Weak and strong convergence of derivations and stability of flows with respect to MGH convergence

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Luigi Ambrosio [*]   Federico Stra [†]   Dario Trevisan [‡]

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Abstract

This paper is devoted to the study of weak and strong convergence of derivations, and of the flows associated to them, when dealing with a sequence of metric measure structures $(X, \mathsf{d}, \mathfrak{m}_n)$, $\mathfrak{m}_n$ weakly convergent to $\mathfrak{m}$. In particular, under curvature assumptions, either only on the limit metric structure $(X, \mathsf{d}, \mathfrak{m})$ or on the whole sequence of metric measure spaces, we provide several stability results.

Contents

[*]Scuola Normale Superiore, luigi.ambrosio@sns.it

[†]Scuola Normale Superiore, federico.stra@sns.it

[‡]Università di Pisa, dario.trevisan@unipi.it

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1 Introduction

In this paper we study convergence of vector fields, more precisely derivations, and Lagrangian flows with respect to measured Gromov-Hausdorff convergence. The literature on the topic of MGH convergence is very wide (see for instance [GMS13], [GR07], [St06], [Sh14], [Vi09]) and the convergence problems can be attacked from different points of view, depending also on the richness of the structure under consideration. Our motivations come from two different directions: the first one is a deeper investigation of the convergence of gradient derivations, particularly in the case when all spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ belong to the same class $\mathrm{RCD}(K, \infty)$ of [AGS14b]. In this respect, our results provide a kind of local version of the Mosco convergence of the 2-Cheeger energies estabilished in [GMS13]. This local version will play a role in a forthcoming paper [AH16], where Mosco convergence is proved for all p-Cheeger energies, also with stability results for BV functions, Cheeger constants and Hessians. The second motivation comes as a natural complement of [AT14], where well-posedness of flows associated to sufficiently regular derivations $\boldsymbol{b}$ (see Theorem 8.1) is proved for the first time in a nonsmooth setting, which includes $\mathrm{RCD}(K, \infty)$ metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$. It is then natural to investigate the stability question (as in the Euclidean theory [AC13]), assuming that the derivations depend on n and that the reference measures $\mathfrak{m}_n$ are variable.

In this paper, when dealing with these problems, we systematically adopt the so-called extrinsic point of view in MGH convergence. Namely, up to an isometric embedding, we assume that neither X nor d depend on n. We assume then that $\mathfrak{m}_n$ are locally uniformly finite and nonnegative Borel measures in (X, d) which weakly converge to $\mathfrak{m}$, namely $\int_X v \, \mathrm{d}\mathfrak{m}_n \to \int_X v \, \mathrm{d}\mathfrak{m}$ for all $v \in C_{\mathrm{bs}}(X)$, where $C_{\mathrm{bs}}(X)$ is the space of bounded continuous functions with bounded support. As illustrated for instance in [GMS13] (where several notions of convergence are carefully compared, for sequences of pointed metric measure spaces), this point of view is not really restrictive, and basically unavoidable when treating Lagrangian questions such as the convergence of flows. Indeed, some coupling between points in the different spaces is necessary to prove convergence of paths to paths, and the simplest way to achieve this is to embed all metric structures into a common one. Still in connection with the metric structure (X, d), we assume only completeness and separability; dropping local compactness assumptions is useful to include in the theory all $\mathrm{RCD}(K, \infty)$ spaces.

Now we pass to a more detailed description of the content of this paper. It consists of a first part where we recall the basic properties of derivations and two more parts, an "Eulerian" one, dealing with the convergence of derivations and particularly of gradient derivations, and a "Lagrangian" part, dealing with the convergence of flows, that can be read almost independently.

Part I. In the seminal paper [W00] the concept of derivation is used to build a good notion of tangent bundle in metric spaces, the main idea being to describe the bundle implicitly through the collection of its ($\mathfrak{m}$-measurable) sections. These sections, called derivations, are linear maps $\boldsymbol{b} : \mathrm{Lip}_{\mathrm{b}}(X) \to L^0(X, \mathfrak{m})$ which satisfy suitable continuity and locality properties. In this paper we almost completely adopt the point of view of [G15b],

where besides linearity one assumes the validity of the inequality

$$|\boldsymbol{b}(f)| \leq h|\mathrm{D}f| \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } f \in \mathrm{Lip}_{\mathrm{b}}(X). \tag{1.1}$$

The only difference is that, since we are dealing with a sequence of measures, we keep $\mathrm{Lip}_{\mathrm{b}}(X)$ as domain of derivations, to have $\mathfrak{m}$-independent domains, instead of the Sobolev space considered in [G15b]. In [G15b] a systematic analysis of first and second order calculus based on this notion is made, including in particular the L^2 duality between tangent and cotangent bundle, see Remark 3.5. The quantity $|\mathrm{D}f|$ in the right hand side of (1.1) is the minimal relaxed slope of [Ch99], which provides integral representation to Cheeger's energy Ch, see Appendix A for a quick introduction to this concept. By duality, the minimal h in (1.1) is denoted by $|\boldsymbol{b}|$ (strictly speaking, we should use the notation $\mathsf{Ch}_{\mathfrak{m}}$, $|\boldsymbol{b}|_{\mathfrak{m}}$ to stress the dependence of these concepts on $\mathfrak{m}$). The L^2 duality estabilished in [G15b] is particularly useful for us to read the Hilbertian character of the norm $|\boldsymbol{b}|$ in terms of the cotangent bundle, i.e. in terms of quadraticity of Cheeger's energy. According to the theory developed in [AGS14a] and [AGS14b], this is the analytically most convenient formulation.

In order to treat weak convergence of derivations from the sequential point of view it is technically useful to consider a countable algebra $\mathscr{A} \subset \mathrm{Lip}_{\mathrm{b}}(X)$ of "test" functions. In the spirit of Gromov's reconstruction theorem [GR07] and many other results of the theory (see also [K04] in connection with Cheeger's coordinates in PI spaces and the more recent work [CKS15]) it is natural to consider the algebra finitely generated by truncations of distance functions. To make it separable, we consider a countable dense set $D \subset X$ and define $\mathscr{A}$ as in (2.3), also considering the subalgebra $\mathscr{A}_{\mathrm{bs}}$ of functions with bounded support. In connection with this choice, denoting by $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ the domain of Ch, we shall also need this approximation result (where $\mathrm{Lip}_{\mathrm{a}}$ stands for the asymptotic Lipschitz constant, see (2.1)): *for all f in $H^{1,2}(X,\mathsf{d},\mathfrak{m})$, there exist $f_n \in \mathscr{A}_{\mathrm{bs}}$ with $f_n \to f$ and $\mathrm{Lip}_{\mathrm{a}}(f_n) \to |\mathrm{D}f|$ in $L^2(X,\mathfrak{m})$.*

Its proof, given in Appendix B, is a further refinement of the techniques and of the results developed in [AGS14a] (see also [ACDM15]), where the approximating functions where chosen, as in [Ch99], in the larger class $\mathrm{Lip}_{\mathrm{b}}(X) \cap L^2(X,\mathfrak{m})$.

Part II. In this part we discuss weak convergence of derivations and criteria for strong convergence. Even though some of our results deal with time-dependent derivations (more natural for the study of flows, in Part III), we present in this introduction only the autonomous case. We say that $\boldsymbol{b}_n$, derivations in $(X,\mathsf{d},\mathfrak{m}_n)$, *weakly* converge in duality with $\mathscr{A}_{\mathrm{bs}}$ to $\boldsymbol{b}$, derivation in $(X,\mathsf{d},\mathfrak{m})$, and write $\boldsymbol{b}_n \xrightarrow{\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$, if

$$\lim_{n\to\infty} \int_X \boldsymbol{b}_n(f)v \,\mathrm{d}\mathfrak{m}_n = \int_X \boldsymbol{b}(f)v \,\mathrm{d}\mathfrak{m} \qquad \forall f \in \mathscr{A}_{\mathrm{bs}}, \ v \in C_{\mathrm{b}}(X). \tag{1.2}$$

Recall that the divergence $\mathrm{div}\,\boldsymbol{b}$ of a derivation is defined by duality, via the formula $\int_X f \,\mathrm{div}\,\boldsymbol{b} \,\mathrm{d}\mathfrak{m} = -\int_X \boldsymbol{b}(f) \,\mathrm{d}\mathfrak{m}$ for all $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$ (as for Ch and $|\boldsymbol{b}|$, we should use the notation $\mathrm{div}_{\mathfrak{m}}$, to stress its dependence on $\mathfrak{m}$). In presence of uniform bounds on $|\boldsymbol{b}_n|$ and on their divergences, Theorem 4.1 provides a sequential compactness result totally independent of regularity assumptions on the metric measure structures. Due to the close relation between vector fields with bounds on divergence and normal currents (see in particular the discussion in [PS14, Appendix A]), this result is reminiscent of the compactness result for normal currents in [AK00]. Notice that the above mentioned approximation result, provided in Appendix B, is necessary to pass to the limit in the

inequality $|\boldsymbol{b}_n(f)| \leq |\boldsymbol{b}_n|_{\mathfrak{m}_n} |\mathrm{D}f|_{\mathfrak{m}_n}$, using as intermediate step the weaker inequality with the asymptotic Lipschitz constant in the right hand side.

When we deal with gradient derivations the bounds on divergence correspond to Laplacian bounds, which could be too restrictive for some applications. For this reason, we exploit the regularizing properties of the heat semigroup P_t. Indeed, under suitable regularity assumptions (only) on the limit metric measure structure $(X, \mathsf{d}, \mathfrak{m})$, which are satisfied by all $\mathrm{RCD}(K, \infty)$ spaces, it is technically convenient to replace in (1.2) the algebra $\mathscr{A}_{\mathrm{bs}}$ by a more regular class of test functions, namely $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}} := \{P_t f : \ t \in \mathbb{Q}_+, \ f \in \mathscr{A}_{\mathrm{bs}}\}$, letting v vary in $C_{\mathrm{bs}}(X)$ (since the support of $P_t f$ may have infinite measure, unless all measures $\mathfrak{m}_n$ are finite). With this new notion of weak convergence, for which we use the notation $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$, we provide another sequential compactness result, see Theorem 4.4, free of divergence bounds; in Remark 4.5 we show that, under uniform bounds on divergence, the two notions of convergence are equivalent.

In addition, for the convergence in duality with $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$, we provide a criterion for strong L^p convergence, namely

$$\lim_{n \to \infty} \int_X |\boldsymbol{b}_n(f)|^p \, \mathrm{d}\mathfrak{m}_n \leq \int_X |\boldsymbol{b}(f)|^p \, \mathrm{d}\mathfrak{m} \qquad \forall f \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}, \tag{1.3}$$

under the assumption that there is no loss of norm in the limit, namely

$$\limsup_{n \to \infty} \int_X |\boldsymbol{b}_n|^p \, \mathrm{d}\mathfrak{m}_n \leq \int_X |\boldsymbol{b}|^p \, \mathrm{d}\mathfrak{m} < \infty.$$

The proof of this criterion, given in Theorem 5.3, is probably the most technical part of this paper: its proof follows closely ideas from the theory of Young measures ([Va90], [AGS08, Section 5.4]), with the extra difficulties due to the fact that we do not have a pointwise description of the tangent bundle (but, via a suitable concept of pre-derivation, we provide a kind of replacement for it).

Finally, we conclude this part by discussing convergence of gradient derivations. If $f_n \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}_n)$ strongly converge in $H^{1,2}$ to $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ (in particular $\limsup_n \mathsf{Ch}_n(f_n) \leq \mathsf{Ch}(f)$) we prove in Theorem 6.4, under a Mosco convergence assumption that the induced derivations $\boldsymbol{b}_{f_n}$ strongly converge in L^2 to $\boldsymbol{b}_f$, meaning that $\boldsymbol{b}_{f_n}(a)$ converge in L^2 to $\boldsymbol{b}_f(a)$ for all $a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$. The Mosco convergence assumption is fulfilled by sequences of $\mathrm{RCD}(K, \infty)$ spaces, as in [GMS13]; these results will play an important role in [AH16].

Part III. In this part we discuss the stability of flows w.r.t. strong convergence of derivations. As in [AT14], we say that $\boldsymbol{X}(t, x)$ is a regular flow relative to a possibly time dependent derivation $\boldsymbol{b}_t$, $t \in (0, T)$, if $\boldsymbol{X}(0, x) = x$, $\boldsymbol{X}(\cdot, x)$ is absolutely continuous in $[0, T]$ for $\mathfrak{m}$-a.e. x and solves the ODE $\gamma' = \boldsymbol{b}_t \circ \gamma$ in the following weak sense:

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \boldsymbol{X}(t, x) = \boldsymbol{b}_t(f)(\boldsymbol{X}(t, x)) \quad \text{for } \mathscr{L}^1 \times \mathfrak{m}\text{-a.e. } (t, x) \in (0, T) \times X, \tag{1.4}$$

for all $f \in \mathrm{Lip}_{\mathrm{b}}(X)$. The adjective "regular" refers, as in the Euclidean theory (see [AC13] and the references therein) to the non-concentration condition $\boldsymbol{X}(t, \cdot)_{\#}\mathfrak{m} \leq C\mathfrak{m}$ for all $t \in (0, T)$, with $C = C(X, \mathfrak{m})$. When the limit structure $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space and the regularity assumptions on $\boldsymbol{b}$ of Theorem 8.1 hold (which ensure uniqueness of the regular flow $\boldsymbol{X}$ relative to $\boldsymbol{b}_t$), we are able to provide in Theorem 8.2 a convergence result for regular flows $\boldsymbol{X}_n$ relative to $\boldsymbol{b}_{n,t}$, assuming strong convergence of $\boldsymbol{b}_{n,t}$ to $\boldsymbol{b}_t$, uniform growth bounds and $\sup_n C(\boldsymbol{X}_n, \mathfrak{m}_n) < \infty$. Here, convergence is undertood as convergence in measure, namely convergence of the $C([0, T]; X)$-valued maps $x \mapsto \boldsymbol{X}_n(\cdot, x)$ to the map

$x \mapsto \boldsymbol{X}(\,\cdot\,,x)$; the notion of convergence in measure can be adapted to our case, where even the reference measures are variable.

The strategy is to prove, via tightness estimates, convergence to a regular generalized flow $\boldsymbol{\pi}$ relative to $\boldsymbol{b}_t$ (see Definition 7.4 for this more general concept of flow) and then use the regularity of $\boldsymbol{b}_t$ to extract a "deterministic" flow out of it. The key step, where strong convergence of the derivations $\boldsymbol{b}_{n,t}$ is involved, is that the ODE condition (1.4) passes to the limit; in the proof of this we use a new principle (see Proposition 7.6) based on the continuity equation which would lead also to the simplification of some proofs of the Euclidean theory (for instance, [AC13, Theorem 12]).

Part I
Preliminary results

2 Notation

Metric concepts. In a metric space (X, d), we denote by $B_r(x)$ and $\bar{B}_r(x)$ the open and closed balls respectively, by $C_{\mathrm{bs}}(X)$ the space of continuous functions with bounded support, by $\mathrm{Lip}_{\mathrm{bs}}(X) \subset C_{\mathrm{bs}}(X)$ the subspace of Lipschitz functions. We use the notation $C_{\mathrm{b}}(X)$ and $\mathrm{Lip}_{\mathrm{b}}(X)$ for bounded continuous and bounded Lipschitz functions respectively.

For a function $f : X \to \mathbb{R}$ we denote by $\mathrm{Lip}(f) \in [0, \infty]$ its Lipschitz constant. We also define the asymptotic Lipschitz constant by

$$\mathrm{Lip}_{\mathrm{a}} f(x) = \inf_{r>0} \mathrm{Lip}\left(f\big|_{B_r(x)}\right) = \lim_{r \to 0^+} \mathrm{Lip}\left(f\big|_{B_r(x)}\right), \tag{2.1}$$

which is upper semicontinuous. If $I \subset \mathbb{R}$ is an interval, we denote by $AC(I; X) \subset C(I; X)$ the space of absolutely continuous maps w.r.t. d, satisfying

$$\mathsf{d}(\gamma(s), \gamma(t)) \leq \int_s^t g(r)\, \mathrm{d}r \qquad \text{for all } s,\, t \in I \text{ with } s \leq t \tag{2.2}$$

for some (nonnegative) $g \in L^1(I)$. We denote by $\mathrm{ev}_t : C(I; X) \to X$ the evaluation map at time t, i.e. $\mathrm{ev}_t(\gamma) = \gamma(t)$.

The metric derivative $|\dot{\gamma}| : \mathring{I} \to [0, \infty]$ of $\gamma \in AC(I; X)$ is the Borel map defined by

$$|\dot{\gamma}|(t) := \limsup_{s \to t} \frac{\mathsf{d}(\gamma(s), \gamma(t))}{|s - t|}$$

and it can be proved (see for instance [AGS08, Theorem 1.1.2]) that the lim sup is a limit $\mathscr{L}^1$-a.e. in I, that $|\dot{\gamma}| \in L^1(I)$ and that $|\dot{\gamma}|$ is the smallest L^1 function with the property (2.2), up to $\mathscr{L}^1$-negligible sets.

The metric algebras $\mathscr{A}$, $\mathscr{A}_{\mathrm{bs}}$. In the sequel we associate to any separable metric space (X, d) a countable dense set $D \subset X$ and the smallest set $\mathscr{A} \subset \mathrm{Lip}_{\mathrm{b}}(X)$ containing

$$\min\{\mathsf{d}(\,\cdot\,, x), k\} \quad \text{with } k \in \mathbb{Q} \cap [0, \infty],\ x \in D, \tag{2.3}$$

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which is a vector space over $\mathbb{Q}$ and is stable under products and lattice operations. It is a countable set and it depends only on the choice of D (we do not emphasize this dependence in our notation). We shall also consider the subalgebra $\mathscr{A}_{\mathrm{bs}}$ of functions with bounded support.

Measure-theoretic notation. The Borel σ-algebra of a metric space (X, d) is denoted by $\mathscr{B}(X)$. The Borel signed measures with finite total variation are denoted by $\mathscr{M}(X)$, while we use the notation $\mathscr{M}^+(X)$, $\mathscr{M}^+_{\mathrm{loc}}(X)$, $\mathscr{P}(X)$ for nonnegative finite Borel measures, Borel measures which are finite on bounded sets and Borel probability measures. For $E \in \mathscr{B}(X)$ the restriction operator $\mu \mapsto \mu \llcorner E$ is defined by $\mu \llcorner E(B) = \mu(B \cap E)$ for all $B \in \mathscr{B}(X)$.

We use the standard notation $L^p(X, \mathfrak{m})$, $L^p_{\mathrm{loc}}(X, \mathfrak{m})$ for the L^p spaces when $\mathfrak{m}$ is non-negative, including also the case $p < 1$ (when $p = 0$ the spaces $L^0(X, \mathfrak{m}) = L^0_{\mathrm{loc}}(X, \mathfrak{m})$ correspond to the class of real-valued $\mathfrak{m}$-measurable functions). Notice that, in this context where no local compactness assumption is made, L^p_{loc} means p-integrability on bounded subsets.

Given metric spaces (X, d_X) and (Y, d_Y) and a Borel map $f : X \to Y$, we denote by $f_\#$ the induced push-forward operator, mapping $\mathscr{P}(X)$ to $\mathscr{P}(Y)$, $\mathscr{M}^+(X)$ to $\mathscr{M}^+(Y)$ and, if the preimage of bounded sets is bounded, $\mathscr{M}^+_{\mathrm{loc}}(X)$ to $\mathscr{M}^+_{\mathrm{loc}}(Y)$. Notice that, for all $\mu \in \mathscr{M}^+(X)$, $f_\# \mu$ is well defined also if f is μ-measurable.

Definition 2.1 (Metric measure space)**.** A *metric measure space* is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m} \in \mathscr{M}^+_{\mathrm{loc}}(X)$.

Convergence of functions and measures. We say that $f_n \in \mathrm{Lip}_{\mathrm{b}}(X)$ converge in the *flat sense* to $f \in \mathrm{Lip}_{\mathrm{b}}(X)$ if $f_n \to f$ pointwise in X and $\sup_n(\sup|f_n| + \mathrm{Lip}(f_n)) < \infty$. If pointwise convergence occurs only $\mathfrak{m}$-a.e., we say that $f_n \to f$ in the $\mathfrak{m}$-*flat sense*.

We say that $\mathfrak{m}_n \in \mathscr{M}_{\mathrm{loc}}(X)$ weakly converge to $\mathfrak{m} \in \mathscr{M}_{\mathrm{loc}}(X)$ if $\int_X v \, \mathrm{d}\mathfrak{m}_n \to \int_X v \, \mathrm{d}\mathfrak{m}$ as $n \to \infty$ for all $v \in C_{\mathrm{bs}}(X)$. When all the measures $\mathfrak{m}_n$ as well as $\mathfrak{m}$ are probability measures, this is equivalent to requiring that $\int_X v \, \mathrm{d}\mathfrak{m}_n \to \int_X v \, \mathrm{d}\mathfrak{m}$ as $n \to \infty$ for all $v \in C_{\mathrm{b}}(X)$. We shall also use the following well-known proposition.

Proposition 2.2. *If* $\mathfrak{m}_n$ *weakly converge to* $\mathfrak{m}$ *in* $\mathscr{M}^+_{\mathrm{loc}}(X)$ *and if*

$$\limsup_{n \to \infty} \int_X \Theta \, \mathrm{d}\mathfrak{m}_n < \infty$$

for some Borel $\Theta : X \to [0, \infty]$, *then*

$$\lim_{n \to \infty} \int_X v \, \mathrm{d}\mathfrak{m}_n = \int_X v \, \mathrm{d}\mathfrak{m} \tag{2.4}$$

for all $v : X \to \mathbb{R}$ *continuous with* $\lim_{\mathsf{d}(x,\bar{x}) \to \infty} |v|(x)/\Theta(x) = 0$ *for some (and thus all)* $\bar{x} \in X$. *If* $\Theta : X \to [0, \infty)$ *is continuous and*

$$\limsup_{n \to \infty} \int_X \Theta \, \mathrm{d}\mathfrak{m}_n \le \int_X \Theta \, \mathrm{d}\mathfrak{m} < \infty,$$

then (2.4) *holds for all* $v : X \to \mathbb{R}$ *continuous with* $|v| \le C\Theta$ *for some constant* C.

For a sequence $(\mathfrak{m}_n) \subset \mathscr{M}^+_{\mathrm{loc}}(X)$ that weakly converges to $\mathfrak{m} \in \mathscr{M}^+_{\mathrm{loc}}(X)$, we introduce, following the presentation in [GMS13], notions of weak and strong convergence for a sequence of Borel functions $f_n : X \to \mathbb{R}$, $n \ge 1$. It is difficult to trace when these concepts, very natural when variable reference measures are involved, have been introduced for the first time; good references in geometric setups are [HU86] and [KS03].

Weak convergence. For $p \in (1, \infty]$, we say that $f_n \in L^p(X, \mathfrak{m}_n)$ converge to $f \in L^p(X, \mathfrak{m})$ weakly (weakly-* if $p = \infty$) in L^p if $f_n \mathfrak{m}_n$ weakly converge to $f \mathfrak{m}$ in $\mathscr{M}_{\mathrm{loc}}(X)$, with

$$\limsup_{n \to \infty} \|f_n\|_{L^p(X, \mathfrak{m}_n)} < \infty. \tag{2.5}$$

In such a situation, one has $\|f\|_{L^p(X, \mathfrak{m})} \leq \liminf_n \|f_n\|_{L^p(X, \mathfrak{m}_n)}$. Moreover, if $p \in (1, \infty)$, any sequence $f_n \in L^p(X, \mathfrak{m}_n)$ such that (2.5) holds admits a subsequence weakly converging to some $f \in L^p(X, \mathfrak{m})$.

We notice that, if two sequences $f_n, g_n \in L^p(X, \mathfrak{m}_n)$ weakly converge in L^p respectively to $f, g \in L^p(X, \mathfrak{m})$ and satisfy $0 \leq f_n \leq g_n h$, $\mathfrak{m}_n$-a.e. in X, where $h : X \to \mathbb{R}^+$ is bounded and upper semicontinuous, then $f \leq gh$, $\mathfrak{m}$-a.e. in X. This follows from duality with $\varphi \in C_{\mathrm{bs}}(X)$, $\varphi \geq 0$, and the fact that h is the pointwise infimum of bounded continuous functions.

Strong convergence. For $p \in (1, \infty)$ we say that $f_n \in L^p(X, \mathfrak{m}_n)$ converge to f strongly in L^p if, in addition to weak convergence in L^p, one has

$$\limsup_n \|f_n\|_{L^p(X, \mathfrak{m}_n)} \leq \|f\|_{L^p(X, \mathfrak{m})}. \tag{2.6}$$

This condition, in the standard setting of a *fixed* Banach space, is reminiscent of the Kadec-Klee property, which yields strong convergence out of weak convergence plus convergence of norms. In our more general situation, one can prove that, under (2.6), weak convergence of $f_n \mathfrak{m}_n \in \mathscr{M}_{\mathrm{loc}}(X)$ to $f \mathfrak{m}$ improves to

$$\lim_{n \to \infty} \int_X v_n f_n \mathfrak{m}_n = \int_X v f \mathfrak{m}, \quad \forall v_n \in L^{p'}(X, \mathfrak{m}_n) \text{ weakly converging in } L^{p'} \text{ to } v, \tag{2.7}$$

where p' is the conjugate exponent of p. Moreover, given two sequences $f_n, g_n \in L^p(X, \mathfrak{m}_n)$ strongly converging in L^p respectively to $f, g \in L^p(X, \mathfrak{m})$, one can show that $f_n + g_n$ strongly converges in L^p to $f + g$. Other improvements, which also motivate the terminology strong convergence, are discussed in Remark 2.6.

In the special case $p = 1$, we say that $f_n \in L^1(X, \mathfrak{m}_n)$ converge to $f \in L^1(X, \mathfrak{m}_n)$ weakly in L^1 if $f_n \mathfrak{m}_n$ weakly converge to $f \mathfrak{m}$ in $\mathscr{M}_{\mathrm{loc}}(X)$. Since in the context of variable measures it seems not easy to formulate an adequate notion of equi-integrability, this notion is strictly weaker than the classical weak L^1 convergence when the measure is fixed. However, sufficient conditions for sequential compactness w.r.t. weak L^1 convergence can be stated also in the case of variable measures.

Lemma 2.3. *The following properties hold.*

(i) *If $|f_n| \leq g_n h_n$ with g_n bounded in L^2 and h_n strongly convergent in L^2, then (f_n) has subsequences $(f_{n(k)})$ weakly convergent in L^1.*

(ii) *If $\sup_n \int_X \Theta(|f_n|) \, \mathrm{d}\mathfrak{m}_n < \infty$ for some $\Theta : [0, \infty) \to [0, \infty]$ with more than linear growth at infinity, then (f_n) has subsequences $(f_{n(k)})$ weakly convergent in L^1. In addition, if Θ is convex and lower semicontinuous,*

$$\int_X \Theta(|f|) \, \mathrm{d}\mathfrak{m} \leq \liminf_{k \to \infty} \int_X \Theta(|f_{n(k)}|) \, \mathrm{d}\mathfrak{m}_{n(k)}.$$

Proof. (i) Let h be the strong L^2 limit of h_n and $L = \sup_n \|g_n\|_{L^2(X, \mathfrak{m}_n)}$. It is easy to check that any limit point μ of $f_n \mathfrak{m}_n$ satisfies $\mu(C) \leq L(\int_C h^2 \, \mathrm{d}\mathfrak{m})^{1/2}$ for all $C \subset X$ closed and bounded, hence $\mu \ll \mathfrak{m}$.

(ii) It follows by a classical duality argument, see for instance [AGS08, Lemma 9.4.3]. $\square$

All such notions of weak and strong convergence have natural local analogues. In particular, we say that f_n converge to f weakly in L^p_{loc} (resp. weakly-* if $p = \infty$) if $f_n v$ converge to $f v$ weakly in L^p (resp. weakly-* if $p = \infty$) for all $v \in C_{\text{bs}}(X)$. In the case of strong L^p_{loc} convergence, $p \in (1, \infty)$, we ask that $f_n \mathfrak{m}_n$ weakly converge to $f \mathfrak{m}$ in $\mathscr{M}_{\text{loc}}(X)$, with

$$\limsup_{n \to \infty} \|f_n\|_{L^p(B_R(\bar{x}), \mathfrak{m}_n)} \leq \|f\|_{L^p(B_R(\bar{x}), \mathfrak{m})} < \infty$$

for some $\bar{x} \in X$ and arbitrarily large R. It follows immediately that (2.7) holds for all v_n weakly converging in $L^{p'}$ to v with $\cup_n \operatorname{supp} v_n$ bounded in X.

Even in the simplest case when $\mathfrak{m}_n = \mathfrak{m}$, another natural notion of convergence for metric-valued maps is convergence in measure.

Convergence in measure for metric space valued maps. Given a metric space (Y, d_Y), we say that Borel functions $f_n : X \to Y$ (locally) converge in measure to a Borel $f : X \to Y$ if, for every $\Phi \in C_{\text{b}}(Y)$ one has that $\Phi(f_n) \mathfrak{m}_n$ weakly converge to $\Phi(f) \mathfrak{m}$, namely

$$\lim_{n \to \infty} \int_X v \Phi(f_n) \, \mathrm{d}\mathfrak{m}_n = \int_X v \Phi(f) \, \mathrm{d}\mathfrak{m} \qquad \forall v \in C_{\text{bs}}(X). \tag{2.8}$$

Notice that, choosing Φ to be constant, already (2.8) encodes the weak convergence of $\mathfrak{m}_n$ to $\mathfrak{m}$, so strictly speaking this should be understood as a convergence of pairs $(\mathfrak{m}_n, f_n)$, where $\mathfrak{m}_n$ are reference measures for f_n. It is also easy to check that, when $\mathfrak{m}_n = \mathfrak{m}$ is fixed, this requirement is equivalent to the usual notion of local convergence in $\mathfrak{m}$-measure.

In the following proposition we provide equivalent formulations of the convergence in measure, closely related to the theory of Young measures [Va90]. To this aim, we introduce the following terminology: we say that a family $(\mu_n) \subset \mathscr{M}^+_{\text{loc}}(X \times Y)$ is locally tight w.r.t. X if, for some $\bar{x} \in X$ and all $R > 0$, the measures $\mu_n \, \llcorner \, B_R(\bar{x}) \times Y$ are tight. Accordingly, we denote by $C_{\text{bsx}}(X \times Y)$ the space

$$C_{\text{bsx}}(X \times Y) := \{ f \in C_{\text{b}}(X \times Y) : \operatorname{supp}(f) \subset B_R(\bar{x}) \times F \text{ for some } \bar{x} \in X, \, R > 0 \}$$

and we say that $\mu_n \xrightarrow{C_{\text{bsx}}(X \times Y)} \mu$ if $\mu_n \to \mu$ in the duality with $C_{\text{bsx}}(X \times Y)$.

Proposition 2.4. *Let (Y, d_Y) be a metric space, let $\mathfrak{m}_n$ be weakly convergent to $\mathfrak{m}$ in $\mathscr{M}^+_{\text{loc}}(X)$ and let $f_n, f : X \to Y$ be Borel. The following conditions are equivalent:*

(a) *f_n converge to f in measure;*

(b) *for any $\Phi \in C_{\text{b}}(Y)$ and any $v_n \in L^1(X, \mathfrak{m}_n)$ weakly converging to $v \in L^1(X, \mathfrak{m})$ in L^1 with $\cup_n \operatorname{supp}(v_n) \subset B_R(\bar{x})$ for some $\bar{x} \in X$ and $R > 0$, one has*

$$\lim_{n \to \infty} \int_X v_n \Phi(f_n) \, \mathrm{d}\mathfrak{m}_n = \int_X v \Phi(f) \, \mathrm{d}\mathfrak{m};$$

(c) *$(\text{Id} \times (\Phi \circ f_n))_\# \mathfrak{m}_n$ converge in duality with $C_{\text{bsx}}(X \times \mathbb{R})$ to $(\text{Id} \times (\Phi \circ f))_\# \mathfrak{m}$ for any $\Phi \in C_{\text{b}}(Y)$.*

Under the assumption that $(\text{Id} \times f_n)_\# \mathfrak{m}_n$ are locally tight w.r.t. X, (a), (b), (c) are also equivalent to $(\text{Id} \times f_n)_\# \mathfrak{m}_n \xrightarrow{C_{\text{bsx}}(X \times Y)} (\text{Id} \times f)_\# \mathfrak{m}$.

Proof. For ease of notation, write $\mu_n := (\mathrm{Id} \times f_n)_\# \mathfrak{m}_n$, and $\mu = (\mathrm{Id} \times f)_\# \mathfrak{m}$. It is obvious that the convergence of μ_n to μ in duality with $C_{\mathrm{bsx}}(X \times Y)$ implies (c), and that (b) implies (a).

We first prove the implications (c) $\implies$ (b) and (a) $\implies$ (c), which provide the equivalence between (a), (b), (c). In the proof of these two implications, without any loss of generality we may assume that Y is a compact interval of $\mathbb{R}$, say $Y = [0,1]$, and that Φ is the identity map (it is sufficient to argue with fixed $\Phi \in C_{\mathrm{b}}(X)$ and replace f_n with $\Phi \circ f_n$, f with $\Phi \circ f$).

(c) $\implies$ (b). Let v_n, v as in (b), with $\sup_n \|v_n\|_{L^1(X, \mathfrak{m}_n)} \leq 1$, and let $w \in C_{\mathrm{bs}}(X)$. By weak convergence of the measures $v_n \mathfrak{m}_n$ to $v\mathfrak{m}$,

$$\lim_{n \to \infty} \int_X v_n f_n \, \mathrm{d}\mathfrak{m}_n = \lim_{n \to \infty} \int_X v_n w \, \mathrm{d}\mathfrak{m}_n + \int_X v_n (f_n - w) \, \mathrm{d}\mathfrak{m}_n = \int v w \, \mathrm{d}\mathfrak{m},$$

because

$$\begin{aligned}
\limsup_{n \to \infty} \left| \int_X v_n (f_n - w) \, \mathrm{d}\mathfrak{m}_n \right| &= \limsup_{n \to \infty} \left| \int_{X \times Y} v_n(x) \, (y - w(x)) \, \mathrm{d}\mu_n(x,y) \right| \\
&\leq \limsup_{n \to \infty} \left\{ M \int_{B_R(\bar{x}) \cap \{|v_n| \leq M\}} |y - w(x)| \, \mathrm{d}\mu_n(x,y) + \mathfrak{m}_n \left(\{|v_n| > M\} \right) \right\} \\
&\leq M \int_{\bar{B}_R(\bar{x}) \times Y} |y - w(x)| \, \mathrm{d}\mu(x,y) + \frac{1}{M} \\
&= M \int_{\bar{B}_R(\bar{x})} |f(x) - w(x)| \, \mathrm{d}\mathfrak{m}(x) + \frac{1}{M},
\end{aligned}$$

and $M > 0$ is arbitrary. By density of $C_{\mathrm{bs}}(X)$ in $L^1_{\mathrm{loc}}(\bar{B}_R(\bar{x}), \mathfrak{m})$, we may optimize the choice of w and then let $M \to \infty$ to conclude.

(a) $\implies$ (c). The family (μ_n) is locally tight w.r.t. X, thanks to the local tightness of $\mathfrak{m}_n$ in X (recall that we are assuming that Y is compact). In addition, any weak limit point σ of (μ_n) in duality with $C_{\mathrm{bsx}}(X \times Y)$ agrees with μ on test functions $\phi(x,y)$ of the form $\psi(x)\Phi(y)$, with $\psi \in C_{\mathrm{bs}}(X)$ and $\Phi \in C_{\mathrm{b}}(Y)$. This class of functions is sufficient to prove that $\sigma = \mu$, hence μ_n weakly converge to μ in duality with $C_{\mathrm{bsx}}(X \times Y)$.

Finally, for general metric spaces (Y, d_Y), if we assume that (μ_n) is locally tight w.r.t. X, the argument used to prove that (a) implies (c) works and provides the weak convergence of μ_n to μ in duality with $C_{\mathrm{bsx}}(X \times Y)$. $\square$

Remark 2.5 (The case of probability measures). In the study of flows, we shall need convergence in measure of $C([0,T]; X)$-valued maps; in this case the reference measures for our maps will be probability measures (actually marginal measures of probabilities defined on $C([0,T]; X)$). In this simpler case the results of Proposition 2.4 can be strengthened. Indeed when all $\mathfrak{m}_n$ as well as $\mathfrak{m}$ are probability measures, an equivalent formulation of the convergence in measure (2.8) is with test functions $v \in C_{\mathrm{b}}(X)$. In addition, stating (b) without any bound on the support of v_n and replacing $C_{\mathrm{bsx}}(X \times Y)$ with $C_{\mathrm{b}}(X \times Y)$, in this modified form (a), (b), (c) are still equivalent; finally, assuming a priori the tightness of μ_n, these notions are equivalent to the weak convergence of μ_n to μ in $\mathscr{P}(X \times Y)$.

In the following remark we compare strong convergence with convergence in measure.

Remark 2.6 (Convergence in measure versus strong L^p convergence). In the case $Y = \mathbb{R}$ it is natural to compare strong convergence in L^p, $p \in (1, \infty)$, with convergence in measure. As in the case of a fixed measure, for sequences convergent in measure and uniformly bounded in L^p_{loc} as in (2.5) for some $p \in (1, \infty)$, one can use the stability of convergence

in measure under truncations and property (b) to show strong convergence in L^q_{loc} for any $q \in (1, p)$ and then (thanks to weak L^p compactness), weak convergence in L^p if the bound is uniform in L^p. Conversely, if f_n converge strongly in L^p, strong convexity of the function $\Theta(z) = |z|^p$ and a detailed analysis of the limit points of the measures $\mu = (\mathrm{Id} \times f_n)_\# \mathfrak{m}_n$ in the duality with $C_{\mathrm{bsx}}(X \times \mathbb{R})$ (see for instance [AGS08, Section 5.4], [GMS13]) show that $f_n \to f$ in measure.

Normed L^∞ modules. Recall that a (real) $L^\infty(X, \mathfrak{m})$-module M is a real vector space with the additional structure of bilinear multiplication by $L^\infty(X, \mathfrak{m})$ functions $\chi : m \in M \mapsto \chi m \in M$, with the associativity property $\chi(\chi' m) = (\chi \chi') m$; in addition, multiplication by $\lambda \in \mathbb{R}$ corresponds to the multiplication by the $L^\infty(X, \mathfrak{m})$ function equal $\mathfrak{m}$-a.e. to λ.

We say that $L^\infty(X, \mathfrak{m})$-module M is a $L^2(X, \mathfrak{m})$-normed module if there exists a "pointwise norm" $|\cdot| : M \to \{f \in L^2(X, \mathfrak{m}) : f \geq 0\}$ satisfying:

(a) $|m + m'| \leq |m| + |m'|$ $\mathfrak{m}$-a.e. in X for all $m, m' \in M$;

(b) $|\chi m| = |\chi||m|$ $\mathfrak{m}$-a.e. in X for all $m \in M$, $\chi \in L^\infty(X, \mathfrak{m})$;

with the property that

$$\|m\| := \left(\int_X |m(x)|^2 \, d\mathfrak{m}(x) \right)^{1/2} \tag{2.9}$$

is a norm in M. Notice that the homogeneity and the subadditivity of $\|\cdot\|$ are obvious consequences of (a), (b).

If M is a $L^2(X, \mathfrak{m})$-normed module, we can define M^* as the set of all continuous $L^\infty(X, \mathfrak{m})$-module isomorphisms $L : M \to L^1(X, \mathfrak{m})$, i.e. linear maps satisfying, for some $h \in L^2(X, \mathfrak{m})$ and all $\chi \in L^\infty(X, \mathfrak{m})$, $m \in M$, the properties

$$L(\chi m) = \chi L(m),$$

$$|L(m)| \leq h|m| \qquad \mathfrak{m}\text{-a.e. in } X.$$

The set M^* can also be given the structure of $L^2(X, \mathfrak{m})$-module, defining $|L|$ as the least function h (in the $\mathfrak{m}$-a.e. sense) satisfying the inequality above.

Finally, we say that a $L^2(X, \mathfrak{m})$-normed module M is a Hilbert module if $(M, \|\cdot\|)$ is a Hilbert space when endowed with the natural norm in (2.9).

The following result has been proved in [G15b]; see Proposition 1.2.21 and Theorem 1.2.24 therein.

Theorem 2.7. *Let $(X, \mathfrak{m})$ be a measure space and let M be a $L^2(X, \mathfrak{m})$-normed Hilbert module. Then the following facts hold.*

(i) The map

$$\langle m, m' \rangle_x := \frac{|m + m'|(x)^2 - |m - m'|(x)^2}{4}$$

provides a $L^\infty(X, \mathfrak{m})$-bilinear and continuous map from $M \times M$ to $L^1(X, \mathfrak{m})$, with $\||m|\|^2_{L^2(X,\mathfrak{m})} = \|\langle m, m \rangle\|_{L^1(X,\mathfrak{m})}$.

(ii) (Riesz theorem) For all $L \in M^*$ there exists $m_L \in M$ such that $|L| = |m_L|$ $\mathfrak{m}$-a.e. in X (in particular $\|L\| = \|m_L\|$) and

$$L(m)(x) = \langle m_L, m \rangle_x \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X, \text{ for all } m \in M.$$

3 Derivations

The following definition is inspired by Weaver [W00], but in the statement of the quantitative locality property we follow more closely the point of view systematically pursued in [G15b]. Recall that $|\mathrm{D}f|$ denotes the minimal relaxed slope, as defined in Appendix A.

Definition 3.1 (Derivations)**.** A *derivation* on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is a linear functional $\boldsymbol{b} : \mathrm{Lip}_{\mathrm{b}}(X) \to L^0(X, \mathfrak{m})$ which, for some $h \in L^0(X, \mathfrak{m})$, satisfies the quantitative locality property

$$|\boldsymbol{b}(f)| \leq h |\mathrm{D}f| \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } f \in \mathrm{Lip}_{\mathrm{b}}(X).$$

The $\mathfrak{m}$-a.e. smallest function h with this property is denoted by $|\boldsymbol{b}|$ and is called (local) norm of $\boldsymbol{b}$. We denote by $\mathrm{Der}(X, \mathsf{d}, \mathfrak{m})$ the space of derivations. For $p \in [1, \infty]$, we denote by $\mathrm{Der}^p(X, \mathsf{d}, \mathfrak{m})$ (resp. $\mathrm{Der}^p_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$) the space of derivations $\boldsymbol{b}$ such that $|\boldsymbol{b}| \in L^p(X, \mathfrak{m})$ (resp. $L^p_{\mathrm{loc}}(X, \mathfrak{m})$).

Notice that, thanks to the locality of the minimal relaxed slope $|\mathrm{D}f|$ on Borel sets, the choice of $\mathrm{Lip}_{\mathrm{b}}(X)$ instead of $\mathrm{Lip}_{\mathrm{bs}}(X)$ is not so relevant, because any functional $\boldsymbol{b} : \mathrm{Lip}_{\mathrm{bs}}(X) \to L^0(X, \mathfrak{m})$ satisfying the locality property can be canonically extended to a derivation $\tilde{\boldsymbol{b}}$, without increasing its norm, by setting

$$\tilde{\boldsymbol{b}}(f)(x) = \boldsymbol{b}(f_R)(x) \qquad \mathfrak{m}\text{-a.e. in } B_R(\bar{x}),$$

where $f_R \in \mathrm{Lip}_{\mathrm{bs}}(X)$ coincides with f in $B_R(\bar{x})$. As a matter of fact, when integrals are involved, we shall more often work with $\mathrm{Lip}_{\mathrm{bs}}(X)$.

As illustrated in details in [G15b], arguments analogous to [AK00] provide the chain rule (with $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz)

$$\boldsymbol{b}(\varphi \circ f) = (\varphi' \circ f)\boldsymbol{b}(f) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f \in \mathrm{Lip}_{\mathrm{b}}(X)$$

and the Leibniz rule

$$\boldsymbol{b}(fg) = \boldsymbol{b}(f)g + f\boldsymbol{b}(g) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f, g \in \mathrm{Lip}_{\mathrm{b}}(X)$$

as a consequence of additivity and locality (in connection with the Leibniz property, see also Remark 3.3 below).

Any $\boldsymbol{b} \in \mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m})$ can be canonically extended to $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ if Ch is quadratic, retaining the condition $|\boldsymbol{b}(f)| \leq |\boldsymbol{b}||\mathrm{D}f|$ using the existence, for all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, of Lipschitz functions f_n with $|\mathrm{D}(f - f_n)| \to 0$ in $L^2(X, \mathfrak{m})$ (so that $\boldsymbol{b}(f_n)$ is a Cauchy sequence in $L^1(X, \mathfrak{m})$). More generally, this extension is possible whenever $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is reflexive.

We can now define pre-derivations by restricting the domain to $\mathscr{A}_{\mathrm{bs}}$ and imposing a weaker locality condition, with $\mathrm{Lip}_{\mathrm{a}}(f)$ in place of $|\mathrm{D}f|$ in the right hand side. As for derivations, thanks to locality on open sets of $\mathrm{Lip}_{\mathrm{a}}$, there is essentially no difference in considering $\mathscr{A}$ or $\mathscr{A}_{\mathrm{bs}}$ as the domain of a pre-derivation.

Definition 3.2 (Pre-derivations)**.** A *pre-derivation* on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathbb{Q}$-linear functional $\boldsymbol{b} : \mathscr{A}_{\mathrm{bs}} \to L^0(X, \mathfrak{m})$ which, for some $h \in L^0(X, \mathfrak{m})$, satisfies the locality property

$$|\boldsymbol{b}(f)| \leq h \, \mathrm{Lip}_{\mathrm{a}}(f) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } f \in \mathscr{A}_{\mathrm{bs}}.$$

Remark 3.3. Any pre-derivation satisfies the Leibniz rule as a consequence of additivity and locality. Indeed, it is sufficient to optimize over rational λ and μ the $\mathfrak{m}$-a.e. inequality

$$|\boldsymbol{b}(fg) - \lambda\boldsymbol{b}(g) - \mu\boldsymbol{b}(f)| \le h\operatorname{Lip}_a(fg - \lambda g - \mu f) = h\operatorname{Lip}_a((f-\lambda)(g-\mu))$$
$$\le h\left[|f-\lambda|\operatorname{Lip}_a(g-\mu) + |g-\mu|\operatorname{Lip}_a(f-\lambda)(\right].$$

Remark 3.4 (Different axiomatizations). Assume that $\boldsymbol{b}: \operatorname{Lip}_b(X) \to L^0(X,\mathfrak{m})$ is a linear map satisfying the weaker locality condition

$$|\boldsymbol{b}(f)| \le h\operatorname{Lip}_a(f) \qquad \mathfrak{m}\text{-a.e. in } X.$$

for some $h \in L^1_{\mathrm{loc}}(X,\mathfrak{m})$ and the continuity w.r.t. flat convergence

$$f_n \to f \text{ in } \operatorname{Lip}_b(X) \text{ implies } \boldsymbol{b}(f_n) \to \boldsymbol{b}(f) \text{ weakly in } L^1_{\mathrm{loc}}(X,\mathfrak{m}). \tag{3.1}$$

Then, $\boldsymbol{b} \in \operatorname{Der}^1(X,\mathsf{d},\mathfrak{m})$ and $|\boldsymbol{b}| \le h$. Indeed, the inequality can be improved, getting $|\mathrm{D}f|$ in the right hand side, using the continuity property and Theorem A.1. Notice that an even weaker locality property will be considered in Lemma 5.2 (ii), and that (3.1) is satisfied when $\operatorname{div}\boldsymbol{b} \in L^1_{\mathrm{loc}}$, see Lemma 3.9.

Remark 3.5 (Derivations as duals of the cotangent bundle). Notice that $\operatorname{Der}(X,\mathsf{d},\mathfrak{m})$ has a natural structure of $L^\infty(X,\mathfrak{m})$-module, and that $\operatorname{Der}^2(X,\mathsf{d},\mathfrak{m})$ has the structure of $L^2(X,\mathfrak{m})$-normed module, provided by the map $\boldsymbol{b} \mapsto |\boldsymbol{b}|$. More specifically, we shall use at some stage that, if $H^{1,2}(X,\mathsf{d},\mathfrak{m})$ is reflexive (this assumption is not needed in [G15b] because Gigli's derivations are already defined on $H^{1,2}(X,\mathsf{d},\mathfrak{m})$), $\operatorname{Der}^2(X,\mathsf{d},\mathfrak{m})$ can be canonically and isometrically identified with the dual of the cotangent bundle $T^*(X,\mathsf{d},\mathfrak{m})$ introduced in [G15b]; in this respect, $\operatorname{Der}^2(X,\mathsf{d},\mathfrak{m})$ can be thought as the collection of all L^2 sections of the tangent bundle of $(X,\mathsf{d},\mathfrak{m})$.

Let us recall first the construction of $T^*(X,\mathsf{d},\mathfrak{m})$ from [G15b]. It arises as the completion of the $L^2(X,\mathfrak{m})$ normed pre-module consisting of finite collections (E_i, f_i) with $E_i \in \mathscr{B}(X)$ partition of X and $f_i \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. Two collections (E_i, f_i) and (F_j, g_j) are identified if $f_i = g_j$ $\mathfrak{m}$-a.e. in $E_i \cap F_j$. The sum of (E_i, f_i) and (F_j, g_j) is defined in the natural way by taking the collection $(E_i \cap F_j, f_i + g_j)$ and the local norm $|(E_i, f_i)| \in L^2(X,\mathfrak{m})$ of (E_i, f_i) is defined by

$$|(E_i, f_i)|(x) := |\mathrm{D}f_i|(x), \quad \mathfrak{m}\text{-a.e. } x \in E_i.$$

Thanks to the locality properties of the minimal relaxed slope, this definition does not depend on the choice of the representative. Multiplication by Borel functions χ taking finitely many values is defined by

$$\chi(E_i, f_i) = (E_i \cap F_j, z_j f_i) \qquad \text{with } \chi = \sum_{j=1}^N z_j \chi_{F_j}$$

and satisfies $|\chi(E_i, F_i)| = |\chi||(E_i, F_i)|$. By completion of this structure we obtain the $L^2(X,\mathfrak{m})$ normed module $T^*(X,\mathsf{d},\mathfrak{m})$.

Now, given any $\boldsymbol{b} \in \operatorname{Der}^2(X,\mathsf{d},\mathfrak{m})$ we can define

$$L_{\boldsymbol{b}}([E_i, f_i]) := \sum_i \chi_{E_i} \boldsymbol{b}(f_i)$$

to obtain a $L^1(X,\mathfrak{m})$-valued functional on the pre-module which satisfies

$$|L_{\boldsymbol{b}}([E_i, f_i])| \le |\boldsymbol{b}||[E_i, f_i]| \qquad \mathfrak{m}\text{-a.e. in } X$$

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and is easily seen to be linear w.r.t. multiplications by Borel functions χ with finitely many values. By completion, $\hat{\boldsymbol{b}}$ extends uniquely to a continuous $L^\infty(X, \mathfrak{m})$-module homomorphism from $T^*(X, \mathsf{d}, \mathfrak{m})$ to $L^1(X, \mathfrak{m})$ with $|L_{\boldsymbol{b}}| \leq |\boldsymbol{b}|$. From the definition of $|\boldsymbol{b}|$ we immediately obtain that $|L_{\boldsymbol{b}}| = |\boldsymbol{b}|$. Conversely, any continuous $L^\infty(X, \mathfrak{m})$-module homomorphism $L : T^*(X, \mathsf{d}, \mathfrak{m}) \to L^1(X, \mathfrak{m})$ induces $\boldsymbol{b} \in \mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m})$, via the formula $\boldsymbol{b}(f) := L([X, f])$.

In the following lemma, to derive a strict convexity property of the norm of derivations, we make the assumption that Ch is quadratic, which amounts to say that the cotangent bundle is a Hilbert $L^2(X, \mathfrak{m})$ module. Of course this convexity could be equally well be taken as an assumption, but we preferred this path (and the use of Remark 3.5) because the quadraticity condition on Ch is the most established assumption in the recent literature on RCD spaces.

Lemma 3.6. *If* Ch *is quadratic, for all* $\boldsymbol{c}$, $\boldsymbol{c}' \in \mathrm{Der}(X, \mathsf{d}, \mathfrak{m})$ *the condition*

$$|\boldsymbol{c} + \boldsymbol{c}'| = |\boldsymbol{c}| + |\boldsymbol{c}'| \qquad \mathfrak{m}\text{-a.e. in } X \tag{3.2}$$

implies

$$\boldsymbol{c}(f) = \lambda(\boldsymbol{c} + \boldsymbol{c}')(f) \qquad \mathfrak{m}\text{-a.e. in } X \text{ for all } f \in \mathrm{Lip}_{\mathrm{b}}(X),$$

with $\lambda = |\boldsymbol{c}|/(|\boldsymbol{c} + \boldsymbol{c}'|)$ *on* $\{|\boldsymbol{c} + \boldsymbol{c}'| > 0\}$ *(thus independent of f).*

Proof. Let $E = \{x \in X : \ \min\{|\boldsymbol{c}|(x), |\boldsymbol{c}'|(x)\} > 0\}$. We obviously need only to show that $\boldsymbol{c}(f) = \lambda(\boldsymbol{c} + \boldsymbol{c}')(f)$ $\mathfrak{m}$-a.e. in E for all $f \in \mathrm{Lip}_{\mathrm{b}}(X)$. Possibly multiplying $\boldsymbol{c}$ and $\boldsymbol{c}'$ by a positive Borel function χ with $\int_X \chi(|\boldsymbol{c}|^2 + |\boldsymbol{c}'|^2) \, \mathrm{d}\mathfrak{m} < \infty$ we see that it is not restrictive to assume that both $\boldsymbol{c}$ and $\boldsymbol{c}'$ belong to $\mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m})$. Now, the assumption that Ch is quadratic yields that the cotangent bundle $T^*(X, \mathsf{d}, \mathfrak{m})$ described in Remark 3.5 is a $L^2(X, \mathfrak{m})$-normed Hilbert module. By the identification introduced in that remark, the same holds for $\mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m})$. Hence, we can use the bilinear map $\langle \, \cdot \, , \, \cdot \, \rangle_x$ on $\mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m}) \times \mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m})$ provided by Theorem 2.7 to write the assumption (3.2) in the form

$$\langle \boldsymbol{c}, \boldsymbol{c}' \rangle_x = |\boldsymbol{c}|(x)|\boldsymbol{c}'|(x) \qquad \mathfrak{m}\text{-a.e. } x \in X.$$

Fix $\bar{x} \in X$, define now $E_n = \{x \in E : \ \min\{|\boldsymbol{c}|(x), |\boldsymbol{c}'|(x)\} \geq 1/n\}$ and set

$$\boldsymbol{c}_{n,R} := \chi_{B_R(\bar{x}) \cap E_n} \frac{1}{|\boldsymbol{c}|} \boldsymbol{c}, \qquad \boldsymbol{c}'_{n,R} := \chi_{B_R(\bar{x}) \cap E_n} \frac{1}{|\boldsymbol{c}'|} \boldsymbol{c}'.$$

Using $L^\infty(X, \mathfrak{m})$ bilinearity gives

$$\langle \boldsymbol{c}_{n,R}, \boldsymbol{c}'_{n,R} \rangle_x = \chi_{B_R(\bar{x}) \cap E_n} \qquad \mathfrak{m}\text{-a.e. in } X.$$

By integration we obtain

$$\int_X |\boldsymbol{c}_{n,R} - \boldsymbol{c}'_{n,R}|^2 \, \mathrm{d}\mathfrak{m} = 0,$$

hence $\boldsymbol{c}_{n,R} = \boldsymbol{c}_{n,R'}$. By approximation with respect to the parameters n and R, this gives $|\boldsymbol{c}'|\boldsymbol{c}(f) = |\boldsymbol{c}|\boldsymbol{c}'(f)$ $\mathfrak{m}$-a.e. on E for all $f \in \mathrm{Lip}_{\mathrm{b}}(X)$ and thus, with a simple algebraic manipulation, the result. $\square$

Definition 3.7 (Divergence of (pre-)derivations)**.** We say that $\boldsymbol{b} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ has divergence in L^1_{loc} if there exists $g \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$ satisfying

$$-\int_X \boldsymbol{b}(f) \, \mathrm{d}\mathfrak{m} = \int_X fg \, \mathrm{d}\mathfrak{m} \qquad \text{for all } f \in \mathrm{Lip}_{\mathrm{loc}}(X). \tag{3.3}$$

The function g is uniquely determined and it will be denoted by $\mathrm{div}\, \boldsymbol{b}$.

The case of a pre-derivation is analogous: the only difference is that $\mathrm{Lip}_{\mathrm{loc}}(X)$ has to be replaced in (3.3) by $\mathscr{A}_{\mathrm{bs}} \subset \mathscr{A}$.

Lemma 3.8. *Let $\boldsymbol{b}$ be a pre-derivation with $h \in L^2_{\mathrm{loc}}(X, \mathfrak{m})$ and $\mathrm{div}\,\boldsymbol{b} \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$. Then there exists a unique extension to $\tilde{\boldsymbol{b}} \in \mathrm{Der}^2_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, with $|\tilde{\boldsymbol{b}}| \le h$ (and same divergence).*

Proof. By the locality property, it is sufficient to build $\boldsymbol{b}$ on $\mathrm{Lip}_{\mathrm{bs}}(X)$. Given $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$, let $f_n \in \mathscr{A}_{\mathrm{bs}}$ be an approximating sequence as provided by Theorem B.1, i.e. such that $\mathrm{Lip}_{\mathrm{a}}(f_n) \to |\mathrm{D}f|$ in $L^2(X, \mathfrak{m})$. Actually, up to truncating f_n, we may assume that $|f_n| \le \|f\|_\infty$, for $n \ge 1$. By the bound $|\boldsymbol{b}(f_n)| \le h\,\mathrm{Lip}_{\mathrm{a}}(f_n)$ and Lemma 2.3 (i), the family $\boldsymbol{b}(f_n)$ has weak limit points in $L^1_{\mathrm{loc}}(X, \mathfrak{m})$ and any weak limit point, denoted $\tilde{\boldsymbol{b}}(f)$, satisfies

$$|\tilde{\boldsymbol{b}}(f)| \le h|\mathrm{D}f|.$$

We only have to show that such limit is uniquely determined by the formula

$$\int_X \varphi \tilde{\boldsymbol{b}}(f)\,\mathrm{d}\mathfrak{m} = -\int_X f\varphi\,\mathrm{div}\,\boldsymbol{b}\,\mathrm{d}\mathfrak{m} - \int_X f\boldsymbol{b}(\varphi)\,\mathrm{d}\mathfrak{m} \qquad \forall \varphi \in \mathscr{A}_{\mathrm{bs}},$$

so that the whole family $\boldsymbol{b}(f_n)$ will be weakly convergent and this will give that $\tilde{\boldsymbol{b}}$ extends $\boldsymbol{b}$ and that the map $f \mapsto \tilde{\boldsymbol{b}}(f)$ is linear. Indeed, by definition of divergence, for any $\varphi \in \mathscr{A}_{\mathrm{bs}}$ one has $f_n\varphi \in \mathscr{A}_{\mathrm{bs}}$ and thus

$$\int_X \varphi \boldsymbol{b}(f_n)\,\mathrm{d}\mathfrak{m} = -\int_X f_n\varphi\,\mathrm{div}\,\boldsymbol{b}\,\mathrm{d}\mathfrak{m} - \int_X f_n\boldsymbol{b}(\varphi)\,\mathrm{d}\mathfrak{m}.$$

By convergence of both sides, the same identity holds with f in place of f_n and $\tilde{\boldsymbol{b}}(f)$ in place of $\boldsymbol{b}(f_n)$. □

Lemma 3.9 (Continuity criterion w.r.t. flat convergence). *Let $\boldsymbol{b} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ and assume that either $\boldsymbol{b} \in \mathrm{Div}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, or $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is reflexive. Then* (3.1) *holds.*

Proof. Let $f_n \to f$ in $\mathrm{Lip}_{\mathrm{b}}(X)$. If $\boldsymbol{b}$ has divergence in L^1_{loc}, we can use the formula

$$\int_X \boldsymbol{b}(f)\phi\,\mathrm{d}\mathfrak{m} = -\int_X \boldsymbol{b}(\phi)f\,\mathrm{d}\mathfrak{m} - \int_X f\phi\,\mathrm{div}\,\boldsymbol{b}\,\mathrm{d}\mathfrak{m} \qquad \phi \in \mathrm{Lip}_{\mathrm{bs}}(X),$$

derived from the Leibniz rule to show the continuity property (3.1). In case of a reflexive Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, assuming (by locality) that the supports of f_n are equibounded, we obtain that $f_n \to f$ weakly in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$; hence, if w is any weak L^1 limit point of $\boldsymbol{b}(f_n)$, we can obtain w as the limit of $\boldsymbol{b}(g_n)$, where g_n are finite convex combinations of f_n converging to f strongly in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. If follows that $\boldsymbol{b}(g_n) \to \boldsymbol{b}(f)$, hence $w = \boldsymbol{b}(f)$. □

In the sequel we shall use the notation

$$\mathrm{Div}^q(X, \mathsf{d}, \mathfrak{m}) := \left\{ \boldsymbol{b} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}) : \ \mathrm{div}\,\boldsymbol{b} \in L^q(X, \mathfrak{m}) \right\}$$

and the analogous notation $\mathrm{Div}^q_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$.

Part II
Weak/strong convergence of derivations

Given $(\mathfrak{m}_n) \subset \mathscr{M}^+_{\mathrm{loc}}(X)$ weakly convergent to $\mathfrak{m} \in \mathscr{M}^+_{\mathrm{loc}}(X)$ and $\boldsymbol{b}_n \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}_n)$, we consider notions of local convergence of $\boldsymbol{b}_n$ to $\boldsymbol{b} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, two weak ones and a strong one. We study these two concepts in the next two sections.

4 Weak convergence of derivations and compactness

We say that $\boldsymbol{b}_n \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}_n)$ weakly converge to $\boldsymbol{b} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ in duality with $\mathscr{A}_{\mathrm{bs}}$, and write $\boldsymbol{b}_n \xrightarrow{\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$ if, for all $f \in \mathscr{A}_{\mathrm{bs}}$, $\boldsymbol{b}_n(f)\mathfrak{m}_n$ weakly converge to $\boldsymbol{b}(f)\mathfrak{m}$ as measures, i.e.

$$\lim_{n\to\infty} \int_X \boldsymbol{b}_n(f) v \, \mathrm{d}\mathfrak{m}_n = \int_X \boldsymbol{b}(f) v \, \mathrm{d}\mathfrak{m} \qquad \forall v \in C_{\mathrm{b}}(X).$$

Theorem 4.1 (Weak compactness of derivations). *Let* $\boldsymbol{b}_n \in \mathrm{Der}^2_{\mathrm{loc}} \cap \mathrm{Div}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}_n)$ *satisfying for some* $\bar{x} \in X$ *and all* $R > 0$ *the conditions*

$$\sup_n \int_{B_R(\bar{x})} |\boldsymbol{b}_n|^2 \, \mathrm{d}\mathfrak{m}_n < \infty, \qquad \sup_n \int_{B_R(\bar{x})} \Theta(|\operatorname{div} \boldsymbol{b}_n|) \, \mathrm{d}\mathfrak{m}_n < \infty$$

for some function $\Theta = \Theta_R : [0, \infty) \to [0, \infty]$ *with more than linear growth at infinity. Then there exists a subsequence* $\boldsymbol{b}_{n(k)}$ *weakly convergent to* $\boldsymbol{b} \in \mathrm{Der}^2_{\mathrm{loc}} \cap \mathrm{Div}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$.

Proof. Let $\varphi \in \mathscr{A}_{\mathrm{bs}}$. Since $|\boldsymbol{b}_n(\varphi)|$ is bounded in L^2, up to a subsequence there exists a weak L^2 limit, that we denote by $\boldsymbol{b}(\varphi)$. By a diagonal argument, we may assume that $\boldsymbol{b}_n(\varphi)$ weakly converge in L^2 to $\boldsymbol{b}(\varphi)$ for every $\varphi \in \mathscr{A}_{\mathrm{bs}}$. We show that $\varphi \mapsto \boldsymbol{b}(\varphi)$ defines a pre-derivation with $\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$, hence by Lemma 3.8 we conclude.

Since $\mathbb{Q}$-linearity is preserved by weak limits we only have to notice that locality of $\boldsymbol{b}$ follows from

$$|\boldsymbol{b}_n(\varphi)| \leq |\boldsymbol{b}_n| \, |\mathrm{D}_n \varphi| \leq |\boldsymbol{b}_n| \, \mathrm{Lip}_{\mathrm{a}}(\varphi), \quad \mathfrak{m}_n\text{-a.e. on } X,$$

where $\varphi \in \mathscr{A}_{\mathrm{bs}}$ and $|\mathrm{D}_n \varphi|$ denotes the minimal relaxed slope of φ with respect to the measure $\mathfrak{m}_n$. Hence, by weak convergence in L^2_{loc} of $|\boldsymbol{b}_n|$ to some $h \in L^2_{\mathrm{loc}}(X, \mathfrak{m})$ (possibly taking a further subsequence) and upper semicontinuity of $\mathrm{Lip}_{\mathrm{a}}(\varphi)$, we conclude that

$$|\boldsymbol{b}(\varphi)| \leq h \, \mathrm{Lip}_{\mathrm{a}}(\varphi), \quad \mathfrak{m}\text{-a.e. on } X.$$

Finally, a similar limiting argument proves that $\operatorname{div} \boldsymbol{b}_n$ has limit points w.r.t. the weak L^1_{loc} convergence, and that limit points correspond to $\operatorname{div} \boldsymbol{b}$. $\square$

Our second notion of weak convergence depends somehow on structural assumptions on the metric measure structures, and actually only on the limit one. Let us assume that the limit metric measure structure $(X, \mathsf{d}, \mathfrak{m})$ satisfies the following two regularity assumptions (recall that $|\mathrm{D}f|$ stands for the minimal relaxed slope, see Appendix A):

(a) Ch is quadratic;

(b) there exists $\omega : (0, \infty) \to [0, \infty)$ such that $\omega(0_+) = 1$ and

$$\mathrm{Lip}_{\mathrm{a}}(P_t f) \leq \omega(t) \sqrt{P_t |\mathrm{D}f|^2} \quad \text{pointwise in supp} \, \mathfrak{m} \tag{4.1}$$

for all $f \in \mathrm{Lip}_{\mathrm{b}}(X) \cap H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and $t > 0$.

These assumptions play a role in Lemma 4.2 and, in a more essential way, in the proof of Lemma 5.2(ii). As illustrated in Appendix A, all these assumptions hold in $\mathrm{RCD}(K, \infty)$ metric measure spaces, in particular (4.1) holds with with $\omega(t) = e^{-Kt}$. Notice that, since $P_t f$ is only defined up to $\mathfrak{m}$-negligible sets, using McShane's extension theorem we can and will improve (b) by asking the validity of (4.1) on the whole of X.

In view of (4.1) it is natural to consider the countable class

$$P_{\mathbb{Q}_+} \mathscr{A}_{\mathrm{bs}} := \{P_t f : \ f \in \mathscr{A}_{\mathrm{bs}}, \ t \in \mathbb{Q}_+\} \subset \mathrm{Lip}_{\mathrm{b}}(X). \tag{4.2}$$

Notice that, thanks to Theorem B.1, $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$ is dense in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Strictly speaking $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$ should be thought as a subset of $\mathrm{Lip}_{\mathrm{b}}(\mathrm{supp}\,\mathfrak{m})$, but we understand that any function in this class is extended to the whole of X, with the same supremum and Lipschitz constant, and we think to this collection as a subset of $\mathrm{Lip}_{\mathrm{b}}(X)$.

Lemma 4.2. *Under assumption (a), any $\boldsymbol{b} \in \mathrm{Der}(X, \mathsf{d}, \mathfrak{m})$ is uniquely determined by its values on $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$.*

Proof. We need to prove that $\boldsymbol{b} \equiv 0$ on $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$ implies $\boldsymbol{b} = 0$. It is not restrictive to assume $\boldsymbol{b} \in \mathrm{Der}^2(X, \mathsf{d}, \mathfrak{m})$. Since, thanks to assumption (a), $P_s f \to f$ in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ as $s \downarrow 0$, we obtain that $\boldsymbol{b} \equiv 0$ on $\mathscr{A}_{\mathrm{bs}}$. Then, Theorem B.1 can be applied. $\qquad\square$

Motivated by Lemma 4.2, we can define another notion of weak convergence.

Definition 4.3. We say that $\boldsymbol{b}_n \in \mathrm{Der}^1(X, \mathsf{d}, \mathfrak{m}_n)$ weakly converge to $\boldsymbol{b} \in \mathrm{Der}^1(X, \mathsf{d}, \mathfrak{m})$ in duality with $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$, and write $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$ if, for all $f \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$, $\boldsymbol{b}_n(f)\mathfrak{m}_n$ weakly converge to $\boldsymbol{b}(f)\mathfrak{m}$ as measures, i.e.

$$\lim_{n \to \infty} \int_X \boldsymbol{b}_n(f) v \, \mathrm{d}\mathfrak{m}_n = \int_X \boldsymbol{b}(f) v \, \mathrm{d}\mathfrak{m} \qquad \forall v \in C_{\mathrm{bs}}(X). \tag{4.3}$$

Notice that in (4.3) the integrals on the left hand side *depend* on the extension of have chosen of f from $\mathrm{supp}\,\mathfrak{m}$ to X; nevertheless, the compactness result presented below shows that still (4.3) makes sense. With this new notion of weak convergence we can remove the bounds on divergence and, at the same time, consider more general growth conditions (not necessarily quadratic) on $\boldsymbol{b}_n$.

Theorem 4.4. *Assume that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the regularity assumptions (a), (b) and let $\boldsymbol{b}_n \in \mathrm{Der}(X, \mathsf{d}, \mathfrak{m}_n)$ be such that*

$$\liminf_{n \to \infty} \int_X \Theta(|\boldsymbol{b}_n|) \, \mathrm{d}\mathfrak{m}_n < \infty$$

for some function $\Theta : [0, \infty) \to [0, \infty]$ with more than linear growth at infinity. Then there exist a subsequence $\boldsymbol{b}_{n(k)}$ and $\boldsymbol{b} \in \mathrm{Der}(X, \mathsf{d}, \mathfrak{m})$ with $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$. In addition, if Θ is convex and lower semicontinuous, one has

$$\int_X \Theta(|\boldsymbol{b}|) \, \mathrm{d}\mathfrak{m} \le \liminf_{k \to \infty} \int_X \Theta(|\boldsymbol{b}_{n(k)}|) \, \mathrm{d}\mathfrak{m}_{n(k)} < \infty.$$

Proof. Let $a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$. Since $|\boldsymbol{b}_n(a)|$ satisfies the assumptions of Lemma 2.3 (ii), up to a subsequence there exists a weak L^1 limit, that we denote by $\boldsymbol{b}(\varphi)$. By a diagonal argument, we may assume that $\boldsymbol{b}_n(a)$ weakly L^1_{loc} converge to $\boldsymbol{b}(a)$ for every $a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$. For the same reason is also not restrictive to assume that $|\boldsymbol{b}_n|$ weakly converge in L^1_{loc} to h.

Since $\mathbb{Q}$-linearity is preserved by weak local limits, taking Lemma 5.2 (ii) into account it is sufficient to prove

$$|\boldsymbol{b}(a)| \le h \, \mathrm{Lip}_{\mathrm{a}}(a) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}.$$

This can be achieved as in the proof of Theorem 4.1 passing to the limit in the inequality $|\boldsymbol{b}_n(a)| \le |\boldsymbol{b}_n| \, \mathrm{Lip}_{\mathrm{a}}(a)$ and using the upper semicontinuity of the asymptotic Lipschitz constant. $\qquad\square$

Remark 4.5 (Comparison of $\boldsymbol{b}_n \xrightarrow{\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$ and $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$). Assume that we are in the setting of both Theorem 4.1 and Theorem 4.4, namely with uniform local equi-integrability bounds on the divergences of $\boldsymbol{b}_n$ and assumptions (a), (b) on the limit metric measure structure. Under this assumption, since both notions of convergence are sequentially compact, we immediately get from Lemma 3.9 that $\boldsymbol{b}_n \xrightarrow{\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$ if and only if $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$.

5 Strong convergence of derivations

Still under the assumptions (a), (b) of the previous section on the metric measure structure, we want to study conditions under which weak convergence in duality with $P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ improves to strong convergence, defined as follows.

Definition 5.1 (Strong convergence of derivations)**.** We say that $\boldsymbol{b}_n \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}_n)$ strongly converge to $\boldsymbol{b} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ if, for all $f \in P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$, $\boldsymbol{b}_n(f)$ converge in measure to $\boldsymbol{b}(f)$. For $p \in (1, \infty)$, if for all $f \in P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ the stronger condition

$$\limsup_{n \to \infty} \int_{B_R(\bar{x})} |\boldsymbol{b}_n(f)|^p \, \mathrm{d}\mathfrak{m}_n \leq \int_{B_R(\bar{x})} |\boldsymbol{b}(f)|^p \, \mathrm{d}\mathfrak{m} < \infty$$

holds for an unbounded family of R's, we say that $\boldsymbol{b}_n$ converge strongly in L^p_{loc} to $\boldsymbol{b}$. Analogously, if

$$\limsup_{n \to \infty} \int_X |\boldsymbol{b}_n(f)|^p \, \mathrm{d}\mathfrak{m}_n \leq \int_X |\boldsymbol{b}(f)|^p \, \mathrm{d}\mathfrak{m} < \infty$$

we say that $\boldsymbol{b}_n \to \boldsymbol{b}$ strongly in L^p.

The terminology is a bit misleading, because strong convergence of $\boldsymbol{b}_n$ should be understood as a "componentwise" strong convergence, where the components of the "vector field" are given by $P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$. As discussed in Remark 2.6, under uniform L^p_{loc} bounds on $\boldsymbol{b}_n(f)$ for some $p > 1$, the convergence in measure of $\boldsymbol{b}_n(f)$ to $\boldsymbol{b}(f)$ yields convergence in L^q_{loc} for all $q \in (1, p)$ and therefore $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$.

The main goal of this section is to provide a sufficient condition for strong convergence. To this aim, it is convenient to extend $\mathscr{A}_{\mathrm{bs}}$ to $\mathscr{A}_*$, defined as follows: $\mathscr{A}_*$ is the smallest algebra, lattice and vector space over $\mathbb{Q}$ containing $\mathscr{A}$ and the set $P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ in (4.2). The set $\mathscr{A}_*$ is still countable and we denote by a the generic element of $\mathscr{A}_*$.

In the following lemma we show how to to build a derivation from a functional $\boldsymbol{c} : \mathscr{A}_* \to L^0(X, \mathfrak{m})$ satisfying a (very) weak locality property and a growth bound with the global Lipschitz constant, see (5.1).

Lemma 5.2. *Assume that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the regularity assumptions (a), (b).*

(i) *Let $\boldsymbol{c} : \mathscr{A}_* \to L^0(X, \mathfrak{m})$ be a linear functional satisfying the locality property $\boldsymbol{c}(a) = 0$ $\mathfrak{m}$-a.e. in any open set U where a is constant, and*

$$|\boldsymbol{c}(a)| \leq h \, \mathrm{Lip}(a) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } a \in \mathscr{A}_* \tag{5.1}$$

for some $h \in L^0(X, \mathfrak{m})$. Then $\boldsymbol{c}$ is a pre-derivation, more precisely, (5.1) holds with $\mathrm{Lip}_a(a)$ in the right hand side.

(ii) *If $\boldsymbol{c} : P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}} \to L^0(X, \mathfrak{m})$ is $\mathbb{Q}$-linear and satisfies*

$$|\boldsymbol{c}(a)| \leq h \, \mathrm{Lip}_a(a) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } a \in P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}, \tag{5.2}$$

for some $h \in L^0(X, \mathfrak{m})$, then there exists a derivation $\boldsymbol{b}$ satisfying $|\boldsymbol{b}| \leq h$ which extends $\boldsymbol{c}$.

Proof. (i) Let D be as in (2.3). We want to improve the bound on $|\boldsymbol{c}(a)|$ to $|\boldsymbol{c}(a)| \leq h \, \mathrm{Lip}_a(a)$ $\mathfrak{m}$-a.e. in X for all $a \in \mathscr{A}_*$. To this aim, we fix $a \in \mathscr{A}_*$ and a bounded open set $U \subset X$ and define L_U to be the Lipschitz constant of the restriction of a to U. For $x_0 \in U$, choose $x_0' \in D$ and $r > 0$ sufficiently small, such that $B_{r/2}(x_0') \subset B_r(x_0) \subset U$,

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$rL_U/2 \in \mathbb{Q}$. Then, for $k \in \mathbb{Q}^+$ larger than $r/2$ and $a_0 \in \mathbb{Q}$, $a_0 \neq a(x_0)$, we consider the function $A = \max\{A_1, A_2\} \in \mathscr{A}_*$, with

$$A_1(x) := \min\{|a(x) - a_0|, \frac{r}{2}L_U\}, \qquad A_2(x) := L_U \min\{\mathsf{d}(x, x_0'), k\}.$$

Since $\mathrm{Lip}(A_1|_U) \leq L_U$, $\mathrm{Lip}(A_2) \leq L_U$ and the set $\{A = A_1\}$ is contained in $B_{r/2}(x_0')$, which is contained in $B_r(x_0)$ and in U, we obtain that $\mathrm{Lip}(A) \leq L_U$. Choosing a_0 sufficiently close to $a(x_0)$ we obtain that $A_1 = |a - a_0|$ in a neighbourhood V of x_0, and choosing x_0' with

$$L_U d(x_0, x_0') < |a(x_0) - a_0|$$

we obtain that $A = |a - a_0|$ in a neighbourhood V of x_0. Hence, locality in $V \cap \{a > a_0\}$ and in $V \cap \{a < a_0\}$ give

$$|\boldsymbol{c}(a)| = |\boldsymbol{c}(A)| \leq h \, \mathrm{Lip}(A) \leq h L_U \qquad \mathfrak{m}\text{-a.e. in } V \setminus \{a = a_0\}.$$

With two different choices of a_0, this proves that for any $x_0 \in U$ there exists a neighbourhood V of x_0 such that $|\boldsymbol{c}(a)| \leq h L_U$ $\mathfrak{m}$-a.e. in V. Since $x_0 \in U$ is arbitrary, and since $\mathfrak{m}$ is concentrated on a σ-compact set, it follows that $|\boldsymbol{c}(a)| \leq h L_U$ $\mathfrak{m}$-a.e. in U. If $\{U_i\}$ is a countable basis for the open sets of (X, d) we can find a $\mathfrak{m}$-negligible set N such that

$$|\boldsymbol{c}(a)|(x) \leq h(x) L_{U_i} \qquad \forall x \in U_i \setminus N, \ \forall i.$$

Since $\mathrm{Lip}_a a(x) = \inf_{U_i \ni x} L_{U_i}$ for all $x \in X$, we conclude.

(ii) First we improve the upper bound (5.2) to

$$|\boldsymbol{c}(a)| \leq h|\mathrm{D}a| \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } a \in P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}. \tag{5.3}$$

Indeed, thanks to assumption (b), one has

$$|\boldsymbol{c}(P_s a)| \leq \omega(s) h \sqrt{P_s|\mathrm{D}a|^2} \quad \mathfrak{m}\text{-a.e. in } X, \text{ for all } s \in \mathbb{Q}^+, \ a \in \mathscr{A}_{\mathrm{bs}}$$

and, writing $a = P_t a'$, we can use the linearity of P_t to get

$$|\boldsymbol{c}(P_s a) - \boldsymbol{c}(a)| = |\boldsymbol{c}(P_t(P_s a' - a'))| \leq \omega(t) h \sqrt{P_t(|\mathrm{D}(P_s a' - a')|^2)}.$$

Then, taking the limit as $\mathbb{Q}^+ \ni s \to 0^+$ of the inequality provides the result. Then, we build $\boldsymbol{b}$ on $\mathrm{Lip}_{\mathrm{bs}}(X)$ by approximation, as a simple consequence of (5.3) and the density of $P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, arguing as in Lemma 3.8: if $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$ and $a_n \in P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ satisfy $|\mathrm{D}(a_n - f)| \to 0$ in $L^2(X, \mathfrak{m})$ and $\mathfrak{m}$-a.e., thanks to (5.3) the sequence $\boldsymbol{b}(a_n)$ converges $\mathfrak{m}$-a.e. and we define $\boldsymbol{b}(f)$ as its limit. This provides a derivation $\boldsymbol{b}$ with $|\boldsymbol{b}| \leq h$ and $\boldsymbol{b} \equiv \boldsymbol{c}$ on $P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$. $\qquad\square$

In the following theorem we show how to improve weak convergence, in duality with $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$, to strong convergence. In connection with the theory of flows, we shall also consider time-dependent derivations and therefore a time averaged version of weak convergence, deriving as a consequence strong convergence in measure, with $\mathscr{L}^1 \times \mathfrak{m}_n$ and $\mathscr{L}^1 \times \mathfrak{m}$ as reference measures. Of course in the simpler autonomous case we get the improvement from weak to strong convergence.

Theorem 5.3. *Assume that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the regularity assumptions (a), (b), that $\boldsymbol{b}_{n,t} \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}_n)$, $\boldsymbol{b}_t \in \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ satisfy*

$$\lim_{n \to \infty} \int_0^T \chi(t) \int_X \boldsymbol{b}_{n,t}(a) v \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t = \int_0^T \chi(t) \int_X \boldsymbol{b}_t(a) v \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t$$

for all $\chi \in C_c(0,T)$, $v \in C_{\mathrm{bs}}(X)$, $a \in P_{\mathbb{Q}_+} \mathscr{A}_{\mathrm{bs}}$, and that

$$\limsup_{n \to \infty} \int_0^T \int_X \Theta(|\boldsymbol{b}_{n,t}|) \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t \leq \int_0^T \int_X \Theta(|\boldsymbol{b}|_t) \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t < \infty \qquad (5.4)$$

with $\Theta : [0, \infty) \to [0, \infty)$ strictly convex and having more than linear growth at infinity. Then $\boldsymbol{b}_{n,t}$ converge strongly to $\boldsymbol{b}_t$ and, if $\Theta(z) = |z|^p$ for some $p > 1$, $\boldsymbol{b}_n$ converge to $\boldsymbol{b}$ strongly in L^p.

Proof. Step 1: the dual $\mathscr{A}'_*$ of $\mathscr{A}_*$. The $\mathbb{Q}$-linear functionals from $\mathscr{A}_*$ to $\mathbb{R}$ can be considered, in some sense, as "pointwise" tangent vectors. More precisely, we denote by $\mathscr{A}'_*$ the class of $\mathbb{Q}$-linear functionals $L : \mathscr{A}_* \to \mathbb{R}$ satisfying $|L(a)| \leq C \, \mathrm{Lip}(a)$ for all $a \in \mathscr{A}_*$, for some $C \in [0, \infty)$. The smallest C will then be denoted by $\|L\|$. We endow $\mathscr{A}'_*$ with the coarser topology that makes all maps $L \mapsto L(a)$, $a \in \mathscr{A}_*$, continuous, so that $L \mapsto \|L\|$ is lower semicontinuous in $\mathscr{A}'_*$ and the sets $\{L \in \mathscr{A}'_* : \|L\| \leq c\}$ are compact for all $c \geq 0$. Since $\mathscr{A}$ is countable we can easily find a distance in $\mathscr{A}'_*$ which induces this topology (on bounded sets).

Step 2: limit of $\boldsymbol{b}_{n,t}$ in the sense of Young. Let us consider the maps $\Sigma_n : (0,T) \times X \to \mathscr{A}'_*$ induced by $\boldsymbol{b}_{n,t}$, namely

$$\Sigma_n(t, x)(a) := \boldsymbol{b}_{n,t}(a)(x)$$

and the push forward measures

$$\sigma_n := (\mathrm{Id} \times \Sigma_n)_\# \mathscr{L}^1 \times \mathfrak{m}_n \in \mathscr{M}^+((0,T) \times X \times \mathscr{A}'_*).$$

Notice that, since $\mathscr{A}_*$ is countable, σ_n is independent of the choice of Borel representatives of $\boldsymbol{b}_{n,t}(a)$, $a \in \mathscr{A}_*$, and that $\|\Sigma_n(t,x)\| \leq |\boldsymbol{b}|_{n,t}(x)$ for $\mathscr{L}^1 \times \mathfrak{m}_n$-a.e. (t,x). Since $\mathscr{L}^1 \times \mathfrak{m}_n$ weakly converge to $\mathscr{L}^1 \times \mathfrak{m}$, they are tight on bounded subsets, hence thanks to Prokhorov theorem and the uniform bound on $\int \Theta(\|L\|) \, \mathrm{d}\sigma_n$, we can assume with no loss of generality that σ_n weakly converge to σ in $(0,T) \times X \times \mathscr{A}'_*$. Using test functions of the form $\psi(t,x)$ we can represent $\sigma = \sigma_{t,x} \otimes (\mathscr{L}^1 \times \mathfrak{m})$, i.e.

$$\int_{(0,T) \times X \times \mathscr{A}'_*} \varphi(t, x, a) \, \mathrm{d}\sigma = \int_0^T \int_X \int_{\mathscr{A}'_*} \varphi(t, x, a) \, \mathrm{d}\sigma_{t,x}(a) \, \mathrm{d}x \, \mathrm{d}t.$$

From now on, for simplicity of notation, we shall omit the integration domain $(0,T) \times X \times \mathscr{A}'_*$.

Let $a \in \mathscr{A}_*$ be constant in an open set $A \subset X$. Using test functions of the form $(t, x, L) \mapsto \chi(t) v(x) |L(a)|$ with $v \in C_{\mathrm{bs}}(X)$ null on $X \setminus A$ we see immediately that $\sigma_{t,x}$ satisfy the following locality property: for $\mathscr{L}^1 \times \mathfrak{m}$-a.e. $(t,x) \in (0,T) \times A$ one has $L(a) = 0$ $\sigma_{t,x}$-a.e. in $\mathscr{A}'_*$. Thanks to this property, defining

$$\tilde{\boldsymbol{b}}_t(a)(x) := \int_{\mathscr{A}'_*} L(a) \, \mathrm{d}\sigma_{t,x}(L) \qquad (5.5)$$

using the inequality $|\tilde{\boldsymbol{b}}_t(a)(x)| \leq \mathrm{Lip}(a) \int_{\mathscr{A}_*} \|L\| \, \mathrm{d}\sigma_{t,x}(L)$ and invoking Lemma 5.2 (ii) we obtain

$$|\tilde{\boldsymbol{b}}_t(a)(x)| \leq \mathrm{Lip}_{\mathrm{a}}(a) \int_{\mathscr{A}_*} \|L\| \, \mathrm{d}\sigma_{t,x}(L) \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for all } a \in \mathscr{A}_* \qquad (5.6)$$

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for $\mathscr{L}^1$-a.e. $t \in (0, T)$.

Set now $\Sigma(t, x)(a) := \boldsymbol{b}_t(a)(x)$, $a \in \mathscr{A}_*$. Our goal is to show that $\sigma_{t,x}$ is a Dirac mass at $\Sigma(t, x)$ for $\mathscr{L}^1 \times \mathfrak{m}$-a.e. (t, x). To this aim, we first observe the following characterization of Dirac masses in $\mathscr{A}'_*$: $\nu \in \mathscr{P}(\mathscr{A}'_*)$ concentrated on $\{L : \|L\| = c\}$ for some $c \geq 0$ is a Dirac mass if and only if, for all $a \in \mathscr{A}_*$, ν is supported in one of the "halfspaces" $\{L : L(a) \geq 0\}$, $\{L : L(a) \leq 0\}$. Since $\mathscr{A}_*$ is countable, this characterization follows by the implication

$$L(a)L'(a) \geq 0 \quad \forall a \in \mathscr{A}_* \implies L = \lambda L' \quad \text{for some } \lambda \geq 0.$$

In turn, the implication above follows by this elementary argument: if $a \in \mathscr{A}_*$ with $L'(a) \neq 0$ and $\lambda = L(a)/L'(a)$, for all $a' \in \mathscr{A}_*$ the functions $s \mapsto L(a+sa')$, $s \mapsto \lambda L'(a+sa')$ coincide at $s = 0$ and have the same sign if and only they coincide, i.e. $L(a') = \lambda L'(a')$.

Now, notice that (5.4) gives

$$\int \Theta(\|L\|) \, d\sigma \leq \liminf_{n \to \infty} \int \Theta(\|L\|) \, d\sigma_n \leq \int_X \Theta(|\boldsymbol{b}|) \, d\mathfrak{m}. \tag{5.7}$$

On the other hand, using test functions $(t, x, L) \mapsto \chi(t)v(x)L(a)$ with $\chi \in C_c^\infty(0, T)$, $v \in C_{\mathrm{bs}}(X)$ and $a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$ (notice that here we use the more than linear growth of Θ) and the convergence of $\boldsymbol{b}_{n,t}$ to $\boldsymbol{b}_t$, passing to the limit as $n \to \infty$ in the identity

$$\int \chi(t)v(x)L(a) \, d\sigma_n(t, x, L) = \int_0^T \chi(t) \int_X \boldsymbol{b}_{n,t}(a)(x)v(x) \, d\mathfrak{m}_n(x) \, dt$$

and using the arbitrariness of χ and v we obtain (with $\tilde{\boldsymbol{b}}_t$ as in (5.5))

$$\tilde{\boldsymbol{b}}_t(a)(x) = \boldsymbol{b}_t(a)(x) \qquad \forall a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}} \tag{5.8}$$

for $\mathscr{L}^1 \times \mathfrak{m}$-a.e. $(t, x) \in (0, T) \times X$. From (5.6) it follows that

$$|\boldsymbol{b}_t(a)|(x) \leq \mathrm{Lip}_{\mathrm{a}}(a) \int_{\mathscr{A}'_*} \|L\| \, d\sigma_x(L) \qquad \forall a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$$

for $\mathscr{L}^1 \times \mathfrak{m}$-a.e. $(t, x) \in (0, T) \times X$, so that Lemma 5.2 (ii) gives

$$|\boldsymbol{b}_t|(x) \leq \int_{\mathscr{A}'_*} \|L\| \, d\sigma_{x,t}(L) \qquad \text{for } \mathscr{L}^1 \times \mathfrak{m}\text{-a.e. } (t, x) \in (0, T) \times X.$$

Combining this information with (5.7) and the strict convexity of Θ we obtain that for $\mathscr{L}^1 \times \mathfrak{m}$-a.e. (t, x) one has

$$\|L\| = |\boldsymbol{b}_t|(x) \qquad \text{for } \sigma_{t,x}\text{-a.e. } L \in \mathscr{A}'_*.$$

Now, let E be a closed set of $\mathscr{A}'_*$ and define $\boldsymbol{c}_t^i : \mathscr{A}_* \to L^0(X, \mathfrak{m})$ by

$$\boldsymbol{c}_t^1(a)(x) := \int_E L(a) \, d\sigma_{t,x}(L), \qquad \boldsymbol{c}_t^2(a)(x) := \int_{\mathscr{A}'_* \setminus E} L(a) \, d\sigma_{t,x}(L),$$

so that $\boldsymbol{c}_t := \boldsymbol{c}_t^1 + \boldsymbol{c}_t^2$ coincides with $\boldsymbol{b}_t$ on $\mathscr{A}_*$. Notice also that $\boldsymbol{c}_t^i$ satisfy the weak locality property (thanks to the locality of $\sigma_{t,x}$) and (5.1) with $h_t^1(x) = \int_E \|L\| \, d\sigma_{t,x}(X)$ and $h_t^2(x) = \int_{\mathscr{A}'_* \setminus E} \|L\| \, d\sigma_{t,x}(X)$. From Lemma 5.2 (i,ii) we obtain that $\boldsymbol{c}_t^i$ induce derivations $\boldsymbol{b}_t^i$ which coincide with $\boldsymbol{c}_t^i$ on $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$ and satisfy

$$|\boldsymbol{b}_t^1|(x) \leq \int_E \|L\| \, d\sigma_{t,x}(L), \quad |\boldsymbol{b}_t^2|(x) \leq \int_{\mathscr{A}'_* \setminus E} \|L\| \, d\sigma_{t,x}(L) \quad \mathscr{L}^1 \times \mathfrak{m}\text{-a.e. in } (0, T) \times X.$$

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Since, thanks to Lemma 5.2 (ii), $\boldsymbol{b}_t$ is uniquely determined by its values on $P_{\mathbb{Q}^+}\mathscr{A}_*$, it follows that $\boldsymbol{b}_t = \boldsymbol{b}_t^1 + \boldsymbol{b}_t^2$ and that $|\boldsymbol{b}_t^1| + |\boldsymbol{b}_t^2| \leq |\boldsymbol{b}_t|$ $\mathscr{L}^1 \times \mathfrak{m}$-a.e. in $(0,T) \times X$. Therefore we can invoke the strict convexity Lemma 3.6 to obtain $\boldsymbol{b}_{t,x}^1(a)\boldsymbol{b}_{t,x}^2(a) \geq 0$ $\mathscr{L}^1 \times \mathfrak{m}$-a.e. for all $a \in \mathscr{A}_*$; by letting E vary in a countable family of closed sets generating the σ-algebra of $\mathscr{A}_*'$, it follows that $\mathscr{L}^1 \times \mathfrak{m}$-a.e. measure $\sigma_{t,x}$ is concentrated on $\{L : \|L\| = |\boldsymbol{b}_{t,x}|\}$ and satisfies the above mentioned criterion for being a Dirac mass. Coming back to (5.5) and (5.8) we obtain that $\sigma_{t,x} = \delta_{\Sigma(t,x)}$.

Step 3. We proved that $(\mathrm{Id} \times \Sigma_n)_{\#}\mathscr{L}^1 \times \mathfrak{m}_n$ weakly converge to $(\mathrm{Id} \times \Sigma)_{\#}\mathscr{L}^1 \times \mathfrak{m}$. Recalling the definitions of Σ_n and Σ, if we consider the continuous map $(t,x,L) \mapsto (t,x,L(a))$ for $a \in P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ fixed, from Proposition 2.2 we obtain that strong convergence of $\boldsymbol{b}_{n,t}(a)$ to $\boldsymbol{b}_t(a)$. In addition, if $\Theta(z) = |z|^p$ for some $p > 1$, from the second part of Proposition 2.2 we get the strong L^p convergence of $\boldsymbol{b}_{n,t}(a)$ to $\boldsymbol{b}_t(a)$. $\qquad\square$

6 Convergence of gradient derivations under Mosco convergence

In this section we use the typical notation of Γ-calculus, namely

$$\Gamma(f,g) := \lim_{\epsilon \downarrow 0} \frac{|\mathrm{D}(f + \epsilon g)|^2 - |\mathrm{D}f|^2}{2\epsilon} \qquad f, g \in H^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

Under the assumption that Ch is quadratic, this is a symmetric and $L^1(X,\mathfrak{m})$-valued bilinear form, with $\Gamma(f,f) = |\mathrm{D}f|^2$ $\mathfrak{m}$-a.e. in X. This fact, proved first in [AGS14b], can now be seen as a particular case of Theorem 2.7. We can canonically associate to any $f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ a *gradient* derivation $\boldsymbol{b}_f = \boldsymbol{b}_{f,\mathfrak{m}} \in \mathrm{Der}^2(X,\mathsf{d},\mathfrak{m})$, defined by

$$\boldsymbol{b}_f(g) := \Gamma(f,g) \qquad g \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$$

(we already consider, as for all derivations in L^2, the extended domain $H^{1,2}(X,\mathsf{d},\mathfrak{m})$).

Lemma 6.1. *Assume that Ch is quadratic, let $f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, $\boldsymbol{b} \in \mathrm{Der}^2(X,\mathsf{d},\mathfrak{m})$ and assume that $\int_X |\boldsymbol{b}|^2 \, \mathrm{d}\mathfrak{m} \leq \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}$ and that $\int_X (\boldsymbol{b}(g) - \boldsymbol{b}_f(g)) \, \mathrm{d}\mathfrak{m} = 0$ for all $g \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. Then $\boldsymbol{b} = \boldsymbol{b}_f$.*

Proof. Since, by the definition of $|\boldsymbol{b}|$, $\boldsymbol{b}(f) \leq |\mathrm{D}f||\boldsymbol{b}|$ and since Hölder's inequality gives

$$\int_X \boldsymbol{b}(f) \, \mathrm{d}\mathfrak{m} = \int_X \Gamma(f,f) \, \mathrm{d}\mathfrak{m} \geq \int_X |\mathrm{D}f||\boldsymbol{b}| \, \mathrm{d}\mathfrak{m}$$

we obtain that

$$\boldsymbol{b}(f) = |\mathrm{D}f|^2 = |\boldsymbol{b}|^2 \qquad \mathfrak{m}\text{-a.e. in } X. \tag{6.1}$$

Now, given any $g \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, the pointwise equalities (6.1) in combination with the pointwise inequalities

$$|\boldsymbol{b}(f)|^2 + 2\varepsilon\boldsymbol{b}(f)\boldsymbol{b}(g) + o(\varepsilon) = |\boldsymbol{b}(f + \varepsilon g)|^2 \leq |\boldsymbol{b}|^2|\mathrm{D}(f + \varepsilon g)|^2 = |\boldsymbol{b}|^2|\mathrm{D}f|^2 + 2\varepsilon|\boldsymbol{b}|^2\Gamma(f,g) + o(\varepsilon)$$

give the result. $\qquad\square$

Definition 6.2 (Mosco convergence). We say that the Cheeger energies $\mathsf{Ch}_n = \mathsf{Ch}_{\mathfrak{m}_n}$ Mosco converge to Ch if both the following conditions hold:

(a) (*Weak*-lim inf). For every $f_n \in L^2(X, \mathfrak{m}_n)$ L^2-weakly converging to $f \in L^2(X, \mathfrak{m})$, one has

$$\mathsf{Ch}(f) \leq \liminf_{n \to \infty} \mathsf{Ch}_n(f_n).$$

(b) (*Strong*-lim sup). For every $f \in L^2(X, \mathfrak{m})$ there exist $f_n \in L^2(X, \mathfrak{m}_n)$, L^2-strongly converging to f with

$$\mathsf{Ch}(f) = \lim_{n \to \infty} \mathsf{Ch}_n(f_n).$$

Such a convergence holds for example if $(X, \mathsf{d}, \mathfrak{m}_n)$ are $\mathrm{RCD}(K, \infty)$ spaces with $\mathfrak{m}_n(B_r(\bar{x})) \leq c_1 e^{c_2 r^2}$, see [GMS13, Theorem 6.8].

In the sequel we also say that $f_n \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}_n)$ are weakly convergent in $H^{1,2}$ to $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ if $f_n \to f$ weakly in L^2 and $\sup_n \mathsf{Ch}_n(f_n)$ is finite. Strong convergence in $H^{1,2}$ is defined by requiring strong L^2 convergence of f_n to f, and $\mathsf{Ch}(f) = \lim_n \mathsf{Ch}_n(f_n)$. We are going to use this well-known consequence of Mosco convergence:

$$\lim_{n \to \infty} \int_X \Gamma_n(f_n, g_n) \, \mathrm{d}\mathfrak{m}_n = \int_X \Gamma(f, g) \, \mathrm{d}\mathfrak{m} \qquad (6.2)$$

for any f_n strongly convergent in $H^{1,2}$ to f and all g_n weakly convergent in $H^{1,2}$ to g.

Indeed, since $f_n + t g_n$ weakly converge in $H^{1,2}$ to $f + tg$ for all $t > 0$, by Mosco convergence we have

$$
\begin{aligned}
\mathsf{Ch}(f) + 2t \int_X \Gamma(f, g) \, \mathrm{d}\mathfrak{m} + t^2 \mathsf{Ch}(g) &= \mathsf{Ch}(f + tg) \\
&\leq \liminf_{n \to \infty} \mathsf{Ch}_n(f_n + t g_n) \\
&= \liminf_{n \to \infty} \mathsf{Ch}_n(f_n) + 2t \int_X \Gamma_n(f_n, g_n) \, \mathrm{d}\mathfrak{m}_n + t^2 \mathsf{Ch}_n(g_n) \\
&\leq \mathsf{Ch}(f) + 2t \liminf_{n \to \infty} \int_X \Gamma_n(f_n, g_n) \, \mathrm{d}\mathfrak{m} + t^2 \limsup_n \mathsf{Ch}_n(g_n).
\end{aligned}
$$

Since $\sup_n \mathsf{Ch}_n(g_n)$ is finite, we may let $t \downarrow 0$ to deduce the lim inf inequality in (6.2); replacing g by $-g$ gives (6.2).

The following simple example shows that even when the functions f_n, g_n are fixed, (6.2) might not hold.

Example 6.3. Take $X = \mathbb{R}^2$ endowed with the Euclidean distance, $f(x_1, x_2) = x_2$ and let

$$\mathfrak{m}_n = n\mathscr{L}^2 \mathsf{L}([0, 1] \times [0, \tfrac{1}{n}]), \qquad \mathfrak{m} = \mathscr{H}^1 \mathsf{L}[0, 1] \times \{0\}.$$

Then, it is easily seen that $\Gamma_n(f, f) = 1$, while $\Gamma(f, f) = 0$.

The next theorem shows that any sequence (f_n) strongly convergent in $H^{1,2}$ to f induces gradient derivations which are strongly converging to the gradient derivation of the limit function.

Theorem 6.4 (Strong convergence of gradient derivations). *Assume (a), (b) of Section 4 on the limit structure $(X, \mathsf{d}, \mathfrak{m})$, that Ch_n are quadratic and that Mosco converge to Ch. Let $f_n \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}_n)$ be strongly convergent in $H^{1,2}$ to $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Then the derivations $\boldsymbol{b}_{f_n}$ strongly converge in L^2 to $\boldsymbol{b}_f$. In addition, $\boldsymbol{b}_{f_n}(g_n)$ weakly converge in L^2 to $\boldsymbol{b}_f(g)$ whenever $g_n \to g$ in $\mathrm{Lip}_b(X)$ with $\cup_n \mathrm{supp}(g_n)$ bounded in X.*

Proof. Set $\boldsymbol{b}_n = \boldsymbol{b}_{f_n}$, $\boldsymbol{b} = \boldsymbol{b}_f$, and let Γ_n be the bilinear form associated to Ch_n. From (6.2) we get

$$\lim_{n\to\infty} \int_X \boldsymbol{b}_n(g)\,\mathrm{d}\mathfrak{m}_n = \int_X \boldsymbol{b}(g)\,\mathrm{d}\mathfrak{m} \qquad \forall g \in \mathrm{Lip}_{\mathrm{bs}}(X). \tag{6.3}$$

By Theorem 4.4 with $\Theta(z) = z^2$ there exist a subsequence $b_{n(k)}$ and $\boldsymbol{c} \in \mathrm{Der}^2(X,\mathsf{d},\mathfrak{m})$ such that $\boldsymbol{b}_{n(k)} \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{c}$ and

$$\int_X |\boldsymbol{c}|^2\,\mathrm{d}\mathfrak{m} \le \liminf_{k\to\infty} \int_X |\boldsymbol{b}_{n(k)}|^2\,\mathrm{d}\mathfrak{m}_{n(k)} \le \int_X |\boldsymbol{b}|^2\,\mathrm{d}\mathfrak{m}.$$

We fix a sequence of cut-off functions $\chi_R \in \mathrm{Lip}_{\mathrm{bs}}(X)$ with $0 \le \chi_R \le 1$, $\mathrm{Lip}(\chi_R) \le 2$ and $\chi_R \equiv 1$ on $B_R(\bar{x})$. To show that $\boldsymbol{b} = \boldsymbol{c}$, let $h \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$ and let us notice that the Young inequality and the $L^2(X,\mathfrak{m})$ integrability of h easily give

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \int_X h|\Gamma_n(f_n,\chi_R)|\,\mathrm{d}\mathfrak{m}_n = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{X\setminus\bar{B}_R(\bar{x})} h|\Gamma_n(f_n,\chi_R)|\,\mathrm{d}\mathfrak{m}_n = 0.$$
$$\tag{6.4}$$

Now, taking (6.3) with $g = h\chi_R$ and (6.4) into account, we can pass to the limit as $n \to \infty$ in the identity

$$\int_X \boldsymbol{b}_n(h)\chi_R\,\mathrm{d}\mathfrak{m}_n = \int_X \Gamma_n(f_n,h\chi_R)\,\mathrm{d}\mathfrak{m}_n - \int_X h\Gamma_n(f_n,\chi_R)\,\mathrm{d}\mathfrak{m}_n$$

to get $\int_X \boldsymbol{c}(h)\chi_R\,\mathrm{d}\mathfrak{m} = \int_X \boldsymbol{b}(h\chi_R)\,\mathrm{d}\mathfrak{m} + \omega_R$, with $\omega_R \to 0$ as $R \to \infty$. Taking the limit w.r.t. R, we obtain $\int_X \boldsymbol{b}(h)\,\mathrm{d}\mathfrak{m} = \int_X \boldsymbol{c}(h)\,\mathrm{d}\mathfrak{m}$ for every $h \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$. This identity extends, by the density of $P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$, to any $h \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$; then, Lemma 6.1 applies and gives $\boldsymbol{b} = \boldsymbol{c}$.

By sequential compactness, this proves that $\boldsymbol{b}_n \xrightarrow{P\mathscr{A}_{\mathrm{bs}}} \boldsymbol{b}$. Strong convergence finally follows from Theorem 5.3, with $\Theta(z) = z^2$.

In order to prove the final part of the statement it is sufficient to apply the Leibniz formula, passing to the limit as $n \to \infty$ in

$$\int_X \boldsymbol{b}_n(g_n)a\,\mathrm{d}\mathfrak{m}_n = \int_X \Gamma_n(f_n,g_na)\,\mathrm{d}\mathfrak{m}_n - \int_X \boldsymbol{b}_n(a)g_n\,\mathrm{d}\mathfrak{m}_n \qquad a \in P_{\mathbb{Q}_+}\mathscr{A}_{\mathrm{bs}}$$

and using (6.2) and the L^2 convergence of $\boldsymbol{b}_n(a)$ to $\boldsymbol{b}(a)$. This proves that $\boldsymbol{b}_{f_n}(g_n)$ weakly converge to $\boldsymbol{b}_f(g)$ in the duality with $P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$, and since this class of test functions uniquely determines the limit, the thesis follows. $\square$

Remark 6.5. In the setting of the analysis of Ricci limit spaces, S. Honda deeply studied in [H11] notions of convergence for gradient derivations $\boldsymbol{b}_{f_n}$ associated to f_n, by looking essentially at the weak convergence of of $\boldsymbol{b}_{f_n}(g_n)$ to $\boldsymbol{b}_f(g)$ when $g_n = d(\cdot, z_n)$, $g = d(\cdot, z)$, with $z_n \to z$ (see in particular Definition 4.18 in [H11], and [H15] for more general tensor fields). In this respect, the final part of the statement of Theorem 6.4 shows the connection between ours and Honda's convergence, under the assumption of strong convergence in $H^{1,2}$ of the f_n to f.

We conclude this section by providing two examples.

Example 6.6 (Convergence of resolvents)**.** Under the assumptions of the previous theorem, let $f_n \in L^2(X,\mathfrak{m}_n)$ be strongly convergent in L^2 to $f \in L^2(X,\mathfrak{m})$, let $\lambda > 0$ and set $u_n = (\lambda - \Delta_n)^{-1}f_n \in D(\Delta_n)$, so that $\Delta_n u_n = \lambda u_n - f_n$, and consider the gradient derivations $\boldsymbol{b}_{u_n}$.

It is then known (see e.g., [GMS13, Corollary 6.10]) that Mosco convergence entails L^2-strong convergence of u_n to $u = (\lambda - \Delta)^{-1} f \in D(\Delta)$, as well as $\lim_n \mathsf{Ch}_n(u_n) = \mathsf{Ch}(u)$. We may choose $\Theta(z) = z^2$, to fulfil the assumptions of Theorem 6.4 and we deduce strong convergence of $\boldsymbol{b}_{u_n}$ to $\boldsymbol{b}_u$.

Example 6.7 (Laplacian eigenvalues)**.** Let us assume that $(X, \mathsf{d}, \mathfrak{m}_n)$ are $\mathrm{RCD}(K, \infty)$ spaces, so that in particular the assumptions of the previous theorem hold. In this case, let us consider normalized eigenfunctions of (minus) the Laplacian operators, i.e. $u_n \in D(\Delta_n)$ which satisfy $\int_X u_n^2 \, \mathrm{d}\mathfrak{m}_n = 1$, and $-\Delta_n u_n = \lambda_n u_n$, for some $\lambda_n \in \mathbb{R}^+$. Assuming in addition that either $K > 0$ or all $\mathfrak{m}_n$ are probability measures, using min-max arguments and Mosco convergence of the Cheeger energies, it has been proved in [GMS13, Theorem 7.8] that, representing the discrete spectra of $-\Delta_n$ as $(\lambda_n^k)_{k \geq 0}$ (in non-decreasing order), for each $k \geq 0$ the eigenvalues λ_n^k converge as $n \to \infty$ to the k-th eigenvalue of λ^k of $-\Delta$, and that the associated eigenfunctions u_n^k L^2-strongly converge to a corresponding eigenfunction u^k, possibly extracting a subsequence (there is no need to extract subsequences if the limit eigenvalue is simple).

Since $\mathsf{Ch}_n(u_n^k) = \lambda_n^k \to \lambda^k$, by Theorem 6.4 we deduce the strong convergence of the gradient derivations $\boldsymbol{b}_{u_n^k}$.

Part III
Flows associated to derivations and their convergence

7   Continuity equations and flows associated to derivations

Given $(\boldsymbol{b}_t)_{t \in (0,T)} \subset \mathrm{Der}^1_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, we now consider the continuity equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(\boldsymbol{b}_t \mu_t) = 0, \\ \mu_0 = \bar{\mu} \end{cases} \tag{CE}$$

and its weak formulation in the space of probability measures $\mu_t = u_t \mathfrak{m}$ absolutely continuous w.r.t. $\mathfrak{m}$.

Definition 7.1 (Weak solutions to the continuity equation)**.** Let $\mu_t = u_t \mathfrak{m} \in \mathscr{P}(X)$, $t \in (0, T)$, and $\bar{\mu} \in \mathscr{P}(X)$. We say that μ_t is a solution to (CE) if

$$\int_0^T \int_X |\boldsymbol{b}_t(f)| \, \mathrm{d}\mu_t \, \mathrm{d}t < \infty \qquad \forall f \in \mathrm{Lip}_{\mathrm{bs}}(X) \tag{7.1}$$

and for every $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$ and $\chi \in C_c^1([0, T))$ one has

$$-\int_0^T \chi'(t) \int_X f \, \mathrm{d}\mu_t \, \mathrm{d}t = \int_0^T \chi(t) \int_X \boldsymbol{b}_t(f) \, \mathrm{d}\mu_t \, \mathrm{d}t + \chi(0) \int_X f \, \mathrm{d}\bar{\mu}. \tag{7.2}$$

Remark 7.2 (Different classes of integrability and test functions)**.** Under the stronger assumption $\mu_t \leq C\mathfrak{m}$, the weak formulation of (CE) makes sense for all test functions $f \in \mathrm{Lip}_{\mathrm{b}}(X)$ assuming the condition

$$\int_0^T \|\boldsymbol{b}_t(f)\|_{L^1 + L^\infty(X, \mathfrak{m})} \, \mathrm{d}t < \infty \qquad \forall f \in \mathrm{Lip}_{\mathrm{b}}(X), \tag{7.3}$$

somehow weaker than (7.1) (a weaker norm, but a larger class of functions). If one has

$$\int_0^T \int_X \Theta(|\boldsymbol{b}_t|)\, \mathrm{d}\mathfrak{m}\, \mathrm{d}t < \infty \tag{7.4}$$

for some $\Theta : [0, \infty) \to [0, \infty]$ with more than linear growth at infinity, then the inequality $z \leq c(1 + \Theta(z))$ immediately yields that (7.3) holds. Furthermore, if we assume (7.4) and $\mu_t \leq C\mathfrak{m}$, it is easy by a truncation argument to pass in the weak formulation of (CE) from $\mathrm{Lip}_{\mathrm{bs}}(X)$ to $\mathrm{Lip}_{\mathrm{b}}(X)$.

We say that a solution μ_t to (CE) is weakly continuous if $t \mapsto \int_X f\, \mathrm{d}\mu_t$ is continuous in $(0, T)$ for all $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$; under this assumption, because of (7.1), the map is absolutely continuous and (7.2) can be written in the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_X f\, \mathrm{d}\mu_t = \int_X \boldsymbol{b}_t(f)\, \mathrm{d}\mu_t \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0, T), \qquad \lim_{t \downarrow 0} \int_X f\, \mathrm{d}\mu_t = \int_X f\, \mathrm{d}\bar{\mu}.$$

A natural class of examples of weakly continuous solutions to (CE) is given by $\mu_t = (\mathrm{ev}_t)_{\#}\boldsymbol{\pi}$ with $\boldsymbol{\pi} \in \mathscr{P}(C([0, T]; X))$, see Remark 7.5 below for more details.

Definition 7.3 (Regular flow relative to $\boldsymbol{b}$). Let $\boldsymbol{X} : [0, T] \times X \to X$ be a Borel map. We say that $\boldsymbol{X}$ is a regular flow relative to $\boldsymbol{b}_t$ if:

(a) for some constant $C = C(\boldsymbol{X}, \mathfrak{m})$, one has $\boldsymbol{X}(t, \cdot)_{\#}\mathfrak{m} \leq C\mathfrak{m}$ for all $t \in [0, T]$;

(b) $\boldsymbol{X}(\cdot, x) \in AC([0, T]; X)$ and $\boldsymbol{X}(0, x) = x$ for $\mathfrak{m}$-a.e. $x \in X$ and, for all $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \boldsymbol{X}(t, x) = \boldsymbol{b}_t(f)(\boldsymbol{X}(t, x)) \qquad \text{for } \mathscr{L}^1 \times \mathfrak{m}\text{-a.e. } (t, x) \in (0, T) \times X.$$

The property of being a regular flow is independent of the choice of representatives $\boldsymbol{b}_t(f)$; to see this, notice that the condition $\boldsymbol{X}(t, \cdot)_{\#}\mathfrak{m} \ll \mathfrak{m}$ for $\mathscr{L}^1$-a.e. $t \in (0, T)$, weaker than (a), and Fubini's theorem give that the set

$$\{(t, x) \in (0, T) \times X : \ (t, \boldsymbol{X}(t, x)) \in N\}$$

is $\mathscr{L}^1 \times \mathfrak{m}$-negligible for any $\mathscr{L}^1 \times \mathfrak{m}$-negligible set $N \subset (0, T) \times X$.

In connection with the stability analysis, the previous definition needs to be extended in order to cover generalized flows, namely flows where branching behaviour is allowed.

Definition 7.4 (Regular generalized flow relative to $\boldsymbol{b}$). Let $\boldsymbol{\pi} \in \mathscr{P}(C([0, T]; X))$. We say that $\boldsymbol{\pi}$ is a regular flow relative to $\boldsymbol{b}_t$ if:

(a) for some constant $C = C(\boldsymbol{\pi}, \mathfrak{m})$, one has $(\mathrm{ev}_t)_{\#}\boldsymbol{\pi} \leq C\mathfrak{m}$ for all $t \in [0, T]$;

(b) $\boldsymbol{\pi}$ is concentrated on $AC([0, T]; X)$ and for all $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) = \boldsymbol{b}_t(f)(\gamma(t)) \text{ for } \mathscr{L}^1 \times \boldsymbol{\pi}\text{-a.e. } (t, \gamma) \in (0, T) \times C([0, T]; X). \tag{7.5}$$

As for Definition 7.3, it is the regularity condition (a), even in the weakened form $(\mathrm{ev}_t)_{\#}\boldsymbol{\pi} \ll \mathfrak{m}$ for $\mathscr{L}^1$-a.e. $t \in (0, T)$, that ensures that property (b) above is independent of the choice of representatives of $\boldsymbol{b}_t(f)$. Obviously any regular flow $\boldsymbol{X}$ induces regular generalized flows $\boldsymbol{\pi}$, given by $\Sigma_{\#}(f\mathfrak{m})$, where f is the density of a probability measure and $\Sigma : X \to C([0, T]; X)$ is given by $\Sigma(x) = \boldsymbol{X}(\cdot, x)$. The converse holds if $\mathfrak{m} \ll (\mathrm{ev}_0)_{\#}\boldsymbol{\pi}$ and if the conditional probability measures $\boldsymbol{\pi}_x$ in $C([0, T]; X)$ induced by ev_0 are Dirac masses $\mathfrak{m}$-a.e. in X; indeed, if this is the case, setting $\boldsymbol{\pi}_x = \delta_{\{\boldsymbol{X}(\cdot, x)\}}$, we recover $\boldsymbol{X}$.

Remark 7.5. Under the mild integrability assumption (7.1) on $\boldsymbol{b}_t$, property (7.5) in Definition 7.4 and the continuity equation are closely related. More precisely, if $\boldsymbol{\pi} \in \mathscr{P}(C([0,T];X))$ satisfies $(\mathrm{ev}_t)_\# \boldsymbol{\pi} \ll \mathfrak{m}$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$ (weaker than (a)) and (7.5), then its marginals $\mu_t = (\mathrm{ev}_t)_\# \boldsymbol{\pi}$, $t \in [0,T]$, are weakly continuous in time and satisfy (CE). Indeed, fix $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$; it is clear that $t \mapsto \int_X f \, \mathrm{d}\mu_t$ is continuous. Moreover, by the Fubini-Tonelli theorem, for $\mathscr{L}^1$-a.e. $t \in (0,T)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t) = \boldsymbol{b}_t(f)(\gamma(t)) \qquad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma \in C([0,T];X)$$

and, by integration, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_X f \, \mathrm{d}\mu_t = \frac{\mathrm{d}}{\mathrm{d}t} \int f(\gamma(t)) \, \mathrm{d}\boldsymbol{\pi}(\gamma) = \int \boldsymbol{b}_t(f)(\gamma(t)) \, \mathrm{d}\boldsymbol{\pi}(\gamma) = \int_X \boldsymbol{b}_t(f) \, \mathrm{d}\mu_t.$$

The following elementary criterion will be useful and provides a converse to Remark 7.5; roughly speaking, it links the validity of the continuity equations for arbitrary modifications $g\boldsymbol{\pi}$, $g \in C_{\mathrm{b}}(C([0,T];X))$, to the property of being concentrated on solutions to the ODE, in the weak sense expressed by condition (b).

Proposition 7.6 (Concentration criterion)**.** *Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,T];X))$ be concentrated on $AC([0,T];X)$ with $(\mathrm{ev}_t)_\# \boldsymbol{\pi} \ll \mathfrak{m}$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$ and assume that $\boldsymbol{b}_t$ satisfy (7.1). Then, the following properties are equivalent:*

(a) for all $f \in \mathrm{Lip}_{\mathrm{bs}}(X)$, (7.5) holds;

(b) for all $g \in C_{\mathrm{b}}(C([0,T];X))$, with $g\boldsymbol{\pi} \in \mathscr{P}(C([0,T];X))$, the curve

$$\mu_t^g := (\mathrm{ev}_t)_\#(g\boldsymbol{\pi}), \quad t \in [0,T],$$

solves the continuity equation (CE) with $\bar{\mu} = \mu_0^g$.

Proof. We already proved the implication from (7.5) to the continuity equation. To show the converse, fix g as in (b). For all $g \in \mathrm{Lip}_{\mathrm{bs}}(X)$, by integration in an interval $[s,t] \subset [0,T]$ we get

$$\int_X f \, \mathrm{d}\mu_t^g - \int_X f \, \mathrm{d}\mu_s^g = \int_s^t \int_X \boldsymbol{b}_r(f) \, \mathrm{d}\mu_r^g \, \mathrm{d}r,$$

so that

$$\int f(\gamma(t)) - f(\gamma(s)) \, \mathrm{d}g\boldsymbol{\pi}(\gamma) = \int \int_s^t \boldsymbol{b}_r(f)(\gamma(r)) \, \mathrm{d}r \, \mathrm{d}g\boldsymbol{\pi}(\gamma),$$

i.e., $\int \left[f(\gamma(t)) - f(\gamma(s)) - \int_s^t \boldsymbol{b}_r(f)(\gamma(r)) \, \mathrm{d}r \right] g(\gamma) \, \mathrm{d}\boldsymbol{\pi}(\gamma) = 0$. Since g varies in a sufficiently large class, we obtain that $f(\gamma(t)) - f(\gamma(s)) = \int_s^t \boldsymbol{b}_r(f)(\gamma(r)) \, \mathrm{d}r$ for $\boldsymbol{\pi}$-a.e. γ, for s and t fixed. Since $f \circ \gamma \in AC([0,T])$ for $\boldsymbol{\pi}$-a.e. γ, by letting s and t vary in $\mathbb{Q} \cap [0,T]$ we obtain, via a density argument

$$f(\gamma(t)) - f(\gamma(s)) = \int_s^t \boldsymbol{b}_r(f)(\gamma(r)) \, \mathrm{d}r \quad \text{for all } s, t \in [0,T] \text{ with } s \le t, \text{ for } \boldsymbol{\pi}\text{-a.e. } \gamma,$$

which yields (7.5). $\square$

Remark 7.7 (Metric speed of a generalized flow). Let us recall that $|\boldsymbol{b}_t|$ provides an upper bound for the metric speed of curves selected by a generalized flow: indeed, [AT14, Lemma 7.4] gives the inequality $|\dot{\gamma}|(t) \le |\boldsymbol{b}_t|(\gamma(t))$ $\mathscr{L}^1$-a.e. in $(0,T)$, for $\boldsymbol{\pi}$-a.e. γ.

Proposition 7.8 (Tightness for generalized flows). *Assume that $\boldsymbol{\pi}_n \in \mathscr{P}(C([0,T];X))$ are regular generalized flows relative to $\boldsymbol{b}_{n,t}$ such that*

$$\lim_{R\to\infty} \sup_n \, (\mathrm{ev}_0)_{\#}\boldsymbol{\pi}_n(X \setminus B_R(\bar{x})) = 0 \tag{7.6}$$

for some $\bar{x} \in X$,

$$\sup_n C(\boldsymbol{\pi}_n, \mathfrak{m}_n) < \infty \quad \text{and} \quad \sup_n \int_0^T \int_X \Theta(|\boldsymbol{b}_{n,t}|) \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t < \infty,$$

with $\Theta : [0,\infty) \to [0,\infty]$ having more than linear growth at infinity. Then the family $\{\boldsymbol{\pi}_n\}$ is tight in $\mathscr{P}(C([0,T];X))$, and any limit point $\boldsymbol{\pi} \in \mathscr{P}(C([0,T];X))$ is concentrated on $AC([0,T];X)$.

Proof. In order to prove tightness, we build a coercive functional $\Psi : C([0,T];X) \to [0,\infty]$ (i.e., a map with relatively compact sublevel sets) such that

$$\sup_n \int_{C([0,T];X)} \Psi(\gamma) \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) < \infty.$$

Let $R_i \uparrow \infty$ with $\mathfrak{m}(\partial B_{R_i}(\bar{x})) = 0$. From the convergence of $\mathfrak{m}_n \llcorner \bar{B}_{R_i}(\bar{x})$ to $\mathfrak{m} \llcorner \bar{B}_{R_i}(\bar{x})$ we obtain coercive functions $\psi_i : \bar{B}_{R_i}(\bar{x}) \to [0,\infty]$ such that

$$\sup_n \int_{\bar{B}_{R_i}(\bar{x})} \psi_i \, \mathrm{d}\mathfrak{m}_n \le 2^{-i}.$$

Setting $\psi_i = 0$ on $X \setminus \bar{B}_{R_i}(\bar{x})$ and $\psi = \sum_{i\ge 1} \psi_i$, we obtain

$$\sup_n \int_X \psi \, \mathrm{d}\mathfrak{m}_n \le 1$$

and, since $\psi \ge \psi_i$ and $R_i \to \infty$, all sets $\{\psi \le t\} \cap \bar{B}_R(\bar{x})$ are relatively compact in X. Using (7.6) we can also find $\varphi : X \to [0,\infty]$ with $\varphi(x) \to \infty$ as $\mathsf{d}(x,\bar{x}) \to \infty$ and $\sup_n \int_X \varphi(\gamma(0)) \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) < \infty$.

Now fix a countable dense set $\{t_j\}$ in $[0,T]$ and define

$$\Psi(\gamma) = \varphi(\gamma(0)) + \sum_{j\in\mathbb{N}} 2^{-j}\psi(\gamma(t_j)) + \int_0^T \Theta(|\dot{\gamma}|(t)) \, \mathrm{d}t$$

if $\gamma \in AC([0,T];X)$, $\Psi(\gamma) = \infty$ otherwise. Thanks to the above-mentioned local compactness property of the sublevel sets of ψ, it is easily seen (as in Ascoli-Arzelà's theorem) that Ψ is coercive. It is now clear, using the condition $(\mathrm{ev}_{t_j})_{\#}\boldsymbol{\pi}_n \le C(\boldsymbol{\pi}_n, \mathfrak{m}_n)\mathfrak{m}_n$, that

$$\sup_n \int_{C([0,T];X)} \left[\varphi(\gamma(0)) + \sum_{j\in\mathbb{N}} 2^{-j}\psi(\gamma(t_j)) \right] \mathrm{d}\boldsymbol{\pi}_n(\gamma) < \infty.$$

In addition

$$\begin{aligned}
\int_{C([0,T];X)} \int_0^T \Theta(|\dot{\gamma}|(t)) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) &\le \int_0^T \int_{C([0,T];X)} \Theta(|\boldsymbol{b}_{n,t}|(\gamma(t))) \, \mathrm{d}\boldsymbol{\pi}_n(\gamma) \, \mathrm{d}t \\
&\le C(\boldsymbol{\pi}_n, \mathfrak{m}_n) \int_0^T \int_X \Theta(|\boldsymbol{b}_{n,t}|) \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t,
\end{aligned}$$

where the first inequality above follows from Remark 7.7. $\qquad\square$

In the following proposition we prove that limits of flows relative to $\boldsymbol{b}_{n,t}$ are flows relative to $\boldsymbol{b}_t$, if $\boldsymbol{b}_{n,t}$ strongly converge according to (7.7). This condition, stated in the minimal form needed for the validity of the proof, is in many cases implied by the notion of strong convergence of the previous sections, for instance in $\mathrm{RCD}(K, \infty)$ spaces the class $\mathcal{D} = P_{\mathbb{Q}^+}\mathscr{A}_{\mathrm{bs}}$ is dense w.r.t. $\mathfrak{m}$-flat convergence and one can use the convergence in measure of $\boldsymbol{b}_{n,t}(f)$ to $\boldsymbol{b}_t(f)$ for $f \in \mathcal{D}$, together with uniform bounds in $L^1 + L^\infty$, to prove (7.7). We do not discuss in generality this point, referring to the specific examples discussed in Section 8.

Proposition 7.9 (Closure theorem). *Assume that $\boldsymbol{\pi}_n \in \mathscr{P}(C([0,T]; X))$ are regular generalized flows relative to $\boldsymbol{b}_{n,t}$ and that:*

(i) $\sup_n C(\boldsymbol{\pi}_n, \mathfrak{m}_n) < \infty$ and $\boldsymbol{\pi}_n$ weakly converge to $\boldsymbol{\pi} \in \mathscr{P}(C([0,T]; X))$, with $\boldsymbol{\pi}$ concentrated on $AC([0,T]; X)$;

(ii) for all f in a class $\mathcal{D} \subset \mathrm{Lip}_{\mathrm{b}}(X)$, dense w.r.t. $\mathfrak{m}$-flat convergence, one has

$$\lim_{n \to \infty} \int_0^T \int_X \boldsymbol{b}_{n,t}(f) v_n \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t = \int_0^T \int_X \boldsymbol{b}_t(f) v \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t \qquad (7.7)$$

whenever $0 \le v_n \le C < \infty$ and $v_n(t, \cdot)\mathfrak{m}_n \in \mathscr{P}(X)$ weakly converge to $v(t, \cdot)\mathfrak{m} \in \mathscr{P}(X)$ for all $t \in (0, T)$;

(iii) $\int_0^T \|\boldsymbol{b}_t(f)\|_{L^1 + L^\infty(X, \mathfrak{m})} \, \mathrm{d}t < \infty$ and $\sup_n \int_0^T \|\boldsymbol{b}_{n,t}(f)\|_{L^1 + L^\infty(X, \mathfrak{m}_n)} \, \mathrm{d}t < \infty$ for all $f \in \mathrm{Lip}_{\mathrm{b}}(X)$;

(iv) either $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is reflexive, or $\int_0^T \int_{B_R(\bar{x})} |\operatorname{div} \boldsymbol{b}_t| \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t < \infty$ for all $R > 0$.

Then $\boldsymbol{\pi}$ is a regular generalized flow relative to $\boldsymbol{b}_t$.

Proof. Fix $g \in C_{\mathrm{b}}(C([0,T]; X))$ with $g\boldsymbol{\pi} \in \mathscr{P}(C([0,T]; X))$, set $g_n = g / \int g \, \mathrm{d}\boldsymbol{\pi}_n$, so that, for n large enough, the measures $\boldsymbol{\pi}_n^{g_n} := g_n \boldsymbol{\pi}_n \in \mathscr{P}(C([0,T]; X))$ are well-defined and weakly converge to $\boldsymbol{\pi}^g := g\boldsymbol{\pi}$.

By Proposition 7.6, applied with π_n and g_n, the marginal measures $\mu_{n,t}^{g_n} := (\mathrm{ev}_t)_\# \boldsymbol{\pi}_n^{g_n}$ solve (CE) with $\bar{\mu} = \mu_{n,0}^{g_n}$, have uniformly bounded (w.r.t. $t \in [0,T]$ and n sufficiently large) densities w.r.t. $\mathfrak{m}_n$,

$$u_{n,t}^{g_n} \le \sup_n C(\boldsymbol{\pi}_n, \mathfrak{m}_n) \|g_n\|_\infty$$

and weakly converge to $\mu_t^g := (\mathrm{ev}_t)_\# \boldsymbol{\pi}^g$, since ev_t is a continuous map. Passing to the limit as $n \to \infty$ in the weak formulation

$$-\int_0^T \chi'(t) \int_X f \, \mathrm{d}\mu_{n,t}^{g_n} \, \mathrm{d}t = \int_0^T \chi(t) \int_X \boldsymbol{b}_{n,t}(f) u_{n,t}^{g_n} \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t + \chi(0) \int_X f \, \mathrm{d}\mu_{n,0}^{g_n}.$$

with $f \in \mathcal{D}$ and using the convergence assumption with $v_n(t, x) = T^{-1} u_{n,t}^{g_n}(x)$ we get

$$-\int_0^T \chi'(t) \int_X f \, \mathrm{d}\mu_t^g \, \mathrm{d}t = \int_0^T \chi(t) \int_X \boldsymbol{b}_t(f) \, \mathrm{d}\mu_t^g \, \mathrm{d}t + \chi(0) \int_X f \, \mathrm{d}\mu_0^g.$$

for all $f \in \mathcal{D}$. By an easy approximation based on assumption (iv) and Lemma 3.9, the density of $\mathcal{D}$ gives that μ_t^g solve (CE) with $\bar{\mu} = \mu_0^g$. Again by Proposition 7.6, we obtain that $\boldsymbol{\pi}$ is a regular generalized flow relative to $\boldsymbol{b}$. $\qquad\square$

Remark 7.10. In Proposition 7.9 the strong convergence property of $\boldsymbol{b}_{n,t}$ expressed by (7.7) can not be replaced, in general, by weak convergence together with convergence of the norms. In fact, we have the following simple counterexample. Consider in $(\mathbb{R}^3, \|\cdot\|_\infty, \mathscr{L}^3)$ the vector fields

$$\boldsymbol{b}_n(x) = w_n(x_1)e_2 + e_3$$

with (weak convergences are meant in the weak-$*$ topology of L^∞)

$$w_n : \mathbb{R} \to \mathbb{R}, \qquad w_n \rightharpoonup 0, \qquad |w_n| = 1, \qquad w_n^+ \rightharpoonup \frac{1}{2}.$$

For instance, we can take $w_n(t) = \operatorname{sign} \sin(2^n t)$. Then we have that $\boldsymbol{b}_n \rightharpoonup e_3$ and $\lim_n |\boldsymbol{b}_n| = |e_3|$. Let us call $\boldsymbol{X}_n$ the flow relative to $\boldsymbol{b}_n$. Then, given any probability density u, we have the following convergence of the associated generalized flows $\eta_n = X_n(\,\cdot\,,x)_\#(u(x)\mathscr{L}^3)$:

$$\int_{\mathbb{R}^3} \delta_{\boldsymbol{X}_n(\,\cdot\,,x)} u(x)\,\mathrm{d}x \xrightarrow{\;\mathscr{P}(C([0,T];\mathbb{R}^3))\;} \int_{\mathbb{R}^3} \frac{1}{2}\left(\delta_{\gamma^+(\,\cdot\,,x)} + \delta_{\gamma^-(\,\cdot\,,x)}\right) u(x)\,\mathrm{d}x$$

where

$$\gamma^\pm(t,x) = (x_1, x_2 \pm \frac{1}{2}t, x_3 + t).$$

Therefore, we have that the generalized flows converge in $\mathscr{P}(C([0,T];\mathbb{R}^3))$ to a probability measure that is not a generalized flow relative to the weak limit e_3.

This example raises a problem about possible extensions of Theorem 5.3 to the case when the norm on derivations is not strictly convex, since oscillations in the vector fields might not be detected by the convergence of the norms.

We end this section with the following stability result for regular flows, under the assumption that for the limit vector field any regular generalized flow is induced by a regular flow, see (7.8). Notice that with a similar proof a similar convergence result holds if we replace $\boldsymbol{X}_n$ by regular generalized flows $\boldsymbol{\pi}_n$.

Proposition 7.11 (Stability for regular flows, general case)**.** *Assume that $\boldsymbol{X}_n$ are regular flows relative to $\boldsymbol{b}_{n,t}$ with $\boldsymbol{b}_{n,t}$ strongly convergent to $\boldsymbol{b}_t$ according to (7.7) of Proposition 7.9,*

$$\sup_n C(\boldsymbol{X}_n, \mathfrak{m}_n) < \infty \quad and \quad \sup_n \int_0^T \int_X \Theta(|\boldsymbol{b}_{n,t}|)\,\mathrm{d}\mathfrak{m}_n\,\mathrm{d}t < \infty,$$

and $\Theta : [0,\infty) \to [0,\infty]$ having more than linear growth at infinity. Assume also that any regular generalized flow $\boldsymbol{\pi}$ relative to $\boldsymbol{b}$ is induced by a regular flow $\boldsymbol{X}$ relative to $\boldsymbol{b}$, i.e.

$$\boldsymbol{\pi} = \int_X \boldsymbol{X}(\,\cdot\,,x)\,\mathrm{d}(\mathrm{ev}_0)_\#\boldsymbol{\pi}. \tag{7.8}$$

Then, $\boldsymbol{X}_n : X \to C([0,T];X)$ converge in measure towards $\boldsymbol{X} : X \to C([0,T];X)$.

Proof. To deduce convergence in measure, we rely on Proposition 2.4, with $Y = C([0,T];X)$. Let $v \in C_{\mathrm{bs}}(X)$ nonnegative with $\int_X v\,\mathrm{d}\mathfrak{m} = 1$, let $\bar{x} \in X$ and $\bar{R} > 0$ large enough so that $\operatorname{supp}(v) \subset B_{\bar{R}}(\bar{x})$. We define $c_n := \int_X v\,\mathrm{d}\mathfrak{m}_n$, which converge to 1 as $n \to \infty$, and, for n large enough, $\boldsymbol{\pi}_n := c_n^{-1} \int_X \boldsymbol{X}(\,\cdot\,,x)v(x)\,\mathrm{d}\mathfrak{m}_n \in \mathscr{P}(C([0,T];X))$. Then, $\boldsymbol{\pi}_n$ is a regular generalized flow, relative to $\boldsymbol{b}_n$, with $C(\boldsymbol{\pi}_n, \mathfrak{m}_n) \leq c_n^{-1}C(\boldsymbol{X}_n, \mathfrak{m}_n) \sup |v|$, hence uniformly bounded in n.

By Proposition 7.8, the family $\{\boldsymbol{\pi}_n\}$ is tight, in $\mathscr{P}(C([0,T];X))$: indeed, the only non trivial condition to check is (7.6), but that expression is 0 for $R > \bar{R}$. Next, by

Proposition 7.9, any limit point $\boldsymbol{\pi}$ is a regular generalized flow relative to $\boldsymbol{b}$: in this case, we have to check condition *(iii)* only, which follows from the inequality

$$|b_{n,t}(f)| \leq |\boldsymbol{b}_{n,t}|\operatorname{Lip}(f) \leq c\operatorname{Lip}(f)(1 + \Theta(|\boldsymbol{b}_{n,t}|)),$$

where $c > 0$ is some constant such that $|z| \leq c(1 + \Theta(z))$ for all $z \in [0, \infty)$. Hence, our assumptions entail then that $\boldsymbol{\pi}$ can be written as in (7.8).

To apply Proposition 2.4 with $Y = C([0,T];X)$, let $\Phi : Y \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be bounded and continuous functions. Then,

$$\int_{X \times Y} g(x)\Phi(\gamma)\,\mathrm{d}(\mathrm{Id} \times \boldsymbol{X}_n)_\#(v\mathfrak{m}_n)(x,\gamma) = \int_X g(x), \Phi(\boldsymbol{X}_n(x,\cdot))v(x)\,\mathrm{d}\mathfrak{m}_n(x)$$

$$(\text{since } X_n(x,0) = x \text{ } \mathfrak{m}_n\text{-a.e. on } X) \qquad = \int_X g(\boldsymbol{X}_n(x,0))\Phi(\boldsymbol{X}_n(x,\cdot))v(x)\,\mathrm{d}\mathfrak{m}_n(x)$$

$$= \int_Y g(\mathrm{ev}_0(\gamma))\Phi(\gamma)\,\mathrm{d}\boldsymbol{\pi}_n(\gamma)$$

$$(\text{as } n \to \infty) \qquad \to \int_Y g(\mathrm{ev}_0(\gamma))\Phi(\gamma)\,\mathrm{d}\boldsymbol{\pi}(\gamma)$$

$$(\text{by the representation (7.8)}) \qquad = \int_X g(\boldsymbol{X}(x,0))\Phi(\boldsymbol{X}(x,\cdot))v(x)\,\mathrm{d}\mathfrak{m}(x)$$

$$= \int_{X \times Y} g(x)\Phi(\gamma)\,\mathrm{d}(\mathrm{Id} \times \boldsymbol{X})_\#(v\mathfrak{m})(x,\gamma),$$

where the limit as $n \to \infty$ follows from the fact that $g(\mathrm{ev}_0)\Phi$ is a bounded continuous function on Y. $\square$

8 Convergence of flows under Ricci curvature bounds

Let us recall the following well-posedness result, from [AT14], that we report here in a form suitable for our purposes. A crucial assumption, leading to well-posedness of the continuity equation, and then to the existence and uniqueness of the regular flow, is on the "deformation" $D^{\mathrm{sym}}\boldsymbol{b}$ of $\boldsymbol{b}$ (corresponding to the symmetric part of derivative in the Euclidean case), which can be defined in a suitable weak sense [AT14, Definition 5.2]. For example, if $\boldsymbol{b}(\cdot) = (\omega + c)\Gamma(f,\cdot)$, for $f \in D(\Delta)$, $\omega \in H^{1,2}(X,\mathsf{d},\mathfrak{m}) \cap L^\infty(X,\mathfrak{m})$, $c \in \mathbb{R}$, then [AT14, Theorem 6.7] gives $\|D^{\mathrm{sym}}\boldsymbol{b}\|_{4,4} < \infty$.

Theorem 8.1 (Well-posedness of flows). *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ space, for some $K \in \mathbb{R}$ and let $\boldsymbol{b} = (\boldsymbol{b}_t)_{t \in (0,T)}$ be a Borel time-dependent derivation with*

$$|\boldsymbol{b}|, \operatorname{div}\boldsymbol{b} \in L^1_t(L^2(X,\mathfrak{m}) + L^\infty(X,\mathfrak{m})), \quad (\operatorname{div}\boldsymbol{b})^- \in L^1_t(L^\infty(X,\mathfrak{m})),$$

$$\text{and} \quad \|D^{\mathrm{sym}}\boldsymbol{b}_t\|_{4,4} \in L^1(0,T).$$

Then, there exists a unique regular flow $\boldsymbol{X}$ relative to $\boldsymbol{b}$ and every regular generalized flow $\boldsymbol{\pi}$ relative to $\boldsymbol{b}$ is induced by $\boldsymbol{X}$ as in (7.8).

Proof. By [AT14, Theorem 4.3, Theorem 5.4], the continuity equation (CE) associated to $\boldsymbol{b}$ has existence and uniqueness of solutions in the class of weakly continuous solutions $\mu_t = u_t\mathfrak{m} \in \mathscr{P}(X)$, $t \in [0,T]$, with $\|u_t\|_{L^\infty(X,\mathfrak{m})}$ bounded in $[0,T]$, so that [AT14, Theorem 8.3] applies, providing existence and uniqueness of a regular flow $\boldsymbol{X}$ relative to $\boldsymbol{b}$. The last statement follows from [AT14, Theorem 8.4]. $\square$

We state and prove our main result concerning stability of flows on converging $\mathrm{RCD}(K,\infty)$ metric measure spaces. Let us stress the fact that bounds on divergence and deformation are assumed only for the limit derivation.

Theorem 8.2 (Stability of regular flows under curvature assumptions). *Let $\boldsymbol{b} = (\boldsymbol{b}_t)_{t \in (0,T)} \in L_t^1(\mathrm{Der}_{\mathrm{loc}}^1(X,\mathsf{d},\mathfrak{m}))$ as in Theorem 8.1. For $n \geq 1$, let $\boldsymbol{b}_n = (\boldsymbol{b}_{n,t})_{t \in (0,T)} \in L_t^1(\mathrm{Der}_{\mathrm{loc}}^1(X,\mathsf{d},\mathfrak{m}_n))$ and assume that*

$$\lim_{n \to \infty} \int_0^T \chi(t) \int_X \boldsymbol{b}_{n,t}(a) v \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t = \int_0^T \chi(t) \int_X \boldsymbol{b}_t(a) v \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t$$

for all $\chi \in C_c(0,T)$, $v \in C_{\mathrm{bs}}(X)$, $a \in P_{\mathbb{Q}_+} \mathscr{A}_{\mathrm{bs}}$, and

$$\limsup_{n \to \infty} \int_0^T \int_X \Theta(|\boldsymbol{b}_{n,t}|) \, \mathrm{d}\mathfrak{m}_n \, \mathrm{d}t \leq \int_0^T \int_X \Theta(|\boldsymbol{b}|_t) \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t < \infty$$

with $\Theta : [0,\infty) \to [0,\infty)$ strictly convex and having more than linear growth at infinity.

Let $\boldsymbol{X}_n$ be regular flows relative to $\boldsymbol{b}_n$, and $\boldsymbol{X}$ be the ($\mathfrak{m}$-a.e. unique) regular flow relative to $\boldsymbol{b}$. If $\sup_n C(\boldsymbol{X}_n, \mathfrak{m}_n) < \infty$, then $\boldsymbol{X}_n : X \to C([0,T];X)$ converge in measure towards $\boldsymbol{X} : X \to C([0,T];X)$.

Proof. The thesis follows from Proposition 7.11, with $\mathcal{D} = P_{\mathbb{Q}_+} \mathscr{A}_{\mathrm{bs}}$. Indeed, it is sufficient to notice that, by Theorem 4.4 and Theorem 5.3 we obtain strong convergence of $\boldsymbol{b}_n$ to $\boldsymbol{b}$, and by Theorem 8.1, we deduce that the representation (7.8) holds true for any generalized flow $\boldsymbol{\pi}$ relative to $\boldsymbol{b}$. $\qquad\square$

Example 8.3 (Convergence of flows associated to resolvents). In the setting of Example 6.6, since the operator $(\lambda - \Delta_n)^{-1}$ are Markovian, if for some constant c one has $|f_n| \leq c \, \mathfrak{m}_n$-a.e. then it follows $|u_n| \leq c \, \mathfrak{m}_n$-a.e. and therefore $\mathrm{div}\, \boldsymbol{b}_{u_n} = \lambda(u_n - f_n) \in L^\infty(X, \mathfrak{m}_n)$ uniformly w.r.t. n. Now, by [AT14, Theorem 6.7], one has $\|D^{\mathrm{sym}} \boldsymbol{b}_{u_n}\|_{4,4} < \infty$. We are in a position to deduce that there exist the $\mathfrak{m}_n$-a.e. unique regular flows $\boldsymbol{X}_n$ relative to $\boldsymbol{b}_{u_n}$ and that, as $n \to \infty$, $\boldsymbol{X}_n$ converge in measure in $C([0,T];X)$ to the $\mathfrak{m}$-a.e. unique flow $\boldsymbol{X}$ relative to $\boldsymbol{b}_u$.

Example 8.4 (Convergence of flows associated to Laplacian eigenfunctions). In the setting of Example 6.7, for k fixed set $\boldsymbol{b}_n = \boldsymbol{b}_{u_n^k}$ and $\boldsymbol{b} = \boldsymbol{b}_{u^k}$. From [AT14, Theorem 6.7] we obtain $\|D^{\mathrm{sym}} \boldsymbol{b}_n\|_{4,4} < \infty$, but in order to obtain the validity of the assumption $(\mathrm{div}\, \boldsymbol{b}_n)^- \in L^\infty(X, \mathfrak{m})$ in Theorem 8.1, the $\mathrm{RCD}(K,\infty)$ assumption is not sufficient, e.g., in case of Gaussian space where the eigenfunctions are Hermite polynomials. To obtain non trivial examples, we may restrict ourselves to metric measure spaces $(X, \mathsf{d}, \mathfrak{m}_n)$ where the heat semigroup P_t^n is ultracontractive, i.e., mapping for $t > 0$ the space $L^2(X, \mathfrak{m}_n)$ into $L^\infty(X, \mathfrak{m}_n)$. Under this assumption, for $t > 0$ one has

$$\Delta_n u_n^k = -\lambda_n^k e^{\lambda_n^k t} P_t^n u_n^k \in L^\infty(X, \mathfrak{m}_n).$$

If we assume that quantitative ultracontractive bounds hold uniformly w.r.t. n, then a unique regular flow $\boldsymbol{X}_n$ relative to $\boldsymbol{b}_n$ is defined and we deduce convergence in measure to the natural limit flow $\boldsymbol{X}$, as $n \to \infty$.

Appendix

A   Minimal relaxed slopes and Cheeger energy

In this section we recall basic facts about minimal relaxed slopes, Sobolev spaces and heat flow in metric measure spaces, see [AGS14a] and [G15a] for a more systematic treat-

ment of this topic. The Cheeger energy $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, \infty]$ is the convex and lower semicontinuous functional defined as follows:

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int_X \mathrm{Lip}_\mathrm{a}^2(f_n) \, \mathrm{d}\mathfrak{m} : \ f_n \in \mathrm{Lip}_\mathrm{b}(X) \cap L^2(X, \mathfrak{m}), \ \|f_n - f\|_2 \to 0 \right\}.$$

The Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is simply defined as the finiteness domain of Ch. It can be proved that $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is Hilbert if Ch is quadratic, and reflexive if (X, d) is doubling. (see [ACDM15]).

In connection with the definition of Ch, for all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ one can consider the collection $RS(f)$ of all functions in $L^2(X, \mathfrak{m})$ larger than a weak $L^2(X, \mathfrak{m})$ limit of $\mathrm{Lip}_\mathrm{a}(f_n)$, with $f_n \to f$ in $L^2(X, \mathfrak{m})$. This collection describes a convex, closed and nonempty set, whose element with smallest $L^2(X, \mathfrak{m})$ norm is called minimal relaxed slope and denoted by $|\mathrm{D}f|$. Because of the minimality property, $|\mathrm{D}f|$ provides an integral representation to Ch and it is not hard to improve weak to strong convergence.

Theorem A.1. *For all $f \in D(\mathsf{Ch})$ one has*

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}$$

and there exist $f_n \in \mathrm{Lip}_\mathrm{b}(X) \cap L^2(X, \mathfrak{m})$ with $f_n \to f$ in $L^2(X, \mathfrak{m})$ and $\mathrm{Lip}_\mathrm{a}(f_n) \to |\mathrm{D}f|$ in $L^2(X, \mathfrak{m})$. In particular, if $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is reflexive, there exist $f_n \in \mathrm{Lip}_\mathrm{b}(X) \cap L^2(X, \mathfrak{m})$ satisfying $f_n \to f$ in $L^2(X, \mathfrak{m})$ and $|\mathrm{D}(f_n - f)| \to 0$ in $L^2(X, \mathfrak{m})$.

Most standard calculus rules can be proved, when dealing with minimal relaxed slopes. For the purposes of this paper the most relevant ones are:

Locality on Borel sets. $|\mathrm{D}f| = |\mathrm{D}g|$ $\mathfrak{m}$-a.e. on $\{f = g\}$ for all $f, g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$;

Pointwise minimality. $|\mathrm{D}f| \leq g$ $\mathfrak{m}$-a.e. for all $g \in RS(f)$;

Degeneracy. $|\mathrm{D}f| = 0$ $\mathfrak{m}$-a.e. on $f^{-1}(N)$ for all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and all $\mathscr{L}^1$-negligible $N \subset \mathbb{R}$ Borel;

Chain rule. $|\mathrm{D}(\phi \circ f)| = |\phi'(f)||\mathrm{D}f|$ for all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and all $\phi : \mathbb{R} \to \mathbb{R}$ Lipschitz with $\phi(0) = 0$.

Another object canonically associated to Ch and then to the metric measure structure is the heat flow P_t, defined as the $L^2(X, \mathfrak{m})$ gradient flow of Ch, according to the Brezis-Komura theory of gradient flows of lower semicontinuous functionals in Hilbert spaces, see for instance [B70]. This theory provides a continuous contraction semigroup. We shall use P_t only in the case when Ch is quadratic, as a regularizing operator. In this special case P_t is also linear (and this property is equivalent to Ch being quadratic) and it is easily seen that

$$\lim_{t \downarrow 0} P_t f = f \qquad \text{for all } f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}).$$

Finally, we describe the class of $\mathrm{RCD}(K, \infty)$ metric measure spaces of [AGS14b], where thanks to the lower bounds on Ricci curvature even stronger properties of P_t can be proved. We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfying the growth bound (for some constants c_1, c_2 and some $\bar{x} \in X$)

$$\mathfrak{m}\big(B_r(\bar{x})\big) \leq c_1 e^{c_2 r^2} \qquad \forall r > 0 \tag{A.1}$$

is a RCD(K, ∞) metric measure space, with $K \in \mathbb{R}$, if Ch is quadratic and if, setting

$$\mathscr{P}_2(X) := \left\{ \mu \in \mathscr{P}(X) : \int_X \mathsf{d}^2(\bar{x}, x) \, d\mathfrak{m}(x) < \infty \right\},$$

the Relative Entropy Functional $\mathrm{Ent}(\mu) : \mathscr{P}_2(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$\mathrm{Ent}(\mu) := \begin{cases} \int_X \rho \log \rho \, d\mathfrak{m} & \text{if } \mu = \rho \mathfrak{m} \ll \mathfrak{m}, \\ \infty & \text{otherwise} \end{cases}$$

is K-convex along Wasserstein geodesics in $\mathscr{P}_2(X)$. See [AGS14b] (dealing with finite reference measures), [AGMR15] (for the σ-finite case) and [AGS15] for various characterizations of this class of spaces. We quote here the following result, which essentially derives from the identification of P_t as the gradient flow of Ent w.r.t. the Wasserstein distance and the contractivity properties with respect to that distance.

Proposition A.2. *In* RCD(K, ∞) *spaces* $(X, \mathsf{d}, \mathfrak{m})$, *for all* $t > 0$ *the semigroup* P_t *maps* $L^2 \cap L^\infty(X, \mathfrak{m})$ *to* $C_{\mathrm{b}}(X)$ *and if* $f \in \mathrm{Lip}(X) \cap H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ *one has*

$$\mathrm{Lip}_{\mathrm{a}}(P_t f) \le e^{-Kt} \sqrt{P_t |\mathrm{D}f|^2} \qquad \text{pointwise in } X.$$

B   An approximation result

In this section we improve Theorem A.1 by showing that the approximating sequence can be chosen in the "canonical" algebra $\mathscr{A}$ generated by the distance functions, and even in the subalgebra $\mathscr{A}_{\mathrm{bs}}$. The deep reason why this is possible is the fact that this is the class of functions appearing in the Hopf-Lax formula, see (B.8).

Theorem B.1. *For all* $f \in D(\mathsf{Ch})$ *there exist* $f_n \in \mathscr{A}_{\mathrm{bs}}$ *with* $f_n \to f$ *in* $L^2(X, \mathfrak{m})$ *and* $\mathrm{Lip}_{\mathrm{a}}(f_n) \to |\mathrm{D}f|$ *in* $L^2(X, \mathfrak{m})$. *In particular, if* $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ *is reflexive, one has the existence of* $f_n \in \mathscr{A}_{\mathrm{bs}}$ *satisfying* $f_n \to f$ *in* $L^2(X, \mathfrak{m})$ *and* $|\mathrm{D}(f_n - f)| \to 0$ *in* $L^2(X, \mathfrak{m})$.

Proof. The proof follows closely the strategy developed in [AGS14a]. We first build a variant of Cheeger's energy by restricting the approximation to functions in $\mathscr{A}_{\mathrm{bs}}$; this construction provides a new minimal relaxed slope, that we denote by $|\mathrm{D}f|_{\mathscr{A}}$ and that we prove to coincide with $|\mathrm{D}f|$ passing through the notion of minimal 2-weak upper gradient $|\mathrm{D}f|_w$. Let us outline the main steps, quoting them also from [ACDM15], whose context is closer to the one of the present paper (indeed, in [AGS14a] more general metric measure structures, with possibly infinite distances, are allowed).

Step 1. (Construction of $\mathsf{Ch}_{\mathscr{A}}$, $|\mathrm{D}f|_{\mathscr{A}}$ and calculus rules). For all $f \in L^2(X, \mathfrak{m})$ we define

$$\mathsf{Ch}_{\mathscr{A}}(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int_X \mathrm{Lip}_{\mathrm{a}}^2(f_n) \, d\mathfrak{m} : \ f_n \in \mathscr{A}_{\mathrm{bs}}, \ f_n \to f \text{ in } L^2(X, \mathfrak{m}) \right\}.$$

It is immediate to check that $\mathsf{Ch}_{\mathscr{A}}$ is a convex and $L^2(X, \mathfrak{m})$-lower semicontinuous functional in $L^2(X, \mathfrak{m})$, with a dense domain (here we use that $\mathscr{A}_{\mathrm{bs}}$ is dense in $L^2(X, \mathfrak{m})$). It is also obvious by the definition that

$$\mathsf{Ch} \le \mathsf{Ch}_{\mathscr{A}} \tag{B.1}$$

where Ch is Cheeger's functional considered in the previous section.

As in [ACDM15, Section 4], for all $f \in D(\mathsf{Ch}_{\mathscr{A}})$ one can define $RS_{\mathscr{A}}(f)$ as the collection of all functions $g \in L^2(X, \mathfrak{m})$ which are larger than a weak $L^2(X, \mathfrak{m})$ limit of $\mathrm{Lip}_{\mathrm{a}}(f_n)$, with

$f_n \in \mathscr{A}_{\mathrm{bs}}$ and $f_n \to f$ in $L^2(X, \mathfrak{m})$. By weak L^2-compactness, this set is not empty and it is not hard to show that it is closed and convex. Its element with minimal $L^2(X, \mathfrak{m})$ norm is defined to be $|\mathrm{D}f|_{\mathscr{A}}$. Since $RS_{\mathscr{A}}(f) \subset RS(f)$ for all $f \in D(\mathsf{Ch}_{\mathscr{A}})$, we can use the pointwise minimality property of $|\mathrm{D}f|$ to refine (B.1) to a pointwise inequality:

$$|\mathrm{D}f| \le |\mathrm{D}f|_{\mathscr{A}} \qquad \mathfrak{m}\text{-a.e. in } X \text{ for all } f \in D(\mathsf{Ch}_{\mathscr{A}}). \tag{B.2}$$

As we already mentioned in the previous section, in connection with $|\mathrm{D}f|$, several properties of $|\mathrm{D}f|_{\mathscr{A}}$ stem from the variational definition, in particular,

(a) there exist $f_n \in \mathscr{A}_{\mathrm{bs}}$ with $f_n \to f$ in $L^2(X, \mathfrak{m})$ and $\mathrm{Lip}_{\mathrm{a}}(f_n) \to |\mathrm{D}f|_{\mathscr{A}}$ strongly in $L^2(X, \mathfrak{m})$;

(b) locality on Borel sets, pointwise minimality, degeneracy and chain rule;

(c) for all $g \in D(\mathsf{Ch}_{\mathscr{A}})$ where $\mathsf{Ch}_{\mathscr{A}}$ has nonempty subdifferential one has

$$-\int_X f \Delta_{\mathscr{A}} g \, \mathrm{d}\mathfrak{m} \le \int_X |\mathrm{D}f|_{\mathscr{A}} |\mathrm{D}g|_{\mathscr{A}} \, \mathrm{d}\mathfrak{m},$$

where $-\Delta_{\mathscr{A}} g$ is the element with smallest norm in the subdifferential of $\mathsf{Ch}_{\mathscr{A}}(g)$, with equality if $f = \varphi(g)$ with φ Lipschitz, nondecreasing, $\varphi(0) = 0$.

Notice that the proof of properties (b), which rests on the decomposability property of $RS_{\mathscr{A}}$

$$\chi g_1 + (1 - \chi) g_2 \in RS_{\mathscr{A}}(f) \qquad \text{for all } g_1, g_2 \in RS_{\mathscr{A}}(f),\, \chi \in \mathscr{A},\, 0 \le \chi \le 1,$$

depends on the fact that the class of functions $\chi \in \mathscr{A}$ with values in $[0, 1]$ is dense in $L^\infty(X, [0, 1])$, w.r.t. convergence in $\mathfrak{m}$-measure. Here the assumption that $\mathscr{A}$ is a lattice plays a role.

Because of property (a), the proof of the theorem will be achieved if we show that, for all $f \in D(\mathsf{Ch})$, one has $f \in D(\mathsf{Ch}_{\mathscr{A}})$ and $|\mathrm{D}f|_{\mathscr{A}} = |\mathrm{D}f|$ $\mathfrak{m}$-a.e. in X. Notice that, because of (B.1) and (B.2), one has $D(\mathsf{Ch}_{\mathscr{A}}) \subset D(\mathsf{Ch})$ and $|\mathrm{D}f| \le |\mathrm{D}f|_{\mathscr{A}}$ $\mathfrak{m}$-a.e. in X for all $f \in D(\mathsf{Ch}_{\mathscr{A}})$, so our main concern will be to prove the converse inclusion and inequality.

For the proof of this statement we can easily reduce ourselves to the case when the support of $\mathfrak{m}$ has finite diameter and $\mathfrak{m}(X) < \infty$. Indeed, notice that $\mathscr{A}$ contains cut-off functions $h_n : X \to [0, 1]$, $n \ge 1$, with $\mathrm{Lip}(h_n) \le 2$, $h_n \equiv 1$ in $B_{n-1}(\bar{x})$ and $h_n \equiv 0$ in $X \setminus B_n(\bar{x})$. Now, fix $f \in D(\mathsf{Ch})$. Denoting by $\mathfrak{m}_n$ the measures $\chi_{B_n(\bar{x})} \mathfrak{m}$, and by Ch_n, $|\mathrm{D}f|_n$, $\mathsf{Ch}_{\mathscr{A},n}$, $|\mathrm{D}f|_{\mathscr{A},n}$ the corresponding relaxed energies and slopes, we obviously have $f \in D(\mathsf{Ch}_n)$ and $|\mathrm{D}f|_n \le |\mathrm{D}f|$ $\mathfrak{m}_n$-a.e. in X. If we are able to show that $f \in D(\mathsf{Ch}_{\mathscr{A},n})$ and $|\mathrm{D}f|_{\mathscr{A},n} \le |\mathrm{D}f|_n$ $\mathfrak{m}_n$-a.e. in X for all n, we are then able to find functions $f_{k,n} \in \mathscr{A}_{\mathrm{bs}}$ with $f_{k,n} \to f$ in $L^2(X, \mathfrak{m}_n)$ as $k \to \infty$

$$\limsup_{k \to \infty} \int_X \mathrm{Lip}_{\mathrm{a}}^2(f_{k,n}) \, \mathrm{d}\mathfrak{m}_n \le \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}_n.$$

Using the inequality $\mathrm{Lip}_{\mathrm{a}}(hg) \le \mathrm{Lip}_{\mathrm{a}}(g) + \mathrm{Lip}(h)g\chi_{\mathrm{supp}(1-h)}$, with $h : X \to [0, 1]$ and $\mathrm{Lip}(h) \le 1$, it is now immediate to build, by a diagonal argument, $f_{k_n,n} h_n \to f$ in $L^2(X, \mathfrak{m})$ with

$$\limsup_{n \to \infty} \int_X \mathrm{Lip}_{\mathrm{a}}^2(f_{k_n,n} h_n) \, \mathrm{d}\mathfrak{m} = \limsup_{n \to \infty} \int_X \mathrm{Lip}_{\mathrm{a}}^2(f_{k_n,n} h_n) \, \mathrm{d}\mathfrak{m}_n \le \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}.$$

Since $f_{k_n,n}h_n \in \mathscr{A}_{\mathrm{bs}}$, this proves that $f \in D(\mathsf{Ch}_{\mathscr{A}})$; in addition, by lower semicontinuity, we obtain $\int_X |\mathrm{D}f|_{\mathscr{A}}^2 \, \mathrm{d}\mathfrak{m} \leq \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}$, whence the equality of the relaxed slopes follows.

So, from now on, we assume $\mathfrak{m}(X) < \infty$ and (possibly replacing X by the support of $\mathfrak{m}$) that (X, d) has finite diameter.

Step 2. (Gradient flow of $\mathsf{Ch}_{\mathscr{A}}$ and computation of the energy dissipation rate). Thanks to the convexity and lower semicontinuity properties of $\mathsf{Ch}_{\mathscr{A}}$ we can define a (possibly nonlinear) continuous semigroup $\mathcal{P}_t$ in $L^2(X, \mathfrak{m})$ by considering the $L^2(X, \mathfrak{m})$ gradient flow of $\mathsf{Ch}_{\mathscr{A}}$, as we did in the previous section for Ch. Notice that it is the density of $D(\mathsf{Ch}_{\mathscr{A}})$ that enables to the define $\mathcal{P}_t$ on the whole of $L^2(X, \mathfrak{m})$. Specifically, $t \mapsto \mathcal{P}_t f$ is locally absolutely continuous in $(0, \infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_t f = \Delta_{\mathscr{A}}(\mathcal{P}_t f) \qquad \mathscr{L}^1\text{-a.e. in } (0, \infty).$$

Then, using the chain rule, an approximation argument (since $z \mapsto z \log z$ is not Lipschitz) and property (c) one can prove the entropy dissipation formula

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_X f_t \log f_t \, \mathrm{d}\mathfrak{m} = \int_{\{f_t > 0\}} \frac{|\mathrm{D}f_t|_{\mathscr{A}}^2}{f_t} \, \mathrm{d}\mathfrak{m}$$

with $f_t = \mathcal{P}_t f$, for all $f \in L^2(X, \mathfrak{m})$ nonnegative with $\int_X f \log f \, \mathrm{d}\mathfrak{m} < \infty$.

Notice that the sign condition is preserved by $\mathcal{P}_t$, as well as the integral of f, namely $\int_X \mathcal{P}_t f \, \mathrm{d}\mathfrak{m} = \int_X f \, \mathrm{d}\mathfrak{m}$. This last property crucially depends on the finiteness assumption of $\mathfrak{m}$ (the finiteness assumption can be relaxed to (A.1), still sufficient for the mass-preserving property). Because of this, if f is a probability density also $\mathcal{P}_t f$ is a probability density and we shall consider the path of measures

$$\mu_t = f_t\mathfrak{m} = \mathcal{P}_t f\mathfrak{m} \qquad t \geq 0. \tag{B.3}$$

Step 3. (Minimal 2-weak upper gradient $|\mathrm{D}f|_w$ and energy dissipation estimate with $|\mathrm{D}f|_w$). We say that $g \in L^2(X, \mathfrak{m})$ is a 2-weak upper gradient if the inequality

$$|f(\gamma(1)) - f(\gamma(0))| \leq \int_0^1 g(\gamma(s))|\dot{\gamma}|(s) \, \mathrm{d}s \tag{B.4}$$

holds $\boldsymbol{\pi}$-a.e. for all 2-test plans $\boldsymbol{\pi}$. Recall that a 2-test plan is a probability measure $\boldsymbol{\pi}$ in $C([0,1]; X)$ concentrated on $AC^2([0,1]; X)$ and satisfying, for some $C = C(\boldsymbol{\pi}) \geq 0$,

$$(\mathrm{ev}_t)_{\#}\boldsymbol{\pi} \leq C\mathfrak{m} \qquad \forall t \in [0,1]. \tag{B.5}$$

Because of the non-concentration condition (B.5), the property of being a 2-weak upper gradient is independent of the choice of the representative of g in $L^2(X, \mathfrak{m})$; furthermore, the class of 2-weak upper gradients is a convex closed set and we can identify, as we did for $|\mathrm{D}f|_{\mathscr{A}}$, the element with minimal $L^2(X, \mathfrak{m})$ norm. This distinguished element, the so-called minimal 2-weak upper gradient, will be denoted by $|\mathrm{D}f|_w$. The stability property of 2-weak upper gradients (a variant of the so-called Fuglede's lemma), the fact that the asymptotic Lipschitz constant is a (2-weak) upper gradient and Theorem A.1 then give

$$|\mathrm{D}f|_w \leq |\mathrm{D}f| \qquad \mathfrak{m}\text{-a.e. in } X \text{ for all } f \in D(\mathsf{Ch}). \tag{B.6}$$

For all probability density $f_0 \in L^2(X, \mathfrak{m})$ having a 2-weak upper gradient, by integrating (B.4) with $f = f_0$ and $g = |\mathrm{D}f_0|_w$ with respect to a suitable test plan provided by

Lisini's superposition theorem [L07], we obtain an estimate on the entropy dissipation rate involving $|Df_0|$ (see [AGS14a, Lemma 5.17]):

$$\int_X f_0 \log f_0 \, \mathrm{d}\mathfrak{m} - \int_X f_t \log f_t \, \mathrm{d}\mathfrak{m} \leq \frac{1}{2} \int_0^t \int_{\{f_0 > 0\}} \frac{|Df_0|_w^2}{f_0^2} f_s \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s + \frac{1}{2} \int_0^t |\dot{\mu}|^2(s) \, \mathrm{d}s \quad \text{(B.7)}$$

for all $t > 0$. Here $\mu_t = f_t \mathfrak{m}$ is any curve in $AC_{\mathrm{loc}}^2([0, \infty); (\mathscr{P}(X), W_2))$ and $|\dot{\mu}|(t)$ is its metric derivative.

Using this sharper energy dissipation estimate we will prove that $f \in D(\mathsf{Ch}_{\mathscr{A}})$ for all f having a 2-weak upper gradient as well as the inequality $|Df|_w = |Df|_{\mathscr{A}}$ in the next, final, step. In combination with (B.6) this provides the converse inequality to (B.2) and then the result.

Step 4. (The equality $|Df|_{\mathscr{A}} = |Df|_w$). Given $h \in L^2(X, \mathfrak{m})$ having a 2-weak upper gradient, we want to prove that $h \in D(\mathsf{Ch}_{\mathscr{A}})$ and that $|Dh|_{\mathscr{A}} = |Dh|_w$ $\mathfrak{m}$-a.e. in X. By the local property of this statement, we can consider with no loss of generality only $h \in L^\infty(X, \mathfrak{m})$ with $\int_X h^2 \, \mathrm{d}\mathfrak{m} = 1$, $h \geq c > 0$ and we set $f = h^2$. Then we consider the path of measures $\mu_t = f_t \mathfrak{m}$ in (B.3), noticing that still $f_t \geq c^2 > 0$.

The refined subsolution property (B.9) of Lemma B.2 can be used together with the integration by parts formula (d) to obtain the so-called Kuwada's lemma (see for instance [ACDM15, Lemma 34]), which yields $\mu_t \in AC^2([0, \infty); (\mathscr{P}(X), W_2))$ and

$$|\dot{\mu}|(s) \leq \int_X \frac{|Df_s|_{\mathscr{A}}^2}{f_s} \, \mathrm{d}\mathfrak{m} \qquad \text{for } \mathscr{L}^1\text{-a.e. } s \in (0, \infty).$$

This, in combination with (B.7) with $f_0 = f$, gives

$$\int_X f \log f \, \mathrm{d}\mathfrak{m} - \int_X f_t \log f_t \, \mathrm{d}\mathfrak{m} \leq \frac{1}{2} \int_0^t \int_X \frac{|Df|_w^2}{f^2} f_s \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s + \frac{1}{2} \int_0^t \int_X \frac{|Df_s|_{\mathscr{A}}^2}{f_s} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s.$$

Hence we deduce

$$\int_0^t 4\mathsf{Ch}_{\mathscr{A}}(\sqrt{f_s}) \, \mathrm{d}s = \frac{1}{2} \int_0^t \int_X \frac{|Df_s|_{\mathscr{A}}^2}{f_s} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s \leq \frac{1}{2} \int_0^t \int_X \frac{|Df|_w^2}{f^2} f_s \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s.$$

Letting $t \downarrow 0$, taking into account the L_{loc}^2-lower semicontinuity of $\mathsf{Ch}_{\mathscr{A}}$ and the fact that $f_t \to f$ in $L^2(X, \mathfrak{m})$, we get $\mathsf{Ch}_{\mathscr{A}}(h) = \mathsf{Ch}_{\mathscr{A}}(\sqrt{f}) \leq \liminf_{t \downarrow 0} \frac{1}{t} \int_0^t \mathsf{Ch}_{\mathscr{A}}(\sqrt{f_s}) \, \mathrm{d}s$. On the other hand, since $|Df|_w^2/f^2$ belongs to $L^1(X, \mathfrak{m})$, the $L^2(X, \mathfrak{m})$ convergence of f_s to f (which entails weak* convergence in $L^\infty(X, \mathfrak{m})$) yields

$$\int_X \frac{|Df|^2}{f} \, \mathrm{d}\mathfrak{m} = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t \int_X \frac{|Df|_w^2}{f^2} f_s \, \mathrm{d}\mathfrak{m} \, \mathrm{d}s.$$

In summary, we proved that $h \in D(\mathsf{Ch}_{\mathscr{A}})$ and that

$$2 \int_X |Dh|_{\mathscr{A}}^2 \, \mathrm{d}\mathfrak{m} \leq \frac{1}{2} \int_X \frac{|Df|_w^2}{f} \, \mathrm{d}\mathfrak{m} = 2 \int_X |Dh|_w^2 \, \mathrm{d}\mathfrak{m},$$

where we used the chain rule once once. Taking the inequality $|Dh| \leq |Dh|_{\mathscr{A}}$ into account, this integral inequality proves the coincidence $\mathfrak{m}$-a.e. in X. $\qquad\square$

Lemma B.2 (Subsolution property). *If $\mathfrak{m}(X) < \infty$, X has finite diameter and $f \in C_b(X)$, the function*

$$Q_t(f) := \inf_{y \in X} f(y) + \frac{1}{2t} \mathsf{d}^2(x, y) \tag{B.8}$$

is locally Lipschitz in $(0, \infty) \times X$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t f + \frac{1}{2} |DQ_t f|_{\mathscr{A}}^2 \leq 0 \qquad \mathscr{L}^1 \times \mathfrak{m}\text{-a.e. in } (0, \infty) \times X. \tag{B.9}$$

Proof. Let $D = \{x_k\}$ be as in (2.3). Let $n \geq 1$ and define

$$Q_t^n f(x) := \min_{1 \leq i \leq n} f(x_i) + \frac{1}{2t} \mathsf{d}^2(x, x_i) \qquad x \in X, \ t > 0.$$

Then $x \mapsto Q_t^n f(x) \in \mathscr{A}$ for all $t > 0$, $Q_t^n f$ is locally Lipschitz in $(0, \infty) \times X$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t^n f + \frac{1}{2} \mathrm{Lip}_{\mathrm{a}}^2(Q_t^n f) \leq 0 \qquad \mathscr{L}^1 \times \mathfrak{m}\text{-a.e. in } (0, \infty) \times X.$$

The proof of this inequality is elementary, see for instance [ACDM15, Theorem 14].

In order to obtain (B.9), we notice that the density of D in X yields that the family $Q_t^n f$ monotonically converges to $Q_t f$ from above. Given $\zeta(t, x) = \chi(t)\psi(x)$, with $\chi \in C_c^1(0, T)$ nonnegative and $\psi \in C_{\mathrm{b}}(X)$, $\inf \psi > 0$, we can pass to the limit in the inequality

$$\int_0^T \int_X \left(-Q_t^n f \frac{\mathrm{d}}{\mathrm{d}t} \zeta + \frac{\zeta}{2} \mathrm{Lip}_{\mathrm{a}}^2(Q_t^n f) \right) \mathrm{d}\mathfrak{m} \, \mathrm{d}t \leq 0$$

to get

$$\int_0^T \int_X \left(-Q_t f \frac{\mathrm{d}}{\mathrm{d}t} \zeta + \frac{\zeta}{2} |\mathrm{D}Q_t f|_{\mathscr{A}}^2 \right) \mathrm{d}\mathfrak{m} \, \mathrm{d}t \leq 0$$

which provides, by the arbitrariness of ζ, (B.9). In this limiting argument we used the lower semicontinuity property

$$\int_X \psi |\mathrm{D}g|_{\mathscr{A}}^2 \, \mathrm{d}\mathfrak{m} \leq \liminf_{n \to \infty} \int_X \psi \, \mathrm{Lip}_{\mathrm{a}}^2(g_n) \, \mathrm{d}\mathfrak{m}_n$$

whenever $g_n \in \mathscr{A}$ and $g_n \to g$ in $L^2(X, \mathfrak{m})$, which is a simple consequence of the $\mathfrak{m}$-a.e. minimality property of the minimal relaxed slope. $\square$

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