

Non-modal stability analysis of magnetohydrodynamic flows for liquid metal blankets of fusion reactors

Original

Non-modal stability analysis of magnetohydrodynamic flows for liquid metal blankets of fusion reactors / Lo Verso, M., Introini, C., Cervi, E., Di Prinzio, M., Caramello, M., Giacobbo, F., Foulon, F., Wang, X., Savoldi, L., Cammi, A.. - In: FUNDAMENTAL PLASMA PHYSICS. - ISSN 2772-8285. - ELETTRONICO. - 15:(2025). [10.1016/j.fpp.2025.100097]

Availability:

This version is available at: 11583/3015145 since: 2026-09-03T12:45:54Z

Publisher:

Elsevier B.V.

Published

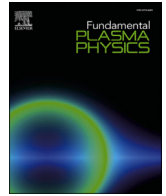
DOI:10.1016/j.fpp.2025.100097

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)



Non-modal stability analysis of magneto-hydrodynamic flow in a single pipe

Matteo Lo Verso^a, Carolina Introini^a, Luciana Barucca^b, Marco Caramello^b,
Matteo Di Prinzio^b, Francesca Giacobbo^a, Laura Savoldi^c, Antonio Cammi^{a,*}

^a Politecnico di Milano, Dept. of Energy, CeSNEF-Nuclear Engineering Division, Nuclear Reactors Group, via La Masa, 34, 20156 Milano, Italy

^b Ansaldo Nucleare SpA, Genova, Italy

^c MATHEP Group, Dept. of Energy "Galileo Ferraris", Politecnico di Torino, Torino, Italy

ARTICLE INFO

Keywords:

Magneto hydrodynamics
Non-modal analysis
Poiseuille flow
Spectral methods

ABSTRACT

A complete understanding of the stability of fluid flows under varying magnetic field profiles is imperative for achieving control of plasma and operating fluids in the blankets of future fusion reactors. In this context, the primary objective of this study is to investigate the influence of varying magnetic profiles on the flow regime of a generic fluid, which is representative of both thermonuclear plasma and conductive fluids within a nuclear fusion reactor. To this aim in this work non-modal stability theory is adopted to perform stability analysis of a magneto-hydrodynamic (MHD) flow in an infinite circular pipe in order to study the effects of the magnetic field on the fluid dynamics of the pipe flow. In particular, the effects on the general stability of two magnetic field profiles are compared with the reference case of a pipe Poiseuille flow without magnetic field. Firstly, the classic modal stability technique is employed to study asymptotical stability. Then, non-modal stability analysis is applied to magneto-hydrodynamic pipe flow to study the system's response for a finite time immediately after a perturbation. Fourier–Chebyshev Petrov–Galerkin spectral method is used to compute the eigenvalues and pseudospectra of the weak formulation associated with the linearised system. Investigations on the dependence of spectra and transient growths on the specific magnetic profiles are conducted for different values of perturbation wave numbers. The obtained results show that in general the magnetic field has an effect of stabilization on the system, which depends on the specific magnetic profile considered. In addition, the non-modal stability analysis reveals that the inclusion of the magnetic field mitigates the effects of perturbations also in the short term, a phenomenon that cannot be seen using only modal stability analysis.

1. Introduction

Magneto-hydrodynamics (MHD) studies the dynamics of electrically conducting fluids under a magnetic field. In nuclear engineering, this theory provides mathematical models which are largely applied, for example, in magnetic confinement fusion (MCF), since thermonuclear plasma can be treated as a conducting fluid confined by an intense magnetic field [1]. Moreover, MHD theory can be applied to the description of conducting liquid metals that flow in the blanket of EU-DEMO-like reactors [2], or molten salts foreseen in the ARC reactor [3]. In these complex scenarios, the plasma must be strictly confined by the imposed magnetic field. At the same time the flow of liquid metals or molten salts in the blanket must be designed to occur permanently in the presence of a magnetic field varying in amplitude and direction in time. Hence,

in order to guarantee the stability of flows under magnetic fields, which is essential for the successful operation of nuclear fusion devices, it becomes necessary to study MHD stability.

For several decades, the conventional approach for stability analysis for engineering problems was represented by modal (or linear) stability analysis theory, which focuses on solving eigenvalue problems associated with the physical systems of equations [4]. This strategy allows analyzing the asymptotical response to perturbations of the system in the long term. However, in some systems characterized by non-normal operators, eigenmodes may exhibit significant transient growth within a finite time, even though they would be damped in the limit $t \rightarrow \infty$. In these cases, the transient effects are completely ignored by modal analysis. For this reason, an alternative theory, called non-modal stability analysis, has been gaining relevance in the last few years. This

* Corresponding author.

E-mail address: antonio.cammi@polimi.it (A. Cammi).

approach was adopted by Schmid [5] to address issues in hydrodynamic stability theory: it was employed to examine the transition Reynolds number for enclosed shear flows, accounting for observed discrepancies between experimental and numerical values [6]. In short, while modal techniques focus solely on the long-term stability of a system, i.e. the asymptotic behaviour of a perturbed system, non-modal methods also consider short-term behaviour within a limited time interval.

For decades, the approach employed in MHD stability analysis has been the modal technique. For example, Lock [7] carried out a linear stability analysis for the case of Hartmann flow with the no-slip condition; Takashima [8] investigated the linear stability of a fully developed pressure-driven steady flow of an electrically conducting fluid between two infinite parallel plates; Potter and Kutchey [9] investigated the modal stability of a plane Hartmann flow for infinitesimal three-dimensional disturbances. Moreover, linear stability analyses have been conducted to study the stability of MHD models representing problems in the nuclear field: for example, Manickam et al. [10] proposed a computational model to analyze the linear stability properties of general toroidal systems in the ideal MHD limit, while Bühler et al. [11] applied linear stability analysis to study the onset of time-dependent instabilities in liquid metal magnetohydrodynamic.

However, stability analysis cannot provide any information regarding short term stability, thus in the last years, non-modal stability analysis has become an object of interest in the nuclear field, as done by Introini et al. [12] where non-modal stability theory is employed to study the non-linear dynamics of the Pressurised Water Reactor (PWR) using a zero-dimensional model. More specifically, some examples of non-modal stability studies in the MHD field include Vetcha et al. [13], which performed a non-modal stability analysis to study the transient energy growth in a Hartmann flow with inter-facial slip; MacTaggart [14], which studied the non-modal onset of the classical tearing instability for viscoresistive MHD flows; and Landreman et al. [15], which demonstrated non-modal linear amplification and nonlinear turbulence in a kinetic plasma system with no unstable eigenmodes.

Following the above examples, in this work for the first time the non-modal stability approach is applied to a magneto-hydrodynamic pipe flow in order to investigate both the short-term and long-term stability. Indeed, the study of modal and non-modal MHD stability represents an intriguing investigation due to the limited understanding of both the long-term and the immediate effects of magnetic fields on the fluid dynamics of conductive fluids. Specifically, the purpose is to begin the investigation from a fundamental case, i.e. the pipe flow, taken as a reference point within the field of hydrodynamics. This choice allows to establish a comparative baseline by comparing the findings with those of a non-magnetic scenario, namely, the pipe Poiseuille flow.

The structure of this paper is reported in the following. Section 2 provides a concise description of non-modal stability theory, putting in evidence its differences compared to conventional linear stability theory. Section 3 describes the adopted model for the MHD pipe flow. Section 4 reports the derivation of the weak formulation for the considered problem and the Petrov-Galerkin discretization to perform the calculation of the eigenvalues associated with the system. Section 5 reports a sensitivity analysis on a dimensionless parameter, the magnetic Reynolds number, and the results obtained by applying the non-modal stability technique compared to the linear stability approach. Finally, Section 6 resumes the main conclusions of the present work.

2. Non-modal stability theory

The initial investigations to understand the instability of a dynamical system relied on linear stability analysis [4], which examines the evolution of infinitesimal perturbations from a reference state assuming an exponential time-dependence. This approach transforms the linear (or linearised) initial value problem into an eigenvalue problem for the growth rates of normal modes, whose temporal behaviour is exponential. The analysis of the eigenvalues of the system describes the

asymptotic behaviour, with a single unstable mode sufficient to classify the reference state as unstable. However, this approach is not always sufficient to explain experimental results, and in several cases, the asymptotic behaviour fails to describe the actual perturbation dynamics [6]. In these cases, the study of short-term evolution is necessary to understand how the system eventually reaches the asymptotic stable state.

To overcome the limitations of modal analysis, a new procedure, called non-modal stability analysis, was introduced. This new approach reveals that transient but significant short-term energy growth may occur even for scenarios that are predicted to have no long-term effects induced by the perturbation. Hence, small disturbances can be amplified to the extent that a transition from a stable to an unstable state (and then back to stable) may occur in short time intervals. For instance, transient growth in hydrodynamics can lead to turbulence, as demonstrated by Rayleigh-Benard convection cases [16]. From the mathematical point of view, the non-modal stability analysis approach differs from modal analysis in that it considers the behaviour of a mixture of normal modes, which may exhibit non-orthogonality and lead to short-term energy growth.

This approach is distinct from traditional Lyapunov stability [17], which is commonly used in the analysis of dynamical systems, including nuclear reactors. Additionally, non-modal stability is also different from other stability techniques like numerical continuation methods, as it does not focus on the final state of the system after perturbations, but instead, on what occurs immediately after the disturbance. Thus, stability is defined more broadly as the response of the system equations to specific input variables such as initial conditions, external disturbances, internal uncertainties, and geometric constraints, and both the time horizon and perturbation shape for the analysis are not predetermined. These differences between modal and non-modal stability analysis approaches from the mathematical point of view are now briefly outlined to better illustrate their diversity.

2.1. Modal stability analysis

A generic linear system of equations can be expressed in matrix form, exploiting the state-space representation described by Nise [18], as:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.\end{aligned}\tag{1}$$

Here, the state of the system is represented by $\mathbf{x}$, the output is represented by $\mathbf{y}$ and the external forcing function is represented by $\mathbf{u}$. The dynamic matrix $\mathbf{A}$ describes the evolution of the system free dynamics, while the control matrix $\mathbf{B}$ expresses the relationship between the system state and its inputs. Matrix $\mathbf{C}$, which links the output to the state of the system, represents the sensors while matrix $\mathbf{D}$, which connects directly the output to the input, represents the direct terms. All four matrices are assumed to be constant over time.

Linear stability analysis is often used to study the behaviour of linear systems under small perturbations compared to a reference state. Moreover, it is possible to apply this analysis to non-linear systems as well, by performing a linearisation around the steady state and then analysing the characteristic roots of the resulting linear system. Given a generic steady-state $\mathbf{x}_0$, infinitesimal perturbations $\delta\mathbf{x}$ around this point are considered, while inputs are assumed either fixed ($\delta\mathbf{u} = 0$) or bounded to maintain BIBO (bounded inputs bounded outputs) stability, as stated by Åström and Murray [19].

The asymptotic stability, which by definition means that all the eigenvalues of the system matrix lie in the left-half of the complex plane, is now considered:

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x}.\tag{2}$$

The asymptotic stability of the solution of Equation (2) can be studied simply by looking at the eigenvalues of $\mathbf{A}$. A continuous-time system

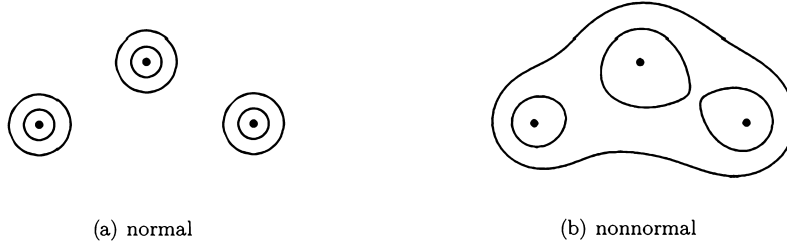


Fig. 1. Schematic representation of pseudospectra for normal (a) and non-normal (b) matrices (see Appendix A). The contours represent the boundary of the pseudo-spectrum for two different values of ϵ . Picture taken from [20].

is considered BIBO stable if and only if the entire spectrum of the dynamic matrix representing the system lies on the left of the imaginary axis of the Gauss plane. This implies that even a single eigenvalue with a positive real part is sufficient to classify the entire system state as unstable. Hence, the dominant eigenvalue for the stability of the system is the one with the largest real part, which is referred to as spectral abscissa. In a stable configuration, the spectral abscissa is connected to the velocity at which the perturbations are suppressed. Naturally, the eigenvalues disposition in the Gauss plane depends on specific parameters of the physical problem and variations in these values may cause one or more eigenvalues to change their location, potentially reaching the right half and compromising the stability of the system. Nevertheless, the spectrum provides information only about the asymptotical response of the system to disturbances. No insight into its short-term response to disturbances is provided. For this reason, the non-modal stability theory deserves equal attention in stability studies.

2.2. Spectra and pseudospectra

Albeit useful, the use of eigenvalue analysis only may lead to incomplete interpretations of the dynamic behaviour of the system, because it focuses only on the asymptotical behaviour of the perturbed system. For this reason, modal analysis should be supported and completed by non-modal stability analyses, which rely on pseudo-spectral analysis (whereas modal techniques belongs to the field of spectral analysis). To elucidate the concept of pseudospectra and how they complete the spectral analysis, one can draw motivation from observations in applied mathematics.

A possible definition of the spectrum of a matrix $\mathbf{A}$ is the complete set of points where the so-called resolvent matrix $(z\mathbf{I} - \mathbf{A})^{-1}$ is not defined. So, examining whether z constitutes an eigenvalue of the matrix $\mathbf{A}$ is equivalent to investigating the singularity of $\|(z\mathbf{I} - \mathbf{A})^{-1}\|$. However, the question of whether a quantity is singular is not always appropriate, as there are circumstances where even a small perturbation can completely change the answer. In these circumstances, it makes more sense to inquire whether $\|(z\mathbf{I} - \mathbf{A})^{-1}\|$ is large. This reasoning leads us to introduce the concept of pseudo-spectrum (for a comprehensive explanation, refer to [20]). Considering an arbitrary positive ϵ , the pseudo-spectrum associated to this parameter, referred to as ϵ -pseudospectrum, delineates the open subset of the complex plane confined by the ϵ^{-1} level curve of the norm of the resolvent, i.e. where $\|(z\mathbf{I} - \mathbf{A})^{-1}\| > \frac{1}{\epsilon}$. The importance of this concept becomes evident in contexts where matrices deviate substantially from normality (for more details refer to Appendix A). In these situations, the norm of the resolvent can be large even when z is distant from every eigenvalue (Fig. 1).

Moreover, from a physical point of view, ϵ can be related to the concept of perturbation. From this prospect, the pseudospectrum represents the set of eigenvalues of some perturbed matrix $\mathbf{A} + \mathbf{E}$, with $\|\mathbf{E}\| < \epsilon$. For further mathematical details on pseudospectra, refer to Appendix A. The concept of pseudospectrum holds essential importance in non-modal stability analysis, upon which this approach is based. In the following section, an elucidation of the concept of non-modal analysis will be provided.

2.3. Non-modal stability analysis

Non-modal stability analysis studies the evolution of the system in the time interval immediately following the perturbation of a given (equilibrium) configuration. Specifically, this analysis aims at identifying the factors that lead to the maximum amplification of the perturbation. The evolution of perturbations depends on system parameters, initial conditions $\mathbf{y}_0$, and external forcing functions, which can be either time-dependent or constant, represented as $\mathbf{u}$. The evolution in time t of the output vector $\mathbf{y}$ of a perturbed system can be represented with the Lagrange formula:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{y}_0 + \mathbf{C} \int_0^t e^{(\mathbf{A}(t-\tau))}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t). \quad (3)$$

The dynamics of the system under consideration are represented by Equation (3). In particular, the first term represents the effect of the initial condition $\mathbf{y}_0$, whereas the second one represents the effect of the external forcing functions $\mathbf{u}$. The analysis of both these factors improves the understanding of how perturbations will behave over time; nevertheless, without losing generality this work focuses on the response to perturbations on the initial conditions and on the parameters of the dynamic matrix $\mathbf{A}$. Hence, Equation (3) becomes:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{y}_0. \quad (4)$$

Non-modal stability analysis focuses on identifying the initial condition and parameter values whose perturbation results in the largest amplification of an appropriately chosen quantity. The maximum amplification of energy $G(t)$ in a defined time interval is considered, which describes the highest energy amplification over all possible initial conditions. This amplification can be written as:

$$G(t) = \max_{\mathbf{y}_0} \frac{\|\mathbf{y}(t)\|^2}{\|\mathbf{y}_0\|^2} = \max_{\mathbf{y}_0} \frac{\|\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{y}_0\|^2}{\|\mathbf{y}_0\|^2} = \|\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\|^2. \quad (5)$$

As evidenced by the last equality of (5), $G(t)$ appears independent on the initial condition. However, it is possible to identify the specific initial perturbation responsible for the maximum amplification by performing Singular Value Decomposition (SVD) of the matrix exponential (for more details, see [20]).

As already explained, linear stability theory focuses on the least stable mode of the dynamic matrix, which is assumed to control the system dynamics. This assumption is equivalent to replacing the norm of $G(t)$ with $g(t) = e^{(2\lambda_R t)}$, where λ_R represents the real part of the dominant (least stable) eigenvalue. This way, only the eigenvalues of $\mathbf{A}$ are considered, and the corresponding eigenfunctions, which are contained in $\mathbf{V}$ from the eigenvalue decomposition $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, are not taken into account. Indeed, by applying the eigenvalue decomposition on the last equality of Equation (5):

$$G(t) = \|\mathbf{C}\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}\mathbf{B}\|^2. \quad (6)$$

Spectral analysis involves analyzing the real part of the dominant eigenvalue and thus characterizing the time evolution of the norm of

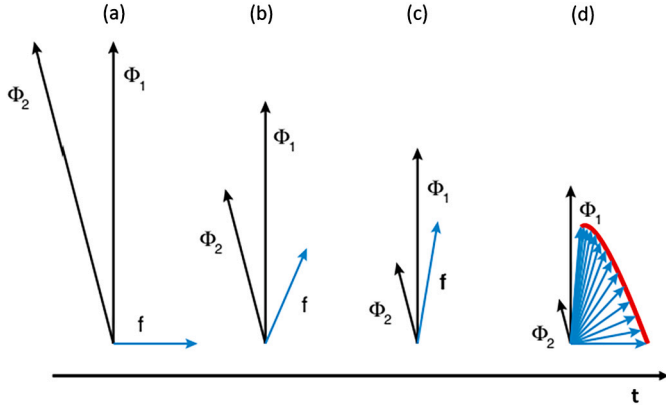


Fig. 2. Example of transient growth using vectors. Picture readapted from [5].

the matrix exponential only for unitary matrices $\mathbf{V}$, as the norm is invariant under similarity transformations. However, many operators are non-unitary nor they are normal, and therefore, their eigenfunctions are non-orthogonal, as explained by Reddy et al. [21]. This means that the behaviour of $G(t)$ may differ from that of $g(t)$, and the study of λ_R may not fully capture the general disturbance behaviour over time. Moreover, the dynamic behaviour of the dominant eigenvalue may not be significant for the behaviour at a finite time.

Fig. 2 provides a simple visual illustration of what may occur in systems characterized by non-normal matrices. It shows how a perturbation of a unit-length initial condition, indicated with $\mathbf{f}$, may lead to a short but significant amplification of energy $G(t)$ for non-normal dynamic matrices $\mathbf{A}$. Indeed, after some time, the eigenvectors ϕ_1 and ϕ_2 of $\mathbf{A}$ exhibit a decrease in their norm (by 25% and 50% respectively); however, the vector $\mathbf{f}$, given by their difference, experiences a transient increase and only in the long run eventually decreases to zero. This example highlights the significant effect of non-orthogonality on short-term growth: if the eigenvectors of $\mathbf{A}$ form a non-orthogonal set, a transient increase of energy can occur even when the dominant eigenvalue is negative. Thus, to fully capture the behaviour of infinitesimal disturbances in time, the norm of the matrix exponential $G(t)$, which contains the full dynamical information, must be considered.

The behaviour of the system for finite time intervals can be studied in the following way. Considering a Taylor expansion of the exponential matrix around $t = 0^+$, expression (4) can be written:

$$\max_y \frac{1}{\|\mathbf{y}\|^2} \frac{d\|\mathbf{y}\|^2}{dt} \Big|_{t=0^+} = \max_y \frac{1}{\|\mathbf{y}\|^2} \times \frac{d}{dt} \Big|_{t=0^+} \left\| \mathbf{C}(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \dots) \mathbf{B} \mathbf{y}_0 \right\|^2 \quad (7)$$

$$= \lambda_{\max}(\mathbf{Q}^H + \mathbf{Q}),$$

where $\mathbf{Q} = \frac{1}{2} \mathbf{C} \mathbf{A} \mathbf{B}$ and the superscript H represents the Hermitian transpose. The quantity $\lambda_{\max}(\mathbf{Q}^H + \mathbf{Q})$ is referred to as the numerical abscissa. From a mathematical point of view, it determines the maximum extension of the numerical range into the right half-plane, which represents an estimate of the eigenvalues of $\mathbf{Q}$, and it is defined as:

$$W(\mathbf{Q}) = \left\{ \frac{\mathbf{x}^H \mathbf{Q} \mathbf{x}}{\mathbf{x}^H \mathbf{x}} \mid \mathbf{x} \in \mathbb{C}^N, \mathbf{x} \neq 0 \right\}, \quad (8)$$

where $\mathbf{x}$ is a generic complex vector of dimension N . From a physical point of view, the numerical abscissa is the energy increase in the time immediately after the perturbation. Moreover, ϵ -pseudospectra can be introduced to gain further understanding of the largest short-time energy growth [20]: they consist in regions in the complex plane, parametrized by ϵ , where the resolvent norm obeys the following relation:

$$\left\| \mathbf{C}(z\mathbf{I} - \mathbf{A}^{-1})\mathbf{B} \right\| > \frac{1}{\epsilon}. \quad (9)$$

Thus, ϵ -pseudo-eigenvalues correspond to the eigenvalues of the matrix $\mathbf{A} + \mathbf{E}$, where $\mathbf{E}$ is the operator of norm ϵ representing the disturbance. Obviously, in the limit $\epsilon \rightarrow 0$, the regular spectrum is retrieved [22]. More mathematical details about the pseudospectra can be found in Appendix A.

3. Model description

As a test case, in this work an infinite circular pipe with an incompressible Newtonian flow subjected to a magnetic field at the inlet is considered. The flow is described with the following non-dimensional, incompressible, viscoresistive magneto-hydrodynamic model [23]:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \frac{N}{Re_m} (\nabla \times \mathbf{B}) \times \mathbf{B}; \quad (10)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \frac{1}{Re_m} \Delta \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u}; \quad (11)$$

$$\nabla \cdot \mathbf{u} = 0; \quad (12)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (13)$$

where $\mathbf{u}$ is the velocity of the fluid, $\mathbf{B}$ is the magnetic field, p is the pressure, Re is the hydraulic Reynolds number, Re_m is the magnetic Reynolds number and N is the Stuart number. The hydraulic Reynolds number represents the ratio between inertial and viscous forces and it is defined as:

$$Re = \frac{\rho U L}{\mu}, \quad (14)$$

where ρ and μ are the density and viscosity of the fluid, while U and L represent, respectively, the reference characteristic velocity and length scale of the flow. The magnetic Reynolds number, instead, is defined as:

$$Re_m = \mu_B \sigma L U, \quad (15)$$

where μ_B denotes the magnetic permeability and σ the electrical conductivity. Finally, the Stuart number represents the ratio of electromagnetic to inertial forces:

$$N = \frac{\sigma L B_0^2}{\rho U}, \quad (16)$$

where B_0 is the characteristic scale of the magnetic field intensity. In the following, the Stuart number will be assumed to be equal to the magnetic Reynolds number to better focus the analysis on the hydraulic and magnetic Reynolds numbers Re and Re_m .

Now, assuming the following equilibrium configuration:

$$p_0 = p_0(r), \quad \mathbf{u}_0 = \bar{w}(r) \mathbf{e}_z, \quad \mathbf{B}_0 = \bar{B}^z(r) \mathbf{e}_z, \quad (17)$$

where $\bar{w}(r)$ and $\bar{B}^z(r)$ are, respectively, the axial components of the velocity and the magnetic field at the stationary state, the steady states system becomes:

$$(\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 = -\nabla p_0 + \frac{1}{Re} \Delta \mathbf{u}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0; \quad (18)$$

$$\mathbf{u}_0 \cdot \nabla \mathbf{B}_0 = \frac{1}{Re_m} \Delta \mathbf{B}_0 + \mathbf{B}_0 \cdot \nabla \mathbf{u}_0. \quad (19)$$

Once $\mathbf{u}_0$ and $\mathbf{B}_0$ are chosen, the background pressure p_0 can be determined from (18). Now, Equations (10) and (11) can be linearised around the steady state defined in Equations (18) and (19):

$$\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_0 = -\nabla p_1 + \frac{1}{Re} \Delta \mathbf{u}_1 + (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 + (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0; \quad (20)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{B}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{B}_0 = \frac{1}{Re_m} \Delta \mathbf{B}_1 + \mathbf{B}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{B}_1 \cdot \nabla \mathbf{u}_0. \quad (21)$$

where the linear expansion $(\mathbf{u}, \mathbf{B}, p) = (\mathbf{u}_0, \mathbf{B}_0, p_0) + (\mathbf{u}_1, \mathbf{B}_1, p_1)$ is used. To complete the set-up of the model, no-slip and perfectly insulating boundary conditions are considered:

$$\mathbf{u}_1|_{\text{wall}} = \mathbf{B}_1|_{\text{wall}} = \mathbf{0}. \quad (22)$$

Now, perturbations¹ are assumed as a sum of complex Fourier modes:

$$\begin{aligned} \mathbf{u}_1 &= \begin{pmatrix} u(r, t) \\ v(r, t) \\ w(r, t) \end{pmatrix} e^{i(n\theta + kz)}, \quad \mathbf{B}_1 = \begin{pmatrix} B^r(r, t) \\ B^\theta(r, t) \\ B^z(r, t) \end{pmatrix} e^{i(n\theta + kz)}, \\ p_1 &= \begin{pmatrix} p^r(r, t) \\ p^\theta(r, t) \\ p^z(r, t) \end{pmatrix} e^{i(n\theta + kz)}, \end{aligned} \quad (23)$$

where $n \in \mathbb{Z}$, $k \in \mathbb{R}$ ² and r , θ , z represent, respectively, the radial, azimuthal and axial coordinates. Substituting these expressions in Equations (20) and (21) and expanding in cylindrical coordinates, the following set of equations is obtained:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial r} + \frac{1}{Re} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{n^2}{r^2} u - k^2 u - \frac{2in}{r^2} v - \frac{u}{r^2} \right) - ik\bar{u}w \\ &\quad - \frac{\partial \bar{B}^z}{\partial r} B^z + \left(ikB^r - \frac{\partial B^z}{\partial r} \right) \bar{B}^z, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{Re} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{n^2}{r^2} v - k^2 v + \frac{2in}{r^2} u - \frac{v}{r^2} \right) \\ &\quad - ik\bar{u}v - \left(i\frac{n}{r} B^z - ikB^\theta \right) \bar{B}^z, \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{n^2}{r^2} w - k^2 w \right) \\ &\quad - u \frac{\partial \bar{w}}{\partial r} - ik\bar{u}w + \frac{\partial \bar{B}^z}{\partial r} B^r, \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial B^r}{\partial t} &= \frac{1}{Re_m} \left(\frac{\partial^2 B^r}{\partial r^2} + \frac{1}{r} \frac{\partial B^r}{\partial r} - \frac{n^2}{r^2} B^r - k^2 B^r - 2i\frac{n}{r^2} B^\theta - \frac{1}{r^2} B^r \right) \\ &\quad + ik\bar{B}^z u - ik\bar{u}B^r, \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{\partial B^\theta}{\partial t} &= \frac{1}{Re_m} \left(\frac{\partial^2 B^\theta}{\partial r^2} + \frac{1}{r} \frac{\partial B^\theta}{\partial r} - \frac{n^2}{r^2} B^\theta - k^2 B^\theta + 2i\frac{n}{r^2} B^r - \frac{1}{r^2} B^\theta \right) \\ &\quad + ik\bar{B}^z v - ik\bar{v}B^\theta, \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{\partial B^z}{\partial t} &= \frac{1}{Re_m} \left(\frac{\partial^2 B^z}{\partial r^2} + \frac{1}{r} \frac{\partial B^z}{\partial r} - \frac{n^2}{r^2} B^z - k^2 B^z \right) + ik\bar{B}^z w \\ &\quad + B^r \frac{\partial \bar{w}}{\partial r} - ik\bar{w}B^z - u \frac{\partial \bar{B}^z}{\partial r}, \end{aligned} \quad (29)$$

together with the solenoidal conditions (Equations (12) and (13)):

$$\frac{\partial u}{\partial r} + i\frac{n}{r} v + ikw + \frac{1}{r} u = 0, \quad (30)$$

$$\frac{\partial B^r}{\partial r} + i\frac{n}{r} B^\theta + ikB^z + \frac{1}{r} B^r = 0. \quad (31)$$

Equations (24) to (29) represent a linear system of partial differential equations (PDEs). In the following section, the selected numerical method for the computation of its spectra and pseudospectra is discussed.

4. Petrov-Galerkin discretization

To solve the MHD model derived in Section 3, a numerical method originally developed for Navier-Stokes presented in [24], [25] and [26]

is adapted. In particular, the system of linear PDEs derived above is written in algebraic form:

$$\left(\frac{\partial \mathbf{u}}{\partial t} \right) = \begin{pmatrix} \mathcal{L}_u \mathbf{u} \\ \mathcal{L}_B \mathbf{B} \end{pmatrix}, \quad (32)$$

where $\mathcal{L}_u$ and $\mathcal{L}_B$ are two linear operators representing the evolution of the fluid velocity and the magnetic field. Equation (32) is then rewritten using the weak-formulation. The weak formulation is a mathematical framework for solving partial differential equations that operates by seeking solutions that satisfy the integral formulation of the governing equations, instead of the so-called ‘strong’ formulation. Weak solutions can have more singular behaviour compared to strong ones, therefore the weak formulation broadens the range of possible solutions, thus facilitating the exploration of complex physical phenomena in various computational domains. The weak problem has weak solutions compared to certain test functions lying in a suitable space. In this procedure the weak formulation is introduced in order to transform the matrix representing the system in a easier matrix to handle for the computation of the spectra and pseudospectra. The weak formulation of Equation (32) is then derived by performing the inner product $\langle \cdot, \cdot \rangle$ with a test function W belonging to a suitable test function space: in particular, W must be divergence-free and null at the wall of the cylinder (due to the constraints of $\mathbf{u}_1$ and $\mathbf{B}_1$). This choice for the test function space allows to eliminate the pressure term from the weak formulation. Indeed an integration by part gives $\langle \nabla p, W \rangle = -\langle p, \nabla \cdot W \rangle = 0$, due to the null divergence property of W . This operation allows the reduction of the number of the variables of the system and to keep the focus of the stability analysis on the initial conditions, since the pressure term is seen as an external forcing.

This work follows the same procedure of [26] for the definition of the test function space (refer to it for a more comprehensive description of the mathematical details of the test functions). Following the derivation, the weak formulation for the problem under consideration is:

$$\left(\left\langle \frac{\partial \mathbf{u}}{\partial t}, W \right\rangle \right) = \left(\left\langle \mathcal{L}_u \mathbf{u}, W \right\rangle \right), \quad (33)$$

where $\mathcal{L}_u \mathbf{u}$ and $\mathcal{L}_B \mathbf{B}$ can be decomposed explicitly as:

$$\begin{aligned} (\mathcal{L}_u \mathbf{u})_r &= \frac{1}{Re} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{n^2}{r^2} u - k^2 u - \frac{2in}{r^2} v - \frac{u}{r^2} \right) - ik\bar{u}w \\ &\quad - \frac{\partial \bar{B}^z}{\partial r} B^z + \left(ikB^r - \frac{\partial B^z}{\partial r} \right) \bar{B}^z; \end{aligned} \quad (34)$$

$$\begin{aligned} (\mathcal{L}_u \mathbf{u})_\theta &= \frac{1}{Re} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{n^2}{r^2} v - k^2 v + \frac{2in}{r^2} u - \frac{v}{r^2} \right) - ik\bar{u}v \\ &\quad - \left(i\frac{n}{r} B^z - ikB^\theta \right) \bar{B}^z; \end{aligned} \quad (35)$$

$$\begin{aligned} (\mathcal{L}_u \mathbf{u})_z &= \frac{1}{Re} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{n^2}{r^2} w - k^2 w \right) - u \frac{\partial \bar{w}}{\partial r} - ik\bar{u}w + \frac{\partial \bar{B}^z}{\partial r} B^r; \end{aligned} \quad (36)$$

$$\begin{aligned} (\mathcal{L}_B \mathbf{B})_r &= \frac{1}{Re_m} \left(\frac{\partial^2 B^r}{\partial r^2} + \frac{1}{r} \frac{\partial B^r}{\partial r} - \frac{n^2}{r^2} B^r - k^2 B^r - 2i\frac{n}{r^2} B^\theta - \frac{1}{r^2} B^r \right) \\ &\quad + ik\bar{B}^z u - ik\bar{u}B^r; \end{aligned} \quad (37)$$

$$\begin{aligned} (\mathcal{L}_B \mathbf{B})_\theta &= \frac{1}{Re_m} \left(\frac{\partial^2 B^\theta}{\partial r^2} + \frac{1}{r} \frac{\partial B^\theta}{\partial r} - \frac{n^2}{r^2} B^\theta - k^2 B^\theta + 2i\frac{n}{r^2} B^r - \frac{1}{r^2} B^\theta \right) \\ &\quad + ik\bar{B}^z v - ik\bar{v}B^\theta; \end{aligned} \quad (38)$$

$$\begin{aligned} (\mathcal{L}_B \mathbf{B})_z &= \frac{1}{Re_m} \left(\frac{\partial^2 B^z}{\partial r^2} + \frac{1}{r} \frac{\partial B^z}{\partial r} - \frac{n^2}{r^2} B^z - k^2 B^z \right) + ik\bar{B}^z w \\ &\quad + B^r \frac{\partial \bar{w}}{\partial r} - ik\bar{w}B^z - u \frac{\partial \bar{B}^z}{\partial r}. \end{aligned} \quad (39)$$

¹ Perturbations refer to variations in the physical quantities (in the fluid or in the applied magnetic field) due to internal or external factors.

² Depending on the assigned value, k can represent streamwise-independent ($k = 0$) or streamwise-dependent ($k \neq 0$) structures.

Table 1
Spectral abscissa for the pipe Poiseuille flow.

	λ_{max}
$k = 0$	-0.0014
$k = 1$	-0.0142

Further details about the numerical method, including the spectral approximation of the spatial derivatives, can be found in [27]. The derived model (Equations (34) to (39)) is the one considered in the following for the stability analysis.

5. Stability analysis

In this Section, an overview of the results of the stability analysis is reported. According to modal analysis, the stability of the MHD model can be determined by analyzing the eigenvalues of the dynamic matrix $\mathbf{A}$, i.e., the eigenvalues of the system (Equation (33)). In the following, two pairs of parameters of the Fourier perturbation have been selected:

$$n = 1, \quad k = 0, \quad \& \quad n = 1, \quad k = 1. \quad (40)$$

It is worth noting that the actual flow comprises significantly more wavenumber combinations than those mentioned above. Nonetheless, these two options represent two of the fundamental constituents of pipe flow dynamics. Specifically, the first option ($k = 0$) represents stream wise-independent modes, while the second one ($k \neq 0$) stands for streamwise-dependent structures. Instead n has been put equal to one for the sake of simplicity. Naturally, the selection has taken into account the constraints on these two wave-numbers: namely, while k can assume any real number, n instead must be an integer [28].

The present study aims to investigate the impact of the magnetic field on the stability of the flow. More specifically, it aims to show how some magnetic field configurations can perturb a fluid regime, and to seek if some particular magnetic profiles can stabilise the entire system. Therefore, two different radial profiles (with $r \in [0, 1]$) for the magnetic field at the steady state, following the cylinder axial direction, have been considered:

$$\text{CASE A : } \mathbf{B}_0 = \left(r - \frac{r^3}{3}\right) \mathbf{e}_z, \quad \text{CASE B : } \mathbf{B}_0 = \left(1 - \frac{r^2}{2}\right) \mathbf{e}_z. \quad (41)$$

The cubic profile represents a typical benchmark in the literature [29], here it is considered with a radial dependence. The parabolic case instead has been selected in order to consider a very different field from the cubic one: indeed it has a quadratic and symmetric shape with its maximum value in the centre of the pipe.

Additionally, a stability analysis of a pipe Poiseuille flow (i.e., a Navier-Stokes model) has been performed, to observe how stability changes when the magnetic field comes into play. In the next sections, the results of these three physical scenarios will be presented.

5.1. Parameter sensitivity

The first part of this investigation consists of identifying the role of the dimensionless magnetic Reynolds number Re_m on the stability of the system. First of all, the spectral abscissa has been calculated for the pipe Poiseuille flow for a reference case with $Re = 10^4$. As explained above, the spectral abscissa represents the dominant eigenvalue of the matrix associated with the problem (i.e. the eigenvalue with the largest real part) and, according to classical stability theory, it determines the general asymptotic stability of the system. Table 1 reports the values of the spectral abscissa (for the pipe Poiseuille flow, hence without magnetic field) obtained for the two selected values of k .

Next, the effect of the magnetic field on the pipe flow will be studied.

Fig. 3 shows the results obtained for the scenarios with the two different magnetic profiles (cubic and parabolic) and the two selected

Table 2
Spectral abscissa for the three considered physical scenarios.

	$B_0 = 0$	$B_0 = r - r^3/3$	$B_0 = 1 - r^2/2$
$k = 0$	-0.0014	-0.0747	-0.0741
$k = 1$	-0.0142	-0.2189	-0.0278

values for k (with the same Reynolds $Re = 10^4$ considered in the pipe Poiseuille flow). When only streamwise-independent structures are considered ($k = 0$) both magnetic profiles tend to stabilize the system in a very similar way as long as Re_m does not take too high values. Indeed, the spectral abscissa assumes a lower (negative) value when compared to the reference case without the magnetic field. This is no longer true when the analysis considers streamwise-dependent perturbations ($k = 1$). Indeed, significant differences arise if different magnetic profiles are imposed. More precisely, the cubic profile keeps the spectral abscissa always at a lower level than the reference one, even for very high values of Re_m . On the other hand, in the parabolic case, the effect of $\mathbf{B}$ is strictly connected to the specific value of the magnetic Reynolds: indeed, the system benefits from the presence of the magnetic field in terms of stability when $Re_m \lesssim 100$ but at the same time, a value of Re_m of several hundred leads to the onset of instabilities, as pointed out by the fact that the spectral abscissa changes sign taking a positive value. This non-monotonic behaviour of the spectral abscissa for this last case will be investigated in the following from a stability point of view to assess whether it has physical meaning (i.e., it corresponds to a region where the magnetic field destabilises the system) or it is caused by numerical errors only.

5.2. Non-modal analysis results

As seen in the previous section, the results of the stability analysis depend on various factors, such as the value assigned to the magnetic Reynolds or the imposed magnetic field shape. In this section, non-modal stability analysis is applied to the proposed MHD problem.

Fig. 4 and 5 show the spectra and the pseudospectra (see Appendix A) for the cases $k = 0$ and $k = 1$ respectively of all the described physical scenarios with $Re = 10^4$ and $Re_m = 100$. The value for the Re_m has been selected considering a point in which the spectral abscissa is close enough to the zero value and to the spectral abscissa of the pipe Poiseuille flow (where energy amplification during the transient is expected to occur), without considering too high values which are not interesting from a physical point of view. Both the spectra and the pseudospectra have been computed with the aid of the Eigtool package in MATLAB [30]. All the eigenvalues have negative real part, indicating asymptotic decay of the perturbation, as suggested by linear stability analysis. Nonetheless, the outer lines of the pseudospectra cross the right half-plane, indicating that energy growth can occur over finite-time intervals. As seen with stability analysis, the two considered magnetic profiles tend to pull the system away from the instability threshold, resulting in an even more negative spectral abscissa (for both the streamwise-independent and streamwise-dependent perturbed configurations): this means that the presence of the magnetic field increases the speed at which the system returns to a stable configuration. Table 2 reports the values of the spectral abscissa in the three physical scenarios.

Moreover, the pseudospectra lines are pushed on the left, indicating that the magnetic field tends to reduce the energy growth during the transient after the perturbation. Indeed, for the pipe Poiseuille flow configuration with $k = 0$, the green contour line (corresponding to the smallest considered perturbation) crosses the imaginary axis, but this does not occur for the cases with the magnetic field, and both the cubic and parabolic profiles keep the green contour to the left of the $Re(\lambda) = 0$ boundary with a very similar effect. For $k = 1$ instead, in the non-magnetic configuration, the green contour appears very close

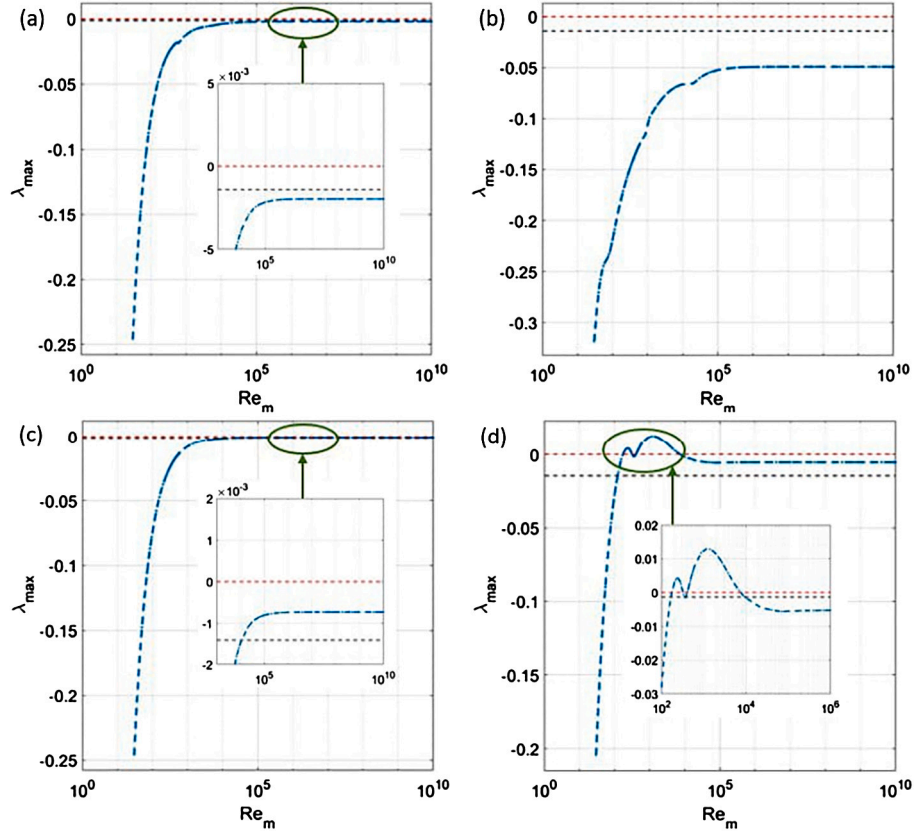


Fig. 3. Spectral abscissa (blue line) for the configuration with the cubic magnetic profile (first row) and the parabolic magnetic profile (second row) in the case $k = 0$ (first column) and $k = 1$ (second column), with $Re = 10^4$. The black line represents the spectral abscissa for the pipe Poiseuille flow, while the red line is the zero axis.

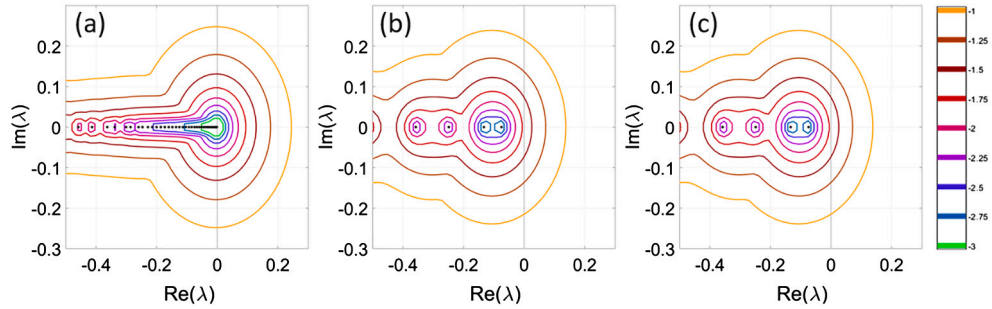


Fig. 4. Spectra (black dots) and pseudospectra (coloured lines) for the three physical scenarios in the case $k = 0$ for different values of the perturbation ϵ . From outside in, the curves correspond to $\epsilon = 10^{-1}, 10^{-1.25}, \dots, 10^{-3}$. Figure (a) represents the pipe Poiseuille flow, while figure (b) and (c) represent the configuration with the cubic and parabolic magnetic profile respectively.

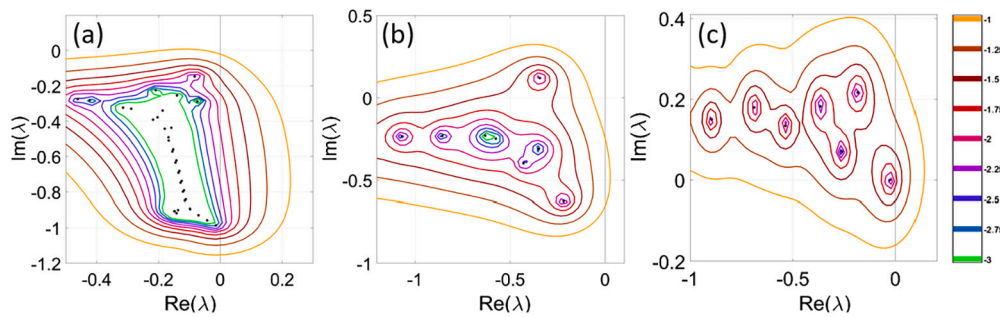


Fig. 5. Spectra (black dots) and pseudospectra (coloured lines) for the three physical scenarios in the case $k = 1$ for different values of the perturbation ϵ . From outside in, the curves correspond to $\epsilon = 10^{-1}, 10^{-1.25}, \dots, 10^{-3}$. Figure (a) represents the pipe Poiseuille flow, while figure (b) and (c) represent the configuration with the cubic and parabolic magnetic profile respectively.

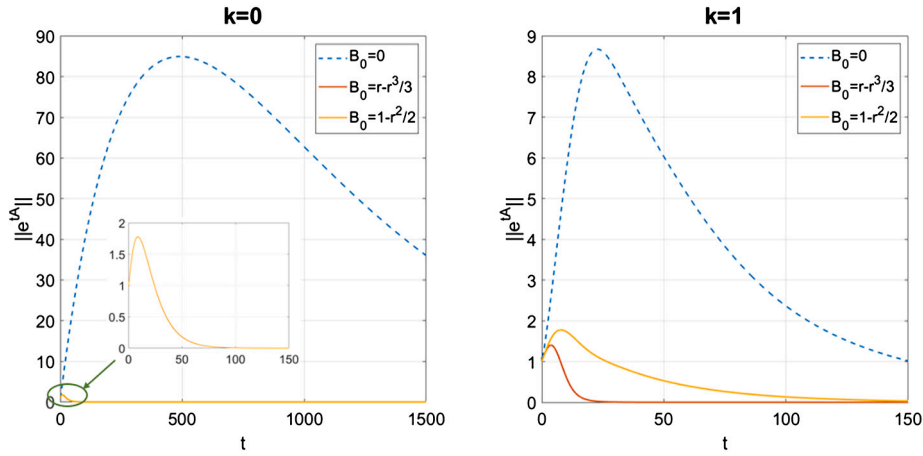


Fig. 6. Evolution of the norm of the exponential matrix for the three physical scenarios with respect to a dimensionless time in the case $k = 0$ (left) and $k = 1$ (right).

to the boundary, while the imposition of a magnetic field can push the line to the left, especially in the scenario with the cubic profile. Thus, it is interesting to notice that, while in the $k = 0$ configuration there is no substantial difference between the cubic and quadratic shape compared to the impact on the stability of the system, in the case $k = 1$ the cubic profile has a major effect on the stabilization of the systems compared to the parabolic one. These results are in agreement with the general considerations derived from the parameter sensitivity and the modal stability analysis, in which the cubic magnetic field shows a stronger effect on the stabilization of the system compared to the parabolic one, especially in the case with $k = 1$.

The subsequent phase of the analysis involves the examination of the system's behaviour during the immediate aftermath of the perturbation. In particular, the focus will be on how the system responds to the disturbance in a short period and how the disturbance evolves in time. Fig. 6 illustrates the evolution of the exponential matrix L_2 -norm, which represents the kinetic energy of the perturbation, i.e. the maximum possible transient growth that can be achieved at a time t optimized over all normalized initial conditions [31].

The dashed line represents the energy evolution for the Poiseuille flow, whereas the red and yellow lines represent instead the energy of the MHD scenarios, with respectively the cubic and parabolic profiles. The figure confirms the observations of the previous analysis about the stabilizing role of the magnetic field. Indeed, in both the cases involving streamwise-independent and streamwise-dependent perturbations, the magnetic field acts as a sort of stabilizing factor to the system: indeed, the MHD scenarios are affected by a much lower energy growth and the perturbation is absorbed by the system in a far less time. In the case $k = 0$, the imposition of the magnetic field results in a drastic reduction of both the peak of the energy function and the time at which the amplification falls to zero. Moreover, while the two considered magnetic profiles have identical effects on the energy amplification of streamwise-independent perturbations, when streamwise-dependent structures are involved the two magnetic shapes impact differently the short-term stability of the system. Indeed, as shown by Fig. 6, the cubic profile turns out to have a stronger effect, enhancing considerably the decreasing rate of the energy amplification compared to the parabolic one. Table 3 reports more details on the energy amplification and on the time required by the system to absorb the perturbation.

Furthermore, an additional stability analysis was applied to the configuration where instability was observed, i.e. the scenario with a parabolic magnetic field and $k = 1$ (see Fig. 3d). Firstly, a preliminary test was conducted to ensure that the instability was not due to numerical issues. The spectral abscissa was recomputed by varying the number of Chebyshev modes, a numerical parameter employed in the spectral approximation (for more details, see [27]). Fig. 7 shows the obtained results, where M represents the specific number of modes employed.

Table 3

The structure of each table is the following. The first row represents the value of the maximum amplification. The second and third row represents respectively the dimensionless time at which the energy is lower than the 10% and the 1% of the maximum amplification.

Energy amplification in the case $k = 0$.			
	$B_0 = 0$	$B_0 = r - r^3/3$	$B_0 = 1 - r^2/2$
Maximum amplification	84.982	1.776	1.777
Time - 10%	2579	51	51
Time - 1%	4176	82	83
Energy amplification in the case $k = 1$.			
	$B_0 = 0$	$B_0 = r - r^3/3$	$B_0 = 1 - r^2/2$
Maximum amplification	8.672	1.403	1.776
Time - 10%	160.5	15.6	89.2
Time - 1%	322.7	25.2	171.9

Table 4

Spectral abscissa for the three considered values of Re_m in the configuration with parabolic magnetic field and $k = 1$.

	$Re_m = 100$	$Re_m = 170$	$Re_m = 1000$
λ_{max}	-0.0278	-0.0007	0.0125

Even when varying the value of this parameter, from $M > 12$, the situation does not change, meaning that the non-monotonic behaviour is not due to numerical approximations (previous results were found with $M = 24$), thus confirming the numerical independency of the results and indicating that the instability found could be connected to the physics of the problem under consideration. Thus, the stability analysis for the region of non-monotonic behaviour was undertaken. For this purpose, two values of Re_m were selected, corresponding to two different situations in Fig. 3. The first selected value is $Re_m = 170$, where the spectral abscissa is still negative but very close to zero; the second value is $Re_m = 1000$, where the spectral abscissa significantly exceeds the stability threshold, reaching the instability region. Table 4 reports the obtained spectral abscissas for the different considered values of Re_m , compared with the previous case with $Re_m = 100$. Fig. 8 shows the pseudospectra in the three cases.

The results confirms the above considerations, showing that for $Re_m = 170$, where the spectral abscissa is negative but very close to zero, the pseudospectrum is slightly shifted to the right but enough for all contour lines associated with the considered ϵ to cross the imaginary axis. This means that even smaller perturbations, compared to $Re_m = 100$, are sufficient to significantly increase the transient growth

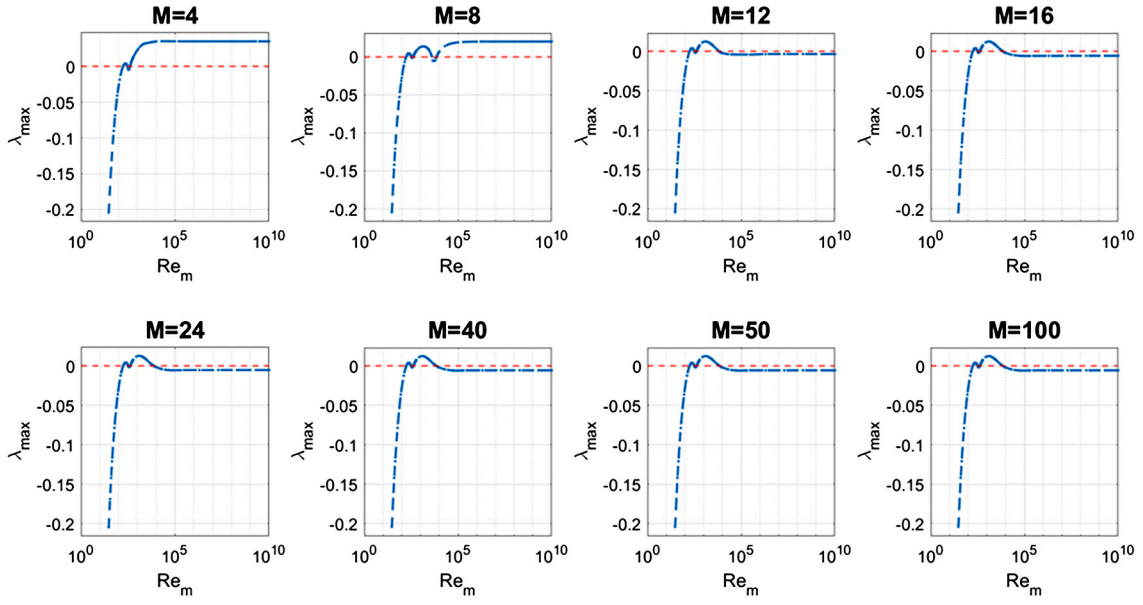


Fig. 7. Spectral abscissa for the scenario with the parabolic magnetic profile in the case $k = 1$ for different values of the number of Chebyshev modes M .

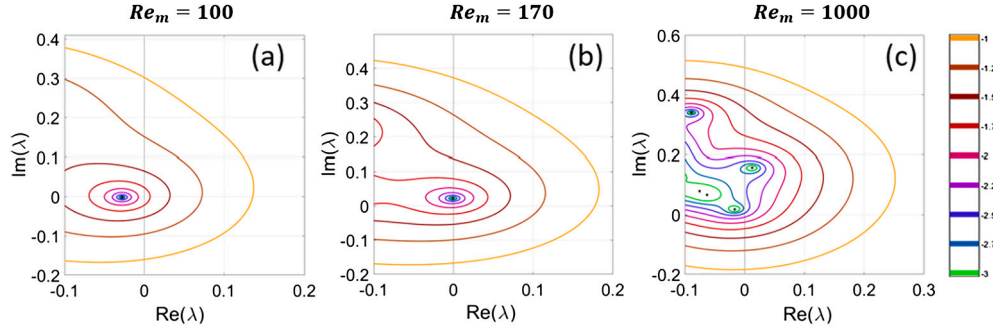


Fig. 8. Spectra (black dots) and pseudospectra (coloured lines) for the three considered values of Re_m in the case with parabolic magnetic field and $k = 1$ for different values of the perturbation ϵ . From outside in, the curves correspond to $\epsilon = 10^{-1}, 10^{-1.25}, \dots, 10^{-3}$. Figure (a), (b) and (c) represent the cases with $Re_m = 100$, $Re_m = 170$ and $Re_m = 1000$ respectively.

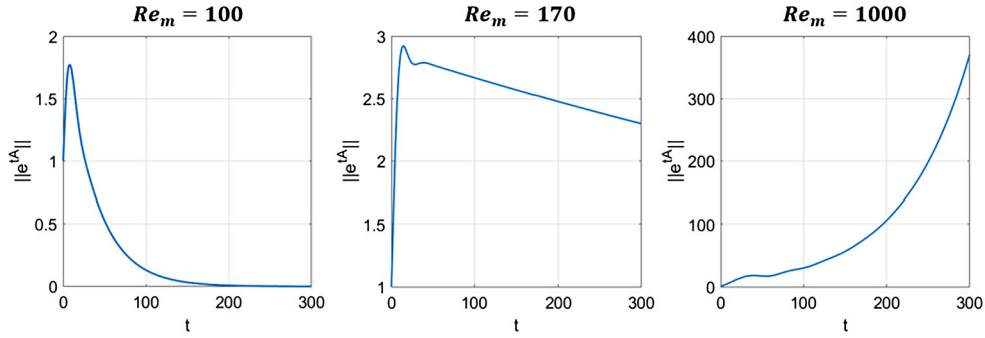


Fig. 9. Evolution of the norm of the exponential matrix for the three considered value of Re_m with respect to a dimensionless time in the configuration with parabolic magnetic field and $k = 1$.

rate. Conversely, when $Re_m = 1000$, the spectral abscissa is positive, indicating that the stability threshold has already been surpassed, and consequently, the entire pseudospectrum significantly exceeds the imaginary axis. Furthermore, the norms of the exponential matrix were calculated in the three cases as a verification of the previous considerations. From Fig. 9, it can be observed that when $Re_m = 170$, the maximum possible energy amplification associated with the perturbation is greater compared to the previous case, and the system takes more time to stabilize. Finally, when $Re_m = 1000$, the maximum amplification explodes towards infinity, confirming instability.

These results demonstrate that high values of Re_m for the case $k = 1$ can dramatically increase the energy associated with the perturbation, even leading to instability. This analysis for high values of Re_m , even if not significant from a practical standpoint (as for the majority of practical applications, Re_m is less than 1), nevertheless allows for important considerations to be deduced. In fact, the addition of a magnetic field with some specific profile, for certain values of Re_m , can have the undesired effect of destabilizing a system that was otherwise stable in a non-magnetic scenario, thus further proving the need for in-depth stability analysis also from the non-modal standpoint.

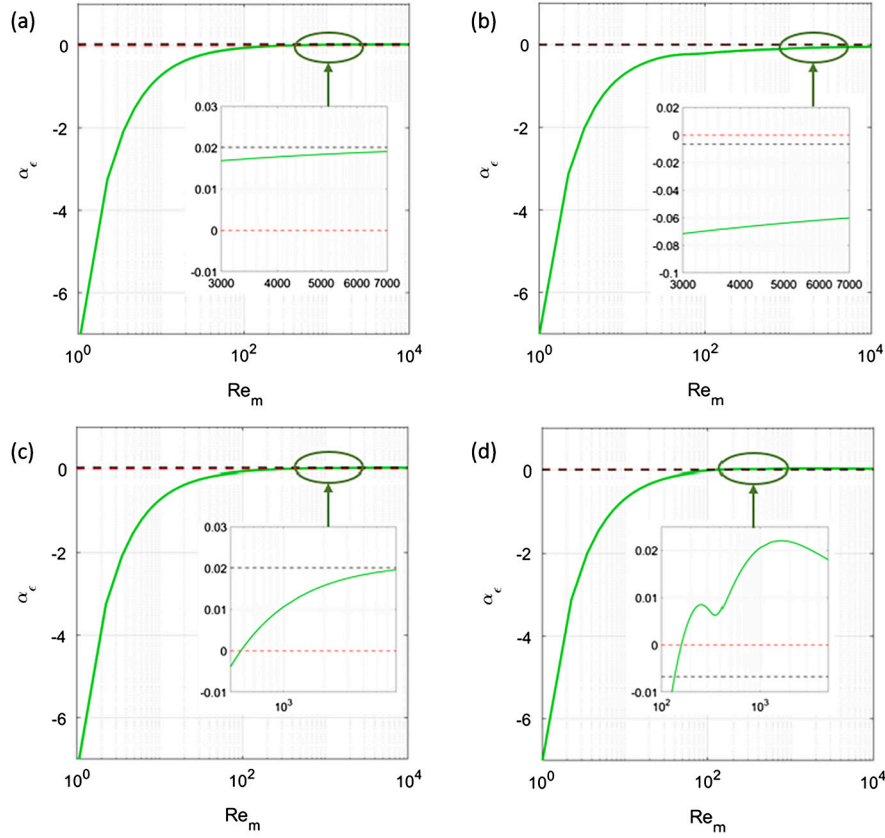


Fig. 10. Pseudo-spectral abscissa (green line) for the configuration with the cubic magnetic profile (first row) and the parabolic magnetic profile (second row) in the case $k = 0$ (first column) and $k = 1$ (second column), with $Re = 10^4$. The black line represents the pseudo-spectral abscissa for the pipe Poiseuille flow, while the red line is the zero axis. All values were obtained by setting $\epsilon = 10^{-3}$.

Finally, an additional investigation on the pseudospectra was conducted, focusing on the boundary of the pseudospectrum related to the smallest considered perturbation ($\epsilon = 10^{-3}$) for different values of Re_m . Fig. 10 presents the obtained results in terms of pseudo-spectral abscissa, i.e. the real part of the rightmost point in the ϵ -pseudospectrum (for the mathematical definition, see Appendix A). The calculated profiles are consistent with the general results presented so far. In particular, it can be observed that generally, the addition of the magnetic field tends to decrease the value of the pseudo-spectral abscissa. Furthermore, it can be noted that, even in this case, for very high values of Re_m , the parabolic magnetic field risks compromising the stability of the system, as evidenced by the fact that the value of the pseudo-spectral abscissa exceeds the value it would have without the magnetic field, thus proving the need for an in-depth modal and non-modal stability analysis for MHD problems. However, these values of Re_m are never encountered in practical applications.

5.3. Discussion of the results

In this work, both modal and non-modal approaches were employed to investigate the stability of the considered MHD pipe flow. The modal analysis results indicate that in general the presence of the magnetic field tends to asymptotically stabilize the system, shifting the spectral abscissa further to the left in the complex plane. However, these results provide insights only into the asymptotic behaviour and lacks information about the immediate post-perturbation evolution. Consequently, certain safety thresholds of importance in nuclear fusion frameworks may be exceeded during the transient, despite modal analysis predicting asymptotic stability. For this reason also the non-modal analysis was exploited, whose importance lies in its ability to quantify energy growth related to the perturbation in the short term, that is, immedi-

ately following the perturbation. The obtained results demonstrate that the system also benefits in the short term from the presence of a magnetic field, experiencing reduced energy amplification in the transient period following the perturbation. From an engineering point of view, this means that the presence of a magnetic field dampens unwanted short-term oscillations in the perturbed quantities, increasing safety. Moreover, the magnitude of both the long-term and short-term benefits depends on the specific applied magnetic profile. Finally, it should be considered that the imposition of a magnetic field does not always benefit the system. Indeed, certain magnetic profiles for specific values of Re_m could lead to instabilities in configurations that were stable in the absence of a magnetic field. Therefore, it proves useful to carry out a stability analysis to understand the actual stabilizing effect of the magnetic field for different scenarios.

6. Conclusions

The present work aimed to perform a stability analysis of a magneto-hydrodynamic problem in a cylindrical geometry. Besides the classical modal stability technique, the non-modal stability method has been exploited as well, to investigate the short-term stability, i.e. the dynamic response to the perturbation at the beginning of the transient. This study represents the first attempt to apply the non-modal approach for studying the MHD stability of a pipe flow scenario. The effect of the magnetic field on the stability of the flow regime has been studied: two different magnetic profiles (a cubic and a parabolic magnetic field) have been compared with a scenario of a pipe Poiseuille flow without any magnetic contribution. Both streamwise-independent and streamwise-dependent modes have been considered. Firstly a parameter sensitivity analysis has been carried out analyzing the value of the spectral abscissa for different values of the dimensionless magnetic Reynolds number (for

a fixed hydraulic Reynolds number). Moreover both a modal and a non-modal stability analysis have been conducted for selected values of the dimensionless numbers.

The results show that in general a magnetic field in the flow direction can have a beneficial role in the stability of the system, but the effect may depend on the specific shape of the field. In particular the cubic profile seems to stabilize the system in a better way with respect to the parabolic one. So the growth of the energy of the perturbed system can be significantly reduced by a wise magnetic profile. Among the future developments of the present work, there is a stability analysis including the energy balance in the proposed MHD model and also the consideration of other profiles of the magnetic fields. Moreover, future studies could verify the results obtained by the stability analysis with a numerical investigation of the evolution of the non-linear model in the different magnetic scenarios.

CRedit authorship contribution statement

Matteo Lo Verso: Writing – original draft, Software, Methodology, Conceptualization. **Carolina Introini:** Writing – review & editing. **Luciana Barucca:** Writing – review & editing. **Marco Caramello:** Writing – review & editing. **Matteo Di Prinzio:** Writing – review & editing. **Francesca Giacobbo:** Writing – review & editing. **Laura Savoldi:** Writing – review & editing. **Antonio Cammi:** Writing – review & editing, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix A. Pseudospectra

The examination of the spectra of an operator may not always provide an accurate understanding of all stability problems. Namely, the behaviour of certain systems can sometimes lead to phenomena that cannot be fully examined by solely looking at the location of the eigenvalues in the Gauss plane. An approach to address this issue is to consider a further matrix attribute, i.e. the pseudospectra of the linear operator. This method is particularly useful in situations where small perturbations can render a generic matrix $\mathbf{A}$ singular. Indeed, investigating whether z is an eigenvalue of $\mathbf{A}$ is equivalent to exploring whether $z\mathbf{I} - \mathbf{A}$ is singular, or more precisely, whether the quantity $\|(z\mathbf{I} - \mathbf{A})^{-1}\|$ is large.

Consider a complex linear operator $\mathbf{A} \in \mathbb{C}^{N \times N}$. Let ϵ be an arbitrary (positive) number. The ϵ -pseudospectrum of $\mathbf{A}$ can be defined as:

$$\sigma_\epsilon(\mathbf{A}) = \{z \in \mathbb{C} : \|(z\mathbf{I} - \mathbf{A})^{-1}\| > \frac{1}{\epsilon}\}. \quad (\text{A.1})$$

From this definition, it seems evident that the spectrum of $\mathbf{A}$ is a subset of the ϵ -pseudospectrum, since for $z \in \sigma(\mathbf{A})$, the resolvent norm $\|(z\mathbf{I} - \mathbf{A})^{-1}\| \rightarrow \infty$. This means that generally matrices exhibit a large resolvent norm for values of z in the proximity of an eigenvalue. However, for non-normal matrices, the resolvent norm can become large even when z is not an eigenvalue. In such cases, the ϵ -pseudospectrum of $\mathbf{A}$ can equivalently be defined as

$$\sigma_\epsilon(\mathbf{A}) = \{z \in \mathbb{C} : \exists \mathbf{E} \in \mathbb{C}^{N \times N} \text{ s.t. } \|\mathbf{E}\| < \epsilon \wedge z \in \sigma(\mathbf{A} + \mathbf{E})\}. \quad (\text{A.2})$$

Simply put, the ϵ -pseudospectrum is the set of eigenvalues of the matrix $\mathbf{A} + \mathbf{E}$, where $\mathbf{E}$ is a small perturbation added to $\mathbf{A}$. Non-normal

operators have certain properties that distinguish them from normal operators. One of these properties is represented by the presence of transient effects in time-dependent dynamical systems. Transient effects are deviations from the long-term behaviour of the system that occur over short time intervals. These transient effects can be caused by the non-linearity of time-varying coefficients, but even in linear (or linearised) problems, non-normality can cause significant transient behaviours. In linear systems, the transient evolution of the system differs from its long-term behaviour due to non-normality. Non-normal operators do not commute with their adjoints, which can lead to amplification or attenuation of certain modes. As a result, the evolution of the system can be very sensitive to small perturbations in the initial conditions or parameters, leading to transient dynamics that can be difficult to predict using traditional eigenvalue analysis.

Pseudospectra provide a way to analyze the growth and decay of solutions of the time-dependent equation $\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$, which cannot be captured by the spectrum examination. In a long time the asymptotic growth rate of $\|e^{\mathbf{A}t}\|$ is determined by the spectral abscissa of $\mathbf{A}$, which is the eigenvalue with the largest real part. Instead, the initial growth rate is determined by the numerical abscissa, i.e.:

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} \|e^{\mathbf{A}t}\| \equiv \omega(\mathbf{A}) := \sup \sigma \left(\frac{1}{2}(\mathbf{A} + \mathbf{A}^*) \right). \quad (\text{A.3})$$

Therefore, eigenvalues provide a lower bound (through the spectral abscissa $\alpha(\mathbf{A})$), while the upper bound is derived from the numerical abscissa $\omega(\mathbf{A})$, i.e.:

$$\begin{aligned} \|e^{\mathbf{A}t}\| &\geq e^{\alpha(\mathbf{A})t}, \\ \|e^{\mathbf{A}t}\| &\leq e^{\omega(\mathbf{A})t}. \end{aligned} \quad (\text{A.4})$$

The main point of interest for pseudospectra is what happens for finite time intervals. Indeed, pseudospectra provide a practical lower bound for intermediate values of time. Indeed, once the Kreiss constant is defined as:

$$K(\mathbf{A}) := \sup \Re(z) \left\| (z\mathbf{I} - \mathbf{A})^{-1} \right\|, \quad (\text{A.5})$$

it is possible to write the following estimate:

$$\sup_{t \geq 0} \|e^{\mathbf{A}t}\| \geq \alpha_\epsilon \left(\frac{1}{\epsilon} \mathbf{A} \right) = K(\mathbf{A}), \quad (\text{A.6})$$

where α_ϵ is the pseudospectral abscissa, defined as the real part of the rightmost point belonging to the ϵ -pseudospectrum, i.e.:

$$\alpha_\epsilon(\mathbf{A}) = \sup_{z \in \mathbb{C}} \{\Re(z) : z \in \sigma_\epsilon(\mathbf{A})\}. \quad (\text{A.7})$$

This inequality provides a way to determine whether transient growth occurs. Specifically, if the pseudospectra cross into the right half-plane, i.e. if

$$\alpha_\epsilon(\mathbf{A}) > \epsilon > 0, \quad (\text{A.8})$$

transient growth is guaranteed to occur.

References

- [1] H. Zohm, G. Gantenbein, A. Isayama, A. Keller, R.J. La Haye, M. Maraschek, A. Mück, K. Nagasaki, S.D. Pinches, E.J. Strait, Mhd limits to tokamak operation and their control, *Plasma Phys. Control. Fusion* 45 (12A) (2003) A163.
- [2] S. Molokov, R. Moreau, K. Moffatt, L. Bühler, Liquid metal magnetohydrodynamics for fusion blankets, in: *Magnetohydrodynamics: Historical Evolution and Trends*, 2007, pp. 171–194.
- [3] G. Ferrero, R. Testoni, M. Zucchetti, Impact assessment of radiative heat transport in arc-class reactor flibe liquid immersion blanket, *Nucl. Sci. Eng.* (2023) 1–16.
- [4] Lord Rayleigh, On the stability, or instability, of certain fluid motions, *Proc. Lond. Math. Soc.* 9 (1880) 57–70.
- [5] P.J. Schmid, Nonmodal stability theory, *Annu. Rev. Fluid Mech.* 39 (2007) 129–162.
- [6] P.J. Schmid, D.S. Henningson, *Stability and Transition in Shear Flows*, Springer-Verlag, New York, 2001.

- [7] R.C. Lock, The stability of the flow of an electrically conducting fluid between parallel planes under a transverse magnetic field, *Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci.* 233 (1192) (1955) 105–125.
- [8] M. Takashima, The stability of the modified plane Poiseuille flow in the presence of a transverse magnetic field, *Fluid Dyn. Res.* 17 (6) (1996) 293.
- [9] M.C. Potter, J.A. Kutchey, Stability of plane Hartmann flow subject to a transverse magnetic field, *Phys. Fluids* 16 (11) (1973) 1848–1851.
- [10] J. Manickam, R.C. Grimm, R.L. Dewar, The linear stability analysis of mhd models in axisymmetric toroidal geometry, *Comput. Phys. Commun.* 24 (3–4) (1981) 355–361.
- [11] L. Bühler, T. Arlt, T. Boeck, L. Braiden, V. Chowdhury, D. Krasnov, C. Mistrangelo, S. Molokov, J. Priede, Magnetically induced instabilities in duct flows, *IOP Conf. Ser., Mater. Sci. Eng.* 228 (2017) 012003, IOP Publishing.
- [12] C. Introini, A. Cammi, F. Giacobbo, Stability analysis of a zero-dimensional model of pwr core using non-modal stability theory, *Ann. Nucl. Energy* 146 (2020) 107624.
- [13] N. Vetcha, S. Smolentsev, M. Abdou, Linear stability analysis for the Hartmann flow with interfacial slip, *Magnetohydrodynamics* 48 (1) (2012) 147–155.
- [14] D. MacTaggart, The non-modal onset of the tearing instability, *J. Plasma Phys.* 84 (5) (2018) 905840501.
- [15] M. Landreman, G.G. Plunk, W. Dorland, Generalized universal instability: transient linear amplification and subcritical turbulence, *J. Plasma Phys.* 81 (5) (2015) 905810501.
- [16] Lord Rayleigh, LIX. On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side, *Philos. Mag.* 32 (192) (1916) 529–546.
- [17] H.K. Khalil, *Nonlinear Systems*, third edition, Patience Hall, 2002, p. 115.
- [18] N.S. Nise, *Control Systems Engineering*, John Wiley & Sons, 2020.
- [19] K.J. Åström, R.M. Murray, *Feedback Systems: An Introduction for Scientists and Engineers*, Princeton University Press, 2021.
- [20] L.N. Trefethen, M. Embree, The behavior of nonnormal matrices and operators, in: *Spectra and Pseudospectra*, 2005.
- [21] S.C. Reddy, P.J. Schmid, D.S. Henningson, Pseudospectra of the Orr–Sommerfeld operator, *SIAM J. Appl. Math.* 53 (1) (1993) 15–47.
- [22] L.N. Trefethen, Pseudospectra of linear operators, *SIAM Rev.* 39 (3) (1997) 383–406.
- [23] U. Müller, L. Bühler, *Magnetofluidynamics in Channels and Containers*, Springer Science & Business Media, 2001.
- [24] A. Leonard, A. Wray, A new numerical method for the simulation of three-dimensional flow in a pipe, in: *Eighth International Conference on Numerical Methods in Fluid Dynamics: Proceedings of the Conference*, Rheinisch-Westfälische Technische Hochschule Aachen, Germany, June 28–July 2, 1982, Springer, 2005, pp. 335–342.
- [25] R.D. Moser, P. Moin, A. Leonard, A spectral numerical method for the Navier–Stokes equations with applications to Taylor–Couette flow, *J. Comput. Phys.* 52 (3) (1983) 524–544.
- [26] A. Meseguer, L.N. Trefethen, Linearized pipe flow to Reynolds number 107, *J. Comput. Phys.* 186 (1) (2003) 178–197.
- [27] A. Meseguer, L.N. Trefethen, A spectral Petrov–Galerkin formulation for pipe flow I: Linear stability and transient growth, Technical report, Oxford University, 2000.
- [28] J.A. Peacock, *Fourier Analysis, Lecture Notes*, 2014.
- [29] R.B. Dahlburg, T.A. Zang, D. Montgomery, M.Y. Hussaini, Viscous, resistive magnetohydrodynamic stability computed by spectral techniques, *Proc. Natl. Acad. Sci. USA* 80 (18) (1983) 5798–5802.
- [30] T.G. Wright, *EigTool (MATLAB software package)*, <http://www.cs.ox.ac.uk/pseudospectra/eigtool/>.
- [31] D. MacTaggart, The tearing instability of resistive magnetohydrodynamics, in: *Topics in Magnetohydrodynamic Topology, Reconnection and Stability Theory*, 2020, pp. 37–67.