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***Abstract*—Networks of coupled nonlinear oscillators are among the recently proposed computation structures that can possibly overcome bottlenecks and limitations of current designs. It has been shown that coupled oscillator networks are capable of solving complex combinatorial optimization problems, such as the MAX-CUT problem and the Boolean Satisfiability (SAT) problem. The goal of this work is to provide a theoretical framework for designing logic gates based on coupled nonlinear oscillators. We show how a simplified model for the network can be derived using the phase reduction technique. The phase deviation equations obtained are then used to design simple networks that achieve the desired phase patterns implementing the corresponding logic gates.**

I. Introduction

For almost a century, the von Neumann architecture has been the standard reference for the design of electronic computing machines, especially for the general purpose ones. Its most basic description encompasses an input module, a central processing unit, a memory bank and an output module. Sets of instructions (programs) can be written in the memory, and the processing unit accesses the program and processes input data performing a sequence of operations, including reading and writing repeatedly the memory, before ending the program and returning the output [1].

The von Neumann architecture represents the ideal hardware design of a Turing machine [1], [2], and it inherits its versatility and limitations. In fact, there is no known design that can be used to create better artificial general purpose computing machines and, at the same time, be completely and deterministically programmable and controllable. The same design naturally reveals its drawbacks, such as the famous *von Neumann bottleneck* [1], dictating that the system throughput is limited by the data transfer between CPU and memory, as the majority of the computation energy is used for the data movement rather than for the actual computation [1].

Recently, novel or rediscovered alternative architectures for computation have been proposed. Artificial intelligence, deep learning and artificial neural networks [3], [4], are driving industry and academia towards new computing architectures specialized for the training or the inference of neural networks [5], [6]. For these applications two main approaches exist: digital, such as Google's TPU [7], and analogue, as the memristor crossbar neural networks on chip [8]–[10].

The quest for non-conventional computing solutions has also recently revived the use of oscillators as building blocks for both von Neumann and non-von Neumann architectures. They were initially introduced independently by Goto [11], [12] and von Neumann [13] in the 1950s. In a network of nonlinear oscillators, information is encoded in the relative phase among the oscillators, and basic logic operations are implemented manipulating these phases, exploiting the network dynamics and couplings [11]–[15]. In the 1960s, machines called Parametrons implementing the Goto design were built in Japan [11], [12]. Parametrons saw some successes, but were soon eclipsed by the rapid growth of digital computers. Today, the interest is renewed not only because there are many other modern and more compact ways to integrate oscillators, ranging from ring oscillators [16] to spin-torque [17] and laser-based [18] structures and beyond [19]–[24], but also because coupled oscillators potentially represent a very low power computing system when employed in von Neumann architectures [15], [25], [26].

In this paper we present a theoretical framework for the analysis of networks of coupled nonlinear oscillators, and the design of logic gates based on this type of dynamical systems. We give a rigorous definition for the phase of nonlinear oscillators of arbitrary order, and we show how a phase equation, defining the time evolution of the oscillators' phases, can be derived from the state equations. The phase equation greatly simplifies the analysis of the network dynamics. In fact for a network composed by N nonlinear oscillators of order n, the problem of finding stable synchronous oscillations in a system of $n \times N$ ordinary differential equations (ODEs), is reduced to the quest of stable equilibrium points in a system of N ODEs. Stability analysis is simplified as well. Finally, we show how the phase equation can be used to design logic gates, implementing a complete set of logic operation. The procedure is based on a proper design of the couplings between the oscillators, in such a way that the stable synchronous states correspond to the desired set of phase relationships among the oscillators. The logic gates proposed are reciprocal, meaning that there is no formal distinction between input and output terminals. For instance for the proposed NOT gate, any of the two terminals can be used as input, the other becoming the output. For a three terminals gate, such as an OR or an AND gate, any pair of terminals can be used as inputs, and the remaining one becomes the output. We also discuss the advantages of reciprocal logic gates in terms of scalability and self-organizing properties of the network.

II. Nonlinear oscillators and phase dynamics

We consider nonlinear oscillators described by the ordinary differential equations (ODEs)

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{x} : \mathbb{R} \mapsto \mathbb{R}^n$ is the state of the oscillator, and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a $\mathcal{C}^1(\Omega \subseteq \mathbb{R}^n)$ vector valued function that describes the oscillator's dynamics. We assume that equation (1) admits of an asymptotically stable T-periodic solution $\mathbf{x}_s(t) = \mathbf{x}_s(t+T)$, corresponding to a limit cycle $\boldsymbol{\gamma}$ in its state space.

The phase of a nonlinear oscillator can be defined introducing the concept of isochrons. Consider a reference initial point on the limit cycle $\mathbf{x}_0 \in \boldsymbol{\gamma}$, and assign phase zero to this point $\phi(\mathbf{x}_0) = 0$. The phase of the solution $\mathbf{x}_s(t)$ with initial condition $\mathbf{x}_s(0) = \mathbf{x}_0$, at any arbitrary time instant is $\phi(\mathbf{x}_s(t)) = 2\pi t/T = \omega t$, where $\omega = 2\pi/T$ is the oscillator free running frequency. The isochron based at $\mathbf{x}_0 \in \boldsymbol{\gamma}$ is defined as

$$I_{\mathbf{x}_0} = \left\{ \mathbf{x}_\alpha(0) \in \mathbb{R}^n/\boldsymbol{\gamma} \ : \ \lim_{t\to+\infty} ||\mathbf{x}_\alpha(t) - \mathbf{x}_0(t)|| = 0 \right\} \tag{2}$$

That is, the isochron $I_{\mathbf{x}_0}$ is the $\mathbb{R}^{n-1}$ dimensional manifold consisting of all the initial conditions $\mathbf{x}_\alpha(0)$, such that the trajectories starting from $\mathbf{x}_\alpha(0)$ eventually meet on $\boldsymbol{\gamma}$ at $\mathbf{x}_0(t)$. We define the phase for points within the basin of attraction of $\boldsymbol{\gamma}$, assigning the same phase to all points belonging to the same isochron, that is, isochrons are the level sets of the scalar field $\phi(\mathbf{x})$.

To give a complete decomposition of the oscillator's state space, we also define a vector valued function $\mathbf{R}(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$, representing an amplitude deviation from the limit cycle. The amplitude deviation is most conveniently measured on the linear subspace locally tangent to the isochrons on the limit cycle. Consider an infinitesimal perturbation added to the vector field, and decompose this perturbation into two components: one along the tangent bundle to the limit cycle $T_{\boldsymbol{\gamma}}$, and the other along the tangent bundle to the isochron $T_{I_{\mathbf{x}_0}}$. As long as the perturbation is infinitesimal, the component tangent to the cycle is responsible for a phase shift, leaving the amplitude unchanged. Conversely, the component tangent to the isochron modifies the amplitude, without modifying the phase. This decomposition allows for a decoupling, up to linear terms, of the phase and amplitude dynamics [27], [28]. The coordinate transformation $\mathbf{x} \mapsto (\phi, \mathbf{R})$ is locally invertible in a small enough neighborhood of the limit cycle, and by their very definitions

$$\frac{d\phi}{dt} = \nabla\phi(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = \omega \tag{3}$$

and $\mathbf{R}(\mathbf{x}) \to 0$ for $t \to +\infty$.

A network composed by N coupled nonlinear oscillators can be conveniently described by the ODEs (from now on, subscripts will identify the oscillator)

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) + \varepsilon \sum_{j=1}^{N} \mathbf{g}_{ij}(\mathbf{x}_i, \mathbf{x}_j) \quad i = 1, \ldots, N \tag{4}$$

where $\varepsilon \ll 1$ is a parameter that measures the coupling strength, and $\mathbf{g}_{ij} : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^n$ are vector fields describing the coupling between the i-th and the j-th oscillators. For the sake of simplicity, we have assumed that all oscillators are identical (the vector field $\mathbf{f}$ is the same for all the oscillators) and that only pairwise couplings occur. The theory can be generalized to more complicated structures composed by nonidentical oscillators and to couplings involving more than two arguments.

Introducing $\phi_i(\mathbf{x}_i)$ and $\mathbf{R}_i(\mathbf{x}_i)$, representing the phase and amplitude of the i-th oscillator, respectively, it is straightforward to derive

$$\frac{d\phi_i}{dt} = \omega + \varepsilon \sum_{j=1}^{N} \Gamma_{i,j}(\phi_i, \phi_j, \mathbf{R}_i, \mathbf{R}_j) \tag{5a}$$

$$\frac{d\mathbf{R}_i}{dt} = \mathbf{F}_i(\phi_i, \mathbf{R}_i) + \varepsilon \sum_{j=1}^{N} \mathbf{G}_{ij}(\phi_i, \phi_j, \mathbf{R}_i, \mathbf{R}_j) \tag{5b}$$

where, taking into account that $\mathbf{x}_i = \mathbf{x}_i(\phi_i, \mathbf{R}_i)$,

$$\Gamma_{ij}(\phi_i, \phi_j, \mathbf{R}_i, \mathbf{R}_j) = \frac{\partial\phi_i}{\partial\mathbf{x}_i} \cdot \mathbf{g}_{ij}(\mathbf{x}_i, \mathbf{x}_j) \tag{6}$$

$$\mathbf{F}_i(\phi_i, \mathbf{R}_i) = \frac{\partial\mathbf{R}_i}{\partial\mathbf{x}_i} \, \mathbf{f}(\mathbf{x}_i) \tag{7}$$

$$\mathbf{G}_{ij}(\phi_i, \phi_j, \mathbf{R}_i, \mathbf{R}_j) = \frac{\partial\mathbf{R}_i}{\partial\mathbf{x}_i} \, \mathbf{g}_{ij}(\mathbf{x}_i, \mathbf{x}_j) \tag{8}$$

and $\partial\mathbf{R}_i/\partial\mathbf{x}_i$ is the Jacobian matrix of partial derivatives.

The asymptotic stability of the limit cycles implies that, for small values of ε, the amplitude deviation remains close to zero. Therefore, it is common assuming $\mathbf{R}_i \approx 0$ for all $i = 1, \ldots, N$, and the phase equation (5a) becomes

$$\frac{d\phi_i}{dt} = \omega + \varepsilon \sum_{j=1}^{N} \Gamma_{ij}(\phi_i, \phi_j) \tag{9}$$

Introducing the *phase deviation* $\psi_i = \phi_i - \omega t$ yields

$$\frac{d\psi_i}{dt} = \varepsilon \sum_{j=1}^{N} \Gamma_{ij}(\psi_i + t, \psi_j + t) \tag{10}$$

For small values of ε, the phase deviation is a slow (nearly constant) variable. Time averaging over one period does not introduce a large error, and leads to the averaged phase deviation equation

$$\frac{d\psi_i}{d\tau} = \sum_{j=1}^{N} \overline{\Gamma}_{ij}(\psi_j - \psi_i) \tag{11}$$

where $\tau = \varepsilon t$ is the slow time, and

$$\overline{\Gamma}_{ij}(\psi_j - \psi_i) = \frac{1}{T} \int_0^T \Gamma_{ij}(\psi_i + t, \psi_j + t) \, dt \tag{12}$$

The coupling function $\overline{\Gamma}_{ij}$ is periodic, as periodic functions are its arguments. Moreover, symmetry considerations imply that, in many practical situations, it is also an odd function, and therefore it can be expanded into a Fourier sine series. If it also low-pass, retaining only the leading terms of the series leads to the celebrated Kuramoto model

$$\frac{d\psi_i}{dt} = \varepsilon \sum_{j=1}^{N} \gamma_{ij} \sin(\psi_j - \psi_i) \tag{13}$$

where the coupling parameters γ_{ij} are the coefficients of the two variables Fourier series.

III. Logic gates based on coupled oscillators

We shall choose an oscillator as a reference, and we shall measure all phase deviations with respect to the its phase ϕ_R, that is

$$\psi_i = \phi_i - \phi_R \tag{14}$$

In logic gates based on nonlinear oscillators, bits of information are encoded into the phase deviations. A phase deviation $\psi_i = 0$ corresponds to logic state 0, whereas $\psi_i = \pi$ correspond to logic state 1. Information is manipulated through the oscillator dynamics and the couplings. The design of the logic gates proceeds as follows:

- Determine the phase pattern (stable equilibrium points of the phase deviation equation) corresponding to the desired truth table;
- Determine the phase deviation equation, and in particular the couplings, providing the desired phase pattern;
- Find the network of coupled oscillators implementing the desired phase deviation equation.

A. NOT gate

Consider the phase deviation equations

$$\frac{d\psi_i}{dt} = \rho \sin(\psi_i - \psi_j) - \gamma \sin(\psi_i - \psi_{D_i}) \tag{15a}$$

$$\frac{d\psi_j}{dt} = \rho \sin(\psi_j - \psi_i) \tag{15b}$$

where ρ and γ are real valued positive parameters, and ψ_{D_i} represents the phase deviation of an external driving signal, used to set the state of oscillator i.

Theorem 1 (NOT gate): The phase deviation equations (15), with $\rho > 0$ and $\gamma > 0$ admit the asymptotically stable equilibrium point $\overline{\psi}_i = \psi_{D_i}$, $\overline{\psi}_j = \psi_{D_i} + \pi$.

Proof: Equations (15) admit the equilibrium points $\overline{\psi}_i = \psi_{D_i} + k_i\pi$, $\overline{\psi}_j = \overline{\psi}_i + k_j\pi$, with $k_i, k_j = 1, 2, \ldots$. The Jacobian matrix at $\overline{\psi}_i = \psi_{D_i}$, $\overline{\psi}_j = \psi_{D_i} + \pi$ has determinant $\det(J) = \rho\gamma$, and trace $\mathrm{T}r(J) = -2\rho - \gamma$ implying that the eigenvalues are both real and negative. By Hartman-Grobman theorem, the equilibrium is locally asymptotically stable. □

Similar arguments show that the other equilibria, for example $\overline{\psi}_i = \overline{\psi}_j = \psi_{D_i}$ or those involving $\overline{\psi}_i = \psi_{D_i} + \pi$, are asymptotically unstable. Therefore the phase deviation equations (15) implements a NOT gate, where i-th oscillator locks in phase with the external driving ψ_{D_i}, and thus with the same logic state, while j-th oscillator locks to the driving with phase shift equal to π, and thus to the opposite logic state.

B. AND gate and OR gate

To form a complete set of logic gates, AND gate and OR gate have to be designed. This can be done designing a MAJORITY gate. MAJORITY gate is a three inputs, one output gate, that realizes either an AND gate, or an OR gate. Consider a network composed by three nonlinear oscillators denoted by i, j and k, subject to three external driving signals. The first driving signal, denoted by ψ_D, is applied to all three oscillators, its role being to determine the behavior of the MAJORITY. For $\psi_D = 0$ the MAJORITY gate works as an AND, while for $\psi_D = \pi$ it works as an OR. The other two drivings are applied to two oscillators, for example oscillators i and j, while oscillator k represents the output terminal. Consider the phase deviation equations

$$\begin{aligned}\frac{d\psi_i}{dt} =& -\gamma_i \sin(\psi_i - \psi_{D_i}) - \gamma \sin(\psi_i - \psi_D)\\ &-\gamma \sum_{m=j,k} \sin(\psi_i - \psi_m)\end{aligned} \tag{16a}$$

$$\begin{aligned}\frac{d\psi_j}{dt} =& -\gamma_j \sin(\psi_j - \psi_{D_j}) - \gamma \sin(\psi_j - \psi_D)\\ &-\gamma \sum_{m=i,k} \sin(\psi_j - \psi_m)\end{aligned} \tag{16b}$$

$$\frac{d\psi_k}{dt} = -\gamma \sin(\psi_k - \psi_D) - \gamma \sum_{m=i,j} \sin(\psi_k - \psi_m)$$

Theorem 2 (AND gate): Consider the phase deviation equation (16), with $\psi_D = 0$ and $\gamma_i = \gamma_j > \gamma > 0$. Then:

1) If $\psi_{D_i} = \psi_{D_j} = 0$, then $\overline{\psi}_i = \overline{\psi}_j = \overline{\psi}_k = 0$ is an asymptotically stable equilibrium point.
2) If $\psi_{D_i} = \pi$, and $\psi_{D_j} = 0$ (respectively $\psi_{D_i} = 0$, and $\psi_{D_j} = \pi$), then $\overline{\psi}_i = \pi$, $\overline{\psi}_j = \overline{\psi}_k = 0$ (respectively $\overline{\psi}_j = \pi$, $\overline{\psi}_i = \overline{\psi}_k = 0$) is an asymptotically stable equilibrium point.
3) If $\psi_{D_i} = \psi_{D_j} = \pi$ then $\overline{\psi}_i = \overline{\psi}_j = \overline{\psi}_k = \pi$ is an asymptotically stable equilibrium point.

Proof: Equilibrium points of the phase deviation equations are readily found. Consider function $V : [0, 2\pi[\times[0, 2\pi[\times[0, 2\pi[\mapsto \mathbb{R}$

$$\begin{aligned}V(\psi_i, \psi_j, \psi_k) =& -(\gamma_i \cos\psi_{D_i} + \gamma \cos\psi_D)(\cos\psi_i + N_i)\\ &-(\gamma_j \cos\psi_{D_j} + \gamma \cos\psi_D)(\cos\psi_j + N_j)\\ &-\gamma \cos\psi_D(\cos\psi_k + N_k) - \gamma[\cos(\psi_i - \psi_j)\\ &+\cos(\psi_i - \psi_k) + \cos(\psi_j - \psi_k)\\ &+N_{ij} + N_{ik} + N_{jk}]\end{aligned} \tag{17}$$

where

$$N_\alpha = \begin{cases} +1 & \text{if} \quad \overline{\psi}_\alpha = \pi \\ -1 & \text{if} \quad \overline{\psi}_\alpha = 0, \end{cases} \tag{18}$$

$$N_{\alpha\beta} = \begin{cases} +1 & \text{if} \quad \overline{\psi}_\alpha - \overline{\psi}_\beta = \pm\pi \\ -1 & \text{if} \quad \overline{\psi}_\alpha - \overline{\psi}_\beta = 0, \end{cases} \tag{19}$$

Function $V(\psi_i, \psi_j, \psi_k)$ is a strict Lyapunov function, thus the equilibrium points mentioned in the theorem are asymptotically stable. □

The stable phase patterns of theorem 2 correspond to the truth table of an AND gate as required. Proving that all other equilibrium points of the phase deviation equations are unstable is more involved, and it is currently under investigation.

Theorem 3 (OR gate): Consider the phase deviation equation (16), with $\psi_D = \pi$ and $\gamma_i = \gamma_j > 2\gamma > 0$. Then:

1) If $\psi_{D_i} = \psi_{D_j} = 0$, then $(\overline{\psi}_i = \overline{\psi}_j = \overline{\psi}_k = 0$ is an asymptotically stable equilibrium point.
2) If $\psi_{D_i} = 0$, and $\psi_{D_j} = \pi$ (respectively $\psi_{D_i} = \pi$, and $\psi_{D_j} = 0$), then $\overline{\psi}_i = 0, \overline{\psi}_j = \overline{\psi}_k = \pi$ (respectively $\overline{\psi}_j = 0, \overline{\psi}_i = \overline{\psi}_k = \pi$) is an asymptotically stable equilibrium point.
3) If $\psi_{D_i} = \psi_{D_j} = \pi$ then $\overline{\psi}_i = \overline{\psi}_j = \overline{\psi}_k = \pi$ is an asymptotically stable equilibrium point.

Proof: The proof is completely analogous to theorem 2. □

Similarly to theorem 2, theorem 3 establishes a set of phase patterns corresponding to the truth table of an OR gate. Again, proving that the desired pattern is the unique stable equilibrium point is very complicated, and require further analysis.

An important aspect to be mentioned is that the logic gates based on coupled nonlinear oscillators are reciprocal.

Definition 1 (Reciprocal logic gate): A logic gate is reciprocal if any terminal can be used as an input or as an output, indifferently.

For example, in the AND gate and in the OR gate oscillators i and j were used as inputs (by the application of the driving signals to set their states), and oscillator k has been used as the output. It is trivial to see, by simple symbol permutation, that the driving signals can be applied to any set $\{i, j\}$, $\{i, k\}$ or $\{j, k\}$, with the remaining oscillator working as the output.

IV. Conclusions

In this contribution we have presented a general framework for the design of logic gates implemented through coupled nonlinear oscillators. The design procedure is based on a phase reduction of the state equations of the coupled oscillators, that allows for a simplified description of the oscillator dynamics as far as the coupling is feeble. By defining an oscillator as a reference to measure phase deviations, we have shown the conditions that allow for the implementation of a complete set of logic operations (NOT, AND and OR). These gates share the important property of reciprocity, meaning that the input oscillators can be chosen at will among the ones forming the gate, thus leaving the remaining oscillator as the output.

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