Propagation of anisotropic Gabor singularities for Schrödinger type equations

J. Evol. Equ. (2024)24:36

https://doi.org/10.1007/s00028-024-00963-w

Journal of Evolution Equations

Check for updates

Propagation of anisotropic Gabor singularities for Schrödinger type equations

MARCO CAPPIELLO, LUIGI RODINO AND PATRIK WAHLBERG

Abstract. We show results on propagation of anisotropic Gabor wave front sets for solutions to a class of evolution equations of Schrödinger type. The Hamiltonian is assumed to have a real-valued principal symbol with the anisotropic homogeneity $a(\lambda x, \lambda^{\sigma}\xi) = \lambda^{1+\sigma} a(x, \xi)$ for $\lambda > 0$ where $\sigma > 0$ is a rational anisotropy parameter. We prove that the propagator is continuous on anisotropic Shubin–Sobolev spaces. The main result says that the propagation of the anisotropic Gabor wave front set follows the Hamilton flow of the principal symbol.

1. Introduction

We prove results on propagation of anisotropic phase space singularities for the initial value Cauchy problem for evolution equations of the form

$$\begin{cases} \partial_t u(t,x) + i a^w(x, D_x) u(t,x) = 0, \quad x \in \mathbf{R}^d, \quad t \in [-T, T] \setminus \{0\}, \\ u(0,\cdot) = u_0. \end{cases} \tag{1.1}$$

Here $T > 0$, $a^w(x, D_x)$ is a Weyl pseudodifferential operator and $u_0 \in \mathscr{S}'(\mathbf{R}^d)$ is a tempered distribution.

The Hamiltonian $a^w(x, D_x)$ is assumed to have real-valued principal symbol a_0. Following the fundamental idea of Hörmander we show that the singularities at time $t \in [-T, T]$ are the singularities of the initial datum u_0 transported by the Hamilton flow χ_t of the principal symbol a_0. The Hamilton flow $(x(t), \xi(t)) = \chi_t(x, \xi)$ is the solution to Hamilton's equation with initial datum $(x, \xi) \in T^*\mathbf{R}^d \setminus \{(0, 0)\}$, that is the solution to the system of ordinary differential equations

$$\begin{cases} x'(t) = \nabla_\xi a_0\,(x(t), \xi(t))\,, \\ \xi'(t) = -\nabla_x a_0\,(x(t), \xi(t))\,, \\ x(0) = x, \\ \xi(0) = \xi. \end{cases}$$

Mathematics Subject Classification: 46F05, 46F12, 35A27, 47G30, 35S05, 35A18, 81S30, 58J47, 47D06
Keywords: Tempered distributions, Global wave front sets, Microlocal analysis, Phase space, Anisotropy, Propagation of singularities, Evolution equations.

Published online: 02 April 2024

The concept of phase space singularities that we use is the anisotropic Gabor wave front set, which is determined by an anisotropy parameter $\sigma > 0$. For $u \in \mathscr{S}'(\mathbf{R}^d)$ the anisotropic Gabor wave front set $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)$ is a σ-conic closed subset of $T^*\mathbf{R}^d \setminus 0$. A σ-conic subset of $T^*\mathbf{R}^d \setminus 0$ contains anisotropic phase space curves of the form

$$\lambda \mapsto (\lambda x, \lambda^{\sigma}\xi) \in T^*\mathbf{R}^d \setminus 0, \quad \lambda > 0, \tag{1.2}$$

if one point of the curve belongs to the subset.

The anisotropic Gabor wave front set $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)$ is defined by means of the short-time Fourier transform $V_{\varphi}u(x,\xi) = \mathscr{F}\left(u\,\overline{\varphi(\cdot - x)}\right)(\xi)$ where $\varphi \in \mathscr{S}(\mathbf{R}^d)\setminus\{0\}$ is a window function. To wit $z_0 = (x_0, \xi_0) \in T^*\mathbf{R}^d\setminus 0$ satisfies $z_0 \notin \mathrm{WF}_{\mathrm{g}}^{\sigma}(u)$ if there exists an open set $U \subseteq T^*\mathbf{R}^d$ such that $z_0 \in U$ and

$$\sup_{(x,\xi)\in U,\ \lambda>0} \lambda^N |V_{\varphi}u(\lambda x, \lambda^{\sigma}\xi)| < +\infty \quad \forall N \geqslant 0.$$

This means that the short-time Fourier transform, which a priori is polynomially upper bounded, decays superpolynomially along curves of the form (1.2) in a neighborhood of z_0. For $u \in \mathscr{S}'(\mathbf{R}^d)$ we have $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u) = \emptyset$ if and only if $u \in \mathscr{S}(\mathbf{R}^d)$ so $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)$ measures globally singular behavior in the sense of lack of smoothness or decay at infinity comprehensively.

We impose the condition that the Hamiltonian $a^w(x, D)$ has a real-valued principal symbol a_0 which satisfies the anisotropic homogeneity

$$a_0(\lambda x, \lambda^{\sigma}\xi) = \lambda^{1+\sigma} a_0(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \quad \lambda > 0. \tag{1.3}$$

This condition turns out to have several beneficial consequences for the problem we study.

First it implies that the Hamilton flow χ_t of a_0 commutes with the anisotropic scaling map

$$T^*\mathbf{R}^d \setminus 0 \ni (x,\xi) \mapsto (\lambda x, \lambda^{\sigma}\xi) \in T^*\mathbf{R}^d \setminus 0$$

for each $\lambda > 0$. This is a natural requirement for propagation results of the form $\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u_0) \subseteq \chi_t \mathrm{WF}_{\mathrm{g}}^{\sigma}(u_0)$, where $\mathscr{K}_t u_0 = e^{-ita^w(x,D)}u_0$ denotes the solution operator (propagator) for (1.1), that we aim for, since $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)$ is σ-conic for all $u \in \mathscr{S}'(\mathbf{R}^d)$.

Secondly if $\sigma > 0$ is rational then condition (1.3) on the principal symbol allows us to prove the main result of this paper, that is the propagation of singularities

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u_0) = \chi_t(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u_0)), \quad t \in [-T, T], \quad u_0 \in \mathscr{S}'(\mathbf{R}^d), \tag{1.4}$$

where $T > 0$.

The term "principal symbol" refers here to the pseudodifferential calculus of anisotropic Shubin symbols [7,27,32]. The symbols exhibit anisotropic behavior on phase space according to the assumed estimates

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta} a(x,\xi)| \lesssim (1 + |x| + |\xi|^{\frac{1}{\sigma}})^{m-|\alpha|-\sigma|\beta|}, \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \alpha, \beta \in \mathbf{N}^d,$$

where again $\sigma > 0$ is a given anisotropy parameter, and $m \in \mathbf{R}$ is the order. These symbol classes are denoted $G^{m,\sigma}$.

In the main result Theorem 8.3, we show (1.4) under the following assumptions. Suppose $k, m \in \mathbf{N} \backslash 0$, $\sigma = \frac{k}{m}$, and let $a \in G^{1+\sigma,\sigma}$, $a \sim \sum_{j=0}^{\infty} a_j$, where $a_0 \in C^\infty(\mathbf{R}^{2d} \backslash 0)$ is real-valued and satisfies (1.3), whereas the lower order terms satisfy $a_j \in G^{(1+\sigma)(1-2j),\sigma}$ for $j \geqslant 1$. An example of a symbol that satisfies the criteria is

$$a(x,\xi) = c\psi(x,\xi)\left(|x|^{2k} + |\xi|^{2m}\right)^{\frac{1}{2}\left(\frac{1}{k}+\frac{1}{m}\right)}$$

where $c \in \mathbf{R} \setminus 0$ and ψ is a smooth function vanishing in a small ball around the origin in $T^*\mathbf{R}^d$.

We show that the solution operator is continuous on anisotropic Shubin–Sobolev spaces. This is of independent interest but also a tool for the proof of Theorem 8.3. The proof of the main result is based on ideas from [16]. More precisely our result is an anisotropic version of [21, Theorem 4.2] which treats propagation of the (isotropic) Gabor wave front set when the principal symbol is real-valued and homogeneous of order two on $T^*\mathbf{R}^d$. With $\sigma = 1$ our result implies a weaker form of [21, Theorem 4.2].

The proof ideas for Theorem 8.3 and [21, Theorem 4.2] are based on Hörmander's proof of [16, Theorem 23.1.4]. This result concerns Hamiltonians with first-order Hörmander type symbols, the continuity concerns classical Sobolev spaces, and the singularities are the classical smooth wave front set. The proof techniques rely on energy estimates, functional analysis and pseudodifferential calculus. Our proofs in this paper are worked out in detail as opposed to the rather brief arguments in [16, Chapter 23.1] and [21].

We also prove the propagation (1.4) for a different type of Hamiltonian of the form $a^w(x, D) = p(D) + \langle v, x\rangle$ where $p \in C^\infty(\mathbf{R}^d)$ is a sum of polynomials of each variable in $\mathbf{R}^d$, with real coefficients, of order $m \geqslant 2$, $v \in \mathbf{R}^d$ is a vector each of whose coordinate is nonzero, and $\sigma = \frac{1}{m-1}$. Since this setup includes the Airy operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x$ when $d = 1$ we say that the corresponding equation (1.1) is of Airy–Schrödinger type. Using results from [31] we also formulate a version of (1.4) in the Gelfand–Shilov space functional framework and corresponding anisotropic wave front sets [26].

Denoting by P_m the principal part of p, we show (1.4) where χ_t is the Hamilton flow of $P_m(\xi)$. This generalizes a particular case of [32, Theorem 5.1] where $v = 0$. Since $P_m(\xi)$ does not depend on x, the Hamiltonian flow for P_m is trivial in the sense that it is constant in time with respect to the dual coordinates as $\chi_t(x,\xi) = (x + t\nabla P_m(\xi), \xi)$. This contrasts to the Hamilton flow in the main result Theorem 8.3 where both space and dual coordinates may depend on time. The techniques we use for Airy–Schrödinger equations are an explicit formula for the Schwartz kernel of the propagator and general results on propagation of singularities from [26,27,31,32].

Our results in this paper fit in a project to investigate globally anisotropic pseudodifferential operators [3,5,7,19,26,27] and propagation of global singularities for evolution equations [24,31,32]. The techniques are inspired from those of pseudodifferential operators defined by symbols that are anisotropic in the dual variables for fixed space coordinates. These ideas have been investigated e.g. in [11,18,22].

A major new feature of our main result Theorem 8.3 as opposed to earlier propagation results [32], is that it admits Hamiltonians that give rise to flows that are non-trivial in the sense that the dynamics involve all phase space coordinates.

Concerning the organization of the paper, Sect. 2 contains notations, background concepts and conventions, and Sect. 3 recalls material on anisotropic Shubin pseudodifferential calculus. Section 4 is devoted to Shubin–Sobolev modulation spaces in the anisotropic context, a recollection of localization operators, and an inequality of sharp Gårding type which is essential. In Sect. 5 we deduce propagation results for Airy–Schrödinger equations. Section 6 treats Hamiltonians that are anisotropically homogeneous as in (1.3) and their Hamilton flows, and in Sect. 7 we show existence and uniqueness of solutions to an inhomogeneous form of (1.1) in anisotropic Shubin–Sobolev spaces for Hamiltonian symbols in $G^{1+\sigma,\sigma}$ with bounded imaginary part and $\sigma > 0$ rational. Then Sect. 8 is dedicated to the main result on propagation of singularities, and finally Sect. 9 consists of a very short discussion of examples.

2. Preliminaries

The unit sphere in $\mathbf{R}^d$ is denoted $\mathbf{S}^{d-1} \subseteq \mathbf{R}^d$. An open ball of radius $r > 0$ centered in $x \in \mathbf{R}^d$ is denoted $\mathrm{B}_r(x)$, and $\mathrm{B}_r(0) = \mathrm{B}_r$. The transpose of a matrix $A \in \mathbf{R}^{d\times d}$ is denoted A^T and the inverse transpose of $A \in \mathrm{GL}(d, \mathbf{R})$ is A^{-T}. We write $f(x) \lesssim g(x)$ provided there exists $C > 0$ such that $f(x) \leqslant C\, g(x)$ for all x in the domain of f and of g. If $f(x) \lesssim g(x) \lesssim f(x)$ then we write $f \asymp g$. We use the partial derivative $D_j = -i\partial_j$, $1 \leqslant j \leqslant d$, acting on functions and distributions on $\mathbf{R}^d$, with extension to multi-indices. The bracket $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ for $x \in \mathbf{R}^d$ satisfies Peetre's inequality with optimal constant [26, Lemma 2.1], that is

$$\langle x + y\rangle^s \leqslant \left(\frac{2}{\sqrt{3}}\right)^{|s|} \langle x\rangle^s \langle y\rangle^{|s|} \qquad x, y \in \mathbf{R}^d, \quad s \in \mathbf{R}. \tag{2.1}$$

We use the normalization of the Fourier transform

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, \mathrm{d}x, \qquad \xi \in \mathbf{R}^d,$$

for $f \in \mathscr{S}(\mathbf{R}^d)$ (the Schwartz space), where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the scalar product on $\mathbf{R}^d$. The conjugate linear action of a distribution u on a test function ϕ is written (u, ϕ), consistent with the L^2 inner product $(\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\,\cdot\,)_{L^2}$ which is conjugate linear in the second argument.

Denote translation by $T_x f(y) = f(y-x)$ and modulation by $M_\xi f(y) = e^{i\langle y,\xi\rangle} f(y)$ for $x, y, \xi \in \mathbf{R}^d$ where f is a function or distribution defined on $\mathbf{R}^d$. If $\varphi \in \mathscr{S}(\mathbf{R}^d)\setminus\{0\}$

then the short-time Fourier transform (STFT) of a tempered distribution $u \in \mathscr{S}'(\mathbf{R}^d)$ is defined by

$$V_\varphi u(x,\xi) = (2\pi)^{-\frac{d}{2}}(u, M_\xi T_x \varphi) = \mathscr{F}(u T_x \overline{\varphi})(\xi), \quad x,\xi \in \mathbf{R}^d.$$

The function $V_\varphi u$ is smooth and polynomially bounded [13, Theorem 11.2.3], that is there exists $k \geqslant 0$ such that

$$|V_\varphi u(x,\xi)| \lesssim \langle (x,\xi)\rangle^k, \quad (x,\xi) \in T^*\mathbf{R}^d. \tag{2.2}$$

We have $u \in \mathscr{S}(\mathbf{R}^d)$ if and only if

$$|V_\varphi u(x,\xi)| \lesssim \langle (x,\xi)\rangle^{-N}, \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \forall N \geqslant 0. \tag{2.3}$$

The transform inverse to the STFT is given by

$$u = (2\pi)^{-\frac{d}{2}} \iint_{\mathbf{R}^{2d}} V_\varphi u(x,\xi) M_\xi T_x \varphi \, \mathrm{d}x \, \mathrm{d}\xi \tag{2.4}$$

provided $\|\varphi\|_{L^2} = 1$, with action under the integral understood, that is

$$(u, f) = (V_\varphi u, V_\varphi f)_{L^2(\mathbf{R}^{2d})} = (V_\varphi^* V_\varphi u, f) \tag{2.5}$$

for $u \in \mathscr{S}'(\mathbf{R}^d)$ and $f \in \mathscr{S}(\mathbf{R}^d)$, cf. [13, Theorem 11.2.5].

According to [13, Corollary 11.2.6] the topology for $\mathscr{S}(\mathbf{R}^d)$ can be defined by the collection of seminorms

$$\mathscr{S}(\mathbf{R}^d) \ni \psi \mapsto \|\psi\|_m := \sup_{z \in \mathbf{R}^{2d}} \langle z \rangle^m |V_\varphi \psi(z)|, \quad m \in \mathbf{N}, \tag{2.6}$$

for any $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$.

The Beurling type Gelfand–Shilov space $\Sigma_\nu^\mu(\mathbf{R}^d)$ is for $\nu, \mu, h > 0$ is defined as the topological projective limit

$$\Sigma_\nu^\mu(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{\nu,h}^\mu(\mathbf{R}^d)$$

where $\mathcal{S}_{\nu,h}^\mu(\mathbf{R}^d)$ is the Banach space of smooth functions that have finite

$$\|f\|_{\mathcal{S}_{\nu,h}^\mu} \equiv \sup_{x \in \mathbf{R}^d,\, \alpha,\beta \in \mathbf{N}^d} \frac{|x^\alpha D^\beta f(x)|}{h^{|\alpha+\beta|} \alpha!^\nu \, \beta!^\mu}$$

norm [12]. The space $\Sigma_\nu^\mu(\mathbf{R}^d)$ is a Fréchet space with respect to the seminorms $\|\cdot\|_{\mathcal{S}_{\nu,h}^\mu}$ for $h > 0$, and $\Sigma_\nu^\mu(\mathbf{R}^d) \neq \{0\}$ if and only if $\nu + \mu > 1$ [23].

If $\nu + \mu > 1$ the topological dual of $\Sigma_\nu^\mu(\mathbf{R}^d)$ is the space of (Beurling type) Gelfand–Shilov ultradistributions [12, Section I.4.3]

$$(\Sigma_\nu^\mu)'(\mathbf{R}^d) = \bigcup_{h>0} (\mathcal{S}_{\nu,h}^\mu)'(\mathbf{R}^d).$$

The space of ultradistributions $(\Sigma_\nu^\mu)'(\mathbf{R}^d)$ may be equipped with several possibly different topologies [31]. In this paper we use exclusively the weak* topology.

The Gelfand–Shilov (ultradistribution) spaces enjoy invariance properties, with respect to translation, dilation, tensorization, coordinate transformation and (partial) Fourier transformation. The Fourier transform extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, from $(\Sigma_\nu^\mu)'(\mathbf{R}^d)$ to $(\Sigma_\mu^\nu)'(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, from $\Sigma_\nu^\mu(\mathbf{R}^d)$ to $\Sigma_\mu^\nu(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

3. Anisotropic Shubin pseudodifferential calculus

In this section we retrieve some essential facts from pseudodifferential calculus of anisotropic Shubin symbols [27,32].

Let $\sigma > 0$. We use the weight function on $(x,\xi) \in T^*\mathbf{R}^d$

$$\theta_\sigma(x,\xi) = 1 + |x| + |\xi|^{\frac{1}{\sigma}}. \tag{3.1}$$

For this weight we have the following inequality of Peetre type [27]. If $s \in \mathbf{R}$ then

$$\theta_\sigma(x+y,\xi+\eta)^s \leqslant C_{\sigma,s}\theta_\sigma(x,\xi)^{|s|}\theta_\sigma(y,\eta)^s, \quad x,y,\xi,\eta \in \mathbf{R}^d. \tag{3.2}$$

When σ is rational, $\sigma = \frac{k}{m}$, $k,m \in \mathbf{N}\backslash 0$, an alternative weight is

$$w_{k,m}(x,\xi) = \left(1 + |x|^{2k} + |\xi|^{2m}\right)^{\frac{1}{2}}. \tag{3.3}$$

Note that

$$w_{k,m} \asymp \theta_\sigma^k. \tag{3.4}$$

The motivation for using $w_{k,m}$ instead of θ_σ^k is that the former is smooth as opposed to the latter.

By [27, Eq. (3.4)] we have for $\sigma > 0$

$$\langle (x,\xi)\rangle^{\min\left(1,\frac{1}{\sigma}\right)} \lesssim \theta_\sigma(x,\xi) \lesssim \langle (x,\xi)\rangle^{\max\left(1,\frac{1}{\sigma}\right)}, \quad (x,\xi) \in T^*\mathbf{R}^d, \tag{3.5}$$

and for $k,m \in \mathbf{N} \setminus 0$

$$\langle (x,\xi)\rangle^{\min(k,m)} \lesssim w_{k,m}(x,\xi) \lesssim \langle (x,\xi)\rangle^{\max(k,m)}, \quad (x,\xi) \in T^*\mathbf{R}^d. \tag{3.6}$$

The anisotropic Shubin symbols are defined as follows.

Definition 3.1. Let $\sigma > 0$ be real and $m \in \mathbf{R}$. The space of (σ-)anisotropic Shubin symbols $G^{m,\sigma}$ of order m consists of functions $a \in C^\infty(\mathbf{R}^{2d})$ that satisfy the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim (1 + |x| + |\xi|^{\frac{1}{\sigma}})^{m-|\alpha|-\sigma|\beta|}, \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \alpha,\beta \in \mathbf{N}^d. \tag{3.7}$$

The space $G^{m,\sigma}$ is a Fréchet space with respect to the seminorms on $a \in G^{m,\sigma}$ indexed by $j \in \mathbf{N}$

$$\|a\|_j = \max_{|\alpha+\beta| \leqslant j} \sup_{(x,\xi)\in\mathbf{R}^{2d}} \theta_\sigma(x,\xi)^{-m+|\alpha|+\sigma|\beta|} \left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right|.$$

If $\sigma = 1$ then $G^{m,\sigma}$ is the space of isotropic Shubin symbols with parameter $\rho = 1$ [20,28]. Recall that the isotropic Shubin symbol of order m and parameter $0 \leqslant \rho \leqslant 1$, denoted $a \in G_\rho^m$, satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \langle (x,\xi) \rangle^{m-\rho|\alpha+\beta|}, \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \alpha, \beta \in \mathbf{N}^d.$$

We have $G^{m,\sigma} \subseteq G_\rho^{m_0}$, where $m_0 = \max(m, m/\sigma)$ and $\rho = \min(\sigma, 1/\sigma)$, and

$$\bigcap_{m\in\mathbf{R}} G^{m,\sigma} = \mathscr{S}(\mathbf{R}^{2d}). \tag{3.8}$$

The following lemma is a tool for verification of membership in $G^{m,\sigma}$.

Lemma 3.2. *If $m \in \mathbf{R}$, $\sigma, r > 0$ and $a \in C^\infty(\mathbf{R}^{2d})$ satisfies*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(\lambda x, \lambda^\sigma \xi) \right| \lesssim \lambda^{m-|\alpha|-\sigma|\beta|}, \ (x,\xi) \in T^*\boldsymbol{R}^d, \ \ |(x,\xi)| = r, \lambda \geqslant 1, \alpha, \beta \in \boldsymbol{N}^d, \tag{3.9}$$

then $a \in G^{m,\sigma}$.

Proof. Let $(y,\eta) \in \mathbf{R}^{2d} \setminus \mathrm{B}_r$. By [27, Section 3] $(y,\eta) = (\lambda x, \lambda^\sigma \xi)$ for a unique $(x,\xi) \in \mathbf{R}^{2d}$ such that $|(x,\xi)| = r$ and $\lambda \geqslant 1$. Combining

$$1 + |y| + |\eta|^{\frac{1}{\sigma}} = 1 + \lambda(|x| + |\xi|^{\frac{1}{\sigma}}) \asymp 1 + \lambda$$

with (3.9) we obtain for any $\alpha, \beta \in \mathbf{N}^d$

$$\left| \partial_y^\alpha \partial_\eta^\beta a(y,\eta) \right| \lesssim (1+\lambda)^{m-|\alpha|-\sigma|\beta|} \lesssim (1 + |y| + |\eta|^{\frac{1}{\sigma}})^{m-|\alpha|-\sigma|\beta|}.$$

The same estimate is trivial for $(y,\eta) \in \mathrm{B}_r$ so referring to (3.7) we may conclude that $a \in G^{m,\sigma}$. □

Corollary 3.3. *If $\sigma > 0$, $m \geqslant 0$ and $a \in C^\infty(\mathbf{R}^{2d})$ is anisotropically homogeneous as*

$$a(\lambda x, \lambda^\sigma \xi) = \lambda^m a(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \lambda > 0, \tag{3.10}$$

then $a \in G^{m,\sigma}$.

For $a \in G^{m,\sigma}$ and $\tau \in \mathbf{R}$ a pseudodifferential operator in the τ-quantization is defined by

$$a_\tau(x,D)f(x) = (2\pi)^{-d}\int_{\mathbf{R}^{2d}} e^{i\langle x-y,\xi\rangle} a((1-\tau)x+\tau y,\xi)\, f(y)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{3.11}$$

for $f \in \mathscr{S}(\mathbf{R}^d)$ when $m < -d\sigma$. The definition extends to $m \in \mathbf{R}$ if the integral is viewed as an oscillatory integral. If $\tau = 0$ we get the Kohn–Nirenberg quantization $a_0(x,D) = a(x,D)$ and if $\tau = \frac{1}{2}$ we have the Weyl quantization $a_{1/2}(x,D) = a^w(x,D)$. The Weyl quantization enjoys a simple formal adjoint relation: $a^w(x,D)^* = \overline{a}^w(x,D)$. We will use exclusively the Weyl quantization in this paper. By [27, Proposition 3.3 (i)] the symbol classes $G^{m,\sigma}$ are homeomorphically invariant under change of quantization parameter $\tau \in \mathbf{R}$, for any $\sigma > 0$ and $m \in \mathbf{R}$. If $a \in G^{m,\sigma}$ then the operator $a^w(x,D)$ acts continuously on $\mathscr{S}(\mathbf{R}^d)$ and extends uniquely by duality to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$ [27,28]. If $a \in \mathscr{S}'(\mathbf{R}^{2d})$ then $a^w(x,D)$ extends to a continuous operator $a^w(x,D): \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}'(\mathbf{R}^d)$. If $a \in \mathscr{S}(\mathbf{R}^{2d})$ then $a^w(x,D)$ is regularizing, in the sense that it is continuous $a^w(x,D): \mathscr{S}'(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d)$ with $\mathscr{S}'(\mathbf{R}^d)$ equipped with the strong topology [6].

If $a \in \mathscr{S}'(\mathbf{R}^{2d})$ then

$$(a^w(x,D)f,g) = (2\pi)^{-d}(a, W(g,f)), \quad f,g \in \mathscr{S}(\mathbf{R}^d), \tag{3.12}$$

where the cross-Wigner distribution [10,13] is defined as

$$W(g,f)(x,\xi) = \int_{\mathbf{R}^d} g(x+y/2)\overline{f(x-y/2)}e^{-i\langle y,\xi\rangle}\mathrm{d}y, \quad (x,\xi) \in \mathbf{R}^{2d}.$$

If $f,g \in \mathscr{S}(\mathbf{R}^d)$ then $W(g,f) \in \mathscr{S}(\mathbf{R}^{2d})$.

Given a sequence of symbols $a_j \in G^{m_j,\sigma}$, $j = 1,2,\ldots$, such that $m_j \to -\infty$ as $j \to \infty$ we write

$$a \sim \sum_{j=1}^{\infty} a_j$$

provided that for any $n \geqslant 2$

$$a - \sum_{j=1}^{n-1} a_j \in G^{\mu_n,\sigma}$$

where $\mu_n = \max_{j\geqslant n} m_j$. By [27, Lemma 3.2] there exists a symbol $a \in G^{m,\sigma}$ where $m = \max_{j\geqslant 1} m_j$ such that $a \sim \sum_{j=1}^{\infty} a_j$ under the stated circumstances. The symbol a is unique modulo $\mathscr{S}(\mathbf{R}^{2d})$.

The bilinear Weyl product $a\#b$ of two symbols $a \in G^{m,\sigma}$ and $b \in G^{n,\sigma}$ is defined as the product of symbols corresponding to operator composition: $(a\#b)^w(x,D) = a^w(x,D)b^w(x,D)$. By [27, Proposition 3.3 (ii)] the Weyl product is continuous $\# :$

$G^{m,\sigma} \times G^{n,\sigma} \to G^{m+n,\sigma}$. The asymptotic expansion formula for the Weyl product [16,28] is

$$a\#b(x,\xi) \sim \sum_{\alpha,\beta \geqslant 0} \frac{(-1)^{|\beta|}}{\alpha!\beta!} 2^{-|\alpha+\beta|} D_x^\beta \partial_\xi^\alpha a(x,\xi)\, D_x^\alpha \partial_\xi^\beta b(x,\xi). \tag{3.13}$$

If $a \in G^{m,\sigma}$ and $b \in G^{n,\sigma}$ then each term in the sum belongs to $G^{m+n-(1+\sigma)|\alpha+\beta|,\sigma}$.

For $\sigma > 0$ a σ-conic subset $\Gamma \subseteq T^*\mathbf{R}^d \setminus 0$ is closed under the operation $T^*\mathbf{R}^d \setminus 0 \ni (x,\xi) \mapsto (\lambda x, \lambda^\sigma \xi)$ for all $\lambda > 0$. By [27, Definition 3.4 and Lemma 3.5] (cf. also [32, Remark 3.4]) it is possible to construct σ-conic open subsets of given points in $T^*\mathbf{R}^d \setminus 0$, and corresponding cutoff functions.

A symbol $a \in G^{m,\sigma}$ is said to be non-characteristic at $z_0 \in T^*\mathbf{R}^d \setminus 0$, if

$$|a(x,\xi)| \geqslant C\theta_\sigma(x,\xi)^m, \quad (x,\xi) \in \Gamma, \quad |(x,\xi)| \geqslant R \tag{3.14}$$

for $C, R > 0$, where $\Gamma \subseteq T^*\mathbf{R}^d \setminus 0$ is an open σ-conic subset containing z_0. The complement in $T^*\mathbf{R}^d \setminus 0$ of the non-characteristic points is called the characteristic set $\mathrm{char}_\sigma(a) \subseteq T^*\mathbf{R}^d \setminus 0$. It is a closed and σ-conic subset of $T^*\mathbf{R}^d \setminus 0$. This is a particular case of [27, Definition 3.8].

In most respects the anisotropic Shubin pseudodifferential calculus for the symbol classes $G^{m,\sigma}$ with $m \in \mathbf{R}$ and $\sigma > 0$ works as the isotropic calculus in [20,28]. In fact [27, Section 3] contains the basics of the anisotropic pseudodifferential calculus, and by [27, Lemma 6.3] and its proof it is possible to construct parametrices for elliptic symbols. Thus if $\sigma > 0$ and $a \in G^{m,\sigma}$ is elliptic in the sense of $\mathrm{char}_\sigma(a) = \emptyset$, that is,

$$|a(x,\xi)| \geqslant C\theta_\sigma(x,\xi)^m, \quad (x,\xi) \in \mathbf{R}^{2d} \setminus \mathrm{B}_R, \tag{3.15}$$

for $C, R > 0$, then there exists an elliptic symbol $b \in G^{-m,\sigma}$ such that

$$a\#b = 1 + r_1, \quad b\#a = 1 + r_2,$$

where $r_1, r_2 \in \mathscr{S}(\mathbf{R}^{2d})$.

The following definition concerns the anisotropic Gabor wave front set $\mathrm{WF}_\mathrm{g}^\sigma(u) \subseteq T^*\mathbf{R}^d \setminus 0$ of $u \in \mathscr{S}'(\mathbf{R}^d)$ [27, Definition 4.1] which is important in this paper. It is a closed and σ-conic subset of $T^*\mathbf{R}^d \setminus 0$ well adapted to the anisotropic Shubin calculus.

Definition 3.4. Suppose $u \in \mathscr{S}'(\mathbf{R}^d)$, $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$, and $\sigma > 0$. Then $z_0 = (x_0, \xi_0) \in T^*\mathbf{R}^d \setminus 0$ satisfies $z_0 \notin \mathrm{WF}_\mathrm{g}^\sigma(u)$ if there exists an open set $U \subseteq T^*\mathbf{R}^d$ such that $z_0 \in U$ and

$$\sup_{(x,\xi) \in U,\ \lambda > 0} \lambda^N |V_\varphi u(\lambda x, \lambda^\sigma \xi)| < +\infty \quad \forall N \geqslant 0. \tag{3.16}$$

If $\sigma = 1$ then $\mathrm{WF}_\mathrm{g}^\sigma(u) = \mathrm{WF}_\mathrm{g}(u)$ that denotes the usual Gabor wave front set [17,25], which is isotropic in phase space. The σ-conic sets are then ordinary cones in $T^*\mathbf{R}^d \setminus 0$, that is sets closed under multiplication with a positive parameter.

The definition of $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)$ does not depend on $\varphi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ [27, Proposition 4.2], and [27, Proposition 4.3 (i)] says that

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\widehat{u}) = \mathcal{J}\mathrm{WF}_{\mathrm{g}}^{\frac{1}{\sigma}}(u), \quad u \in \mathscr{S}'(\mathbf{R}^d), \tag{3.17}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \in \mathbf{R}^{2d \times 2d} \tag{3.18}$$

is the matrix that defines the symplectic group [10].

By [32, Proposition 3.5] we may express the anisotropic Gabor wave front set of $u \in \mathscr{S}'(\mathbf{R}^d)$ as

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(u) = \bigcap_{a \in G^{m,\sigma}:\ a^w(x,D)u \in \mathscr{S}} \mathrm{char}_\sigma(a) \tag{3.19}$$

for any $m \in \mathbf{R}$. For $\sigma > 0$, $m \in \mathbf{R}$, $a \in G^{m,\sigma}$ and $u \in \mathscr{S}'(\mathbf{R}^d)$ we have the microlocal and microelliptic inclusions

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(a^w(x,D)u) \subseteq \mathrm{WF}_{\mathrm{g}}^{\sigma}(u) \subseteq \mathrm{WF}_{\mathrm{g}}^{\sigma}(a^w(x,D)u) \bigcup \mathrm{char}_\sigma(a)$$

(cf. [27, Proposition 5.1 and Theorem 6.4] which are stated slightly more generally).

At a few occasions we will use anisotropic wave front sets in the Gelfand–Shilov functional framework. The Gelfand–Shilov wave front set of $u \in (\Sigma_\nu^\mu)'(\mathbf{R}^d)$ with $\nu + \mu > 1$ is based on the following facts. If $\varphi \in \Sigma_\nu^\mu(\mathbf{R}^d)\backslash 0$ then

$$|V_\varphi u(x,\xi)| \lesssim e^{r(|x|^{\frac{1}{\nu}} + |\xi|^{\frac{1}{\mu}})}$$

for some $r > 0$, and $u \in \Sigma_\nu^\mu(\mathbf{R}^d)$ if and only if

$$|V_\varphi u(x,\xi)| \lesssim e^{-r(|x|^{\frac{1}{\nu}} + |\xi|^{\frac{1}{\mu}})}$$

for all $r > 0$. See e.g., [30, Theorems 2.4 and 2.5]. The ν, μ-Gelfand–Shilov wave front set $\mathrm{WF}^{\nu,\mu}(u) \subseteq T^*\mathbf{R}^d\backslash 0$ is defined as follows.

Definition 3.5. Let $\nu, \mu > 0$ satisfy $\nu + \mu > 1$, and suppose $\varphi \in \Sigma_\nu^\mu(\mathbf{R}^d)\backslash 0$ and $u \in (\Sigma_\nu^\mu)'(\mathbf{R}^d)$. Then $(x_0, \xi_0) \in T^*\mathbf{R}^d\backslash 0$ satisfies $(x_0, \xi_0) \notin \mathrm{WF}^{\nu,\mu}(u)$ if there exists an open set $U \subseteq T^*\mathbf{R}^d \setminus 0$ containing (x_0, ξ_0) such that

$$\sup_{\lambda > 0,\ (x,\xi) \in U} e^{r\lambda} |V_\varphi u(\lambda^\nu x, \lambda^\mu \xi)| < \infty, \quad \forall r > 0.$$

The requested decay is thus exponential rather than superpolynomial as for $\mathrm{WF}_{\mathrm{g}}^{\sigma}$. The ν, μ-Gelfand–Shilov wave front set is a closed and μ/ν-conic subset of $T^*\mathbf{R}^d \setminus 0$ [26].

The next result identifies powers of the weight $w_{k,m}$ defined in (3.3) as anisotropic Shubin symbols.

Lemma 3.6. *Let $k, m \in \mathbf{N} \setminus 0$ and $\sigma = \frac{k}{m}$. If $n \in \mathbf{R}$ then $w_{k,m}^n \in G^{nk,\sigma}$.*

Proof. To simplify notation we write $w = w_{k,m}$. It is clear that $w \in C^\infty(\mathbf{R}^{2d})$ and that w is positive everywhere. We claim that for $\alpha, \beta \in \mathbf{N}^d$ we can write

$$\partial_x^\alpha \partial_\xi^\beta w^n(x,\xi) = \left(w(x,\xi)^2\right)^{\frac{n}{2}-|\alpha+\beta|} p_{\alpha,\beta}(w,x,\xi) \tag{3.20}$$

where $p_{\alpha,\beta}$ are polynomials of the form

$$p_{\alpha,\beta}(w,x,\xi) = \sum_{2j+\frac{|\gamma|}{k}+\frac{|\kappa|}{m} \leqslant \left(2-\frac{1}{k}\right)|\alpha|+\left(2-\frac{1}{m}\right)|\beta|} c_{j,\gamma,\kappa} w^{2j} x^\gamma \xi^\kappa \tag{3.21}$$

with real coefficients $c_{j,\gamma,\kappa}$ for $(j,\gamma,\kappa) \in \mathbf{N} \times \mathbf{N}^d \times \mathbf{N}^d$.

In fact the claim follows from an induction argument with respect to $|\alpha+\beta|$, starting with

$$\partial_{x_\ell}\left(\left(w(x,\xi)^2\right)^{\frac{n}{2}}\right) = nk|x|^{2(k-1)} x_\ell \left(w(x,\xi)^2\right)^{\frac{n}{2}-1}$$

and

$$\partial_{\xi_\ell}\left(\left(w(x,\xi)^2\right)^{\frac{n}{2}}\right) = nm|\xi|^{2(m-1)} \xi_\ell \left(w(x,\xi)^2\right)^{\frac{n}{2}-1}$$

for $1 \leqslant \ell \leqslant d$.

Next we estimate a generic monomial in (3.21), using $2j + \frac{|\gamma|}{k} + \frac{|\kappa|}{m} \leqslant \left(2-\frac{1}{k}\right)|\alpha| + \left(2-\frac{1}{m}\right)|\beta|$, as

$$\left|w^{2j} x^\gamma \xi^\kappa\right| \leqslant w(x,\xi)^{2j+\frac{|\gamma|}{k}+\frac{|\kappa|}{m}} \leqslant w(x,\xi)^{\left(2-\frac{1}{k}\right)|\alpha|+\left(2-\frac{1}{m}\right)|\beta|}.$$

Inserting into (3.20) and exploiting (3.4) finally give for any $\alpha, \beta \in \mathbf{N}^d$ the estimate

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta w^n(x,\xi)\right| &= w(x,\xi)^{n-2|\alpha+\beta|} \left|p_{\alpha,\beta}(w,x,\xi)\right| \\ &\lesssim w(x,\xi)^{n-\frac{1}{k}|\alpha|-\frac{1}{m}|\beta|} \\ &\asymp \theta_\sigma(x,\xi)^{nk-|\alpha|-\sigma|\beta|}.\end{aligned}$$

□

Suppose $\sigma > 0$ is rational, that is $\sigma = \frac{k}{m}$ with $k, m \in \mathbf{N} \setminus 0$. In [7, Proposition 4.2] the authors identify the symbol class $a \in G^{n,\sigma}$ with $n \in \mathbf{R}$ as the Weyl–Hörmander symbol class [16, Chapter 18.4]

$$G^{n,\sigma} = S(h_g^{-\frac{n}{1+\sigma}}, g) \tag{3.22}$$

defined by the metric

$$g = \frac{\mathrm{d}x^2}{\left(1+|x|^{2k}+|\xi|^{2m}\right)^{\frac{1}{k}}} + \frac{\mathrm{d}\xi^2}{\left(1+|x|^{2k}+|\xi|^{2m}\right)^{\frac{1}{m}}}$$

and the weight $h_g^{-\frac{n}{1+\sigma}}$. Here h_g is the so called Planck function associated to g [7,16]. The Planck function is according to [7, Remark 2.4]

$$h_g(x,\xi) = \left(1 + |x|^{2k} + |\xi|^{2m}\right)^{-\frac{1}{2}\left(\frac{1}{k}+\frac{1}{m}\right)} = \left(w_{k,m}(x,\xi)\right)^{-\left(\frac{1}{k}+\frac{1}{m}\right)}. \tag{3.23}$$

From this two conclusions follows: First we observe that h_g satisfies the so-called uncertainty principle

$$h_g(x,\xi) \leqslant 1 \quad \forall (x,\xi) \in T^*\mathbf{R}^d, \tag{3.24}$$

and secondly by Lemma 3.6 we have $h_g \in G^{-1-\sigma,\sigma}$.

4. Globally anisotropic Shubin–Sobolev spaces, localization operators and a sharp Gårding inequality

In this paper we will often use the following parametrized family of Hilbert modulation spaces. These spaces also have an independent interest. Proposition 4.2 complements the anisotropic Shubin pseudodifferential calculus in [27].

Definition 4.1. Let $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$. The anisotropic Shubin–Sobolev modulation space $M_{\sigma,s}(\mathbf{R}^d)$ with anisotropy parameter $\sigma > 0$ and order $s \in \mathbf{R}$ is the Hilbert subspace of $\mathscr{S}'(\mathbf{R}^d)$ defined by the norm

$$\|u\|_{M_{\sigma,s}} = \left(\iint_{\mathbf{R}^{2d}} |V_\varphi u(x,\xi)|^2 \, \theta_\sigma(x,\xi)^{2s} \, \mathrm{d}x \, \mathrm{d}\xi \right)^{\frac{1}{2}}. \tag{4.1}$$

For any $\sigma > 0$ we have $M_{\sigma,0}(\mathbf{R}^d) = L^2(\mathbf{R}^d)$ [13], and $M_{\sigma,s_1}(\mathbf{R}^d) \subseteq M_{\sigma,s_2}(\mathbf{R}^d)$ is a continuous inclusion when $s_1 \geqslant s_2$. It holds

$$\mathscr{S}(\mathbf{R}^d) = \bigcap_{s\in\mathbf{R}} M_{\sigma,s}(\mathbf{R}^d), \quad \mathscr{S}'(\mathbf{R}^d) = \bigcup_{s\in\mathbf{R}} M_{\sigma,s}(\mathbf{R}^d), \tag{4.2}$$

and $\{\| \cdot \|_{M_{\sigma,s}}, s \geqslant 0\}$ is a family of seminorms that defines the Fréchet space topology on $\mathscr{S}(\mathbf{R}^d)$ [13].

The next continuity result is a natural generalization of the isotropic Shubin calculus. More precisely it generalizes [20, Proposition 1.5.5] and [28, Theorem 25.2].

Proposition 4.2. *Let $\sigma > 0$ and $m, s \in \mathbf{R}$. If $a \in G^{m,\sigma}$ then*

$$a^w(x,D) : M_{\sigma,s+m}(\mathbf{R}^d) \to M_{\sigma,s}(\mathbf{R}^d) \tag{4.3}$$

is continuous.

Proof. By a small modification of the proof of [4, Proposition 3.2] it follows that $a \in G^{m,\sigma}$ if and only if

$$\left|\partial_{z_1}^\alpha \partial_{z_2}^\beta \mathcal{T}_\varphi a(z,\zeta)\right| \lesssim \theta_\sigma(z)^{m-|\alpha|-\sigma|\beta|}\langle\zeta\rangle^{-N}, \quad z,\zeta\in\mathbf{R}^{2d}, \quad \alpha,\beta\in\mathbf{N}^d, \quad N\geqslant 0, \tag{4.4}$$

where $z=(z_1,z_2)$ with $z_1,z_2\in\mathbf{R}^d$, $g\in\mathscr{S}(\mathbf{R}^{2d})\backslash 0$, and where $\mathcal{T}_\varphi u$ is defined by

$$\mathcal{T}_\varphi u(x,\xi) = (2\pi)^{-\frac{d}{2}}(u, T_x M_\xi \varphi) = e^{i\langle x,\xi\rangle} V_\varphi u(x,\xi), \quad x,\xi\in\mathbf{R}^d,$$

for $u\in\mathscr{S}'(\mathbf{R}^d)$ and $\varphi\in\mathscr{S}(\mathbf{R}^d)\setminus 0$. In fact in the original proof [4] we only have to replace the weight $\langle\cdot\rangle$ used there by θ_σ, take into account the behavior with respect to derivatives of $a\in G^{m,\sigma}$ with respect to z_1 and z_2 respectively, and use (3.2).

Let $\varphi\in\mathscr{S}(\mathbf{R}^d)\setminus 0$ and set $\Phi = W(\varphi,\varphi)\in\mathscr{S}(\mathbf{R}^{2d})\backslash 0$. If $u\in\mathscr{S}'(\mathbf{R}^d)$ then by [27, Eq. (5.3)] we have

$$|V_\varphi(a^w(x,D)u)(z)| \lesssim \int_{\mathbf{R}^{2d}} |V_\varphi u(z-w)| \left|V_\Phi a\left(z-\frac{w}{2},\mathcal{J}w\right)\right| \mathrm{d}w. \tag{4.5}$$

We obtain from (4.4), (3.2) and (3.5) the estimates

$$\begin{aligned}\left|V_\Phi a\left(z-\frac{w}{2},\mathcal{J}w\right)\right| &\lesssim \theta_\sigma(z-w)^m\theta_\sigma(w)^{|m|}\langle w\rangle^{-N}\\ &\lesssim \theta_\sigma(z-w)^m\langle w\rangle^{-\left(N-|m|\max(1,\frac{1}{\sigma})\right)}, \quad z,w\in\mathbf{R}^{2d}, \quad N\geqslant 0.\end{aligned}$$

Combining this with (4.5), Minkowski's inequality and again (3.2) yields

$$\begin{aligned}\|a^w(x,D)u\|_{M_{\sigma,s}} &= \left\|V_\varphi(a^w(x,D)u)\,\theta_\sigma^s\right\|_{L^2(\mathbf{R}^{2d})}\\ &\lesssim \left\|\int_{\mathbf{R}^{2d}} |V_\varphi u(\cdot-w)| \left|V_\Phi a\left(\cdot-\frac{w}{2},\mathcal{J}w\right)\right| \mathrm{d}w\,\theta_\sigma(\cdot)^s\right\|_{L^2(\mathbf{R}^{2d})}\\ &\lesssim \left\|\int_{\mathbf{R}^{2d}} |V_\varphi u(\cdot-w)|\,\theta_\sigma(\cdot-w)^{m+s}\langle w\rangle^{-\left(N-(|m|+|s|)\max(1,\frac{1}{\sigma})\right)}\mathrm{d}w\right\|_{L^2(\mathbf{R}^{2d})}\\ &\lesssim \left\|V_\varphi u\,\theta_\sigma^{m+s}\right\|_{L^2(\mathbf{R}^{2d})}\\ &= \|u\|_{M_{\sigma,m+s}}\end{aligned}$$

provided $N\geqslant 0$ is sufficiently large. $\square$

Let $\varphi\in\mathscr{S}(\mathbf{R}^d)$ satisfy $\|\varphi\|_{L^2}=1$. A localization operator A_a with symbol $a\in\mathscr{S}'(\mathbf{R}^{2d})$ is defined as

$$(A_a f, g) = (aV_\varphi f, V_\varphi g) = (V_\varphi^* a V_\varphi f, g), \quad f,g\in\mathscr{S}(\mathbf{R}^d), \tag{4.6}$$

that is $A_a = V_\varphi^* a V_\varphi$. Then $A_a:\mathscr{S}(\mathbf{R}^d)\to\mathscr{S}'(\mathbf{R}^d)$ is continuous. We will assume that φ is a Gaussian.

By [14, Theorem 1.1] we have for any $\sigma>0$ and $s\in\mathbf{R}$

$$\|A_{\theta_\sigma^s}u\|_{L^2(\mathbf{R}^d)} \asymp \|u\|_{M_{\sigma,s}(\mathbf{R}^d)} \tag{4.7}$$

which means that $A_{\theta_\sigma^s}: M_{\sigma,s}(\mathbf{R}^d)\to L^2(\mathbf{R}^d)$ is an isometry.

If $a\in\mathscr{S}'(\mathbf{R}^{2d})$ and φ is a Gaussian on $\mathbf{R}^d$ we have $A_a = b^w(x,D)$ where $b=a*\psi$ with ψ a Gaussian on $\mathbf{R}^{2d}$ [20, Proposition 1.7.9]. If $\sigma>0$ and $a\in G^{m,\sigma}$ then also $b\in G^{m,\sigma}$, and a real-valued implies that also b is real-valued [20, Theorem 1.7.10].

In Sect. 7 we will need the following inequality of sharp Gårding type.

Lemma 4.3. *Let $k, m \in \mathbf{N} \setminus 0$ and $\sigma = \frac{k}{m}$. If $a \in G^{2(1+\sigma),\sigma}$ and $a \geqslant 0$ then there exists $c > 0$ such that*

$$(a^w(x, D) f, f) \geqslant -c\|f\|_{L^2}^2, \quad f \in \mathscr{S}(\mathbf{R}^d). \tag{4.8}$$

Proof. By (3.22) we have $G^{2(1+\sigma),\sigma} = S(h_g^{-2}, g)$, where the Planck function h_g is defined by (3.23) and satisfies the uncertainty principle (3.24). The conclusion is now a consequence of the Fefferman–Phong inequality [16, Theorem 18.6.8]. □

5. Propagation of anisotropic Gabor wave front sets for evolution equations of Airy–Schrödinger type

In this section we consider the evolution equation

$$\begin{cases} \partial_t u(t,x) + i\,(p(D_x) + \langle v, x\rangle)\, u(t,x) = 0, \quad x \in \mathbf{R}^d, \quad t \in \mathbf{R}, \\ \qquad\qquad\qquad\qquad\qquad u(0,\cdot) = u_0. \end{cases} \tag{5.1}$$

Here $v = (v_1, \ldots, v_d) \in \mathbf{R}^d$ is a vector with nonzero entries: $v_j \neq 0$, $1 \leqslant j \leqslant d$, and $p : \mathbf{R}^d \to \mathbf{R}$ is a polynomial with real coefficients of order $m \geqslant 2$ which is a sum of one variable polynomials, that is

$$p(\xi) = \sum_{j=1}^{d} p_j(\xi_j), \quad \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbf{R}^d, \tag{5.2}$$

where

$$p_j(\xi_j) = \sum_{k=0}^{m_j} c_{j,k}\xi_j^k, \quad c_{j,k} \in \mathbf{R}, \quad c_{j,m_j} \neq 0, \tag{5.3}$$

and $\max_{j=1}^d \deg p_j = \max_{j=1}^d m_j = m$. The principal part of p is

$$P_m(\xi) = \sum_{j\in\{1,\ldots,d\}:\ m_j=m} c_{j,m}\xi_j^m. \tag{5.4}$$

We say that the equation (5.1) is of Airy–Schrödinger type, since when $d = 1$ a particular case of the Hamiltonian is the operator

$$a(x, D) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x$$

which defines the Airy equation $a(x, D) f = 0$. This equation is satisfied by the Airy function [16, Chapter 7.6].

First we deduce the explicit solution $u(t, x) = \mathscr{K}_t u_0(x)$ to (5.1) defined by the propagator $\mathscr{K}_t$, and in particular an expression for the Schwartz kernel K_t of $\mathscr{K}_t$ for each $t \in \mathbf{R}$. Let q_j be primitive polynomials of p_j:

$$q_j' = p_j, \quad 1 \leqslant j \leqslant d. \tag{5.5}$$

If $u_0 \in \mathscr{S}(\mathbf{R}^d)$ then the solution to (5.1) is given by

$$\begin{aligned} u(t,x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{i\left(\langle x, \xi - vt\rangle + \sum_{j=1}^d v_j^{-1}(q_j(\xi_j - tv_j) - q_j(\xi_j))\right)} \widehat{u}_0(\xi) \mathrm{d}\xi \\ &= e^{-it\langle x, v\rangle} \mathscr{F}^{-1}\left(e^{i\varphi_t}\widehat{u}_0\right)(x) = M_{-tv}\mathscr{F}^{-1}\left(e^{i\varphi_t}\widehat{u}_0\right)(x) \end{aligned} \tag{5.6}$$

where

$$\varphi_t(\xi) = \sum_{j=1}^d v_j^{-1}\left(q_j(\xi_j - tv_j) - q_j(\xi_j)\right). \tag{5.7}$$

This can be confirmed by insertion of (5.6) into (5.1).

The solution operator

$$\mathscr{K}_t f = M_{-tv}\mathscr{F}^{-1}\left(e^{i\varphi_t}\widehat{f}\right) \tag{5.8}$$

is unitary on $L^2(\mathbf{R}^d)$, and since $\varphi_{-t}(\xi) = -\varphi_t(\xi + tv)$ we obtain for $f, g \in \mathscr{S}(\mathbf{R}^d)$

$$\begin{aligned} (\mathscr{K}_t f, g) &= \left(\widehat{f}, e^{-i\varphi_t}\mathscr{F}(M_{tv}g)\right) = \left(\widehat{f}, e^{-i\varphi_t}T_{tv}\widehat{g}\right) = \left(\widehat{f}, T_{tv}\left(e^{-i\varphi_t(\cdot+tv)}\widehat{g}\right)\right) \\ &= \left(f, \mathscr{F}^{-1}\left(T_{tv}\left(e^{i\varphi_{-t}}\widehat{g}\right)\right)\right) = \left(f, M_{tv}\mathscr{F}^{-1}\left(e^{i\varphi_{-t}}\widehat{g}\right)\right) = (f, \mathscr{K}_{-t}g) \end{aligned}$$

so $\mathscr{K}_t^* = \mathscr{K}_{-t} = \mathscr{K}_t^{-1}$. If $t_1, t_2 \in \mathbf{R}$ then

$$\varphi_{t_1}(\xi - t_2 v) + \varphi_{t_2}(\xi) = \varphi_{t_1+t_2}(\xi)$$

which gives

$$\begin{aligned} \mathscr{K}_{t_1}\mathscr{K}_{t_2} f &= M_{-t_1 v}\mathscr{F}^{-1}\left(e^{i\varphi_{t_1}}\mathscr{F}\left(M_{-t_2 v}\mathscr{F}^{-1}\left(e^{i\varphi_{t_2}}\widehat{f}\right)\right)\right) \\ &= M_{-t_1 v}\mathscr{F}^{-1}\left(e^{i\varphi_{t_1}}T_{-t_2 v}\left(e^{i\varphi_{t_2}}\widehat{f}\right)\right) \\ &= M_{-t_1 v}\mathscr{F}^{-1}\left(T_{-t_2 v}\left(e^{i\left(\varphi_{t_1}(\cdot - t_2 v) + \varphi_{t_2}\right)}\widehat{f}\right)\right) \\ &= M_{-(t_1+t_2)v}\mathscr{F}^{-1}\left(e^{i\varphi_{t_1+t_2}}\widehat{f}\right) = \mathscr{K}_{t_1+t_2} f \end{aligned}$$

so the map $\mathbf{R} \ni t \mapsto \mathscr{K}_t$ is in fact a one-parameter group of unitary operators.

The Schwartz kernel of the solution operator $\mathscr{K}_t$ is

$$\begin{aligned} K_t(x,y) &= (2\pi)^{-d}\int_{\mathbf{R}^d} e^{i(\langle x, \xi - vt\rangle - \langle y, \xi\rangle + \varphi_t(\xi))}\mathrm{d}\xi \\ &= (2\pi)^{-\frac{d}{2}} e^{-it\langle x, v\rangle}\mathscr{F}^{-1}\left(e^{i\varphi_t}\right)(x-y) \\ &= (2\pi)^{-\frac{d}{2}} e^{-it\langle x, v\rangle}\left(1 \otimes \mathscr{F}^{-1}e^{i\varphi_t}\right) \circ \kappa^{-1}(x,y) \in \mathscr{S}'(\mathbf{R}^{2d}) \end{aligned} \tag{5.9}$$

where $\kappa \in \mathbf{R}^{2d\times 2d}$ is the matrix defined by $\kappa(x,y) = (x+\frac{y}{2}, x-\frac{y}{2})$ for $x, y \in \mathbf{R}^d$. We note that $\mathscr{K}_t$ acts continuously on $\mathscr{S}(\mathbf{R}^d)$ for any $t \in \mathbf{R}$, and extends uniquely to a continuous linear operator on $\mathscr{S}'(\mathbf{R}^d)$ by

$$(\mathscr{K}_t u, \varphi) := (u, \mathscr{K}_{-t}\varphi), \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \varphi \in \mathscr{S}(\mathbf{R}^d).$$

For each $1 \leqslant j \leqslant d$ we have

$$\begin{aligned} q_j(\xi_j - tv_j) - q_j(\xi_j) &= \sum_{k=0}^{m_j} \frac{c_{j,k}}{k+1}\left((\xi_j - tv_j)^{k+1} - \xi_j^{k+1}\right) \\ &= \sum_{k=0}^{m_j}\left(-c_{j,k} t v_j \xi_j^k + \frac{c_{j,k}}{k+1}\sum_{n=2}^{k+1}\binom{k+1}{n}(-tv_j)^n \xi_j^{k+1-n}\right). \end{aligned}$$

Hence the phase function $\varphi_t(\xi)$ is a polynomial of order m with highest order term

$$\varphi_{t,m}(\xi) = -t \sum_{j\in\{1,\ldots,d\}:\, m_j=m} c_{j,m}\xi_j^m = -t P_m(\xi).$$

We may now give a result which generalizes a particular case of [32, Theorem 5.1]. More precisely, in the quoted result the polynomial p is arbitrary with real coefficients, whereas here we assume the particular "separable" form (5.2). On the other hand, in Theorem 5.1 below we allow a vector $v \in \mathbf{R}^d$ with nonzero entries. The result uses the Hamilton flow corresponding to the principal part $P_m(\xi)$ of the polynomial $p(\xi)$, that is

$$\chi_t(x,\xi) = (x + t\nabla P_m(\xi), \xi), \quad t \in \mathbf{R}, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0. \tag{5.10}$$

Theorem 5.1. *Let p be a polynomial with real coefficients defined by (5.2), (5.3), of order $m = \max_{j=1}^d \deg p_j \geqslant 2$, with principal part P_m defined by (5.4). Denote the Hamilton flow of $P_m(\xi)$ as in (5.10). Suppose $\mathscr{K}_t : \mathscr{S}'(\mathbf{R}^d) \to \mathscr{S}'(\mathbf{R}^d)$ is the solution operator for the evolution equation (5.1), with Schwartz kernel (5.9) where φ_t is defined by (5.5) and (5.7). Then*

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u) = \chi_t\left(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)\right), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma = \frac{1}{m-1}, \tag{5.11}$$

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u) = \mathrm{WF}_{\mathrm{g}}^{\sigma}(u), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}), \quad \sigma < \frac{1}{m-1}. \tag{5.12}$$

Proof. By [27, Theorems 7.1 and 7.2] we have

$$\begin{aligned} \mathrm{WF}_{\mathrm{g}}^{m-1}\left(e^{i\varphi_t}\right) &\subseteq \{\big(x, \nabla\varphi_{t,m}(x)\big) \in \mathbf{R}^{2d} : x \in \mathbf{R}^d \setminus 0\} \\ &= \{(x, -t\nabla P_m(x)) \in \mathbf{R}^{2d} : x \in \mathbf{R}^d \setminus 0\}, \qquad (5.13) \\ \mathrm{WF}_{\mathrm{g}}^{\sigma}\left(e^{i\varphi_t}\right) &\subseteq \left(\mathbf{R}^d \setminus 0\right) \times \{0\}, \quad \sigma > m-1. \end{aligned}$$

Combining (5.13) with [27, Eq. (4.6) and Proposition 4.3 (i)], cf. (3.17), gives

$$\begin{aligned} \mathrm{WF}_{\mathrm{g}}^{\frac{1}{m-1}}\left(\mathscr{F}^{-1}e^{i\varphi_t}\right) &\subseteq \{(t\nabla P_m(x), x)) \in \mathbf{R}^{2d} : x \in \mathbf{R}^d \setminus 0\}, \\ \mathrm{WF}_{\mathrm{g}}^{\sigma}\left(\mathscr{F}^{-1}e^{i\varphi_t}\right) &\subseteq \{0\} \times \left(\mathbf{R}^d \setminus 0\right), \quad \sigma < \frac{1}{m-1}. \end{aligned} \tag{5.14}$$

Now (5.9), [27, Corollary 5.2 and Proposition 4.3 (ii)], [32, Proposition 3.2], [27, Proposition 5.3 (iii)] and (5.14) yield if $\sigma = \frac{1}{m-1}$

$$\begin{aligned} &\mathrm{WF}_{\mathrm{g}}^{\sigma}(K_t) = \mathrm{WF}_{\mathrm{g}}^{\sigma}(\left(1 \otimes \mathscr{F}^{-1}e^{i\varphi_t}\right) \circ \kappa^{-1}) \\ &= \begin{pmatrix} \kappa & 0 \\ 0 & \kappa^{-T} \end{pmatrix} \mathrm{WF}_{\mathrm{g}}^{\sigma}\left(1 \otimes \mathscr{F}^{-1}e^{i\varphi_t}\right) \\ &\subseteq \{(\kappa(x_1, x_2), \kappa^{-T}(\xi_1, \xi_2)) \in T^*\mathbf{R}^{2d} : \\ &\qquad (x_1, \xi_1) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(1) \cup \{0\},\ (x_2, \xi_2) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{F}^{-1}e^{i\varphi_t}) \cup \{0\}\} \setminus 0 \\ &= \{(\kappa(x_1, t\nabla P_m(x_2)), \kappa^{-T}(0, x_2) \in T^*\mathbf{R}^{2d} :\ x_1, x_2 \in \mathbf{R}^d\} \setminus 0 \\ &= \left\{\left(x_1 + t\frac{1}{2}\nabla P_m(x_2), x_1 - t\frac{1}{2}\nabla P_m(x_2), x_2, -x_2\right) \in T^*\mathbf{R}^{2d} :\ x_1, x_2 \in \mathbf{R}^d\right\} \setminus 0 \\ &= \left\{(x_1 + t\nabla P_m(x_2), x_1, x_2, -x_2) \in T^*\mathbf{R}^{2d} :\ x_1, x_2 \in \mathbf{R}^d\right\} \setminus 0. \end{aligned}$$

Since $m \geqslant 2$ we have $\nabla P_m(0) = 0$. Hence $\mathrm{WF}_{\mathrm{g}}^{\sigma}(K_t)$ does not contain points of the form $(x, 0, \xi, 0)$ nor of the form $(0, x, 0, -\xi)$ for any $(x, \xi) \in T^*\mathbf{R}^d \setminus 0$. We may therefore apply [32, Theorem 4.4] which gives for $u \in \mathscr{S}'(\mathbf{R}^d)$

$$\begin{aligned} \mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u) \subseteq &\mathrm{WF}_{\mathrm{g}}^{\sigma}(K_t)' \circ \mathrm{WF}_{\mathrm{g}}^{\sigma}(u) \\ = &\{(x, \xi) \in T^*\mathbf{R}^d :\ \exists (y, \eta) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(u),\ (x, y, \xi, -\eta) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(K_t)\} \\ \subseteq &\{(x_1 + t\nabla P_m(x_2), x_2) \in T^*\mathbf{R}^d :\ (x_1, x_2) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(u)\} \\ = &\chi_t\left(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)\right). \end{aligned} \tag{5.15}$$

Since $\mathscr{K}_t^{-1} = \mathscr{K}_{-t}$ and $\chi_t^{-1} = \chi_{-t}$ we may strengthen (5.15) into

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u) = \chi_t\left(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)\right), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma = \frac{1}{m-1}.$$

We have proved (5.11).

Likewise if $\sigma < \frac{1}{m-1}$ then again (5.9), [27, Corollary 5.2 and Proposition 4.3 (ii)], [32, Proposition 3.2], [27, Proposition 5.3 (iii)] and (5.14) yield

$$\begin{aligned} \mathrm{WF}_{\mathrm{g}}^{\sigma}(K_t) &\subseteq \{(\kappa(x_1, x_2), \kappa^{-T}(\xi_1, \xi_2)) \in T^*\mathbf{R}^{2d} : \\ &\qquad (x_1, \xi_1) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(1) \cup \{0\},\ (x_2, \xi_2) \in \mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{F}^{-1}e^{i\varphi_t}) \cup \{0\}\} \setminus 0 \\ &\subseteq \{(\kappa(x_1, 0), \kappa^{-T}(0, x_2) \in T^*\mathbf{R}^{2d} :\ x_1, x_2 \in \mathbf{R}^d\} \setminus 0 \\ &= \left\{(x_1, x_1, x_2, -x_2) \in T^*\mathbf{R}^{2d} :\ x_1, x_2 \in \mathbf{R}^d\right\} \setminus 0. \end{aligned}$$

Again [32, Theorem 4.4] gives

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u) = \mathrm{WF}_{\mathrm{g}}^{\sigma}(u), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma < \frac{1}{m-1},$$

which proves (5.12). □

Remark 5.2. The conclusion from (5.11) and (5.12) is that the propagation of singularities for the equation (5.1) works exactly as when $v = 0$, as described in [32, Theorem 5.1]. The Hamiltonian in (5.1) is $a(x,\xi) = p(\xi) + \langle v, x\rangle$, but the propagation of singularities follows the Hamiltonian flow of $P_m(\xi)$. Note that $a_0(x,\xi) = P_m(\xi)$ satisfies the anisotropic homogeneity

$$a_0(\lambda x, \lambda^\sigma \xi) = \lambda^{1+\sigma} a_0(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \lambda > 0,$$

if $\sigma = \frac{1}{m-1}$, so $a_0 \in G^{1+\sigma,\sigma}$ according to Corollary 3.3.

If we decompose the polynomial p as

$$p(\xi) = P_m(\xi) + \sum_{j=0}^{m-1} P_j(\xi)$$

where each term $P_j(\xi)$ is homogeneous of degree j for $0 \leqslant j \leqslant m$, then each term P_j satisfies

$$P_j(\lambda^\sigma \xi) = \lambda^{\frac{j}{m-1}} P_j(\xi), \quad \xi \in \mathbf{R}^d, \quad \lambda > 0, \quad 0 \leqslant j \leqslant m.$$

Thus $a - a_0 = \sum_{j=0}^{m-1} P_j + \langle v, x\rangle = \sum_{j=0}^{m-1} b_j$ with terms b_j, considered as functions on $(x,\xi) \in T^*\mathbf{R}^d$, of homogeneities

$$b_j(\lambda x, \lambda^\sigma \xi) = \lambda^{\frac{j}{m-1}} b_j(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \lambda > 0, \quad 0 \leqslant j \leqslant m-1. \tag{5.16}$$

The terms b_j have all smaller order $\frac{j}{m-1} = j\sigma$ of anisotropic homogeneity than the principal part $a_0 = P_m$ which has order $1 + \sigma = m\sigma$, and which governs the propagation of singularities. Thus one may see $a - a_0$ as lower order perturbations of the Hamiltonian that do not affect propagation of singularities. Note that the Hamiltonian term $\langle v, x\rangle$ satisfies (5.16) with $j = m-1$.

As a complementary result we formulate a version of Theorem 5.1 in the framework of Beurling type Gelfand–Shilov spaces $\Sigma_\nu^\mu(\mathbf{R}^d)$ for $\nu + \mu > 1$ and their dual ultradistribution spaces $\left(\Sigma_\nu^\mu\right)'(\mathbf{R}^d)$.

Theorem 5.3. *Let p be a polynomial with real coefficients defined by (5.2), (5.3), of order $m = \max_{j=1}^d \deg p_j \geqslant 2$, with principal part P_m defined by (5.4). Denote the Hamilton flow of $P_m(\xi)$ as in (5.10). Suppose $\mathscr{K}_t : \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d)$ is the solution operator for the evolution equation (5.1), with Schwartz kernel (5.9) where φ_t is defined by (5.5) and (5.7).*

If $\nu \geqslant \mu(m-1) > 1$ then $\mathscr{K}_t$ is continuous on $\Sigma_\nu^\mu(\mathbf{R}^d)$, extends uniquely to a continuous linear operator on $\left(\Sigma_\nu^\mu\right)'(\mathbf{R}^d)$, and

$$\mathrm{WF}^{\nu,\mu}(\mathscr{K}_t u) = \chi_t\left(\mathrm{WF}^{\nu,\mu}(u)\right), \quad t \in \mathbf{R}, \quad u \in \left(\Sigma_\nu^\mu\right)'(\mathbf{R}^d), \quad \nu = \mu(m-1) > 1, \tag{5.17}$$

$$\mathrm{WF}^{\nu,\mu}(\mathscr{K}_t u) = \mathrm{WF}^{\nu,\mu}(u), \quad t \in \mathbf{R}, \quad u \in \left(\Sigma_\nu^\mu\right)'(\mathbf{R}^d), \quad \nu > \mu(m-1) > 1. \tag{5.18}$$

Proof. In the Gelfand–Shilov functional framework we have, similar to (5.13), by [31, Theorems 6.1 and 6.2] if $\nu > \frac{1}{m-1}$

$$\begin{aligned} \mathrm{WF}^{\nu,\nu(m-1)}\left(e^{i\varphi_t}\right) &\subseteq \{(x, -t\nabla P_m(x)) \in \mathbf{R}^{2d} : x \in \mathbf{R}^d \setminus 0\}, \\ \mathrm{WF}^{\nu,\mu}\left(e^{i\varphi_t}\right) &\subseteq \left(\mathbf{R}^d \setminus 0\right) \times \{0\}, \quad \mu > \nu(m-1). \end{aligned} \tag{5.19}$$

As before [26, Eq. (3.8) and Proposition 3.6 (i)] give

$$\begin{aligned} \mathrm{WF}^{\nu(m-1),\nu}\left(\mathscr{F}^{-1}e^{i\varphi_t}\right) &\subseteq \{(t\nabla P_m(x), x) \in \mathbf{R}^{2d} : x \in \mathbf{R}^d \setminus 0\}, \\ \mathrm{WF}^{\mu,\nu}\left(\mathscr{F}^{-1}e^{i\varphi_t}\right) &\subseteq \{0\} \times \left(\mathbf{R}^d \setminus 0\right), \quad \mu > \nu(m-1). \end{aligned} \tag{5.20}$$

From [31, Proposition 4.5] and [26, Proposition 3.6 (ii), Corollary 6.4 and Proposition 7.1 (iii)] we obtain if $\nu = \mu(m-1) > 1$

$$\mathrm{WF}^{\nu,\mu}(K_t) \subseteq \left\{(x_1 + t\nabla P_m(x_2), x_1, x_2, -x_2) \in T^*\mathbf{R}^{2d} : \ x_1, x_2 \in \mathbf{R}^d\right\} \setminus 0,$$

and if $\nu > \mu(m-1) > 1$

$$\mathrm{WF}^{\nu,\mu}(K_t) \subseteq \left\{(x_1, x_1, x_2, -x_2) \in T^*\mathbf{R}^{2d} : \ x_1, x_2 \in \mathbf{R}^d\right\} \setminus 0.$$

At this point [31, Theorem 5.5] yields the following two final conclusions: If $\nu \geqslant \mu(m-1) > 1$ then $\mathscr{K}_t$ is continuous on $\Sigma_\nu^\mu(\mathbf{R}^d)$ and extends uniquely to a continuous linear operator on $\left(\Sigma_\nu^\mu\right)'(\mathbf{R}^d)$, and the propagation of singularities follows (5.17) and (5.18). $\square$

Again the overall conclusion is that propagation of singularities works as if $v = 0$.

5.1. Fourier transformation of the evolution equation

Next we take the Fourier transform $\mathscr{F}u(t,\cdot)$. If we denote this Fourier transform for simplicity still by $u(t,\cdot)$, then we obtain from (5.1) the evolution equation

$$\begin{cases} \partial_t u(t,x) + i\left(-\langle v, D_x\rangle + p(x)\right)u(t,x) = 0, \quad x \in \mathbf{R}^d, \quad t \in \mathbf{R}, \\ \hfill u(0,\cdot) = u_0. \end{cases} \tag{5.21}$$

where again $v \in \mathbf{R}^d$ is a vector with nonzero entries: $v_j \neq 0$, $1 \leqslant j \leqslant d$.

Referring to (5.7) and (5.8) the solution is now for $u_0 \in \mathscr{S}(\mathbf{R}^d)$

$$\begin{aligned} u(t,x) = \widetilde{\mathscr{K}_t} u_0 = \mathscr{F} \mathscr{K}_t \mathscr{F}^{-1} u_0 \quad &= T_{-tv}\left(e^{i\varphi_t} u_0\right)(x) \\ &= e^{i\varphi_t(x+tv)} u_0(x+tv) \\ &= e^{-i\varphi_{-t}(x)} u_0(x+tv). \end{aligned} \tag{5.22}$$

The solution operator $\widetilde{\mathscr{K}_t}$ is continuous on $\mathscr{S}(\mathbf{R}^d)$ and extends to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$. Now (5.11) combined with [27, Proposition 4.3 (i)] give

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\widetilde{\mathscr{K}_t} u_0) = \widetilde{\chi}_t\left(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u_0)\right), \quad t \in \mathbf{R}, \quad u_0 \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma = m-1, \tag{5.23}$$

where

$$\widetilde{\chi}_t(x,\xi) = \mathcal{J}\chi_t(-\mathcal{J})(x,\xi) = (x, \xi - t\nabla P_m(x)), \quad t \in \mathbf{R}, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \tag{5.24}$$

and χ_t is defined by (5.10). This is the Hamilton flow corresponding to the principal part $P_m(x)$ of the polynomial $p(x)$. We also obtain

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\widetilde{\mathscr{K}_t} u_0) = \mathrm{WF}_{\mathrm{g}}^{\sigma}(u_0), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma > m-1. \tag{5.25}$$

These considerations, combined with a similar discussion in the Gelfand–Shilov framework, may be summarized as follows.

Theorem 5.4. *Let p be a polynomial with real coefficients defined by (5.2), (5.3), of order $m = \max_{j=1}^{d} \deg p_j \geqslant 2$, with principal part P_m defined by (5.4). Denote the Hamilton flow of $P_m(x)$ as in (5.24). Suppose $\widetilde{\mathscr{K}_t} : \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{R}^d)$ is the solution operator (5.22), where φ_t is defined by (5.5) and (5.7), for the evolution equation (5.21). Then*

$$\begin{aligned} &\mathrm{WF}_{\mathrm{g}}^{\sigma}(\widetilde{\mathscr{K}_t} u) = \widetilde{\chi}_t\left(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)\right), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma = m-1, \\ &\mathrm{WF}_{\mathrm{g}}^{\sigma}(\widetilde{\mathscr{K}_t} u) = \mathrm{WF}_{\mathrm{g}}^{\sigma}(u), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \quad \sigma > m-1. \end{aligned}$$

If $\nu \geqslant \mu(m-1) > 1$ then $\widetilde{\mathscr{K}_t}$ is continuous on $\Sigma_\mu^\nu(\mathbf{R}^d)$, extends uniquely to a continuous linear operator on $\left(\Sigma_\mu^\nu\right)'(\mathbf{R}^d)$, and

$$\begin{aligned} &\mathrm{WF}^{\mu,\nu}(\widetilde{\mathscr{K}_t} u) = \widetilde{\chi}_t\left(\mathrm{WF}^{\mu,\nu}(u)\right), \quad t \in \mathbf{R}, \quad u \in \left(\Sigma_\mu^\nu\right)'(\mathbf{R}^d), \quad \nu = \mu(m-1) > 1, \\ &\mathrm{WF}^{\mu,\nu}(\widetilde{\mathscr{K}_t} u) = \mathrm{WF}^{\mu,\nu}(u), \quad t \in \mathbf{R}, \quad u \in \left(\Sigma_\mu^\nu\right)'(\mathbf{R}^d), \quad \nu > \mu(m-1) > 1. \end{aligned}$$

The conclusion from Theorem 5.4 is that the propagation of singularities for (5.21) works again exactly as when $v = 0$, in both the tempered Schwartz distribution and the Gelfand–Shilov ultradistribution frameworks, respectively.

Remark 5.5. Consider the Hamiltonian $a(x,\xi) = p(x) - \langle v, \xi \rangle$ in the equation (5.21), with $\deg p = m$. The propagation of $\mathrm{WF}_{\mathrm{g}}^{\sigma}$ with $\sigma = m-1$ is governed by $a_0(x,\xi) = P_m(x)$ which satisfies

$$a_0(\lambda x, \lambda^\sigma \xi) = \lambda^m P_m(x) = \lambda^{1+\sigma} a_0(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \lambda > 0.$$

Thus $a_0 \in G^{1+\sigma,\sigma} = G^{m,m-1}$ according to Corollary 3.3. This is similar to Remark 5.2. In Sect. 6 we will study more general Hamiltonians that satisfy this type of anisotropic homogeneity.

When the Hamiltonian Weyl symbol $a(x,\xi) = p(\xi)$ is a polynomial in ξ of the form (5.2) then the Hamilton flow is as in (5.10) that is

$$\chi_t(x,\xi) = (x + t\nabla P_m(\xi), \xi), \quad t \in \mathbf{R}, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \tag{5.26}$$

where P_m is the principal part of p. When instead the Weyl symbol depends on x, $a(x,\xi) = p(x)$, with the same assumptions on p, we obtain the Hamilton flow (5.24) that is

$$\chi_t(x,\xi) = (x, \xi - t\nabla P_m(x)), \quad t \in \mathbf{R}, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0. \tag{5.27}$$

Define for $\sigma > 0$ the anisotropic scaling map $\Lambda_\sigma(\lambda) : T^*\mathbf{R}^d \to T^*\mathbf{R}^d$ as

$$\Lambda_\sigma(\lambda)(x,\xi) = (\lambda x, \lambda^\sigma \xi), \quad (x,\xi) \in T^*\mathbf{R}^d, \quad \lambda > 0. \tag{5.28}$$

For suitable $\sigma > 0$ the Hamilton flows (5.26) and (5.27) commute with $\Lambda_\sigma(\lambda)$ for all $\lambda > 0$. In fact, if χ_t is defined by (5.26) and $\sigma = \frac{1}{m-1}$ then for $\lambda > 0$

$$\chi_t(\lambda x, \lambda^\sigma \xi) = (\lambda x + t\nabla P_m(\lambda^\sigma \xi), \lambda^\sigma \xi) = (\lambda(x + t\nabla P_m(\xi)), \lambda^\sigma \xi) = \Lambda_\sigma(\lambda)\chi_t(x,\xi).$$

Likewise if χ_t is defined by (5.27) and $\sigma = m-1$ then for $\lambda > 0$

$$\chi_t(\lambda x, \lambda^\sigma \xi) = (\lambda x, \lambda^\sigma \xi - t\nabla P_m(\lambda x)) = (\lambda x, \lambda^\sigma(\xi - t\nabla P_m(x))) = \Lambda_\sigma(\lambda)\chi_t(x,\xi).$$

Thus in both cases the Hamilton flow χ_t commutes with anisotropic scaling Λ_σ

$$\chi_t \Lambda_\sigma(\lambda) = \Lambda_\sigma(\lambda)\chi_t, \quad \lambda > 0, \quad t \in \mathbf{R}, \tag{5.29}$$

suppressing the variables $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$.

Remark 5.6. The commutativity (5.29) means that the considered Hamilton flows are consistent with the propagation inclusion that we aim for, namely

$$\mathrm{WF}_{\mathrm{g}}^{\sigma}(\mathscr{K}_t u) \subseteq \chi_t\left(\mathrm{WF}_{\mathrm{g}}^{\sigma}(u)\right), \quad t \in \mathbf{R}, \quad u \in \mathscr{S}'(\mathbf{R}^d), \tag{5.30}$$

for the solution operator (propagator) $\mathscr{K}_t$ of a Schrödinger type evolution equation.

Indeed the inclusion (5.30) requires that the image of χ_t of any $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u) \subseteq T^*\mathbf{R}^d \setminus 0$ for $u \in \mathscr{S}'(\mathbf{R}^d)$ contains a closed σ-conic subset of $T^*\mathbf{R}^d \setminus 0$. It is not known if for any closed σ-conic subset of $\Gamma \subseteq T^*\mathbf{R}^d\backslash 0$ there exists $u \in \mathscr{S}'(\mathbf{R}^d)$ such that $\mathrm{WF}_{\mathrm{g}}^{\sigma}(u) = \Gamma$ except when $\sigma = 1$. In fact if $\sigma = 1$ then [29, Theorem 6.1] answers the question affirmatively. Nevertheless it seems reasonable to ask that the image of χ_t of any closed σ-conic subset of $T^*\mathbf{R}^d \setminus 0$ contains a closed σ-conic subset of $T^*\mathbf{R}^d \setminus 0$. Then in particular a σ-conic curve of the form

$$R_{x,\xi} = \{(\lambda x, \lambda^{\sigma}\xi) \in T^*\mathbf{R}^d \setminus 0,\ \lambda > 0\}$$

where $(x, \xi) \in \mathbf{S}^{2d-1}$, must be mapped into another such curve, that is, $\chi_t R_{x,\xi} = R_z$ where $z \in \mathbf{S}^{2d-1}$.

6. Anisotropically homogeneous Hamiltonians and their flows

Given a Hamiltonian $a : \mathbf{R}^{2d} \setminus 0 \to \mathbf{R}$ of class C^{∞}, Hamilton's system of equations is

$$\begin{cases} x'(t) = \nabla_{\xi} a\,(x(t), \xi(t))\,, \\ \xi'(t) = -\nabla_{x} a\,(x(t), \xi(t))\,, \\ x(0) = x, \\ \xi(0) = \xi, \end{cases} \tag{6.1}$$

for initial datum $(x, \xi) \in T^*\mathbf{R}^d\backslash 0$ and $t \in (-T, T)$ with $T > 0$. By the Picard–Lindelöf theorem there is a unique solution $(x(t), \xi(t)) = \chi_t(x, \xi)$, $\chi_t : \mathbf{R}^{2d}\backslash 0 \to \mathbf{R}^{2d}\backslash 0$, $t \in (-T, T)$, for some $T > 0$. It is called the Hamiltonian flow. In general the maximal T depends on (x, ξ). The map $(-T, T) \ni t \to \chi_t$ satisfies $\chi_{t_1+t_2} = \chi_{t_1}\chi_{t_2}$ and $\chi_t^{-1} = \chi_{-t}$ [1]. The solution χ_t is a symplectomorphism on $T^*\mathbf{R}^d$ for fixed $t \in (-T, T)$ [8], C^1 with respect to t, and hence a C^1 diffeomorphism on $T^*\mathbf{R}^d$. If the level sets of a are compact then the solution $\chi_t(x, \xi)$ extends to all $t \in \mathbf{R}$ [1]. Using the matrix (3.18) we may write the differential equation in (6.1) as

$$\begin{pmatrix} x'(t) \\ \xi'(t) \end{pmatrix} = \mathcal{J}\nabla_{x,\xi} a\,(x(t), \xi(t))\,. \tag{6.2}$$

Suppose the solution $\chi_t : \mathbf{R}^{2d}\backslash 0 \to \mathbf{R}^{2d}\backslash 0$ is well defined for $t \in (-T, T)$ with the parameter $T > 0$ valid for all initial data $(x, \xi) \in T^*\mathbf{R}^d\backslash 0$. The assumption $a \in C^{\infty}(\mathbf{R}^{2d}\backslash 0)$ and [15, Theorem V.4.1] imply that

$$(-T, T) \times \mathbf{R}^{2d} \setminus 0 \ni (t, x, \xi) \mapsto \partial_x^{\alpha}\partial_{\xi}^{\beta}\chi_t(x, \xi) \in \mathbf{R}^{2d} \setminus 0 \in C((-T, T) \times \mathbf{R}^{2d} \setminus 0) \quad \forall \alpha, \beta \in \mathbf{N}^d, \tag{6.3}$$

and in particular $\chi_t \in C^{\infty}(\mathbf{R}^{2d}\backslash 0, \mathbf{R}^{2d}\backslash 0)$ for each $t \in (-T, T)$.

The next lemma will be used in the proofs of Proposition 6.2 and its converse Proposition 6.4.

Lemma 6.1. *If $\sigma > 0$ and $a \in C^{\infty}(\mathbf{R}^{2d} \setminus 0)$ is real-valued then*

$$a(\lambda x, \lambda^{\sigma}\xi) = \lambda^{\sigma+1} a(x, \xi), \quad (x, \xi) \in T^*\mathbf{R}^d \setminus 0, \quad \lambda > 0, \tag{6.4}$$

holds if and only if

$$\lim_{(x,\xi)\to(0,0)} a(x,\xi) = 0, \tag{6.5}$$

$$\nabla_x a(\lambda x, \lambda^\sigma \xi) = \lambda^\sigma \nabla_x a(x,\xi), \quad (x,\xi) \in T^* \boldsymbol{R}^d \setminus 0, \quad \lambda > 0, \quad \textit{and} \tag{6.6}$$

$$\nabla_\xi a(\lambda x, \lambda^\sigma \xi) = \lambda \nabla_\xi a(x,\xi), \quad (x,\xi) \in T^* \mathbf{R}^d \setminus 0, \quad \lambda > 0, \tag{6.7}$$

hold.

Proof. It is immediate to see that (6.4) implies (6.6) and (6.7). Since any $(y,\eta) \in T^*\mathbf{R}^d \setminus 0$ can be written as $(y,\eta) = (\lambda x, \lambda^\sigma \xi)$ for a unique $\lambda > 0$ and a unique $(x,\xi) \in \mathbf{S}^{2d-1}$ [27, Section 3], also (6.5) follows from (6.4).

Assume on the other hand (6.5), (6.6) and (6.7). Let $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$ and define the function $f(t) = a(tx, t\xi)$ for $t > 0$. Then we have for $0 < \varepsilon < 1$

$$a(x,\xi) = f(1) = \int_\varepsilon^1 f'(t)\,\mathrm{d}t + f(\varepsilon) = \int_\varepsilon^1 \langle \nabla_{x,\xi} a(t(x,\xi)), (x,\xi) \rangle \,\mathrm{d}t + a(\varepsilon(x,\xi))$$

which gives for $\lambda > 0$, using (6.6) and (6.7),

$$\begin{aligned} a(\lambda x, \lambda^\sigma \xi) &= \int_\varepsilon^1 \langle \nabla_{x,\xi} a(t(\lambda x, \lambda^\sigma \xi)), (\lambda x, \lambda^\sigma \xi) \rangle \,\mathrm{d}t + a(\varepsilon(\lambda x, \lambda^\sigma \xi)) \\ &= \lambda^{\sigma+1} \int_\varepsilon^1 \langle \nabla_{x,\xi} a(t(x,\xi)), (x,\xi) \rangle \,\mathrm{d}t + a(\varepsilon(\lambda x, \lambda^\sigma \xi)) \\ &= \lambda^{\sigma+1} \Big(a(x,\xi) - a(\varepsilon(x,\xi)) \Big) + a(\varepsilon(\lambda x, \lambda^\sigma \xi)). \end{aligned}$$

The claim (6.4) now follows from the limit as $\varepsilon \to 0^+$ using the assumption (6.5). □

In the following result we show that the Hamilton flow commutes with anisotropic scaling for Hamiltonians with the anisotropic homogeneity (6.4).

Proposition 6.2. *Let $\sigma > 0$, and suppose $a \in C^\infty(\mathbf{R}^{2d} \setminus 0)$ is real-valued and satisfies*

$$a(\lambda x, \lambda^\sigma \xi) = \lambda^{\sigma+1} a(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \quad \lambda > 0. \tag{6.8}$$

Then there exists $T > 0$ such that the Hamilton flow $\chi_t(x,\xi)$ defined by the function a is well defined for $t \in [-T, T]$ uniformly for all $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$, and χ_t satisfies

$$\chi_t(\Lambda_\sigma(\lambda)(x,\xi)) = \Lambda_\sigma(\lambda)\chi_t(x,\xi), \quad \lambda > 0, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \quad t \in [-T, T], \tag{6.9}$$

where $\Lambda_\sigma(\lambda) : T^*\mathbf{R}^d \to T^*\mathbf{R}^d$ is defined in (5.28).

Proof. The assumption (6.8) and Lemma 6.1 give the anisotropic homogeneities

$$\begin{aligned} &\nabla_x a(\lambda x, \lambda^\sigma \xi) = \lambda^\sigma \nabla_x a(x,\xi), \\ &\nabla_\xi a(\lambda x, \lambda^\sigma \xi) = \lambda \nabla_\xi a(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \quad \lambda > 0, \end{aligned}$$

which can be written as

$$\nabla_{x,\xi} a(\lambda x, \lambda^\sigma \xi) = \Lambda_{\frac{1}{\sigma}}(\lambda^\sigma) \nabla_{x,\xi} a(x, \xi). \tag{6.10}$$

This gives $\lim_{(x,\xi)\to(0,0)} \nabla_{x,\xi} a(x, \xi) = 0$. Set

$$M = \sup_{0<|(x,\xi)|\leqslant\frac{3}{2}} |\nabla_{x,\xi} a(x, \xi)| < +\infty.$$

If $(x, \xi) \in \mathbf{S}^{2d-1}$ then by the Picard–Lindelöf theorem [15, Theorem II.1.1] the Hamilton flow stays in the ball $\chi_t(x, \xi) \in \overline{\mathrm{B}_{\frac{1}{2}}(x, \xi)}$ if $-T \leqslant t \leqslant T$ and $T = \frac{1}{2M}$. Thus there exists $T > 0$ such that the Hamilton flow $\chi_t : \mathbf{S}^{2d-1} \to \mathbf{R}^{2d} \setminus 0$ is well defined for $-T \leqslant t \leqslant T$ uniformly over $\mathbf{S}^{2d-1}$.

Let $(x, \xi) \in T^*\mathbf{R}^d \setminus 0$ and set $(x(t), \xi(t)) = \chi_t(x, \xi)$. For $T_0 > 0$ sufficiently small we have $(x(t), \xi(t)) \in T^*\mathbf{R}^d \setminus 0$ for $t \in [-T_0, T_0]$. From (6.2), (6.10) and

$$\mathcal{J} \Lambda_{\frac{1}{\sigma}}(\lambda^{-\sigma}) = \Lambda_\sigma(\lambda^{-1}) \mathcal{J} \tag{6.11}$$

we obtain

$$\begin{aligned} \frac{d}{dt}\chi_t(x, \xi) = \begin{pmatrix} x'(t) \\ \xi'(t) \end{pmatrix} &= \mathcal{J} \nabla_{x,\xi} a\,(x(t), \xi(t)) \\ &= \mathcal{J} \Lambda_{\frac{1}{\sigma}}(\lambda^{-\sigma}) \nabla_{x,\xi} a\left(\lambda x(t), \lambda^\sigma \xi(t)\right) \\ &= \Lambda_\sigma(\lambda^{-1}) \mathcal{J} \nabla_{x,\xi} a\left(\lambda x(t), \lambda^\sigma \xi(t)\right) \end{aligned}$$

which may be written

$$(\lambda x'(t), \lambda^\sigma \xi'(t)) = \mathcal{J} \nabla_{x,\xi} a\left(\lambda x(t), \lambda^\sigma \xi(t)\right).$$

Thus $(\lambda x(t), \lambda^\sigma \xi(t))$ solves (6.1) for $t \in [-T_0, T_0]$ with initial datum $(\lambda x, \lambda^\sigma \xi)$, for any $\lambda > 0$. If we choose $\lambda > 0$ such that $|(\lambda x, \lambda^\sigma \xi)| = 1$ then the solution is well defined for $t \in [-T, T]$ by the first part of the proof. The solution $(\lambda x(t), \lambda^\sigma \xi(t))$ hence extends to $t \in [-T, T]$ for all $\lambda > 0$. By the uniqueness of the solution we have $\chi_t(\lambda x, \lambda^\sigma \xi) = (\lambda x(t), \lambda^\sigma \xi(t))$. It follows that the Hamilton flow $\chi_t : \mathbf{R}^{2d} \setminus 0 \to \mathbf{R}^{2d} \setminus 0$ is well defined in the interval $t \in [-T, T]$ uniformly over the phase space $\mathbf{R}^{2d} \setminus 0$. In conclusion we have

$$\Lambda_\sigma(\lambda)\chi_t(x, \xi) = (\lambda x(t), \lambda^\sigma \xi(t)) = \chi_t(\lambda x, \lambda^\sigma \xi) = \chi_t(\Lambda_\sigma(\lambda)(x, \xi))$$

for $(x, \xi) \in T^*\mathbf{R}^d \setminus 0$, $\lambda > 0$ and $t \in [-T, T]$. $\square$

Remark 6.3. With the assumptions of Proposition 6.2, for any $t \in [-T, T]$ we have

$$\lim_{(x,\xi)\to(0,0)} \chi_t(x, \xi) = 0.$$

In fact this is an immediate consequence of (6.9). So defining $\chi_t(0, 0) = (0, 0)$ we could extend the Hamilton flow as a continuous bijection $\chi_t : \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ for $t \in [-T, T]$. By [15, Theorem V.4.1] we know that $\chi_t \in C^\infty(\mathbf{R}^{2d} \setminus 0, \mathbf{R}^{2d} \setminus 0)$ but we cannot extend the smoothness to the new domain point $(0, 0)$.

Next we show a converse of Proposition 6.2.

Proposition 6.4. *Let $a \in C^\infty(\mathbf{R}^{2d} \setminus 0)$ be real-valued and suppose*

$$\lim_{(x,\xi)\to(0,0)} a(x,\xi) = 0.$$

Suppose the solution $\chi_t(x,\xi)$ to (6.1) is well defined for $t \in [-T,T]$ for some $T > 0$ for all $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$. If $\sigma > 0$ and (6.9) holds true then a satisfies the homogeneity

$$a(\lambda x, \lambda^\sigma \xi) = \lambda^{\sigma+1} a(x,\xi), \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \quad \lambda > 0. \tag{6.12}$$

Proof. For $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$ we denote $(x(t),\xi(t)) = \chi_t(x,\xi)$. Formula (6.9) means that the solution to (6.1) with (x,ξ) replaced by $(\lambda x, \lambda^\sigma \xi)$ for $\lambda > 0$ is $\Lambda_\sigma(\lambda)\chi_t(x,\xi) = (\lambda x(t), \lambda^\sigma \xi(t))$.

Let $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$. From (6.2) and (6.9) we obtain for any $\lambda > 0$

$$\begin{aligned}
\mathcal{J}\nabla_{x,\xi} a\,(x(t),\xi(t)) &= \frac{d}{dt}\chi_t(x,\xi) = \frac{d}{dt}\Big(\Lambda_\sigma(\lambda^{-1})\chi_t\,(\Lambda_\sigma(\lambda)(x,\xi))\Big) \\
&= \Lambda_\sigma(\lambda^{-1})\frac{d}{dt}\left(\chi_t\,(\Lambda_\sigma(\lambda)(x,\xi))\right) \\
&= \Lambda_\sigma(\lambda^{-1})\mathcal{J}\nabla_{x,\xi} a\left(\lambda x(t), \lambda^\sigma \xi(t)\right).
\end{aligned}$$

With aid of (6.11) and $\mathcal{J}^{-1} = -\mathcal{J}$ this gives

$$\begin{aligned}
\nabla_{x,\xi} a\,(x(t),\xi(t)) &= -\mathcal{J}\Lambda_\sigma(\lambda^{-1})\mathcal{J}\nabla_{x,\xi} a\left(\lambda x(t), \lambda^\sigma \xi(t)\right) \\
&= \Lambda_{\frac{1}{\sigma}}(\lambda^{-\sigma})\nabla_{x,\xi} a\left(\lambda x(t), \lambda^\sigma \xi(t)\right).
\end{aligned}$$

For $t = 0$ we get

$$\begin{aligned}
\nabla_x a(\lambda x, \lambda^\sigma \xi) &= \lambda^\sigma \nabla_x a(x,\xi), \\
\nabla_\xi a(\lambda x, \lambda^\sigma \xi) &= \lambda \nabla_\xi a(x,\xi)
\end{aligned}$$

which together with the assumption $\lim_{(x,\xi)\to(0,0)} a(x,\xi) = 0$ is equivalent to (6.12) by Lemma 6.1. □

We note that a function a that satisfies (6.12) is determined by its values on the unit sphere $\mathbf{S}^{2d-1}$, and

$$a(x,\xi) = \lambda_\sigma^{\sigma+1}(x,\xi)\, a(p_\sigma(x,\xi)), \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0,$$

where $\lambda_\sigma : \mathbf{R}^{2d} \setminus 0 \to \mathbf{R}_+$ and $p_\sigma : \mathbf{R}^{2d} \setminus 0 \to \mathbf{S}^{2d-1}$ are smooth functions defined in [27, Section 3].

Examples of Hamiltonians that satisfy $a \in C^\infty(\mathbf{R}^{2d} \setminus 0)$ and (6.12) are the homogeneous polynomials that depend on either x or ξ (but not both) studied in Sect. 5 (cf. Remarks 5.2 and 5.5), that is

$$a(x,\xi) = P_m(x), \quad \sigma = m - 1,$$

$$a(x,\xi) = P_m(\xi), \quad \sigma = \frac{1}{m-1},$$

where $m \in \mathbf{N}$ and $m \geqslant 2$. Other examples are

$$a(x,\xi) = c\left(|x| + |\xi|^{\frac{1}{\sigma}}\right)^{\sigma+1},$$

where $\sigma > 0$ and $c \in \mathbf{R} \setminus 0$, and

$$a(x,\xi) = c\left(|x|^{2k} + |\xi|^{2m}\right)^{\frac{1}{2}\left(\frac{1}{k}+\frac{1}{m}\right)},$$

with $k, m \in \mathbf{N}\setminus 0$, $\sigma = \frac{k}{m}$ and $c \in \mathbf{R}\setminus 0$.

Note that the Hamiltonians

$$a(x,\xi) = c_1|x|^{\sigma+1} + c_2|\xi|^{1+\frac{1}{\sigma}}$$

with $c_1, c_2 \in \mathbf{R}$ and $\sigma > 0$,

$$a(x,\xi) = c_1|x|^{2k} + c_2|\xi|^{1+\frac{1}{2k-1}}$$

with $k \in \mathbf{N}\setminus 0$, $\sigma = 2k-1$ and $c_1, c_2 \in \mathbf{R}$, and

$$a(x,\xi) = c_1|x|^{1+\frac{1}{2k-1}} + c_2|\xi|^{2k}$$

with $\sigma = \frac{1}{2k-1}$ and $c_1, c_2 \in \mathbf{R}$, all satisfy (6.12). But none of them satisfy $a \in C^\infty(\mathbf{R}^{2d}\setminus 0)$.

The final result in this section will be useful in Sect. 8. It says that the $G^{m,\sigma}$ property of a symbol is preserved under composition with a Hamiltonian flow that satisfies the anisotropic scaling commutativity (6.9). We need a cutoff function $\psi_\delta(x,\xi) = \varphi(|x|^2 + |\xi|^2) \in C^\infty(\mathbf{R}^{2d})$ where $\varphi \in C^\infty(\mathbf{R})$, $0 \leqslant \varphi \leqslant 1$, $\varphi(t) = 0$ for $t \leqslant \frac{\delta^2}{4}$ and $\varphi(t) = 1$ for $t \geqslant \delta^2$ for a given $\delta > 0$. Thus $\psi_\delta\big|_{\mathrm{B}_{\frac{\delta}{2}}} \equiv 0$ and $\psi_\delta\big|_{\mathbf{R}^{2d}\setminus \mathrm{B}_\delta} \equiv 1$.

Proposition 6.5. *Let $\sigma, \delta, T > 0$, and suppose $\chi_t \in C^\infty(\mathbf{R}^{2d}\setminus 0, \mathbf{R}^{2d}\setminus 0)$ for $-T \leqslant t \leqslant T$ is a Hamiltonian flow that satisfies the anisotropic scaling commutativity (6.9). If $a \in G^{m,\sigma}$ then $b_t = \psi_\delta(a \circ \chi_t) \in G^{m,\sigma}$ uniformly for all $-T \leqslant t \leqslant T$.*

Proof. Let $(x,\xi) \in T^*\mathbf{R}^d$ satisfy $|(x,\xi)| \geqslant \delta$ and let $\lambda \geqslant 1$. From (6.9) we obtain

$$b_t(\lambda x, \lambda^\sigma \xi) = a\left(\chi_t(\lambda x, \lambda^\sigma \xi)\right) = a\left(\Lambda_\sigma(\lambda)\chi_t(x,\xi)\right) = a\left(\lambda \chi_{t,1}(x,\xi), \lambda^\sigma \chi_{t,2}(x,\xi)\right)$$

decomposing $\chi_t = (\chi_{t,1}, \chi_{t,2})$ into its two $\mathbf{R}^d$ component functions. For $1 \leqslant k \leqslant d$ we denote by $\chi_{t,j,k}$ the component with index k of $\chi_{t,j}$ for $j = 1, 2$.

We claim that for $|(x,\xi)| > \delta$, $\lambda \geqslant 1$, and $\alpha, \beta \in \mathbf{N}^d$ we have

$$\begin{aligned} &\partial_x^\alpha \partial_\xi^\beta \Big(a\left(\lambda \chi_{t,1}(x,\xi), \lambda^\sigma \chi_{t,2}(x,\xi)\right) \Big) \\ &\quad = \sum_{|\gamma+\kappa| \leqslant |\alpha+\beta|} \lambda^{|\gamma|+\sigma|\kappa|} \left(\partial_x^\gamma \partial_\xi^\kappa a\right) \Big(\Lambda_\sigma(\lambda)\chi_t(x,\xi)\Big) f_{\gamma,\kappa}(x,\xi) \end{aligned} \tag{6.13}$$

where $f_{\gamma,\kappa} \in C^\infty(\mathbf{R}^{2d} \setminus 0)$ are smooth functions. In fact the claim follows by induction with respect to $|\alpha + \beta|$, starting with $|\alpha + \beta| = 1$, as follows. With $e_k \in \mathbf{N}^d$ denoting the standard basis vector, $1 \leqslant k \leqslant d$, we may write $\partial_{x_k} a(x, \xi) = \partial_x^{e_k} a(x, \xi)$ and $\partial_{\xi_k} a(x, \xi) = \partial_\xi^{e_k} a(x, \xi)$. If $|\alpha + \beta| = 1$ we have either

$$\begin{aligned}
&\partial_{x_j} \Big(a \left(\lambda \chi_{t,1}(x, \xi), \lambda^\sigma \chi_{t,2}(x, \xi) \right) \Big) \\
&\quad = \sum_{k=1}^d \left(\lambda \left(\partial_x^{e_k} a \right) \Big(\Lambda_\sigma(\lambda) \chi_t(x, \xi) \Big) \frac{\partial \chi_{t,1,k}}{\partial x_j}(x, \xi) + \lambda^\sigma \left(\partial_\xi^{e_k} a \right) \Big(\Lambda_\sigma(\lambda) \chi_t(x, \xi) \Big) \frac{\partial \chi_{t,2,k}}{\partial x_j}(x, \xi) \right)
\end{aligned}$$

or

$$\begin{aligned}
&\partial_{\xi_j} \Big(a \left(\lambda \chi_{t,1}(x, \xi), \lambda^\sigma \chi_{t,2}(x, \xi) \right) \Big) \\
&\quad = \sum_{k=1}^d \left(\lambda \left(\partial_x^{e_k} a \right) \Big(\Lambda_\sigma(\lambda) \chi_t(x, \xi) \Big) \frac{\partial \chi_{t,1,k}}{\partial \xi_j}(x, \xi) + \lambda^\sigma \left(\partial_\xi^{e_k} a \right) \Big(\Lambda_\sigma(\lambda) \chi_t(x, \xi) \Big) \frac{\partial \chi_{t,2,k}}{\partial \xi_j}(x, \xi) \right)
\end{aligned}$$

for $1 \leqslant j \leqslant d$. Thus (6.13) holds when $|\alpha + \beta| = 1$. The induction step follows straight-forwardly. It follows that (6.13) holds for all $\alpha, \beta \in \mathbf{N}^d$, $|(x, \xi)| > \delta$, and $\lambda \geqslant 1$, as claimed.

We fix $r > \delta$ and consider any $(x, \xi) \in T^* \mathbf{R}^d$ such that $|(x, \xi)| = r$. Using (6.13), the assumption $a \in G^{m,\sigma}$ and

$$\inf_{\substack{|t| \leqslant T \\ |(x,\xi)| = r}} |\chi_t(x, \xi)| > 0, \qquad \sup_{\substack{|t| \leqslant T \\ |(x,\xi)| = r}} |\chi_t(x, \xi)| < \infty,$$

we estimate for $\alpha, \beta \in \mathbf{N}^d$

$$\begin{aligned}
\lambda^{|\alpha| + \sigma|\beta|} \left| \left(\partial_x^\alpha \partial_\xi^\beta b_t \right)(\lambda x, \lambda^\sigma \xi) \right| &= \left| \partial_x^\alpha \partial_\xi^\beta \Big(b_t(\lambda x, \lambda^\sigma \xi) \Big) \right| \\
&= \left| \partial_x^\alpha \partial_\xi^\beta \Big(a \left(\Lambda_\sigma(\lambda) \chi_t(x, \xi) \right) \Big) \right| \\
&\lesssim \sum_{|\gamma + \kappa| \leqslant |\alpha + \beta|} \lambda^{|\gamma| + \sigma|\kappa|} \left| \left(\partial_x^\gamma \partial_\xi^\kappa a \right) \Big(\Lambda_\sigma(\lambda) \chi_t(x, \xi) \Big) \right| \\
&\lesssim \sum_{|\gamma + \kappa| \leqslant |\alpha + \beta|} \lambda^{|\gamma| + \sigma|\kappa|} \theta_\sigma \left(\lambda \chi_{t,1}(x, \xi), \lambda^\sigma \chi_{t,2}(x, \xi) \right)^{m - |\gamma| - \sigma|\kappa|} \\
&\lesssim \lambda^m
\end{aligned}$$

for all $\lambda \geqslant 1$. The conclusion $b_t \in G^{m,\sigma}$ uniformly for all $-T \leqslant t \leqslant T$ is now a consequence of Lemma 3.2. $\square$

7. Solutions to a class of Schrödinger type equations with anisotropic Hamiltonians

In the sequel we use $k, m \in \mathbf{N} \setminus 0$ and $\sigma = \frac{k}{m}$. We consider in this section first the Cauchy problem

$$\begin{cases} \partial_t u(t, x) + i a^w(x, D) u(t, x) = f(t, x), \quad x \in \mathbf{R}^d, \quad 0 < t \leqslant T, \\ \qquad u(0, \cdot) = u_0, \end{cases} \tag{7.1}$$

where $T > 0$ and $a \in G^{1+\sigma,\sigma}$. Later we will extend the time domain to $[-T, T]$.

Simplifying notation we set $M_s = M_{\sigma,s}(\mathbf{R}^d)$ for $s \in \mathbf{R}$ and $a^w = a^w(x, D)$. The main purpose of the section is to show existence and uniqueness of solutions to (7.1) considering $u(t, \cdot)$ as a function of t with values in M_s spaces.

We will need the following lemma which says that $C^1([0, T], \mathscr{S})$ is dense in $C([0, T], M_\mu) \cap C^1([0, T], M_\nu)$ for any $\mu, \nu \in \mathbf{R}$.

Lemma 7.1. *If $\mu, \nu \in \mathbf{R}$ and $u \in C([0, T], M_\mu) \cap C^1([0, T], M_\nu)$ then there exists a sequence $(u_n)_{n\geqslant 1} \subseteq C^1([0, T], \mathscr{S})$ such that*

$$\lim_{n\to+\infty} \sup_{0\leqslant t\leqslant T} \|u_n(t, \cdot) - u(t, \cdot)\|_{M_\mu} = 0, \tag{7.2}$$

$$\lim_{n\to+\infty} \sup_{0\leqslant t\leqslant T} \|\partial_t u_n(t, \cdot) - \partial_t u(t, \cdot)\|_{M_\nu} = 0. \tag{7.3}$$

Proof. Let $\varphi \in \mathscr{S}(\mathbf{R}^d)$ satisfy $\|\varphi\|_{L^2} = 1$. We use the approximations (cf. [13])

$$u_n(t, \cdot) = V_\varphi^* \chi_n V_\varphi u(t, \cdot) \in \mathscr{S}(\mathbf{R}^d), \quad 0 \leqslant t \leqslant T,$$

where χ_n is the indicator function of the ball $\mathrm{B}_n \subseteq \mathbf{R}^{2d}$, $n \in \mathbf{N}\backslash 0$.

By [13, Eq. (11.29)] we have on the one hand

$$\left|V_\varphi \left(u_n(t, \cdot) - u_n(\tau, \cdot)\right)\right| \leqslant \left(\chi_n \left|V_\varphi \left(u(t, \cdot) - u(\tau, \cdot)\right)\right|\right) * |V_\varphi\varphi| \tag{7.4}$$

and on the other hand, using (2.5) in the form $V_\varphi^* V_\varphi = \mathrm{id}_{\mathscr{S}'}$,

$$\left|V_\varphi \left(u_n(t, \cdot) - u(t, \cdot)\right)\right| \leqslant \left((1 - \chi_n) \left|V_\varphi u(t, \cdot)\right|\right) * |V_\varphi\varphi|. \tag{7.5}$$

With $m \geqslant 0$ we write using (3.2) and (3.5)

$$\begin{aligned} \langle z\rangle^m &\lesssim \theta_\sigma(z)^{m \max(1,\sigma)} \lesssim \theta_\sigma(z-w)^{m \max(1,\sigma)}\theta_\sigma(w)^{m \max(1,\sigma)} \\ &\leqslant \theta_\sigma(z-w)^{m \max(1,\sigma)+|\mu|+\mu}\theta_\sigma(w)^{m \max(1,\sigma)}, \quad z, w \in \mathbf{R}^{2d}, \end{aligned}$$

which inserted into (7.4) gives by means of the Cauchy–Schwarz inequality, again (3.5) and (2.3)

$$\begin{aligned} &\langle z\rangle^m \left|V_\varphi \left(u_n(t, \cdot) - u_n(\tau, \cdot)\right)(z)\right| \\ &\lesssim \left(\chi_n \theta_\sigma^{m \max(1,\sigma)+|\mu|}\theta_\sigma^\mu \left|V_\varphi \left(u(t, \cdot) - u(\tau, \cdot)\right)\right|\right) * \left(\theta_\sigma^{m \max(1,\sigma)}|V_\varphi\varphi|\right)(z) \\ &\leqslant \sup_{\mathbf{R}^{2d}} \left(\chi_n \theta_\sigma^{m \max(1,\sigma)+|\mu|}\right) \left\|\theta_\sigma^\mu \left|V_\varphi \left(u(t, \cdot) - u(\tau, \cdot)\right)\right|\right\|_{L^2(\mathbf{R}^{2d})} \left\|\theta_\sigma^{m \max(1,\sigma)}|V_\varphi\varphi|\right\|_{L^2(\mathbf{R}^{2d})} \\ &\lesssim \|u(t, \cdot) - u(\tau, \cdot)\|_{M_\mu}. \end{aligned}$$

Referring to the assumption $u \in C([0, T], M_\mu)$ and to the seminorms (2.6) this shows that $u_n \in C([0, T], \mathscr{S})$, and $u_n \in C^1([0, T], \mathscr{S})$ follows similarly from $\partial_t u_n(t, \cdot) = V_\varphi^* \chi_n V_\varphi \partial_t u(t, \cdot)$, replacing μ with ν and using the assumption $u \in C^1([0, T], M_\nu)$.

From (7.5) and Young's inequality we obtain, again using (3.2), (2.3) and (3.5),

$$\begin{aligned}\|u_n(t,\cdot)-u(t,\cdot)\|_{M_\mu} &= \left\|\theta_\sigma^\mu \left|V_\varphi\left(u_n(t,\cdot)-u(t,\cdot)\right)\right|\right\|_{L^2(\mathbf{R}^{2d})}\\ &\lesssim \left\|\left((1-\chi_n)\theta_\sigma^\mu \left|V_\varphi u(t,\cdot)\right|\right) * \left(\theta_\sigma^{|\mu|}|V_\varphi\varphi|\right)\right\|_{L^2(\mathbf{R}^{2d})}\\ &\lesssim \left\|(1-\chi_n)\theta_\sigma^\mu\, V_\varphi u(t,\cdot)\right\|_{L^2(\mathbf{R}^{2d})}\left\|\theta_\sigma^{|\mu|}V_\varphi\varphi\right\|_{L^1(\mathbf{R}^{2d})}\\ &\lesssim \left\|(1-\chi_n)\theta_\sigma^\mu\, V_\varphi u(t,\cdot)\right\|_{L^2(\mathbf{R}^{2d})} := f_n(t).\end{aligned}$$

Note the monotonicity $f_n(t) \geqslant f_{n+1}(t)$ for each $n \in \mathbf{N}\backslash 0$, and by the assumption $u \in C([0,T], M_\mu)$ and dominated convergence we get $\lim_{n\to\infty} f_n(t) = 0$ for each $t \in [0,T]$. For each $n \in \mathbf{N}\setminus 0$ we have $f_n \in C([0,T])$. In fact

$$\begin{aligned}|f_n(t)-f_n(\tau)| &= \left|\left\|(1-\chi_n)\theta_\sigma^\mu V_\varphi u(t,\cdot)\right\|_{L^2(\mathbf{R}^{2d})} - \left\|(1-\chi_n)\theta_\sigma^\mu V_\varphi u(\tau,\cdot)\right\|_{L^2(\mathbf{R}^{2d})}\right|\\ &\leqslant \left\|(1-\chi_n)\theta_\sigma^\mu\left(V_\varphi u(t,\cdot)-V_\varphi u(\tau,\cdot)\right)\right\|_{L^2(\mathbf{R}^{2d})}\\ &\leqslant \left\|\theta_\sigma^\mu\left(V_\varphi u(t,\cdot)-V_\varphi u(\tau,\cdot)\right)\right\|_{L^2(\mathbf{R}^{2d})}\\ &= \|u(t,\cdot)-u(\tau,\cdot)\|_{M_\mu}\end{aligned}$$

so $f_n \in C([0,T])$ follows from the assumption $u \in C([0,T], M_\mu)$. Now it follows from Dini's theorem that $f_n(t) \to 0$ uniformly for $t \in [0,T]$ as $n \to \infty$. This means that we have shown (7.2), and (7.3) follows in the same fashion. $\square$

Remark 7.2. We note that Lemma 7.1 is true also when we replace the interval $[0,T]$ with $[-T,T]$.

By (3.4) we have $w_{k,m}^{1/k} \asymp \theta_\sigma$ when $\sigma = \frac{k}{m}$ and $k, m \in \mathbf{N}\backslash 0$. Combining this with (4.1), (4.7) and [14, Theorem 1.1] it follows that if $s \in \mathbf{R}$ then the symbol θ_σ^s for the localization operator (4.7) that defines the isometry $M_s \to L^2$ can be replaced by $w_{k,m}^{s/k}$. We denote for simplicity this localization operator by $A_s = A_{w_{k,m}^{s/k}}$. We will need the following auxiliary result.

Lemma 7.3. *Let $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$ and $a \in G^{1+\sigma,\sigma}$, and suppose that*

$$\operatorname{Im} a(x,\xi) \leqslant C_1, \quad (x,\xi) \in T^*\mathbf{R}^d,$$

for $C_1 > 0$. If $s \in \mathbf{R}$ then $A_s\, a^w A_s^{-1} = b^w$ where $b \in G^{1+\sigma,\sigma}$ and

$$\operatorname{Im} b(x,\xi) \leqslant C_2, \quad (x,\xi) \in T^*\mathbf{R}^d,$$

for some $C_2 > 0$.

Proof. By Lemma 3.6 we have $w_{k,m}^{s/k} \in G^{s,\sigma}$ for the symbol of A_s. From [20, Theorem 1.7.10] it follows that $A_s = a_1^w$ where $a_1 \in G^{s,\sigma}$ is real-valued, cf. Sect. 4.

The symbol $w_{k,m}^{s/k}$ for A_s is positive everywhere and elliptic, cf. (3.15). By the proof of [2, Theorem 8.2] (cf. also [20, Proposition 1.7.12]), slightly modified to the anisotropic calculus, it follows that A_s is invertible on $\mathscr{S}$, and $A_s^{-1} = c^w$ where $c \in G^{-s,\sigma}$. From

$$(A_s f, f) = \int_{\mathbf{R}^{2d}} w_{k,m}^{s/k}(z) |V_\varphi f(z)|^2 \, \mathrm{d}z > 0$$

for all $f \in \mathscr{S} \setminus 0$ it follows that $(c^w f, f) > 0$ for all $f \in \mathscr{S} \setminus 0$ which implies that c is a real-valued symbol. Indeed we have

$$\begin{aligned} 2i\,((\operatorname{Im} c)^w f, f) &= (c^w f, f) - (\overline{c}^w f, f) \\ &= (c^w f, f) - (f, c^w f) = (c^w f, f) - \overline{(c^w f, f)} = 0 \end{aligned}$$

for all $f \in \mathscr{S}$, which by polarization yields

$$\begin{aligned} 4((\operatorname{Im} c)^w f, g) &= \big((\operatorname{Im} c)^w (f+g), f+g\big) - \big((\operatorname{Im} c)^w (f-g), f-g\big) \\ &\quad + i\,\big((\operatorname{Im} c)^w (f+ig), f+ig\big) - i\,\big((\operatorname{Im} c)^w (f-ig), f-ig\big) = 0 \end{aligned}$$

for all $f, g \in \mathscr{S}$. This implies $\operatorname{Im} c \equiv 0$.

Finally from $b^w = A_s\, a^w A_s^{-1} = a_1^w a^w c^w$ and (3.13) we obtain

$$b = a_1 \# a \# c = a_1\, a\, c + b_1$$

where $b_1 \in G^{0,\sigma}$ is bounded. Thus since $a_1\, c \in G^{0,\sigma}$ is also bounded we get

$$\operatorname{Im} b = (\operatorname{Im} a)\, a_1\, c + \operatorname{Im} b_1 \leqslant C_1 \sup_{\mathbf{R}^{2d}} (a_1\, c) + \sup_{\mathbf{R}^{2d}} \operatorname{Im} b_1 = C_2 < \infty$$

for some $C_2 > 0$. □

Remark 7.4. The proof of Lemma 7.3 shows that from the added assumption

$$|\operatorname{Im} a(x,\xi)| \leqslant C_1, \quad (x,\xi) \in T^* \mathbf{R}^d,$$

follows the stronger conclusion

$$|\operatorname{Im} b(x,\xi)| \leqslant C_2, \quad (x,\xi) \in T^* \mathbf{R}^d.$$

The following two results Lemma 7.5 and Theorem 7.9 are detailed adaptations of [16, Lemma 23.1.1 and Theorem 23.1.2] from the calculus of Hörmander symbols to the anisotropic Shubin calculus.

Lemma 7.5. *Let $k, m \in \mathbf{N} \backslash 0$, $\sigma = \frac{k}{m}$, $a \in G^{1+\sigma,\sigma}$, and suppose that*

$$\operatorname{Im} a(x,\xi) \leqslant C, \quad (x,\xi) \in T^* \mathbf{R}^d, \tag{7.6}$$

for $C > 0$. If $s \in \mathbf{R}$, $u \in C([0,T], M_{s+1+\sigma}) \cap C^1([0,T], M_s)$ then

$$f(t) = \partial_t u(t,\cdot) + i a^w u(t,\cdot) \in C([0,T], M_s), \tag{7.7}$$

and there exists $c > 0$ such that

$$\|u(t)\|_{M_s} \lesssim e^{ct} \|u(0)\|_{M_s} + \int_0^t e^{c(t-\tau)} \|f(\tau)\|_{M_s} \mathrm{d}\tau \tag{7.8}$$

for $0 \leqslant t \leqslant T$.

Proof. First we prove the result for $s = 0$. The assumptions, $M_{1+\sigma} \subseteq L^2$ and Proposition 4.2 imply

$$\begin{cases} t \mapsto u(t, \cdot) \in C([0,T], L^2), \\ t \mapsto \partial_t u(t, \cdot) \in C([0,T], L^2), \\ t \mapsto a^w u(t, \cdot) \in C([0,T], L^2) \end{cases} \implies \quad t \mapsto f(t) \in C([0,T], L^2).$$

The conclusion (7.7) follows.

By Lemma 7.1 we may replace L^2 by $\mathscr{S}$ above. The assumptions $a \in G^{1+\sigma,\sigma}$ and $\operatorname{Im} a(x,\xi) \leqslant C$ make Lemma 4.3 applicable. Combining with (3.12) and the fact that $W(g,g)$ is real-valued [13] we get for $g \in \mathscr{S}(\mathbf{R}^d)$

$$\begin{aligned} \operatorname{Re}(i a^w g, g) &= -\operatorname{Im}(a^w g, g) = -(2\pi)^{-d} \operatorname{Im}(a, W(g,g)) = -(2\pi)^{-d} (\operatorname{Im} a, W(g,g)) \\ &= -((\operatorname{Im} a)^w g, g) = ((C - \operatorname{Im} a)^w g, g) - C \|g\|_{L^2}^2 \geqslant -(b + C) \|g\|_{L^2}^2 \end{aligned}$$

where $b > 0$. If $0 \leqslant t \leqslant T$ and $\mu \in \mathbf{R}$ this gives, writing $u(t) = u(t, \cdot)$ for brevity,

$$\begin{aligned} \partial_t \left(e^{-2\mu t} \|u(t)\|_{L^2}^2 \right) &= 2e^{-2\mu t} \left(\operatorname{Re}\, (\partial_t u(t), u(t)) - \mu \|u(t)\|_{L^2}^2 \right) \\ &= 2e^{-2\mu t} \left(\operatorname{Re}\, (f(t), u(t)) - \operatorname{Re}\big((ia + \mu)^w u(t), u(t)\big) \right) \\ &\leqslant 2e^{-2\mu t} \operatorname{Re}\, (f(t), u(t)) \end{aligned}$$

provided $\mu \geqslant b + C$.

Integration gives for any $0 \leqslant \nu \leqslant t$

$$\begin{aligned} e^{-2\mu\nu} \|u(\nu)\|_{L^2}^2 &\leqslant \|u(0)\|_{L^2}^2 + 2 \int_0^\nu e^{-2\mu\tau} \|f(\tau)\|_{L^2} \, \|u(\tau)\|_{L^2} \, \mathrm{d}\tau \\ &\leqslant \|u(0)\|_{L^2}^2 + 2 \int_0^t e^{-2\mu\tau} \|f(\tau)\|_{L^2} \, \|u(\tau)\|_{L^2} \, \mathrm{d}\tau \\ &\leqslant \|u(0)\|_{L^2}^2 + 2M(t) \int_0^t e^{-\mu\tau} \|f(\tau)\|_{L^2} \, \mathrm{d}\tau \end{aligned}$$

with

$$M(t) = \sup_{0 \leqslant \tau \leqslant t} e^{-\mu\tau} \|u(\tau)\|_{L^2}.$$

Thus

$$\left(M(t) - \int_0^t e^{-\mu\tau} \|f(\tau)\|_{L^2} \, \mathrm{d}\tau \right)^2 \leqslant \left(\|u(0)\|_{L^2} + \int_0^t e^{-\mu\tau} \|f(\tau)\|_{L^2} \, \mathrm{d}\tau \right)^2$$

which yields

$$e^{-\mu t} \|u(t)\|_{L^2} \leqslant M(t) \leqslant \|u(0)\|_{L^2} + 2 \int_0^t e^{-\mu\tau} \|f(\tau)\|_{L^2} \, \mathrm{d}\tau.$$

We have now shown (7.8) for $s = 0$ and $c = \mu$.

Next let $s \in \mathbf{R}$ and $u \in C([0,T], M_{s+1+\sigma}) \cap C^1([0,T], M_s)$. Then $\partial_t u, a^w u \in C([0,T], M_s)$, again appealing to Proposition 4.2, and the conclusion (7.7) follows. By the proof of Lemma 7.3 we know that $A_s = a_1^w$ with $a_1 \in G^{s,\sigma}$. Proposition 4.2 yields

$$\begin{aligned} & A_s u \in C([0,T], M_{\sigma+1}) \cap C^1([0,T], L^2), \quad a^w A_s u \in C([0,T], L^2) \\ \Longrightarrow \quad & \partial_t A_s u + i a^w A_s u \in C([0,T], L^2). \end{aligned}$$

By Lemma 7.3 the symbol $b \in G^{1+\sigma,\sigma}$ defined by $b^w = A_s\, a^w A_s^{-1}$ satisfies $\operatorname{Im} b \leqslant C_2$ for some $C_2 > 0$. The inequality (7.8) with $a = b$ and $s = 0$ thus gives

$$\|A_s u(t)\|_{L^2} \lesssim e^{ct} \|A_s u(0)\|_{L^2} + \int_0^t e^{c(t-\tau)} \|\partial_t A_s u(\tau) + i b^w A_s u(\tau)\|_{L^2} \mathrm{d}\tau$$

which finally yields

$$\begin{aligned} \|u(t)\|_{M_s} \asymp \|A_s u(t)\|_{L^2} & \lesssim e^{ct} \|A_s u(0)\|_{L^2} \\ & \quad + \int_0^t e^{c(t-\tau)} \|A_s \left(\partial_t u(\tau) + i A_s^{-1} b^w A_s u(\tau) \right) \|_{L^2} \mathrm{d}\tau \\ & \asymp e^{ct} \|u(0)\|_{M_s} + \int_0^t e^{c(t-\tau)} \|\partial_t u(\tau) + i a^w u(\tau)\|_{M_s} \mathrm{d}\tau. \end{aligned}$$

□

Remark 7.6. If we strengthen the assumption (7.6) with a lower bound as

$$-C \leqslant \operatorname{Im} a(x,\xi) \leqslant C, \quad (x,\xi) \in T^* \mathbf{R}^d, \tag{7.9}$$

for $C > 0$, then the time direction may be reversed in Lemma 7.5. More precisely the lower bound in (7.9) yields the estimate

$$\operatorname{Re}(i a^w g, g) \leqslant (b + C) \|g\|_{L^2}^2, \quad g \in \mathscr{S}.$$

Straightforward modifications of the argument in the proof for the case $s = 0$ leads to the estimate

$$\|u(-t)\|_{L^2} \lesssim e^{ct} \|u(0)\|_{L^2} + \int_{-t}^0 e^{c(t+\tau)} \|f(\tau)\|_{L^2} \, \mathrm{d}\tau$$

for $c > 0$ and $0 \leqslant t \leqslant T$. Taking into account Remark 7.4 a statement replacing (7.8) can then be formulated as follows. If $s \in \mathbf{R}$, $u \in C([-T,T], M_{s+1+\sigma}) \cap C^1([-T,T], M_s)$ then

$$\|u(t)\|_{M_s} \lesssim e^{c|t|} \|u(0)\|_{M_s} + \int_{|\tau| \leqslant |t|} e^{c(|t|+|\tau|)} \|f(\tau)\|_{M_s} \mathrm{d}\tau$$

for $-T \leqslant t \leqslant T$, where $c > 0$.

The final tool for the proof of existence and uniqueness of a solution to (7.1) we need is the following approximation lemma.

Lemma 7.7. *Let $s \in \mathbf{R}$. If $f \in L^1([0,T], M_s)$ then there exists a sequence $(f_n)_{n \geqslant 1} \subseteq C_c^\infty((0,T), \mathscr{S}(\mathbf{R}^d))$ such that*

$$\lim_{n \to +\infty} \| f_n - f \|_{L^1([0,T],M_s)} = 0. \tag{7.10}$$

Proof. Since $C([0,T], M_s) \subseteq L^1([0,T], M_s)$ is dense we may assume $f \in C([0,T], M_s)$, and by Lemma 7.1 we may assume $f \in C([0,T], \mathscr{S})$. Thus we have

$$\lim_{n \to +\infty} \sup_{|\tau| \leqslant \frac{1}{2}} \int_{\frac{\tau+1}{n}}^{T+\frac{\tau-1}{n}} \left\| f\left(t - \frac{\tau}{n}\right) - f(t) \right\|_{M_s} \mathrm{d}t = 0. \tag{7.11}$$

We regularize f with respect to $t \in [0,T]$ as

$$f_n(t) = \psi_n * (f \chi_n)(t) \in C_c^\infty((0,T), \mathscr{S}(\mathbf{R}^d))$$

where $\chi_n \in C_c^\infty(\mathbf{R})$ is the indicator function for the interval $[\frac{1}{n}, T - \frac{1}{n}] \subseteq \mathbf{R}$, $\psi \in C_c^\infty(\mathbf{R})$, $\psi \geqslant 0$, $\operatorname{supp} \psi \subseteq [-\frac{1}{2}, \frac{1}{2}]$, $\int_{\mathbf{R}} \psi(x)\,\mathrm{d}x = 1$, and $\psi_n(x) = n\psi(nx)$.

Writing

$$f_n(t) - f(t) = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \psi_n(\tau)\Big(\big(f(t-\tau) - f(t)\big)\chi_n(t-\tau) + f(t)\big(\chi_n(t-\tau) - 1\big)\Big)\,\mathrm{d}\tau$$

we may estimate

$$\int_0^T \| f_n(t) - f(t) \|_{M_s} \,\mathrm{d}t \leqslant I_n + J_n$$

where

$$\begin{aligned} I_n &= \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \psi_n(\tau) \int_0^T \| f(t-\tau) - f(t) \|_{M_s} \chi_n(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \psi_n(\tau) \int_{\tau+\frac{1}{n}}^{T+\tau-\frac{1}{n}} \| f(t-\tau) - f(t) \|_{M_s} \,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(\tau) \int_{\frac{\tau+1}{n}}^{T+\frac{\tau-1}{n}} \left\| f\left(t - \frac{\tau}{n}\right) - f(t) \right\|_{M_s} \mathrm{d}t\,\mathrm{d}\tau \\ &\longrightarrow 0, \quad n \to +\infty \end{aligned}$$

using (7.11). Finally

$$J_n = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \psi_n(\tau) \int_0^T \| f(t) \|_{M_s} \left(1 - \chi_n(t-\tau)\right)\,\mathrm{d}t\,\mathrm{d}\tau$$

$$
\begin{aligned}
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(\tau) \int_0^T \|f(t)\|_{M_s} \left(1 - \chi_n\left(t - \frac{\tau}{n}\right)\right) \mathrm{d}t\, \mathrm{d}\tau \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(\tau) \left(\int_0^{\frac{\tau+1}{n}} + \int_{T+\frac{\tau-1}{n}}^T \right) \|f(t)\|_{M_s} \,\mathrm{d}t\, \mathrm{d}\tau \\
&\leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(\tau) \left(\int_0^{\frac{3}{2n}} + \int_{T-\frac{3}{2n}}^T \right) \|f(t)\|_{M_s} \,\mathrm{d}t\, \mathrm{d}\tau \\
&= \left(\int_0^{\frac{3}{2n}} + \int_{T-\frac{3}{2n}}^T \right) \|f(t)\|_{M_s} \,\mathrm{d}t \\
&\longrightarrow 0, \quad n \to +\infty.
\end{aligned}
$$

We have shown (7.10). □

Remark 7.8. Again we note (cf. Remark 7.2) that Lemma 7.7 is true also when we replace the interval $[0, T]$ with $[-T, T]$.

We have now finally arrived at a point where we may state and prove the existence and uniqueness of solutions to (7.1) that are continuous on the spaces M_s.

Theorem 7.9. *Let $T > 0$, $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$, $a \in G^{1+\sigma,\sigma}$, suppose*

$$
\operatorname{Im} a(x, \xi) \leqslant C, \quad (x, \xi) \in T^*\mathbf{R}^d,
$$

for $C > 0$, and let $s \in \mathbf{R}$. If $u_0 \in M_s$ and $f \in L^1([0, T], M_s)$, then the equation (7.1) has a unique solution $u \in C([0, T], M_s)$.

Proof. First we assume $u_0 \in \mathscr{S}$ and $f \in C_c^\infty((0, T), \mathscr{S}(\mathbf{R}^d))$.

Let $\psi \in C_c^\infty((0, T) \times \mathbf{R}^d)$. With $\psi(t) = \psi(t, \cdot)$ we have $\psi(t), \partial_t \psi(t), \overline{a}^w \psi(t) \in C((0, T), \mathscr{S})$. Let $\nu \in \mathbf{R}$. Lemma 7.5 applied to $t \mapsto \psi(T - t, \cdot)$ and $-\overline{a}$ gives

$$
\begin{aligned}
\sup_{0 \leqslant t \leqslant T} \|\psi(t)\|_{M_{-\nu}} &\lesssim \int_0^T \| -\partial_t \psi(T - t, \cdot) - i\overline{a}^w \psi(T - t, \cdot)\|_{M_{-\nu}} \,\mathrm{d}t \\
&= \int_0^T \|\partial_t \psi(t, \cdot) + i\overline{a}^w \psi(t, \cdot)\|_{M_{-\nu}} \,\mathrm{d}t.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\left| \int_0^T (f(t), \psi(t)) \,\mathrm{d}t \right| &\leqslant \|f\|_{L^1([0,T],M_\nu)} \|\psi\|_{L^\infty([0,T],M_{-\nu})} \lesssim \sup_{0 \leqslant t \leqslant T} \|\psi(t)\|_{M_{-\nu}} \\
&\lesssim \int_0^T \|\partial_t \psi(t, \cdot) + i\overline{a}^w \psi(t, \cdot)\|_{-\nu} \,\mathrm{d}t.
\end{aligned}
$$

Thus

$$
L^1((0, T], M_{-\nu}) \ni -\partial_t \psi(t, \cdot) - i\overline{a}^w \psi(t, \cdot) \mapsto \int_0^T (f(t), \psi(t)) \,\mathrm{d}t
$$

is an anti-linear continuous functional. By the Hahn–Banach theorem it can be extended to a functional on $L^1((0,T], M_{-\nu})$. From [9, Theorem IV.1 and Corollary IV.4] we know that the dual space of $L^1((0,T], M_{-\nu})$ can be identified with $L^\infty((0,T], M_\nu)$ through the natural pairing. Hence there exists $u \in L^\infty((0,T], M_\nu) \subseteq \mathscr{D}'((0,T) \times \mathbf{R}^d)$ such that

$$\begin{aligned}\int_0^T (f(t), \psi(t))\,\mathrm{d}t &= \int_0^T (u(t), -\partial_t \psi(t,\cdot) - i\overline{a}^w \psi(t,\cdot))\,\mathrm{d}t \\ &= \int_0^T (\partial_t u(t) + i a^w u(t), \psi(t))\,\mathrm{d}t.\end{aligned}$$

It follows from this argument that $\partial_t u + i a^w u = f$ in $\mathscr{D}'((0,T) \times \mathbf{R}^d)$. From $u \in L^\infty((0,T], M_\nu)$, $a \in G^{1+\sigma,\sigma}$ and Proposition 4.2 it follows that $\partial_t u \in L^\infty((0,T], M_{\nu-(1+\sigma)})$. If we set $g(0) = u_0$ and

$$g(t) = \int_0^t \partial_t u(\tau)\,\mathrm{d}\tau + u_0, \quad 0 < t \leqslant T,$$

then $g \in C([0,T], M_{\nu-(1+\sigma)})$, and it follows from Lebesgue's differentiation theorem for Bochner integrals [9, Theorem II.2.9] that $g'(t) = \partial_t u(t)$ for almost all $t \in [0,T]$.

If $\psi \in C_c^\infty((0,T) \times \mathbf{R}^d)$ then we obtain from this

$$(u, \partial_t \psi) = -(\partial_t u, \psi) = -\int_{\mathbf{R}^d} \int_0^T \partial_t g(t,x) \overline{\psi(t,x)}\,\mathrm{d}t\,\mathrm{d}x = (g, \partial_t \psi)$$

which shows that $u = g \in C([0,T], M_{\nu-(1+\sigma)})$. Now $\partial_t u + i a^w u = f$ and $a \in G^{1+\sigma,\sigma}$ yields $u \in C^1([0,T], M_{\nu-2(1+\sigma)})$. We may now apply Lemma 7.5 and conclude

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\|_{M_{\nu-2(1+\sigma)}} \lesssim \|u_0\|_{M_{\nu-2(1+\sigma)}} + \int_0^T \|f(t)\|_{M_{\nu-2(1+\sigma)}}\,\mathrm{d}t.$$

Since $\nu \in \mathbf{R}$ is arbitrary we get the following conclusion. If $u_0 \in \mathscr{S}$ and $f \in C_c^\infty((0,T), \mathscr{S}(\mathbf{R}^d))$ then for any $\nu \in \mathbf{R}$ there exists a solution

$$u \in C([0,T], M_{\nu+1+\sigma}) \cap C^1([0,T], M_\nu)$$

to (7.1) such that

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\|_{M_\nu} \lesssim \|u_0\|_{M_\nu} + \int_0^T \|f(t)\|_{M_\nu}\,\mathrm{d}t. \tag{7.12}$$

If $u_0 \in M_s$ and $f \in L^1([0,T], M_s)$ we take sequences $(u_n)_{n=1}^\infty \subseteq \mathscr{S}$ and $(f_n)_{n=1}^\infty \subseteq C_c^\infty((0,T), \mathscr{S}(\mathbf{R}^d))$ such that $\|u_n - u_0\|_{M_s} \to 0$ and $\|f_n - f\|_{L^1([0,T],M_s)} \to 0$ as $n \to +\infty$. The former is possible due to [13, Proposition 11.3.4], and the latter thanks to Lemma 7.7. By the first part of the proof there exists a sequence

$(u_n(t))_{n=1}^{+\infty} \subseteq C([0,T], M_{s+1+\sigma})$ such that $\partial_t u_n(t) + i a^w u_n(t) = f_n(t)$ and $u_n(0) = u_n$ for each $n \geqslant 1$. By (7.12) with $\nu = s$ the sequence $(u_n(t))_n$ is a Cauchy sequence in $C([0,T], M_s)$. It follows that $(\partial_t u_n(t))_{n=1}^{+\infty} \subseteq C([0,T], M_s)$ is a Cauchy sequence in $L^1([0,T], M_{s-1-\sigma})$.

The sequence $(u_n(t))_n$ converges in $C([0,T], M_s)$ to $u(t) \in C([0,T], M_s)$, and the sequence $(\partial_t u_n(t))_n$ converges in $L^1([0,T], M_{s-1-\sigma})$ to $v(t) \in L^1([0,T], M_{s-1-\sigma})$, $v + i a^w u = f$ in $L^1([0,T], M_{s-1-\sigma})$, and $u(0) = u_0$.

If $\psi \in C_c^\infty((0,T) \times \mathbf{R}^d)$ then

$$
\begin{aligned}
(\partial_t u, \psi) = -(u, \partial_t \psi) &= -\lim_{n \to \infty} \int_0^T (u_n(t), \partial_t \psi(t,\cdot))\,\mathrm{d}t \\
&= \lim_{n \to \infty} \int_0^T (\partial_t u_n(t), \psi(t,\cdot))\,\mathrm{d}t \\
&= \int_0^T (v(t), \psi(t,\cdot))\,\mathrm{d}t = (v, \psi)
\end{aligned}
$$

which shows that $v = \partial_t u$ in $L^1([0,T], M_{s-1-\sigma})$. We conclude that $\partial_t u + i a^w u = f$ in $L^1([0,T], M_{s-1-\sigma})$, $u(0) = u_0$, $u \in C([0,T], M_s)$, and

$$
\sup_{0 \leqslant t \leqslant T} \|u(t)\|_{M_s} \lesssim \|u_0\|_{M_s} + \int_0^T \|f(t)\|_{M_s}\,\mathrm{d}t.
$$

It remains to prove the uniqueness of the solution. Suppose $u \in C([0,T], M_s)$, $\partial_t u + i a^w u = 0$ and $u(0) = 0$. Then $u \in C^1([0,T], M_{s-1-\sigma})$ by Proposition 4.2, and thus by Lemma 7.5 we have $u(t) = 0$ in $M_{s-1-\sigma}$, which implies $u(t) = 0$ in M_s, for each $t \in [0,T]$. $\square$

By Remarks 7.2, 7.6, 7.8 and straightforward modifications in the proof of Theorem 7.9 we may strengthen the assumption on $\operatorname{Im} a$, reverse the time direction and obtain results for the equation

$$
\begin{cases}
\partial_t u(t,x) + i a^w(x,D) u(t,x) = f(t,x), \quad x \in \mathbf{R}^d, \quad t \in [-T,T] \setminus 0, \\
u(0,\cdot) = u_0.
\end{cases}
\tag{7.13}
$$

Corollary 7.10. *Let $T > 0$, $k, m \in \mathbf{N} \setminus 0$, $\sigma = \frac{k}{m}$, $a \in G^{1+\sigma,\sigma}$, suppose*

$$
|\operatorname{Im} a(x,\xi)| \leqslant C, \quad (x,\xi) \in T^*\mathbf{R}^d,
$$

for $C > 0$, and let $s \in \mathbf{R}$. If $u_0 \in M_s$ and $f \in L^1([-T,T], M_s)$, the equation (7.13) has a unique solution $u \in C([-T,T], M_s)$.

Since

$$
\begin{aligned}
L^1([-T,T], \mathscr{S}(\mathbf{R}^d)) &= \bigcap_{s \in \mathbf{R}} L^1([-T,T], M_s), \\
C([-T,T], \mathscr{S}(\mathbf{R}^d)) &= \bigcap_{s \in \mathbf{R}} C([-T,T], M_s)
\end{aligned}
$$

we get the following corollary taking into account (4.2).

Corollary 7.11. *Let $T > 0$, $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$, $a \in G^{1+\sigma,\sigma}$, and suppose*

$$|\mathrm{Im}\, a(x,\xi)| \leqslant C, \quad (x,\xi) \in T^*\mathbf{R}^d,$$

for $C > 0$. If $f \in L^1([-T,T], \mathscr{S}(\mathbf{R}^d))$ and $u_0 \in \mathscr{S}(\mathbf{R}^d)$ then the unique solution to (7.13) satisfies $u \in C([-T,T], \mathscr{S}(\mathbf{R}^d))$.

Finally we state a result dual to Corollary 7.11.

Corollary 7.12. *Let $T > 0$, $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$, $a \in G^{1+\sigma,\sigma}$, and suppose*

$$|\mathrm{Im}\, a(x,\xi)| \leqslant C, \quad (x,\xi) \in T^*\mathbf{R}^d,$$

for $C > 0$. If $u_0 \in \mathscr{S}'$ then by (4.2) there exists $s \in \mathbf{R}$ such that $u_0 \in M_s$. If $f \in L^1([-T,T], M_s)$ then the unique solution to (7.13) satisfies $u \in C([-T,T], M_s)$.

8. Propagation of anisotropic Gabor wave front sets for Schrödinger type equations

The following lemma is an anisotropic version of [6, Lemma 3.6]. Its proof is similar so we omit it (cf. also [27, Lemma 3.2]). The lemma will be used in the proof of Proposition 8.2 which is essential for our main result Theorem 8.3.

Lemma 8.1. *Suppose $\sigma > 0$, $r_j(t) \in C([-T,T], G^{m_j,\sigma})$ for $j \geqslant 0$, where $(m_j)_{j=0}^{\infty} \subseteq \mathbf{R}$ is decreasing, $[-T,T] \ni t \mapsto \partial_t r_j(t)(z)$ is continuous for each $z \in T^*\mathbf{R}^d$, and $\partial_t r_j(t) \in L^\infty([-T,T], G^{m_j,\sigma})$ for all $j \geqslant 0$. Then there exists $r(t) \in C([-T,T], G^{m_0,\sigma})$ such that for any $n \geqslant 1$*

$$r(t) - \sum_{j=0}^{n-1} r_j(t) \in C([-T,T], G^{m_n,\sigma}).$$

We write $r(t) \sim \sum_{j=0}^{\infty} r_j(t)$.

Note that $r(t)$ is unique modulo an element in $C([-T,T], \mathscr{S}(\mathbf{R}^{2d}))$.

If $r_j(t) \in L^\infty([-T,T], G^{m_j,\sigma})$ for $j \geqslant 0$ we abuse the notation $r(t) \sim \sum_{j=0}^{\infty} r_j(t)$ to mean

$$r(t) - \sum_{j=0}^{n-1} r_j(t) \in L^\infty([-T,T], G^{m_n,\sigma})$$

for $n \geqslant 1$. In this interpretation $r(t)$ is unique modulo an element in $L^\infty([-T,T], \mathscr{S}(\mathbf{R}^{2d}))$. Thus in Lemma 8.1 it holds $\partial_t r(t) \sim \sum_{j=0}^{\infty} \partial_t r_j(t)$ in the latter sense.

In the next result we use the cutoff function ψ_δ introduced prior to Proposition 6.5.

Proposition 8.2. *Let $\delta > 0$, $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$, and suppose that $a \in G^{1+\sigma,\sigma}$, $a \sim \sum_{j=0}^{\infty} a_j$, where $a_0 \in C^\infty(\mathbf{R}^{2d}\backslash 0)$ is real-valued,*

$$a_0(\lambda x, \lambda^\sigma \xi) = \lambda^{1+\sigma} a_0(x,\xi), \quad \lambda > 0, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \tag{8.1}$$

and $a_j \in G^{(1+\sigma)(1-2j),\sigma}$ for $j \geqslant 1$. The Hamiltonian flow $\chi_t : T^*\mathbf{R}^d\backslash 0 \to T^*\mathbf{R}^d\backslash 0$ corresponding to the Hamiltonian a_0 is then defined for $-T \leqslant t \leqslant T$ with $T > 0$.

If $q_0 \in G^{0,\sigma}$ then there exists a function $t \mapsto q(t)$ such that $q(0) = \psi_\delta q_0$,

$$q(t) \in C([-T,T], G^{0,\sigma}), \tag{8.2}$$

$$q(t) \sim \sum_{j=0}^{\infty} q_j(t), \quad q_j(t) \in C([-T,T], G^{-2j(1+\sigma),\sigma}), \tag{8.3}$$

$$\partial_t q(t) \sim \sum_{j=0}^{\infty} \partial_t q_j(t), \quad \partial_t q_j(t) \in L^\infty([-T,T], G^{-2j(1+\sigma),\sigma}), \tag{8.4}$$

$$q_0(t)(x,\xi) = \psi_\delta(x,\xi) q_0(\chi_{-t}(x,\xi)), \quad (x,\xi) \in T^*\mathbf{R}^d, \quad t \in [-T,T], \tag{8.5}$$

and $r(t) \in L^\infty([-T,T], \mathscr{S}(\mathbf{R}^{2d}))$ where

$$r(t)^w = q(t)^w \left(\partial_t + i a^w\right) - \left(\partial_t + i a^w\right) q(t)^w.$$

Proof. The claim that the Hamiltonian flow $\chi_t(x,\xi) \in T^*\mathbf{R}^d\backslash 0$ corresponding to the Hamiltonian a_0 is defined for $-T \leqslant t \leqslant T$ with the same parameter $T > 0$ for all initial data $(x,\xi) \in T^*\mathbf{R}^d \setminus 0$ is a consequence of Proposition 6.2.

We will design $q(t)$ such that (8.2), (8.3), (8.4) and (8.5) are satisfied, and, noting that $\partial_t q(t)^w = q(t)^w \partial_t + (\partial_t q(t))^w$,

$$r(t) = i\left(q(t)\# a - a\# q(t)\right) - \partial_t q(t) \in L^\infty([-T,T], \mathscr{S}(\mathbf{R}^{2d})). \tag{8.6}$$

By [16, Theorem 18.5.4] we have

$$\begin{aligned} & i\left(q(t)\# a - a\# q(t)\right)(x,\xi) \\ & \sim \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j+1)!2^{2j}} \left(\langle \partial_x, \partial_\eta\rangle - \langle \partial_y, \partial_\xi\rangle\right)^{2j+1} q(t)(x,\xi) a(y,\eta)\Big|_{(y,\eta)=(x,\xi)} \\ & \sim \{q(t), a\}(x,\xi) \\ & \quad + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j+1)!2^{2j}} \left(\langle \partial_x, \partial_\eta\rangle - \langle \partial_y, \partial_\xi\rangle\right)^{2j+1} q(t)(x,\xi) a(y,\eta)\Big|_{(y,\eta)=(x,\xi)} \end{aligned} \tag{8.7}$$

where we use the Poisson bracket notation

$$\{q(t), a\} = \langle \nabla_\xi q(t), \nabla_x a\rangle - \langle \nabla_x q(t), \nabla_\xi a\rangle = \langle \mathcal{J} \nabla_{x,\xi} q(t), \nabla_{x,\xi} a\rangle.$$

If we introduce for $j \geqslant 0$ the bilinear differential operator

$$\{f, g\}_j(x,\xi) = (-1)^j \left(\langle \partial_x, \partial_\eta\rangle - \langle \partial_y, \partial_\xi\rangle\right)^j f(x,\xi) g(y,\eta)\Big|_{(y,\eta)=(x,\xi)} \tag{8.8}$$

then $\{f, g\}_1 = \{f, g\}$, so $\{f, g\}_j$ extends the Poisson bracket to higher order differential operators. Note that for $j \geqslant 0$

$$a \in G^{m,\sigma}, \quad b \in G^{n,\sigma} \quad \Longrightarrow \quad \{a, b\}_j \in G^{m+n-j(1+\sigma),\sigma}. \tag{8.9}$$

The notation (8.8) allows us to abbreviate (8.7) as

$$i\,(q(t)\#a - a\#q(t)) \sim \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!2^{2j}} \{q(t), a\}_{2j+1}.$$

Inserting $a \sim \sum_{j=0}^{\infty} a_j$ and $q(t) \sim \sum_{j=0}^{\infty} q_j(t)$, and collecting terms of order $j \geqslant 0$ gives

$$i\,(q(t)\#a - a\#q(t)) \sim \sum_{j=0}^{\infty} \sum_{k+n+m=j} \frac{(-1)^m}{(2m+1)!2^{2m}} \{q_k(t), a_n\}_{2m+1} \tag{8.10}$$

since $\{q_k(t), a_n\}_{2m+1} \in C([-T, T], G^{-2j(1+\sigma),\sigma})$ when $k + n + m = j$.

The remainder (8.6) can now be written $r(t) \sim \sum_{j=0}^{\infty} r_j(t)$ as an asymptotic sum in $L^\infty([-T, T], G^{0,\sigma})$ with

$$r_j(t) = \sum_{k+n+m=j} \frac{(-1)^m}{(2m+1)!2^{2m}} \{q_k(t), a_n\}_{2m+1} - \partial_t q_j(t) \in L^\infty([-T, T], G^{-2j(1+\sigma),\sigma}) \tag{8.11}$$

for $j \geqslant 0$. In the proof we show how to pick $\{q_j(t)\}_{j=0}^{\infty}$ with the stated properties so that $r(t) \in L^\infty([-T, T], \mathscr{S}(\mathbf{R}^{2d}))$.

Set for $(x, \xi) \in T^*\mathbf{R}^d$ and $t \in [-T, T]$

$$q_0(t)(x, \xi) = \psi_\delta(x, \xi) q_0(\chi_{-t}(x, \xi))$$

so that (8.5) is satisfied. The purpose of the factor ψ_δ is to make $q_0(t)$ a well defined smooth function also around $(0, 0) \in T^*\mathbf{R}^d$ where χ_{-t} may not be smooth. For each $(x, \xi) \in T^*\mathbf{R}^d$, $t \mapsto q_0(t)(x, \xi) \in C([-T, T])$. By Proposition 6.5 we have $q_0(t) \in G^{0,\sigma}$ uniformly for all $t \in [-T, T]$.

We write $q_0(t)(\chi_t(x, \xi)) = \psi_\delta(\chi_t(x, \xi)) q_0(x, \xi)$. Then differentiation with respect to t, $\partial_t \chi_t(x, \xi) = \mathcal{J} \nabla a_0(\chi_t(x, \xi))$ (cf. (6.2)) and the Chain Rule give for $(x, \xi) \in T^*\mathbf{R}^d \backslash 0$

$$\begin{aligned}
\{a_0, \psi_\delta\}(\chi_t(x, \xi)) q_0(x, \xi) &= \langle \nabla \psi_\delta(\chi_t(x, \xi)), \mathcal{J} \nabla a_0(\chi_t(x, \xi)) \rangle q_0(x, \xi) \\
&= \langle \nabla \psi_\delta(\chi_t(x, \xi)), \partial_t \chi_t(x, \xi) \rangle q_0(x, \xi) \\
&= (\partial_t q_0(t))\,(\chi_t(x, \xi)) + \langle \nabla q_0(t)(\chi_t(x, \xi)), \mathcal{J} \nabla a_0(\chi_t(x, \xi)) \rangle \\
&= (\partial_t q_0(t))\,(\chi_t(x, \xi)) - \{q_0(t), a_0\}(\chi_t(x, \xi)).
\end{aligned} \tag{8.12}$$

Thus for all $(x, \xi) \in T^*\mathbf{R}^d$

$$\partial_t q_0(t)(x, \xi) = \{q_0(t), a_0\}(x, \xi) - \{\psi_\delta, a_0\}(x, \xi) q_0(\chi_{-t}(x, \xi)).$$

Note that the right hand side is supported in $\mathbf{R}^{2d} \setminus \mathrm{B}_{\frac{\delta}{2}}$, and the second term is compactly supported in $\overline{\mathrm{B}}_\delta \subseteq T^*\mathbf{R}^d$.

The function $[-T, T] \ni t \mapsto \partial_t q_0(t)(x, \xi)$ is continuous for each $(x, \xi) \in T^*\mathbf{R}^d$. Indeed the continuity of the first term

$$[-T, T] \ni t \mapsto \{q_0(t), a_0\}(x, \xi) = -\langle \nabla q_0(t)(x, \xi), \mathcal{J} \nabla a_0(x, \xi) \rangle$$

is a consequence of (6.3) and the chain rule, and the continuity of the second term has been verified above.

From $q_0(t) \in G^{0,\sigma}$ and (8.9) it now follows that $\partial_t q_0(t) \in L^\infty([-T, T], G^{0,\sigma})$, and then the continuity of $[-T, T] \ni t \mapsto \partial_t q_0(t)(x, \xi)$ and the mean value theorem gives $q_0(t) \in C([-T, T], G^{0,\sigma})$. By (8.11) we have

$$r_0(t) = \{q_0(t), a_0\} - \partial_t q_0(t) = \{\psi_\delta, a_0\} q_0 \circ \chi_{-t} \in L^\infty([-T, T], G^{0,\sigma})$$

which implies that supp $r_0(t) \subseteq \overline{\mathrm{B}}_\delta \subseteq T^*\mathbf{R}^d$ for all $t \in [-T, T]$ so in fact we have $r_0(t) \in L^\infty([-T, T], C_c^\infty)$. This means that the principal symbol of $r(t)$ vanishes: $r_0(t) \sim 0$.

Next we eliminate the second highest order term in (8.11) $r_1(t) \in L^\infty([-T, T], G^{-2(1+\sigma),\sigma})$ by a proper choice of $q_1(t) \in C([-T, T], G^{-2(1+\sigma),\sigma})$. The term in $C([-T, T], G^{-2(1+\sigma),\sigma})$ in (8.10) is

$$\{q_0(t), a_1\} + \{q_1(t), a_0\} - \frac{1}{24}\{q_0(t), a_0\}_3.$$

Define

$$\rho_1(t) = \{q_0(t), a_1\} - \frac{1}{24}\{q_0(t), a_0\}_3 \in C([-T, T], G^{-2(1+\sigma),\sigma}) \tag{8.13}$$

so that $\rho_1(t) + \{q_1(t), a_0\}$ is the term in $C([-T, T], G^{-2(1+\sigma),\sigma})$ in (8.10). Define

$$q_1(t)(\chi_t(x, \xi)) = \int_0^t \rho_1(\tau)(\chi_\tau(x, \xi))\, \mathrm{d}\tau \tag{8.14}$$

or equivalently

$$q_1(t)(x, \xi) = \int_0^t \rho_1(\tau)(\chi_{\tau - t}(x, \xi))\, \mathrm{d}\tau.$$

From (8.13) and Proposition 6.5 it follows that $q_1(t) \in G^{-2(1+\sigma),\sigma}$ uniformly for all $t \in [-T, T]$. We differentiate (8.14) with respect to t which gives if $(x, \xi) \in T^*\mathbf{R}^d \setminus 0$

$$\begin{aligned} \rho_1(t)(\chi_t(x, \xi)) &= (\partial_t q_1(t))(\chi_t(x, \xi)) + \langle \nabla q_1(t)(\chi_t(x, \xi)), \partial_t \chi_t(x, \xi) \rangle \\ &= (\partial_t q_1(t))(\chi_t(x, \xi)) - \{q_1(t), a_0\}(\chi_t(x, \xi)). \end{aligned}$$

Thus $\partial_t q_1(t) - \{q_1(t), a_0\} = \rho_1(t)$ which implies $\partial_t q_1(t) \in L^\infty([-T, T], G^{-2(1+\sigma),\sigma})$.

Let $(x,\xi) \in T^*\mathbf{R}^d$ be fixed. We know that $[-T,T] \ni t \mapsto \rho_1(t)(x,\xi)$ is continuous, and the continuity of

$$[-T,T] \ni t \mapsto \{q_1(t), a_0\}(x,\xi) = -\langle \nabla q_1(t)(x,\xi), \mathcal{J}\nabla a_0(x,\xi)\rangle$$

is a consequence of the continuity of

$$[-T,T] \ni t \mapsto \partial_z q_1(t)(x,\xi) = \int_0^t \partial_z \left(\rho_1(\tau)(\chi_{\tau-t}(x,\xi))\right) \mathrm{d}\tau$$

for $z = x_j$ and $z = \xi_j$ for all $1 \leqslant j \leqslant d$. In turn, the latter is a consequence of

$$\begin{aligned}\partial_z \left(q_1(t+s) - q_1(t)\right)(x,\xi) = &\int_t^{t+s} \partial_z \left(\rho_1(\tau)(\chi_{\tau-t-s}(x,\xi))\right) \mathrm{d}\tau \\ &+ \int_0^t \partial_z \Big(\rho_1(\tau)(\chi_{\tau-t-s}(x,\xi)) - \rho_1(\tau)(\chi_{\tau-t}(x,\xi))\Big) \mathrm{d}\tau,\end{aligned}$$

the chain rule, and again (6.3).

It follows that $[-T,T] \ni t \mapsto \partial_t q_1(t)(x,\xi)$ is continuous for each $(x,\xi) \in T^*\mathbf{R}^d$. Combining this with $\partial_t q_1(t) \in L^\infty([-T,T], G^{-2(1+\sigma),\sigma})$ we may conclude that $q_1(t) \in C([-T,T], G^{-2(1+\sigma),\sigma})$. Referring to (8.11) this implies that $r_1(t) \in L^\infty([-T,T], G^{-2(1+\sigma),\sigma})$ and

$$\begin{aligned} r_1(t) &= \{q_1(t), a_0\} + \{q_0(t), a_1\} - \frac{1}{24}\{q_0(t), a_0\}_3 - \partial_t q_1(t) \\ &= \{q_1(t), a_0\} + \rho_1(t) - \partial_t q_1(t) = 0 \end{aligned}$$

which shows that the choice of $q_1(t)$ in (8.14) indeed eliminates $r_1(t) \in L^\infty([-T,T], G^{-2(1+\sigma),\sigma})$.

In a similar way we construct $q_j(t) \in C([-T,T], G^{-2j(1+\sigma),\sigma})$ for $j \geqslant 2$ using $\{q_k(t)\}_{k=0}^{j-1}$, by defining

$$\rho_j(t) = \sum_{k+n+m=j,\ k<j} \frac{(-1)^m}{(2m+1)!2^{2m}} \{q_k(t), a_n\}_{2m+1} \in C([-T,T], G^{-2j(1+\sigma),\sigma}) \tag{8.15}$$

and

$$q_j(t)(\chi_t(x,\xi)) = \int_0^t \rho_j(\tau)(\chi_\tau(x,\xi))\, \mathrm{d}\tau. \tag{8.16}$$

As before $\partial_t q_j(t) \in L^\infty([-T,T], G^{-2j(1+\sigma),\sigma})$, $q_j(t) \in C([-T,T], G^{-2j(1+\sigma),\sigma})$, and $\partial_t q_j(t) - \{q_j(t), a_0\} = \rho_j(t)$, which yields $r_j(t) \in L^\infty([-T,T], G^{-2j(1+\sigma),\sigma})$ (cf. (8.11)) and

$$r_j(t) = \sum_{k+n+m=j} \frac{(-1)^m}{(2m+1)!2^{2m}} \{q_k(t), a_n\}_{2m+1} - \partial_t q_j(t)$$

$$= \{q_j(t), a_0\} + \rho_j(t) - \partial_t q_j(t) = 0.$$

So $r_j(t) \sim 0$ for all $j \geqslant 0$ which means that

$$r(t) \in \bigcap_{j\geqslant 0} L^\infty([-T,T], G^{-2j(1+\sigma),\sigma}) = L^\infty([-T,T], \mathscr{S}(\mathbf{R}^{2d})).$$

Finally defining $q(t)$ by (8.3), Lemma 8.1 shows that (8.2) and (8.4) hold. The claim $q(0) = \psi_\delta q_0$ is a consequence of $q_0(0) = \psi_\delta q_0$ and $q_j(0) = 0$ for $j \geqslant 1$. □

Combining Corollaries 7.10 and 7.11 with Proposition 8.2 we obtain our main result about propagation of anisotropic Gabor singularities for the evolution equation

$$\begin{cases} \partial_t u(t,x) + i a^w(x,D) u(t,x) = 0, \quad x \in \mathbf{R}^d, \quad t \in [-T,T] \setminus 0, \\ u(0,\cdot) = u_0. \end{cases} \tag{8.17}$$

Theorem 8.3. *Let $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$, and suppose that $a \in G^{1+\sigma,\sigma}$, $a \sim \sum_{j=0}^\infty a_j$, where $a_0 \in C^\infty(\mathbf{R}^{2d}\backslash 0)$ is real-valued,*

$$a_0(\lambda x, \lambda^\sigma \xi) = \lambda^{1+\sigma} a_0(x,\xi), \quad \lambda > 0, \quad (x,\xi) \in T^*\mathbf{R}^d \setminus 0, \tag{8.18}$$

and $a_j \in G^{(1+\sigma)(1-2j),\sigma}$ for $j \geqslant 1$. The Hamiltonian flow $\chi_t : T^*\mathbf{R}^d\backslash 0 \to T^*\mathbf{R}^d\backslash 0$ corresponding to the Hamiltonian a_0 is then defined for $-T \leqslant t \leqslant T$ with $T > 0$. If $u_0 \in \mathscr{S}'(\mathbf{R}^d)$ then (8.17) has a unique solution denoted $\mathscr{K}_t u_0$, and we have

$$\mathrm{WF}_{\mathrm{g}}^\sigma(\mathscr{K}_t u_0) = \chi_t \mathrm{WF}_{\mathrm{g}}^\sigma(u_0), \quad t \in [-T,T].$$

Proof. By Proposition 6.2 there exists $T > 0$ such that the Hamiltonian flow $\chi_t : T^*\mathbf{R}^d\backslash 0 \to T^*\mathbf{R}^d\backslash 0$ corresponding to the Hamiltonian a_0 is well defined for $-T \leqslant t \leqslant T$.

By (4.2) there exists $s \in \mathbf{R}$ such that $u_0 \in M_s$. From Corollary 7.10 we obtain the existence of a unique solution $u(t) = \mathscr{K}_t u_0 \in C([-T,T], M_s)$ to (8.17).

Let $z_0 \in T^*\mathbf{R}^d \backslash \left(\mathrm{WF}_{\mathrm{g}}^\sigma(u_0) \cup \{0\}\right)$. We may assume that $|z_0| = 1$. By (3.19) with $m = 0$ there exists $q_0 \in G^{0,\sigma}$ such that $q_0^w u_0 \in \mathscr{S}$ and $z_0 \notin \mathrm{char}_\sigma(q_0)$. By (3.14) there exists a σ-conic neighborhood $\Gamma \subseteq T^*\mathbf{R}^d\backslash 0$ such that $z_0 \in \Gamma$, and $|q_0(x,\xi)| \geqslant C > 0$ when $(x,\xi) \in \Gamma \setminus \mathrm{B}_r$ for some $r > 0$.

Let $0 < \delta \leqslant |\chi_t(z_0)|$ for all $t \in [-T,T]$. By Proposition 8.2 there exists $q(t) \in C([-T,T], G^{0,\sigma})$ such that $q(t) \sim \sum_{j=0}^\infty q_j(t)$ with $q_j(t) \in C([-T,T], G^{-2j(1+\sigma),\sigma})$ for $j \geqslant 0$, $q_0(t)(x,\xi) = \psi_\delta(x,\xi) q_0(\chi_{-t}(x,\xi))$ and $r(t) \in L^\infty([-T,T], \mathscr{S}(\mathbf{R}^{2d}))$ where

$$r(t)^w = q(t)^w \left(\partial_t + i a^w\right) - \left(\partial_t + i a^w\right) q(t)^w.$$

This gives

$$0 = q(t)^w \left(\partial_t + i a^w\right) u(t) = \left(\partial_t + i a^w\right) q(t)^w u(t) + r(t)^w u(t)$$

that is

$$\left(\partial_t + i a^w\right) q(t)^w u(t) = -r(t)^w u(t).$$

Set $f(t) = -r(t)^w u(t)$. By (3.8) and Proposition 4.2 we have for any $m \in \mathbf{R}$

$$\sup_{|t| \leqslant T} \|f(t)\|_{M_{m+s}} \lesssim \sup_{|t| \leqslant T} \|u(t)\|_{M_s} < \infty$$

which by (4.2) implies that $f \in L^\infty([-T, T], \mathscr{S}(\mathbf{R}^d)) \subseteq L^1([-T, T], \mathscr{S}(\mathbf{R}^d))$.

Thus $q(t)^w u(t)$ solves the equation (7.13), and for the initial value we have

$$q(0)^w u(0) = q_0(0)^w u(0) = (\psi_\delta q_0)^w u_0 = q_0^w u_0 + ((\psi_\delta - 1) q_0)^w u_0 \in \mathscr{S}$$

due to $\psi_\delta - 1 \in C_c^\infty(\mathbf{R}^{2d})$. At this point we may apply Corollary 7.11 which gives $q(t)^w u(t) \in \mathscr{S}(\mathbf{R}^d)$ for all $t \in [-T, T]$. We note that $q_0(t)(\chi_t(x, \xi)) = q_0(x, \xi)$ if $|\chi_t(x, \xi)| \geqslant \delta$, and $\chi_t \Gamma \subseteq T^* \mathbf{R}^d \backslash 0$ is a σ-conic neighborhood of $\chi_t(z_0)$, which is a consequence of Proposition 6.2. This implies that $\chi_t(z_0) \notin \operatorname{char}_\sigma(q(t))$, since the lower order terms $\{q_j(t)\}_{j \geqslant 1}$ in $q(t)$ decay on $T^* \mathbf{R}^d$. By (3.19) this means that $\chi_t(z_0) \notin \mathrm{WF}_{\mathrm{g}}^\sigma(u(t)) = \mathrm{WF}_{\mathrm{g}}^\sigma(\mathscr{K}_t u_0)$. We have shown

$$\mathrm{WF}_{\mathrm{g}}^\sigma(\mathscr{K}_t u_0) \subseteq \chi_t \mathrm{WF}_{\mathrm{g}}^\sigma(u_0), \quad t \in [-T, T].$$

The opposite inclusion follows from $\mathscr{K}_t^{-1} = \mathscr{K}_{-t}$ and $\chi_t^{-1} = \chi_{-t}$. □

Remark 8.4. In Theorem 5.1 the Hamiltonian has by Remark 5.2 the form

$$a = \sum_{j=0}^m a_j$$

where a_j is real-valued for all $0 \leqslant j \leqslant m$, $\sigma = \frac{1}{m-1}$, $a_0 \in G^{1+\sigma, \sigma}$ satisfies (8.18), and $a_j \in G^{\sigma(m-j), \sigma} = G^{(1+\sigma)\left(1 - \frac{j}{m}\right), \sigma} \subseteq G^{1, \sigma}$ for $1 \leqslant j \leqslant m$.

In Theorem 8.3 on the other hand $\sigma = \frac{k}{m}$ and the Hamiltonian is $a \sim \sum_{j=0}^\infty a_j$, a_0 is again real-valued and satisfies (8.18), and $a_j \in G^{(1+\sigma)(1-2j), \sigma} \subseteq G^{-(1+\sigma), \sigma}$ for $j \geqslant 1$.

Comparing Theorem 5.1 and Theorem 8.3 we may conclude that the former is not a particular case of the latter, due to the different assumptions on the perturbation $a - a_0$ of the Hamiltonian.

9. Examples

Let again $\psi_\delta(x, \xi) = \varphi(|x|^2 + |\xi|^2) \in C^\infty(\mathbf{R}^{2d})$ where $\varphi \in C^\infty(\mathbf{R})$, $0 \leqslant \varphi \leqslant 1$, $\varphi(t) = 0$ for $t \leqslant \frac{\delta^2}{4}$ and $\varphi(t) = 1$ for $t \geqslant \delta^2$ for a given $\delta > 0$. Thus $\psi_\delta\big|_{\mathrm{B}_{\frac{\delta}{2}}} \equiv 0$ and $\psi_\delta\big|_{\mathbf{R}^{2d} \backslash \mathrm{B}_\delta} \equiv 1$.

Example 9.1. Let $\delta > 0$, $c \in \mathbf{R}\backslash 0$, $k, m \in \mathbf{N}\backslash 0$, $\sigma = \frac{k}{m}$, and set

$$a(x,\xi) = c\psi_\delta(x,\xi)\left(|x|^{2k} + |\xi|^{2m}\right)^{\frac{1}{2}\left(\frac{1}{k}+\frac{1}{m}\right)}.$$

Then

$$a(\lambda x, \lambda^\sigma \xi) = \lambda^{1+\sigma} a(x,\xi), \quad \lambda \geqslant 1, \quad (x,\xi) \in T^*\mathbf{R}^d, \quad |(x,\xi)| \geqslant \delta,$$

and $a \in G^{1+\sigma,\sigma}$. Theorem 8.3 applies to this Hamiltonian.

Example 9.2. Let $c_1, c_2 \in \mathbf{R} \setminus 0$, $k \in \mathbf{N} \setminus 0$, and set

$$a(x,\xi) = \psi_\delta(x,\xi)\left(c_1|x|^{\frac{2k}{2k-1}} + c_2|\xi|^{2k}\right).$$

With $\sigma = \frac{1}{2k-1}$ we have

$$a(\lambda x, \lambda^\sigma \xi) = \lambda^{1+\sigma} a(x,\xi), \quad \lambda \geqslant 1, \quad (x,\xi) \in T^*\mathbf{R}^d, \quad |(x,\xi)| \geqslant \delta.$$

However, we note that the singularity (non-smoothness) of the term $|x|^{\frac{2k}{2k-1}} = |x|^{1+\sigma}$ at the origin is not annihilated by the cutoff function ψ_δ unless $k = 1$. For this purpose, we would need a cutoff function that depends on x only. But this type of cutoff function does not fit into the calculus with $G^{m,\sigma}$ symbols. So $a \notin G^{1+\sigma,\sigma}$ and we cannot apply Theorem 8.3 to this Hamiltonian.

Acknowledgements

We thank the anonymous referee whose careful comments have spurred us to improve the manuscript. The first author is partially supported by the INDAM-GNAMPA project CUP E53C23001670001. This work is partially supported by the MIUR project “Dipartimenti di Eccellenza 2018-2022” (CUP E11G18000350001).

Funding Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement.

Data availability Not applicable.

Declarations

Conflict of interest There are no conflicts of interests.

REFERENCES

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York Heidelberg Berlin, 1978.

[2] P. Boggiatto, E. Buzano and L. Rodino, *Global Hypoellipticity and Spectral Theory*, Akademie Verlag Mathematical Research **92**, Berlin, 1996.

[3] M. Cappiello, T. Gramchev and L. Rodino, *Entire extensions and exponential decay for semilinear elliptic equations*, J. Anal. Math. **111** (2010), 339–367.

[4] M. Cappiello, R. Schulz and P. Wahlberg, *Conormal distributions in the Shubin calculus of pseudodifferential operators*, J. Math. Phys. **59** (2) 021502 (2018).

[5] M. Cappiello, T. Gramchev, S. Pilipović and L. Rodino, *Anisotropic Shubin operators and eigenfunction expansions in Gelfand–Shilov spaces*, J. Anal. Math. **138** (2019), 857–870.

[6] M. Cappiello, R. Schulz and P. Wahlberg, *Shubin type Fourier integral operators and evolution equations*, J. Pseudo-Differ. Oper. Appl. **11** (1) (2020) 119–139.

[7] M. Chatzakou, J. Delgado and M. Ruzhansky, *On a class of anharmonic oscillators*, J. Math. Pures Appl. **153** (2021) 1–29.

[8] M. A. de Gosson, *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, Birkhäuser, Basel, 2011.

[9] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Mathematical Surveys and Monographs **15** AMS, Providence, Rhode Island, 1977.

[10] G. B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, 1989.

[11] G. Garello and A. Morando, *m-Microlocal elliptic pseudodifferential operators acting on* $L^p_{\mathrm{loc}}(\Omega)$, Math. Nachr. **289** (2016), 1820–1837.

[12] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. II, Academic Press, New York London, 1968.

[13] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[14] K. Gröchenig and J. Toft, *Isomorphism properties of Toeplitz operators and pseudo-differential operators between modulation spaces*, J. Anal. Math. **114** (2011), 255–283.

[15] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston Basel Stuttgart, 1982.

[16] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. I, III, IV, Springer, Berlin, 1990.

[17] L. Hörmander, *Quadratic hyperbolic operators*, Microlocal Analysis and Applications, Lecture Notes in Math. **1495**, Eds. L. Cattabriga, L. Rodino, pp. 118–160, Springer, 1991.

[18] R. Lascar, *Propagation des singularités d'équations pseudodifférentielles quasi homogènes*, Ann. Inst. Fourier Grenoble **27** (1977), 79–123.

[19] J. Martin, *Spectral inequalities for anisotropic Shubin operators*, arXiv:2205.11868 [math.AP] (2022).

[20] F. Nicola and L. Rodino, *Global Pseudo-Differential Calculus on Euclidean Spaces*, Birkhäuser, Basel, 2010.

[21] F. Nicola and L. Rodino, *Propagation of Gabor singularities for semilinear Schrödinger equations*, Nonlinear Differential Equations Appl. **22** (6) (2015), 1715–1732.

[22] C. Parenti and L. Rodino, *Parametrices for a class of pseudo differential operators I,II*, Ann. Mat. Pura Appl. **125** (4) (1980), 221–254 and 255–278.

[23] A. Petersson, *Fourier characterizations and non-triviality of Gelfand-Shilov spaces, with applications to Toeplitz operators*, J. Fourier Anal. Appl. **29** (3) 29 (2023).

[24] K. Pravda-Starov, L. Rodino and P. Wahlberg, *Propagation of Gabor singularities for Schrödinger equations with quadratic Hamiltonians*, Math. Nachr. **291** (1) (2018), 128–159.

[25] L. Rodino and P. Wahlberg, *The Gabor wave front set*, Monaths. Math. **173** (4) (2014), 625–655.

[26] L. Rodino and P. Wahlberg, *Microlocal analysis of Gelfand–Shilov spaces*, Ann. Mat. Pura Appl. **202** (5) (2023), 2379–2420.

[27] L. Rodino and P. Wahlberg, *Anisotropic global microlocal analysis for tempered distributions*, Monatsh. Math. **202** (2) (2023), 397–434.

[28] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer, Berlin Heidelberg, 2001.

[29] R. Schulz and P. Wahlberg, *Equality of the homogeneous and the Gabor wave front set*, Comm. Partial Differential Equations **42** (5) (2017), 703–730.

[30] J. Toft, *The Bargmann transform on modulation and Gelfand–Shilov spaces, with applications to Toeplitz and pseudo-differential operators*, J. Pseudo-Differ. Oper. Appl. **3** (2) (2012), 145–227.

[31] P. Wahlberg, *Propagation of anisotropic Gelfand–Shilov wave front sets*, J. Pseudo-Differ. Oper. Appl. **14** (1) 7 (2023).

[32] P. Wahlberg, *Propagation of anisotropic Gabor wave front sets*, arXiv:2301.03190 [math.AP] (2022).

Marco Cappiello and Luigi Rodino
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 Turin
Italy
E-mail: marco.cappiello@unito.it

Luigi Rodino
E-mail: luigi.rodino@unito.it

Patrik Wahlberg
Dipartimento di Scienze Matematiche
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Turin
Italy
E-mail: patrik.wahlberg@polito.it

Accepted: 24 February 2024