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A Simple Method to Calculate Random-Coding Union Bounds for Ultra-Reliable Low-Latency Communications

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***Abstract*—Ultra-Reliable Low-Latency Communications are based on very short codes whose performances cannot be evaluated correctly by using the Shannon capacity formula holding for asymptotically large code lengths. The use of Random-Coding Union Bounds (RCUB's) has been suggested in the literature as an alternative for this application. Unfortunately, their calculation is difficult and the Gaussian approximation may lead to erroneous results. The saddlepoint approximation has been proposed as an alternative to overcome the limitations of the Gaussian approximation. Though this technique is valid in many cases, situations exist where the exact calculation provides different results. A simple numerical technique is proposed in this letter to calculate numerically the exact value of the RCUB. Its accuracy is compared to that of the Gaussian and saddlepoint approximations in some cases of interest.**

***Index Terms*—Ultra-Reliable Low-Latency Communications. Random-Coding Union Bounds. Gaussian approximation. Saddlepoint approximation. Gauss-Chebyshev Quadrature Rules.**

I. INTRODUCTION

One of the requirements of Ultra-Reliable Low-Latency Communications (URLLC) is to limit the system outage probability below 10^{-5} (or, equivalently, to achieve a reliability of 99.999%) with a latency of 1 ms in order to be applicable in mission-critical use cases such as smart grids, intelligent transport systems, and remote surgery [1], [2]. As noticed in the recent work [3], the latency requirements prevent resorting to time and/or frequency diversity so that spatial diversity remains the only option. Massive Multiple-Input Multiple-Output (MIMO) communication systems [4] offer this diversity level but several results representing the foundations for these systems are based on the assumption that the transmitted codewords have infinite length, which becomes questionable for URLLC scenarios [5].

Recently, Östman *et al.* [3] proposed to resort to the evaluation of an upper bound to the finite block length error probability deriving from random coding union bounds with a parameter s (RCUBs) introduced in [6], adapted for the massive MIMO setup. This work improved the quality of the calculation of the RCUBs by resorting to the saddlepoint approximation instead than to the Gaussian approximation, earlier suggested in [7]. The resulting error probability bounds are applied in [3] to characterize several massive MIMO scenarios with pilot contamination and different linear receivers.

Since these works from the literature have evidenced the importance of a precise evaluation of the RCUBs, we present a method to calculate them exactly, which provides better accuracy than both the Gaussian and the saddlepoint approximations. This exact method can also be applied to find the RCUB of some coded modulation systems, where the generalized information density and the Moment Generating Function (MGF) can be derived.

The letter is organized as follows. Random-Coding Upper Bounds are reviews in the first part of Section II. The saddlepoint approximation is introduced in Section II-A resorting to results from [3], [6]. Then, the Gaussian approximation is derived in Section II-B and some basic notation is established. Next, a preliminary Moment-Generating Function is characterized analytically along with its convergence region in Section II-C. The analytic derivation is summarized in Section II-D along with the numerical computation reported in Appendix A. Section III provides numerical results showing the differences between the approximations and the actual values in selected cases. Finally, the complexities of the approximations and of the exact calculations are discussed in Section III-A.

II. CALCULATION OF RANDOM-CODING UPPER BOUNDS

We consider a communication channel characterized by

$$Y_k = gX_k + Z_k, \qquad k = 1, \dots, n. \tag{1}$$

Here, X_k is the k-th transmitted symbol and $Z_k \sim \mathcal{CN}(0, 1)$ is the k-th additive noise sample. The channel gain g is assumed to be constant through the transmission of n consecutive symbols, which corresponds to the quasi-static fading channel assumption [8]. The receiver has an estimate $\hat{g}$ of the channel gain, which is uses to estimate the transmitted codeword $\boldsymbol{x} = (X_1, \dots, X_n)$ by the received signal vector $\boldsymbol{y} = (Y_1, \dots, Y_n)$ by using the approximate Maximum-Likelihood (ML) metric

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}\in\mathcal{C}} \|\boldsymbol{y} - \hat{g}\boldsymbol{x}\|^2, \tag{2}$$

where $\mathcal{C}$ is the channel code used for the transmission. We consider a Gaussian random ensemble for $\mathcal{C}$ and assume that the codeword symbols are distributed as $\mathcal{CN}(0, \rho)$. The Random Coding Upper bound with parameter s (RCUBs) to the error probability can be obtained as the following expression:

$$\begin{aligned}\varepsilon_{\mathsf{RCUB}}(s) &\triangleq P\left\{\sum_{k=1}^{n} \imath_s(X_k, Y_k) \le \ln\frac{|\mathcal{C}|-1}{U}\right\} \\ &\ge P(\hat{\boldsymbol{x}} \ne \boldsymbol{x})\end{aligned} \tag{3}$$

where U is an independent random variable uniformly distributed over $(0,1)$ and, for $k=1,\ldots,n$,

$$\imath_s(X_k,Y_k) = -s|Y_k-\hat{g}X_k|^2 + \frac{s|Y_k|^2}{1+s\rho|\hat{g}|^2} + \ln(1+s\rho|\hat{g}|^2) \tag{4}$$

is the generalized information density defined in [6]. Now, we consider different types of approximations we can resort to in order to evaluate (3).

A. Saddlepoint approximation

In this section we summarize the results from [3, Th. 2] with our notation. First, we define the cumulant generating function (CGF) of the generalized information density as

$$\begin{aligned}\kappa_s(\zeta) &\triangleq \ln \mathbb{E}[\exp(-\zeta \imath_s(X_k,Y_k))] \\ &= -\ln D_s(\zeta) - \zeta\ln(1+s\rho|\hat{g}|^2),\end{aligned} \tag{5}$$

for $k=1,\ldots,n$, where $D_s(\zeta)$ is derived in Section II-C and defined in eq. (23). We also define the code rate as

$$R \triangleq \frac{\ln|\mathcal{C}|}{n} \quad \frac{\text{nats}}{\text{symbol}}. \tag{6}$$

Then, we solve the equation

$$\begin{aligned}-\kappa_s'(\zeta) &= \frac{\rho\alpha_s+\gamma_s+2\zeta\rho(\alpha_s\gamma_s-|\beta_s|^2)}{D_s(\zeta)} + \ln(1+s\rho|\hat{g}|^2) \\ &= R,\end{aligned} \tag{7}$$

where $\alpha_s, \beta_s, \gamma_s$ are defined in (14), and define $\tilde{\zeta}_s$ its solution. According to [3, Th. 2], we have three possible cases:

1) $\tilde{\zeta}_s < 0$: In this case,

$$\begin{aligned}\varepsilon_{\mathsf{RCUB}}(s) = 1 &- \mathsf{e}^{n[\kappa_s(\tilde{\zeta}_s)+\tilde{\zeta}_s R]} \\ &\times [\Psi_{n,\tilde{\zeta}_s}(-\tilde{\zeta}_s) - \Psi_{n,\tilde{\zeta}_s}(1-\tilde{\zeta}_s) + o(n^{-1/2})]\end{aligned} \tag{8}$$

with

$$\Psi_{n,\tilde{\zeta}_s}(u) \triangleq \mathsf{e}^{n\frac{u^2}{2}\kappa_s''(\tilde{\zeta}_s)} Q(u\sqrt{n\kappa_s''(\tilde{\zeta}_s)}). \tag{9}$$

2) $0 \le \tilde{\zeta}_s \le 1$: In this case,

$$\begin{aligned}\varepsilon_{\mathsf{RCUB}}(s) &= \mathsf{e}^{n[\kappa_s(\tilde{\zeta}_s)+\tilde{\zeta}_s R]} \\ &\times [\Psi_{n,\tilde{\zeta}_s}(\tilde{\zeta}_s) + \Psi_{n,\tilde{\zeta}_s}(1-\tilde{\zeta}_s) + o(n^{-1/2})].\end{aligned} \tag{10}$$

3) $\tilde{\zeta}_s > 1$: In this case,

$$\begin{aligned}\varepsilon_{\mathsf{RCUB}}(s) &= \mathsf{e}^{n[\kappa_s(1)+R]} \\ &\times [\tilde{\Psi}_n(1,1) + \tilde{\Psi}_n(0,-1) + O(n^{-1/2})]\end{aligned} \tag{11}$$

with

$$\begin{aligned}\tilde{\Psi}_n(a_1,a_2) &\triangleq \mathsf{e}^{na_1[-\kappa_s'(1)-R+\kappa_s''(1)/2]} \\ &\times Q\left(a_1\sqrt{n\kappa_s''(1)} - na_2\frac{\kappa_s'(1)+R}{\sqrt{n\kappa_s''(1)}}\right).\end{aligned} \tag{12}$$

B. Gaussian approximation

A direct approach for the approximation of the RCUBs (3), originally proposed in [7], consists of approximating the sum of the mutual information densities $\imath_s(X_k,Y_k)$ by a Gaussian random variable and using the Q function to calculate it. We define the random variables

$$\begin{aligned}\Delta_{s,k} &\triangleq -s|Y_k-\hat{g}X_k|^2 + \frac{s|Y_k|^2}{1+s\rho|\hat{g}|^2} \\ &= \alpha_s|X_k|^2 + 2\Re(\beta_s X_k Z_k^*) + \gamma_s|Z_k|^2\end{aligned} \tag{13}$$

for $k=1,\ldots,n$, where

$$\begin{cases}\alpha_s & \triangleq -s|g-\hat{g}|^2 + \dfrac{s|g|^2}{1+s\rho|\hat{g}|^2} \\ \beta_s & \triangleq -s(g-\hat{g})^* + \dfrac{sg^*}{1+s\rho|\hat{g}|^2} \\ \gamma_s & \triangleq -\dfrac{s^2\rho|\hat{g}|^2}{1+s\rho|\hat{g}|^2}\end{cases} \tag{14}$$

We can see that[1]

$$\mathbb{E}[\imath_s(X_k,Y_k)] = \alpha_s\rho + \gamma_s + \ln(1+s\rho|\hat{g}|^2) \tag{15}$$

$$\mathbb{V}[\imath_s(X_k,Y_k)] = \alpha_s^2\rho^2 + \gamma_s^2 + 2\rho|\beta_s|^2 \tag{16}$$

for $k=1,\ldots,n$. Defining

$$\Delta_s \triangleq \sum_{k=1}^{n} \imath_s(X_k,Y_k) + \ln\left[\frac{U}{|\mathcal{C}|-1}\right], \tag{17}$$

we can easily check that the RCUBs becomes

$$\varepsilon_{\mathsf{RCUB}}(s) = P\{\Delta_s \le 0\}. \tag{18}$$

Then, using the facts that $\mathbb{E}[\ln U] = -1$, $\mathbb{V}[\ln U] = 1$, we get

$$\begin{cases}\mathbb{E}[\Delta_s] & = n[\alpha_s\rho+\gamma_s+\ln(1+s\rho|\hat{g}|^2)] \\ & \quad -\ln(|\mathcal{C}|-1) - 1 \\ \mathbb{V}[\Delta_s] & = n[\alpha_s^2\rho^2+\gamma_s^2+2\rho|\beta_s|^2] + 1\end{cases} \tag{19}$$

Finally, we obtain the Gaussian approximation:[2]

$$\varepsilon_{\mathsf{RCUB}}(s) \approx Q\left(\frac{\mathbb{E}[\Delta_s]}{\sqrt{\mathbb{V}[\Delta_s]}}\right). \tag{20}$$

C. Moment generating function of $\tilde{\Delta}_{s,k}$

The Moment Generating Function[3] (MGF) of the random variable $\Delta_{s,k}$, for $k=1,\ldots,n$, is given by[4]

$$\begin{aligned}\Phi_{\tilde{\Delta}_s}(\zeta) &\triangleq \mathbb{E}[\exp(-\zeta\tilde{\Delta}_s] \\ &= \int_{\mathbb{C}^2} \frac{1}{\pi^2\rho} \mathsf{e}^{-(\zeta\alpha_s+\frac{1}{\rho})|x|^2 - 2\zeta\Re(\beta_s x z^*) - (\zeta\gamma_s+1)|z|^2} dxdz \\ &= \frac{1}{(1+\zeta\rho\alpha_s)(1+\zeta\gamma_s) - \zeta^2\rho|\beta_s|^2}\end{aligned} \tag{21}$$

[1] $\mathbb{V}[X]$ denotes the variance of the random variable X.

[2] According to [14], the Q-function can be approximated by

$$Q(x) \approx \frac{1}{12}\exp\left(-\frac{x^2}{2}\right) + \frac{1}{4}\exp\left(-\frac{2x^2}{3}\right)$$

This approximation can be used to reduce the complexity of the calculation of the Q function, if required, and yield a moderate accuracy degradation in the resulting RCUB.

[3] This is also the Laplace transform of the probability density function.

[4] Here we can apply the integral $\int_{\mathbb{C}^n}\exp(-\boldsymbol{z}^{\mathsf{H}}\boldsymbol{A}^{-1}\boldsymbol{z})d\boldsymbol{z} = \det(\pi\boldsymbol{A})$ for an $n\times n$ positive definite Hermitian matrix $\boldsymbol{A}$.

Hereafter, $\tilde{\Delta}_s$ denotes a random variable distributed as any $\Delta_{s,k}$ (for $k = 1, \ldots, n$). Therefore, $\Phi_{\tilde{\Delta}_s}(\zeta) = \Phi_{\Delta_{s,k}}(\zeta)$ for all $k = 1, \ldots, n$.

The expression in (21) holds if and only if the quadratic form in the argument of the exponential inside the integral is negative definite. Hence, the following inequalities must be satisfied:

$$\begin{cases} 1 + \zeta\rho\alpha_s > 0 \\ \zeta < \dfrac{1}{|\gamma_s|} \\ (1+\zeta\rho\alpha_s)(1+\zeta\gamma_s) > \zeta^2\rho|\beta_s|^2. \end{cases} \tag{22}$$

Now, let us define the denominator of (21) as

$$D_s(\zeta) \triangleq 1 + (\rho\alpha_s + \gamma_s)\zeta + \rho(\alpha_s\gamma_s - |\beta_s|^2)\zeta^2. \tag{23}$$

We can see that the second-degree coefficient satisfies

$$\rho(\alpha_s\gamma_s - |\beta_s|^2) = -\frac{s^2\rho|\hat{g}|^2}{1+s\rho|\hat{g}|^2} < 0. \tag{24}$$

Thus, $D_s(\zeta)$ has two real roots of opposite signs:

$$\zeta_s^{\pm} \triangleq \frac{\rho\alpha_s + \gamma_s \pm \sqrt{(\rho\alpha_s+\gamma_s)^2 + 4\rho(|\beta_s|^2 - \alpha_s\gamma_s)}}{2\rho(|\beta_s|^2 - \alpha_s\gamma_s)}. \tag{25}$$

The condition $D_s(\zeta) > 0$ implies that $\zeta \in (\zeta_s^-, \zeta_s^+)$.[5] Since

$$\begin{cases} D\left(-\dfrac{1}{\gamma_s}\right) &= -\dfrac{\rho|\beta_s|^2}{\gamma_s^2} &< 0, \\ D\left(-\dfrac{1}{\rho\alpha_s}\right) &= -\dfrac{|\beta_s|^2}{\rho\alpha_s^2} &< 0, \end{cases} \tag{26}$$

the convergence conditions (22) are satisfied when ζ belongs to the following MGF convergence region:

$$\mathcal{R}_{\tilde{\Delta}} \triangleq \{\zeta : \zeta_s^- < \Re(\zeta) < \zeta_s^+\}. \tag{27}$$

D. Chernoff bound and exact derivation

The main contribution of this section is the derivation of a simple numerical algorithm based on Chebyshev quadrature rules for the calculation of the RCUBs. In addition to that, we derive the simple Chernoff upper bound to the same quantity. Let us recall that the RCUBs $\varepsilon_{\mathsf{RCUB}}(s)$ is given by equation (18), where Δ_s is defined in (17). Since

$$P(\Delta_s \le 0) = \mathbb{E}[u(-\Delta_s)] \le \mathbb{E}[\exp(-\zeta\Delta_s)], \tag{28}$$

for any $\zeta \ge 0$, we have obtained the simple upper Chernoff bound to the RCUBs:

$$\varepsilon_{\mathsf{RCUB}}(s) \le \min_{\zeta \ge 0} \Phi_{\Delta_s}(\zeta). \tag{29}$$

To proceed with the exact derivation of the RCUBs, we define the MGF of the random variable Δ defined in (17).

$$\Phi_{\Delta_s}(\zeta) \triangleq \frac{1}{1-\zeta}\Phi_{\tilde{\Delta}_s}(\zeta)^n \left(\frac{|\mathcal{C}|-1}{(1+s\rho|\hat{g}|^2)^n}\right)^{\zeta}. \tag{30}$$

The convergence region of (30) is the intersection of the region $\{\zeta : \Re(\zeta) < 1\}$ (because of the factor $\mathbb{E}[U^{-\zeta}] = \frac{1}{1-\zeta}$) with the convergence region (27), i.e.,

$$\mathcal{R}_{\Delta_s} \triangleq \{\zeta : \zeta_s^- < \Re(\zeta) < \min(1, \zeta_s^+)\}. \tag{31}$$

[5]This two extremes are labeled as $\underline{\zeta}$ and $\bar{\zeta}$, respectively, in [3].

To calculate $\varepsilon_{\mathsf{RCUB}}(s)$, we exploit the method summarized in Appendix A, based on the Chebyshev quadrature rules. The result follows by applying eq. (38) after choosing the constant c in the convergence region of $\frac{\Phi_{\Delta_s}(\zeta)}{\zeta}$, which is the intersection of the convergence region (31) and the positive real part of the complex plane. So, for example, we can take $c = \frac{1}{2}\min(1, \zeta_s^+)$, where ζ_s^+ is defined in (25). To increase the numerical stability of (38) we can take c as the real value in $\mathcal{R}_\Delta$ minimizing $\Phi_{\Delta_s}(c)$.

III. Numerical results

We consider some numerical examples characterizing the differences between the approximations and the exact calculation proposed in this work: i) The Gaussian approximation. ii) The Saddlepoint approximation. iii) The exact analytic method supported by the numerical calculation through the Chebyshev quadrature rules. iv) The Chernoff bound. To this purpose, we shall consider four scenarios characterized by the channel gain $g = 1$, the estimated gain $\hat{g}$, the SNR ρ, expressed in dB, the codeword length n, the code rate R, which is assumed to be proportional to the achievable rate by a factor $\breve{\alpha}$, i.e.,

$$R = \breve{\alpha}\log_2(1+\rho). \tag{32}$$

The quality of the approximation depends very much on the parameters considered and we are going to focus on the cases when either the Gaussian or the Saddlepoint approximations do not provide numerically accurate results. In such cases, the exact analytic method represents a basic resource to obtain valid results in any framework of interest.

The Chernoff bound is always well above the RCUBs so that it is not a useful approximation in this framework.

In the following we shall refer by the name RCUB to the minimum over the parameter s of the RCUBs, i.e.,

$$\varepsilon_{\mathsf{RCUB}} \triangleq \min_{s\ge 0} \varepsilon_{\mathsf{RCUB}}(s). \tag{33}$$

Fig. 1 shows several plots of the RCUB versus SNR in the cases described by the following parameters:

$$g = \hat{g} = 1, n = 30, 50, \breve{\alpha} = 0.1.$$

We can see that the exact curve is close to the Gaussian approximation for lower SNR and to the saddlepoint approximation for higher SNR. For intermediate SNR's neither the Gaussian nor the saddlepoint approximation are very accurate. Comparing the curves corresponding to the two code lengths considered ($n = 30$ and 50), we notice that the gap between the exact RCUB and its approximations (Gaussian and saddlepoint) widens as the code length gets shorter. Fig. 2 extends the results of Fig. 1 to the case $\breve{\alpha} = 0.2$. The results can be interpreted similarly but the gap between exact values and approximation is narrower in this case. Fig. 3 considers a sample case of mismatched detection with $\hat{g} = 1.3$ and $\breve{\alpha} = 0.1$. In this case, we notice a wider gap between exact values and approximations. Finally, the dependency of the RCUB on the rate logarithmic factor $\breve{\alpha}$ is illustrated in Fig. 4. As expected, increasing $\breve{\alpha}$ increases the RCUB for a given SNR.

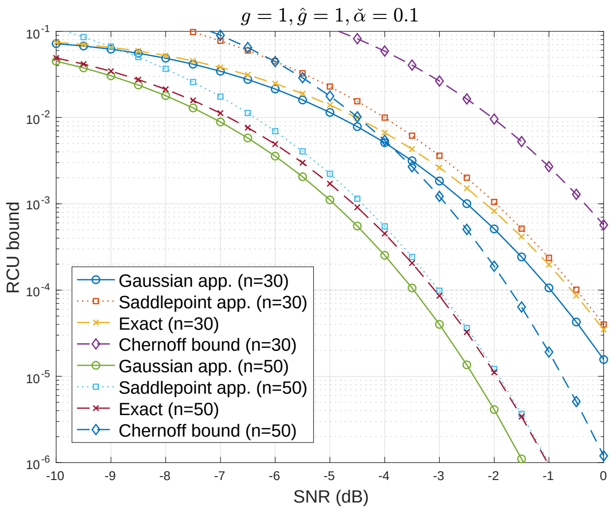

Fig. 1. Plot of the RCUB versus SNR with the following parameters: $g = \hat{g} = 1, n = 30, 50, \breve{\alpha} = 0.1$. The curves report the Gaussian and Saddlepoint approximations, the Exact analytic evaluation, and the Chernoff bound.

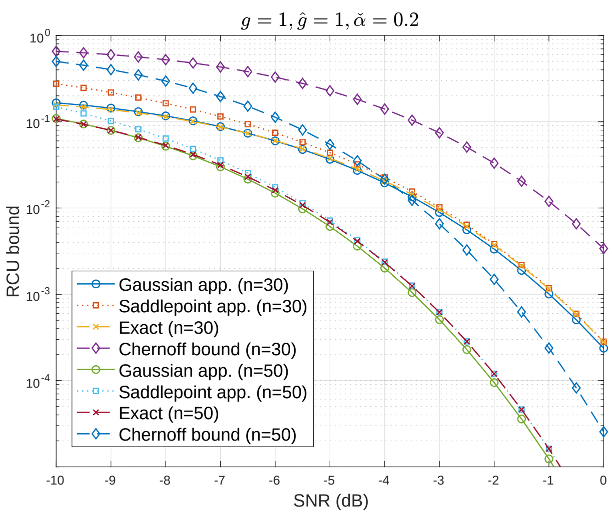

Fig. 2. Same as Fig. 1 but $\breve{\alpha} = 0.2$.

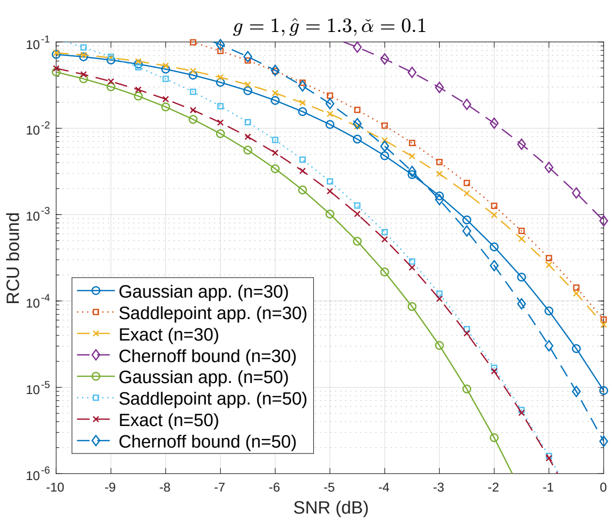

Fig. 3. Same as Fig. 1 but $\hat{g} = 1.3$ (mismatched).

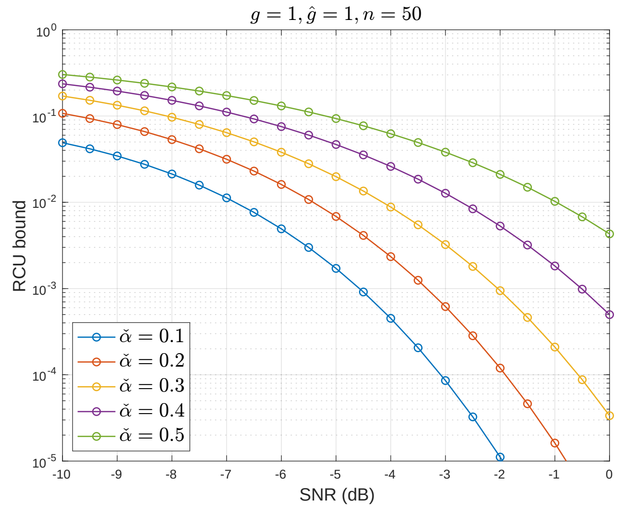

Fig. 4. Plot of the RCUB versus SNR with the following parameters: $g = \hat{g} = 1, n = 50, \breve{\alpha} = 0.1, 0.2, 0.3, 0.4, 0.5$.

A. *Complexity*

We notice that both the Gaussian and the Saddlepoint approximations require the calculation of one or more Q functions, contrary to the Chernoff bound and the Chebyshev approximation of the exact value. The complexity of the proposed method stands in the minimization of $\Phi_{\Delta_s}(\zeta)$ over the real part of $\mathcal{R}_{\Delta_s}$, yielding the optimum value of c, and on a number of additional calculations of $\Phi_{\Delta_s}(c(1 + \mathrm{j}\,\tau_k))$. In the above numerical results we used $\nu = 64$. Summarizing, the exact calculation requires more elementary operations but doesn't involve the calculation of any Q functions.

Appendix A
Evaluation of $P(\Delta < 0)$

These results have been derived in [9], [10] and are reported here for easy reference. Let the MGF of Δ be defined as

$$\Phi_\Delta(\zeta) \triangleq \mathbb{E}[\exp(-\zeta\Delta)] = \int_{-\infty}^{\infty} f_\Delta(x) \mathrm{e}^{-\zeta x} dx. \tag{34}$$

We assume that the pdf $f_\Delta(x)$ has no masses and admits the Laplace transform in a certain convergence region. Then,

$$\begin{aligned} P(\Delta < 0) &= \int_{-\infty}^{\infty} f_\Delta(x) u(-x) dx \\ &= [f_\Delta(x) * u(x)]_{x=0} \\ &= \frac{1}{\mathrm{j}\,2\pi} \int_{c-\mathrm{j}\,\infty}^{c+\mathrm{j}\,\infty} \frac{\Phi_\Delta(\zeta)}{\zeta} d\zeta \end{aligned} \tag{35}$$

where $u(x) = 1$ for $x > 0$ and 0 otherwise, with Laplace transform $1/\zeta$ (for $\Re(\zeta) > 0$), and we applied the standard properties of Laplace transforms [11]. The real number c is in the convergence region of $\frac{\Phi_\Delta(\zeta)}{\zeta}$, which is the intersection of the convergence region of $\Phi_\Delta(\zeta)$ with the convergence region of $1/\zeta$, i.e., $\{\zeta : \Re(\zeta) > 0\}$. Expanding the integral in (35) we get

$$\begin{aligned} P(\Delta < 0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Phi_\Delta(c+j\omega)}{c+j\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c\Re\{\Phi_\Delta(c+j\omega)\} + \omega\Im\{\Phi_\Delta(c+j\omega)\}}{c^2+\omega^2} d\omega \end{aligned} \tag{36}$$

Applying the change of variables $\omega = c\sqrt{1-x^2}/x$ to (36) yields

$$\begin{aligned} P(\Delta < 0) = \frac{1}{2\pi}\int_{-1}^{1} &\left\{\Re\left[\Phi_\Delta\left(c + \mathrm{j}\, c\frac{\sqrt{1-x^2}}{x}\right)\right]\right. \\ &\left. + \frac{\sqrt{1-x^2}}{x}\Im\left[\Phi_\Delta\left(c + \mathrm{j}\, c\frac{\sqrt{1-x^2}}{x}\right)\right]\right\} \frac{dx}{\sqrt{1-x^2}} \end{aligned} \tag{37}$$

Finally, applying the Gauss-Chebyshev quadrature rule [12, 25.4.38] with ν nodes, we get:

$$\begin{aligned} P(\Delta < 0) = &\frac{1}{2\nu}\sum_{k=1}^{\nu}\left\{\Re[\Phi_\Delta(c(1+\mathrm{j}\,\tau_k))]\right. \\ &\left. + \tau_k\,\Im[\Phi_\Delta(c(1+\mathrm{j}\,\tau_k))]\right\} + R_\nu \end{aligned} \tag{38}$$

where $\tau_k = \tan((k-1/2)\pi/\nu)$ and $R_\nu \to 0$ as $\nu \to \infty$. More precisely, [12, 25.4.38], we have

$$R_\nu = \frac{\pi}{(2\nu)!2^{2\nu-1}} f^{(2\nu)}(\xi) \tag{39}$$

with $\xi \in (-1, 1)$ and

$$\begin{aligned} f(x) \triangleq &\Re\left[\Phi_\Delta\left(c + \mathrm{j}\, c\frac{\sqrt{1-x^2}}{x}\right)\right] \\ &+ \frac{\sqrt{1-x^2}}{x}\Im\left[\Phi_\Delta\left(c + \mathrm{j}\, c\frac{\sqrt{1-x^2}}{x}\right)\right]. \end{aligned} \tag{40}$$

The error term can also be derived as a contour integral and bounded by the method described in [13].

Appendix B
Example of calculation of $\varepsilon_{\mathsf{RCUB}}(s)$

We provide in this appendix a simple computational example with the following parameters:

$$s = 1, \rho = 1, g = \hat{g} = 1, n = 50, \check{\alpha} = 0.1.$$

It is immediate to obtain $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{1}{2}, \gamma_1 = -\frac{1}{2}$ and

$$\Phi_{\Delta_1}(\zeta) = \frac{\exp(-31.223372\zeta)}{(1-\zeta)(1-\frac{1}{2}\zeta^2)^{50}}$$

From eq. (25), we get $\zeta_{1,\pm} = \pm\sqrt{2}$, so that the ROC in (31) becomes $\mathcal{R}_{\Delta_1} = \{\zeta : -\sqrt{2} < \Re(\zeta) < 1\}$. The ROC of $\Phi_{\Delta_1}(\zeta)/\zeta$ must be sought over $\zeta \in (0,1)$ and is found at $c = 0.508789$. Next, we calculate the $\Phi_{\Delta_1}(c(1+\mathrm{j}\,\tau_k))$ for $k = 1, \ldots, \nu = 2^6$ and the expression in (38) to obtain

$$\varepsilon_{\mathsf{RCUBs}}(1) = 2.1899 \cdot 10^{-5}.$$

Notice that this is not the RCUB reported in Fig. 1 since it has not been minimized over s but is specific for $s = 1$.

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