amounting to several tens of thousands of euros or dollars [3]. However, the unit cost per chip remains high, typically ranging from 100 to 1000 euros, due to the limited number of dies and lack of economies of scale. In contrast, dedicated full-wafer or full-reticle production runs require initial investments of several hundred thousand euros or dollars, but drastically reduce the cost per chip to values of around 10 euros or less. In large-scale production involving hundreds or thousands of wafers, the unit cost can decrease further to below one euro, making silicon photonics competitive in high-volume markets [3].

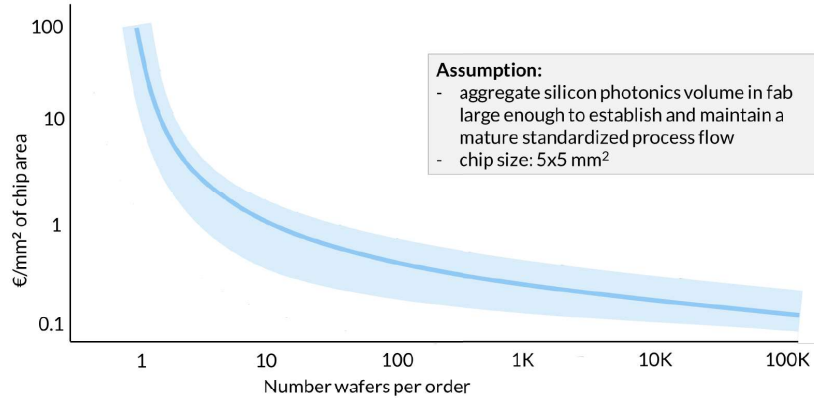


Fig. 1.1 Unit-area cost (per mm^2) of silicon photonic integrated circuits fabricated in a high-volume CMOS foundry with a mature process platform [3].

A commonly used cost metric is the cost per unit chip area, which can reach values of approximately one euro or dollar per square millimetre at moderate production volumes and may be reduced to around one tenth of a euro or dollar per square millimetre in high-volume manufacturing scenarios [3], as shown Fig. 1.1. A key advantage of silicon photonics is the ability to leverage existing CMOS foundries and high-quality silicon-on-insulator (SOI) wafers, which offer higher production yields than platforms based on III–V materials [3]. Nevertheless, recent studies indicate that the overall cost of integrated photonic systems is currently dominated by packaging rather than chip fabrication. One of the structural advantages of silicon photonics is the ability to make use of current CMOS foundries and high-quality silicon-on-insulator (SOI) wafers [3]. These provide a higher production yield than platforms based on III–V materials. However, recent studies suggest that packaging currently dominates the overall cost of integrated photonic systems rather than chip fabrication. In particular, achieving high-precision alignment between optical fibres and on-chip waveguides is one of the main challenges to industrial scalability. Additional cost contributions arise from integrating III–V-based laser sources and inefficient use of chip area in advanced CMOS nodes, often making heterogeneous integration approaches and multi-wafer architectures a more cost-effective solution.

1.1.3 Device building blocks

Silicon photonic integrated circuits (PICs) comprise a set of passive and active components that enable the generation, manipulation and detection of optical signals directly on a silicon substrate, as shown Fig.1.2. These elements form the basis of the silicon photonics platform and are designed to work together, offering high bandwidth, low energy consumption and high integration density.

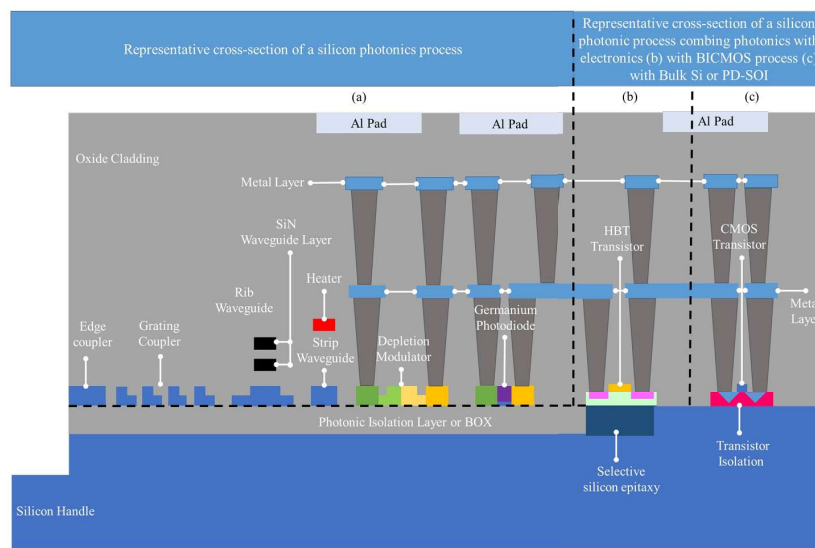


Fig. 1.2 Silicon photonics wafer platforms have been developed to allow for photonic functionalities, either with or without monolithic integration of electronic components[3].

Surface grating couplers (SGCs) and edge couplers are the two primary solutions commonly adopted to enable optical coupling between external components and integrated photonic circuits (PICs). SGCs are periodic diffractive structures, typically integrated with a waveguide, that allow light to be injected into or extracted from the silicon photonic (SiPh) platform. This coupling mechanism is referred to as vertical coupling because the optical signal is transferred through a fiber placed above the chip surface, usually with a specific tilt angle. In contrast, edge coupling operates within the plane of the photonic circuit, where light is coupled directly into the waveguide through the chip facet.

Active functionalities in silicon photonics are primarily provided by optical modulators, which encode electrical data onto optical carriers. In general, optical modulation can be implemented either through thermally controlled Mach–Zehnder interferometers (MZIs) or through plasma-dispersion-based devices. The former relies on heating effects to modify the refractive index, but their performance is limited by the slow diffusion of heat within the structure, resulting in relatively low modulation speeds. In contrast, plasma dispersion modulators exploit carrier-induced refractive index changes and they are more complex to implement, but enable significantly faster operation, supporting data rates of several tens of gigabits per second.

Photodetectors perform the complementary operation of converting optical signals back into electrical currents. Because silicon is transparent at telecommunications wavelengths, the standard solution for efficient photodetection is to integrate germanium onto silicon, due to its favorable absorption properties in the infrared range used in optical communications.

Light sources are one of the platform’s major technological challenges, as silicon is an inefficient light emitter due to its indirect bandgap. Consequently, laser integration usually involves the use of III–V materials, such as indium phosphide, combined with silicon via heterogeneous or hybrid integration techniques. Dedicated coupling structures enable optical input/output, including surface-coupling grating couplers, which facilitate wafer-level testing, and edge couplers, which generally provide higher bandwidth and lower coupling losses.

The waveguides form the foundation of a photonic chip by routing light between its various functional components. Depending on their geometry, they are primarily classified as either strip or rib waveguides. Strip waveguides provide strong optical field confinement and enable very tight bending radii, making them ideal for high-integration-density layouts. Rib waveguides, on the other hand, are characterised by lower propagation losses and are therefore favoured for integrating active devices, offering a better balance between optical confinement and electrical performance. More broadly, waveguides and passive photonic circuits form the technological basis of the entire silicon photonics platform.

Waveguides	[unit]	2024	2030	2035	2040
Waveguide loss	dB/cm	1-0.2	0.2-0.05	0.05-0.001	0.001
Effective index		1.8 - 4	1.8 - 4	1.8 - 4	1.8 - 4
Power	mW	30-100mW	100mW	200mW	500mW
Dimensional uniformity	nm	1nm	0.7nm	0.5nm	0.3nm
Material System	Core /Clad	Si, SiNx, multilayer	multilayer	multilayer	multilayer

Fig. 1.3 The table presents a summary of the principal characteristics of passive waveguides in silicon photonics platforms [3].

As illustrated in Fig. 1.3, recent roadmaps clearly delineate the trajectory for their performance evolution up to 2040 for the waveguide, as outlined in IPSRI’s 2024 report. Propagation losses, in particular, are expected to be reduced from current values of approximately 0.2–1 dB/cm towards nearly negligible levels of around 0.001 dB/cm [3]. This is essential for large-scale, complex photonic circuits with long optical paths. In addition, future developments are expected to shift towards multilayer photonic architectures to enhance design flexibility, routing density, and overall performance. Another key performance metric is optical power processing, with target levels rising to several 30-100 mW up to around 500 mW, to support advanced sensing applications and emerging photonic computing paradigms [3].

In addition to these core components, PICs incorporate various passive devices, including optical filters, arrayed waveguide gratings, power splitters and ring resonators. These are used for applications such as wavelength management, signal routing and sensing. Finally, modern silicon photonic systems increasingly rely on close integration with CMOS control electronics, either monolithically or through advanced packaging solutions. This minimises latency and power consumption, enabling scalable, system-level implementations.

1.2 Microring resonator

A microring resonators (MRR) operate through evanescent coupling between a bus waveguide and a circular optical cavity, where light is coupled from the straight waveguide and circulates within the ring, experiencing constructive interference when the round-trip optical phase matches an integer multiple of 2π . As a result, only wavelengths satisfying the resonance condition are transmitted or filtered, enabling wavelength-selective control. Within photonic integrated circuits, microring resonators are fundamental building blocks thanks to their compact size and ability to implement a variety of functions, such as wavelength filtering, modulation, sensing and optical feedback in more complex systems, as shown Fig. 1.4.

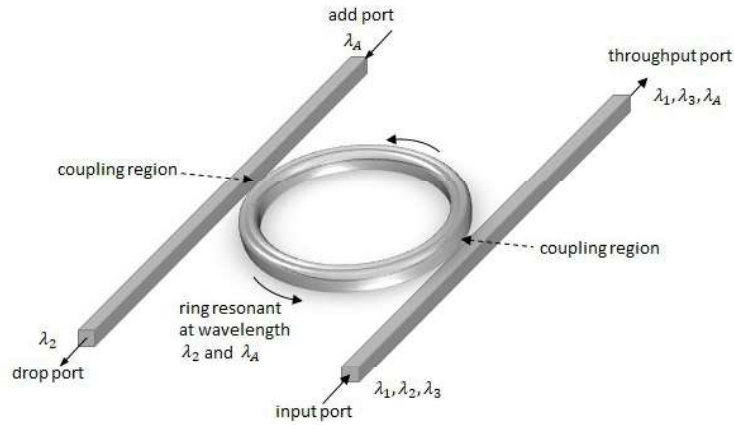


Fig. 1.4 Sketch of add-drop filter based on a ring resonator [4].

The performance of MRRs is strongly influenced by the material platform. Semiconductor-Insulator-Semiconductor Capacitor (SiSCAP) technology enables the production of ultra-compact resonators with sub-millimetre radii thanks to the high refractive index contrast, which ensures strong optical confinement and enhanced light–matter interaction.

However, silicon demonstrates significant nonlinear effects at telecommunications wavelengths, such as two-photon absorption (TPA), free-carrier absorption (FCA) and thermo-optic nonlinearities, as well as a large thermo-optic coefficient. These factors lead to resonance instability, additional losses, and complex dynamic behaviour, often necessitating active thermal stabilisa-

tion. Alternative material platforms have been developed to address some of these limitations. Silicon nitride (SiN) resonators offer ultra-low propagation losses, negligible nonlinear absorption, and a wide transparency window extending from the visible to the mid-infrared, enabling high-Q devices with improved thermal robustness. In contrast, polycrystalline silicon (poly-Si) offers improved compatibility with CMOS processes and facilitates monolithic electronic–photonic integration. However, it is susceptible to increased scattering losses due to material inhomogeneities.

Nonlinear effects in silicon resonators are frequently evaluated as a disadvantage in applications requiring stable and linear operation. However, they can also be exploited for advanced functionalities, such as all-optical signal processing and frequency comb generation. Consequently, it is imperative to accurately model the interplay between nonlinear, thermal, and dispersive effects to facilitate the design and optimisation of resonant photonic devices.

In the relevant literature, several models have been proposed to describe nonlinear effects in silicon ring resonators. Typically, these models assume strip waveguide cross-sections and adopt lumped-parameter approaches [5]. However, when more complex geometries, such as rib waveguides, are considered, the physical parameters become inherently spatially distributed, and simplified models may fail to accurately capture the device behaviour. In this context, this thesis focuses on the development of a modelling framework for silicon ring resonators with complex cross-sections, with a view to mitigating nonlinear effects. The proposed approach accounts for the spatial distribution of optical and physical parameters within the waveguide, enabling a more accurate representation of the device physics. The model has been subjected to experimental validation, thereby demonstrating its efficacy in predicting the behaviour of resonators designed to reduce nonlinear distortion.

Furthermore, MRRs are of key importance in more complex photonic architectures, such as hybrid integrated tunable lasers. In these systems, III-V semiconductor gain sections are combined with passive silicon-based circuits, where resonant structures provide optical feedback and wavelength selectivity. This underlines the critical importance of accurate resonator design and modelling.

1.3 Tunable hybrid integrated lasers

The integration of on-chip light sources remains a significant research challenge in silicon photonics, primarily due to the indirect bandgap of silicon, which renders it an inefficient light emitter. In order to overcome this limitation, a number of integration strategies have been developed, including hybrid, monolithic, and heterogeneous approaches. The integration of these components is achieved through the co-packaging of pre-fabricated III-V lasers, such as InP-based edge emitters, onto silicon photonic circuits, typically through the process of flip-chip bonding. Despite the relative sophistication of this method, it necessitates precise and frequently expensive active alignment. Monolithic integration, predicated on the direct epitaxial growth of III-V materials on silicon, is an ideal solution in terms of scalability and integration density. However, this approach is complicated by lattice mismatch, thermal expansion differences, and high growth temperatures that are incompatible with standard CMOS processes. Heterogeneous integration via wafer bonding is regarded as the most promising approach for large-scale manufacturing, combining III-V gain materials with silicon and enabling wafer-level alignment. The techniques employed encompass direct, adhesive (e.g., DVS-BCB), and metal bonding, in addition to the emerging micro-transfer printing technique, which has been demonstrated to facilitate efficient chiplet integration. Nevertheless, challenges pertaining to contamination, yield, and reliability persist, with the expectation of fully mature wafer-scale solutions within the next decade, likely in the back-end-of-line for CMOS compatibility.

In the context of hybrid laser architectures, the selection of the passive feedback platform is of critical importance in determining the overall performance of the device. Silicon nitride has been shown to be particularly well suited for external cavities due to its ultra-low losses and negligible nonlinear absorption. This enables narrow-linewidth emission and high spectral purity. However, the relatively low index contrast results in larger device footprints and reduced free spectral range, thereby limiting the achievable tuning range. Indeed, a model for the study of a hybrid laser with a silicon nitride feedback cavity has been presented in the following reference: [6], as shown Fig. 1.5.

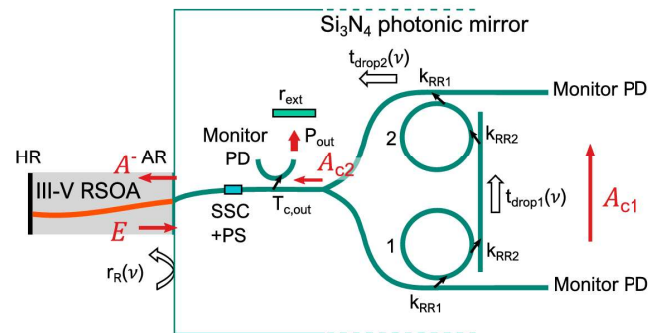


Fig. 1.5 A schematic representation of the III-V/SiN hybrid tunable integrated laser structure [6].

In contrast, silicon permits the creation of more compact cavities with larger FSR and enhanced tunability. However, its strong nonlinear and thermo-optic effects have the potential to introduce instabilities and degrade laser performance.

This thesis investigates hybrid integrated tunable lasers employing silicon or silicon nitride external cavities, with particular emphasis on the interaction from the external mirror, carrier dynamics, and material nonlinearities.

1.4 Thesis organization

This thesis is organized as follows.

- Chapter 2 presents a comprehensive modeling framework for the analysis of microring resonators with complex waveguide geometries. Particular attention is devoted to the interplay between nonlinear optical effects, such as two-photon absorption, free-carrier dynamics and thermal phenomena. The proposed approach provides an efficient and accurate tool for predicting the behavior of ring resonators. The model is validated through experimental measurements performed on two microring resonators featuring highly complex cross-section geometries.

- Chapter 3 investigates hybrid integrated tunable lasers based on external cavities incorporating silicon nitride resonators. The dynamical behavior of the devices is analyzed, with particular emphasis on the emergence of oscillatory regimes and comb-like spectral features. A time-delayed algebraic model is employed to interpret the experimental observations and to identify the underlying physical mechanisms.
- Chapter 4 extends the analysis to silicon-based external cavities, with the aim of understanding the impact of stronger nonlinear effects on laser performance. A comparative study between silicon and silicon nitride platforms is carried out, highlighting the trade-offs between compactness, tunability, and dynamical stability.
- Chapter 5 summarizes the main results of this work and presents the concluding remarks. The overall structure and key outcomes of the thesis are schematically illustrated in the figure below.

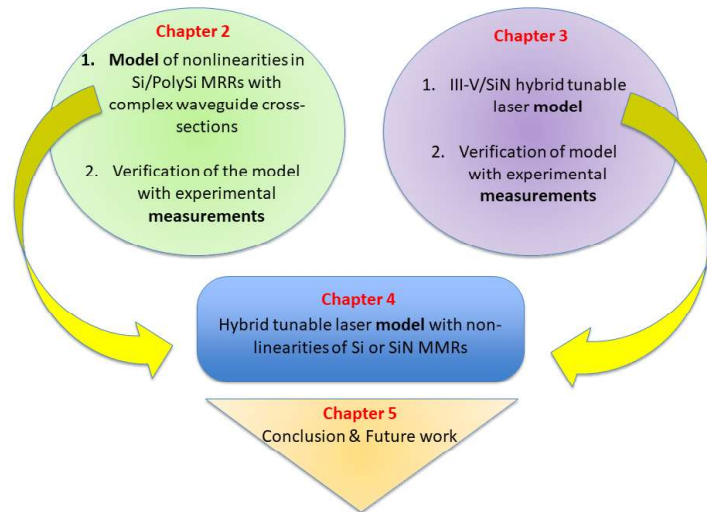


Fig. 1.6 Outline of the thesis structure

Chapter 2

Nonlinear modelling in ring resonators with complex waveguide cross-sections.

Microring resonators (MRRs) represent key components in silicon photonic integrated circuits and are widely employed in a variety of applications. Their functionalities span wavelength filtering, sensing, optical switching, and modulation [7], [8]. Nevertheless, the intrinsic nonlinear behavior of silicon, even at relatively moderate input power levels, can significantly affect the performance of high-Q MRRs [9], [10].

This chapter introduces a modeling and simulation approach capable of accounting for both nonlinear and thermal effects in silicon MRRs with arbitrarily complex waveguide cross-sections. The proposed method is applied to the design of MRRs based on the SISCAP (semiconductor-insulator-semiconductor capacitor) platform [11]. This platform offers advantages in reducing both cost and energy consumption of optical transceivers for interconnect applications, while still supporting high data rates and scalability [12]. Within the SISCAP technology, the silicon layer thickness is typically around 100 nm. As a consequence, rib waveguides—commonly adopted in thicker platforms (e.g., 220 nm or 500 nm [13]) are not suitable due to excessive bending losses for radii below $10\mu\text{m}$. Alternative solutions for achieving very high-Q and high-finesse MRRs with reduced nonlinearities have been

recently proposed [14], although they still rely on thicker crystalline silicon layers.

The main contribution of this work is the development of a rigorous self-consistent model for analyzing the nonlinear response of MRRs with complex waveguide geometries. The proposed methodology is particularly suited for structures that cannot be approximated as simple lumped strip waveguides. Although alternative approaches, such as finite-difference time-domain (FDTD) simulations, can be used, they typically require significantly longer computation times.

As an example, we performed FDTD simulations using RSoft FullWAVE on a strip waveguide ring with a radius of $3\text{ }\mu\text{m}$ to extract its transmission response. This method was selected because it is among the most suitable in RSoft for incorporating nonlinear effects. However, when including dispersive materials and Kerr nonlinearity, the simulation time is approximately one hour. Additionally, incorporating FCA effects within commercial tools remains challenging.

In contrast, this approach relies on COMSOL Multiphysics simulations, achieving computation times between 2 and 15 minutes on the same hardware, depending on the complexity of the waveguide cross-section. The simulations incorporate nonlinear effects (TPA and FCA) as well as self-heating, by sweeping the circulating power over 12 different values using a finite element method.

2.1 Nonlinear effects in silicon

The dominant mechanisms responsible for nonlinear loss and dispersion in silicon, illustrated in Fig. 2.1, are two-photon absorption (TPA) and free-carrier absorption (FCA). TPA involves the simultaneous absorption of two photons, leading to the generation of an electron-hole pair. Subsequently, FCA occurs when these carriers absorb additional photons, increasing their energy within the conduction and valence bands. The generated free carriers (FCs) eventually relax toward lower energy states or recombine with holes through Shockley-Read-Hall (SRH) and other non-radiative recombination

processes, releasing energy in the form of heat (self-heating). This temperature rise, combined with nonlinear absorption and refractive index changes, alters the transmission characteristics of the MRR, resulting in a resonance wavelength shift and a degradation of the quality factor [15].

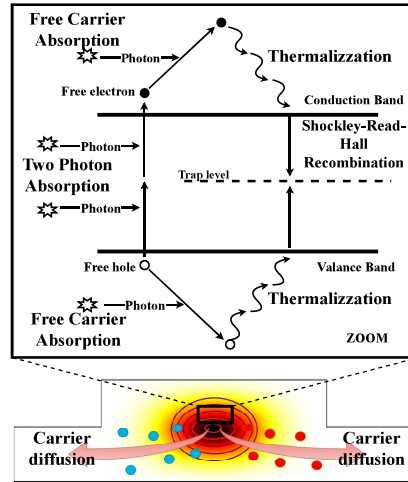


Fig. 2.1 Illustration of the main nonlinear effects in silicon, involving transitions between valence and conduction bands.

In the case of rib waveguides or more complex geometries (i.e., non-uniform cross-sections), free carriers can spatially diffuse within the silicon, while heat generation due to SRH recombination follows the carrier distribution, as depicted in Fig. 2.1 for a rib structure. Under these conditions, an accurate evaluation of the MRR performance degradation—both in terms of quality factor and transmission spectrum—requires two-dimensional simulations. These simulations must include TPA-induced carrier generation, carrier transport, non-radiative recombination, as well as heat generation and dissipation.

Such a self-consistent framework provides spatial distributions of both carrier density and temperature across the waveguide, enabling the extraction of variations in modal loss and effective refractive index. Existing models addressing nonlinear effects in MRRs typically rely on simplified lumped strip waveguide approximations, thus neglecting the influence of the actual waveguide geometry. Furthermore, they do not account for carrier diffusion

in rib structures nor for heterogeneous cross-sections involving different materials (e.g., silicon and polysilicon) or doping profiles.

2.2 2D Model

In this section, the complete modelling framework employed in this thesis is described. The methodology used to determine the spatial distribution of free carriers within the waveguide cross-section includes the associated thermal effects in a self-consistent manner. Then, the expressions used to evaluate the variations of modal loss and effective refractive index are introduced. Finally, the nonlinear transmission response of the MRR is derived as a function of the input wavelength and the optical power injected into the bus waveguide.

2.2.1 Modelling carrier photo-generation, transport and self-heating

Carrier photo-generation due to two-photon absorption (TPA) is described by the volumetric generation rate G_{ph} , its proof can be found in Appendix A [16]:

$$G_{ph}(x, y) = \frac{P_c^2}{\hbar \cdot \omega} \cdot \beta_{TPA} \frac{n^2}{Z_0^2 \cdot 8 \cdot P_\mu^2} |\mathcal{E}(x, y)|^4 \quad (2.1)$$

where P_c represents the optical power circulating inside the resonator, corresponding to the power confined within a generic cross-section of the ring. In addition, $\hbar$ denotes Planck's constant, ω is the angular frequency, and Z_0 represents the impedance of free space (377Ω). The parameter n is the refractive index of the medium where TPA occurs (silicon or polysilicon), while β_{TPA} is the corresponding absorption coefficient [15]. The normalization constant P_μ is defined as in [16]. The quantities $\mathcal{E}(x, y)$ and $\mathcal{H}(x, y)$ represent the normalized electric and magnetic field distributions of the fundamental guided mode, assuming single-mode propagation.

The electromagnetic field profiles are first computed using a finite element method (FEM) and normalized with respect to the maximum electric

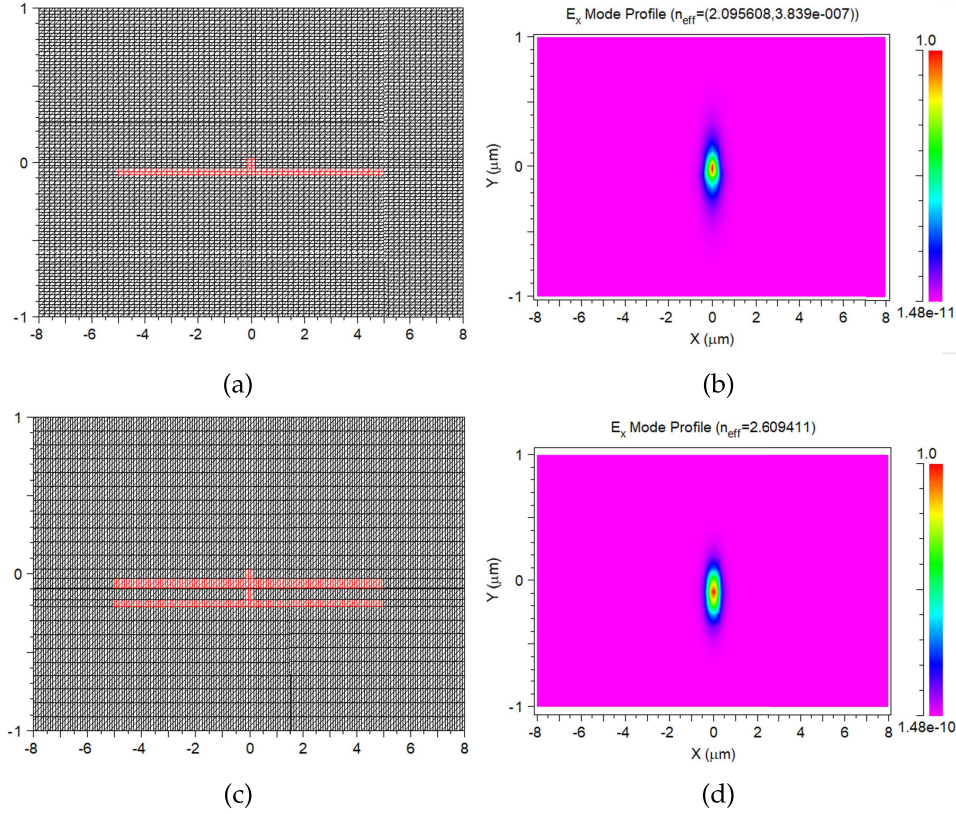


Fig. 2.2 Examples of mesh construction using the RSoft CAD Layout tool: (a) a silicon rib waveguide (shown in red); (c) silicon (bottom) and polysilicon (top) rib waveguides. The black mesh is the simulation area in silicon oxide. (b) Normalized electric field profile ($V/\mu m^2$) for a rib waveguide; (d) corresponding profiles for silicon and polysilicon rib waveguides, with perfectly matched layers (PMLs) applied on all boundaries of the simulation domain. For further details, see Appendix A.

field amplitude, using the RSoft Cad Layout tool, as shown in Fig. 2.2; for further details, see Appendix A. These profiles are subsequently imported into COMSOL Multiphysics, where the semiconductor transport equations and the heat transfer equations are solved simultaneously through a finite element formulation, ensuring a self-consistent solution.

The generated electron-hole pairs propagate throughout the silicon or polysilicon region according to the drift-diffusion formalism [17]:

$$\nabla^2 \Phi(x, y) = \frac{q}{\epsilon} (p(x, y) - n(x, y) - N_a^- - f_t \cdot N_t) \quad (2.2)$$

$$\begin{aligned}\nabla(-D_n(x,y) \cdot \nabla n(x,y) + \mu_n(x,y) \cdot n(x,y) \cdot \nabla \Phi(x,y)) = \\ = G_{ph}(x,y) - R_n(x,y)\end{aligned}\quad (2.3)$$

$$\begin{aligned}\nabla(-D_p(x,y) \cdot \nabla p(x,y) - \mu_p(x,y) \cdot p(x,y) \cdot \nabla \Phi(x,y)) = \\ = G_{ph}(x,y) - R_p(x,y)\end{aligned}\quad (2.4)$$

Equation (2.2) is Poisson's equation, linking the electrostatic potential Φ to the local charge distribution. In this formulation, q is the elementary charge, ϵ is the permittivity, and N_a^- denotes the doping concentration. The free electron density $n(x,y)$ and free hole density $p(x,y)$ are treated as unknown variables of the system (2.2)–(2.11). Trap-related effects are described through the occupancy probability f_t and the volumetric trap density N_t . Surface states at the silicon/oxide interface, characterized by a density N_s , are assumed to be uniformly distributed. These are incorporated into the model through an equivalent bulk trap density [18]:

$$N_t = N_s \cdot \frac{2(W + h)}{W \cdot h} \quad (2.5)$$

The transport equations (2.3)–(2.4) are solved under steady-state conditions. Both carrier mobility and diffusion coefficients depend on the local temperature distribution $T(x,y)$. In particular, the diffusion coefficient is expressed as $D_{n,p}(x,y) = \mu_{n,p} k_B T(x,y)$, where k_B is the Boltzmann constant. The mobility dependence on temperature is modeled as [19]:

$$\mu_{n,p}(x,y) = \mu_{n0,p0} \left(\frac{T(x,y)}{T_{ref}} \right)^{-\alpha_n} \quad (2.6)$$

where μ_{n0} and μ_{p0} are the carrier mobilities at the reference temperature $T_{ref} = 300$ K, and $\alpha_n = 2.33$ is an experimentally determined coefficient for silicon.

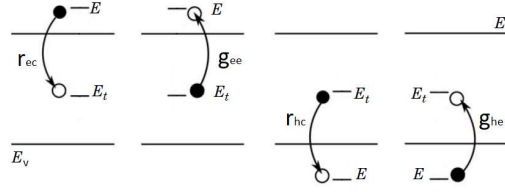


Fig. 2.3 The SRH recombination comprises four distinct processes. The first of these is the capture of the electron from the conductive band to the trap level, denoted by r_{ec} . The second process is the emission of the electron from the trap level to the conductive band, denoted by g_{ee} . In a similar way, it can be demonstrated that r_{hc} and g_{he} are, respectively, the capture and emission of a hole between the trap level and the valence band.

Carrier recombination ($R_{n,p}(x,y)$) is described through the Shockley-Read-Hall (SRH) non-radiative mechanism, which depends on both trap density and trap energy level E_t .

The trap energy level E_t is assumed to lie at mid-gap, as illustrated in Fig. 2.3. The recombination rates are defined as the net balance between capture and emission processes, namely $R_n = r_{ec} - g_{ee}$ and $R_p = r_{hc} - g_{he}$. The full system is implemented within the COMSOL Semiconductor Module [20]. Under steady-state conditions, $R_n = R_p$ and can be written as [20], [21], [22]:

$$R_n(x,y) = C_n \cdot N_t \cdot n(x,y) (1 - f_t) \left(1 - e^{\frac{E_{ft} - E_{fn}}{k_b T}} \right) \quad (2.7)$$

$$R_p(x,y) = C_p \cdot N_t \cdot p(x,y) f_t \left(1 - e^{\frac{E_{fp} - E_{ft}}{k_b T}} \right) \quad (2.8)$$

$$f_t = \frac{1}{1 + g_D \cdot e^{\frac{E_t - E_{ft}}{k_b T}}} \quad (2.9)$$

where $C_{n,p}$ denotes the electron/hole capture probability, defined as

$C_{n,p} = \langle \sigma_{n,p} \rangle v_{n,p}^{th}$. Here, $\langle \sigma_{n,p} \rangle$ represents the average capture cross section

for electrons or holes, while $v_{n,p}^{th}$ is their thermal velocity. The parameter g_D is the degeneracy factor of the energy level and is equal to 1.

E_{fn} and E_{fp} correspond to the quasi-Fermi levels of free electrons and holes in the conduction and valence bands, respectively. E_{ft} indicates the quasi-Fermi level associated with the electron population in the trap [22], and the electron occupancy of the trap (f_t) follows the Fermi–Dirac distribution.

Energy dissipation associated with carrier thermalization and SRH recombination leads to heat generation, which in turn modifies the local temperature distribution. The resulting thermal behavior depends both on the ring geometry and on the vertical separation between the waveguide and the substrate, as illustrated in Fig. 2.4.

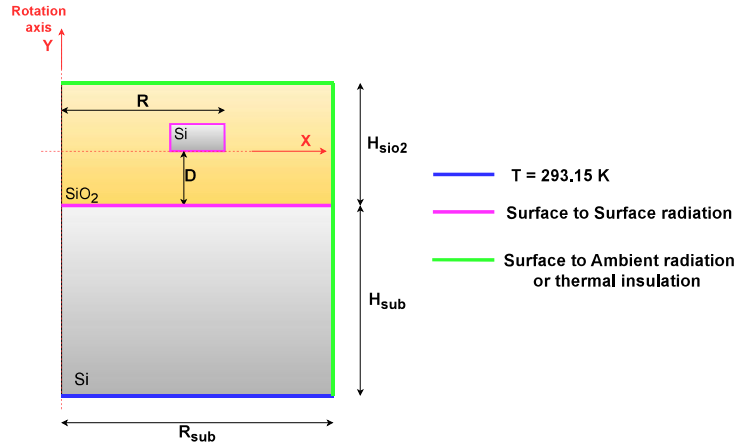


Fig. 2.4 Cross-sectional view of a silicon ring resonator, where the coating consists of silicon oxide deposited on a silicon substrate. The initial temperature of the entire device is set equal to the ambient temperature. For each interface, the applied boundary condition is specified; for further details, see Appendix A.

Thermalization and Shockley-Read-Hall (SRH) recombination processes of free electron-hole pairs lead to an energy loss, resulting in the generation of heat and a localized increase in temperature. In this case we consider two factors that influence the temperature variation and heat dissipation: the ring radius and the distance between the silicon waveguide and the silicon substrate.

The heat equation is expressed as:

$$-\nabla(d_z \cdot k_c \nabla T(x, y)) = Q(x, y) \cdot d_z \quad (2.10)$$

k_c denotes the thermal conductivity of the material, while d_z represents the ring length, given by $d_z = 2\pi R$, with R being the ring radius. Due to the circular symmetry of the structure, the temperature distribution can be evaluated using a 2D axisymmetric approach rather than a full 3D model. This simplification is valid because the distribution of free carriers remains uniform along the z -direction, which is perpendicular to the waveguide cross-section. $Q(x, y)$ is the heat source inside the waveguide core [23], [24]:

$$Q(x, y) = \frac{|J_n|^2}{\Sigma_n} + \frac{|J_p|^2}{\Sigma_p} + Q_{FCA}(x, y) + R_n \cdot 2\hbar\omega \quad (2.11)$$

In Eq. (2.11), the first two contributions on the right-hand side account for the heat produced by Joule heating. Here, J_n and J_p denote the electron and hole current densities, respectively, while Σ_n and Σ_p correspond to the electrical conductivities of electrons and holes. These quantities are defined as $\Sigma_n = q \cdot n(x, y) \cdot \mu_n$ and $\Sigma_p = q \cdot p(x, y) \cdot \mu_p$ [25].

The third term on the right-hand side describes the heat arising from the thermal relaxation of high-energy free carriers as they relax to the bottom of the conduction and valence bands, meaning that all optical power absorbed through free-carrier absorption (FCA) is ultimately converted into heat. Accordingly, Q_{FCA} represents the spatially averaged optical power density absorbed via FCA within the MRR [15].

$$Q_{FCA}(x, y) = \rho_{abs}(x, y) \cdot I_{opt}(x, y) \quad (2.12)$$

with

$$\rho_{abs}(x, y) = P_c(\alpha_0 + \Delta\alpha_{FC}(x, y))(1 + t^2) \quad (2.13)$$

where α_0 represents the intrinsic linear propagation loss in the silicon microring, while $\Delta\alpha_{FC}(x, y)$ denotes the spatially varying additional loss induced by free-carrier absorption [26], [27]. The parameter t is the transmis-

sion coefficient of the microring resonator coupler, defined as $t = \sqrt{1 - k^2}$, where k is the coupling coefficient of the MRR. Finally, I_{opt} describes the normalized transverse field distribution of the guided optical mode:

$$I_{opt}(x, y) = \frac{\text{Re}\{e(x, y) \times h(x, y)^*\} \cdot \hat{z}}{2 \cdot P_\mu} \quad (2.14)$$

The final contribution on the right-hand side of Eq. (2.11) accounts for heat generation associated with non-radiative recombination processes. In particular, when an electron-hole pair recombines through a trap-assisted mechanism, the energy released corresponds to $2\hbar\omega$, which originates from the absorption of two photons via TPA.

2.2.2 Variation of modal loss and effective refractive index

The evaluation of the variation of the effective refractive index and the modal loss in the waveguide begins with the determination of the perturbations in the real and imaginary components of permittivity [28], [29]. These are expressed as $\Delta\epsilon_{r,i}(x, y) = 2 \cdot \epsilon_0 \cdot n \cdot \Delta n_{r,i}(x, y)$, where ϵ_0 is the vacuum permittivity. The term $\Delta n_r(x, y)$ represents the spatial variation of the real refractive index induced by FCD and thermal effects, whereas $\Delta n_i(x, y)$ accounts for the change in the imaginary part associated with FCA. The detailed derivation of these relations is reported in Appendix A.

The corresponding variation of the effective refractive index due to FCD is given by:

$$\Delta n_{eff, FCD} = \frac{c \cdot \epsilon_0 \cdot n \int \int_\infty \Delta n_{FCD}(x, y) \cdot |e(x, y)|^2 dx dy}{\int \int_\infty \text{Re}\{e(x, y) \times h(x, y)^*\} \cdot \hat{z} dx dy}, \quad (2.15)$$

From this formulation it follows that, when the carrier density is assumed to be uniform across the entire waveguide cross-section (as is typically the case for a silicon strip waveguide), the expression simplifies to $\Delta n_{eff, FCD} = \Gamma \cdot \Delta n_{FCD}$ [26], [27]. Here, Γ denotes the optical confinement factor within the silicon core, while Δn_{FCD} is defined as [26]:

$$\Delta n_{FCD}(x, y) = p_f \cdot n(x, y)^q + r_f \cdot p(x, y)^s \quad (2.16)$$

The variation of refractive index cause to self-heating is:

$$\Delta n_{eff,T} = \frac{c \cdot \epsilon_0 \cdot n \int \int_{\infty} \frac{dn}{dT} \cdot \Delta T(x, y) \cdot |e(x, y)|^2 dx dy}{\int \int_{\infty} \text{Re}\{e(x, y) \times h(x, y)^*\} \cdot \hat{z} dx dy} \quad (2.17)$$

where $\frac{dn}{dT}$ is the material thermo-optic coefficient (silicon or polySi).
The optical modal loss due to FC is [29]:

$$\Delta \alpha = \frac{2c \cdot \epsilon_0 \cdot n \int \int_{\infty} \frac{\Delta \alpha_{FC}(x, y) \lambda}{4\pi} \cdot |e(x, y)|^2 dx dy}{\int \int_{\infty} \text{Re}\{e(x, y) \times h(x, y)^*\} \cdot \hat{z} dx dy} \quad (2.18)$$

where $\Delta \alpha_{FC}(x, y)$ is defined as [26]:

$$\Delta \alpha_{FC}(x, y) = a_f \cdot n(x, y)^b + c_f \cdot p(x, y)^d \quad (2.19)$$

the experimental parameters a_f , b , c_f , d , p_f , q , r_f and s in eqs. (2.16) and (2.19) depend on the material and on the wavelength (in this specific case is $\lambda = 1.3 \mu m$); they are documented in Table 2.1. The system of equations ranging from Eq. (2.1) to Eq. (2.19) is simultaneously solved within COMSOL and RSoft in order to determine the modal loss and the change in effective refractive index, assuming a fixed circulating power P_c in the ring resonator.

2.2.3 Modelling of the ring transmission coefficient

The power-dependent variations of modal attenuation and effective refractive index introduced in previous Section can now be incorporated into the nonlinear model of the microring resonator (MRR), allowing the evaluation of the transmission response as a function of the input bus power and wavelength. The MRR transmission coefficient is given by [15]:

$$T_{thr} = \left| \frac{t \left(1 - k^2 \cdot a(P_c) \cdot e^{j\theta(P_c)} \right)}{1 - t^2 \cdot a(P_c) \cdot e^{j\theta(P_c)}} \right|^2 \quad (2.20)$$

In this expression, a denotes the round-trip amplitude attenuation, while θ represents the accumulated phase shift experienced by the optical field inside the resonator. Both quantities exhibit a nonlinear dependence on the circulating power P_c , which is determined through:

$$P_c = P_{bus} \cdot \frac{k^2(1 - \eta^2)}{|1 - t^2 \cdot a(P_c) \cdot e^{j\theta(P_c)}|^2} \quad (2.21)$$

where P_{bus} is the optical power launched into the bus waveguide.

The total phase accumulation per round trip can be written as:

$$\theta(P_c) = \theta_0 + \frac{n_g}{c}(\omega - \omega_0)L + \Delta\theta(P_c) \quad (2.22)$$

Here, θ_0 is the linear-phase contribution evaluated at the reference angular frequency ω_0 , n_g is the group refractive index, and L is the resonator circumference. The term $\Delta\theta(P_c)$ accounts for additional phase shifts induced by nonlinearities and thermal effects, and it is expressed as:

$$\Delta\theta(P_c) = \frac{2\pi L}{\lambda_0} \Delta n_{eff}(P_c) \quad (2.23)$$

where $\Delta n_{eff}(P_c)$ represents the effective refractive index change arising from both free-carrier effects and temperature variations as a function of the circulating power.

The single-round-trip field attenuation is described by:

$$a(P_c) = e^{-\frac{\alpha_{eff}(P_c)L}{2}} \quad (2.24)$$

where α_{eff} is the total effective propagation loss, including both linear and nonlinear contributions, defined as:

$$\alpha_{eff}(P_c) = \alpha_0 + \alpha_{rad} + \Delta\alpha(P_c) \quad (2.25)$$

In this context, α_0 accounts for intrinsic losses such as scattering, surface-state absorption, and residual doping, while α_{rad} represents bending-related radiation losses due to coupling of light into the cladding region and it is defined as: $\alpha_{rad} = \beta_{TPA} \frac{|E_c|^2}{A_{eff}}$

Finally, by numerically solving the implicit nonlinear Eq. (2.21) for a given input bus power and wavelength, the circulating power P_c can be determined, enabling the computation of the corresponding MRR transmission through Eq. (2.20).

2.3 Results of the numerical simulations

This section presents the numerical and physical analysis of five different microring resonator (MRR) geometries, namely Si Strip, Si/PolySi Strip, Si Rib, Si/PolySi 4 wings Rib, and Si/PolySi 3 wings Rib, whose cross-sections are illustrated in Fig. 2.5.

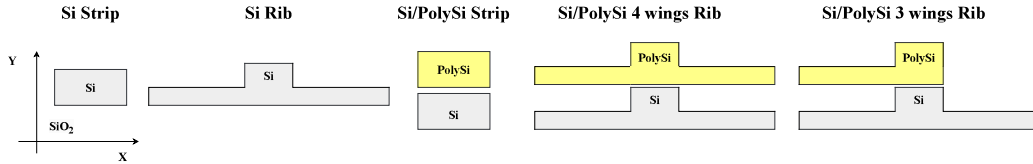


Fig. 2.5 Cross-sections of silicon and polysilicon waveguides analyzed in this study, based on the geometries provided by the SISCAP platform.

The simulations are carried out using material parameters for silicon and polysilicon summarized in Table 2.1. These values are taken from previously reported experimental characterizations and validated numerical models.

Before analyzing the nonlinear and thermal response, the linear guiding properties are evaluated through bend-loss calculations. Figure 2.6 reports the bend loss as a function of radius for all considered waveguide geometries. From these results, a minimum usable radius is identified by imposing a threshold of 1 dB/cm bend loss.

Table 2.1 The parameters utilised in RSoft-Comsol simulations for each waveguide cross-section.

Parameter	Silicon	Polysilicon	Unit
a_f	$3.48 \cdot 10^{-22}$	$5.2 \cdot 10^{-20}$	cm^2 [26], [27]
b	1.229	1.167	- [26], [27]
c_f	$1.02 \cdot 10^{-19}$	$1.2 \cdot 10^{-19}$	cm^2 [26], [27]
d	1.089	1.109	- [26], [27]
p_f	$2.98 \cdot 10^{-22}$	$6.6 \cdot 10^{-22}$	cm^2 [26], [27]
q	1.016	1.011	- [26], [27]
r_f	$1.25 \cdot 10^{-18}$	$2.4 \cdot 10^{-18}$	cm^2 [26], [27]
s	0.835	0.838	- [26], [27]
n	3.45	3.45	- [27]
β_{TPA}	$15 \cdot 10^{-12}$	$15 \cdot 10^{-12}$	m/W [27]
μ_{n0}	1450	55.9	$cm^2/(V \cdot s)$ [27], [30]
μ_{p0}	500	45	$cm^2/(V \cdot s)$ [27], [30]
N_t	$5.454 \cdot 10^{16}$	$5.36 \cdot 10^{17}$	$1/cm^3$ [27]
$\langle \sigma_n \rangle$	$2.18 \cdot 10^{-15}$	$8.11 \cdot 10^{-16}$	cm^2 [27]
$\langle \sigma_p \rangle$	$1.09 \cdot 10^{-16}$	$2.388 \cdot 10^{-15}$	cm^2 [27]
v_n^{th}	$2.3 \cdot 10^7$	$2.3 \cdot 10^7$	cm^2/s [30]
v_p^{th}	$1.65 \cdot 10^7$	$1.65 \cdot 10^7$	cm^2/s [30]
$\frac{dn}{dT}$	$1.84 \cdot 10^{-4}$	$1.84 \cdot 10^{-4}$	K^{-1} [15]
λ	1.3	1.3	μm
k_c	148	148	$W/(m \cdot K)$ [31]
α_0	1.02	8	dB/cm [27]
N_a^-	$1 \cdot 10^{15}$	$1 \cdot 10^{15}$	$1/cm^3$

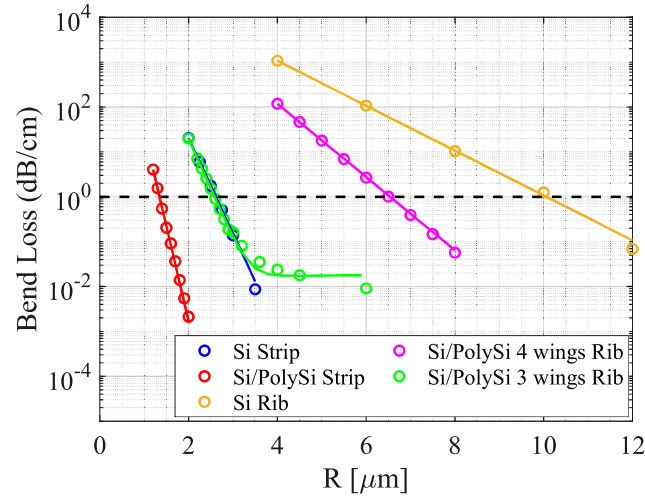


Fig. 2.6 Bending losses versus radius for four types of MRRs with different cross-sections; solid lines indicate exponential fits to the numerical data.

The analysis shows that reliable operation for radii below $6\mu m$ with negligible bending losses (< 1 dB/cm) is not achievable for the Si rib and Si/PolySi 4 wings rib waveguides, due to insufficient optical confinement. In contrast, the introduction of a polysilicon layer improves confinement in both strip and rib geometries. As a consequence, the Si rib structure is excluded from the subsequent nonlinear study. The remaining three configurations are therefore designed with a fixed radius of $3.5\mu m$, a coupling coefficient $k^2 = 0.0055$, and a resulting cold-cavity quality factor of approximately $Q \approx 30000$.

To better understand nonlinear behavior at the device level, the carrier generation and transport mechanisms are first examined. Figure 2.7 presents the optical generation rate at $P_c = 100$ for the three considered structures, namely the Si strip, the Si/PolySi strip, and the Si/PolySi 3 wing rib waveguide, as obtained from COMSOL simulations. The photogenerated carriers are initially confined within the waveguide core in all cases. Subsequently, for the rectangular waveguides, the carriers redistribute uniformly across the entire cross-section, whereas in the rib waveguide they exhibit lateral diffusion into the wing regions, as further illustrated in Fig. 2.8.

Indeed, as shown in Fig. 2.8, carrier concentration is generally higher in silicon regions compared to polysilicon. This behavior is attributed to

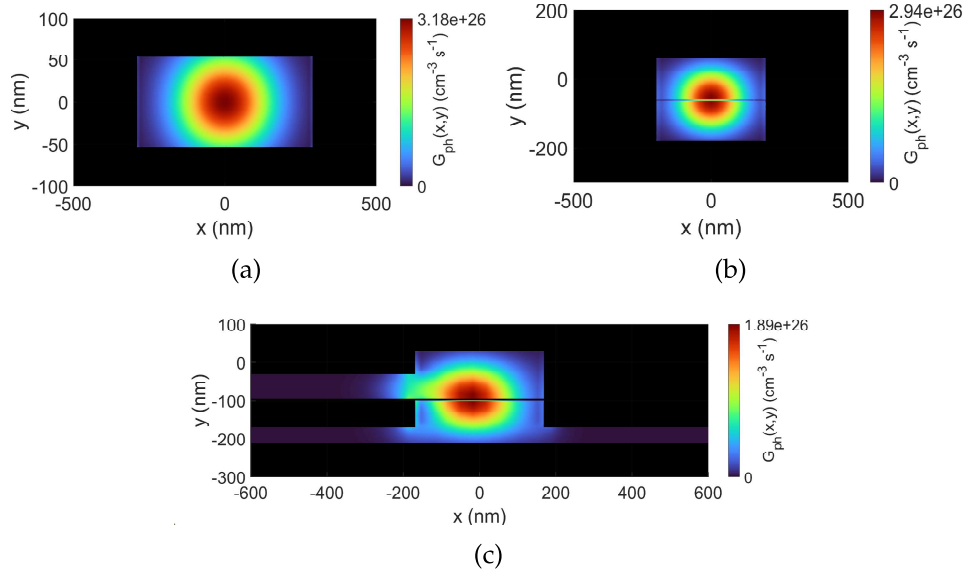


Fig. 2.7 Example of photo-generation rate ($G_{ph}(x, y)$) with circulating power of 100 mW in three waveguides: (a) Si Strip, (b) Si/PolySi Strip and (c) Si/PolySi 3 wings Rib.

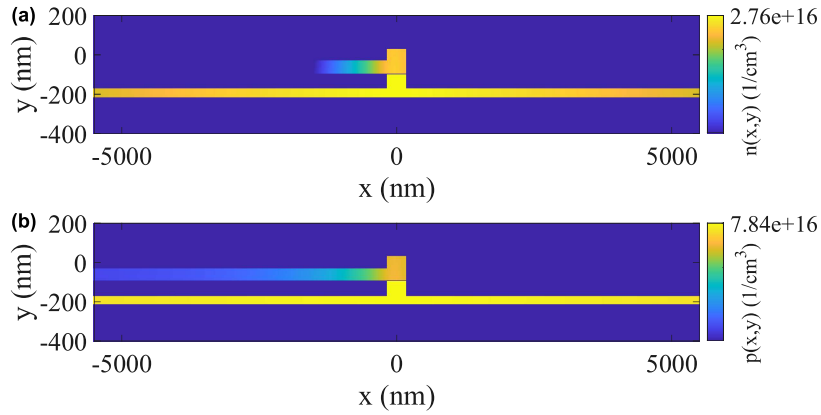


Fig. 2.8 Example of carrier distributions: (a) electron concentration ($n(x, y)$) and (b) hole concentration ($p(x, y)$) in the Si/PolySi three-wing rib waveguide, resulting from the photon generation shown in Fig. 2.7.

the larger SRH recombination rate in polysilicon, which is associated with a higher density of defect states and consequently a shorter diffusion length. A quantitative comparison of carrier distributions along the x-axis at $y = -190$ nm is reported in Fig. 2.9a.

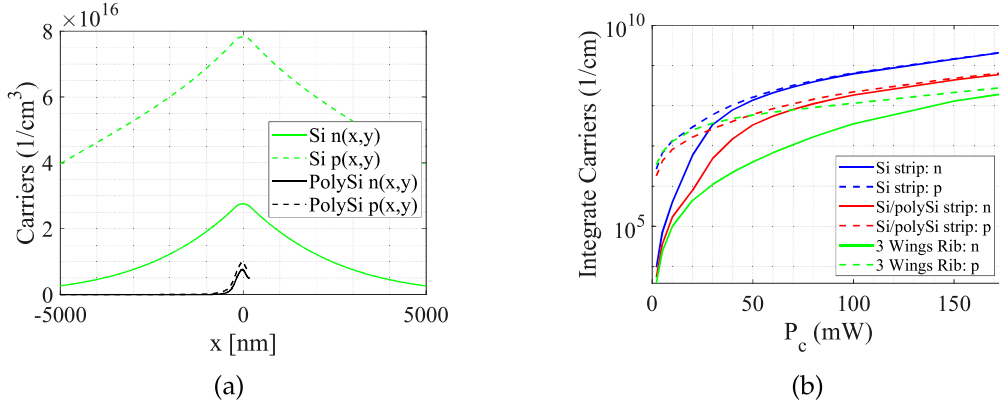


Fig. 2.9 (a) Carrier distribution along the x -direction at $y = -62, \text{nm}$ and $y = -190, \text{nm}$ for the case shown in Fig. 2.8b Integrated carrier density as a function of the circulating power for each waveguide under study.

Since FCA is directly linked to the accumulated carrier density, Fig. 2.9b compares the total carrier population across the three waveguide designs under identical circulating power conditions. This quantity is obtained by integrating $n(x,y)$ and $p(x,y)$ over an effective region corresponding to 90% of the optical mode area.

Thermal effects are then investigated, showing that the Si/PolySi 3 wings rib structure exhibits the lowest temperature rise for identical input power levels. This improvement is mainly due to the presence of lateral wings, which enhance heat spreading and reduce thermal localization in the waveguide core. Figure 2.10 compares the resulting temperature distributions for the three geometries, while Fig. 2.11a reports the maximum core temperature as a function of circulating power.

Overall heating is found to be dominated by free-carrier thermalization, which is further quantified in Fig. 2.11b through the integrated FCA heat source. In this context, the reduced carrier population in the Si/PolySi 3 wings rib directly translates into lower thermal load and weaker nonlinear feedback. For circulating powers below 80 mW, however, the Si/PolySi strip shows slightly higher heating than the Si strip due to increased intrinsic absorption losses in polysilicon.

Nonlinear optical effects are then summarized in Fig. 2.12, where the variations of modal loss ($\Delta\alpha$), refractive index shift due to free carriers

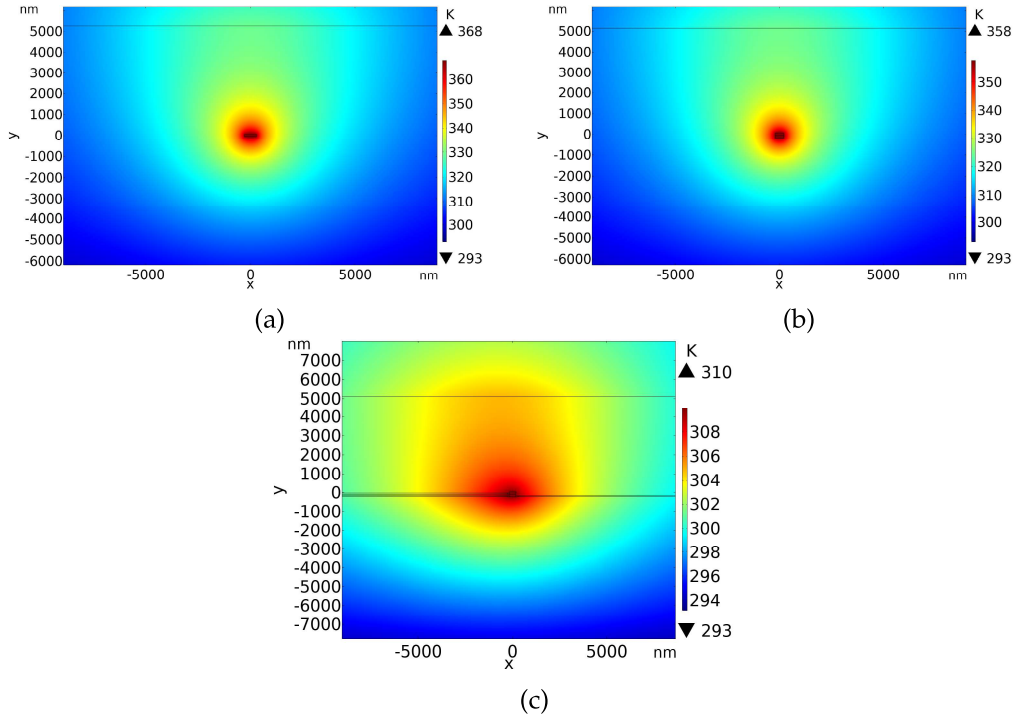


Fig. 2.10 Example of temperature increase due to self-heating at $P_c = 100, \text{mW}$ for (a) Si Strip, (b) Si/PolySi Strip and (c) Si/PolySi 3 wings Rib.

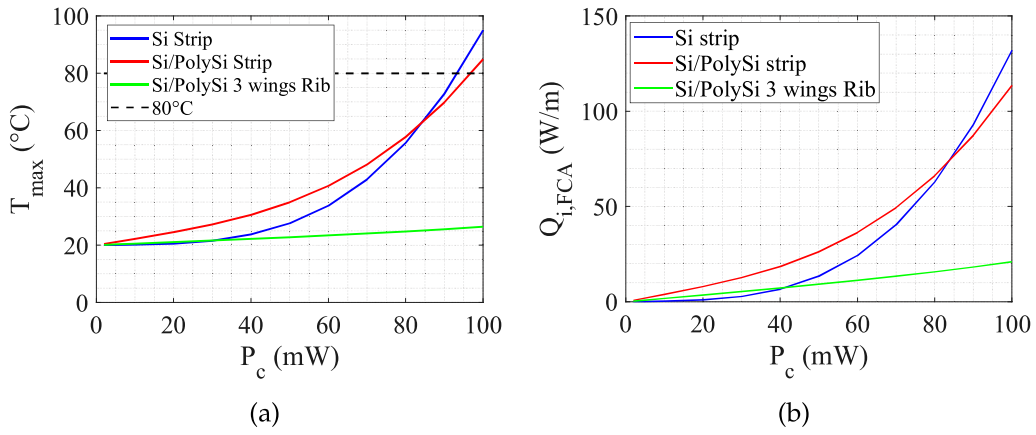


Fig. 2.11 (a) Maximum temperature reached by the MRRs versus circulating power for each structure. (b) FCA-induced heat source integrated across the waveguide cross-section.

$(\Delta n_{eff,FCD})$, and thermo-optic contribution $(\Delta n_{eff,T})$ are reported as functions of circulating power. In all cases, the thermo-optic contribution dominates

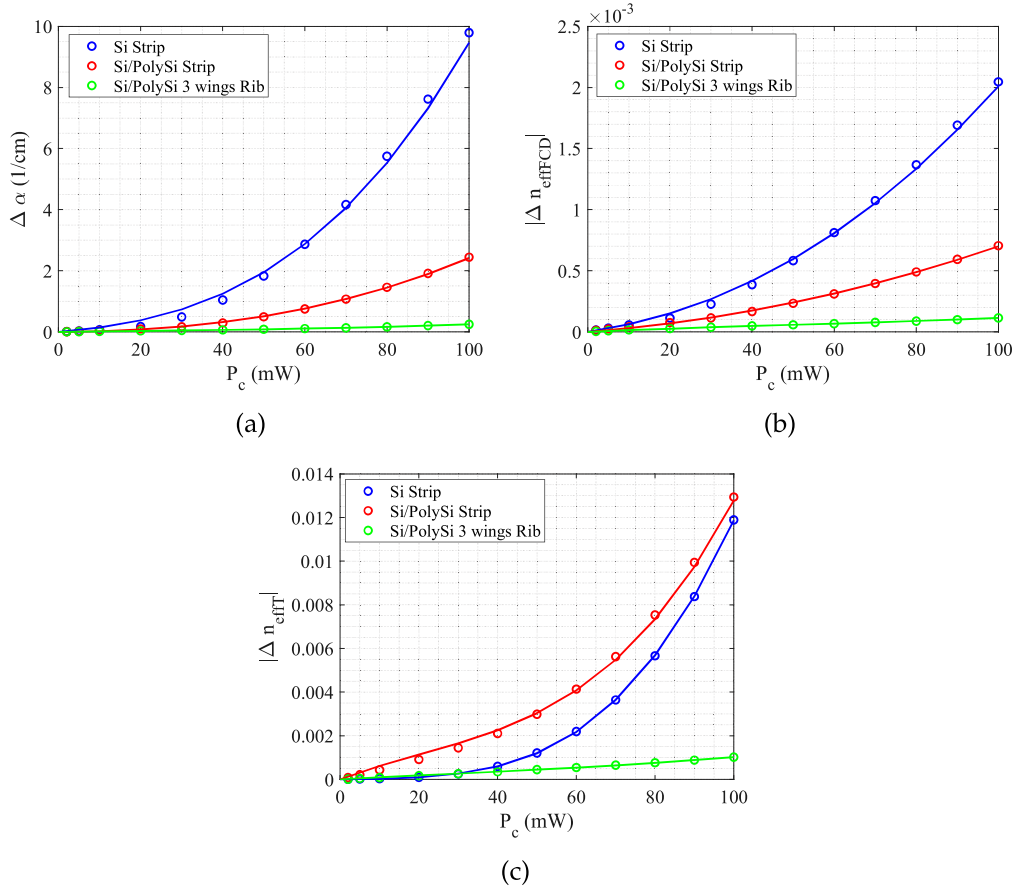


Fig. 2.12 Variations in modal loss and effective refractive index as a function of P_c are shown for each MMR cross-section examined in this study. The continuous curves correspond to polynomial fits of the numerical data indicated by the markers.

over free-carrier dispersion, leading to an overall red shift of the resonances. Among the investigated structures, the Si/PolySi 3 wings rib consistently exhibits the weakest nonlinear response.

These parameter variations are subsequently fitted using third-order polynomials as functions of circulating power. The resulting expressions are used as input for the transmission model introduced in Section 2.2.3, which enables calculation of the full MRR spectral response. The simulated transmission spectra for the three configurations are reported in Fig. 2.13, highlighting a reduced nonlinear wavelength shift in the Si/PolySi 3 wings rib structure.

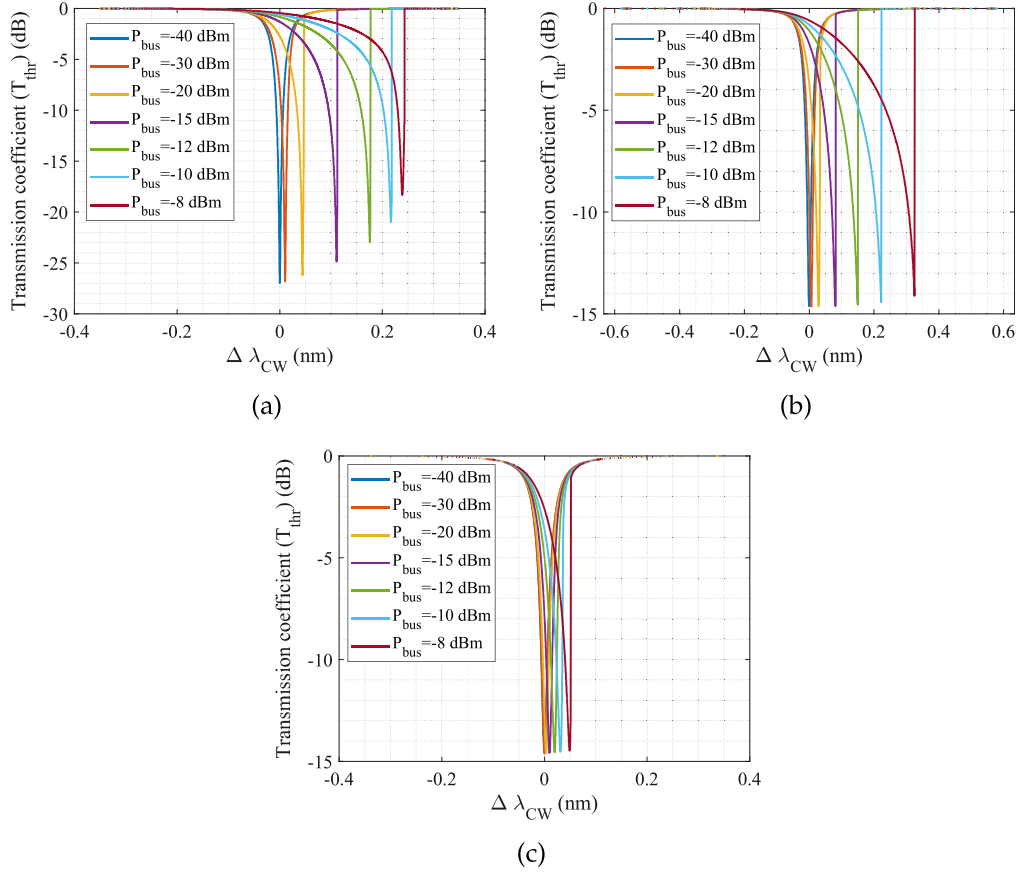


Fig. 2.13 Transmission spectra (in dB) for: (a) Si strip waveguide, (b) Si/PolySi strip configuration, and (c) Si/PolySi 3-wing rib structure, evaluated for different bus input power levels, with a radius of $3.5\ \mu\text{m}$ and $k^2 = 0.0055$.

The impact of nonlinearities on device performance is further quantified through the degradation of the quality factor, reported in Fig. 2.14a. The quality factor is defined as the ratio between the resonant wavelength and the full-width at half-maximum (FWHM) of the transmission peak, and is evaluated only within the stable operating regime (excluding bistability and oscillatory regimes). The Si/PolySi 3 wings rib consistently maintains a higher quality factor under increasing power, enabling operation at power levels approximately 10 dB higher (for a 10% Q degradation) compared to the standard Si strip waveguide.

To complement the quality factor analysis, an equivalent free-carrier lifetime is introduced to capture recombination dynamics in the optical mode volume. For electrons, it is defined as:

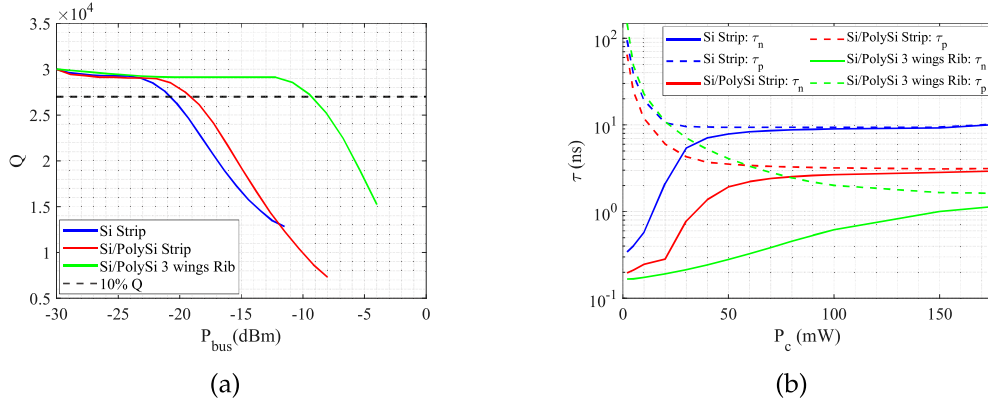


Fig. 2.14 (a) Reduction of 10% in the quality factor as a function of input bus power for the investigated structures. The microring resonator has a radius of $3.5 \mu\text{m}$ and a coupling coefficient of $k^2 = 0.0055$. (b) Effective electron and hole lifetimes as a function of the circulating power.

$$\tau_{eq,n} = \frac{\int \int_{A_{eq}} n(x,y) dx dy}{\int \int_{A_{eq}} G_{ph}(x,y) dx dy} \quad (2.26)$$

where A_{eq} represents the effective region containing 90% of the optical mode energy. An analogous expression holds for holes using $p(x,y)$.

The resulting lifetimes, shown in Fig. 2.14b, reveal opposite trends for electrons and holes. This difference arises from the distinct capture cross sections, where hole trapping is roughly an order of magnitude lower than electron trapping [18]. At low power, this leads to longer hole lifetimes, while at higher power the increasing trap occupation modifies recombination dynamics, increasing electron lifetime and reducing hole lifetime. In all configurations, polysilicon-based structures exhibit shorter carrier lifetimes at high power due to enhanced defect-assisted recombination and stronger diffusion effects.

2.4 Comparison with experiments

This section compares the model described above with experimental measurements provided by Cisco Systems. The data were obtained from the characterization of two ring resonators with different waveguide cross-sections,

namely silicon and polysilicon. The comparison is used to assess the accuracy of the model in predicting the optical response of the devices and to evaluate its validity across different material platforms.

2.4.1 Ring with Si/PolySi rib waveguides

The first experimental validation is carried out on a Si/PolySi two-waveguide rib microring resonator, as shown in Fig. 2.15, with a radius $r = 10\mu\text{m}$ (bend losses of fundamental mode: 0.038 dB/cm so they are negligible) and estimated coupling coefficients $k_{1,2}^2 = 0.132$ for the two bus waveguides.

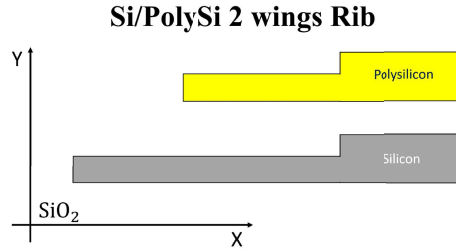


Fig. 2.15 Cross-sectional view of the complex waveguide geometry: Si/PolySi 2 wings Rib

Sixteen transmission spectra were provided, recorded as a function of the bus waveguide input power in the range from -6 dBm to 9 dBm, over a wavelength span from 1280 nm to 1320 nm, as illustrated in Fig. 2.16(a), which reports two representative measurements. For each resonance peak, the evolution of both the resonance wavelength and the transmitted power is analysed in order to investigate how nonlinear effects are influenced by increasing bus power.

From Figs. 2.16(b) and 2.16(c), it can be observed that, as the bus power increases, the transmission spectrum undergoes a clear red shift. This behaviour, i.e., a shift towards longer wavelengths, indicates that thermal effects are dominant in this regime, with self-heating being the prevailing nonlinear mechanism affecting the resonator response.

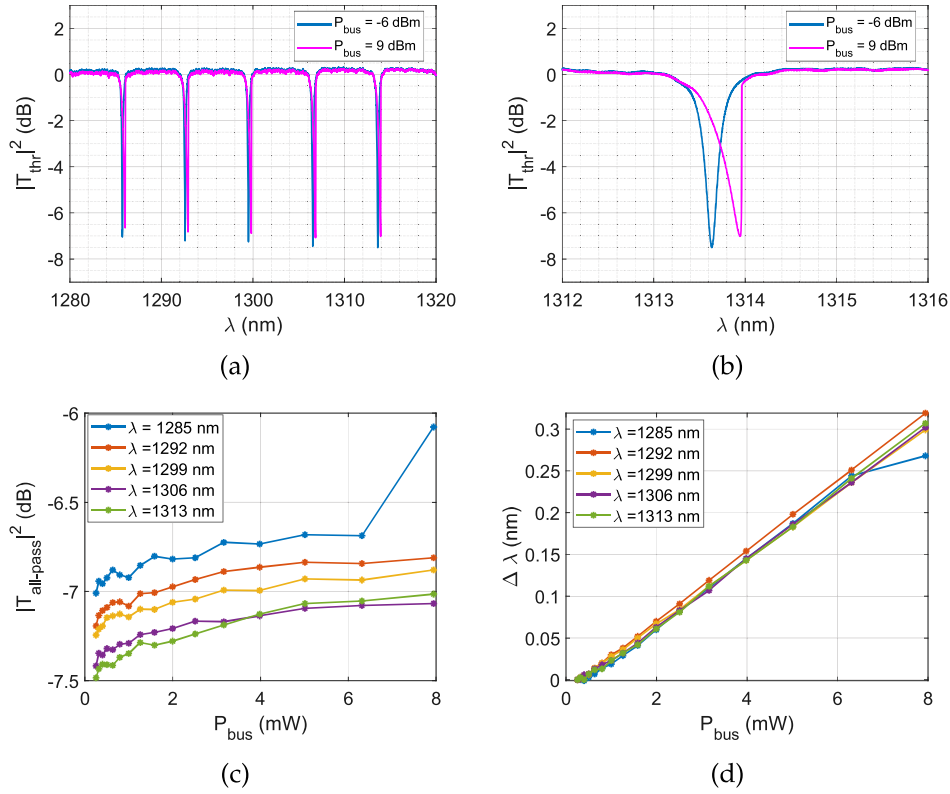


Fig. 2.16 (a) Measured transmission spectra of the ring resonator for different input powers in the bus waveguide (-6 dBm and 9 dBm); (b) zoomed view around 1313 nm. (c) Variation in transmitted power (in dB) for each resonance peak as a function of the bus input power. (d) Corresponding wavelength shift of each resonance peak as a function of the bus input power.

Subsequently, the device was simulated using COMSOL to extract the power-dependent variations of $\Delta\alpha$, $\Delta n_{\text{eff,FCD}}$, and $\Delta n_{\text{eff,T}}$, which were then incorporated into the transmission model described in Section 2.2.3. Fig. 2.17(a) and 2.17(b) report the simulated transmission coefficient and the circulating power as a function of the bus input power. It can be observed that the transmission spectrum exhibits a red shift, while the circulating power shows a nonlinear response with a hysteresis behavior toward longer wavelengths, consistent with a forward wavelength sweep.

Finally, Fig. 2.18 compares the experimental measurements with the model predictions. A good agreement is observed in terms of the resonance wavelength shift. This result is obtained by assuming the following bulk trap densities: $N_{t,\text{Si}} = 4.889 \times 10^{16} \text{ cm}^{-3}$ and $N_{t,\text{poly}} = 5.36 \times 10^{17} \text{ cm}^{-3}$.

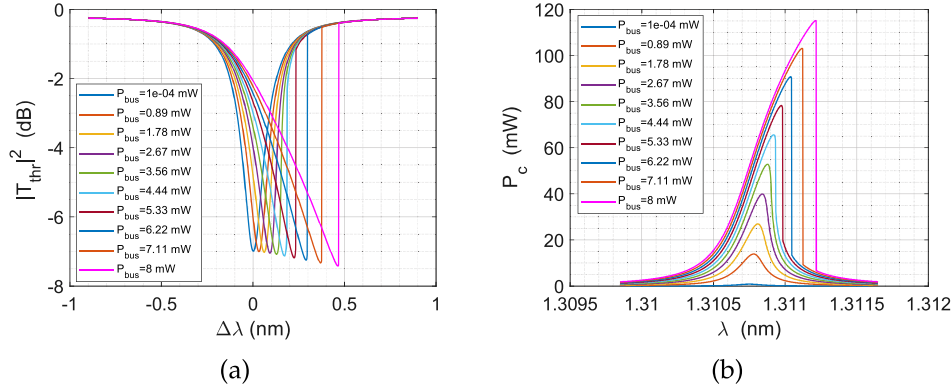


Fig. 2.17 (a) Transmission coefficient and circulating power (b) as a function of wavelength for each input bus power.

On the other hand, the agreement in terms of transmitted power variation remains limited and calls for further refinement of the model. With the current set of parameters, an increase in the circulating power leads to a critical coupling condition due to the associated rise in losses. Addressing this issue requires additional information on the device parameters, such as the doping profile of the waveguide, which was not available and would need to be provided by the company.

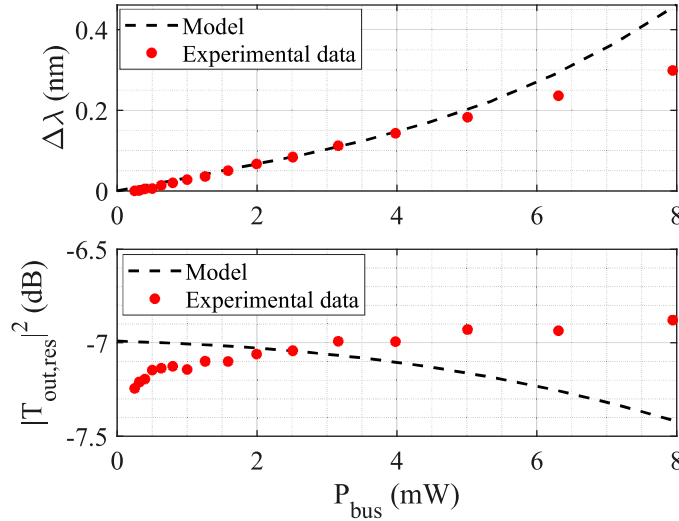


Fig. 2.18 Comparison between measured and modeled variations of the resonant wavelength (a) and transmission coefficients (b) as a function of CW power for an MRR based on a Si/PolySi 2 wings rib waveguide with a radius of $10\mu m$.

2.4.2 Ring with Si/PolySi strip waveguides

The second experimental validation is performed on a Si/PolySi strip microring resonator operating at 1550 nm, with radius $r = 2\ \mu\text{m}$ and estimated coupling coefficients $k_1^2 = 0.08$ and $k_2^2 = 0.09$ corresponding to the two bus waveguides [27]. The same data processing of the experimental measurements and the simulation procedure are repeated for this second device, as previously described in the preceding section.

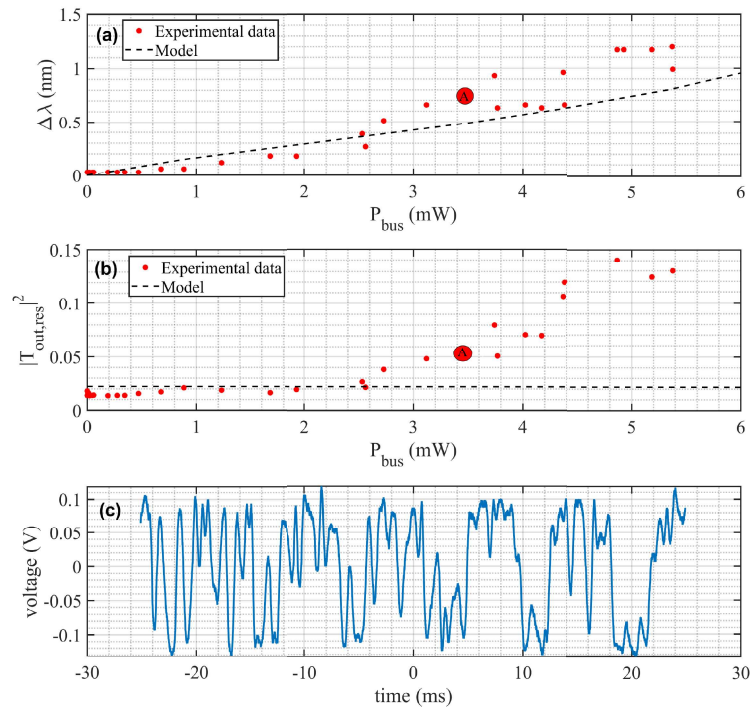


Fig. 2.19 Comparison between measured and modeled variations of the resonant wavelength (a) and transmission coefficients (b) as a function of CW power for an MRR based on a Si/PolySi strip waveguide with a radius of $2\ \mu\text{m}$. Temporal fluctuations of the CW power at the through port are also shown for a bus power of 3.5 mW (point A in Fig. 14(a)-(b)).

Optical characterization is performed by injecting a tunable laser into the bus waveguide and measuring the through-port transmission. The spectrum is obtained by sweeping the laser wavelength across a selected resonance λ_0 , from shorter to longer wavelengths. The measurements are carried out using an Agilent 81980A tunable laser source with a sweep rate of 5 nm/s [15].

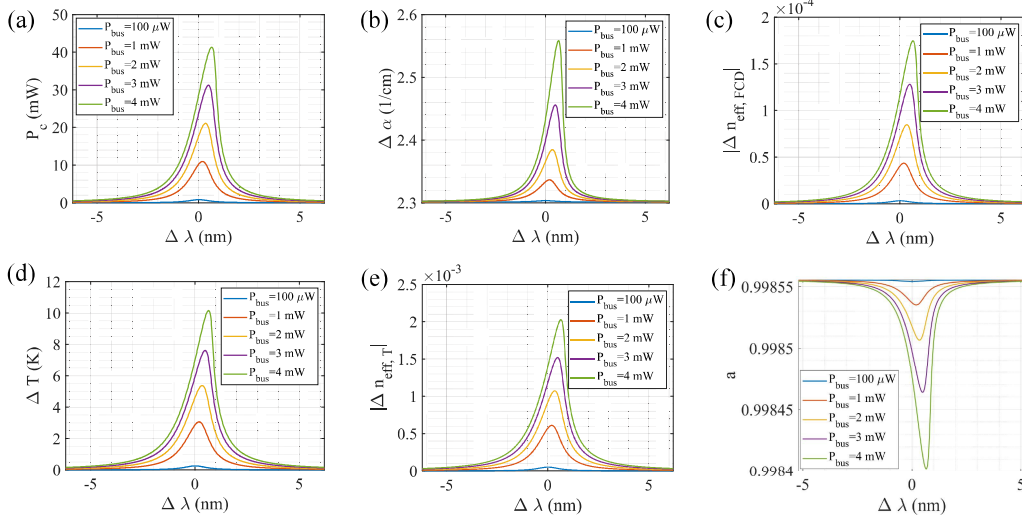


Fig. 2.20 Effect of the bus power on selected figures of merit of the ring resonator as a function of wavelength: (a) circulating power P_c , (b) change in modal losses $\Delta\alpha$, (c) effective refractive index variation due to free-carrier dispersion $\Delta n_{\text{eff}, \text{FCD}}$, (d) temperature increase induced by self-heating ΔT , (e) refractive index variation due to thermal effects $\Delta n_{\text{eff}, T}$, and (f) propagation loss a .

Figure 2.19(a) compares the measured and simulated resonance shift as a function of bus power under forward wavelength sweep conditions. To achieve agreement between model and experiment, trap densities of $4.89 \cdot 10^{16} \text{ cm}^{-3}$ in silicon and $5.36 \cdot 10^{17} \text{ cm}^{-3}$ in polysilicon are assumed, together with surface trap densities of $2.18 \cdot 10^{11} \text{ cm}^{-2}$ and $2.59 \cdot 10^{12} \text{ cm}^{-2}$, respectively.

Figure 2.19(b) shows the corresponding transmission at resonance as a function of input power. Good agreement is obtained up to approximately 4 mW, while deviations at higher power are attributed to the onset of self-oscillations, where the system no longer supports a steady-state transmission spectrum. In this regime, the measured signal represents a time-averaged oscillatory response rather than a static operating point, making direct comparison with the steady-state model invalid.

A progressive approach of the input wavelength toward resonance results in an increase of intracavity power (Fig. 2.20(a)), which in turn enhances nonlinear effects. This leads first to stronger free-carrier generation and a corresponding reduction in refractive index via FCD (Fig. 2.20(c)), producing an initial blue shift. Simultaneously, carrier recombination induces heating

(Fig. 2.20(d)), which increases the refractive index through the thermo-optic effect (Fig. 2.20(e)). Since the thermal contribution dominates, the overall resonance shift is toward longer wavelengths. Despite these strong internal dynamics, the extinction ratio remains nearly unchanged (Fig. 2.20(f)), confirming that propagation losses remain close to unity within the studied range.

Overall, Fig. 2.20 summarizes the interplay between optical, electrical, and thermal effects governing the nonlinear response of the MRR up to 4 mW of bus power. Above this threshold, deviations arise due to the transition into self-oscillatory operation, where the steady-state model is no longer applicable.

2.5 Comparison between lumped and distributed model

The proposed distributed model provides a more physically detailed description of silicon and polysilicon microring resonators by explicitly accounting for the spatial distributions of optical absorption, free carriers, and temperature. Although lumped models can often reproduce the overall stationary response after suitable calibration, the distributed formulation significantly reduces the reliance on geometry-dependent effective parameters and enables the investigation of local physical phenomena that cannot be directly accessed with lumped approaches.

In lumped models, the complex spatial behaviour of the device is represented through a limited number of averaged quantities. As a consequence, several effective parameters must be introduced to account for the underlying physical processes. Typical examples include the effective thermal resistance, the effective mode area, and averaged carrier-related quantities, all of which generally require calibration for each device geometry [15].

A representative example is the modeling of two-photon absorption (TPA), where the nonlinear loss is commonly expressed as $\alpha_{\text{rad}} = \frac{\beta_{\text{TPA}}}{A_{\text{eff}}} P_c$, where A_{eff} is the effective optical mode area. The definition of A_{eff} depends both on the optical mode profile and the geometry of the waveguide [15].

In particular, for rib waveguides, no unique definition of the effective mode area exists, and different formulations may lead to different estimations of the nonlinear losses. Consequently, the use of effective geometric parameters introduces an additional source of uncertainty that must be compensated through calibration.

In contrast, the proposed distributed model explicitly incorporates the actual device geometry and solves the coupled optical, electrical, and thermal equations over the waveguide cross-section. Therefore, lumped quantities such as an effective mode area or an effective thermal resistance are no longer required, since the optical absorption, heat generation, carrier transport, and temperature fields are directly obtained from the governing physical equations. This approach naturally captures the interaction between the different physical mechanisms while preserving their spatial dependence.

Another important advantage of the distributed formulation is its capability to account for non-uniform physical phenomena. Heat generation is not assumed to be uniformly distributed over the device but is locally determined from the optical absorption profile, leading to a more realistic prediction of the temperature distribution and the associated dissipated power. Likewise, the model explicitly computes the spatial distribution of free carriers and their diffusion within the semiconductor.

Furthermore, the distributed model provides considerable flexibility in the description of carrier trapping mechanisms. Surface and bulk trap densities can be directly assigned to their physical locations within the computational domain, making it possible to investigate the influence of complex geometries and fabrication-dependent interfaces without introducing additional geometrical averaging. This feature is particularly advantageous for silicon photonic devices, where surface effects often dominate the carrier dynamics.

It should be noted that the proposed model still requires material-dependent parameters, such as the trap density and the trap energy level, whose values depend on the fabrication process and therefore cannot be determined a priori. These parameters must be identified from experimental measurements. However, unlike the effective quantities introduced in lumped formulations, they retain a direct physical meaning and represent intrinsic material prop-

erties rather than parameters introduced to compensate for the geometrical simplifications of the model. Consequently, once identified, they can be consistently employed to analyze different operating conditions without redefining geometry-dependent effective quantities.

The main differences between lumped and distributed modeling approaches are summarized in Table 2.2.

Table 2.2 Comparison between lumped and spatially resolved modeling approaches.

Aspect	Lumped model	Distributed model
Geometry	Effective parameters	Actual geometry
Spatial information	Not available	Spatial resolution
Thermal behavior	Thermal resistance	Heat diffusion equation
Optical losses	A_{eff} required	Optical field calculation
Heat generation	Spatial average	Local distribution
Carrier transport	Average lifetime	Diffusion and recombination
Trap effects	Required calibration	Required calibration
Calibration	Geometry-dependent fitting	Physical parameters
Outputs	Global quantities	Global + spatial maps

Overall, lumped models remain valuable tools for compact simulations because of their simplicity and computational efficiency, particularly when only global stationary quantities are of interest, as shown in Fig. 2.21. However, the distributed approach provides a more comprehensive physical description by explicitly accounting for the interaction between optical, electrical, and thermal phenomena and by resolving their spatial distributions. Although these capabilities already improve the physical interpretation of stationary operating conditions, they are expected to become even more relevant in the analysis of transient phenomena, where spatial carrier diffusion and heat propagation play a fundamental role.

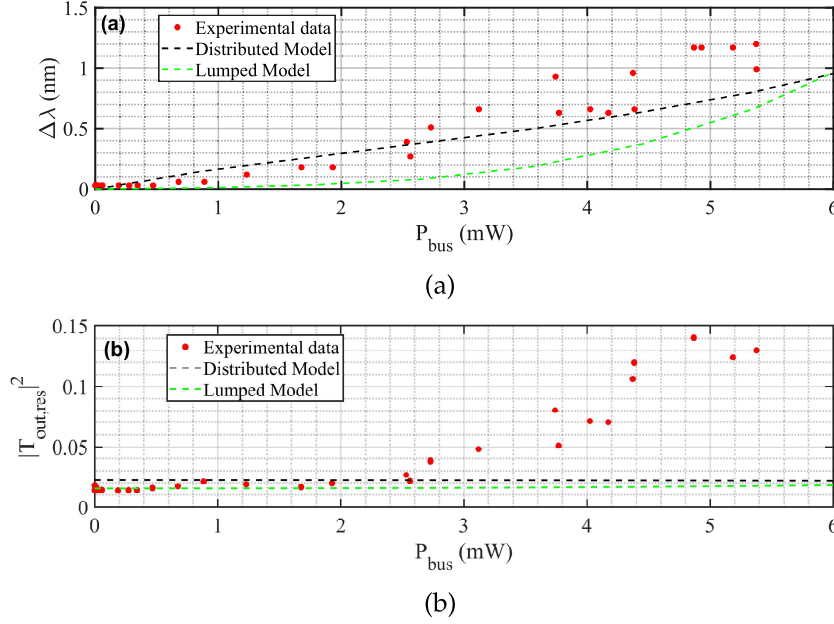


Fig. 2.21 (a) and (b) Comparison between the lumped (in green) and distributed model (in black) for the device analysed in Fig. 2.19.

2.6 Conclusion

In this work, we introduced a modelling framework capable of evaluating the two-dimensional spatial distribution of both free carriers and temperature in microring resonators (MRRs) with complex waveguide cross-sections within the SISCAP platform. By leveraging the finite element method (FEM), the proposed approach achieves a substantial reduction in computational time compared to more resource-intensive techniques such as FDTD.

A key strength of this methodology lies in its applicability to geometries that cannot be approximated as standard strip waveguides. While conventional lumped models are well suited for simple structures, extending them to rib-based or more intricate cross-sections is nontrivial due to the underlying geometry. In these cases, the spatial dynamics of carrier diffusion play a fundamental role and must be explicitly resolved, making a distributed modelling approach such as the one presented here essential.

Through this framework, we identified an optimized waveguide configuration that mitigates both nonlinear effects and thermal accumulation. Among the investigated solutions, the Si/PolySi 3 wings rib geometry emerges as the most effective. Its improved performance originates from a combination of enhanced lateral carrier spreading and more efficient heat dissipation enabled by the rib structure. At the same time, the inclusion of polysilicon contributes to reduced bend losses and shorter carrier lifetimes, due to increased Shockley–Read–Hall (SRH) recombination associated with defect states in the material.

Overall, these results highlight the versatility and robustness of the proposed model as a powerful design tool for high-Q MRRs, enabling accurate analysis and optimization across a wide range of materials and geometrical configurations.

Chapter 3

Modelling and characterization of a hybrid integrated tunable laser

3.1 Instabilities and multimode dynamics in lasers

Hybrid integration of III–V gain sections onto silicon or silicon nitride photonics platforms combines the efficient gain and modulation properties of III–V semiconductors with the low-loss, wavelength-selective feedback provided by passive silicon nitride (SiN) waveguides, which act as external cavity mirrors. This configuration enables narrow-linewidth continuous-wave (CW) operation and tunable emission through Vernier-effect filters [32].

In this chapter, we focus on the dynamic behaviour of a hybrid integrated tunable laser that was characterised experimentally at the University of Twente. Although the device was originally designed for stable CW emission, we observed multimode optical spectra with nearly equidistant lines under specific operating conditions. These initial observations suggested the possible formation of an optical frequency comb (OFC), in analogy with numerical predictions for similar external-cavity configurations [33].

However, a detailed analysis of the intensity modulation (IM) response, optical spectra and radio-frequency (RF) measurements revealed that the device operates as a self-oscillator driven by undamped relaxation oscillations.

tions. The comb-like spectral appearance does not correspond to a stabilised mode-locking mechanism; rather, it arises from intrinsic oscillatory dynamics. Self-pulsing only emerges for large asymmetric detuning between the lasing frequency and the SiN feedback peak, a regime that is only partially accessible experimentally. It is important to note that the self-pulsing mechanism functions without the presence of saturable absorber sections. The present platform is distinct from alternative integrated platforms, such as Q-switching or Fano-laser-based pulsing [34], [35].

During my PhD I have contributed to the analysis of these lasers and I have used a time-delayed algebraic model to support these observations. This is the same model that was previously used to study CW stability and back-reflection sensitivity [6], [36]. Here, the model reproduces the onset of oscillatory instabilities and the resulting spectral features, including line spacing and envelope. As described in this chapter, this analysis confirms that relaxation oscillations underlie the observed dynamics.

3.2 Modelling

A system of time-domain algebraic equations is utilised to represent the hybrid laser, a method that is particularly suitable for describing the shift from continuous-wave (CW) emission to multimode operation, as well as for identifying the mechanisms responsible for instability. The present formulation incorporates delayed feedback, competition among modes, and nonlinear interactions between longitudinal modes. The model is defined by the following equations: [6], [33], [36].

$$E(t) = \frac{e^{i(\omega_s - \omega_0)\tau_{in}} \exp(f(\omega_s, N_s))}{r_{eff}(\omega_s)} A^-(t - \tau_{in}) + F(t) \quad (3.1)$$

$$\frac{dA_{c1}(t)}{dt} = \gamma_{c1} t_{tr} E(t - \tau_{in, SiN}/2) e^{-i\frac{\Delta\phi}{2}} - \gamma_{t1} A_{c1}(t) \quad (3.2)$$

$$\frac{dA_{c2}(t)}{dt} = \gamma_{c2} A_{c1}(t) - \gamma_{t2} A_{c2}(t) \quad (3.3)$$

$$\frac{dN(t)}{dt} = \frac{\eta_i I}{qV} - \frac{N(t)}{\tau_N} - v_g g_N \ln\left(\frac{N(t)}{N_0}\right) \sigma \frac{|E(t)|^2}{V} \quad (3.4)$$

with

$$A^-(t - \tau_{in}) = t_{tr} e^{-i\frac{\Delta\phi}{2}} A_{c2}(t - \tau_{in} - \tau_{in, SiN}/2)$$

$$f(\omega_s, N_s) = \frac{1}{\tau_{in}} \int_{t-\tau_{in}}^t L\left(1 + i\alpha_H \frac{\omega_s}{\omega_0}\right) g_N \ln\left(\frac{N(\bar{t})}{N_s}\right) d\bar{t}$$

and effective reflectivity for the SiN feedback mirror:

$$r_{eff}(\omega) = \frac{\gamma_{c1} \gamma_{c2} t_{tr}^2}{(\gamma_{t1} + i\Delta\omega)(\gamma_{t2} + i\Delta\omega)} e^{-i\Delta\phi} e^{-i\Delta\omega \tau_{in, SiN}}, \quad (3.5)$$

In this model, $E(t)$ represents the complex electric field at the output of the reflective semiconductor optical amplifier (RSOA), as shown in Fig. 3.1. The term $A_{c1}(t)$ denotes the external optical field evaluated at the drop port of microring resonator 1, while $A_{c2}(t)$ corresponds to the optical field after propagation through microring resonator 2. The temporal evolution of the carrier population inside the RSOA active region is described by the carrier density $N(t)$.

The round-trip time of the internal RSOA cavity is denoted by τ_{in} , whereas $\tau_{in, SiN}$ accounts for the additional time delay introduced by propagation through the bus waveguide sections of the SiN circuit. The phase accumulation induced by the actively controlled phase section is modeled through the phase shift parameter $\Delta\phi$.

The effective reflectivity of the external cavity is characterized by the coefficients $\gamma_{c1,2}$ and $\gamma_{t1,2}$. In particular, $\gamma_{c1,2}$ represent the coupling rates of the optical field from the bus waveguide into microrings 1 and 2, respectively,

while $\gamma_{t1,2}$ quantify the corresponding loss rates of the optical field due to intrinsic and coupling-related dissipation mechanisms [36], [37].

$$\gamma_{c1,2} = \frac{v_{g1,2} k_{RR} (1 - \eta)}{2\pi R_{1,2} t_{RR} A_{loss1,2}^{1/4}} \quad (3.6)$$

$$\gamma_{t1,2} = \frac{v_{g1,2} (1 - t_{RR} A_{loss1,2}^{1/2})}{2\pi R_{1,2} t_{RR} A_{loss1,2}^{1/2}} \quad (3.7)$$

where the transmission coefficient, denoted by t_t , is defined as the transmission through the spot size converter, denoted by t_{SSC} , and the MRR-to-SOA transmission of the MZI output coupler between the SSC and the rings. This is expressed as follows: $t_{tr} = t_{SSC} \sqrt{1 - T_{C,out}}$.

Finally, the term $F(t)$ is introduced as a Langevin noise source, which models the contribution of spontaneous emission coupled into the optical mode of interest.

The following solutions for the steady (CW) states are obtained by solving the oscillation condition for modulus and phase:

$$\Delta\omega_s = \omega_s - \omega_0 = \frac{v_g}{2L} \left[\alpha_H (-\log(r_{HF}|r_R(\omega_s)|)) + \alpha_i L - 2m\pi + \phi_R(\omega_s) \right] \quad (3.8)$$

$$N_s = N_0 \exp \left[\frac{-\log(r_{HR}|r_{eff}(\omega_s)|) + \alpha_i L}{L g_N} \right] \quad (3.9)$$

where ω_s is the angular frequency associated with the continuous-wave (CW) steady-state solution and they are associated with different values of m . A reference angular frequency $\omega_0 = 2\pi c / \lambda_0$ is introduced, corresponding to the wavelength λ_0 at which the effective reflectivity of the external cavity reaches its maximum. Frequency detuning is therefore defined as $\Delta\omega = \omega - \omega_0$. $\Delta\omega$ is associated with discrete values of threshold carrier densities N_s . The minimum of N_s corresponds to the lasing mode.

Additional details regarding the underlying assumptions of the model, the normalization of the variables, and the specific values of the simulation and device parameters are provided in previous publications [6], [33], [36].

3.3 Experimental Setup

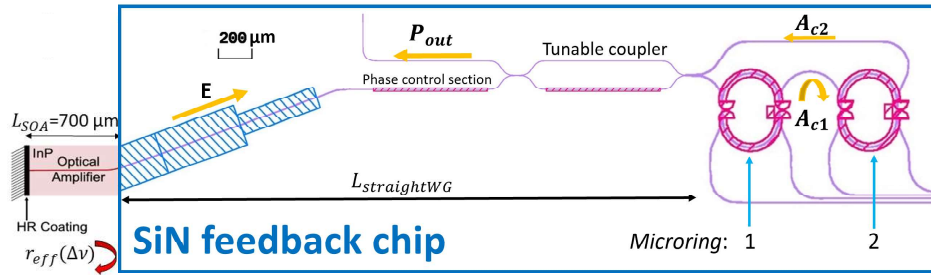


Fig. 3.1 Layout of the hybrid III-V/SiN integrated tunable laser.

The hybrid integrated laser analyzed in this work is depicted in Fig. 3.1 and was designed and fabricated in the group of Prof. Boller at University of Twente (The Netherlands), where its main specifications are summarized in Table 3.1. The device comprises an InP chip with high-reflection/anti-reflection (HR/AR) facet coatings, manufactured by the Fraunhofer Heinrich Hertz Institute (HHI, Berlin, Germany). The chip integrates a semiconductor optical amplifier (SOA) [38], [39] operating around 1550 nm and offering a gain bandwidth of roughly 130 nm.

Table 3.1 Main parameters of hybrid tunable integrated laser

Parameter	Symbol	Value
SOA		
Length	L_{SOA}	700 μm
Effective refractive index	n_{eff}	3.15
Group refractive index	n_g	3.6
Modal gain coefficient	g_N	30 cm^{-1}
Linewidth Enhancement Factor	α_H	5
SiN mirror		
Circumference of MRR 1	C_1	850 μm
Circumference of MRR 2	C_2	860 μm
Coupling coefficient	k^2	0.06
Quality factor of MRRs	Q	~ 40000
Bus waveguides length	$L_{straightWG}$	5200 μm
Waveguide propagation losses	α_0	0.1 dB/cm
Waveguide effective refractive index	n_{effSiN}	1.53
Waveguide group refractive index	$n_{g,SiN}$	1.715
Power output coupler coefficient	$T_{c,out}$	0.8
Spot size converter	t_{SSC}	-2 dB
Working wavelength	λ_0	1.55 μm

Its active region is based on a multi-quantum-well architecture designed to deliver high optical gain together with low noise performance across the C-band. To suppress parasitic reflections at the chip facet, the InP waveguide is tilted by 9° with respect to the low-reflectivity interface. The chip is edge-coupled to a silicon nitride (SiN) photonic circuit fabricated by LioniX International, which incorporates a pair of coupled microring resonators (MRRs). Using the Vernier effect, this configuration achieves a narrow and strongly frequency-selective effective reflectivity, exhibiting a full width at half maximum (FWHM) of 2.67 GHz, as shown in Fig. 3.3. The waveguides are realized using the TriPleX Asymmetric Double Stripe (ADS) platform [40], characterized by $\text{Si}_3\text{N}_4/\text{SiO}_2$ core structures. This technology enables compact bending radii while preserving ultra-low propagation losses (below 0.1 dB/cm), a key requirement for implementing high-quality-factor resonators. A dedicated phase section (PS) control, integrated with thin-film thermoelectric heaters analogous to those employed for the MRRs, allows

precise adjustment of the emission frequency with respect to the Vernier filter maximum. Through thermo-optic control, the laser wavelength can be tuned continuously over several nanometers, achieving sub-GHz resolution. This is achieved by the dissipation of power by the heaters, which results in local temperature variations and, consequently, modifications to the effective refractive index of the waveguide. The devices are mounted on a Peltier module and temperature-stabilized at 23 °C by means of a TEC controller (ILX Lightwave LDT-5910). The experimental setup has been reported previously in [32], [38] and it is illustrated in Fig. 3.2. This set-up was implemented by the University of Twente, measurements were performed by the University of Twente and made available for my PhD project in the framework of an informal collaboration. The main components used for the measurements are summarized as follows:

- Optical spectra were acquired using an optical spectrum analyzer (OSA, Coherent–Finisar WaveAnalyzer 1500S) with a 150 MHz resolution bandwidth.
- Radio-frequency (RF) spectra were obtained via a high-speed photodiode (Discovery Semiconductors DSC30S) connected to an electrical spectrum analyzer (ESA, Keysight Technologies N9000B CXA).
- For small-signal intensity modulation measurements, the DC injection current from a precision current source (ILX Lightwave LDX-3207) was combined with the RF signal from a vector network analyzer (VNA, Keysight Technologies P9375A) using a Bias-T (Mini-Circuits ZFBT-4R2GW, 0.1 MHz–4.2 GHz). The intensity modulation (IM) response was obtained from the VNA S21 parameter.

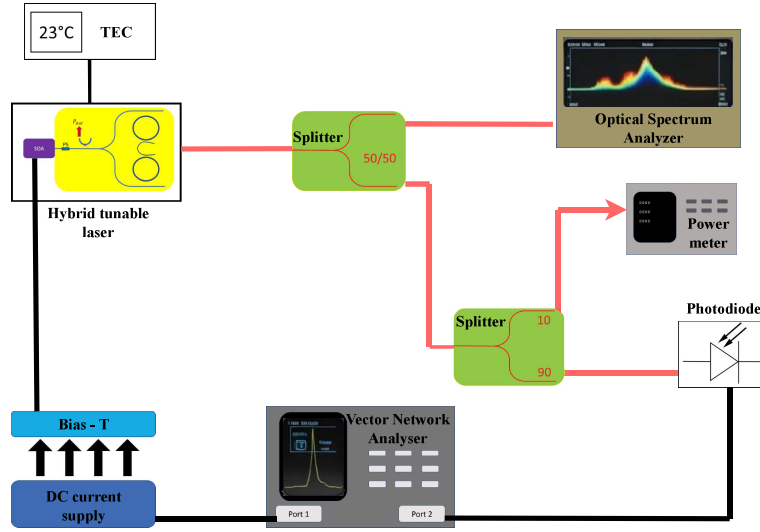


Fig. 3.2 Experimental setup at Univeristy of Twente for the measurements: optical spectrum, RF spectrum and IM response.

3.4 Measurements and numerical simulations: onset of self-oscillator

In the following section, I analyse three operating conditions preceding the onset of instability. These conditions are fixed at a bias current of 195 mA, while varying the voltage applied to the phase section: 0 V, 4 V, and 8 V to detune the lasing mode with respect to the peak of the effective reflection coefficient of the SiPh mirror. The scope of these conditions is to investigate how the system approaches the unstable regime as the effective detuning is progressively modified. As a preliminary step, the numerical model is calibrated by identifying the simulated detuning value that reproduces the experimental measurements at 195 mA. The matching procedure is performed sequentially, with the intensity modulation (IM) response being compared first, followed by the optical spectrum, and finally the radio-frequency (RF) spectrum. This approach guaranties the uniformity between the observed dynamical characteristics and the predictions of the time-delayed model.

The numerical simulation is performed using the time-delayed model presented in section 3.2, with the parameters reported in Table 3.1, but prop-

agation losses in the SiN waveguide are neglected due to their negligible magnitude in fabricated devices. After identifying the appropriate detuning value through comparison with the experimental data, we proceed to examine the steady-state solutions and their relation to the external cavity free spectral range (FSR) and to the distance between the spectral lines observed experimentally. This comparison provides insight into the physical origin of the emerging oscillatory dynamics and clarifies whether the line spacing is determined by cavity resonances or by intrinsic relaxation-driven mechanisms.

3.4.1 Solutions for steady state

In order to compare the experimental measurements of optical and RF spectrum with the numerical simulations, the simulated laser frequency detunings, relative to the reflectivity peak, are identified as follows: $\Delta\nu_s = -0.62$ GHz at a PS voltage of 0 V (blue), $\Delta\nu_s = -1.33$ GHz at 4 V (magenta), and $\Delta\nu_s = -1.42$ GHz at 8 V (green). From a theoretical point of view, I analyze the steady-state solutions of oscillation condition are defined in equations 3.8 and 3.9 and then shown in Fig.3.3a.

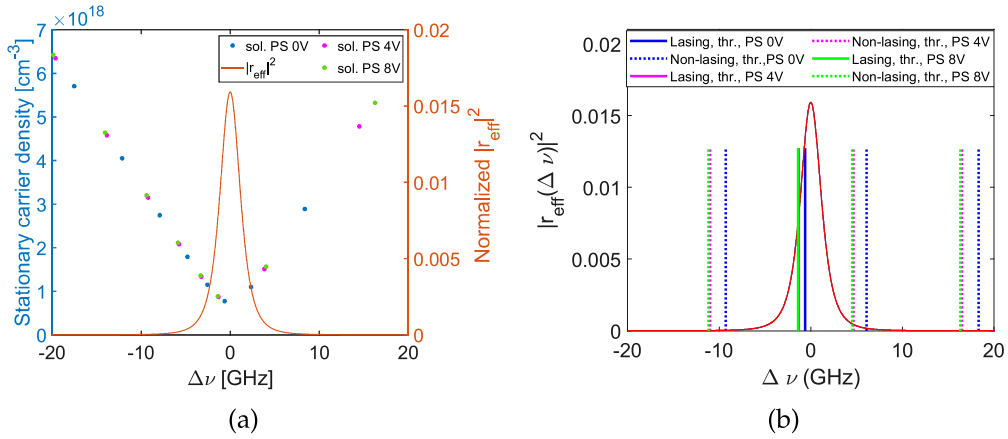


Fig. 3.3 (a) The graph illustrates the steady states solutions of the oscillation condition, in circles, and the associated carrier density at threshold (on the left), instead the red line represents the narrowband effective reflectivity of the SiN passive mirror, with a FWHM of 2.67 GHz. (b) The lasing mode at threshold is the solid line, while the dotted lines are non-lasing longitudinal modes at threshold.

Each circle corresponds to a cavity mode with simulated frequency detuning $\Delta\nu_s$ relative to the reflectivity peak, together with the associated carrier density N_s required to reach threshold. The lasing mode (M_0) is the minimum of the cavity mode and it is therefore characterized by a detuning $\Delta\nu_s$, while the remaining solutions display an asymmetric distribution with respect to M_0 . This asymmetry arises from the non-zero linewidth enhancement factor of the RSOA ($\alpha_H = 5$) and from the nonlinear dependence of the reflectivity phase. Instead, Fig. 3.3b shows the steady-state solutions that were calculated with the carrier density clamped at the threshold, where the lasing mode at threshold is represented by the solid line, while the non-lasing longitudinal modes at threshold are shown as dotted lines, whose carrier density is fixed at the minimum value. As can be seen, the free spectral range (FSR) at threshold is 6 GHz, which is significantly higher than both the RO frequency and the distance between the lines in optical spectrum.

Furthermore, in these graphs we can see that the frequency-selective reflectivity has a Lorentzian shape with an FWHM of 2.67 GHz, meaning that this reflectivity is very narrow, so even a small detuning (~ 1 GHz) shift causes instability.

3.4.2 Intensity modulation response

Fig.3.4 shows, in red line, the measured IM responses with a limit to 5 GHz, because the bandwidth constraint of the bias-T used to supply both the DC bias and the modulation current. Consequently, this cut-off does not reflect the intrinsic modulation capability of the laser and should not be used as a reference when comparing with the numerical results. The experimental curves exhibit a resonance peak at 3.24 GHz that becomes increasingly pronounced as the phase-section voltage increases. The reduced damping of the oscillations suggests that the device progressively approaches a self-pulsing regime at higher PS voltage values. The corresponding numerical simulations are also plotted in black curves, and we find that the simulated detunings are $\Delta\nu_s = -0.62$ GHz in (a), -1.28 GHz in (b) and -1.30 GHz in (c) relative to the reflectivity peak. A slight variation (within $\pm 7\%$) of the simulated detuning was introduced with respect to the value used for the optical spectrum, in Fig. 3.5, since the optical and IM measurements were

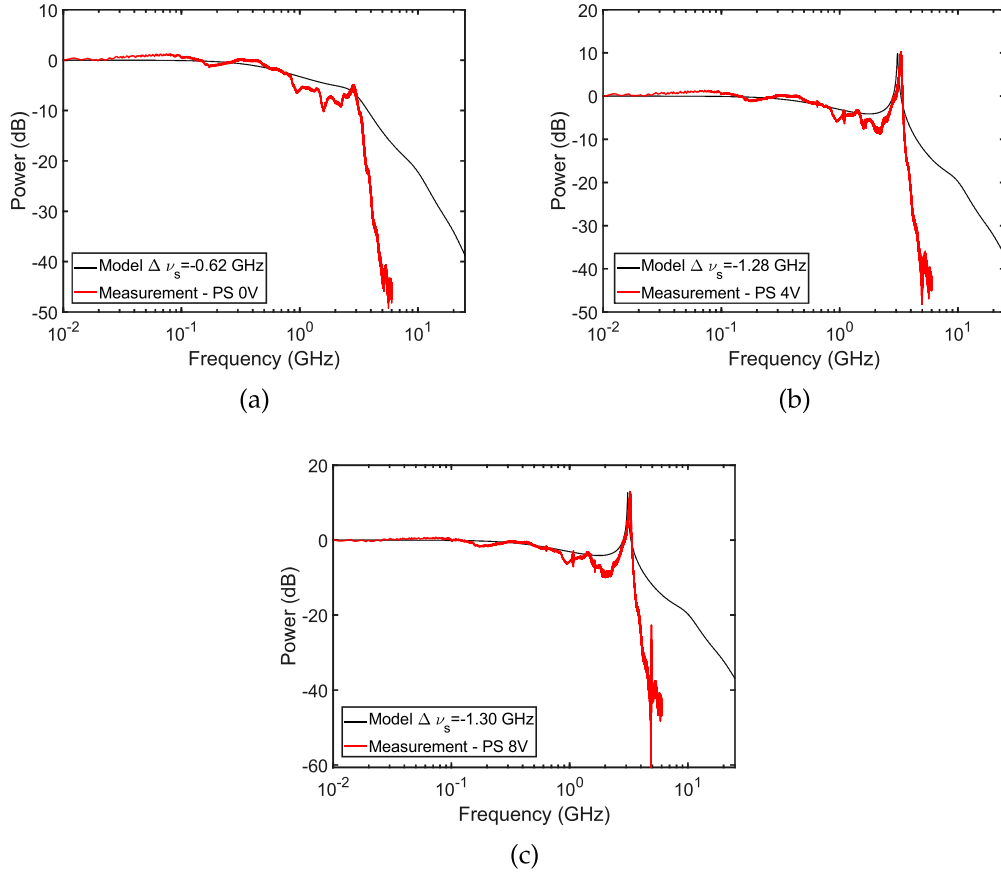


Fig. 3.4 Simulated (black) and measured (red) intensity modulation responses (IMRs) at a bias current of 195 mA under single-mode operation. In (a), the PS voltage is 0 V, corresponding to a simulated detuning of $\Delta\nu_s = -0.62$ GHz and stable CW emission. In (b), the PS voltage is 4 V with $\Delta\nu_s = -1.29$ GHz. In (c), the PS voltage is 8 V and $\Delta\nu_s = -1.30$ GHz, close to the CW instability threshold and the onset of self-pulsing.

performed at different times. Despite the experimental modulation response decreasing rapidly above 5 GHz, the simulated IM curves demonstrate a minor additional feature around 10 GHz. The secondary peak under consideration is attributed to the phenomenon of photon-photon resonance. To reproduce the measured IM response more realistically, a first-order low-pass filter (0.5 GHz cutoff) was applied to the simulated output power modulation in order to account for bandwidth limitations introduced by the experimental setup (including the photodiode, VNA, bias-T, and associated electrical connections).

3.4.3 Optical spectrum

Under the same operating conditions as the IM responses shown in Fig. 3.4, the Fig. 3.5 displays the measured (red) and simulated (blue) optical spectra. The bias current is fixed at 195 mA, while in the simulations the laser frequency detuning are $\Delta\nu_s = -0.62$ GHz in (a), -1.33 GHz in (b) and -1.42 GHz in (c) relative to the reflectivity peak.

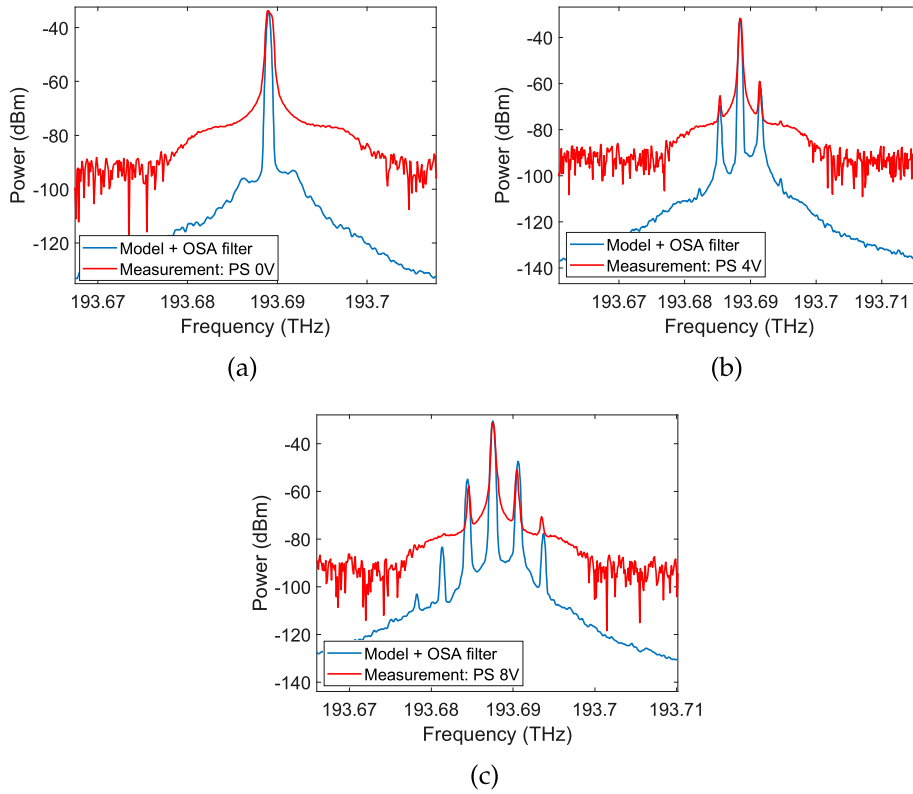


Fig. 3.5 Measured (red) and simulated (blue) optical spectra obtained under the same operating conditions as in Fig. 3.4. In the simulations, the detuning is varied within a $\pm 7\%$ range relative to the value considered in Fig. 3.4. The experimental phase-section (PS) voltage is set to 0 V in (a), 4 V in (b), and 8 V in (c). The measurements are limited to a dynamic range between 0 and -90 dBm due to technical noise constraints.

In the numerical results, the finite resolution bandwidth of the optical spectrum analyzer (OSA) is taken into account by applying a 150 MHz resolution filter. As the lasing frequency progressively moves away from the maximum of the SiN effective reflectivity, the sidebands around the

main lasing peak become more pronounced. In the pulsing regime, these sidebands develop into distinct spectral lines with a spacing of approximately 3.1 GHz, which is consistent with the relaxation oscillation (RO) frequency reported in Fig. 3.4. An asymmetry between the lower- and higher-frequency sidebands is observed in both experimental and simulated spectra. This effect originates from the asymmetric detuning between the laser emission frequency and the reflectivity peak of the passive feedback mirror.

3.4.4 Radio-frequency spectrum

In our model, the conversion from optical power fluctuations at the photodetector to the electrical power measured by the spectrum analyzer is treated consistently in both time and frequency domains. The analyzer measures a sinusoidal voltage characterized by a root-mean-square value $V_{\text{RMS}} = V_{\text{peak}}/\sqrt{2}$, so that the electrical power delivered to a load resistance R_L is $P_E = V_{\text{RMS}}^2/R_L$. Considering small fluctuations around the mean value, the corresponding electrical power variation is $\delta P_E(t) = \delta V_{\text{RMS}}^2(t)/R_L$. The photodetector output current propagates through a transmission line of impedance R_{cable} and is terminated by the analyzer input impedance R_L , leading to a voltage division such that $\delta V_{\text{RMS}}(t) = \frac{R_L}{R_{\text{cable}} + R_L} R_{\text{cable}} \delta i(t)$. Under impedance-matching conditions ($R_{\text{cable}} = R_L = 50\Omega$), this reduces to $\delta V_{\text{RMS}}(t) = \frac{1}{2} R_L \delta i(t)$. The photocurrent fluctuation is proportional to the optical power fluctuation through the responsivity $\mathcal{R}$, defined as $\mathcal{R} = 0.7 \text{ A/W}$, yielding $\delta i(t) = \mathcal{R} \delta P_O(t)$. Substituting into the electrical power expression gives $\delta P_E(t) = \left(\frac{1}{2} R_L \mathcal{R} \delta P_O(t) \right)^2 / R_L$, which simplifies to $\delta P_E(t) = \frac{R_L \mathcal{R}^2}{4} \delta P_O^2(t)$, thereby establishing the quadratic relation between optical power fluctuations and measured electrical power fluctuations. In the frequency domain, intensity noise is described by the power spectral density (PSD), expressed in W^2/Hz . The spectrum analyzer provides power measurements in dBm, which are converted to watts via $P_E[\text{W}] = 10^{(P_{\text{dBm}} - 30)/10}$. Since this power is integrated over the resolution bandwidth (RBW, equal to 300 MHz), the double-sided electrical PSD is $S_{P_E}(f) = 10^{(P_{\text{dBm}} - 30)/10} / \text{RBW}$. Using the previously derived proportionality between electrical and optical power, the optical power spectral density follows as $S_{P_O}(f) = \frac{2 \cdot 4}{R_L \cdot \mathcal{R}^2} S_{P_E}(f)$,

yielding the final expression $S_{P_O}(f) = \frac{2.4}{R_L \cdot \mathcal{R}^2} \frac{10^{(P_{dBm} - 30)/10}}{\text{RBW}}$. For numerical simulations, the optical PSD is estimated from the discrete time series $\delta P_O(t_n)$ using the Fast Fourier Transform, leading to the double-sided estimator $S_{P_O}(f) = |\text{FFT}\{\delta P_O(t)\}|^2 \Delta t / N$. This formalism guarantees full consistency between experimental RF measurements and numerical modeling of optical power fluctuations.

As illustrated in Fig. 3.6, a detailed comparison is presented between the measured and simulated RF spectra. In this instance, the initial harmonic of frequency 3.1 GHz appears within the experimental RF spectrum, a phenomenon that is congruent with the results of our simulations.

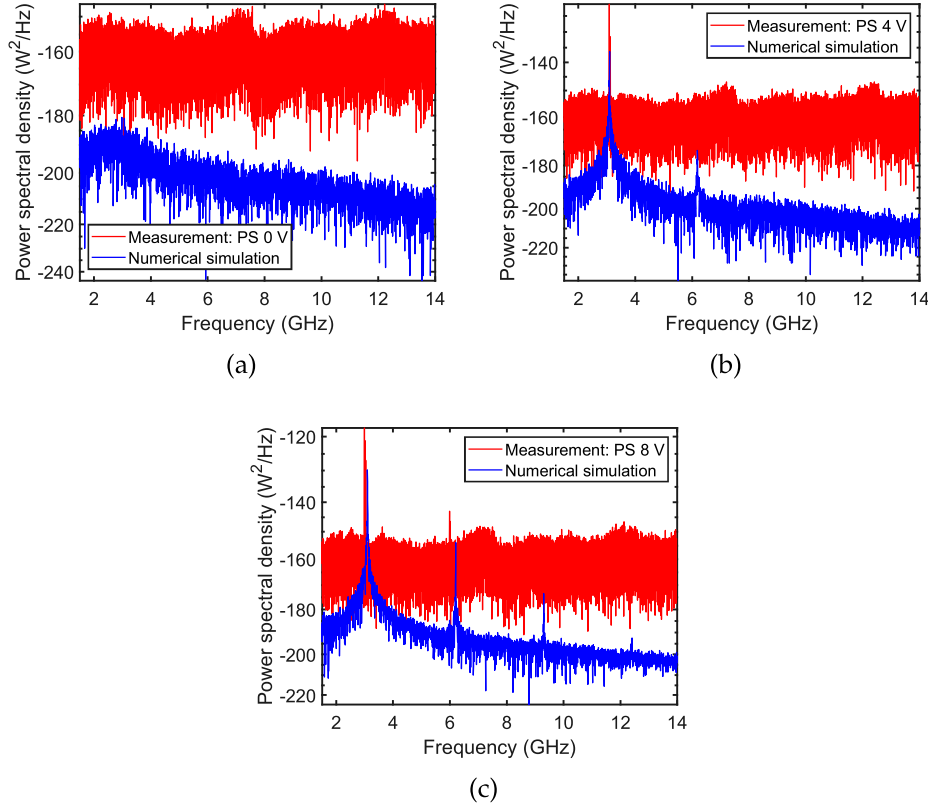


Fig. 3.6 The red (measured) and blue (simulated) curves show the RF spectra under the same operating conditions as in Fig. 3.5.

Also in this RF spectrum, we can observe that as the absolute value of the detuning increases, the laser transitions from a single-mode regime to a pulsing regime. In fact, the SMSR of the first harmonic increases due to the sidebands becoming more pronounced and beating with the main lasing

peak. In the case of Fig. 3.6(c), two additional harmonics appear, further confirming the onset of the multimode regime.

3.4.5 Carrier density, instantaneous frequency, output power intensity

Figure 3.7 illustrates the time evolution, between 100 and 102 ns, of three key quantities obtained by the simulations in the same PS conditions discussed in the previous paragraph.

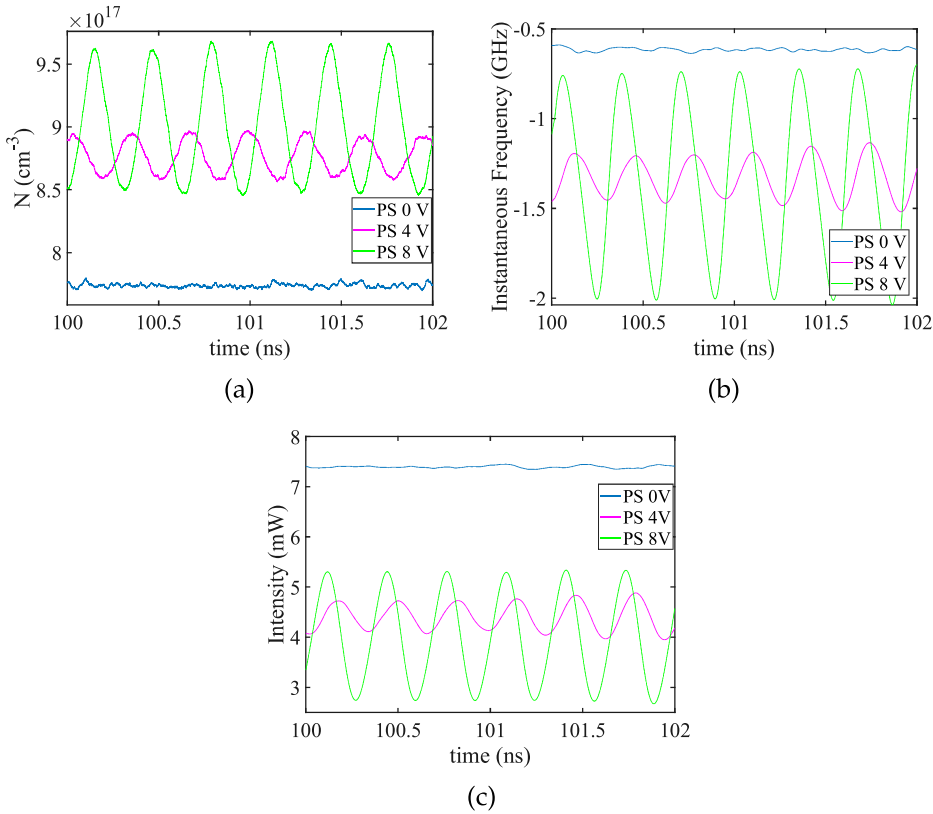


Fig. 3.7 Time evolution (100–102 ns) of: (a) carrier density N , (b) instantaneous frequency (GHz), and (c) output optical intensity (mW) for three values of the PS voltage: 0 V (blue), 4 V (magenta), and 8 V (green).

The laser behaviour strongly depends on the applied PS voltage. At PS = 0 V, the carrier density N , instantaneous frequency, and output optical intensity exhibit only minimal, random fluctuations, characteristic of a continuous-wave (CW) regime. Instead the PS voltage increases to 4 V and 8

V, clear periodic oscillations emerge in all three quantities, with the largest amplitude observed at 8 V, indicating the onset of a self-oscillatory regime. This shows that increasing the PS voltage enhances dynamic modulation: the carrier density, instantaneous frequency, and optical intensity all display progressively stronger periodic variations, reflecting resonance and beating phenomena typical of sustained oscillations.

In summary, the laser remains in the CW regime when only stochastic fluctuations are present, while the appearance of well-defined oscillations marks the transition to the self-oscillatory regime, which occurs when the laser detuning is applied toward the lower-frequency side the Vernier mirror resonance.

3.5 Approaching self-pulsing regime

In this section, I analyze the laser behavior when it is already operating in the self-oscillating regime, approaching the self-pulsating regime. The measurements were performed with a bias current of 180 mA and a phase-section voltage of 7.42 V, considering both the optical spectrum and the RF spectrum. Using the model, I first perform a comparison with the experimental results and then carry out a theoretical analysis of the steady-state solutions, the output optical intensity, the carrier dynamics, and the instantaneous frequency, in order to gain a deeper understanding of the self-pulsation phenomenon.

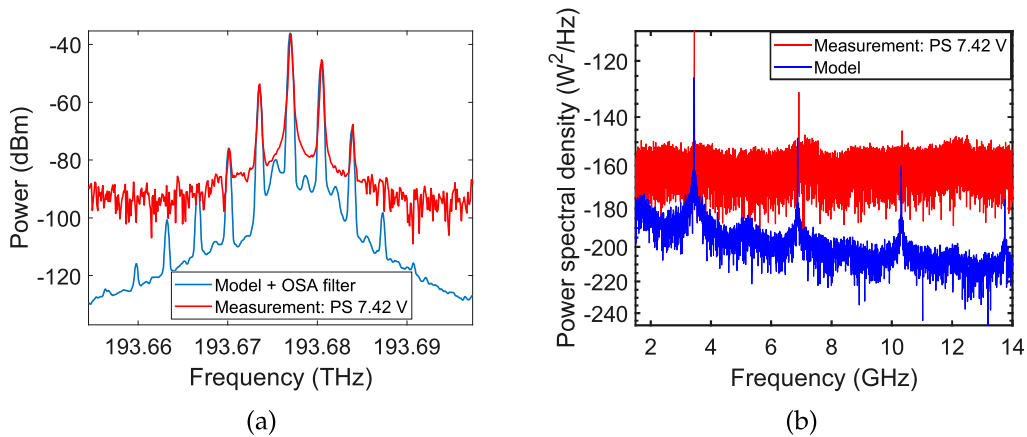


Fig. 3.8 Experimental results: optical spectrum and RF spectrum at 180 mA.

The numerical simulations, shown as blue curves and in good agreement with the experimental measurements, indicate a simulated detuning of $\Delta\nu_s = -1.57$ GHz. It is worth noting that, in the experiments, any change in the bias current also affects the PS voltage required to achieve the same or a larger laser frequency detuning. Therefore, for the experimental spectra shown in Fig. 3.8, the PS voltage was slightly adjusted to 7.42 V to optimize both the number and the intensity of the side modes.

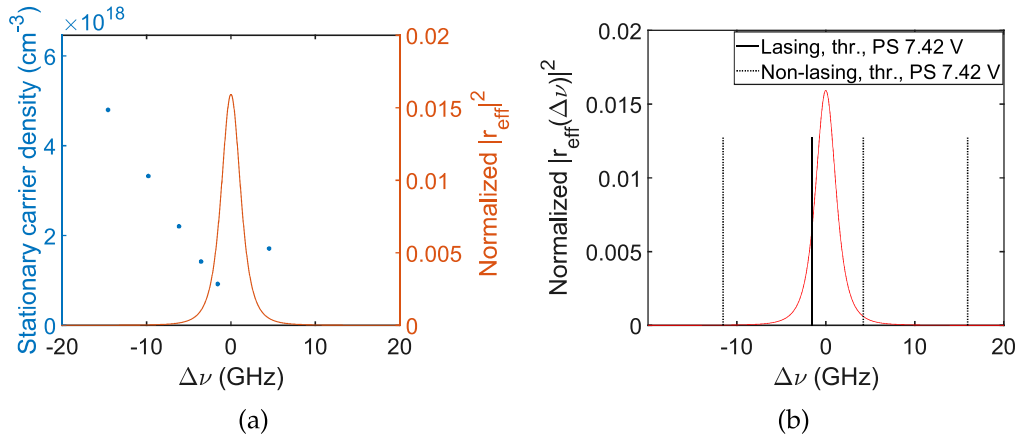


Fig. 3.9 Steady-state solutions under different operating conditions: 180 mA of bias current and -1.57 GHz of detuning. In (a) the solutions are associated with carrier density, while in (b) the carrier density is fixed at minimum value.

We observe the emergence of a pronounced pulsation regime approaching self-pulsing, with a repetition rate of 3.4 GHz, as reported in Fig. 3.8(a). In Fig. 3.8(b), the first and second harmonics are clearly above the noise floor, demonstrating that this hybrid laser can produce pulsations at a frequency comparable to the relaxation oscillation (RO) frequency, while remaining below the cavity free spectral range (FSR), which is approximately 6 GHz as shown in Fig. 3.9(b).

In order to understand the origin of the spectral broadening observed in the laser, it is useful to analyse the effects of amplitude modulation (AM) and frequency modulation (FM) separately. This distinction allows us to isolate the fundamental contributions, which, in real lasers, are instead coupled due to the internal dynamics of the device. Specifically, amplitude modulation describes the temporal variations in the emitted optical power, while frequency modulation is associated with variations in the optical phase

and thus in the instantaneous frequency of the field. It is important to note that these two mechanisms are not independent of one another. In fact, they are intrinsically linked via the Henry factor, which in our case is equal to 5. This introduces a coupling between variations in gain and refractive index. However, the separate analysis of AM and FM provides a useful tool for interpreting the spectral behaviour of the system both qualitatively and quantitatively. Fig. 3.10(a) shows the optical spectrum in the case of amplitude modulation only, with the frequency set to its average value. Fig. 3.10(b), on the other hand, shows the optical spectrum in the case of frequency modulation only, with the amplitude of the signal maintained at its mean value.

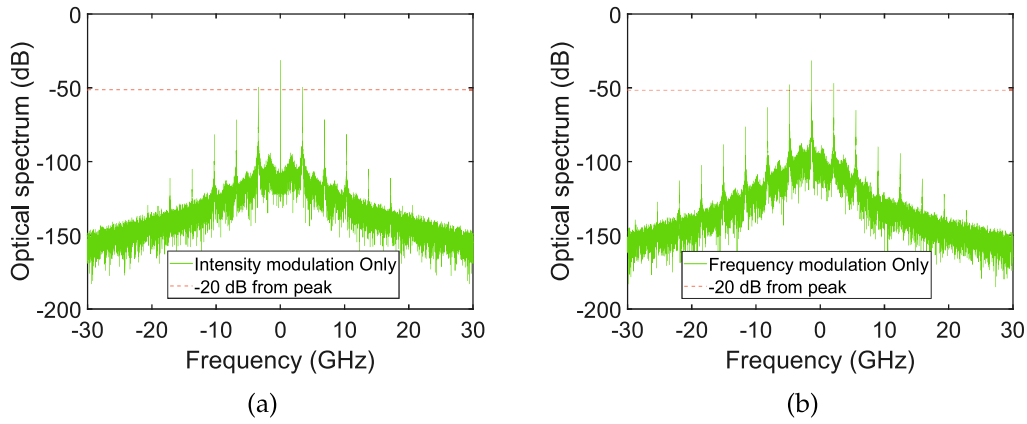


Fig. 3.10 (a) The optical spectrum in the case of amplitude modulation only.(b) The optical spectrum in the case of frequency modulation only.

In this particular case of a pulsed regime, the optical spectrum with amplitude modulation generates a significantly broadened spectrum. This is due to the presence of multiple harmonic components associated with the non-linearity of the temporal signal. Concurrently, the amplitude-phase coupling factor (α) introduces a dynamic chirp that further contributes to spectral broadening. Consequently, in a non-linear pulsed regime, both amplitude and frequency modulation can generate a comparable spectral broadening (a bandwidth of approximately 7 GHz). This suggests that measuring the bandwidth, for example at -20 dB from the peak, is not a sufficient criterion for distinguishing the dominant contribution between AM and FM. A correct interpretation of the signal therefore necessitates a more

in-depth analysis, incorporating consideration of the temporal dynamics of the signal and the evolution of the instantaneous frequency.

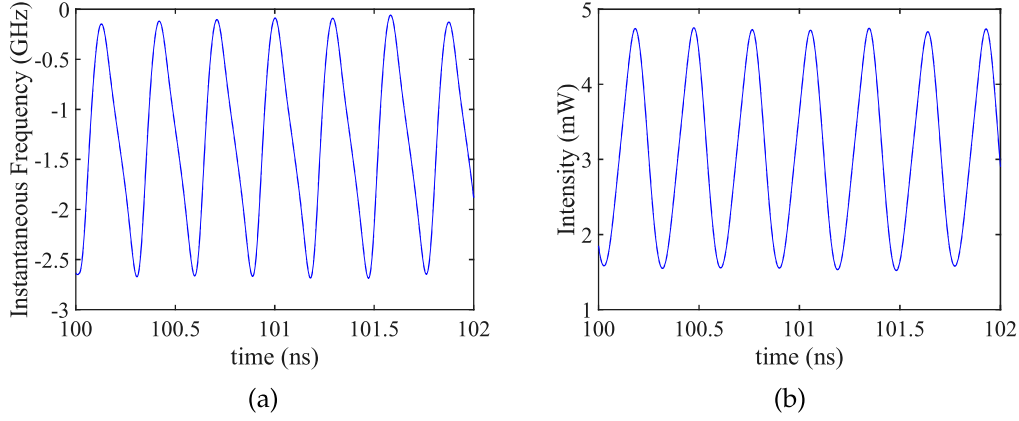


Fig. 3.11 Time evolution (100–102 ns) of: (a) instantaneous frequency (GHz), and (b) output optical intensity (mW) under operating conditions: 180 mA of bias current and -1.57 GHz of detuning.

A detailed analysis of the output optical intensity temporal evolution reveals a self-sustained oscillatory regime, characterised by regular, periodic oscillations, as illustrated in Figure 3.11(a). Analysis of the observed waveforms indicates that these oscillations are approximately sinusoidal, suggesting a stable dynamic regime. However, even in the presence of quasi-sinusoidal temporal dynamics, the laser exhibits nonlinear effects that primarily manifest themselves in the spectral domain. In addition, the optical intensity presents a modulation index $m = \frac{P_{max} - P_{min}}{P_{max} + P_{min}} = 0.54$, indicating an acceptable modulation contrast.

Furthermore, the analysis of the instantaneous frequency, shown in Figure 3.11(b), reveals variations synchronised with the intensity dynamics, confirming the presence of a significant dynamic chirp. This behaviour is a direct consequence of the amplitude-phase coupling described by the Henry factor.

The combined observation of these quantities demonstrates that the system cannot be described as either purely amplitude-modulated or purely frequency-modulated. Instead, the laser dynamics result from the interplay between amplitude and phase modulation, where the two components co-exist and continuously influence one another. In this context, the observed

spectral broadening is attributed to the combined effects of the nonlinear oscillatory dynamics and the chirp induced by amplitude-phase coupling.

The observed temporal waveform is approximately sinusoidal and therefore differs from the well-defined short pulses typically associated with passive or active mode-locking. Nevertheless, the observed dynamics are consistent with a dispersive self-Q-switching (DQS) regime rather than with conventional coherent mode-locking.

Following a detailed comparison with previously reported frequency-selective-feedback lasers, the observed dynamics can be interpreted within the framework of DQS. In particular, the temporal waveforms closely resemble those reported in this Ref. [41], where relaxation oscillations become progressively undamped owing to the interaction between carrier-induced wavelength shifts and a narrow frequency-selective reflector.

In the present device, the lasing wavelength is intentionally positioned on the steep slope of the SiN reflectivity spectrum [42]. Consequently, fluctuations in the optical intensity modify the carrier density in the RSOA which, through the Henry factor, produces a shift of the lasing wavelength. Because the reflector exhibits a steep wavelength-dependent reflectivity, this wavelength shift changes the effective optical feedback experienced by the cavity [42]. The resulting dynamic modulation of the effective cavity losses (or equivalently the cavity Q-factor) creates a feedback mechanism that progressively undamps the relaxation oscillations, eventually leading to a stable DQS regime [42].

Although the temporal intensity waveform remains close to sinusoidal, this does not preclude the identification of the observed dynamics as dispersive self-Q-switching. Unlike conventional mode-locked lasers, where short pulses originate from phase locking among longitudinal modes, the oscillations observed here arise from a relaxation-oscillation instability sustained by frequency-selective optical feedback. This interpretation is consistent with both the experimental observations and the time-delayed numerical model, which reproduces the measured dynamics without requiring any explicit mode-locking mechanism.

Furthermore, the last analysis carried out shows that varying the SOA current alone is not sufficient to induce the transition to this dynamical regime.

Instead, the onset of the oscillations is strongly governed by the laser-reflector detuning, which determines the stability of the frequency-selective feedback. When the lasing wavelength is appropriately positioned along the steep reflectivity slope of the SiN mirror, the carrier-induced wavelength shifts modulate the effective optical feedback, progressively reducing the damping of the relaxation oscillations until a stable dispersive self-Q-switching regime is established.

3.5.1 Amplitude and phase noise of the spectral optical lines

The spectral and temporal stability of optical lines generated in the optical spectrum can be characterized through two indicators, namely the Mode Partition Noise (MPN) and the Differential Phase Noise (DPN), which quantify the temporal fluctuations of the line intensities and the relative phase variations between adjacent lines.

MPN is a theoretical parameter that describes how the intensities of all longitudinal optical modes fluctuate, namely, the random redistribution of optical power among the longitudinal modes of the laser cavity. DPN is defined as the random fluctuations in the phase difference between the longitudinal modes of a multimode laser. These fluctuations are primarily attributable to spontaneous emission coupled into the lasing lines, which induces phase diffusion.

For this purpose, the optical spectrum of the emitted field is filtered in order to retain only the spectral lines within a power range of 20 dB from the maximum spectral intensity. Let $P_q(t)$ denote the temporal amplitude of the q -th spectral line and $\Delta\Phi_q(t)$ the phase difference between two adjacent lines. The corresponding temporal mean values are defined as [43], [44], [45]

$$\mu_{P_q} = \langle P_q(t) \rangle, \quad \mu_{\Delta\Phi_q} = \langle \Delta\Phi_q(t) \rangle, \quad (3.10)$$

where $\langle \cdot \rangle$ denotes the time average. The fluctuations around these mean values are then quantified through the standard deviations

$$\sigma_{P_q} = \sqrt{\left\langle \left(P_q(t) - \mu_{P_q} \right)^2 \right\rangle}, \quad (3.11)$$

$$\sigma_{\Delta\Phi_q} = \sqrt{\left\langle \left(\Delta\Phi_q(t) - \mu_{\Delta\Phi_q} \right)^2 \right\rangle}. \quad (3.12)$$

In particular, $\sigma_{\delta\phi}$ is defined as the standard deviation of the differential phase fluctuations between adjacent comb lines, while σ_P represents the root mean square (RMS) of the intensity noise for each comb line. Notably, for an ideal self-pulsing, these quantities are expected to be extremely small, less than 0.1 [45].

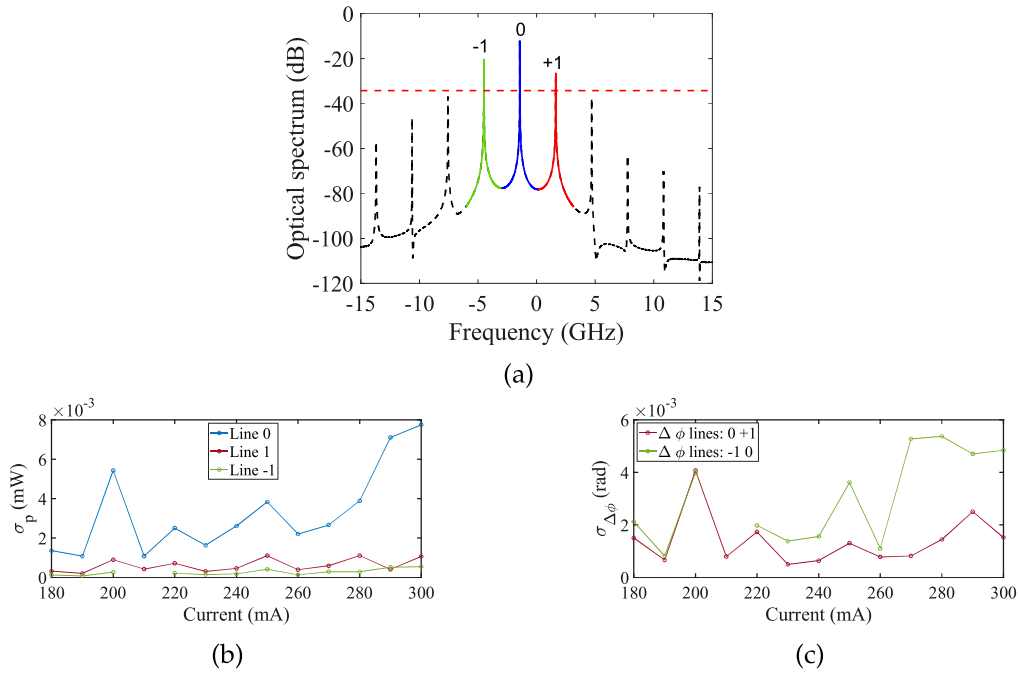


Fig. 3.12 (a) Optical spectra showing the primary mode (blue) and adjacent side modes (red and green) within the -20 dB bandwidth, each selectively filtered with a rectangular window. (b) Mode partition noise (MPN) of the main line (Line 0) and first-order side lines (Line ± 1) versus bias current. (c) Differential phase noise of the corresponding spectral components (Line 0, Line +1, Line -1) over the 180–300 mA current range.

We evaluate the two indicators as a function of bias current, as shown in Fig. 3.12. In panel (a), the optical spectrum highlights all lines within

the -20 dB bandwidth; the main laser line is shown in blue, while the first-order sidebands appear in red and green. Each line was isolated using a rectangular filter for separate noise analysis. Panels (b) and (c) report the extracted indicators for these three spectral components over the current range 180–300 mA. Notably, the main laser line experiences the largest power drop with increasing current, yet only three dominant lines remain visible within this bandwidth. The very low noise values obtained confirm the high spectral quality and stability of the laser across the investigated operating range.

3.6 Design Improvements

A possible design improvement for the hybrid laser under investigation can be identified by comparing its architecture with the device reported in [6].

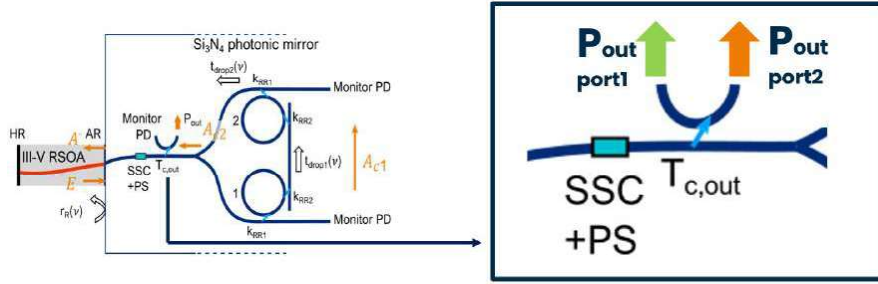


Fig. 3.13 Schematic for comparing the two output ports

In the two configurations, the optical output is extracted from different ports, as illustrated in Fig. 3.13. In the laser analyzed in this work, the output power is taken from Port 1, corresponding to the optical power reflected by the two narrowband Vernier rings. In the design proposed in [6], instead, the output is directly extracted from the SOA section (Port 2). This architectural difference leads to several trade-offs in device performance.

As can be observed in the optical spectrum reported in Fig. 3.14, the output taken from port 2 exhibits a larger number of spectral lines within the -20 dB bandwidth. In particular, three lines are present at port 1, while five lines can be observed at port 2. Moreover, a significantly higher output

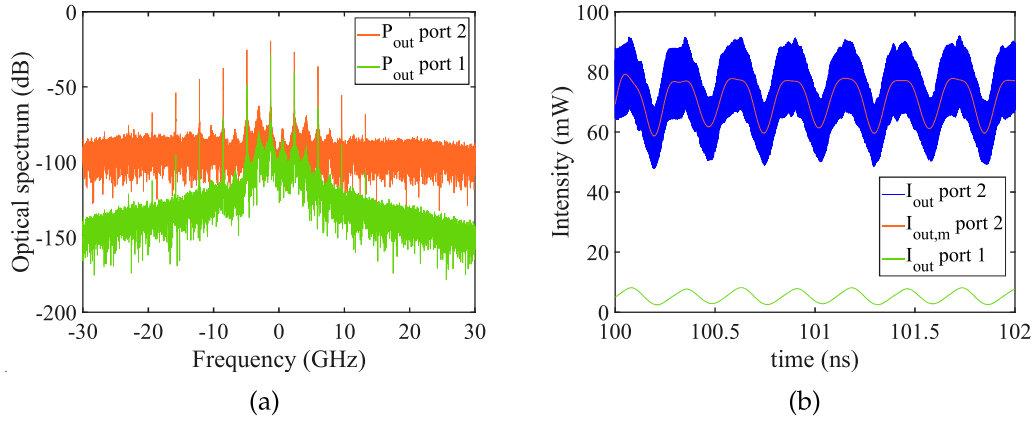


Fig. 3.14 (a) Optical spectrum at port 1 (green) compared with that at port 2 (orange) at same conditions. (b) Output intensity comparison between the two ports.

power is obtained when the signal is extracted from the SOA, increasing from approximately 5 mW to about 65 mW. In this configuration, the spectral properties at port 2 are further investigated by separately considering pure amplitude modulation and pure frequency modulation, as illustrated in Fig. 3.15.

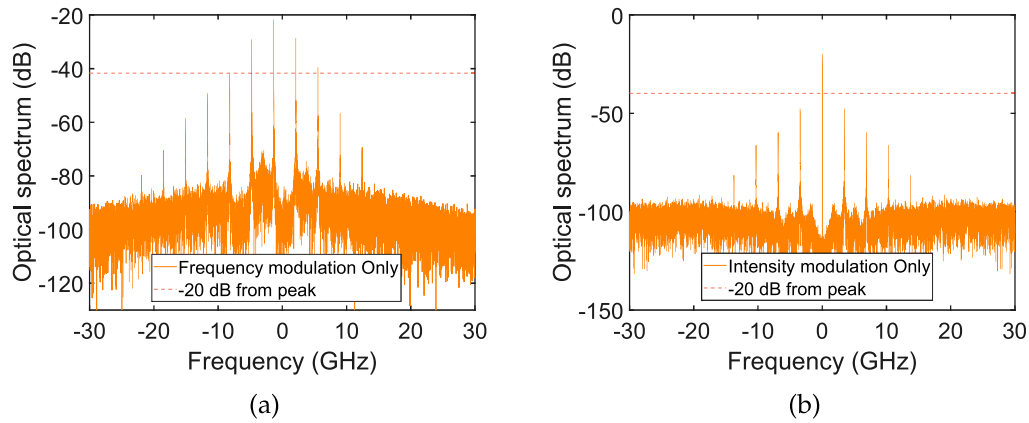


Fig. 3.15 (a) The optical spectrum in the case of amplitude modulation only. (b) The optical spectrum in the case of frequency modulation only for the port 2.

The results clearly indicate that pure AM produces a spectrum that is essentially not broadened, whereas FM leads to a significant spectral broadening on the order of several gigahertz. This suggests that frequency modulation plays a dominant role. However, the observed spectral broadening is not

due to an independent frequency modulation process. Rather, it originates from a dynamic chirp induced by intensity oscillations through a relatively large Henry factor. In summary, in the self-pulsing regime considered, frequency modulation dominates the spectral broadening. Nevertheless, it should not be interpreted as an independent effect; instead, it arises as a direct consequence of the coupled dynamics between the amplitude and the phase of the optical field.

From a dynamic point of view, the comparison between the two output ports highlights the critical role played by the relative weight of amplitude and frequency modulation. In the case of port 2, where the spectral broadening is predominantly governed by frequency modulation and the amplitude modulation component remains comparatively weak, the laser operates in a more regular and stable regime. This regime is characterized by approximately sinusoidal temporal oscillations of the intensity and by an overall more controllable dynamics. In contrast, at port 1, where the modulation of the amplitude contributes significantly to the spectral broadening, the laser exhibits a more pronounced nonlinear behavior. The presence of multiple harmonic components in the spectrum reflects a complex temporal dynamics, associated with stronger deviations from sinusoidal oscillations. While such a regime may be advantageous for applications requiring broadband emission or multi-frequency generation, it generally entails reduced stability and increased sensitivity to perturbations.

The stability of the spectral lines at port 2 is further investigated through the analysis of the line stability indicators, as shown in Fig.3.16. By comparing these results with those obtained for port 1 in Fig.3.12, it can be observed that the values of the mode partition noise are several orders of magnitude higher. In particular, the MPN values approach the threshold of 0.1, which represents the upper limit associated with unstable modal behavior. This indicates that a significant fraction of the optical power of the lasing mode is redistributed among the sidebands, resulting in a degradation of the modal intensity stability.

On the contrary, the differential phase noise remains within the same order of magnitude as that observed for port 1. This suggests that, despite the increased intensity fluctuations, the relative phase between the spectral lines

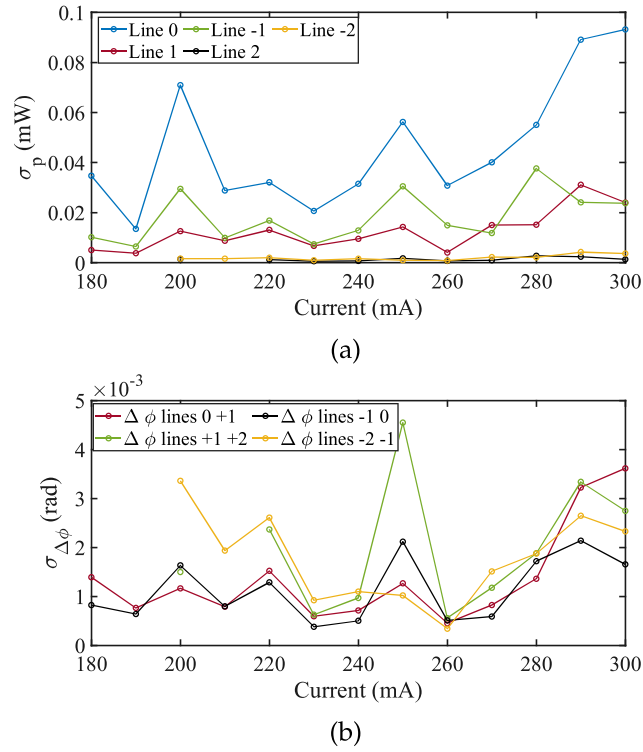


Fig. 3.16 In (a), Mode Partition Noise, and in (b), Differential Phase Noise, for each spectral line as the bias current is varied from 180 mA to 300 mA, simulated at the output of port 2.

is still preserved. Therefore, the results indicate that extracting the optical signal directly from the SOA mainly affects the amplitude stability of the modes, while the phase coherence between the lines remains substantially unchanged.

Overall, these results highlight a fundamental design trade-off in the hybrid laser architecture: extracting the optical signal directly from the SOA significantly increases the available output power, but at the expense of a substantial degradation in modal intensity stability, while the phase coherence between the spectral lines remains largely preserved. Within the operating conditions considered, this trade-off suggests that a regime in which spectral broadening is primarily governed by frequency modulation, with only a limited contribution from amplitude modulation, is preferable. Such a configuration ensures a more stable and controllable dynamical behavior, while

simultaneously facilitating a clearer interpretation of the underlying physical mechanisms.

3.7 Model improvements: resonance detuning effects

In this hybrid laser based on the Vernier effect, spectral selection is achieved using two ring resonators with slightly different free spectral ranges (FSRs) because the circumference lengths of the two rings are different. In this configuration, maximum transmission occurs when the resonances of the two resonators are spectrally aligned, thereby achieving the Vernier effect and allowing the laser operating wavelength to be selected efficiently. To control and optimise the alignment of the resonances, each resonator is fitted with an integrated heater. The power dissipated by the heaters causes a local temperature variation, which modifies the effective refractive index of the waveguide through the thermo-optic effect. This variation results in a shift in the resonance frequencies of the rings, allowing their spectral position to be tuned with high precision. By controlling the power applied to the heaters appropriately, the resonances of the two resonators can be shifted until they coincide at the desired wavelength.

3.7.1 Modelling

The model has been modified to also take into account the presence of a small misalignment of a few GHz between the resonances of the two rings. Such a mismatch may arise from various practical factors, such as limitations in the stability or resolution of the thermal control, local temperature non-uniformities within the device or fabrication tolerances of the photonic structures. From a spectral point of view, this misalignment results in an overall response that is not perfectly flat, but instead exhibits slight curvature (or 'bumps') towards the edges of the spectrum, which consequently results in an asymmetric spectral response. This behaviour is consistent with experi-

mental observations, where the measured spectra show a slightly rounded shape at the extremities of the transmission band.

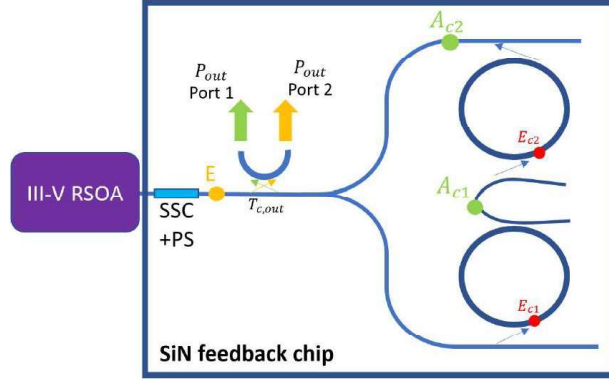


Fig. 3.17 Schematic layout of hybrid integrated tunable laser showing the model variables.

In the model, Eqs. 3.2 and 3.3 are reformulated as Eqs. 3.13 and 3.14 by incorporating the dependence on the circulating fields within the rings, E_{c1} and E_{c2} , as illustrated in Fig. 3.17. Additionally, Eqs. 3.15 and 3.16 are introduced to describe the time-dependent behavior of the circulating fields, accounting for nonlinear effects in each ring, i.e., when the resonances are detuned.

$$A_{c1}(t) = ik_1 A_{\text{loss1}}^{1/2} e^{i\beta_0 L_1/2} E_{c1}(t - \tau_1) \quad (3.13)$$

$$A_{c2}(t) = ik_2 A_{\text{loss2}}^{1/2} e^{i\beta_0 L_2/2} E_{c2}(t - \tau_2) \quad (3.14)$$

where factor i denotes a 90° phase shift between the through field and the cross-coupled field. The transmitted field (through) maintains the phase of the input signal. The coupled field (cross) is instead in phase quadrature, i.e. phase-shifted by $\pi/2$ with respect to the transmitted field. This behaviour is intrinsic to a lossless directional coupler, where the two outputs are phase-shifted by 90° to ensure the conservation of power (unitarity) of the

system. The half-roundtrip propagation phase is defined as $e^{\beta_0 L_{1,2}/2}$, where the propagation constant is given by $\beta_0 = \frac{2\pi n_{\text{eff, SiN}}}{\lambda_0}$. In addition, the delayed fields, $E_{c1,2}(t - \tau_{1,2})$, account for the finite round-trip time of light in the ring, where $\tau_{1,2} = \frac{L}{v_g} = \frac{2\pi R_{1,2}}{c} n_g$.

The coupled differential equations describe the temporal evolution of the complex electric field amplitudes $E_{c1}(t)$ and $E_{c2}(t)$ inside two optically coupled ring resonators.

$$\begin{aligned} \frac{dE_{c1}(t)}{dt} = & -\frac{1}{\tau_{g1}} \left(\frac{k_1}{t_1^2 A_{\text{loss1}}} t_{\text{ssc}}^2 [1 - T_{C,\text{out}}] E(t - \tau_{\text{eff}}) e^{-i\Delta\phi} \right. \\ & + \left(\frac{1}{t_1^2 A_{\text{loss1}}} - 1 \right) E_{c1}(t) \\ & \left. - j \frac{2\pi R_1}{c} n_g \delta\omega_{r1} E_{c1}(t) \right) \end{aligned} \quad (3.15)$$

$$\begin{aligned} \frac{dE_{c2}(t)}{dt} = & -\frac{1}{\tau_{g2}} \left(\frac{k_2}{t_2^2 A_{\text{loss2}}} k_1 A_{\text{loss1}} E_{c1}(t - \tau_1) \right. \\ & + \left(\frac{1}{t_2^2 A_{\text{loss2}}} - 1 \right) E_{c2}(t) \\ & \left. - j \frac{2\pi R_2}{c} n_g \delta\omega_{r2} E_{c2}(t) \right) \end{aligned} \quad (3.16)$$

where each equation is normalised with respect to $\tau_{g,1,2}$, which is defined as $\tau_{g,1,2} = \frac{L}{v_g} = \frac{2\pi R_{1,2}}{c} n_g$.

The term $\frac{dE_{ci}(t)}{dt}$ represents the temporal variation of the intracavity field, resulting from the balance between gain and losses along the optical path within the ring.

The terms proportional to k_1 and k_2 represent the coupling between each ring resonator and the bus waveguide. More specifically, in the equation 3.15, the bus waveguide field coincides with the outgoing delay field from the RSOA, i.e., $E(t - \tau_{\text{eff}})$, which acts as the input to the first ring. In contrast,

in the equation 3.16, the bus waveguide field is given by A_{c1} , which directly drives the second resonator.

The contributions of the equation

$$\left(\frac{1}{t_i^2 A_{\text{loss}}} - 1 \right) E_{ci}(t)$$

account for intracavity losses, where they are defined as $A_{\text{loss}} = e^{-\alpha_0 2\pi R/2}$.

The complex terms are proportional to

$$-j \frac{2\pi R_i}{c} n_g \delta\omega_{ri} E_{ci}(t)$$

describe the possible frequency detuning with respect to the resonance condition, where $\Delta\omega_r$ defines two slightly misaligned resonances.

Finally the equation of effective reflectivity for the SiN feedback mirror is expressed as:

$$r_{eff}(\omega) = ik^4 a^{3/2} e^{i\phi_0 + i\omega\tau_{g2}} \frac{t_{sc}^2 (1 - T_{C,out}) e^{-i\Delta\phi - i\omega\tau_{eff}}}{\left[1 - t^2 a \left(1 - \frac{i\Delta\omega}{v_g} L - \frac{i\delta\omega_{r1}}{v_g} \right) \right] \left[1 - t^2 a \left(1 - \frac{i\Delta\omega}{v_g} L - \frac{i\delta\omega_{r2}}{v_g} \right) \right]} \quad (3.17)$$

For further details and a derivation of the equations in the new model, see Appendix B.

3.7.2 Measurements and numerical simulations

In Fig. 3.18(a), I compare the measurement taken with a bias current of 195 mA and a PS voltage of 0 V with the numerical simulation carried out using the new model, in which $\delta\omega_{r1}$ is 0 GHz and $\delta\omega_{r2}$ is 0.5 GHz, resulting in a resonance detuning $\delta\omega_r = 0.5$ GHz. We can see that the simulation follows the same trend as the experimental measurement. In Fig. 3.18 (b), the effective reflectivity and the lasing mode shift to lower frequencies compared to the case without effects. As can be seen in Fig. 3.19, the effective reflectivity

shifts and the peak drops as the resonance detuning increases, resulting in two peaks and laser instability.

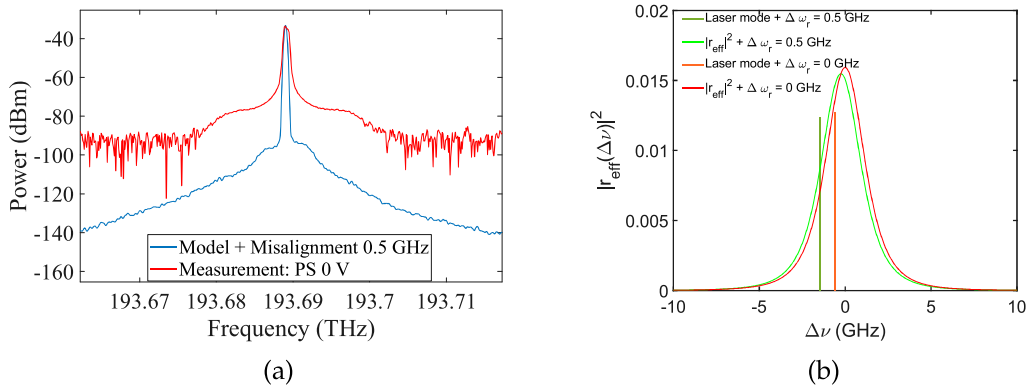


Fig. 3.18 (a) Measurement (in red) and simulation (in blue) taking into account a resonance detuning of 0.5 GHz between the two resonances, under conditions of 195 mA bias current and PS 0 V. (B) Curve of the effective reflectivity and the respective lasing mode without (in green) and with the effect of resonance detuning (in red).

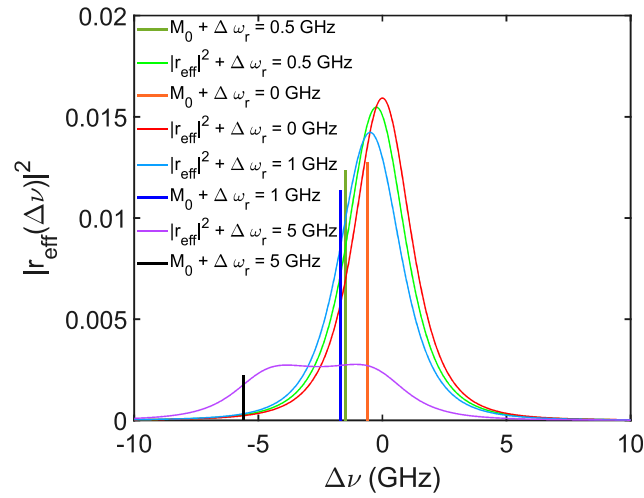


Fig. 3.19 Curves of the effective reflectivity and the respective lasing modes, varying the resonance detuning from 0.5 to 5 GHz.

3.8 Discussion and conclusion

Although the optical spectra display a comb-like structure featuring nearly equidistant lines, the underlying physical mechanism differs fundamentally from that of a conventional optical frequency comb produced by passive or active mode-locking in lasers. In mode-locked lasers, longitudinal cavity modes are phase-locked, yielding ultrashort pulses characterized by high mutual coherence, low phase noise, and a repetition rate set by the cavity round-trip time.

In contrast, the self-pulsing regime observed here arises from the undamping of relaxation oscillations, amplified by asymmetric detuning of the lasing frequency with respect to the reflectivity peak of the narrowband external SiN mirror, together with nonlinear wave mixing processes (primarily four-wave mixing) mediated by carrier dynamics in the RSOA. The resulting pulses occur on a picosecond timescale and do not require phase locking among cavity modes. Consequently, the repetition rate—approximately 3–3.4 GHz—matches the intrinsic relaxation oscillation frequency of the device rather than a cavity free spectral range. This interpretation is supported by several key observations: when the intensity modulation response shows a pronounced relaxation oscillation peak; then strong RF tones appear together with their harmonics. The experimental behaviour agrees excellently with time-delayed simulations that incorporate no explicit mode-locking mechanism.

The observed self-pulsing regime can therefore be classified as a dispersive self-Q-switching behaviour induced by frequency-selective feedback [41]. The narrow linewidth of the SiN frequency-selective reflector, being significantly smaller than the longitudinal mode spacing, suppresses multimode operation and decreases the probability of mechanisms relying on intermodal beating or four-wave mixing. Instead, the observed oscillatory dynamics are consistent with a relaxation-oscillation-driven instability mediated by the interaction between carrier dynamics and frequency-selective optical feedback.

This work demonstrates that hybrid integrated lasers on SiN platforms can function as compact, efficient sources of autonomous self-pulsing regimes,

achieved simply through careful engineering of the passive feedback circuit. No external modulators, saturable absorbers, or complex photonic-crystal structures [35], [41] are needed. The narrowband reflectivity provided by the SiN mirror not only promotes stable operation but also reduces sensitivity to external optical feedback [6], [36]. By integrating detailed experimental characterization with time-delayed algebraic modeling, it has been possible to comprehensive understanding of the hybrid laser dynamics: self-pulsing emerges from the undamping of relaxation oscillations, combined with asymmetric detuning from the reflectivity peak and carrier-mediated nonlinear effects.

The demonstrated device provides a compact platform for exploring frequency-selective-feedback-induced nonlinear dynamics and for generating controlled optical intensity modulation. Potential applications in areas such as waveform shaping [46], micromachining [47], and other precision photonic systems may be envisioned; however, further optimization of the cavity design and operating conditions would be required to achieve pulse characteristics comparable to those of conventional mode-locked sources.

In conclusion, this study establishes that hybrid integrated lasers based on a SiN platform can operate as compact and efficient sources of autonomous self-pulsing output enabled purely by passive intracavity feedback engineering. The observed regime relies on the narrowband effective reflectivity of the SiN mirror, which ensures operational stability and reduced sensitivity to perturbations, while producing oscillatory dynamics governed by the intrinsic relaxation behaviour of the device rather than by conventional mode-locking mechanisms.

Chapter 4

Modelling of hybrid integrated tunable laser with non-linear effects of ring resonators.

4.1 Optical non-linearity of silicon nitride and silicon

In the previous chapter, the photonic mirror was initially developed using silicon nitride (SiN) with a silicon dioxide (SiO_2) coating. This material offers several advantages: it is compatible with CMOS fabrication processes; it has a wider transparency range than silicon, allowing it to cover more of the visible spectrum; it exhibits low propagation losses, and it is largely unaffected by nonlinear effects due to its wide bandgap. These properties make SiN ideal for creating mirrors in integrated narrow-linewidth laser systems. However, there are also some drawbacks. The refractive index contrast between SiN and SiO_2 is relatively small, leading to higher bending losses unless large ring radii of around $100\text{ }\mu\text{m}$ are used. Consequently, the device footprint increases and the resonator's free spectral range (FSR) decreases. A smaller FSR limits the tunability of the laser. Additionally, the thermo-optic coefficient of silicon is around seven times greater than that of SiN, making SiN more challenging to tune thermally.

For these reasons in some cases Si MRR could be preferred with respect to SiN ones. This chapter focuses on simulating an integrated silicon laser cavity with the aim of understanding the impact of these nonlinear effects on the laser performance.

4.2 Modelling including non-linear effects in MRRs

This section introduces a new model by combining the frameworks presented in Chapters 2 and 3. In particular, the time-domain system is described by a set of algebraic equations given by 3.1, 3.4, 3.13, and 3.14, together with the additional relations reported below; the proof can be found in Appendix B.

$$\begin{aligned} \frac{dE_{c1}(t)}{dt} = & -\frac{1}{\tau_{g1}} \left(\frac{k_1}{\tau_1^2 a_{\text{loss}1}} t_{sc}^2 [1 - T_{0,out}] E(t - \tau_{eff}) e^{-i\Delta\phi} \right. \\ & + \left(\frac{1}{\tau_1^2 a_{\text{loss}1}} - 1 \right) E_{c1}(t) \\ & \left. - j \frac{2\pi R_1}{c} \omega_0 \Delta n_{\text{eff}1} E_{c1}(t) \right) \end{aligned} \quad (4.1)$$

$$\begin{aligned} \frac{dE_{c2}(t)}{dt} = & -\frac{1}{\tau_{g2}} \left(\frac{k_2}{\tau_2^2 a_{\text{loss}2}} k_1 a_{\text{loss}1} E_{c1}(t - \tau_1) \right. \\ & + \left(\frac{1}{\tau_2^2 a_{\text{loss}2}} - 1 \right) E_{c2}(t) \\ & \left. - j \frac{2\pi R_2}{c} \omega_0 \Delta n_{\text{eff}2} E_{c2}(t) \right) \end{aligned} \quad (4.2)$$

A key element of the model is the definition of this term

$$-j \frac{2\pi R_i}{c} \omega_0 \Delta n_{\text{eff}} E_{ci}(t)$$

that depend on the effective refractive index variation, Δn_{eff} , in general is non-linear in the sense that it depends on the circulating optical power in the ring. Then, based on the material of the ring, this term can be a non-linear

Kerr term (in SiN) and/or a FCD term in Si. When the cavity is fabricated from silicon nitride, the dominant nonlinear contribution is the Kerr effect. In this case, Δn_{eff} is defined according to the following expression:

$$\Delta n_{\text{eff,Kerr}} = \frac{\Gamma n_2}{A_{\text{eff}}} |E_{c1}|^2 \quad (4.3)$$

where n_2 is the Kerr coefficient in silicon nitride and it is equal to $2.5 \cdot 10^{-19} \text{m}^2/\text{W}$ [48]. Conversely, when the mirror cavity is made of silicon, the nonlinear behaviour is primarily governed by free-carrier absorption (FCA) and two-photon absorption (TPA). The corresponding expressions for Δn_{eff} have already been derived and discussed in Chapter 2.

$$\Delta n_{\text{eff,Si}} = \Delta n_{\text{eff,T}} - \Delta n_{\text{eff,FCD}} \quad (4.4)$$

In this particular case, the passive circuit is implemented using a silicon strip waveguide, leading to the following equations, whose parameters are lumped—i.e., they are uniform across the entire cross-section and independent of spatial variables.

$$\Delta n_{\text{eff,FCD}} = \Gamma \cdot (p_f \cdot n_e^q + r_f \cdot p_e^s) \quad (4.5)$$

where Γ is the optical confinement factor in the silicon core. Instead, the variation of refractive index due to self-heating is defined as follows.

$$\Delta n_{\text{eff,T}} = \Gamma \frac{dn_{\text{si}}}{dT} Z_T |E_c|^2 (1 - a)(1 + t^2 a) \quad (4.6)$$

where $\frac{dn}{dT}$ is the material thermo-optic coefficient (silicon or polySi). Finally, the propagation loss of the optical field per round trip is expressed as:

$$a = e^{\frac{-\alpha_{\text{eff}} L}{2}} \quad (4.7)$$

where α_{eff} is the effective loss due to linear and nonlinear effects and is given by the sum of four terms: α_0 denotes the intrinsic linear losses due to scattering, surface-state absorption, and slight residual doping; α_{rad} describes the bending losses, representing the portion of light that radiates into the

cladding, and is negligible because, in this case, the radii are very large. α_{TPA} is the TPA loss, and $\Delta\alpha_{FC}(x, y)$ is the optical modal loss due to FC [26]. The last two terms are defined by the following equations.

$$\alpha_{rad} = \beta_{TPA} \frac{|E_c|^2}{A_{eff}} \quad (4.8)$$

$$\Delta\alpha_{FC} = \Gamma(a_f \cdot n_e^b + c_f \cdot p_e^d) \quad (4.9)$$

the experimental parameters a_f , b , c_f , d , p_f , q , r_f and s in eqs. (4.5) and (4.9) depend on the material and they are wavelength dependent quantities ($\lambda = 1.55 \mu m$ in this case); they are reported in Table 4.1. Finally n_e and p_e are, respectively, the electron and hole concentrations in MRR.

In this framework, the temporal evolution of the excess carrier concentrations can be described by Eq. 4.10. The right-hand side of these equations represents the recombination rates for electrons (R_e) and holes (R_h) [49].

$$\begin{aligned} \frac{\partial n_e}{\partial t} &= G_{ph} - \frac{1}{\tau_{n0}} \left(\frac{(n_0 + n_1 + n_e)(n_e - p_e)}{N_f} + \frac{n_e n_1}{n_1 + n_0} \right) \\ \frac{\partial p_e}{\partial t} &= G_{ph} - \frac{1}{\tau_{p0}} \left(\frac{(p_0 + p_1 + p_e)(p_e - n_e)}{N_f} + \frac{p_e p_1}{p_1 + p_0} \right) \end{aligned} \quad (4.10)$$

where photo-generation rate $G_{ph} = \frac{\beta_{TPA} P_c^2}{2\hbar\omega A A_{eff}}$. The parameters $\tau_{n0} = (N_f \sigma_n v_n)^{-1}$ and $\tau_{p0} = (N_f \sigma_p v_p)^{-1}$ correspond to the minimum carrier capture times when the trap states are empty. Here, σ_n and σ_p denote the capture cross sections, while v_n and v_p are the thermal velocities of electrons and holes.

Finally, n_0 and p_0 represent the equilibrium electron and hole densities in the absence of traps, while $n_1 = n_0 \exp\left(\frac{E_t - \psi_f}{k_B T}\right)$ and $p_1 = p_0 \exp\left(\frac{\psi_f - E_t}{k_B T}\right)$ depend on the relative position between the trap energy level and the Fermi level.

4.3 Kerr effect

As discussed in the previous section, the laser was implemented using TriPleX technology [40], a platform in which the effects of the Kerr non-linearity are negligible due to reduced modal confinement. This results in a significant fraction of the optical mode extending into the surrounding SiO₂ cladding.

In contrast, let us consider a hybrid laser whose feedback circuit is realized using a thicker strip waveguide, characterized by an effective area $A_{\text{eff}} = 0.7, \mu\text{m}^2$ and a confinement factor $\Gamma = 0.9$. The optical and RF spectra shown in Fig. 3.5(b) were analyzed by comparing the cases with and without the inclusion of the Kerr effect. As reported in Fig. 4.1, the peak of the lasing mode remains essentially unchanged in terms of both power and frequency. However, a clear reduction in the amplitude of the sideband peaks is observed when the Kerr effect is taken into account.

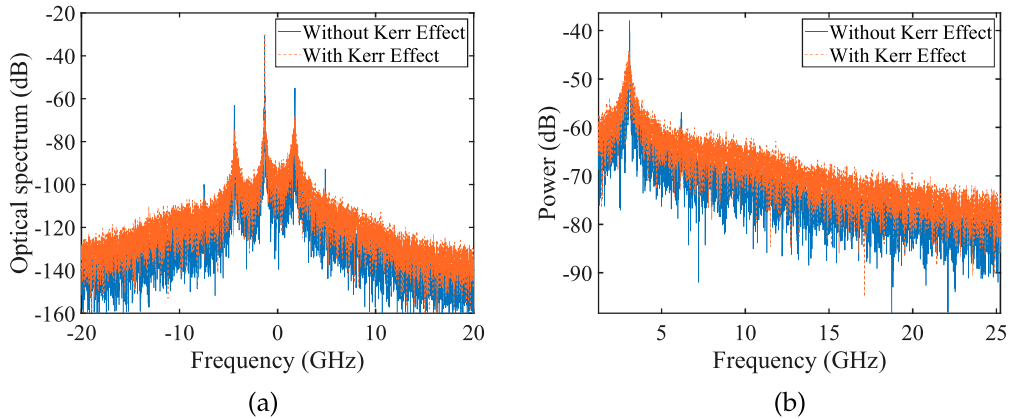


Fig. 4.1 (a) Simulated optical spectrum and (b) simulated RF spectrum, under the same condition of Fig. 3.5(b). The blue curves correspond to the linear case, while the orange curves include Kerr effect

Consistently, the optical spectrum reveals a degradation of the side-mode suppression ratio (SMSR), which decreases from 44 dB in the absence of Kerr nonlinearity to 32 dB when the Kerr effect is included. This behavior indicates that Kerr nonlinearity does not significantly affect the main lasing mode, while it plays a crucial role in the mechanisms responsible for sideband generation, effectively limiting their suppression.

4.4 Design of hybrid silicon integrated tunable laser

In this section, I analyze the impact of Si non-linearity by considering a tunable laser designed with Si MRR in [6].

Table 4.1 Main parameters of hybrid silicon integrated tunable laser [6], [26], [27]

Parameter	Symbol	Value
SOA		
Length	L_{SOA}	1000 μm
Effective refractive index	n_{eff}	3.15
Group refractive index	n_g	3.6
Modal gain coefficient	g_N	30 cm^{-1}
Linewidth Enhancement Factor	α_H	5
Si mirror		
Radius of MRR 1	R_1	20 μm
Radius of MRR 2	R_2	23 μm
Coupling coefficient	k^2	0.06
Bus waveguides length	$L_{straightWG}$	2300 μm
Waveguide propagation losses	α_0	1 dB/cm
Waveguide effective refractive index	n_{effSi}	2.02
Waveguide group refractive index	$n_{g,Si}$	3.79
Thermal impedance	Z_T	4000 K/W
Power output coupler coefficient	$T_{c,out}$	0.8
Spot size converter	t_{SSC}	-2 dB
Working wavelength	λ_0	1.54 μm
Effective area	A_{eff}	0.07 μm^2
Confinement factor	Γ	0.9
Experimental coefficients:	a_f	$8.8 \cdot 10^{-21} \text{ cm}^2$
	b	1.167
	c_f	$5.84 \cdot 10^{-20} \text{ cm}^2$
	d	1.109
	p_f	$5.4 \cdot 10^{-22} \text{ cm}^2$
	q	1.011
	r_f	$1.53 \cdot 10^{-18} \text{ cm}^2$
	s	0.838

The schematic layout of the hybrid integrated laser under examination is illustrated in Fig. 3.17, where the passive circuit of the laser cavity is fabricated in silicon, with its main specifications summarized in Table 4.1. In this case, the circuit's output port is port 2. A comparison of the passive circuit made of nitrided silicon with the new circuit reveals a significant reduction in dimensions. The radii, which were $100\ \mu\text{m}$, have been reduced to approximately $20\ \mu\text{m}$, while the length of the straight guides has been halved. However, it should be noted that losses have increased significantly.

4.5 Numerical Simulations

In the following section, I analyse two operating conditions in which the laser is in a single-mode regime. These conditions are a bias current of 90 mA and 250 mA, with a detuning of -0.22 GHz in both cases.

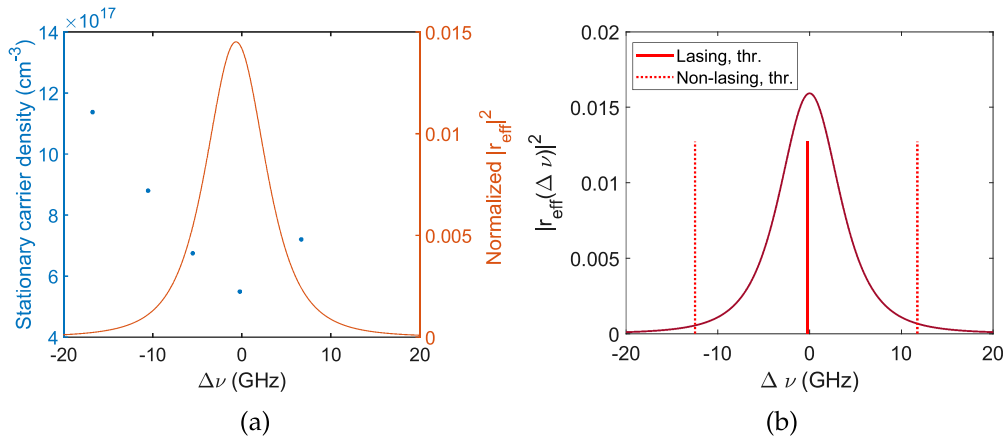


Fig. 4.2 (a) The graph shows the steady-state solutions of the oscillation condition with the associated carrier density at threshold is reported on the left axis. The solid line represents the narrowband effective reflectivity of the Si passive mirror. (b) The lasing mode at threshold is shown as a solid line, while the dotted lines represent non-lasing longitudinal modes at threshold.

The numerical simulation is performed using the time-delayed model presented in section 4.2, with the parameters reported in Table 4.1, I proceed to compare the steady-state solutions, their relation to the external cavity free spectral range (FSR), optical and RF spectrum with and without no

linear effects (TPA and FCA) of ring resonator to understand the impact of these effects. Fig. 4.2 shows that the narrowband effective reflectivity of the Si passive mirror has a FWHM of 7.66 GHz and the FSR is equal to 11.5 GHz. As shown in Fig. 4.3, as the bias current increases, the optical spectrum exhibits sidebands with a spacing of approximately 3–5 GHz, while the SMSR decreases from 98 dB to 80 dB. In addition, the RF spectrum in Fig. 4.3(b) reveals the presence of multiple harmonics. These features indicate the onset of laser instabilities, which ultimately lead to a reduction in the emitted optical power.

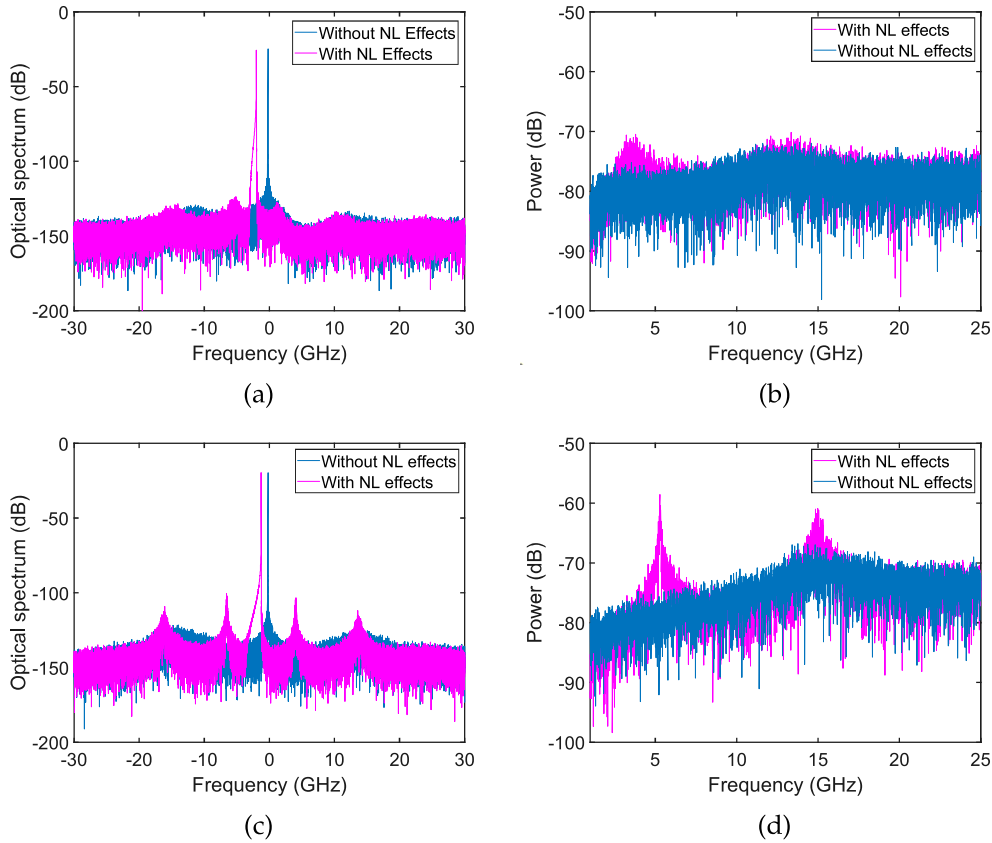


Fig. 4.3 (a) Simulated optical spectrum and (b) simulated RF spectrum. The blue curves correspond to the linear case, while the magenta curves include nonlinear effects. The bias current is 90 mA in (a) and (b), and 250 mA in (c) and (d).

A more detailed analysis of the output optical power as a function of current (see Fig. 4.4(a)) shows that, when nonlinear effects are taken into account, the emitted power is reduced compared to the linear case. This

behavior can be attributed to the increase in optical modal losses due to two-photon absorption with increasing current, along with a similar trend observed in the effective refractive index associated with free carriers. These contributions play a significant role in the simulations and strongly influence the overall device performance, as shown in Fig. 4.4(c,d).

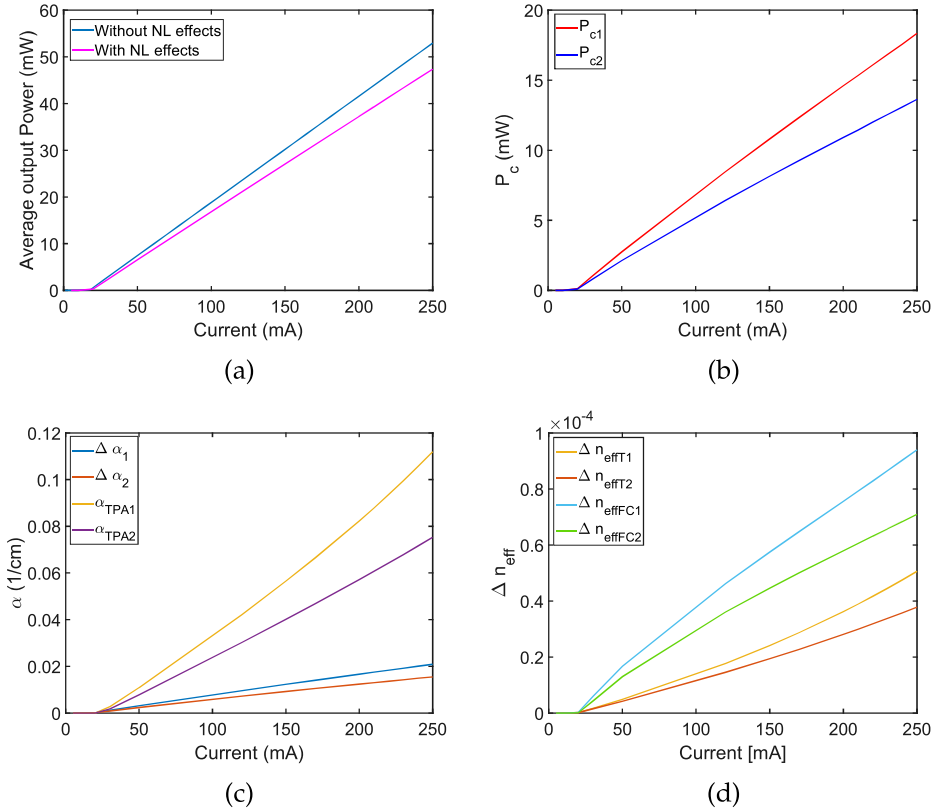


Fig. 4.4 (a) Average output optical power as a function of current, where blue represents the case without nonlinear effects and magenta includes nonlinear effects. (b) Average circulating power as a function of current. (c) Average modal and radiation losses as a function of current in the presence of nonlinear effects. Subscripts 1 and 2 denote the parameters corresponding to MMR 1 and MMR 2, respectively. (d) Average effective refractive index as a function of current, including free-carrier and thermal effects.

A closer inspection of Fig. 4.4(a) further reveals that the lasing threshold remains nearly unchanged, while the slope efficiency decreases in the presence of nonlinear effects, indicating that nonlinearities primarily affect the energy conversion efficiency rather than the onset of lasing.

Figure 4.4(b) illustrates the increase in circulating power within the two rings as the injection current rises. Since radiation losses are directly dependent on the circulating power, higher intracavity power leads to increased losses in the rings, which in turn contributes to the reduction of the output optical power. As shown in Fig. 4.4(c), TPA losses (α_{TPA1} and α_{TPA2}) represent the dominant loss mechanism, while variations in optical modal losses are comparatively smaller. This indicates that internal nonlinear processes, such as two-photon absorption, significantly limit the laser performance.

Furthermore, Fig. 4.4(d) shows that the effective refractive index variation increases with current, but exhibits fluctuations at higher injection levels. These fluctuations are consistent with the instabilities observed in the RF spectrum and the appearance of sidebands, confirming that refractive index changes induced by nonlinear effects have a strong impact on the laser dynamics.

4.6 Discussion and conclusion

In conclusion, the hybrid laser based on a silicon nitride mirror cavity does not exhibit significant Kerr nonlinear effects. This is also confirmed by the simulations, where neither the optical nor the RF spectra show noticeable variations attributable to Kerr-induced phenomena. As a result, the nonlinear contribution of silicon nitride can be considered negligible, and the laser operation remains essentially unaffected by Kerr effects.

In contrast, the hybrid laser with a silicon cavity offers the advantage of enabling much more compact structures, achieving a Free Spectral Range (FSR) of approximately 12 GHz. In the silicon nitride case, the FSR is smaller due to the longer cavity length. However, silicon exhibits a significantly larger thermo-optic coefficient compared to silicon nitride. However, simulations suggest that thermal effects are negligible in comparison to free-carrier effects, which represent the dominant mechanism.

The primary disadvantage associated with the silicon-based laser is the diminished single-mode operating range in comparison to the silicon nitride case. It has been demonstrated that the device in question becomes unstable

at lower injection currents, thus highlighting a heightened sensitivity to nonlinear effects.

Chapter 5

Conclusion

The thesis has focused on the modelling, analysis, and understanding of non-linear dynamics in silicon-based photonic integrated systems, with particular emphasis on microring resonators (MRRs) and hybrid integrated tunable lasers employing silicon and silicon nitride platforms. The central objective of this work was to develop physically sound and computationally efficient modelling tools capable of capturing the interplay between optical, carrier-induced, and thermal effects. The tools were then applied to elaborate the mechanisms by which nonlinear behaviour, stability, and dynamical regimes are influenced in advanced integrated photonic devices. A significant contribution of this thesis is the integration of detailed numerical modelling with experimental validation, thereby facilitating both predictive capability and physical insight. By addressing nonlinear effects across a range of device architectures, from single MRRs to fully hybrid laser cavities, this thesis provides a unified framework for understanding how geometry, material choice, and laser mirror design influence device performance and dynamical operation.

5.1 Summary of main results

The primary significant result of this thesis is the development of a distributed two-dimensional modelling framework for microring resonators with complex waveguide cross-sections, implemented within the SISCAP simulation

platform. The proposed approach utilises finite element methods to enable the spatially resolved evaluation of free-carrier generation, diffusion, recombination, and self-heating, while maintaining a significantly reduced computational intensity compared to fully electromagnetic time-domain techniques such as FDTD. This is a significant advantage when working with realistic device geometries, where simplified lumped models are unable to capture essential spatial effects. The modelling framework has been demonstrated to be a particularly powerful tool when analysing non-standard waveguide architectures, such as rib-based Si/PolySi configurations. In such cases, it has been shown that carrier transport and heat dissipation play a dominant role in determining nonlinear response. Through the implementation of systematic simulations and the comparison of results with experimental measurements, it has been demonstrated that waveguide geometry has a significant impact on nonlinear losses, self-heating, and resonance stability. In the course of the present study, a range of solutions was investigated. The geometry of three-wing ribs with a Si/PolySi structure was identified as a compromise that offers optimal performance in a number of areas. This is due to the fact that it reduces nonlinear distortion and improves thermal handling, and this is achieved through enhanced lateral carrier spreading and efficient heat dissipation. The validity of the proposed modelling framework is further confirmed through direct comparison with experimental measurements, thereby demonstrating its potential as a robust and versatile design tool for high-Q MRRs across a wide range of materials and geometrical configurations.

The secondary significant result of this thesis is a detailed experimental and theoretical investigation of the observed self-pulsing regime. It demonstrates that, despite the presence of a comb-like optical spectrum, the underlying physical mechanism differs fundamentally from conventional mode-locking. Instead, the phenomenon of self-pulsing has been demonstrated to originate from the undamping of relaxation oscillations. These oscillations are driven by asymmetric detuning between the lasing frequency and the effective reflectivity peak of the SiN mirror, and are reinforced by carrier-mediated nonlinear wave-mixing processes within the reflective semiconductor optical amplifier (RSOA). Time-delayed numerical simulations demonstrate excellent agreement with experimental observations across op-

tical, radio-frequency, and time-domain measurements. The repetition rate of the optical oscillations is found to be determined by the intrinsic relaxation oscillation frequency of the active device rather than by the cavity free spectral range. This provides a clear physical interpretation of the observed dynamics. It is important to note that this work demonstrates that stable, autonomous self-pulsing operation can be achieved purely through passive cavity design, without the need for saturable absorbers, external modulators, or complex photonic-crystal structures.

The final part of the thesis extends the modelling approach to hybrid laser architectures incorporating microring resonators and explicitly includes nonlinear effects within the passive cavity. Silicon nitride and silicon-based resonator cavities have different nonlinear behaviours. In the silicon nitride platform, for the considered waveguide designs, Kerr nonlinearities are found to be negligible, as confirmed by both numerical simulations and experimental spectra, leading to robust and stable laser operation. Conversely, silicon-based cavities facilitate considerably more compact designs with larger free spectral ranges; nevertheless, their operation is predominantly influenced by free-carrier effects and loss (as compared to thermal nonlinearities); that limit the output power at large currents.

The results presented in this thesis offer a comprehensive physical demonstration of nonlinear phenomena in integrated photonic platforms based on silicon and silicon nitride. I have that accurate, spatially resolved modelling of carriers and temperature is essential for predicting device behaviour in high Q resonators and hybrid laser systems with complex geometries. Simplified approaches may correctly capture qualitative trends, but are insufficient for reliable design and optimisation when nonlinearities play a critical role.

5.2 Future work

Future work could involve experimental measurements to validate the hybrid laser model, accounting for the nonlinear effects of silicon ring resonators in the laser cavity mirror. This approach will help to improve model accuracy and operating limits. A secondary direction involves the extension of the distributed microring resonator model to p-i-n junction-based modu-

lators. The incorporation of electrically driven carrier injection, depletion, and drift effects would facilitate an accurate description of nonlinear and transient behaviour in high-speed MRR modulators, thereby supporting the co-design of resonant and electrically active components in advanced silicon photonic integrated circuits.

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Appendix A

Proofs of equations

Photo-Generation rate demonstration

This section presents the derivation of the photo-generation rate as a function of distributed parameters, specifically accounting for its dependence on the size of the individual mesh elements used in the finite element method (FEM) simulations. This formulation is essential for accurately resolving the carrier distribution across the cross-section of complex waveguide structures.

Starting from Maxwell's curl equations in the time domain:

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \quad (\text{A.1})$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \quad (\text{A.2})$$

where the electric displacement field $\mathbf{D}$ is decomposed into a linear contribution, characterized by the refractive index n , and a nonlinear polarization term $\mathbf{P}^{nl}$,

$$\mathbf{D}(\mathbf{r}, t) = \epsilon_0 n^2 \mathbf{E} + \mathbf{P}^{nl}(\mathbf{r}, t) \quad (\text{A.3})$$

and the nonlinear polarization is modeled via a third-order susceptibility tensor, accounting for the medium's cubic response with full temporal dispersion [50],

$$\mathbf{P}^{(3)}(\mathbf{r}, t) = \varepsilon_0 \chi^{(3)} : \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \quad (\text{A.4})$$

while the magnetic induction field $\mathbf{B}$ is directly proportional to the magnetic field $\mathbf{H}$, with a proportionality constant equal to the magnetic permeability of free space μ_0 .

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t) \quad (\text{A.5})$$

The optical signal in the μ th mode of the waveguide is modelled using the slowly varying envelope approximation (SVEA) with a carrier frequency ω_c . Throughout this thesis, the waveguides under consideration are assumed to be single-mode.

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \left\{ A_\mu(z, t) \frac{\mathcal{E}_\mu(x, y, \omega_c)}{\sqrt{\mathcal{P}_\mu}} e^{j(\omega_c t - \beta_\mu(\omega_c)z)} \right\} \quad (\text{A.6})$$

$$\mathbf{H}(\mathbf{r}, t) = \text{Re} \left\{ A_\mu(z, t) \frac{\mathcal{H}_\mu(x, y, \omega_c)}{\sqrt{\mathcal{P}_\mu}} e^{j(\omega_c t - \beta_\mu(\omega_c)z)} \right\} \quad (\text{A.7})$$

where $A_\mu(z, t)$ represents the complex envelope of the field and its dimension is $\sqrt{W}$. The vector mode profiles of the electric and magnetic fields are respectively $\mathcal{E}_\mu(x, y, \omega_c)$ and $\mathcal{H}_\mu(x, y, \omega_c)$. Finally, $\mathcal{P}_\mu$ is the power normalization of the mode profiles and it is define as:

$$\mathcal{P}_\mu \delta_{\mu, \nu} = \frac{1}{4} \iint_{-\infty}^{+\infty} [(\mathcal{E}_\mu \times \mathcal{H}_\nu^*) + (\mathcal{E}_\nu^* \times \mathcal{H}_\mu)] \cdot \mathbf{e}_z dx dy \quad (\text{A.8})$$

Here and in the next step, the arguments (x, y, ω_c) were omitted. I substitute the modal decomposition of the field into Maxwell's equations A.1 and A.2 and multiply both sides by the conjugate electric or magnetic field. The following step will be to focus on solving the two equations on the left-hand side. For further details on the mathematical steps, see reference [16].

$$\begin{aligned}
[\nabla \times \mathbf{H}(\mathbf{r}, t)] \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} &= \left[A_\mu(z, t) e^{j(\omega_c t - \beta_\mu z)} \nabla \times \frac{\mathcal{H}_\mu}{\sqrt{\mathcal{P}_\mu}} \right. \\
&\quad \left. + \nabla A_\mu(z, t) e^{j(\omega_c t - \beta_\mu z)} \times \frac{\mathcal{H}_\mu}{\sqrt{\mathcal{P}_\mu}} \right] \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} \\
[\nabla \times \mathbf{E}(\mathbf{r}, t)] \cdot \frac{\mathcal{H}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} &= \left[A_\mu(z, t) e^{j(\omega_c t - \beta_\mu z)} \nabla \times \frac{\mathcal{E}_\mu}{\sqrt{\mathcal{P}_\mu}} \right. \\
&\quad \left. + \nabla A_\mu(z, t) e^{j(\omega_c t - \beta_\mu z)} \times \frac{\mathcal{E}_\mu}{\sqrt{\mathcal{P}_\mu}} \right] \cdot \frac{\mathcal{H}_\mu^*}{4\sqrt{\mathcal{P}_\mu}}
\end{aligned}$$

After a series of simplifications and calculations, the final equations on the left-hand side are as follows:

$$\begin{aligned}
[\nabla \times \mathbf{H}(\mathbf{r}, t)] \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} &= \frac{\partial A_\mu(z, t)}{\partial z} e^{j(\omega_c t - \beta_\mu z)} \cdot \frac{-\mathcal{E}_\mu^* \times \mathcal{H}_\mu \hat{\mathbf{z}}}{4\mathcal{P}_\mu} \\
[\nabla \times \mathbf{E}(\mathbf{r}, t)] \cdot \frac{\mathcal{H}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} &= \frac{\partial A_\mu(z, t)}{\partial z} e^{j(\omega_c t - \beta_\mu z)} \cdot \frac{\mathcal{E}_\mu \times \mathcal{H}_\mu^* \hat{\mathbf{z}}}{4\mathcal{P}_\mu}
\end{aligned}$$

The difference between the magnetic field and the electric field is calculated on the left-hand side.

$$[\nabla \times \mathbf{H}(\mathbf{r}, t)] \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} - [\nabla \times \mathbf{E}(\mathbf{r}, t)] \cdot \mathcal{H}_\mu^* \frac{1}{4\sqrt{\mathcal{P}_\mu}} = \quad (\text{A.9})$$

$$\frac{\partial A_\mu(z, t)}{\partial z} e^{j(\omega_c t - \beta_\mu z)} \cdot \frac{\mathcal{E}_\mu \times \mathcal{H}_\mu^* + \mathcal{E}_\mu^* \times \mathcal{H}_\mu}{4\mathcal{P}_\mu} \cdot \hat{\mathbf{z}} \quad (\text{A.10})$$

I substitute the modal decomposition of the field into Maxwell's equations A.1 and A.2 and multiply both sides by the conjugate electric or magnetic field. The following step will be to focus on solving the two equations on the

right-hand side.

$$\begin{aligned} \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} &= \epsilon_0 n^2 \frac{\partial}{\partial t} \left(A_\mu(z, t) e^{j(\omega_c t - \beta_\mu z)} \frac{\mathcal{E}_\mu \cdot \mathcal{E}_\mu^*}{\mathcal{P}_\mu} \right) + \frac{\partial \mathbf{P}^{nl}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} \\ \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{H}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} &= \mu_0 \frac{\partial}{\partial t} \left(A_\mu(z, t) e^{j(\omega_c t - \beta_\mu z)} \frac{\mathcal{H}_\mu \cdot \mathcal{H}_\mu^*}{\mathcal{P}_\mu} \right) \end{aligned}$$

The difference between the magnetic field and the electric field is calculated on the right-hand side.

$$\begin{aligned} &\frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} - \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{H}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} = \\ &-\frac{\partial A_\mu(z, t)}{\partial t} e^{j(\omega_c t - \beta_\mu z)} \left(\mu_0 \frac{\mathcal{H}_\mu \cdot \mathcal{H}_\mu^*}{4\mathcal{P}_\mu} + \epsilon_0 n^2 \frac{\mathcal{E}_\mu \cdot \mathcal{E}_\mu^*}{4\mathcal{P}_\mu} \right) - \frac{\partial \mathbf{P}^{nl}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} \end{aligned}$$

If the integral is applied over the entire cross-section, the definition of stored energy per unit length takes the following form:

$$W_t = \iint_{-\infty}^{+\infty} (\mu_0 \mathcal{H}_\mu \cdot \mathcal{H}_\mu^* + \epsilon_0 n^2 \mathcal{E}_\mu \cdot \mathcal{E}_\mu^*) dx dy$$

where $P = W_t \cdot v_g = W_t \cdot \frac{\partial \omega}{\partial \beta}$

by replacing the definition of group speed:

$$\frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} - \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{H}_\mu^*}{4\sqrt{\mathcal{P}_\mu}} = -\frac{\partial A_\mu(z, t)}{\partial t} e^{j(\omega_c t - \beta_\mu z)} \frac{\partial \omega}{\partial \beta} - \frac{\partial \mathbf{P}^{nl}(\mathbf{r}, t)}{\partial t} \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{\mathcal{P}_\mu}}$$

By replacing the equation A.4 of nonlinear polarization. In many practically relevant situations, only the core or the cladding material exhibits a $\chi^{(3)}$ nonlinearity, which is typically isotropic. The third-order nonlinear susceptibility tensor $\chi^{(3)}$ can therefore be considered negligible outside a nonlinear interaction region D_{inter} (with refractive index n_{inter}), while it remains finite or constant within D_{inter} [16]. Moreover, $\chi^{(3)}$ can be approximated by a scalar quantity $\tilde{\chi}^{(3)}$, such that $\tilde{\chi}^{(3)} : \mathcal{E}_\mu \mathcal{E}_\mu \mathcal{E}_\mu^* = \tilde{\chi}^{(3)} |\mathcal{E}_\mu|^2 \mathcal{E}_\mu^*$. [16]

Then it considers the stationary regime, assuming vanishing temporal derivatives ($\partial/\partial t = 0$), which makes the envelope time-independent.

$$-\mathbf{P}^{nl}(\mathbf{r}) \cdot \frac{\mathcal{E}_\mu^*}{4\sqrt{P_\mu}} = -j|A_\mu(z)|^2 A_\mu(z) \frac{3\omega_c \epsilon_0}{16P_\mu^2} \left(\chi^{(3)} |\mathcal{E}_\mu|^4 \right)$$

By combining the left-hand side and the right-hand side, I obtain the following final equation.

$$\begin{aligned} \frac{\partial A_\mu(z, t)}{\partial z} e^{j(\omega t - \beta_\mu z)} \cdot \frac{\mathcal{E}_\mu \times \mathcal{H}_\mu^* + \mathcal{E}_\mu^* \times \mathcal{H}_\mu}{4\mathcal{P}_\mu} \cdot \hat{\mathbf{z}} = \\ - \frac{\partial A_\mu(z, t)}{\partial t} e^{j(\omega t - \beta_\mu z)} \frac{\partial \omega}{\partial \beta} \\ - j|A_\mu(z)|^2 A_\mu(z) \frac{3\omega_c \epsilon_0}{16P_\mu^2} \left(\chi^{(3)} |\mathcal{E}_\mu|^4 \right) \end{aligned}$$

As before, it considers the stationary regime, assuming vanishing temporal derivatives ($\partial/\partial t = 0$), which makes the envelope time-independent.

$$\frac{\partial A_\mu(z)}{\partial z} \cdot \frac{(\mathcal{E}_\mu \times \mathcal{H}_\mu^* + \mathcal{E}_\mu^* \times \mathcal{H}_\mu) \cdot \hat{\mathbf{z}}}{4\mathcal{P}_\mu} = -j|A_\mu|^2 A_\mu \cdot \frac{3\omega_c \epsilon_0}{16P_\mu^2} \left(\chi^{(3)} |\mathcal{E}_\mu|^4 \right)$$

The imaginary part of the third-order susceptibility is substituted, which accounts for two-photon absorption (TPA) losses.

$$\text{Im}\{\chi^{(3)}\} = \frac{n^2 \beta_{TPA}}{3k_0 Z_0} \quad (\text{A.11})$$

where it is substituted into the following equation.

$$\frac{\partial}{\partial z} A_\mu(z) \frac{(\mathcal{E}_\mu \times \mathcal{H}_\mu^* + \mathcal{E}_\mu^* \times \mathcal{H}_\mu) \cdot \hat{\mathbf{z}}}{4\mathcal{P}_\mu} = -|A_\mu(z)|^2 A_\mu(z) \frac{3\omega_c \epsilon_0}{16 \cdot P_\mu^2} \cdot \frac{n^2 \beta_{TPA}}{3k_0 Z_0} |\mathcal{E}_\mu|^4$$

where the nonlinear waveguide parameter $k(x, y)$ is defined as:

$$k(x, y) = \frac{\omega_c \epsilon_0}{16P_\mu^2} \frac{n^2 \beta_{TPA}}{k_0 Z_0} |\mathcal{E}_\mu(x, y)|^4 \quad (\text{A.12})$$

Starting from the definition of optical power in terms of the modal envelope, $P \propto |A_\mu(z)|^2 = A_\mu(z)A_\mu^*(z)$.

$$\frac{\partial P(x,y,z)}{\partial z} = \left(A_\mu(z) \frac{\partial A_\mu^*(z)}{\partial z} + A_\mu^*(z) \frac{\partial A_\mu(z)}{\partial z} \right)$$

Applying the product rule to explicitly account for the evolution of both the envelope and its complex conjugate. Substituting the previously derived propagation equation $\frac{\partial A_\mu(z)}{\partial z} = -k(x,y)|A_\mu(z)|^2 A_\mu(z)$ and its conjugate.

$$\begin{aligned} \frac{\partial P(x,y,z)}{\partial z} &= A_\mu(z)(-k(x,y)|A_\mu(z)|^2 A_\mu^*(z)) + A_\mu^*(z)(-k(x,y)|A_\mu(z)|^2 A_\mu(z)) = \\ &= -2k(x,y)|A_\mu(z)|^4 \end{aligned}$$

Collecting terms shows that the power decay scales with the fourth power of the field amplitude, characteristic of two-photon absorption (TPA).

$$\frac{\partial P(x,y,z)/A}{\partial t} = -v_g 2k(x,y)|A_\mu(z)|^4$$

Using the relation $dz = v_g dt$, it transforms the spatial attenuation into a temporal rate equation, where v_g is the group velocity.

$$P(z) - P(z + \Delta z) = -2k(x,y)|A_\mu(z)|^4 \Delta z$$

Integrating over a finite propagation step Δz gives the discrete power loss due to nonlinear absorption.

The optical power is related to the photon flux S_{tot} by multiplying the photon energy hc/λ and accounting for the propagation at the group velocity.

$$P(x,y,z) = \frac{S_{tot} hc}{v_g \lambda}$$

Where the temporal variation of photon number is obtained by converting power variation into photon flux through the photon energy.

$$\frac{\partial S_{tot}}{\partial t} = \frac{\Delta z}{v_g} \frac{1}{\frac{hc}{\lambda}}$$

Substituting the previously derived expression for power loss, linking photon depletion directly to nonlinear absorption.

$$\frac{\partial P(x,y,z)}{\partial t} = \frac{\Delta z}{\frac{hc}{\lambda}} (-2k(x,y)|A_\mu(z)|^4)$$

The carrier generation rate is proportional to the absorbed photon flux, while recombination is modeled via a lifetime τ .

$$\frac{\partial N_{tot}}{\partial t} = \frac{2k(x,y)|A_\mu(z)|^4 \Delta z}{\frac{hc}{\lambda}} - \frac{N_{tot}}{\tau}$$

where the total number of carriers is expressed as the carrier density times the silicon volume. $N_{tot} = N_{cell} \cdot V_{Si}$ The interaction volume is defined by the modal cross-sectional area A and the propagation length Δz , so $V_{Si} = A \cdot \Delta z$. Instead N_{cell} is the number of carrier per unit cell

Rewrite the rate equation in terms of carrier density and impose steady-state conditions ($\partial N_{cell} / \partial t = 0$).

$$\frac{\partial N_{cell}}{\partial t} = \frac{2k(x,y)|A_\mu(z)|^4 \Delta z}{\frac{hc}{\lambda}} - \frac{N_{cell} A \Delta z}{\tau} = 0$$

Solving for the steady-state carrier density, showing its proportionality to the fourth power of the field amplitude.

$$N_{cell} = \frac{2k(x,y)|A_\mu(z)|^4 \tau}{\frac{hc}{\lambda} A}$$

Expressing the carrier density at the unit-cell level by explicitly resolving the transverse area ($\Delta x \Delta y$).

$$N_{cell} = \frac{2k(x,y)|A_\mu(z)|^4}{\frac{hc}{\lambda}} \frac{\tau}{(\Delta x \Delta y)}$$

The local carrier generation rate is defined as the ratio between the number of carriers and their lifetime.

$$G_{cell} = \frac{2k(x,y)|A_\mu(z)|^4}{\frac{hc}{\lambda}} \frac{1}{(\Delta x \Delta y)}$$

Finally, substituting the explicit expression of k in terms of the two-photon absorption coefficient β_{TPA} and modal field distribution (eq. A.12), obtaining the generation rate as a function of both material and modal parameters.

$$G_{cell}(x,y) = \frac{|A_\mu(z)|^4}{\frac{hc}{\lambda}} \beta_{TPA} \frac{n^2}{Z_0^2 8 P_\mu^2} |\mathcal{E}_\mu(x,y)|^4 \quad (\text{A.13})$$

where $|A_\mu(z)|^2$ is equal to circulating power P_c , so I obtain the equation 2.1. In the proposed framework, G_{cell} is used to denote the spatially resolved carrier generation rate. It can be demonstrated that, by construction, the surface integral of G over the entire cross-sectional domain of a rectangular waveguide is equivalent to the corresponding lumped formulation of the total generation rate. That is, $G_{ph,tot}$ is equivalent to the following equation:

$$G_{ph,tot} = \frac{1}{A} \int_A (G_{cell}) dx dy = \frac{\beta_{TPA} P_c^2}{2\hbar\omega A A_{eff}} \quad (\text{A.14})$$

It is important to note that the resulting generation profile is determined by the local electric field distribution of the guided mode. This eliminates the need to introduce additional quantities such as the effective area A_{eff} and the geometrical waveguide area A . This aspect is particularly advantageous for complex geometries (e.g., rib waveguides), where the definition of such parameters is not uniquely prescribed and may lead to inconsistencies.

Variation of modal loss and effective refractive index demonstrations

As discussed in chapter 2, the proposed model is characterized by distributed parameters across the entire waveguide cross-section, allowing for a spatially resolved description of the physical quantities involved. In this framework, the generated free carriers (FC) are not treated as lumped quantities but are described locally, and they contribute to variations in the silicon refractive index through both free-carrier dispersion (FCD) and self-heating effects.

The local changes in absorption and refractive index induced by free carriers are therefore defined as $\Delta\alpha_{\text{FCA,D}}(x,y)$ and $\Delta n_{\text{eff,FCD,D}}(x,y)$, respectively.

To determine the corresponding variation in the complex propagation constant of the waveguide, the analysis is carried out starting from the perturbation of the complex relative permittivity of the medium, expressed as $\Delta\varepsilon = \Delta\varepsilon_r + j\Delta\varepsilon_i$. Accordingly, the variation of the guided-mode propagation constant is given by:

$$\Delta\beta_{r,i} = \frac{\omega_0 \iint_{\infty} \Delta\varepsilon_{r,i} \cdot \mathcal{E}^*(x,y) \cdot \mathcal{E}(x,y) dx dy}{P(x,y)} \quad (\text{A.15})$$

where the normalization power is defined as:

$$P(x,y) = 2 \iint_{\infty} \text{Re}\{\mathcal{E}(x,y) \times \mathcal{H}^*(x,y)\} \cdot \mathbf{e}_z dx dy \quad (\text{A.16})$$

The variation of the complex electric permittivity due to FCA and FCD is defined as:

$$\Delta\varepsilon_{r,i}(x,y) = \varepsilon_0 \left(\left(n_{si} + \Delta n_{\text{FCD}}(x,y) + j\Delta\alpha_{\text{FC}}(x,y) \frac{\lambda}{4\pi} \right)^2 - n_{si}^2 \right) \quad (\text{A.17})$$

it is possible to approximate as:

$$\Delta\epsilon_r(x, y) \approx 2\epsilon_0 n_{si} \cdot \Delta n_{eff, FCD}(x, y) \quad (A.18)$$

$$\Delta\epsilon_i(x, y) \approx 2\epsilon_0 n_{si} \cdot \frac{\Delta\alpha_{FC}(x, y) \lambda}{4\pi} \quad (A.19)$$

Substituting the two equations for the real and imaginary electric permittivity into the equation for the propagation constant, we obtain the following equations:

$$\Delta\beta_r = \frac{2\omega_0 \cdot \epsilon_0 \cdot n_{si} \int \int_{-\infty}^{+\infty} \Delta n_{FCD}(x, y) \cdot |\mathcal{E}(x, y)|^2 dx dy}{2 \int \int_{-\infty}^{+\infty} \text{Re}\{\mathcal{E}(x, y) \times \mathcal{H}(x, y)^*\} \cdot \mathbf{e}_z dx dy} \quad (A.20)$$

$$\Delta\beta_i = \frac{2\omega_0 \epsilon_0 n_{si} \int \int_{-\infty}^{+\infty} \frac{\Delta\alpha_{FC}(x, y) \lambda}{4\pi} |\mathcal{E}(x, y)|^2 dx dy}{2 \int \int_{-\infty}^{+\infty} \text{Re}\{(\mathcal{E}(x, y) \times \mathcal{H}(x, y)^*) \cdot \mathbf{e}_z\} dx dy} \quad (A.21)$$

Finally, the effective refractive index variation is defined as:

$$\Delta n_{eff, FCD} = \frac{\lambda}{2\pi} \Delta\beta_r = - \frac{c \cdot \epsilon_0 \cdot n_{si} \int \int_{-\infty}^{+\infty} \Delta n_{FCD}(x, y) \cdot |\mathcal{E}(x, y)|^2 dx dy}{\int \int_{-\infty}^{+\infty} \text{Re}\{\mathcal{E}(x, y) \times \mathcal{H}(x, y)^*\} \cdot \mathbf{e}_z dx dy} \quad (A.22)$$

and the modal loss is:

$$\Delta\alpha = 2\Delta\beta_i = \frac{2c \epsilon_0 n_{si} \int \int_{-\infty}^{+\infty} \frac{\Delta\alpha_{FC}(x, y) \lambda}{4\pi} |\mathcal{E}(x, y)|^2 dx dy}{\int \int_{-\infty}^{+\infty} \text{Re}\{(\mathcal{E}(x, y) \times \mathcal{H}(x, y)^*) \cdot \mathbf{e}_z\} dx dy} \quad (A.23)$$

Optical simulation on RSoft CAD Layout

To construct a finite element model, two main steps are required:

- **Modeling:** this step involves transforming the physical system into a mathematical representation, focusing on a limited set of relevant

variables while neglecting the others. For complex systems, the domain is first divided into subsystems and then discretized into finite elements, on which the mathematical model is applied. Unlike analytical approaches, it is sufficient to use a model that accurately describes the relatively simple geometry of each element.

The analysis starts from the source-free harmonic form of the vector wave equation, considering an arbitrary, anisotropic, and lossy medium with PML boundary conditions. The finite element method (FEM) does not directly solve this equation; instead, it relies on a variational (functional) formulation derived from the differential operator of the original wave equation. In two dimensions, over the domain A , the functional is given by:

$$F(E) = \int_A \left[(\nabla \times E)^* \bar{s}^{-1} (\nabla \times E) - k_0^2 E^* \cdot \bar{\epsilon} \cdot E \right] dA$$

The electric field is expressed as $E(x, y, z) = E(x, y) e^{-j\beta z}$, where β is the propagation constant along the z -direction, $\bar{s}$ denotes the coordinate stretching tensor, and $\bar{\epsilon}$ represents the dielectric material.

The domain is then partitioned into a set of finite elements, each associated with a simple local basis that vanishes outside the element. The global solution is obtained by summing the contributions from all elements. In edge-based hybrid FEM formulations, the transverse components of the field are expanded using a vector basis:

$$E_T(x, y) e^{-j\beta z} = \sum_{i=1}^n N_i E_{Ti} = \sum_{i=1}^n (U_i \hat{x} + V_i \hat{y}) E_{Ti}$$

where E_{Ti} denotes the value of the electric field along each edge. A similar representation is used for the longitudinal component.

- **Discretization:** this step consists of dividing space (or time) to obtain a model with a finite number of degrees of freedom. The transition from a continuous domain to a discrete one inevitably introduces an approximation error in the exact solution of the mathematical model.

This error can be properly evaluated if a suitable mathematical model is available for the entire structure. FemSIM is a finite element method implemented within the RSoft CAD-Layout environment, capable of computing the modes of 1D, 2D, or 3D structures by automatically generating a non-uniform hybrid mesh. This mesh is composed of both triangular and rectangular elements with constant refractive index, defining the simulation domain together with the PML regions.

Thermal simulation on COMSOL Multiphysics

In COMSOL Multiphysics, several types of boundary conditions can be applied to the heat equation. Among these, the most relevant are prescribed temperature and heat flux conditions.

- **Fixed temperature:** this boundary condition enforces a fixed temperature value on a surface:

$$T = T_0$$

In this work, the ambient temperature is imposed at the bottom of the silicon substrate as well as along the external boundaries of the surrounding air domain.

- **Fixed heat flux:** this condition defines the heat flow across a boundary:

$$-\mathbf{n} \cdot \mathbf{q} = q_0$$

where:

- $\mathbf{q}$ is the conductive heat flux vector (SI unit: W/m^2), defined as $\mathbf{q} = -k\nabla T$, with k being the thermal conductivity of the material,
- $\mathbf{n}$ is the outward normal vector to the boundary,
- q_0 represents the inward heat flux (SI unit: W/m^2).

In general, q_0 may include contributions from multiple heat transfer mechanisms, such as convection and radiation. In this study, we consider the case $q_0 = 0$, corresponding to thermal insulation. This condi-

tion is applied at the external boundaries in contact with air, effectively creating an adiabatic system.

- **Surface-to-surface radiation:** this boundary condition accounts for radiative heat exchange between surfaces. Radiation is modeled as an energy transfer between boundaries and heat sources, assuming that the medium itself does not participate. Since the materials involved are not perfectly transparent, they are treated as semi-transparent. Diffuse surfaces are assumed, meaning that radiation is emitted uniformly in all directions. The boundary condition is expressed as:

$$-\mathbf{n} \cdot \mathbf{q} = \varepsilon (G - e_b(T))$$

where ε is the surface emissivity, G is the irradiation, and $e_b(T)$ is the total hemispherical emissive power of a black body. Both the radiative heat flux and the corresponding source term are defined as positive in the direction of the outward normal.

The emissivity ε quantifies the efficiency of a material in emitting thermal radiation, relative to an ideal black body. A perfect black body has $\varepsilon = 1$, whereas real materials satisfy $0 < \varepsilon < 1$. In our case, both silicon (waveguide or ring) and the silicon oxide coating on the substrate are characterized by $\varepsilon = 0.7$ [21].

- **Surface-to-ambient radiation:** this condition describes heat exchange between the structure and the surrounding environment. The net inward radiative heat flux is given by:

$$-\mathbf{n} \cdot \mathbf{q} = \varepsilon \sigma (T_{\text{amb}}^4 - T^4)$$

where σ is the Stefan–Boltzmann constant and T_{amb} is the ambient temperature.

Thermal analyses were performed using a 2D axisymmetric model for the ring resonators, which in this case is equivalent to a full 3D simulation. In contrast, straight waveguides were analyzed using a 2D model. In both cases, the finite element method (FEM) was employed.

Appendix B

Mathematical proofs of the equations of the hybrid laser model

B.0.1 Demonstration of the power circulating in the time domain

The intracavity field in the ring just before the coupler is denoted by E_c and the field after one round trip is denoted by E'_c . During propagation along the ring, the field undergoes an amplitude attenuation described by the factor A_{loss} and accumulates a phase shift given by $e^{-i\beta L}$. Consequently, the field following a single round trip can be expressed as follows:

$$E'_c = A_{loss} \cdot e^{-i\beta L} E_c, \quad (\text{B.1})$$

where L is the circumference of the ring and $\beta(\omega)$ is the propagation constant.

At the coupling region, the standard directional coupler relation applies.

$$E_c = tE'_c + ikE_{bus}, \quad (\text{B.2})$$

where E_{bus} is the input bus field, t is the self-coupling coefficient and k is the cross-coupling coefficient, which satisfy the normalization condition $t^2 + k^2 = 1$. By substituting the expression of E'_c into this equation, one obtains

$$E_c = tA_{loss} \cdot e^{-i\beta L} E_c + ikE_{bus}. \quad (\text{B.3})$$

The previous equation can be rearranged by collecting terms on the left-hand side, which results in the following equation:

$$E_c - tA_{loss}e^{-i\beta L}E_c = ikE_{bus}. \quad (B.4)$$

$$(1 - tA_{loss}e^{-i\beta L})E_c = ikE_{bus}, \quad (B.5)$$

This expression represents the exact frequency-domain equation of the ring resonator.

$$E_c(\omega) = \frac{-kE_{bus}(\omega)}{1 - t^2A_{loss}e^{i\beta(\omega)L}}. \quad (B.6)$$

The focus is now directed towards frequencies in proximity to a resonance frequency, denoted by the symbol, ω_0 , and written as $\omega = \omega_0 + \Delta\omega$. In this regime, it is possible to expand the propagation phase to first order as follows:

$$\beta(\omega)L \simeq \beta_0L + \frac{L}{v_g}\Delta\omega, \quad (B.7)$$

where v_g denotes the group velocity.

At resonance, the phase satisfies $\beta_0L = 2\pi m$, which implies $e^{-i\beta_0L} = 1$. If the ring is perturbed by a variation of the effective refractive index Δn_{eff} , in addition to a detuning between rings $\delta\omega_r$, the propagation phase becomes

$$\beta L = 2\pi m + \frac{\omega_0}{c}L\Delta n_{\text{eff}} + \frac{L}{v_g}(\Delta\omega + \delta\omega). \quad (B.8)$$

The equation can be obtained by expanding the exponential to first order:

$$e^{-i\beta L} \simeq 1 - i\frac{\omega_0}{c}L\Delta n_{\text{eff}} - i\frac{L}{v_g}(\Delta\omega + \delta\omega). \quad (B.9)$$

Substitution of this approximation into the cavity equation results in

$$\left[1 - tA_{loss} \left(1 - i\frac{\omega_0}{c}L\Delta n_{\text{eff}} - i\frac{L}{v_g}(\Delta\omega + \delta\omega) \right) \right] E_c = ikE_{bus}. \quad (B.10)$$

The expansion of the preceding equation results in the following:

$$(1 - t^2 A_{loss})E_c + it^2 A_{loss} \frac{\omega_0}{c} L \Delta n_{\text{eff}} E_c + it^2 A_{loss} \frac{L}{v_g} \Delta \omega E_c + it^2 A_{loss} \frac{L}{v_g} \delta \omega E_c = -k E_{\text{bus}}. \quad (\text{B.11})$$

Isolating the term proportional to $\Delta \omega$,

$$i \Delta \omega \left(\frac{L}{v_g} \right) E_c = -\frac{k}{t^2 A_{loss}} E_{\text{bus}} - \frac{1 - t^2 A_{loss}}{t^2 A_{loss}} E_c - i \frac{\omega_0}{c} L \Delta n_{\text{eff}} E_c - i \frac{L}{v_g} \delta \omega E_c. \quad (\text{B.12})$$

Finally, by exploiting the Fourier transform property

$$\mathcal{F}^{-1} [i \Delta \omega E_c(\Delta \omega)] = \frac{dE_c(t)}{dt}, \quad (\text{B.13})$$

and introducing the round-trip group delay $\tau_g = \frac{L}{v_g}$, results in the following time-domain dynamic equation for the intracavity field:

$$\frac{dE_c(t)}{dt} = \frac{1}{\tau_g} \left[-\frac{k}{t^2 A_{loss}} E_{\text{bus}}(t) - \frac{1 - t^2 A_{loss}}{t^2 A_{loss}} E_c(t) - i \frac{\omega_0}{c} L \Delta n_{\text{eff}} E_c(t) - i \frac{L}{v_g} \delta \omega_r E_c(t) \right]. \quad (\text{B.14})$$

B.0.2 Demonstration of the effective reflectivity with resonance detuning effect

The following mathematical derivation is utilised to calculate the effective reflectivity for the SiN feedback mirror, with the consideration of resonance detuning (the misalignment between the resonances of the two ring resonators). Reflectivity is defined as the ratio of the A_{c2} field to the E field. In order to calculate this ratio, it is first necessary to define A_{c2} and then express E explicitly.

$$A_{c2}(\omega) = ik A_{loss2}^{1/2} e^{i\phi(\omega)} E_{c2}(\omega) \quad (\text{B.15})$$

By explicitly writing $E_{c2}(\omega)$, the expression becomes

$$A_{c2}(\omega) = ikA_{loss2}^{1/2}e^{i\phi_0+i\omega\tau_{g2}}\frac{kA_{c1}(\omega)}{1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r2}}{v_g}\right)} \quad (\text{B.16})$$

By inserting the expression for $A_{c1}(\omega)$, the result becomes

$$A_{c2}(\omega) = ikA_{loss2}^{1/2}e^{i\phi_0+i\omega\tau_{g2}}\frac{k^2aE_{c1}(\omega)e^{-i\omega\tau_{g1}}}{\left[1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r2}}{v_g}\right)\right]\left[1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r1}}{v_g}\right)\right]} \quad (\text{B.17})$$

After grouping constants and phase factors, the expression can be rewritten as

$$A_{c2}(\omega) = ik^4A_{loss1,2}^{3/2}e^{i\phi_0+i\omega\tau_{g2}}\frac{t_{ssc}^2(1-T_{cout})e^{-i\Delta\phi-i\omega\tau_{eff}}E(\omega)}{\left[1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r2}}{v_g}\right)\right]\left[1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r1}}{v_g}\right)\right]} \quad (\text{B.18})$$

The effective reflection coefficient can then be defined as

$$r_{eff}(\omega) = ik^4A_{loss1,2}^{3/2}e^{i\phi_0+i\omega\tau_{g2}}\frac{t_{ssc}^2(1-T_{cout})e^{-i\Delta\phi-i\omega\tau_{eff}}}{\left[1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r2}}{v_g}\right)\right]\left[1-t^2a\left(1-\frac{i\Delta\omega}{v_g}L-\frac{i\delta\omega_{r1}}{v_g}\right)\right]} \quad (\text{B.19})$$

where $\delta\omega_{r1}$ and $\delta\omega_{r2}$ are, respectively, the resonance detuning for the first and second ring resonator.