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Some Topics in Diophantine Approximation

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Abstract

This thesis explores different topics arising from the field of Diophantine approximation.

The first part is devoted to an old problem in number theory: the explicit construction of new transcendental numbers. Classical continued fractions have long served as a powerful tool in this direction, exploiting fundamental results in Diophantine approximation, such as Roth's theorem. By contrast, their multidimensional generalization—introduced by Jacobi and later developed by Perron in response to a question of Hermite—remains largely unexplored.

In a joint work with Murru and Romeo, we establish new transcendence criteria for multidimensional continued fractions generated by the Jacobi–Perron algorithm. We prove that expansions characterized by either sufficiently rapid growth or specific quasi-periodic structures in their partial quotients are transcendental. These results extend classical one-dimensional constructions due to Liouville and to Maillet–Baker, to the multidimensional setting.

This part of the thesis provides a self-contained exposition of these original contributions, while reviewing the necessary background on Diophantine approximation and the theory of classical and multidimensional continued fractions.

The second part concerns the distribution of real sequences modulo 1. A seminal result by Weyl states that for an irrational real number α, the sequence $\alpha, 2\alpha, 3\alpha, \ldots$ is uniformly distributed modulo 1, meaning the proportion of terms $(\{na\})_{n\geq 0}$ falling within any subinterval of $[0,1)$ is equal to the length of that subinterval, where $\{\cdot\}$ denotes the fractional part. Many other sequences are known to enjoy the same property. For instance, geometric progressions of the form $(\lambda\alpha^n)_{n\geq 0}$ are uniformly distributed modulo 1 for fixed $\lambda > 0$ and almost every $\alpha > 1$, or conversely, for fixed $\alpha > 1$ and almost every $\lambda > 0$. However, when both α and λ are fixed, the problem becomes significantly more challenging. A theorem of Pisot and Vijayaraghavan states that, if α is algebraic, then such a sequence has finitely many limit points modulo 1 if and only if α is a Pisot number and $\lambda \in \mathbb{Q}(\alpha)$.

A problem arising from this context is whether, given a Pisot number α, one can find λ such that the geometric progression modulo 1 has a prescribed number of limit points. This question is closely related to the problem of identifying linear recurrences with a fixed characteristic polynomial that exhibit a predetermined number of residues modulo an integer.

The aim of the second part is to examine these questions in greater depth and to present recent results obtained in joint work with Sanna.

Contents

Introduction

God made the integers, all else is the work of man.

Leopold Kronecker

The problem of approximating real quantities by rational numbers is as old as mathematics itself. Long before it was recognized as a theoretical issue, this question arose naturally from practical needs in geometry, astronomy, and measurement. Ancient civilizations such as the Egyptians and Babylonians were already confronted with quantities that could not be expressed exactly as ratios of integers, yet had to be handled numerically for concrete applications.

Over time, this perspective has become increasingly modern, reaching into purely theoretical aspects and evolving into what is known today as *Diophantine approximation*, a branch of number theory that, in its most naive formulation, studies how real numbers can be approximated by rationals or, more generally, when certain inequalities admit integer solutions.

The intuitive problem behind Diophantine approximation can be refined into a precise mathematical objective: given a real number α, one seeks to quantify the error incurred when approximating α by a rational p/q. This is achieved by comparing the difference $|\alpha - p/q|$ with a function of the denominator q, which serves intuitively as a natural measure of the "complexity" of p/q. Understanding how rapidly this error may decay as q grows, and whether certain bounds for such error hold for infinitely many rationals, lies at the heart of the subject.

A well known result by Dirichlet [26] in 1842 provided a first general answer to this class of problems, paving the way for subsequent developments. His theorem states that for any real number α and any real $Q > 1$, there exist integers p and q such that $1 \leq q < Q$ and

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qQ}.$$

Although the proof is a simple application of the pigeonhole principle, the consequences of this result are far-reaching. A simple corollary already implies a sharp distinction between rational and irrational numbers basing on the accuracy with which they can be approximated: if α is irrational, then there exist *infinitely many* pairs of relatively prime integers p and q such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}, \tag{1}$$

while this statement fails for rational α. In this sense, irrational numbers are precisely those real numbers that admit infinitely many "good" rational approximations, that is, approximations of the form p/q whose error $|\alpha - p/q|$ decays at least quadratically with the denominator q.

The appearance of the specific function $1/q^2$ bounding the approximation error naturally leads to two problems. On the one hand, one may be interested in effectively producing such good rational approximations of an irrational number. On the other hand, one is led to the notion of *badly approximable numbers*, namely real numbers for which the bound $1/q^2$ in (1) cannot be replaced by any function of q that decreases faster than $1/q^2$. More formally, a number α is said to be *badly approximable* if there is a constant $c = c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}$$

for every rational p/q. This prompts the question of how to identify and characterize numbers that are, in this precise sense, resistant to rational approximation.

Remarkably, both problems admit elegant and complete solutions by employing the theory of continued fractions, which have been one of the most elementary yet powerful tools in number theory since they were devised in the sixteenth century.

Formally, given $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{Z}_{>0}$, an expression of the form

$$[a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

is called a *(simple) continued fraction*, and the integers a_i are its *partial quotients*. Associated with such an expression is a (possibly finite) sequence of rational numbers $(p_n/q_n)_{n\geq 0}$, known as the sequence of *convergents*, defined by truncating the continued fraction at stage n:

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Here p_n, q_n are relatively prime integers and $q_n > 0$ for every integer n. If the continued fraction is infinite (that is, the sequence of partial quotients is infinite), then it can be proved that the sequence of convergents converges to an irrational number α, which is said to be *represented* by the continued fraction $[a_0; a_1, a_2, \ldots]$ and, we write

$$\alpha = [a_0; a_1, a_2, \ldots].$$

If the continued fraction is finite, we take α as the rational number coinciding with the last convergent of the sequence. Moreover, through a generalization of the Euclidean division algorithm, every real number α admits a representation in terms of continued

fractions, which is called the *continued fraction of* α.

The numerators and denominators of the convergents satisfy the following recurrence relations:

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \tag{2}$$
$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \qquad n \geq 0, \tag{3}$$

and initial conditions: $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, $q_0 = 1$. This also shows that $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ are strictly increasing sequences.

Continued fractions possess remarkable approximation properties. In fact, this was historically one of the main motivations for their introduction. With the above notation, one has for every n,

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{a_{n+1}q_n^2} \tag{4}$$
$$\leq \frac{1}{q_n^2}. \tag{5}$$

Thus, continued fractions provide a canonical and effective way to produce good approximations. Furthermore, by (4) one readily observes that continued fractions with rapidly growing partial quotients exhibit exceptionally strong approximation properties. Conversely, a slower growth rate of the partial quotients results in a less accurate approximation by the convergents. In fact, a real number is badly approximable if and only if the partial quotients in its continued fraction are bounded (see [77, Ch. 1, Theorem 5.F]). Furthermore, a classical theorem of Lagrange states that a real number is a quadratic irrational if and only if the sequence of partial quotients in its continued fraction is ultimately periodic. As a consequence, all quadratic irrationals are badly approximable.

Indeed, a deeper phenomenon emerges from this elementary analysis: the quality of rational approximations to a real number is closely linked to its arithmetic nature. Rational numbers are exactly those that admit only finitely many good approximations; quadratic irrationalities turn out to be badly approximable; and more refined examples could be made beyond these cases. In particular, this connection became even more striking in the mid-nineteenth century, when Liouville uncovered a fundamental link between Diophantine approximation and transcendence theory, opening a new and fertile line of research that remains active today.

The central objects of transcendence theory are, of course, transcendental numbers. To define them, one first introduces algebraic numbers: a real number α is algebraic of some degree d if d is the smallest degree for a monic irreducible polynomial $P(X)$ with rational coefficients such that $P(\alpha) = 0$. This polynomial is unique and is called the *minimal polynomial*. Then, a number that is not algebraic is transcendental.

Liouville's Theorem provides then a criterion to distinguish between algebraic and transcendental numbers. It asserts that if a number admits infinitely many "too good" rational approximations, then it cannot be algebraic. Formally, it states that if α is an algebraic number of degree d, then there exists a constant $c = c(\alpha) > 0$

such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^d}$$

for every rational p/q distinct from α. Consequently, any real number that is approximated by rationals with sufficiently rapid decay of the error cannot be algebraic. Liouville used this idea to construct the first explicit examples of transcendental numbers.

For instance, consider

$$\alpha = \sum_{k\geq 1} \frac{1}{10^{k!}}.$$

Defining for every integer $n \geq 1$,

$$q_n = 10^{n!}, \quad p_n = q_n \sum_{k=1}^{n} \frac{1}{10^{k!}},$$

one obtains

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| = \sum_{k=n+1}^{+\infty} \frac{1}{10^{k!}} < \frac{1}{(10^{n!})^n} = \frac{1}{q_n^n}.$$

Hence, given any degree d and positive constant c, the inequality

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{c}{q_n^d},$$

holds for n sufficiently large. By Liouville's Theorem, α must therefore be transcendental.

Historically, this was a breakthrough. Although Euler had formally introduced the notion of transcendental numbers in 1748, no explicit examples were known for nearly a century. Even Euler himself could only formulate conjectures. It was only one century later that Liouville [56, 57] proved the existence of transcendental numbers by providing an explicit construction. Interestingly, the transcendence of familiar constants followed only later: Hermite [42] proved the transcendence of e in 1873, and Lindemann [55] showed that π is transcendental in 1882. Around the same period, Cantor [21] proved that transcendental numbers are not rare. Indeed, he showed that the set of algebraic numbers is countable, whereas the set of real numbers is uncountable. This implies that the set of transcendental numbers is uncountable, and in particular that almost all real numbers are transcendental.

Liouville's ideas also suggest a continued fraction approach to transcendence. By constructing continued fractions whose partial quotients grow sufficiently rapidly, one obtains transcendental numbers via the approximation properties discussed above. Consider the following construction. Pick an arbitrary integer a_0 and consider $p_0 = a_0$ and $q_0 = 1$. Pick then a_1 strictly greater than 1 and compute the first convergent $p_1/q_1 = [a_0; a_1]$. Proceeding inductively, for each $n \geq 0$ choose

$$a_{n+1} > q_n^n. \tag{6}$$

In this way we obtain a continued fraction $\alpha = [a_0; a_1, a_2, \dots]$. By (4),

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{a_{n+1}q_n^2} < \frac{1}{q_n^{n+2}}$$

for every n. Hence Liouville's Theorem implies again that α is transcendental. The key insight is that such continued fractions, often referred to as *Liouville-type* continued fractions, produce rational approximations that are far too strong to be compatible with algebraicity.

This raises further questions. Are there other ways to obtain transcendental numbers through continued fractions? Which growth conditions on the partial quotients are sufficient to ensure transcendence? These problems have motivated a rich and still active area of research.

If one focuses on the rate of growth of the partial quotients, a question naturally concerns the exponent of q_n in (6): can it be reduced? In a certain sense, the answer is yes, and it arises from refinements of Liouville's Theorem – more precisely, from refinements of a weaker form of it.

Indeed, Liouville's Theorem immediately implies that if α is a real algebraic number of degree d, then for every $\kappa > d$ the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\kappa} \tag{7}$$

has only finitely many rational solutions p/q.

During the 20th century, this bound was progressively sharpened in the case $d \geq 2$. In 1909 Thue [88] proved that it is sufficient to require $\kappa > \frac{d}{2} + 1$ to ensure that (7) still admits only finitely many rational solutions. Then, in 1921 Siegel [82] showed that the conclusion already holds if

$$\kappa > \min_{s\in\{1,\dots,d-1\}}\Big(s + \frac{d}{s+1}\Big),$$

and in particular if $k > 2\sqrt{d}$. Dyson [34] and Gelfond [38] independently improved this to $k > \sqrt{2d}$. Finally, Roth [69] established the definitive result: (7) has only finitely many solutions whenever $\kappa > 2$. In view of (5), Roth's Theorem is optimal, as for $\kappa = 2$ continued fractions provide an explicit method to produce infinitely many solutions.

One readily observes that a construction analogous to Liouville-type continued fractions can also be carried out using Roth's Theorem. In particular, if the growth condition in (6) is replaced by

$$a_{n+1} > q_n^\varepsilon, \tag{8}$$

for some fixed $\varepsilon > 0$, then the convergents p_n/q_n satisfy

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{a_{n+1}q_n^2} < \frac{1}{q_n^{2+\varepsilon}},$$

for infinitely many n. Such an approximation is incompatible with Roth's Theorem for algebraic numbers, and therefore the resulting continued fraction represents again a transcendental number.

Although condition (8) on the growth of the partial quotients is substantially weaker than (6), it still forces them to be unbounded. This leads to investigate whether unbounded growth for partial quotients is unavoidable.

Furthermore, the study of transcendental continued fractions with bounded partial quotients is closely connected to another long-standing question. It is widely conjectured that the continued fraction of any irrational algebraic number α is either eventually periodic — which, as is well known, happens precisely when α is a quadratic irrational — or else contains partial quotients of arbitrarily large size. This problem appears to have been first raised by Khintchine [47], but, despite some heuristic insights, our present understanding of the phenomenon remains rather limited (further discussions on the topic can be found in [7, 81, 90].)

A natural preliminary step toward clarifying the conjecture is the construction of explicit transcendental continued fractions with bounded partial quotients. Indeed, if one expects that every non-periodic continued fraction with bounded partial quotients must represent a transcendental number, then it becomes essential to produce at least some concrete examples. Such constructions may provide guidance and partial confidence in support of the conjecture. This prompted various mathematicians after Liouville to explore advanced Diophantine approximation techniques to develop new classes of transcendental continued fractions. The contributions by Maillet [59] are especially significant, as he was the first to construct explicit examples with bounded partial quotients.

In fact Maillet established that transcendence can be deduced from the presence of sufficiently long repetitions in the sequence of partial quotients. More precisely, he proved that if there are infinitely many indices n_i such that

$$a_{n_i} = a_{n_i+1} = \cdots = a_{n_i+\lambda(i)-1},$$

where $\lambda(i)$ is greater than a suitable function of q_{n_i}, then $\alpha = [a_0, a_1, \dots]$ is transcendental, provided that the continued fraction is not eventually periodic. Actually, Maillet proved a more general statement in which one does not require the repetition of a single partial quotient, but rather the recurrence of an entire finite block of consecutive partial quotients. Because of these patterns, continued fractions of this kind are generally referred to as *quasi-periodic*.

The argument used by Maillet to prove his results relies on an extension of Liouville's Theorem that bounds how closely an algebraic number can be approximated by quadratic irrationals. More general, given two distinct algebraic numbers α and β of degree d and e, respectively, Liouville's inequality provides the following useful lower estimate for the distance between α and β:

$$|\alpha - \beta| \geq \frac{k}{\mathrm{H}(\alpha)^e \, \mathrm{H}(\beta)^d} = \frac{c}{\mathrm{H}(\beta)^d}, \tag{9}$$

where $c = \frac{k}{\mathrm{H}(\alpha)^e}$ and $k = k(d, e)$ is a positive constant depending only on the algebraic degrees of α and β, and $\mathrm{H}(\,\cdot\,)$ indicates the *naive height* of an algebraic number, that is the maximum of the absolute values of the coefficients in its minimal polynomial over the integers; this polynomial is obtained by multiplying the monic minimal polynomial by the least common multiple of its denominators. See [16, Corollary A.2] for an explicit value of k.

In fact, under a technical assumption on the growth of $\lambda(i)$, Maillet proved that the sequence of quadratic irrationals α_i defined as having the following eventually periodic continued fraction

$$[a_0; a_1, \ldots, a_{n_i-1}, a_{n_i}, a_{n_i}, a_{n_i}, \ldots]$$

provide infinitely many "too good" approximations to α and, in virtue of the general inequality by Liouville (9), α must be transcendental.

In his 1962 work [11], Baker used a version of Roth's Theorem for number fields obtained by LeVeque [54] to strongly improve the results of Maillet and make them simpler and more explicit. LeVeque's theorem for a quadratic number field $\mathbb{Q}(\sqrt{\Delta})$ states that, if α does not belong to $\mathbb{Q}(\sqrt{\Delta})$, then for any $\varepsilon > 0$ there exists a positive constant $c = c(\alpha, \Delta, \varepsilon)$ such that

$$|\alpha - \beta| > \frac{c}{\mathrm{H}(\beta)^{2+\varepsilon}},$$

for every β in $\mathbb{Q}(\sqrt{\Delta})$.

Baker's main idea was to remark that when infinitely many of the quadratic approximations found by Maillet lie in a same quadratic number field, one can favourably replace the use of Liouville's inequality by the one of LeVeque's theorem. Then, by exploiting an old result by Davenport and Roth [24] on the rate of increase of the denominators q_n of the convergents to an algebraic number, he was able to obtain a stronger result.

The work of Baker was so influential that it opened the way to numerous improvements and generalizations. A particularly noteworthy development in this direction was achieved more recently by Adamczewski and Bugeaud [4], who revisited the analysis of quasi-periodic continued fractions from a different perspective. Their approach consists in applying Schmidt's Subspace Theorem to refine the result of Davenport and Roth, which, in turn, leads to strengthened versions of Baker's theorems. Several transcendence criteria were also established for other classes of continued fractions, such as *sturmian* [8], *stammering* [3] and *palindromic* [5]. Significant progress along these lines was also achieved in [6]. Finally, Bugeaud addressed the case of the so-called *automatic* continued fractions in [18], thus generalizing also a previous work by Queffélec [68].

These advances have renewed interest in the subject and various generalizations of these constructions have since been explored. In particular, several authors have investigated analogous continued fraction constructions in the setting of p-adic numbers [22, 53, 58, 64].

Part of this thesis is devoted to extending Liouville-type and quasi-periodic constructions to the framework of "higher-dimensional" continued fractions. The problem of such a generalization is not recent; rather, it originates from classical questions concerning the characterization of algebraic numbers via periodicity properties of continued fractions.

Lagrange's theorem for classical continued fractions states that a real number has an eventually periodic continued fraction if and only if it is a quadratic irrational. Multidimensional continued fractions (MCFs) were first introduced by Jacobi [43, 44] in response to a question posed by Hermite [41, 66]. The goal was to obtain an analogue of Lagrange's theorem for cubic irrationals. In particular, Hermite asked for a method of representing algebraic irrationalities by means of periodic sequences that can highlight algebraic properties and possibly provide rational approximations. Jacobi's work was later generalized by Perron [65] to treat algebraic irrationalities of arbitrary degree, leading to the so-called *Jacobi–Perron algorithm* for MCFs. This algorithm takes as input an m-tuple of real numbers $(\alpha_1 \dots, \alpha_m)$ and produces as output m sequences of integers

$$[(a_n^{(1)})_{n\geq 0}, \dots, (a_n^{(m)})_{n\geq 0}],$$

which are again referred to as sequences of partial quotients. In the case $m = 1$, the algorithm coincides with the classical one for continued fractions.

Although the algorithm shows, also in the multidimensional case, that periodicity of the sequences of partial quotients is a sufficient condition for the algebraicity of the α_i (see Section 1.4.5), Hermite's original problem – namely, to characterize algebraic irrationals of degree greater than two in terms of periodicity – remains open (see [45, 46] for some recent progress).

As a side remark, it is worth noting that multidimensional continued fractions have also been developed and investigated in the p-adic setting [60, 70].

Part I of this thesis investigates the interplay between multidimensional continued fractions, Diophantine approximation, and transcendence theory. Its structure is as follows.

Chapter 1 contains some useful preliminaries. In particular, Section 1.1 offers a targeted review of the classical theory of continued fractions. This section is structured specifically to establish a consistent framework for comparison with the (typically weaker) properties found in the multidimensional setting. For completeness, Section 1.3 presents the main results of Baker on quasi-periodic continued fractions, together with the necessary technical tools and a revision of certain results in Diophantine approximation. Section 1.4 is devoted to multidimensional continued fractions (MCFs) obtained via the Jacobi-Perron algorithm and their fundamental properties. This section serves as a useful reference, as the primary elementary results on MCFs are collected in Perron's monograph [65], which currently remains available only in German. Chapter 2 contains original results obtained in joint work with Murru and Romeo [1], in which a systematic study of transcendental continued fractions in higher dimensions is initiated. In this chapter, drawing inspiration from Liouville's ideas as well as the work of Maillet and Baker, several transcendence

criteria for general MCFs are established. More refined results are obtained in the two-dimensional case of Jacobi's algorithm. To the best of our knowledge, the only previous contributions closely related to this topic are due to Tamura [86, 87] , where only specific MCFs were considered.

The methods developed in this work aim to deepen the understanding of multidimensional continued fractions associated with both algebraic and transcendental numbers. Some of the results are of independent interest and may be read separately from the rest of the chapter.

Section 2.1 presents new auxiliary results on MCFs that are used in subsequent sections. Section 2.2 establishes a bound on the height of cubic irrationals possessing a periodic Jacobi MCF. Section 2.3 and Section 2.4 are devoted to proving transcendence results for, respectively, Liouville-type and quasi-periodic multidimensional continued fractions. An estimate for the growth of the denominators of convergents of an MCF is also provided. The final section discusses open problems and possible directions for future research.

The second part of the thesis deals with a different topic, still within the framework of Diophantine approximation. As already noted, a fundamental question in the field concerns the quality of approximation of a real number α by rationals p/q. More precisely, one studies when and how the quantity

$$\left|\alpha - \frac{p}{q}\right| \tag{10}$$

is small compared with a suitable function of q. Equivalently, this amounts to estimating the expression $|q\alpha - p|$ as p and $q > 0$ vary over the integers.

A different class of problems arises when one introduces an additional real number $\beta \neq 0$ and considers the behaviour of

$$|q\alpha - p - \beta|, \tag{11}$$

as p and q vary.

If (10) is small, then $q\alpha$ lies very close to an integer p; in other words, the fractional part of $q\alpha$ approaches zero (or 1). This setting is referred to as *homogeneous approximation*. By contrast, when (11) is small, the fractional part of $q\alpha$ approaches that of β. This is known as *inhomogeneous approximation*.

A typical question is whether results from the homogeneous theory extend to the inhomogeneous setting. A first significant difference is that there is no inhomogeneous analogue, even in weak form, of Dirichlet's Theorem – that is, of the statement guaranteeing, for every $Q > 1$, the existence of integers p and q with $0 < q < Q$ and $|q\alpha - p| < Q^{-1}$ (see [23, Ch. III]).

Nevertheless, a theorem of Minkowski asserts that if α is irrational and not of the form $m\beta + n$ for integers m, n, then for any real number β there exist integers p and

q such that

$$|q\alpha - \beta - p| < \frac{1}{4q}.$$

In particular, there are integers q for which the fractional part $\{q\alpha\}$ is arbitrarily close to any prescribed $\beta \in [0,1)$. Equivalently, the sequence $(\{n\alpha\})_{n\geq 0}$ is dense in the interval $[0,1)$.

In fact, a much stronger statement holds. Weyl [92] proved that this sequence is *uniformly distributed* modulo 1. Informally, this means that the proportion of terms falling into any subinterval $I \subseteq [0,1]$ tends to the length of I. More precisely, a sequence $(s_n)_{n\geq 1}$ of real numbers is uniformly distributed modulo 1 if

$$\lim_{n\to+\infty} \frac{\Big|\Big\{\{s_k\} : k = 1,\dots,n\Big\}\Big| \cap I}{n} = \text{length}(I),$$

for every subinterval $I \subseteq [0,1]$. Weyl was the first to introduce exponential sums as a tool in number theory, and he used them to establish a powerful criterion for uniformly distribution. Specifically, he showed that a sequence $(s_n)_{n\geq 1}$ is uniformly distributed modulo 1 if and only if

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i s_n} = 0. \tag{12}$$

An immediate application of (12), shows that the sequence $(n\alpha)_{n\geq 1}$ is uniformly distributed modulo 1 if and only if α is irrational.

The study of a sequence of real numbers $(s_n)_{n\ geq 1}$ modulo 1, that is, the analysis of how the fractional parts $\{s_n\}$ are distributed within the interval $[0,1)$ has numerous applications. On the one hand, it is of practical importance: as an example, sampling from an uniformly distributed sequence can simulate a uniformly distributed random variable, which is relevant for the generation of pseudorandom numbers. On the other hand, the theory has significant applications within number theory itself.

For instance, it is known (see [51]) that a real number λ is normal in an integer base b – meaning that every block of k digits occurs with frequency $1/b^k$ in its base-b expansion – if and only if the sequence $(\lambda b^n)_{n\geq 1}$ is uniformly distributed modulo 1. Thus, tools from the theory of uniform distribution modulo 1 apply directly to questions of normality.

A classical problem arises when b is replaced by an arbitrary real number $\alpha > 1$, and one studies the distribution modulo 1 of the sequence $(\lambda\alpha^n)_{n\geq 1}$. It is known that for fixed $\alpha > 1$, the sequence $(\lambda\alpha^n)_{n\geq 1}$ is uniformly distributed modulo 1 for all $\lambda > 0$ but a set of Lebesgue measure zero ([92]). Conversely, for fixed λ, the same holds for all $\alpha > 1$ but a set of Lebesgue measure zero ([39, 48]). Results of this kind are known as metric results.

However, the problem becomes extremely hard when both λ and α are fixed. It is not known whether $(\pi^n)_{n\geq 1}$, $(e^n)_{n\geq 1}$ or $(\{(3/2)^n\})_{n\geq 1}$ are uniformly distributed modulo 1.

Specifically, the distribution of $(\{(3/2)^n\})_{n\geq 1}$ is significant due to its connections with Waring's problem. Let $g(n)$ be the minimum integer n such that every positive integer can be written as the sum of n-th powers of natural numbers. An explicit formula for $g(n)$ is known ([62] and previous works), but it depends on inequalities involving $\{(3/2)^n\}$. For example, it is shown in [25] that, for $n \geq 6$ if

$$\left\{(3/2)^n\right\} < 1 - (3/4)^n, \tag{13}$$

then

$$g(n) = 2^n + \left\lfloor (3/2)^n \right\rfloor - \left\{(3/2)^n\right\} - 2.$$

No counterexample to (13) is known; the inequality has been verified for all $n \leq 471,600,000$ [50]. Furthermore, the distribution of $(\{(3/2)^n\})_{n\geq 1}$ is also related to the Collatz $3x+1$ conjecture [52].

Actually, despite the abundance of metric results, no explicit example is known of a real $\alpha > 1$ such that $(\alpha^n)_{n\geq 1}$ is uniformly distributed modulo 1. Only elements of the exceptional sets of measure zero arising in the metric theorems are known.

Among these, Pisot numbers are of particular interest. A Pisot number is a real algebraic integer (that is, an algebraic number whose minimal polynomial has integer coefficients) greater than 1, all of whose conjugates—namely, the other roots of its minimal polynomial—have absolute value strictly less than 1. It was proved [67, 89] that, if $\alpha > 1$ is algebraic and $\lambda > 0$, then the sequence $(\{\lambda\alpha^n\})_{n\geq 1}$ has only finitely many limit points if and only if α is a Pisot number and $\lambda \in \mathbb{Q}(\alpha)$.

Further questions concerning the sequence of fractional parts $(\{\lambda\alpha^n\})_{n\geq 1}$ where α a Pisot number have been investigated more recently in [27, 28, 29, 30, 31, 80]. For comprehensive references on these topics, see [13, 17].

Closely related to the previous discussion is a complementary problem investigated by Zaïmi, Selatnia and Zekraoui [80], and more recently considered by Dubickas and Novikas [33]: given a Pisot number α and a positive integer k, is it possible to find a real number λ such that $(\{\lambda\alpha^n\})_{n\geq 1}$ has exactly k limit points?

They partially resolved this question in the special case where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio. In fact, this result is derived as a corollary of a more general theorem established by Dubickas and Novikas concerning linear recurrence sequences modulo an integer. This result asserts that, for every $r \in \mathbb{N}$ there exist positive integers x_1, x_2, m such that the sequence $\boldsymbol{x} = (x_n)_{n\geq 0}$ defined by the Fibonacci recurrence

$$x_{n+2} = x_{n+1} + x_n,$$

for all n, has exactly r distinct residues modulo m. Furthermore, they proved that one can choose x_1, x_2, M so that none of those r distinct residues modulo m is zero if and only if $r \geq 4$.

In a joint paper with Sanna [2], a generalization of the result of Dubickas and Novikas was recently obtained, extending it to a broader class of recurrence sequences. As a consequence, a new corollary was derived concerning the limit points of certain sequences of fractional parts of powers, thus further extending the original result of

Dubickas and Novikas.

Part II of this thesis is devoted to an exposition of our results. In particular, Chapter 3 presents a detailed account of the original contributions contained in [2]. Section 3.1 collects preliminaries and auxiliary results, while Section 3.2 and Section 3.3 contain the proof of the main theorem and its corollary. Lastly, Section 3.4 concludes with a discussion of open questions related to the topic.

Part I

Transcendence via Continued Fractions

1 Preliminaries

Chapter Contents

1.1 Continued fractions

This section briefly introduces continued fractions and their fundamental properties for the subsequent discussion on quasi-periodic continued fractions and transcendence theory. These concepts are also essential when considering the multidimensional case, where many classical properties no longer hold. For an elementary introduction to the classical theory of continued fractions, we refer the reader to [47, 63].

Let $x_0, x_1, x_2, \dots$ be *real* numbers with $x_i > 0$ for every $i \geq 1$. A finite continued fraction is an expression of the form

$$[x_0; x_1, \dots, x_n] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}}}. \tag{1.1}$$

More generally, we call any expression of the above form, or of the form

$$[x_0; x_1, \dots] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots}}} = \lim_{n \to +\infty} [x_0; x_1, \dots, x_n], \tag{1.2}$$

a continued fraction, provided that the limit exists. For simplicity, we shall write expressions of the form (1.1) (resp., (1.2)) using the more compact notation

$$x_0 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{\dots} + \frac{1}{x_n} \qquad \left(\text{resp., } x_0 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{\dots}\right).$$

One is typically interested in the case where the x_i are integers. Before specializing to this setting, we establish some general lemmas.

Let $\boldsymbol{x} = (x_0, x_1, x_2, \dots)$ be any (finite or infinite) sequence of real numbers with $x_i > 0$ for $i \geq 1$. If $\boldsymbol{x}$ is finite, let $N = N(\boldsymbol{x})$ denote its length minus 1, where the length is defined as the number of elements in the sequence. If $\boldsymbol{x}$ is an infinite (resp. finite) sequence, for any integer $n \geq 1$ (resp. $n = 1, \dots, N(\boldsymbol{x})$), define

$$c_n(\boldsymbol{x}) = [x_0; x_1, \dots, x_n]. \tag{1.3}$$

Lemma 1.1. *For every $\boldsymbol{x}$, set*

$$p_{-1}(\boldsymbol{x}) = 1, \quad q_{-1}(\boldsymbol{x}) = 0, \quad p_0(\boldsymbol{x}) = x_0, \quad q_0(\boldsymbol{x}) = 1.$$

For every $n \geq 1$, define

$$p_n(\boldsymbol{x}) = x_n p_{n-1}(\boldsymbol{x}) + p_{n-2}(\boldsymbol{x}), \quad \textit{and} \quad q_n(\boldsymbol{x}) = x_n q_{n-1}(\boldsymbol{x}) + q_{n-2}(\boldsymbol{x}). \tag{1.4}$$

Then, $c_n(\boldsymbol{x}) = p_n(\boldsymbol{x})/q_n(\boldsymbol{x})$.

Proof. The proof is by induction, see [47, Theorem 1]. □

Lemma 1.2. *For every* $n \geq 1$,

$$q_n(\boldsymbol{x})p_{n-1}(\boldsymbol{x}) - p_n(\boldsymbol{x})q_{n-1}(\boldsymbol{x}) = (-1)^n, \tag{1.5}$$

and

$$q_n(\boldsymbol{x})p_{n-2}(\boldsymbol{x}) - p_n(\boldsymbol{x})q_{n-2}(\boldsymbol{x}) = (-1)^{n-1}x_n. \tag{1.6}$$

Proof. This is an immediate consequence of Lemma 1.1 (cf. [47, Theorem 2]). □

As an immediate consequence, we get the following result.

Corollary 1.3. *For every* $n \geq 1$,

$$c_{n-1}(\boldsymbol{x}) - c_n(\boldsymbol{x}) = \frac{(-1)^n}{q_n(\boldsymbol{x})q_{n-1}(\boldsymbol{x})}, \tag{1.7}$$

and

$$c_{n-2}(\boldsymbol{x}) - c_n(\boldsymbol{x}) = \frac{(-1)^{n-1}x_n}{q_n(\boldsymbol{x})q_{n-2}(\boldsymbol{x})}. \tag{1.8}$$

Proof. The results follow from (1.5) and (1.6) by dividing both sides by the appropriate denominators. □

1.1.1 Simple continued fractions

A natural first question in the theory of continued fractions is whether every real number α can be written in the form $\alpha = [a_0; a_1, \dots]$, where a_i are integers and $a_i \geq 1$ for $i \geq 1$, and whether such a representation is unique. Continued fractions of this type are called *simple* (or *arithmetic*) continued fractions. Unless otherwise specified, we shall refer to them simply as "continued fractions" throughout this discussion.

We begin with the case where α is rational, say $\alpha = u_0/u_1$, with $u_1 > 0$. By Euclidean division, we write $u_0 = a_0u_1 + u_2$, for some integers a_0, u_2 with $0 \leq u_2 < u_1$. Hence,

$$\alpha = \frac{u_0}{u_1} = a_0 + \frac{u_2}{u_1} = a_0 + \frac{1}{\frac{u_1}{u_2}}.$$

Since $u_1 > u_2$, we can write $u_1 = a_1u_2 + u_3$ for some non-negative integers a_1 and $u_3 < u_2$. Repeating the process, we construct integers $a_0, a_1, a_2, \dots$, with $a_i \geq 1$ for

all $i \geq 1$, obtaining expressions of the form

$$\alpha = a_0 + \frac{1}{a_1 + \dfrac{1}{\ldots + \dfrac{1}{a_{i-1} + \dfrac{u_{i+1}}{u_i}}}}.$$

Since the sequence $(u_i)_i$ is strictly decreasing, the process terminates and there exists a first index N such that $u_{N+1} = 1$ and $u_{N+2} = 0$, which precludes performing the division between u_{N+1} and u_{N+2}. In this case, we have $u_N = a_N \cdot 1 + 0 = a_N$, and

$$\begin{aligned}
\alpha &= a_0 + \frac{1}{a_1 + \dfrac{1}{\ldots + \dfrac{1}{a_{N-1} + \dfrac{u_{N+1}}{u_N}}}} \\
&= a_0 + \frac{1}{a_1 + \dfrac{1}{\ldots + \dfrac{1}{a_{N-1} + \dfrac{1}{u_N}}}} \\
&= a_0 + \frac{1}{a_1 + \dfrac{1}{\ldots + \dfrac{1}{a_{N-1} + \dfrac{1}{a_N}}}} \\
&= [a_0; a_1, \ldots, a_N].
\end{aligned}$$

Thus every rational number admits a finite continued fraction. Moreover, since N was the first index for which $u_{N+1} = 1$, this means that $a_N = u_N \geq 2$, and one also has the alternative representation

$$\alpha = [a_0; a_1, \ldots, a_N - 1, 1]. \tag{1.9}$$

Conversely, (1.9) can be reduced to the previous form, so we obtain the following result.

Theorem 1.4. *Any rational number can be expressed as a finite continued fraction of the form*

$$[a_0; a_1, \ldots, a_N],$$

where the a_i are integers and $a_1, \ldots, a_N \geq 1$. Further, the last term a_N can be always modified so as to make the number of terms in the continued fraction either even or odd.

Example 1.5. Let us apply the above procedure to the number $\alpha = 11/4$.

$$\frac{11}{4} = 2 + \frac{3}{4} = 2 + \frac{1}{\frac{4}{3}} = 2 + \frac{1}{1 + \frac{1}{3}} = [2; 1, 3].$$

The same construction extends to any real number α. Define

$$a_0 = \lfloor \alpha \rfloor \quad \text{and} \quad \alpha_1 = \frac{1}{\alpha - \lfloor \alpha \rfloor},$$

when $\alpha \notin \mathbb{Z}$, otherwise put $a_0 = \alpha$ and the algorithm stops. Iterating, we define

$$a_n = \lfloor \alpha_n \rfloor \quad \text{and} \quad \alpha_{n+1} = \frac{1}{\alpha_n - \lfloor \alpha_n \rfloor},$$

whenever $\alpha_n \notin \mathbb{Z}$, otherwise $a_n = \alpha_n$ and we stop. At every step n we have $\alpha_n = a_n + 1/\alpha_{n+1}$ and

$$\alpha = a_0 + \frac{1}{a_1 + \dfrac{1}{\ldots + \dfrac{1}{a_{n-1} + \dfrac{1}{\alpha_n}}}}. \tag{1.10}$$

If α is irrational, then every α_n is also irrational, thus the process never terminates, producing an infinite sequence $a_0, a_1, a_2, \ldots$. Further, since $\alpha_n > 1$ for every $n \geq 1$, then $a_1, a_2, \cdots \geq 1$. If α is rational, the algorithm coincides with the Euclidean algorithm as above and provides a finite sequence.

The numbers a_i are called the partial quotients and α_i the complete quotients of α.

Example 1.6. Let us compute the first partial quotients of π.

$$\pi = 3 + 0,14159... = 3 + \frac{1}{\frac{1}{0,14159...}} = 3 + \frac{1}{7,06264...} = 3 + \frac{1}{7 + 0,06264...}$$
$$3 + \frac{1}{7 + \frac{1}{\frac{1}{0,06264...}}} + = 3 + \frac{1}{7 + \frac{1}{15,99659...}} = 3 + \frac{1}{7 + \frac{1}{15+0,99659...}} = [3; 7, 15, \ldots].$$

1.1.2 Convergents and properties

Definition 1.7. Let α be an irrational number (resp. a rational number). Let $a_0, a_1, \ldots$ (resp. $a_0, a_1, \ldots, a_N$) be the sequence of partial quotients of α. For any integer $n \geq 1$ (resp. $n = 1, \ldots, N$), the rational number

$$c_n = [a_0; a_1, \ldots, a_n] \tag{1.11}$$

is called the n-th convergent of α.

From now on, statements involving "for all $n \geq 1$" are understood to hold for all indices for which the quantities involved are defined.

Remark 1.8. Observe simply that, if α_n is the n-th complete quotient, then by the iterative construction the algorithm applied to α_n provides the sequence

$$[a_{n+1}, a_{n+2}, \dots].$$

The convergents of a continued fractions play a central role in Diophantine approximation. In this setting, we denote by p_n, q_n the sequences defined in Lemma 1.1 for the sequence of partial quotients, so that $c_n = p_n/q_n$. In matrix notation, the recursion given in Lemma 1.1 can be written as

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \dots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}.$$

Lemma 1.2 implies that p_n and q_n are coprime, so each convergent $c_n = p_n/q_n$ is in lowest terms.

Since q_n is positive for every n, Theorem 1.3 immediately implies the following properties about the distribution of the convergents.

Theorem 1.9. *The convergents of even index $(c_{2k})_{k\geq 0}$ form an increasing sequence, while those of odd index $(c_{2k+1})_{k\geq 0}$ form a decreasing sequence.*

Further, for every $h, k \geq 0$ we have

$$c_{2h} < c_{2k+1}.$$

Finally, for every $n \geq 2$ the convergent c_n lies between the two preceding convergents, namely c_{n-1} and c_{n-2}.

Proof. Since $q_n > 0$ for every n, the first claim follows from (1.8) by taking $n = 2k$ and $n = 2k + 1$, and examining the sign of $(-1)^n$.

For the second claim, assume $h \geq k$. Taking $n = 2h + 1$ in (1.7), we obtain $c_{2h} < c_{2h+1}$. By the first part, $c_{2h+1} < c_{2k+1}$, since $h \geq k$. The case $h \leq k$ is analogous.

The final statement follows from the previous two by distinguishing the cases where n is even or odd. □

Lemma 1.10. *Let $a_0, a_1, \dots, a_{n+1}$ be the first partial quotients of α. If n is even, then*

$$c_n < \alpha \leq c_{n+1}.$$

If n is odd,

$$c_{n+1} \leq \alpha < c_n.$$

Proof. See [63, Section 3.6]. □

1.1.3 Convergence

We are now ready to prove that irrational numbers can be expanded as a (simple) continued fraction.

Theorem 1.11. *Let α be an irrational number, and let $a_0, a_1, \dots$ be the sequence of partial quotients obtained via the algorithm above. Then,*

$$\alpha = [a_0; a_1, a_2 \dots] = \lim_{n \to +\infty} [a_0; a_1, \dots, a_n].$$

Proof. We show that the sequence of convergents converges to α. First, the limits

$$\lim_{k \to +\infty} c_{2k} = l_1 \qquad \text{and} \qquad \lim_{k \to +\infty} c_{2k+1} = l_2$$

exist, since by Theorem 1.9 the two sequences are monotonic. Moreover, $l_1 \leq l_2$ and by Lemma 1.10 we have $l_1 \leq \alpha \leq l_2$. Finally, (1.7) implies that $l_1 = l_2$. Hence the sequence $(c_n)_{n \geq 1}$ converges to α. □

Theorem 1.4 and Theorem 1.11 show that every real number α admits an arithmetic continued fraction. The finiteness of such a continued fraction characterizes the rationality of α.

Theorem 1.12. *Let $\boldsymbol{x} = (x_0, x_1, \dots)$ an infinite sequence of integers with $x_i \geq 1$ for all $i \geq 1$. Then,*

$$[x_0; x_1, x_2, \dots]$$

converges to an irrational number.

Proof. We first prove that the limit

$$\lim_{n \to +\infty} [x_0; x_1, \dots, x_n]$$

exists. But, since $q_n(\boldsymbol{x}) > 0$ for every n, convergence follows exactly as in the proof of Theorem 1.11.

Let $\alpha = [x_0; x_1, \dots]$. Suppose, by contradiction, that α is rational. Then, by the algorithm,

$$\alpha = [a_0; a_1, \dots, a_N],$$

for some integers $a_0, a_1, \dots, a_N$ with $a_i \geq 1$ for $i \geq 1$ and $a_N \geq 2$.

Hence,

$$a_0 + \frac{1}{a_1 + \ldots + \frac{1}{a_N}} = x_0 + \frac{1}{[x_1; x_2, \ldots]}. \tag{1.12}$$

Arguing as before, one checks that for every i, the limit

$$\lim_{n \to +\infty} [x_i, \ldots, x_n]$$

exists and is positive. In fact, for $n \geq 1$, the limit is strictly greater than 1, since $[x_{i+1}, x_{i+2}, \ldots] > 0$ and

$$[x_i, x_{i+1}, \ldots] = x_i + \frac{1}{[x_{i+1}, x_{i+2}, \ldots]}.$$

Thus, both sides of (1.12) have the same integer part, so $a_0 = x_0$ and $[a_1; \ldots, a_N] = [x_1; x_2, \ldots]$. Repeating the argument inductively yields $a_i = x_i$ for $i = 0, \ldots, N$, and

$$0 = \frac{1}{[x_{N+1}, x_{N+2}, \ldots]},$$

which is a contradiction. Hence α is irrational. □

Corollary 1.13. *Every rational number admits only finite continued fractions.*

1.1.4 Uniqueness

We now address the uniqueness of the simple continued fraction of a real number. The following theorem covers the case of a rational number, whose continued fraction is unique up to an ambiguity in the last partial quotient.

Theorem 1.14. *Let α be a rational number, and let $a_0, a_1, \ldots, a_N$ be its partial quotients, so that $\alpha = [a_0; a_1, \ldots, a_N]$. Let also $x_0, x_1, \ldots, x_M$ be integers with $x_i \geq 1$ for $i = 1, \ldots, M$, and suppose that*

$$\alpha = [x_0; x_1, \ldots, x_M].$$

Then, one of the following holds:

(i) $N = M$ and $a_i = x_i$ for all $i = 0, \ldots, N$;

(ii) $M = N + 1$, $a_i = x_i$ for $i = 1, \ldots, N - 1$, $x_{N+1} = 1$ and $a_N = x_N + 1$;

(iii) $N = M + 1$, $a_i = x_i$ for $i = 1, \ldots, M - 1$, $a_{M+1} = 1$ and $x_M = a_M + 1$;

Proof. Write

$$a_0 + \cfrac{1}{a_1 + \ldots + \cfrac{1}{a_N}} = x_0 + \cfrac{1}{x_1 + \ldots + \cfrac{1}{x_M}}. \tag{1.13}$$

Since a_0 and x_0 are both the integer part of α, they must be equal. Then, (1.13) implies that

$$a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_N}} = x_1 + \cfrac{1}{x_2 + \ldots + \cfrac{1}{x_M}}.$$

Arguing inductively, and using that $[a_i; \ldots, a_N]$ and $[x_i; \ldots, x_M]$ are rationals greater than 1, we obtain $a_i = x_i$ for all $i = 1, \ldots, L-2$, where $L = \min(M, N)$.

If $M = N$, then $a_{N-1} + 1/a_N = x_{N-1} + 1/x_N$ and, since all the partial quotients a_i and x_i are integers, one checks immediately that $a_{N-1} = x_{N-1}$ and $a_N = x_N$. Suppose $M > N$. Then,

$$a_{N-1} + \cfrac{1}{a_N} = x_{N-1} + \cfrac{1}{x_N + \ldots + \cfrac{1}{x_M}}.$$

Since $[x_N, \ldots, x_M] > 1$, either $a_N = 1$ and $a_{N-1} + \cfrac{1}{a_N} = x_{N-1}$, or $a_N \geq 2$ and $a_{N-1} = x_{N-1}$. The first case yields

$$0 = x_N + \cfrac{1}{x_{N+1} + \ldots + \cfrac{1}{x_M}},$$

a contradiction. Hence, it must be $a_{N-1} = x_{N-1}$ and

$$a_N = x_N + \cfrac{1}{x_{N+1} + \ldots + \cfrac{1}{x_M}}.$$

Since both a_N and x_N are integers, the tail $\cfrac{1}{x_{N+1} + \ldots + \cfrac{1}{x_M}}$ must also be an integer, which occurs only if $M = N + 1$ and $x_{N+1} = 1$.

If $M < N$, by changing the role of all the partial quotients a_i and x_i, then the same argument as before shows that $N = M + 1$, $a_{M+1} = 1$ and $x_M = a_M + 1$. □

We can now consider the case of an irrational number.

Theorem 1.15. *Let α be an irrational number, and let $a_0, a_1, \ldots$ be its partial quotients, so that $\alpha = [a_0; a_1, \ldots]$. Let also $x_0, x_1, \ldots$ be integers with $x_i \geq 1$ for $i \geq 1$, and suppose*

$$\alpha = [x_0; x_1, x_2, \ldots].$$

Then, $a_i = x_i$ for all $i \geq 0$.

Proof. For every $i \geq 1$, set $\xi_i = [x_i; x_{i+1}, \dots]$, so that $\xi_i > 1$. Then, $\alpha = x_0 + 1/\xi_1$. On the other hand, $\alpha = a_0 + 1/\alpha_1$, where $\alpha_1 > 1$. Hence, $x_0 = a_0$, since both are the integer parts of α.

Proceeding inductively, suppose that $a_k = x_k$ for all $k \leq n-1$. Then, $\alpha_n = \xi_n$ and

$$a_n + \frac{1}{\alpha_{n+1}} = x_n + \frac{1}{\xi_{n+1}}.$$

Therefore, $a_n = x_n$. This completes the induction and the proof. □

1.1.5 Periodic continued fractions

So far we have seen a characterization of rational and irrational numbers in terms of the finiteness of their continued fractions. For infinite continued fraction, a further characterization is given by periodic continued fractions.

Definition 1.16. An infinite continued fraction $[a_0; a_1, a_2, \dots]$ is said to be (ultimately) periodic if there exist integers $t > 0$ and $k \geq 0$ such that

$$a_{n+t} = a_n,$$

for every $n \geq k$. The finite sequence $a_k, \dots, a_{k+t-1}$ is called the period, and the integer t is called the length of the period.

If $k = 0$, the continued fraction is said to be purely periodic. If $k > 0$, the finite sequence of partial quotients $a_0, \dots, a_{k-1}$ is called the pre-period.

A periodic continued fraction with period $a_k, \dots, a_{k+t-1}$ and pre-period $a_0, \dots, a_{k-1}$ is generally denoted as

$$[a_0; a_1, \dots, a_{k-1}, \overline{a_k, \dots, a_{k+t-1}}].$$

Such continued fractions provide a complete characterization of quadratic irrationals.

Theorem 1.17. *Let α be a real number such that its continued fraction $[a_0; a_1, \dots]$ is ultimately periodic. Then, α is an algebraic number of degree 2.*

The proof is easy and formal; we briefly outline it, as the same argument will be used later for periodic multidimensional continued fractions (see Theorem 2.14.)

Proof. Suppose first that the continued fraction of α is purely periodic with length of the period t. Then, $\alpha_t = \alpha$ and we can write

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_{t-1} + \cfrac{1}{a_0 + \cfrac{1}{\ldots}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_{t-1} + \cfrac{1}{\alpha}}}}.$$

Rearranging the expression, we see that α satisfies a polynomial equation of degree 2 with integer coefficients. Moreover, this polynomial must be irreducible, since the continued fraction is infinite and therefore α is irrational.

If the continued fraction is not purely periodic, we write

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_{t-1} + \cfrac{1}{\beta}}}}, \tag{1.14}$$

where β has a purely periodic continued fraction. The conclusion follows from the first part of the proof, since β is an algebraic irrational of degree 2, and hence the same holds for α by virtue of (1.14). □

Remark 1.18. By (1.14), any two periodic continued fractions sharing the same period necessarily lie in the same number field.

The converse direction of Theorem 1.17, which is considerably more difficult, was proved by Lagrange in 1770:

Theorem 1.19. *If α is an algebraic number of degree 2, then its continued fraction is ultimately periodic.*

Proof. See [63, Chapter 4]. □

1.2 Basic results in Diophantine approximation

As mentioned above, continued fractions represent a fundamental tool in Diophantine approximation. We begin this section by presenting a classical approximation result due to Dirichlet, which arises naturally from the theory of continued fractions. We then recall Liouville's Theorem to illustrate how Dirichlet's bound is, in a sense, optimal for algebraic numbers. We then discuss a strong generalization of this result, leading to Roth's celebrated theorem. Finally, we review some of the key consequences of Roth's result, focusing on its extensions and quantitave versions.

1.2.1 Approximating via continued fractions

A standard reference for this part is again [47], specifically Chapter 2.

In Lemma 1.10, it was shown that a real number α always lies between two consecutive convergents. We now show that the approximation given by convergents improves at each step.

Lemma 1.20. *Let $\alpha = [a_0; a_1, \dots]$ (resp. $\alpha = [a_0; a_1, \dots, a_N]$). Then, for every $n \geq 0$ (resp. for every $n = 1, \dots, N-1$), it holds*

$$\alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}.$$

Proof. Write

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_n + \cfrac{1}{\alpha_{n+1}}}}} = [a_0; a_1, \dots, a_n, \alpha_{n+1}].$$

The result then follows from Lemma 1.1 by taking $\boldsymbol{x} = (a_0, a_1, \dots, a_n, \alpha_{n+1})$. □

Theorem 1.21. *Let $\alpha = [a_0; a_1, \dots]$ (resp. $\alpha = [a_0; a_1, \dots, a_N]$). Then, for every $n \geq 1$ (resp. for every $n = 1, \dots, N-1$), it holds*

$$|\alpha - c_n| < |\alpha - c_{n-1}|.$$

Proof. By Lemma 1.20,

$$\alpha \left(\alpha_{n+1} q_n + q_{n-1}\right) = \left(\alpha_{n+1} p_n + p_{n-1}\right)$$

for every $n \geq 1$ (resp. for every $n = 1, \dots, N-1$). Dividing both sides by $\alpha_{n+1} q_n$ and rearranging, we obtain

$$\left|\alpha - \frac{p_n}{q_n}\right| = \left|\frac{q_{n-1}}{\alpha_{n+1} q_n}\right| \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right|.$$

Since $\alpha_{n+1} \geq 1$ and $q_n > q_{n-1} > 0$, we have

$$0 < \left|\frac{q_{n-1}}{\alpha_{n+1} q_n}\right| < 1,$$

which proves the claim. □

We can also obtain a precise upper bound on the error $|\alpha - c_n|$.

Theorem 1.22. *Let $\alpha = [a_0; a_1, \dots]$ (resp. $\alpha = [a_0; a_1, \dots, a_N]$). Then,*

$$\frac{1}{2q_{n+1}q_n} < \left|\alpha - c_n\right| < \frac{1}{q_{n+1}q_n},$$

for every $n \geq 1$ (resp. for every $n = 1, \dots, N-1$).

Proof. We know that α lies between c_n and c_{n+1}, hence $|\alpha - c_n| < |c_{n+1} - c_n|$. Moreover, by Theorem 1.3,

$$|c_n - c_{n+1}| = \frac{1}{q_{n+1}q_n}.$$

Finally, by Theorem 1.21 α is closer to c_{n+1} than to c_n, hence

$$|\alpha - c_n| > \frac{|c_{n+1} - c_n|}{2} = \frac{1}{2q_{n+1}q_n}.$$ □

The following theorem is often regarded as a weak formulation of Dirichlet's Theorem. It follows directly from the main result in [26], which was established independently of the theory of continued fractions. Here it is presented as a consequence of Theorem 1.22.

Theorem 1.23. *If α is irrational, there exist infinitely many pairs of coprime integers (p, q), with $q > 0$, such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{1.15}$$

Proof. Since α is irrational, its continued fraction is infinite, hence there are infinitely many convergents $c_n = p_n/q_n$. The result then follows from Theorem 1.22. □

Remark 1.24. Theorem 1.23 does not hold if α is rational. Indeed, let $\alpha = u/v$, with $v > 0$. If $p/q \neq \alpha$, then

$$\left|\alpha - \frac{p}{q}\right| = \left|\frac{u}{v} - \frac{p}{q}\right| = \left|\frac{qu - pv}{vq}\right| \geq \frac{1}{vq}.$$

Hence, (1.15) can be satisfied by only finitely many pairs (p, q) of coprime integers.

1.2.2 Roth's Theorem

For algebraic numbers of degree 2, Liouville's Theorem ensures that the exponent of q in (1.15) cannot be improved, otherwise only finitely many solutions (p, q) would remain. More broadly, the result is the following.

Theorem 1.25 (Liouville's Theorem)**.** *Let α be a real algebraic number of degree $d \geq 2$. Then, there exists a constant $c(\alpha) > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^d}, \tag{1.16}$$

for every pair of integers (p, q), with $q > 0$.

Proof. See [77, Chapter 5]. □

Corollary 1.26. *Let α be a real algebraic number of degree d. Then, for every real number $\kappa > d$,*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\kappa} \tag{1.17}$$

has only finitely many solutions in integers (p, q) with $q > 0$.

Proof. If (p, q) satisfies (1.16), then by comparison with (1.17) we obtain

$$\frac{c(\alpha)}{q^d} < \frac{1}{q^\kappa}.$$

Hence, only finitely many values of q are possible. If there were infinitely many solutions (p, q), then for some fixed $\bar{q}$ there would be infinitely many integers p such that

$$\left|\alpha - \frac{p}{\bar{q}}\right| < \frac{1}{\bar{q}^\kappa},$$

which is impossible since

$$\sup_p \left|\alpha - \frac{p}{\bar{q}}\right| = +\infty.$$

This yields a contradiction and the corollary follows. □

Roth's Theorem shows that, for algebraic numbers of any degree, the exponent 2 in (1.15) is optimal. More precisely:

Theorem 1.27 (Roth's Theorem)**.** *Let α be an algebraic number. Then, for every $\varepsilon > 0$, there exist only finitely many pairs of integers (p, q), with $q > 0$, such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}. \tag{1.18}$$

Proof. See [69] or [77, Chapter 5]. □

Remark 1.28. Theorem 1.27 can be stated equivalently by saying that, for a given algebraic number α and every $\varepsilon > 0$, there exists a positive constant $c(\alpha, \varepsilon)$ for which

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{2+\varepsilon}},$$

for all integers p, q with $q > 0$.

The previous result has a natural application to transcendence. As already pointed out in the introduction, one can construct a sequence of partial quotients as follows: choose $a_0 \in \mathbb{Z}$ and $a_1 \in \mathbb{Z}_{\geq 1}$, and compute $p_1 = a_1 p_0 + p_{-1}$, $q_1 = a_1 q_0 + q_{-1}$, where the initial conditions p_0, q_0, p_{-1}, q_{-1} are as in Lemma 1.1. Fix $\varepsilon > 0$, and for every $n \geq 2$ pick a_n such that

$$a_n > q_{n-1}^{\varepsilon}.$$

This defines an infinite continued fraction $[a_0; a_1, a_2, \dots]$ which, by Theorem 1.12, converges to an irrational number α. Applying Theorem 1.22, we get

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}q_n} = \frac{1}{(a_{n+1}q_n + q_{n-1})q_n} < \frac{1}{a_{n+1}q_n^2} < \frac{1}{q_n^{2+\varepsilon}},$$

for all $n \geq 2$. Hence, by Theorem 1.27, α is transcendental.

For such a construction, the denominators q_n of the convergents grow very rapidly. Indeed,

$$q_n > a_n q_{n-1} + q_{n-2} > q_{n-1}^{(1+\varepsilon)} > q_{n-2}^{(1+\varepsilon)^2} > \cdots > q_2^{(1+\varepsilon)^{n-2}} \geq 2^{(1+\varepsilon)^{n-2}},$$

and thus, for n sufficiently large,

$$\log\log q_n > (n-2)\log(1+\varepsilon) + \log\log 2 > cn, \tag{1.19}$$

for some constant $c > 0$.

Liouville's Theorem implies that such a growth rate of $(q_n)_{n\geq 0}$ is, in general, sufficient to guarantee transcendence. To see this, observe that, if α is algebraic of degree $d \geq 2$, then for every $k > d$,

$$\frac{c(\alpha)}{q_{n-1}^k} < \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_n q_{n-1}}.$$

Hence,

$$q_n < q_{n-1}^{k-1}\, c(\alpha)^{-1}.$$

Proceeding inductively, one arrives at

$$q_n < q_0^{(k-1)^n}\ c(\alpha)^{-[(k-1)^n+(k-1)^{n-1}+\cdots+1]} = c(\alpha)^{-\frac{(k-1)^{n+1}}{k-2}},$$

since $q_0 = 1$. Taking the logarithm, it follows

$$\log q_n < \frac{\log c(\alpha)}{2-k}(k-1)^{n+1}.$$

Without loss of generality, one can now assume $0 < c(\alpha) < 1$. This allows one to take the logarithm again and obtain an upper bound on the growth of $(q_n)_{n\geq 0}$ of the form

$$\log\log q_n < c'n,$$

for a suitable positive constant c' depending on α.

A more precise estimate on the growth of the denominators of the convergents to an algebraic number α was given by Davenport and Roth [24].

Theorem 1.29. *Let α be algebraic, then there exists a positive constant $c = c(\alpha)$ such that for all n,*

$$\log\log q_n < \frac{c(\alpha)\, n}{\sqrt{\log(n)}}. \tag{1.20}$$

Proof. See Theorem 3 in [24]. □

This result is a key tool in the proof of Baker's theorems on quasi-periodic continued fractions (see Section 1.3).

As an immediate corollary of Theorem 1.29, we get a transcendence criterion.

Corollary 1.30. *If α is an irrational number such that*

$$\limsup_{n\to+\infty} \frac{\log\log q_n \sqrt{\log n}}{n} = +\infty,$$

then α is transcendental.

Davenport and Roth actually derived Theorem 1.29 from an estimate on the number of "exceptional" solutions to (1.18).

Theorem 1.31. *If α is algebraic of degree $d \geq 2$ and $0 < \varepsilon < 1/3$, then the number $\mathcal{N}(\alpha, \varepsilon)$ of solutions of (1.18) in coprime integers p, q, with $q > 0$, is such that*

$$\mathcal{N}(\alpha, \varepsilon) < c_1 \exp(c_2 \varepsilon^{-2}), \tag{1.21}$$

for some positive constants c_1, c_2 depending only on α.

As we will briefly discuss in Section 2.5, any qualitative improvement of (1.21) leads to an improvement of (1.20), and can be used to obtain stronger results than those presented in Section 1.3.

For the sake of completeness, we conclude this section by mentioning a natural extension of classical Diophantine approximation, namely the problem of simultaneous approximation. Rather than studying how well a single real number α can be approximated by a rational number p/q, simultaneous approximation considers a collection of real numbers $(\alpha_1, \dots, \alpha_n)$ and seeks to approximate them using rational numbers that share the exact same denominator q. Most of the results discussed above admit simultaneous analogues (see [77, Chapters 2 and 6]). The following theorem, for instance, reduces to Roth's Theorem 1.27 when $n = 1$.

Theorem 1.32. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be real algebraic numbers such that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$. For any $\varepsilon > 0$, there exist only finitely many n-tuples of integers $(p_1, \dots, p_n, q)$ with $q > 0$ such that*

$$\max_{1 \le i \le n} \left| \alpha_i - \frac{p_i}{q} \right| < \frac{1}{q^{1+\frac{1}{n}+\varepsilon}}.$$

Proof. See [77, Chapter 6]. □

Finally, the following theorem, due to Schmdit [76], provides a further generalization of Roth's Theorem and has many important applications in Diophantine approximation.

Theorem 1.33 (Subspace Theorem)**.** *Let $m \ge 2$ and let*

$$L_i(\boldsymbol{X}) = \alpha_{i,1} X_1 + \cdots + \alpha_{i,m} X_m \quad (i = 1, \dots, m)$$

be m linear forms with complex algebraic coefficients. Suppose that these forms are linearly independent over $\mathbb{C}$, and let $\varepsilon > 0$. Then, there are finitely many proper subspaces $T_1, \dots, T_s$ of $\mathbb{Q}^m$ such that every $\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ with

$$|L_1(\boldsymbol{x}) \cdots L_m(\boldsymbol{x})| < ||\boldsymbol{x}||^{-\varepsilon}$$

lies in $T_1 \cup \cdots \cup T_s$.

Remark 1.34. Theorem 1.32 can be easily derived from Theorem 1.33 by taking

$m = n + 1$ and

$$\begin{aligned} L_i(\boldsymbol{X}) &= \alpha_i X_{n+1} - X_i \quad (i = 1, \dots, n), \\ L_{n+1}(\boldsymbol{X}) &= X_{n+1}. \end{aligned}$$

See again [77, Chapter 6] for details.

1.2.3 Approximating by algebraic numbers

As noted in the introduction, there is a natural generalization of central questions in Diophantine approximation, namely what happens when we approximate α not only by rational numbers. Indeed, since a rational number is an algebraic number of degree one, it is natural to study how well α can be approximated by algebraic numbers of degree at most n, or by algebraic numbers lying in a prescribed number field. A standard reference for this topic is surely [16].

To pursue this approach, we need to introduce a function that encodes the "complexity" of an algebraic number. This is done through the notion of the *height* of an algebraic number. There are several possible definitions (see [15] for a monographic reference on heights); here we consider what is often called the "naive height".

Definition 1.35. Let η be an algebraic number of degree d, and let

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$$

be its minimal polynomial over $\mathbb{Z}$ (which is the minimal polynomial cleared of denominators). The height of η is defined as

$$\mathrm{H}(\eta) = \max\{|a_1|, \dots, |a_n|\}.$$

For a rational number written in its lowest terms $\eta = p/q$, its height is simply $\mathrm{H}(p/q) = \max\{|p|, |q|\}$.

Liouville's Theorem can be stated by using the notion of height.

Theorem 1.36. *Let α be an algebraic number of degree $d \geq 2$. Then, there exists a constant $c(\alpha) > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{\mathrm{H}(p/q)^d},$$

for every pair of integers (p, q), with $q > 0$.

Remark 1.37. Theorem 1.25 and Theorem 1.36 are equivalent. Suppose first that Theorem 1.25 holds. It suffices to prove Theorem 1.36 for pairs of *coprime* integers

(p, q). In this case, $\mathrm{H}(p/q) \geq q$, hence

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^d} > \frac{c(\alpha)}{\mathrm{H}(p/q)^d}.$$

Conversely, assume that Theorem 1.36 holds, and let (p, q) be coprime integers with $q > 0$. If $q \geq |p|$, then $\mathrm{H}(p/q) = q$ and

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{\mathrm{H}(p/q)^d} = \frac{c(\alpha)}{q^d}.$$

Suppose now that $|p| > q$. Arguing by contradiction, assume that there exists a sequence of positive constants $(c_i)_{i\geq 1}$ such that $c_i \to 0$ and, for all i, there exists (p_i, q_i) satisfying

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{c_i}{q_i^d}.$$

The sets $\{q_1, q_2, \dots\}$ and $\{p_1, p_2, \dots\}$ cannot both be finite; otherwise $\limsup_i |\alpha - p_i/q_i| > 0$, while $\limsup_i c_i/q_i^d = 0$, a contradiction. Furthermore, we must also rule out also the case in which $\{q_1, q_2, \dots\}$ is finite and $\{p_1, p_2, \dots\}$ is infinite, or else we would have $\limsup_i |\alpha - p_i/q_i| = +\infty$ and $\limsup_i c_i/q_i^d = 0$, again a contradiction. Hence, $\limsup_i q_i = +\infty$. Therefore, there are infinitely many indices i for which p_i/q_i has the same integer part $\lfloor\alpha\rfloor$ as α. Write $p_i/q_i = \lfloor\alpha\rfloor + r_i/q_i$, where $r_i < q_i$. Setting $\beta = \alpha - \lfloor\alpha\rfloor$, we obtain infinitely many solutions r_i/q_i to

$$\left|\beta - \frac{r_i}{q_i}\right| < \frac{c_i}{q_i^d} = \frac{1}{\mathrm{H}(r_i/q_i)^d},$$

contradicting Theorem 1.36 applied to the algebraic number β.

Roth's Theorem also admits an equivalent formulation in terms of height.

Theorem 1.38. *Let α be an algebraic number and fix $\varepsilon > 0$. Then, the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\mathrm{H}(p/q)^{2+\varepsilon}} \tag{1.22}$$

holds for only finitely many pairs of integers (p, q), with $q > 0$.

Now we can generalize Liouville's Theorem to the case where the approximation is made by algebraic numbers. This result is often referred to as Liouville's inequality, as it provides a lower bound for the distance between two distinct algebraic numbers.

Theorem 1.39. *Let α be an algebraic number of degree $d \geq 1$. Then, for every algebraic number η of degree $f \geq 1$ and distinct from α, it holds*

$$|\alpha - \eta| \geq \frac{c(\alpha, f)}{\mathrm{H}(\eta)^d},$$

where

$$c(\alpha, f) = \frac{2(d+1)^{-f/2}(f+1)^{-d/2} \times \max\{2^{-d}(d+1)^{-(f-1)/2}, 2^{-f}(f+1)^{-(d-1)/2}\}}{\mathrm{H}(\alpha)^f}.$$

The following theorem, stated by Roth [69] and proved by LeVeque [54], extends Theorem 1.27 to algebraic numbers lying in the same number field.

Theorem 1.40. *Let K be a number field and α a real algebraic number not in K. Then, for every $\varepsilon > 0$, there exists a positive constant $c = c(\alpha, \varepsilon, K)$ such that*

$$|\alpha - \eta| > \frac{c}{\mathrm{H}(\eta)^{2+\varepsilon}}, \tag{1.23}$$

for all η in K.

The following extension of Dirichlet's Theorem 1.23 shows that the exponent $2+\varepsilon$ of $\mathrm{H}(\eta)$ in (1.23) cannot be replaced by any number strictly smaller than 2.

Theorem 1.41. *Let K be a real algebraic number field and α a real number not in K. Then, there is a constant $c = c(\alpha, K)$ such that*

$$|\alpha - \eta| < \frac{c}{\mathrm{H}(\eta)^2},$$

for infinitely many η in K.

Approximation by algebraic numbers can also be studied in another direction. Instead of studying approximating α by elements of a fixed number field K, one may consider approximation by algebraic numbers of degree at most f.

Wirsing [93] addressed this problem and obtained an exponent depending only on f.

Theorem 1.42. *Let α be a real algebraic number and $f \geq 1$ an integer. Then, for every $\varepsilon > 0$, there exists a positive constant $c = (\alpha, f, \varepsilon)$ such that*

$$|\alpha - \eta| > \frac{c}{\mathrm{H}(\eta)^{2f+\varepsilon}}, \tag{1.24}$$

for any algebraic number $\eta \neq \alpha$ of degree at most f.

Observe that Theorem 1.42 reduces exactly to Roth's Theorem when $f = 1$. In [74, 75] Schmidt proved that the exponent $2f+\varepsilon$ in (1.24) can be replaced by $f+1+\varepsilon$.

Theorem 1.43. *Let α be a real algebraic number and $f \geq 1$ be an integer. Then, for every $\varepsilon > 0$, there exists a positive constant $c = c(\alpha, f, \varepsilon)$ such that*

$$|\alpha - \eta| > \frac{c}{\mathrm{H}(\eta)^{f+1+\varepsilon}},$$

for any algebraic number $\eta \neq \alpha$ of degree at most f.

1.3 Quasi-periodic continued fractions

In this section, we present some of the results obtained by Baker [11] in 1962 on transcendence properties of quasi-periodic continued fractions. These results will serve as the basis for the multidimensional generalizations developed in Chapter 2.

We begin by introducing the main object of study.

Definition 1.44. Let $(\lambda_k)_{k\geq 0}$, $(r_k)_{k\geq 0}$, $(n_k)_{k\geq 0}$ be sequences of positive integers, such that

$$n_{k+1} \geq n_k + r_k\lambda_k,$$

for all k. Let $(a_k)_{k\geq 0}$ be a sequence of integers with $a_k > 0$ for all $k \geq 1$, and suppose that

$$a_m = a_{m+r_k},$$

for every m such that $n_k \leq m \leq n_k + (\lambda_k - 1)r_k - 1$. Then, $[a_0; a_1, a_2, \dots]$ is called *quasi-periodic*.

A quasi-periodic continued fraction can therefore be written in the form

$$\alpha = [a_0; \dots, a_{n_0-1}, \overline{a_{n_0}, \dots, a_{n_0+r_0-1}}^{\lambda_0}, \dots, \overline{a_{n_1}, \dots, a_{n_1+r_1-1}}^{\lambda_1}, \dots],$$

where

$$\overline{a_{n_k}, \dots, a_{n_k+r_k-1}}^{\lambda_k}$$

denotes the block of partial quotients $a_{n_k}, \dots, a_{n_k+r_k-1}$ repeated λ_k times.

Remark 1.45. Periodic continued fractions are clearly a special case of quasi-periodic ones. In the transcendence results that follow, this case will be excluded, as we know that periodic continued fractions are algebraic.

Observe also that Theorem 1.44 formally includes all continued fractions: indeed, the definition becomes trivial if one takes $\lambda_k = 1$ for all k. However, this case is not of particular interest.

1.3.1 Some useful lemmas

The three following lemmas play an important role for the proof of Baker's main theorems. In Chapter 2, we will present their generalization to the multidimensional setting.

Lemma 1.46. *Let $\alpha = [a_0; a_1, \dots]$, $\beta = [b_0; b_1, \dots]$ with $a_i = b_i$ for all $i = 0, \dots, m$. Then,*

$$|\alpha - \beta| < 1/q_m^2,$$

where q_m is the denominator of the m-th convergent of α.

Proof. Since the first $m+1$ partial quotients of α and β coincide, it follows from Lemma 1.1 that they also share the m-th convergent $c_m = p_m/q_m$. Then, we have $|\alpha - c_m| < 1/q_m^2$ and $|\beta - c_m| < 1/q_m^2$. Moreover, by Lemma 1.10, the quantities $\alpha - c_m$ and $\beta - c_m$ have the same sign, which depends only on the parity of m. Hence, α and β both lie either in the interval $(c_m - 1/q_m^2, c_m)$ or in the interval $(c_m, c_m + 1/q_m^2)$. The claim follows immediately. □

Lemma 1.47. *Consider the following periodic continued fraction*

$$\eta = [0; b_1, \dots, b_{k-1}, \overline{b_k, \dots, b_{k+t-1}}].$$

Then, η is a quadratic irrational and

$$\mathrm{H}(\eta) < 2q_{k+t-1}^2.$$

Proof. Let η_i denote, as usual, the i-th complete quotient. Applying Lemma 1.1 to the sequences $(0, b_1, \dots, b_{k-1}, \eta_k)$ and $(0, b_1, \dots, b_{k+t-1}, \eta_{k+t-1})$, we obtain

$$\eta = \frac{\eta_k p_{k-1} + p_{k-2}}{\eta_k q_{k-1} + q_{k-2}} = \frac{\eta_{k+t} p_{k+t-1} + p_{k+t-2}}{\eta_{k+t} q_{k+t-1} + q_{k+t-2}}. \tag{1.25}$$

By Theorem 1.8 and the periodicity of η, we have $\eta_k = \eta_{k+t}$. Hence, (1.25) becomes

$$\eta = \frac{\eta_k p_{k-1} + p_{k-2}}{\eta_k q_{k-1} + q_{k-2}} = \frac{\eta_k p_{k+t-1} + p_{k+t-2}}{\eta_k q_{k+t-1} + q_{k+t-2}}.$$

Rearranging the expressions and eliminating η_k, we get the quadratic equation

$$P\eta^2 + Q\eta + R = 0,$$

where

$$\begin{aligned} P &= q_{k-2}q_{k+t-1} - q_{k-1}q_{k+t-2}, \\ Q &= q_{k-1}p_{k+t-2} + p_{k-1}q_{k+t-2} - p_{k-2}q_{k+t-1} - q_{k-2}p_{k+t-1}, \\ R &= p_{k-2}p_{k+t-1} - p_{k-1}p_{k+t-2}. \end{aligned}$$

Since $b_0 = 0$, a straightforward induction shows that $p_n \leq q_n$ for all $n \geq 0$. Therefore,

$$\mathrm{H}(\eta) \leq \max\{|P|, |Q|, |R|\} < 2q_{k+t-1}^2,$$

which is the thesis. □

Lemma 1.48. *Given $\alpha = [a_0; a_1, a_2, \dots]$, we have*

$$q_n \geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}.$$

Furthermore, if $a_n \leq A$ for all n, then

$$q_n \leq \left(\frac{A+\sqrt{A^2+4}}{2}\right)^{n}.$$

Proof. It suffices to recall that $q_{n+1} = a_{n+1}q_n + q_{n-1}$ and argue by induction. □

1.3.2 A transcendence criterion for quasi-periodic continued fractions

Theorem 1.49. *Let*

$$\alpha = [a_0; \dots, a_{n_0-1}, \overline{a_{n_0}, \dots, a_{n_0+r_0-1}}^{\lambda_0}, \dots, \overline{a_{n_1}, \dots, a_{n_1+r_1-1}}^{\lambda_1}, \dots]$$

be a non-periodic quasi-periodic continued fraction. Let C be a positive constant such that

$$r_k < Cn_k, \tag{1.26}$$

for every k. Assume that

$$\limsup_{k\to+\infty} \frac{(\log \lambda_k)\sqrt{\log n_k}}{n_k} = +\infty. \tag{1.27}$$

Then, α is transcendental.

The proof relies on two main theoretical results. On the one hand, we use Liouville's inequality in its general form for algebraic numbers (see Theorem 1.39). On the other hand, we invoke the result of Davenport and Roth (Theorem 1.29) concerning the growth of the denominators of the convergents of algebraic numbers.

Proof. Suppose, by contradiction, that α is algebraic of some degree d. Since the continued fraction of α is infinite and non-periodic by assumption, we necessarily have $d > 2$.

For every k, define

$$\eta_k = [a_0; a_1, \dots, a_{n_k-1}, \overline{a_{n_k}, \dots, a_{n_k+r_k-1}}].$$

Then, η_k is algebraic of degree 2 and so $\eta_k \neq \alpha$. Hence, by Theorem 1.39, we have

$$|\alpha - \eta_k| \geq \frac{c}{\mathrm{H}(\eta_k)^d}.$$

By Lemma 1.47, we have $\mathrm{H}(\eta_k) < 2q^2_{n_k+r_k-1}$. Therefore,

$$|\alpha - \eta_k| > \frac{c_1}{q^{2d}_{n_k+r_k-1}}, \tag{1.28}$$

for all k, where c_1 is a positive constant independent of k (as are all constants c_i below). On the other hand, since η_k and α share the first $n_k + r_k\lambda_k$ partial quotients, by Lemma 1.46 we obtain

$$|\alpha - \eta_k| < 1/q^2_{n_k+r_k\lambda_k-1}. \tag{1.29}$$

Combining (1.28) and (1.29), we deduce

$$q_{n_k+r_k\lambda_k-1} < c_2 q^d_{n_k+r_k-1}.$$

Now, our goal is to use this inequality to derive a condition on the growth of λ_k. Applying Lemma 1.48, we get

$$\Big((1+\sqrt{5})/2\Big)^{n_k+r_k\lambda_k-2} < c_2 q^d_{n_k+r_k-1}.$$

Taking logarithms, it follows that for k sufficiently large,

$$\lambda_k < n_k + r_k\lambda_k - 2 < c_3 \log q_{n_k+r_k-1}.$$

In particular,

$$\log \lambda_k < c_4 \log\log q_{n_k+r_k-1}.$$

Assuming that α is algebraic, we may apply Theorem 1.29 to obtain

$$\log \lambda_k < \frac{c_5(n_k + r_k - 1)}{\sqrt{\log(n_k + r_k - 1)}}.$$

Using the technical condition (1.26), this implies

$$\log \lambda_k < \frac{c_6 n_k}{\sqrt{\log n_k}},$$

but this contradicts the growth condition (1.27), and the proof is complete. □

1.3.3 The case of bounded partial quotients

Observe that in Theorem 1.49 the only growth condition concerns the length of the blocks, while no assumption is made on the growth of partial quotients, which may in fact be bounded. In this setting, the condition (1.27) can be relaxed, provided that the lengths r_k of the repeated blocks are also uniformly bounded.

Theorem 1.50. *Let*

$$\alpha = [a_0; \dots, a_{n_0-1}, \overline{a_{n_0}, \dots, a_{n_0+r_0-1}}^{\lambda_0}, \dots, \overline{a_{n_1}, \dots, a_{n_1+r_1-1}}^{\lambda_1}, \dots]$$

be a non-periodic quasi-periodic continued fraction. Suppose that there exists an integer $r \geq 1$ such that

$$r_k \leq r,$$

for all k. Assume further that there exists a positive constant A such that

$$a_n \leq A, \tag{1.30}$$

for all n, and

$$\limsup_{k\to+\infty} \frac{\lambda_k}{n_k} > B, \tag{1.31}$$

where

$$B = 2\frac{\log\Big((A + \sqrt{A^2+4})/2\Big)}{\log\Big((1+\sqrt{5})/2\Big)} - 1. \tag{1.32}$$

Then, α is transcendental.

The proof by Baker relies on the generalization of Roth's Theorem to number fields given in Theorem 1.38.

Proof. By (1.31), there exist an infinite set $\mathcal{S}_1$ of indices k and a constant $\delta > 0$ such that

$$\lambda_k > (B + \delta)n_k, \tag{1.33}$$

for all $k \in \mathcal{S}_1$.

Since the sequence of (r_k) is bounded, the pigeonhole principle ensures the existence of an integer r and an infinite subset $\mathcal{S}_2 \subseteq \mathcal{S}_1$ such that $r_k = r$ for all k in $\mathcal{S}_2$. Moreover, the set of finite sequences

$$a_{n_k}, \ldots, a_{n_k+r-1} \quad (k \in \mathcal{S}_2)$$

is finite, since the partial quotients are finitely many by condition (1.30). Therefore, at least one of these finite sequences occurs infinitely often. More precisely, there exist an infinite subset $\mathcal{S}_3 \subseteq \mathcal{S}_2$ and integers $b_1, \ldots, b_r$ such that

$$a_{n_k} = b_1, \quad a_{n_k+1} = b_2, \quad \ldots \quad a_{n_k+r-1} = b_r,$$

for all k in $\mathcal{S}_3$.

Arguing by contradiction, assume that α is algebraic of degree d. Since its continued fraction is infinite and non-periodic, it follows that $d > 2$.

Now, for every k in $\mathcal{S}_3$, define

$$\begin{aligned}\eta_k &= [a_0; a_1, \ldots, a_{n_k-1}, \overline{a_{n_k}, \ldots, a_{n_k+r_k-1}}] \\ &= [a_0; a_1, \ldots, a_{n_k-1}, \overline{b_1, \ldots, b_r}].\end{aligned}$$

Each η_k is algebraic of degree 2 and, by Lemma 1.47, satisfies

$$\mathrm{H}(\eta_k) < 2q^2_{n_k+r-1}.$$

Since all η_k share the same period, they belong to the same quadratic number field K (see Theorem 1.18). As α is not a quadratic irrational, and in particular $\alpha \notin K$, we may apply Theorem 1.40. Hence, for every $\varepsilon > 0$ and every $k \in \mathcal{S}_3$

$$|\alpha - \eta_k| > \frac{c}{\mathrm{H}(\eta_k)^{2+\varepsilon}},$$

for some constant $c > 0$ independent of k. By Lemma 1.47, $\mathrm{H}(\eta_k) < 2q^2_{n_k+r-1}$, and

we obtain

$$|\alpha - \eta_k| > \frac{c_1}{q_{n_k+r-1}^{2(2+\varepsilon)}}, \tag{1.34}$$

for a suitable constant $c_1 > 0$. However, by Lemma 1.46, for every k in $\mathcal{S}_3$

$$|\alpha - \eta_k| < \frac{1}{q_{n_k+r\lambda_k-1}^2}. \tag{1.35}$$

Combining inequalities (1.34) and (1.35), we get

$$q_{n_k+r\lambda_k-1} < c_2 q_{n_k+r-1}^{2+\varepsilon},$$

for some constant c_2 independent of k. By Lemma 1.48, this implies

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n_k+r\lambda_k-2} < c_2\left(\frac{A+\sqrt{A^2+4}}{2}\right)^{(2+\varepsilon)(n_k+r-1)}.$$

Using (1.32), we deduce that for a suitable constant $c_3 > 0$,

$$n_k + r\lambda_k < c_3 + \frac{B+1}{2}(2+\varepsilon)n_k.$$

Hence, for infinitely many k in $\mathcal{S}_3$,

$$\lambda_k < r\lambda_k < c_3 + \left(\frac{B+1}{2}(2+\varepsilon) - 1\right)n_k = c_3 + \left(B + \frac{B+1}{2}\varepsilon\right)n_k.$$

Comparing with (1.33), we obtain

$$0 < \delta < \frac{c_3}{n_k} + \frac{B+1}{2}\varepsilon,$$

which yields a contradiction for k sufficiently large in $\mathcal{S}_3$ and ε sufficiently small. This concludes the proof. □

1.4 Multidimensional continued fractions

1.4.1 Bidimensional continued fractions

Let $\boldsymbol{x}^{(1)} = (x_0^{(1)}, x_1^{(1)}, \dots)$ and $\boldsymbol{x}^{(2)} = (x_0^{(2)}, x_1^{(2)}, \dots)$ be two sequences of real numbers, with $x_k^{(1)} > 0$ for all $k \geq 1$. Let m, n be positive integers such that $m \leq n$. Set

$\xi_n^{(1)} = x_n^{(1)}$ and $\xi_m^{(2)} = x_m^{(2)}$, and for every $k = 1, \dots, m-1$, define

$$\xi_k^{(1)} = x_k^{(1)} + \frac{\xi_{k+1}^{(2)}}{\xi_{k+1}^{(1)}},$$
$$\xi_k^{(2)} = x_k^{(2)} + \frac{1}{\xi_{k+1}^{(1)}}.$$

If $m < n$, then for $k = m, \dots, n-1$ we additionally define

$$\xi_k^{(1)} = x_k^{(1)} + \frac{1}{\xi_{k+1}^{(1)}}.$$

A multidimensional continued fraction (MCF) of dimension 2 (or simply a bidimensional continued fraction) is an expression of the form

$$[(x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)}), (x_0^{(2)}, x_1^{(2)}, \dots, x_m^{(2)})] = (\xi_0^{(1)}, \xi_0^{(2)}), \tag{1.36}$$

where

$$\xi_0^{(1)} = x_0^{(1)} + \frac{\xi_1^{(2)}}{\xi_1^{(1)}} = x_0^{(1)} + \frac{x_1^{(2)} + \dfrac{1}{\xi_2^{(1)}}}{x_1^{(1)} + \dfrac{\xi_2^{(2)}}{\xi_2^{(1)}}} = x_0^{(1)} + \cfrac{x_1^{(2)} + \cfrac{1}{x_2^{(1)} + \cfrac{\xi_3^{(2)}}{\xi_3^{(1)}}}}{x_1^{(1)} + \cfrac{x_2^{(2)} + \cfrac{1}{\xi_3^{(1)}}}{x_2^{(1)} + \cfrac{\xi_3^{(2)}}{\xi_3^{(1)}}}} = \dots,$$

$$\xi_0^{(2)} = x_0^{(2)} + \frac{1}{\xi_1^{(1)}} = x_0^{(2)} + \frac{1}{x_1^{(1)} + \dfrac{\xi_2^{(2)}}{\xi_2^{(1)}}} = x_0^{(2)} + \cfrac{1}{x_1^{(1)} + \cfrac{x_2^{(2)} + \cfrac{1}{\xi_3^{(1)}}}{x_2^{(1)} + \cfrac{\xi_3^{(2)}}{\xi_3^{(1)}}}} = \dots.$$

At stage n, the expression involves only finitely many operations on the entries $x_0^{(1)}, \dots, x_n^{(1)}$ and $x_0^{(2)}, \dots, x_m^{(2)}$.

We also refer to expressions of the form

$$\lim_{n \to +\infty} [(x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)}), (x_0^{(2)}, x_1^{(2)}, \dots, x_m^{(2)})],$$

or

$$\lim_{n,m\to+\infty}[(x_0^{(1)},x_1^{(1)},\dots,x_n^{(1)}),(x_0^{(2)},x_1^{(2)},\dots,x_m^{(2)})],$$

as multidimensional continued fractions, provided that the limits exist.

Since multidimensional continued fractions can be much less intuitive than classical continued fractions, it is useful to give an example.

Example 1.51. Consider the sequences $\boldsymbol{x}^{(1)}=(2,2,3)$ and $\boldsymbol{x}^{(2)}=(1,2,1)$. Then,

$$\xi_0^{(1)}=2+\frac{\xi_1^{(2)}}{\xi_1^{(1)}}=2+\frac{2+\dfrac{1}{\xi_2^{(1)}}}{2+\dfrac{\xi_2^{(2)}}{\xi_2^{(1)}}}=2+\frac{2+\dfrac{1}{3}}{2+\dfrac{1}{3}}=3,$$

$$\xi_0^{(2)}=1+\frac{1}{\xi_1^{(1)}}=1+\frac{1}{2+\dfrac{\xi_2^{(2)}}{\xi_2^{(1)}}}=1+\frac{1}{2+\dfrac{1}{3}}=\frac{10}{7},$$

and we write $(3,10/7)=[(2,2,3),(1,2,1)]$.

Given two sequences $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ as above, and integers m,n with $0\le m\le n$, define

$$\gamma_{n,m}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})=[(x_0^{(1)},x_1^{(1)},\dots,x_n^{(1)}),(x_0^{(2)},x_1^{(2)},\dots,x_m^{(2)})]. \tag{1.37}$$

The following lemma is the multidimensional analogue of Lemma 1.1.

Lemma 1.52. *Set*

$$\begin{bmatrix}A_{-3}^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) & A_{-2}^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) & A_{-1}^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})\\ A_{-3}^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) & A_{-2}^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) & A_{-1}^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})\\ C_{-3}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) & C_{-2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) & C_{-1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},$$

and let $m\ge 0$ be an integer. For every integer $k\ge 0$, define recursively:

(i) if $k=0,\dots,m$, set

$$\begin{aligned}A_k^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})&=x_k^{(1)}A_{k-1}^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})+x_k^{(2)}A_{k-2}^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})+A_{k-3}^{(1)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}),\\ A_k^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})&=x_k^{(1)}A_{k-1}^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})+x_k^{(2)}A_{k-2}^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})+A_{k-3}^{(2)}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}),\\ C_k(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})&=x_k^{(1)}C_{k-1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})+x_k^{(2)}C_{k-2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})+C_{k-3}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)});\end{aligned}$$

(ii) if $k \geq m+1$, set

$$\begin{aligned} A_k^{(1)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) &= x_k^{(1)} A_{k-1}^{(1)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) + A_{k-2}^{(1)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}), \\ A_k^{(2)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) &= x_k^{(1)} A_{k-1}^{(2)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) + A_{k-2}^{(2)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}), \\ C_k(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) &= x_k^{(1)} C_{k-1}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) + C_{k-2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}). \end{aligned}$$

Then, if $n \geq m$,

$$\gamma_{n,m}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = \left(\frac{A_n^{(1)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}{C_n(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}, \frac{A_n^{(2)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}{C_n(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})} \right).$$

Proof. We first treat the case $n = m$ by induction.

For the base case $m = 0$, the claim follows by direct computation.

Assume now that the statement holds for all sequences up to index m, that is

$$\gamma_{k,k}(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) = \left(\frac{A_k^{(1)}(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})}{C_k(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})}, \frac{A_k^{(2)}(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})}{C_k(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})} \right),$$

for any pair of sequences $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})$, and $k = 0, \ldots, m$. We want to prove

$$\gamma_{m+1,m+1}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = \left(\frac{A_{m+1}^{(1)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}{C_{m+1}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}, \frac{A_{m+1}^{(2)}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})}{C_{m+1}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})} \right).$$

Define the sequences $\bar{\boldsymbol{x}}^{(1)}, \bar{\boldsymbol{x}}^{(2)}$, where $\bar{x}_j^{(i)} = x_j^{(i)}$ for every $j \neq m$, $i = 1, 2$, and

$$\begin{aligned} \bar{x}_m^{(1)} &= x_m^{(1)} + \frac{x_{m+1}^{(2)}}{x_{m+1}^{(1)}}, \\ \bar{x}_m^{(2)} &= x_m^{(2)} + \frac{1}{x_{m+1}^{(1)}}. \end{aligned}$$

Then,

$$\begin{aligned} \gamma_{m+1,m+1}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) &= [(x_0^{(1)}, \ldots, x_m^{(1)}, x_{m+1}^{(1)}), (x_0^{(2)}, \ldots, x_m^{(2)}, x_{m+1}^{(2)})] \\ &= \left[\left(x_0^{(1)}, \ldots, x_m^{(1)} + \frac{x_{m+1}^{(2)}}{x_{m+1}^{(1)}} \right), \left(x_0^{(2)}, \ldots, x_m^{(2)} + \frac{1}{x_{m+1}^{(1)}} \right) \right] \\ &= \gamma_{m,m}(\bar{\boldsymbol{x}}^{(1)}, \bar{\boldsymbol{x}}^{(2)}). \end{aligned}$$

Applying the inductive hypothesis to the modified sequences yields the desired recur-

rence relations at level $m+1$. In fact,

$$\begin{aligned} A^{(1)}_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) &= \left(x^{(1)}_m + \frac{x^{(2)}_{m+1}}{x^{(1)}_{m+1}}\right)A^{(1)}_{m-1}(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) \\ &\qquad + \left(x^{(2)}_m + \frac{1}{x^{(1)}_{m+1}}\right)A^{(1)}_{m-2}(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) + A^{(1)}_{m-3}(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) \\ &= A^{(1)}_m(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + \frac{x^{(2)}_{m+1}A^{(1)}_{m-1}(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) + A^{(1)}_{m-2}(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)})}{x^{(1)}_{m+1}} \\ &= A^{(1)}_m(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + \frac{x^{(2)}_{m+1}A^{(1)}_{m-1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + A^{(1)}_{m-2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})}{x^{(1)}_{m+1}}. \end{aligned}$$

Hence,

$$\begin{aligned} x^{(1)}_{m+1}A^{(1)}_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) &= x^{(1)}_{m+1}A^{(1)}_m(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + x^{(2)}_{m+1}A^{(1)}_{m-1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + A^{(1)}_{m-2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) \\ &= A^{(1)}_{m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}). \end{aligned}$$

By the same argument we can prove

$$\begin{aligned} x^{(1)}_{m+1}A^{(2)}_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) &= x^{(1)}_{m+1}A^{(2)}_m(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + x^{(2)}_{m+1}A^{(2)}_{m-1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + A^{(2)}_{m-2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) \\ &= A^{(2)}_{m+1}(\boldsymbol{x}^{(2)},\boldsymbol{x}^{(2)}), \end{aligned}$$

and

$$\begin{aligned} x^{(1)}_{m+1}C_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) &= x^{(1)}_{m+1}C_m(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + x^{(2)}_{m+1}C_{m-1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) + C_{m-2}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) \\ &= C_{m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}), \end{aligned}$$

which yield

$$\begin{aligned} \gamma_{m+1,m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)}) = \gamma_{m,m}(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)}) &= \left(\frac{A^{(1)}_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)})}{C_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)})}, \frac{A^{(2)}_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)})}{C_m(\bar{\boldsymbol{x}}^{(1)},\bar{\boldsymbol{x}}^{(2)})}\right) \\ &= \left(\frac{A^{(1)}_{m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})}{C_{m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})}, \frac{A^{(2)}_{m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})}{C_{m+1}(\boldsymbol{x}^{(1)},\boldsymbol{x}^{(2)})}\right). \end{aligned}$$

The case $n \geq m+1$ is handled similarly, with suitably modified sequences $\bar{\boldsymbol{x}}^{(1)}$ and $\bar{\boldsymbol{x}}^{(2)}$. In particular, the proof relies on a double induction on m and n, where the key observation is that

$$\begin{aligned} &[(x^{(1)}_0,\ldots,x^{(1)}_m,\ldots,x^{(1)}_n,x^{(1)}_{n+1}),(x^{(2)}_0,\ldots,x^{(2)}_m)] = \\ &\qquad = \left[\left(x^{(1)}_0,\ldots,x^{(1)}_n + \frac{1}{x^{(1)}_{n+1}}\right),\left(x^{(2)}_0,\ldots,x^{(2)}_m\right)\right], \end{aligned}$$

from which the sequences $\bar{\boldsymbol{x}}^{(1)}$ and $\bar{\boldsymbol{x}}^{(2)}$ are readily obtained. □

1.4.2 Jacobi's algorithm

As mentioned in the Introduction, multidimensional continued fractions were first introduced by Jacobi [43] in 1848, in response to a question posed by Hermite. More precisely, Hermite asked whether it is possible to obtain a notion of periodicity for cubic irrationalities, that is, whether such numbers can be represented through periodic sequences (see [41, 66]). Jacobi proposed an algorithmic approach to this problem, which was later further developed and generalized by Perron [65] in 1907.

In this section, we recall some basic facts about Jacobi's algorithm and the bidimensional continued fractions arising from it. For a more comprehensive treatment of this algorithm, as well as of the generalization that will be introduced in Section 1.4.4, we refer the reader to [12, 79].

Given $\alpha_0^{(1)}, \alpha_0^{(2)} \in \mathbb{R}$, Jacobi's algorithm produces two sequences of integers $(a_n^{(1)})_{n\geq 0}$ and $(a_n^{(2)})_{n\geq 0}$ via the recursions

$$\begin{cases} a_n^{(1)} = \lfloor \alpha_n^{(1)} \rfloor \\ a_n^{(2)} = \lfloor \alpha_n^{(2)} \rfloor \\ \alpha_{n+1}^{(1)} = \dfrac{1}{\alpha_n^{(2)} - a_n^{(2)}} \\ \alpha_{n+1}^{(2)} = \dfrac{\alpha_n^{(1)} - a_n^{(1)}}{\alpha_n^{(2)} - a_n^{(2)}} \end{cases} \qquad (n \geq 0). \tag{1.38}$$

Observe that the algorithm can be carried out as long as $\alpha_n^{(2)} \notin \mathbb{Z}$ for all n, so that no division by zero occurs and the procedure does not present any interruptions. This situation is usually referred to as the *simple case*, in which both sequences $(a_n^{(1)})_{n\geq 0}$ and $(a_n^{(2)})_{n\geq 0}$ are infinite. In this case, using the notation introduced in (1.36), for every $n \geq 0$ we may write

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, \ldots, a_n^{(1)}, \alpha_{n+1}^{(1)}), (a_0^{(2)}, \ldots, a_n^{(2)}, \alpha_{n+1}^{(2)})],$$

and, for every $k \geq 0$,

$$(\alpha_{n+1}^{(1)}, \alpha_{n+1}^{(2)}) = [(a_{n+1}^{(1)}, \ldots, a_{n+k}^{(1)}, \alpha_{n+k+1}^{(1)}), (a_{n+1}^{(2)}, \ldots, a_{n+k}^{(2)}, \alpha_{n+k+1}^{(2)})].$$

If the condition that leads to the simple case fails, let n_0 be the smallest index such that $\alpha_{n_0}^{(2)} \in \mathbb{Z}$. Then, the terms $a_n^{(2)}$ are no longer defined for $n > n_0$, and the

sequence $(a_n^{(1)})_{n>n_0}$ is obtained by evaluating the classical continued fraction of $\alpha_{n_0}^{(1)}$. In this case, the algorithm provides two sequences of integers, one of which is finite. Accordingly, we may write

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, a_1^{(1)}, \dots, a_{n_0}^{(1)}, \alpha_{n_0+1}^{(1)}), (a_0^{(2)}, a_1^{(2)}, \dots, a_{n_0}^{(2)})],$$

where

$$\alpha_{n_0+1}^{(1)} = [a_{n_0+1}^{(1)}; a_{n_0+2}^{(1)}, \dots].$$

If the sequence $(a_n^{(1)})_{n\geq 0}$ is infinite, we say that the algorithm exhibits one interruption; if both sequences are finite, we say that it exhibits two interruptions. The latter case occurs precisely when $\alpha_{n_0}^{(1)}$ is rational, by the classical theory of continued fractions. We illustrate this in the following example.

Example 1.53. Let us expand into multidimensional continued fraction the pair

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = \left(\frac{7}{5}, \frac{1}{3}\right)$$

using Jacobi's algorithm.

$$\begin{cases} a_0^{(1)} = \lfloor \frac{7}{5} \rfloor = 1 \\ a_0^{(2)} = \lfloor \frac{1}{3} \rfloor = 0 \\ \alpha_1^{(1)} = \dfrac{1}{\frac{1}{3} - 0} = 3 \\ \alpha_1^{(2)} = \dfrac{\frac{7}{5} - 1}{\frac{1}{3} - 0} = \frac{6}{5} \end{cases} \qquad \begin{cases} a_1^{(1)} = \lfloor 3 \rfloor = 3 \\ a_1^{(2)} = \lfloor \frac{6}{5} \rfloor = 1 \\ \alpha_2^{(1)} = \dfrac{1}{\frac{6}{5} - 1} = 5 \\ \alpha_2^{(2)} = \dfrac{3 - 3}{\frac{6}{5} - 1} = 0 \end{cases} \qquad \begin{cases} a_2^{(1)} = \lfloor 5 \rfloor = 5 \\ a_2^{(2)} = \lfloor 0 \rfloor = 0. \end{cases}$$

Hence,

$$\left(\frac{7}{5}, \frac{1}{3}\right) = [(1, 3, 5), (0, 1, 0)].$$

The terms $a_n^{(i)}$ produced by Jacobi's algorithm are called the i-th *partial quotients*, while the $\alpha_n^{(i)}$ are called the i-th *complete quotients*.

While for classical continued fractions the finiteness of the sequence of partial quotients characterizes rational numbers, in the multidimensional setting the finiteness of one or both sequences encodes information about rational relations between $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$.

Theorem 1.54. *Jacobi's algorithm applied to* $(\alpha_0^{(1)}, \alpha_0^{(2)})$ *exhibits* $l = 1, 2$ *interruptions if and only if there exist* l *linearly independent relations over* $\mathbb{Z}$ *among* $1, \alpha_0^{(1)}, \alpha_0^{(2)}$.

Proof. See [65] for the "only if" direction, and [35] for the "if" direction. □

From (1.38), a necessary condition for two infinite sequences of integers $\boldsymbol{a}^{(1)}$, $\boldsymbol{a}^{(2)}$ to arise from Jacobi's algorithm applied to some pair $(\alpha_0^{(1)}, \alpha_0^{(2)})$ is that, for all $n \geq 1$,

$$\begin{gathered} a_n^{(1)} \geq 1, \quad a_n^{(2)} \geq 0, \quad a_n^{(1)} \geq a_n^{(2)}, \\ \text{and } a_{n+1}^{(2)} \geq 1 \quad \text{if} \quad a_n^{(1)} = a_n^{(2)}. \end{gathered} \tag{1.39}$$

Perron [65] showed that such conditions (1.39) are also sufficient; they are therefore referred to as the *admissibility conditions*.

Remark 1.55. If $\boldsymbol{a}^{(2)} = (a_0^{(2)}, \dots, a_m^{(2)})$ is finite, the admissibility conditions are slightly different. To the best of our knowledge, these conditions do not appear in the literature, so we briefly state them here. Let

$$\boldsymbol{a}^{(1)} = (a_0^{(1)}, \dots, a_N^{(1)}),$$

where $N \in \{m, m+1, \dots\} \cup \{\infty\}$, and set

$$\alpha_m^{(1)} = [a_m^{(1)}; a_{m+1}^{(1)}, \dots, a_N^{(1)}].$$

We treat first the case $m \geq 2$, which is the most delicate one. It is quite easy to see that the following conditions are necessary for $(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$ to arise from Jacobi's algorithm

(a) $a_n^{(1)} \geq 1$, for $n = 1, \dots, N$,

(b) $a_N^{(1)} > 1$ if N is finite,

(c) $a_n^{(2)} \geq 0$, $a_n^{(1)} \geq a_n^{(2)}$, for $n = 1, \dots, m$,

(d) $\alpha_m^{(1)} > a_m^{(2)}$,

(e) $a_{n+1}^{(2)} \geq 1$ if $a_n^{(1)} = a_n^{(2)}$, for $m > 1$ and $n = 1, \dots, m-1$,

(f) $a_{m-1}^{(1)} > a_{m-1}^{(2)}$ if $a_m^{(2)} = 1$,

(g) $a_{m-1}^{(1)} > 1$ if $a_m^{(2)} = 0$.

In fact, these conditions are also sufficient, and we now prove this. First, define

$$\alpha_{m-1}^{(1)} = a_{m-1}^{(1)} + \frac{a_m^{(2)}}{\alpha_m^{(1)}}, \qquad \alpha_{m-1}^{(2)} = a_{m-1}^{(2)} + \frac{1}{\alpha_m^{(1)}}.$$

Since $\alpha_m^{(1)} > 1$ by conditions (a) and (b), we have $a_{m-1}^{(2)} = \lfloor \alpha_{m-1}^{(2)} \rfloor$. Moreover, by (d), $a_m^{(2)}/\alpha_m^{(1)} < 1$, and therefore $a_{m-1}^{(1)} = \lfloor \alpha_{m-1}^{(1)} \rfloor$. We also claim that $\alpha_{m-1}^{(2)}/\alpha_{m-1}^{(1)} \notin \mathbb{Z}$. If not, this ratio would be equal to 1 by condition (c), which is impossible by (f). Next, define

$$\alpha_{m-2}^{(1)} = a_{m-2}^{(1)} + \frac{\alpha_{m-1}^{(2)}}{\alpha_{m-1}^{(1)}}, \qquad \alpha_{m-2}^{(2)} = a_{m-2}^{(2)} + \frac{1}{\alpha_{m-1}^{(1)}}.$$

As before, $a_{m-2}^{(2)} = \lfloor \alpha_{m-2}^{(2)} \rfloor$, otherwise we would have $\alpha_{m-1}^{(1)} = 1$, which would imply $a_{m-1}^{(1)} = 1$ and $a_m^{(2)} = 0$, contradicting condition (g). Since $\alpha_{m-1}^{(2)}/\alpha_{m-1}^{(1)} \notin \mathbb{Z}$, it follows that $a_{m-2}^{(1)} = \lfloor \alpha_{m-2}^{(1)} \rfloor$. Furthermore, $\alpha_{m-2}^{(2)}/\alpha_{m-2}^{(1)} \notin \mathbb{Z}$, for otherwise the ratio would again be equal to 1, forcing $\alpha_{m-1}^{(2)} = 1$, which contradicts $\alpha_m^{(1)} > 1$. We can now proceed by induction. For every $i = 2, \ldots, m$, define

$$\alpha_{m-i}^{(1)} = a_{m-i}^{(1)} + \frac{\alpha_{m-i+1}^{(2)}}{\alpha_{m-i+1}^{(1)}}, \qquad \alpha_{m-i}^{(2)} = a_{m-i}^{(2)} + \frac{1}{\alpha_{m-i+1}^{(1)}}.$$

Repeating the previous arguments, one proves that $a_{m-i}^{(1)} = \lfloor \alpha_{m-i}^{(1)} \rfloor$, $a_{m-i}^{(2)} = \lfloor \alpha_{m-i}^{(2)} \rfloor$ and $a_{m-i+1}^{(2)}/\alpha_{m-i+1}^{(1)} \notin \mathbb{Z}$, $\alpha_{m-i+1}^{(1)} \neq 1$. Hence, applying Jacobi's algorithm to the pair $(\alpha_0^{(1)}, \alpha_0^{(2)})$ we recover, by construction, the original pair of sequences $(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$.

Similar arguments show that, when $m = 0$, conditions (a) and (b) are sufficient to ensure that the given sequences are admissible for Jacobi's algorithm. In the case $m = 1$, one must additionally require condition (c).

Example 1.56. The sequences considered in Example 1.51 are not admissible, since condition (f) is not satisfied. Indeed, applying Jacobi's algorithm to the pair $(3, 10/7)$, one readily checks that the resulting sequences of partial quotients are not $(2, 2, 3)$ and $(1, 2, 1)$. In fact,

$$\left(3, \frac{10}{7}\right) = [(3, 2, 3), (1, 0)].$$

As in the classical theory, we now introduce convergents. Using (1.37), we give the following definition.

Definition 1.57. Let $\boldsymbol{a}^{(1)}$, $\boldsymbol{a}^{(2)}$ be two sequences of partial quotients obtained by applying Jacobi's algorithm to a pair $(\alpha_0^{(1)}, \alpha_0^{(2)})$. If both sequences are finite, say $\boldsymbol{a}^{(1)} = (a_0^{(1)}, a_1^{(1)}, \ldots, a_n^{(1)})$ and $\boldsymbol{a}^{(2)} = (a_0^{(2)}, a_1^{(2)}, \ldots, a_m^{(2)})$, then for every $k = 0, \ldots, n$ we define

$$\gamma_k(\alpha_0^{(1)}, \alpha_0^{(2)}) = \begin{cases} \gamma_{k,k}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}) & \text{if} \quad k \leq m, \\ \gamma_{k,m}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}) & \text{if} \quad k > m. \end{cases}$$

If $\boldsymbol{a}^{(1)}$ is infinite and $\boldsymbol{a}^{(2)} = (a_0^{(2)}, a_1^{(2)}, \ldots, a_m^{(2)})$ is finite, then for every $k \geq 0$ we define

$$\gamma_k(\alpha_0^{(1)}, \alpha_0^{(2)}) = \begin{cases} \gamma_{k,k}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}) & \text{if} \quad k \leq m, \\ \gamma_{k,m}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}) & \text{if} \quad k > m. \end{cases}$$

Finally, if both sequences of partial quotients are infinite, for every $k \geq 0$ we set

$$\gamma_k(\alpha_0^{(1)}, \alpha_0^{(2)}) = \gamma_{k,k}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}).$$

In all cases, $\gamma_k = \gamma_k(\alpha_0^{(1)}, \alpha_0^{(2)})$ is called the k-th convergent of $(\alpha_0^{(1)}, \alpha_0^{(2)})$.

Remark 1.58. If both sequences of partial quotients $\boldsymbol{a}^{(1)} = (a_0^{(1)}, a_1^{(1)}, \ldots, a_n^{(1)})$ and $\boldsymbol{a}^{(2)} = (a_0^{(2)}, a_1^{(2)}, \ldots, a_m^{(1)})$ are finite, then by (1.36) and (1.38), we may write

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}] = [(a_0^{(1)}, a_1^{(1)}, \ldots, a_n^{(1)}), (a_0^{(2)}, a_1^{(2)}, \ldots, a_m^{(1)})],$$

and refer to this expression as the (Jacobi) multidimensional continued fraction of $(\alpha_0^{(1)}, \alpha_0^{(2)})$.

Let us denote by $A_n^{(i)} = A_n^{(i)}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$, $C_n = C_n(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$ the sequences introduced in Lemma 1.52. Then,

$$\gamma_k = \left(\frac{A_k^{(1)}}{C_k}, \frac{A_k^{(2)}}{C_k} \right).$$

In matrix notation, when both sequences of partial quotients are infinite, the recurrences given in Lemma 1.52 can be written as

$$\begin{bmatrix} a_0^{(1)} & 1 & 0 \\ a_0^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1^{(1)} & 1 & 0 \\ a_1^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k^{(1)} & 1 & 0 \\ a_k^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_k^{(1)} & A_{k-1}^{(1)} & A_{k-2}^{(1)} \\ A_k^{(2)} & A_{k-1}^{(2)} & A_{k-2}^{(2)} \\ C_k & C_{k-1} & C_{k-2} \end{bmatrix}, \tag{1.40}$$

for all $k \geq 0$. Moreover, for all k and $i = 1, 2$, we have

$$\alpha_0^{(i)} = \frac{\alpha_k^{(1)} A_{k-1}^{(i)} + \alpha_k^{(2)} A_{k-2}^{(i)} + A_{k-3}^{(i)}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}}. \tag{1.41}$$

We now extend the notion of multidimensional continued fraction to the case of infinite sequences of partial quotients, by proving the existence of the limit

$$\lim_{n \to +\infty} \gamma_n(\alpha_0^{(1)}, \alpha_0^{(2)}).$$

The following results show that the convergents associated with Jacobi's algorithm converge to the initial pair $(\alpha_0^{(1)}, \alpha_0^{(2)})$. The proofs can be found in [12], but we include them here to introduce notation that will be used later.

Theorem 1.59 ([12], Theorem 1)**.** *Let $\boldsymbol{a}^{(1)}$, $\boldsymbol{a}^{(2)}$ be sequences of integers satisfying the admissibility conditions* (1.39)*, and assume that $\boldsymbol{a}^{(1)}$ is infinite. Setting $A_k^{(i)} = A_k^{(i)}(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$, $C_k = C_k(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$ we have that the limits*

$$\lim_{k\to+\infty} \frac{A_k^{(1)}}{C_k} \quad \textit{and} \quad \lim_{k\to+\infty} \frac{A_k^{(2)}}{C_k} \tag{1.42}$$

exist and are finite.

Proof. We prove the statement only for the convergents $(A_k^{(1)}/C_k)_{k\geq 0}$ and in the case where both sequences $\boldsymbol{a}^{(1)}$, $\boldsymbol{a}^{(2)}$ are infinite; the other cases are analogous.

Define

$$m_k = \min\left(\frac{A_k^{(1)}}{C_k}, \frac{A_{k+1}^{(1)}}{C_{k+1}}, \frac{A_{k+2}^{(1)}}{C_{k+2}}\right), \quad M_k = \max\left(\frac{A_k^{(1)}}{C_k}, \frac{A_{k+1}^{(1)}}{C_{k+1}}, \frac{A_{k+2}^{(1)}}{C_{k+2}}\right),$$

for all $k \geq 0$. From the recurrence relations we obtain

$$\frac{A_k^{(1)}}{C_k} = \frac{a_k^{(1)} A_{k-1}^{(1)} + a_k^{(2)} A_{k-2}^{(1)} + A_{k-3}^{(1)}}{C_k} = s_k \frac{A_{k-1}^{(1)}}{C_{k-1}} + t_k \frac{A_{k-2}^{(1)}}{C_{k-2}} + u_k \frac{A_{k-3}^{(1)}}{C_{k-3}}, \tag{1.43}$$

where

$$s_k = \frac{a_k^{(1)} C_{k-1}}{C_k}, \quad t_k = \frac{a_k^{(2)} C_{k-2}}{C_k}, \quad u_k = \frac{C_{k-3}}{C_k}.$$

Using (1.40) and the admissibility conditions (1.39), one checks that

$$s_k + t_k + u_k = 1, \quad \frac{t_k}{s_k} \leq 1, \quad \frac{u_k}{s_k} \leq 1, \quad s_k > 1/3,$$

for all $k \geq 0$. Now, substituting $k+3$ to k in (1.43) and recalling the definitions of m_k, we have

$$\frac{A_{k+3}^{(1)}}{C_{k+3}} \geq s_{k+3} m_k + t_{k+3} m_k + u_{k+3} m_k = m_k,$$

and similarly

$$\frac{A_{k+3}^{(1)}}{C_{k+3}} \leq M_k,$$

for all $k \geq 0$. Hence,

$$m_k \leq \min\left(\frac{A_{k+1}^{(1)}}{C_{k+1}}, \frac{A_{k+2}^{(1)}}{C_{k+2}}, \frac{A_{k+3}^{(1)}}{C_{k+3}}\right) = m_{k+1}.$$

Similarly, $M_{k+1} \leq M_k$. Therefore,

$$m_k \leq m_{k+1} \leq M_{k+1} \leq M_k,$$

for all $k \geq 0$. Thus, the sequence $(m_k)_{k\geq 0}$ is non-decreasing and bounded by M_0, while the sequence $(M_k)_{k\geq 0}$ is non-increasing and bounded by m_0. It follows that

$$\lim_{k\to+\infty} m_k = m, \quad \lim_{k\to+\infty} M_k = M,$$

for some $M, m \in \mathbb{R}$. Hence, for all $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$, we have

$$\frac{A_k^{(1)}}{C_k}, \frac{A_{k+1}^{(1)}}{C_{k+1}}, \frac{A_{k+2}^{(1)}}{C_{k+2}} > m - \varepsilon$$

and

$$\begin{aligned}\frac{A_{k+3}^{(1)}}{C_{k+3}} &> (t_{k+3} + u_{k+3})(m-\varepsilon) + s_{k+3} \cdot \frac{A_{k+2}^{(1)}}{C_{k+2}} = \\ &= (1 - s_{k+3})(m-\varepsilon) + s_{k+3} \cdot \frac{A_{k+2}^{(1)}}{C_{k+2}} \geq m - \varepsilon + s_{k+3}\left(\frac{A_{k+2}^{(1)}}{C_{k+2}} - m\right).\end{aligned}$$

Hence,

$$\frac{A_{k+3}^{(1)}}{C_{k+3}} - m > s_{k+3}\left(\frac{A_{k+2}^{(1)}}{C_{k+2}} - m\right) - \varepsilon > \frac{1}{3}\left(\frac{A_{k+2}^{(1)}}{C_{k+2}} - m\right) - \varepsilon,$$

from which

$$\frac{A_{k+3+h}^{(1)}}{C_{k+3+h}} - m > \frac{1}{3^{h+1}}\left(\frac{A_{k+2}^{(1)}}{C_{k+2}} - m\right) - \epsilon\left(1 + \frac{1}{3} + \ldots + \frac{1}{3^h}\right), \tag{1.44}$$

for all $h \geq 0$. Now, let $k_1 > k_0 + 3$ and consider M_{k_1}. Then, there exists $j \in \{0, 1, 2\}$ such that

$$M_{k_1} = \frac{A_{k_1+j}^{(1)}}{C_{k_1+j}}.$$

Let now $k = k_1 + j - 2 > k_0$ and consider m_{k+3}. Then, there exists $h \in \{0, 1, 2\}$ such

that

$$m_{k+3} = \frac{A^{(1)}_{k+3+h}}{C_{k+3+h}}.$$

By (1.44), we have

$$\begin{aligned} 0 \quad = m - m > m_{k+3} - m = \frac{A^{(1)}_{k+3+h}}{C_{k+3+h}} - m &> \frac{1}{3^{h+1}} \left(\frac{A^{(1)}_{k+2}}{C_{k+2}} - m \right) - \varepsilon \left(1 + \frac{1}{3} + \ldots + \frac{1}{3^h} \right) \\ &= \frac{1}{3^{h+1}} \left(M_{k_1} - m \right) - \varepsilon \left(1 + \frac{1}{3} + \ldots + \frac{1}{3^h} \right) \\ &\geq \frac{1}{3^{h+1}} (M - m) - \varepsilon \left(1 + \frac{1}{3} + \ldots + \frac{1}{3^h} \right). \end{aligned}$$

It follows that

$$\epsilon > \frac{M - m}{3^{h+1}(1 + 1/3 + \ldots + 1/3^h)} > \frac{M - m}{3}.$$

Since ϵ is arbitrarily small we must have $m = M$, and we are done. □

Remark 1.60. If $\boldsymbol{a}^{(2)}$ is finite, the argument is similar, but for sufficiently large k the recurrences to be considered are

$$\begin{aligned} A^{(1)}_k &= a^{(1)}_k A^{(1)}_{k-1} + A^{(1)}_{k-2} \\ \text{and} \quad C_k &= a^{(1)}_k C_{k-1} + C_{k-2}, \end{aligned}$$

instead of

$$\begin{aligned} A^{(1)}_k &= a^{(1)}_k A^{(1)}_{k-1} + a^{(2)}_k A^{(1)}_{k-2} + A^{(1)}_{k-3} \\ \text{and} \quad C_k &= a^{(1)}_k C_{k-1} + a^{(2)}_k C_{k-2} + C_{k-3}, \end{aligned}$$

and the proof must be adapted accordingly.

The existence of the limits (1.42) amounts to saying that $[\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}]$ is a bidimensional continued fraction, and we may write

$$(\alpha^{(1)}_0, \alpha^{(2)}_0) = [\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}],$$

for some real numbers $\alpha^{(1)}_0$, $\alpha^{(2)}_0$.

Now we are ready to prove the convergence of Jacobi's algorithm.

Theorem 1.61. *Let $(\alpha^{(1)}_0, \alpha^{(2)}_0)$ be a pair of real numbers, and let $\boldsymbol{a}^{(1)}$,$\boldsymbol{a}^{(2)}$ be the*

sequences of partial quotients. Then,

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}]. \tag{1.45}$$

Proof. We again assume that both sequences are infinite, the other cases following similarly by using Theorem 1.60.

Let $i \in \{1, 2\}$ and denote by $l^{(i)}$ the limit of $(A_k^{(i)}/C_k)_{k\geq 0}$. From (1.41), we obtain

$$\alpha_0^{(i)} = \frac{\alpha_k^{(1)} A_{k-1}^{(i)} + \alpha_k^{(2)} A_{k-2}^{(i)} + A_{k-3}^{(i)}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}} \tag{1.46}$$

$$= f_k \cdot \frac{A_{k-1}^{(i)}}{C_{k-1}} + g_k \cdot \frac{A_{k-2}^{(i)}}{C_{k-2}} + h_k \cdot \frac{A_{k-3}^{(i)}}{C_{k-3}}, \tag{1.47}$$

where

$$\begin{aligned} f_k &= \frac{C_{k-1}\alpha_k^{(1)}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}}, \\ g_k &= \frac{C_{k-2}\alpha_k^{(2)}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}}, \\ h_k &= \frac{C_{k-3}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}}. \end{aligned}$$

Observe that

$$f_k + g_k + h_k = 1. \tag{1.48}$$

Let $\varepsilon > 0$, then for j sufficiently large we have

$$l^{(i)} - \varepsilon < \frac{A_j^{(i)}}{C_j} < l^{(i)} + \varepsilon.$$

Hence, by (1.46) and (1.48), we get

$$l^{(i)} - \varepsilon < \alpha_0^{(i)} < l^{(i)} + \varepsilon.$$

Since ε is arbitrarily small, we conclude $\alpha_0^{(i)} = l^{(i)}$. □

An expression of the form (1.45) is called the Jacobi's MCF of $(\alpha_0^{(1)}, \alpha_0^{(1)})$.

We now address the uniqueness of Jacobi's MCF. For simplicity, we restrict to the case of infinite sequences of partial quotients; however, the argument can be readily adapted to MCFs with one or more interruptions, provided that one takes into account the possible exceptional cases, as in the classical theory of continued

fractions (see Theorem 1.14).

Lemma 1.62. *Let*

$$\begin{aligned}\boldsymbol{x}^{(1)} &= (x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)})\\ \boldsymbol{x}^{(2)} &= (x_0^{(2)}, x_1^{(2)}, \dots, x_n^{(2)})\\ \boldsymbol{y}^{(1)} &= (y_0^{(1)}, y_1^{(1)}, \dots, y_n^{(1)})\\ \boldsymbol{y}^{(2)} &= (y_0^{(2)}, y_1^{(2)}, \dots, y_n^{(2)})\end{aligned}$$

be sequences of real numbers with $x_k^{(i)}, y_k^{(i)} \in \mathbb{Z}$ *for* $k = 0, \dots, n-1$ *and* $i = 1, 2$*, and* $x_n^{(1)}, y_n^{(1)} > 1$*. Assume that both pairs* $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$*,* $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})$ *satisfy the admissibility conditions* (1.39)*, and that*

$$[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}] = [\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}].$$

Then,

$$x_k^{(i)} = y_k^{(i)} \quad \text{for } k = 0, \dots, n \quad \text{and } i = 1, 2.$$

Proof. We argue by induction on n. For the case $n = 1$,

$$[(x_0^{(1)}, x_1^{(1)}), (x_0^{(2)}, x_1^{(2)})] = [(y_0^{(1)}, y_1^{(1)}), (y_0^{(2)}, y_1^{(2)})],$$

that is,

$$\begin{aligned}x_0^{(1)} + \frac{x_1^{(2)}}{x_1^{(1)}} &= y_0^{(1)} + \frac{y_1^{(2)}}{y_1^{(1)}},\\ x_0^{(2)} + \frac{1}{x_1^{(1)}} &= y_0^{(2)} + \frac{1}{y_1^{(1)}}.\end{aligned}$$

From the second equation, since $x_0^{(2)}, y_0^{(2)} \in \mathbb{Z}$ and $x_1^{(1)}, y_1^{(1)} > 1$, we obtain

$$x_0^{(2)} = y_0^{(2)} \quad \text{and} \quad x_1^{(1)} = y_1^{(1)}.$$

Substituting into the first equation and using the admissibility conditions (which imply $\frac{x_1^{(2)}}{x_1^{(1)}}, \frac{y_1^{(2)}}{y_1^{(1)}} \leq 1$) combined with the fact that $x_0^{(1)}, y_0^{(1)}$ are integers, we conclude

$$x_0^{(1)} = y_0^{(1)} \quad \text{and} \quad x_1^{(2)} = y_1^{(2)}.$$

Assume that the claim holds up to $n-1$ and suppose

$$[(x_0^{(1)}, \dots, x_n^{(1)}), (x_0^{(2)}, \dots, x_n^{(2)})] = [(y_0^{(1)}, \dots, y_n^{(1)}), (y_0^{(2)}, \dots, y_n^{(2)})], \tag{1.49}$$

where $(x_0^{(i)}, \ldots, x_n^{(i)})_{i=1,2}$, $(y_0^{(i)}, \ldots, y_n^{(i)})_{i=1,2}$ satisfy the hypothesis of the lemma. Rewriting both sides of (1.49), we obtain

$$\left[\left(x_0^{(1)}, \ldots, x_{n-1}^{(1)} + \frac{x_n^{(2)}}{x_n^{(1)}}\right), \left(x_0^{(2)}, \ldots, x_{n-1}^{(2)} + \frac{1}{x_n^{(1)}}\right)\right] = \\ = \left[\left(y_0^{(1)}, \ldots, y_{n-1}^{(1)} + \frac{y_n^{(2)}}{y_n^{(1)}}\right), \left(y_0^{(2)}, \ldots, y_{n-1}^{(2)} + \frac{1}{y_n^{(1)}}\right)\right].$$

One may easily check that these sequences still satisfy the assumptions of the lemma. Hence, by the inductive hypothesis,

$$x_0^{(i)} = y_0^{(i)}, \ldots, x_{n-2}^{(i)} = y_{n-2}^{(i)},$$

and

$$x_{n-1}^{(1)} + \frac{x_n^{(2)}}{x_n^{(1)}} = y_{n-1}^{(1)} + \frac{y_n^{(2)}}{y_n^{(1)}}, \quad x_{n-1}^{(2)} + \frac{1}{x_n^{(1)}} = y_{n-1}^{(2)} + \frac{1}{y_n^{(1)}}.$$

Applying the same argument as in the base case, we deduce

$$x_{n-1}^{(1)} = y_{n-1}^{(1)}, \quad x_{n-1}^{(2)} = y_{n-1}^{(2)}, \quad x_n^{(1)} = y_n^{(1)}, \quad x_n^{(2)} = y_n^{(2)}.$$

This completes the induction. □

Theorem 1.63. *Let*

$$\boldsymbol{a}^{(1)} = (a_0^{(1)}, a_1^{(1)}, \ldots)$$
$$\boldsymbol{a}^{(2)} = (a_0^{(2)}, a_1^{(2)}, \ldots)$$
$$\boldsymbol{b}^{(1)} = (b_0^{(1)}, b_1^{(1)}, \ldots)$$
$$\boldsymbol{b}^{(2)} = (b_0^{(2)}, b_1^{(2)}, \ldots)$$

be sequences of integers satisfying the admissibility conditions (1.39)*. If*

$$[\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}] = [\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}],$$

then

$$a_n^{(i)} = b_n^{(i)} \quad \textit{for all } n \geq 0 \quad \textit{and } i = 1, 2.$$

Proof. For every $n \geq 0$, we can write

$$\left[\left(a_0^{(1)}, \dots, a_{n-1}^{(1)} + \frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right), \left(a_0^{(2)}, \dots, a_{n-1}^{(2)} + \frac{1}{\alpha_n^{(1)}}\right)\right] = \\ = \left[\left(b_0^{(1)}, \dots, b_{n-1}^{(1)} + \frac{\beta_n^{(2)}}{\beta_n^{(1)}}\right), \left(b_0^{(2)}, \dots, b_{n-1}^{(2)} + \frac{1}{\beta_n^{(1)}}\right)\right],$$

where $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ are the complete quotients associated with the two MCFs. The claim then follows from Lemma 1.62. □

Remark 1.64. The previous results clarify why the admissibility conditions (1.39) are sufficient for two integer sequences to arise from Jacobi's algorithm applied to a suitable pair of real numbers. Indeed, given $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}$ satisfying (1.39), by Theorem 1.59 we have

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}],$$

for some real numbers $(\alpha_0^{(1)}, \alpha_0^{(2)})$. On the other hand, if $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}$ are the sequences of partial quotients produced by Jacobi's algorithm applied to $(\alpha_0^{(1)}, \alpha_0^{(2)})$, then by Theorem 1.61,

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}].$$

Finally, by Theorem 1.63 we conclude

$$\boldsymbol{a}^{(1)} = \boldsymbol{b}^{(1)} \quad \text{and} \quad \boldsymbol{a}^{(2)} = \boldsymbol{b}^{(2)}.$$

Analogous conclusions hold in the presence of interruptions in the algorithm.

1.4.3 Multidimensional continued fractions

A natural extension of bidimensional continued fractions can be formulated in arbitrary dimension m.

Let $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}$ be m sequences of real numbers, with $x_k^{(1)} > 0$ for all $k \geq 1$. Let n be a positive integer, and set $\xi_n^{(i)} = x_n^{(i)}$ for every $i = 1, \dots, m$. Then, for $k \leq n-1$, define inductively

$$\xi_k^{(i-1)} = x_k^{(i-1)} + \frac{\xi_{k+1}^{(i)}}{\xi_{k+1}^{(1)}} \quad i = 2, \dots, m,$$

$$\xi_k^{(m)} = x_k^{(m)} + \frac{1}{\xi_{k+1}^{(1)}}.$$

A multidimensional continued fraction of dimension m is then an expression of the form

$$[\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}] = (\xi_0^{(1)}, \dots, \xi_0^{(m)}), \tag{1.50}$$

where

$$\xi_0^{(i-1)} = x_0^{(i-1)} + \frac{\xi_1^{(i)}}{\xi_1^{(1)}} = \dots \quad \text{for } i = 2, \dots, m,$$
$$\xi_0^{(m)} = x_0^{(m)} + \frac{1}{\xi_1^{(1)}} = \dots .$$

As in the case $m = 2$, we also refer to expressions of the form

$$\lim_{n \to +\infty} [(x_0^{(1)}, \dots, x_n^{(1)}), \dots, (x_0^{(m)}, \dots, x_n^{(m)})]$$

as multidimensional continued fractions, provided that the limit exists.

Remark 1.65. In Section 1.4.1, we considered bidimensional continued fractions defined by sequences $\boldsymbol{x}^{(i)}$ of possibly different lengths n, m. A similar construction could be carried out in the multidimensional setting as well, but the notation would become considerably heavier. For this reason, we restrict here to sequences of the same length.

As before, we define

$$\gamma_n(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) = [\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}].$$

The analogue of Lemma 1.52 in this general setting is given by the following lemma.

Lemma 1.66. *For $i = 1, \dots, m$, define*

$$A_{-n}^{(i)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) = \delta_{i,n}, \quad n = 1, \dots, m+1$$
$$A_n^{(i)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) = \sum_{j=1}^{m} x_n^{(j)} A_{n-j}^{(i)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) + A_{n-m-1}^{(i)}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}), \quad n \geq 0,$$

where $\delta_{i,n}$ is the usual Kronecker delta, and

$$C_{-m-1}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) = 1,$$
$$C_{-n}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) = 0, \quad \text{for } n = 1, \dots, m$$
$$C_n(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) = \sum_{j=1}^{m} x_n^{(j)} C_{n-j}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}) + C_{n-m-1}(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}), \quad n \geq 0.$$

Then,

$$\gamma_n(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m)}) = \left(\frac{A_n^{(1)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m)})}{C_n(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m)})},\dots,\frac{A_n^{(m)}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m)})}{C_n(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(m)})}\right).$$

1.4.4 Jacobi–Perron algorithm

The Jacobi–Perron algorithm generalizes Jacobi's algorithm to m dimensions. It acts on an m-tuple of real numbers $(\alpha_0^{(1)},\dots,\alpha_0^{(m)})$ and produces m sequences of integers

$$\boldsymbol{a}^{(1)} = (a_n^{(1)})_{n\geq 0},\ \dots,\ \boldsymbol{a}^{(m)} = (a_n^{(m)})_{n\geq 0}$$

defined recursively by

$$\begin{cases} a_n^{(i)} = \lfloor \alpha_n^{(i)} \rfloor, \quad i = 1,\dots,m \\ \alpha_{n+1}^{(1)} = \frac{1}{\alpha_n^{(m)} - a_n^{(m)}} \\ \alpha_{n+1}^{(i)} = \frac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}}, \quad i = 2,\dots,m, \end{cases} \qquad (n \geq 0).$$

Observe that, when $m = 2$, this reduces to Jacobi's algorithm.

As in the bidimensional case, the algorithm may exhibit interruptions. If $\alpha_{n_0}^{(m)}$ is an integer for some index n_0, the procedure stops at that stage and continues with the $(m-1)$-tuple $(\alpha_{n_0}^{(1)},\dots,\alpha_{n_0}^{(m-1)})$. Iterating this process, one eventually reaches the classical continued fraction algorithm when only one coordinate remains.

The relationship between the number of interruptions and the number of independent rational relations among $1,\alpha_0^{(1)},\dots,\alpha_0^{(m)}$ is studied in [35]. In particular, if $1,\alpha_0^{(1)},\dots,\alpha_0^{(m)}$ are linearly independent over $\mathbb{Q}$, then no interruption occurs, that is, all sequences of partial quotients are infinite. The converse, however, does not hold in general.

For simplicity, in the remainder of this section we restrict to the case where no interruption occurs (the *simple case*).

The integers $a_n^{(i)}$ and the real numbers $\alpha_n^{(i)}$ are still called, respectively, the *partial quotients* and the *complete quotients*. For every $n \geq 0$, we have

$$(\alpha_0^{(1)},\dots,\alpha_0^{(m)}) = [(a_0^{(1)},\dots,a_n^{(1)},\alpha_{n+1}^{(1)}),\dots,(a_0^{(m)},\dots,a_n^{(m)},\alpha_{n+1}^{(m)})], \tag{1.51}$$

and, for every $k \geq 0$,

$$(\alpha_{n+1}^{(1)},\dots,\alpha_{n+1}^{(m)}) = [(a_{n+1}^{(1)},\dots,a_{n+k}^{(1)},\alpha_{n+k+1}^{(1)}),\dots,(a_{n+1}^{(m)},\dots,a_{n+k}^{(m)},\alpha_{n+k+1}^{(m)})]. \tag{1.52}$$

The partial quotients satisfy the following admissibility conditions:

$$\begin{aligned}
&a_n^{(i)} \geq 0 \quad \text{for } i = 1, \dots, m \\
&a_n^{(1)} \geq 1 \\
&(a_n^{(1)}, a_{n+1}^{(2)}) \succeq (a_n^{(m)}, 1) \\
&(a_n^{(1)}, a_{n+1}^{(2)}, a_{n+2}^{(3)}) \succeq (a_n^{(m-1)}, a_{n+1}^{(m)}, 1) \\
&\qquad\qquad \vdots \\
&(a_n^{(1)}, a_{n+1}^{(2)}, \dots, a_{n+m-2}^{(m-1)}, a_{n+m-1}^{(m)}) \succeq (a_n^{(2)}, a_{n+1}^{(3)}, \dots, a_{n+m-2}^{(m)}, 1).
\end{aligned} \tag{1.53}$$

for all $n \geq 1$, where $\succeq$ denotes the lexicographic order (see again [35]).

We also define the convergents as

$$\gamma_n = \gamma_n(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = \gamma_n(\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)}).$$

By Lemma 1.66, setting $A_n^{(i)} = A_n^{(i)}(\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)})$ and $C_n = C_n(\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)})$, we obtain

$$\gamma_n = \left(\frac{A_n^{(1)}}{C_n}, \dots, \frac{A_n^{(m)}}{C_n} \right).$$

Moreover, for every $i = 1, \dots, m$ and $k \geq 0$, one has

$$\alpha_0^{(i)} = \frac{\alpha_k^{(1)} A_{k-1}^{(i)} + \cdots + \alpha_k^{(m)} A_{k-m}^{(i)} + A_{k-m-1}^{(i)}}{\alpha_k^{(1)} C_{k-1} + \cdots + \alpha_k^{(m)} C_{k-m} + C_{k-m-1}}. \tag{1.54}$$

For every $k \geq 0$, the following matrix equality holds

$$\begin{aligned}
\begin{bmatrix} a_k^{(1)} & 1 & 0 & \cdots & 0 \\ a_k^{(2)} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_k^{(m)} & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \alpha_{k+1}^{(1)} \\ \alpha_{k+1}^{(2)} \\ \vdots \\ \alpha_{k+1}^{(m)} \\ 1 \end{bmatrix} &= \begin{bmatrix} a_k^{(1)} \alpha_{k+1}^{(1)} + \alpha_{k+1}^{(2)} \\ a_k^{(2)} \alpha_{k+1}^{(1)} + \alpha_{k+1}^{(3)} \\ \vdots \\ a_k^{(m)} \alpha_{k+1}^{(1)} + 1 \\ \alpha_{k+1}^{(1)} \end{bmatrix} = \\
&= \alpha_{k+1}^{(1)} \begin{bmatrix} a_k^{(1)} + \frac{\alpha_{k+1}^{(2)}}{\alpha_{k+1}^{(1)}} \\ \vdots \\ a_k^{(m)} + \frac{1}{\alpha_{k+1}^{(1)}} \\ 1 \end{bmatrix} = \alpha_{k+1}^{(1)} \begin{bmatrix} \alpha_k^{(1)} \\ \vdots \\ \alpha_k^{(m)} \\ 1 \end{bmatrix},
\end{aligned} \tag{1.55}$$

Iterating (1.55), we obtain the identity

$$\begin{bmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{bmatrix} = \frac{1}{\alpha_1^{(1)}\alpha_2^{(1)}\dots\alpha_{k+1}^{(1)}} M_0 M_1 \cdots M_k \begin{bmatrix} \alpha_{k+1}^{(1)} \\ \vdots \\ \alpha_{k+1}^{(m)} \\ 1 \end{bmatrix}, \tag{1.56}$$

where, for every n,

$$M_n = \begin{bmatrix} a_n^{(1)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(m)} & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{1.57}$$

Finally, as in Theorem 1.59 and Theorem 1.61, one can show that, for every $i = 1, \dots, m$,

$$\lim_{n\to+\infty} \frac{A_n^{(i)}}{C_n} = \alpha_0^{(i)}.$$

This allows us to write

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), \dots, (a_0^{(m)}, a_1^{(m)}, \dots)].$$

An expression of this form is called a *Jacobi–Perron multidimensional continued fraction*; hereafter, we simply refer to such expressions as *multidimensional continued fractions*.

Also in the multidimensional setting, one can prove that every m-tuple of real numbers admits a unique representation as a multidimensional continued fraction satisfying the admissibility conditions (1.53).

1.4.5 Periodic multidimensional continued fractions

Generalizing Theorem 1.16, we give the following definition.

Definition 1.67. A multidimensional continued fraction $[\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)}]$ is said to be (ultimately) periodic if each $\boldsymbol{a}^{(i)}$ is infinite, and there exist integers $t > 0$ and $k \geq 0$ such that, for every $n \geq k$,

$$a_{n+t}^{(i)} = a_n^{(i)}.$$

In this case we will write

$$\left[\left(a_0^{(1)},\dots,a_{k-1}^{(1)},\overline{a_k^{(1)},\dots,a_{k+t-1}^{(1)}}\right),\dots,\left(a_0^{(m)},\dots,a_{k-1}^{(m)},\overline{a_k^{(m)},\dots,a_{k+t-1}^{(m)}}\right)\right].$$

If $k=0$, the multidimensional continued fraction is said to be purely periodic.

The periodicity of the multidimensional continued fraction of an m-tuple $(\alpha_0^{(1)},\dots,\alpha_0^{(m)})$ encodes information about the algebraicity of the numbers $\alpha_0^{(i)}$. To study this connection, we first introduce some preliminary notions on matrices.

We say that a real matrix with non-negative (resp., positive) entries is *non-negative* (resp., *positive*). A non-negative square matrix M of dimension n can be interpreted as the adjacency matrix of a directed graph $\mathcal{G}(M)$ with n nodes: there is an edge from node i to node j if and only if $M_{i,j}>0$.

A *path* from node i to node j is a finite sequence of edges connecting i to j, and its *length* is the number of edges in the sequence. It is well known that if $(M^k)_{i,j}>0$ for some integer $k\geq 1$, then there exists at least one path of length k from i to j.

A non-negative square matrix $M=(M_{i,k})$ of dimension n is *primitive* if there exists an integer $l\geq 1$ such that M^l is positive. Equivalently, M is primitive if there exists $l\geq 1$ such that for every pair of nodes (i,j) there is a path of length l from i to j.

Given a non-negative square matrix M, define a matrix $B(M)$ of the same dimension by

$$B(M)_{i,j}=\begin{cases}0 & \text{if} \quad M_{i,j}=0\\ 1 & \text{if} \quad M_{i,j}>0.\end{cases}$$

Then, M is positive if and only if $B(M)$ is the matrix with all entries equal to 1, which we denote by U. Moreover, M is primitive if and only if $B(M)$ is primitive, and for any non-negative matrices M,N one has

$$B(MN)=B\left(B(M)B(N)\right).$$

We now recall a classical result.

Lemma 1.68 (Perron-Frobenius Theorem)**.** *Let $M=(M_{i,j})$ be a $n\times n$ primitive matrix. Then,*

(i) There exists a positive real eigenvalue ρ of M that is dominant, i.e. $\rho>|\tau|$ for any other eigenvalue τ.

(ii) The eigenspace associated to ρ is one-dimensional and it is generated by an eigenvector $v=(v_1,\dots,v_n)$ with strictly positive entries.

(iii) Any other eigenvector of M with positive entries is a positive multiple of v.

Proof. See [37] for the original proof. □

Lemma 1.69. *Suppose that $\alpha_0^{(1)}, \dots, \alpha_0^{(m)} \geq 1$, and let $M_0, \dots, M_k$ be as in (1.56). Set $M(k) = M_0 \cdots M_k$. Then, $M(k)$ is primitive.*

Proof. Since $\alpha_0^{(1)}, \dots, \alpha_0^{(m)}$ are real numbers greater than 1, all matrices $M_0, \dots, M_k$ are non-negative, hence so is their product $M(k)$. Define $N_h = M_h - D_h$, where

$$D_h = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_h^{(2)} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_h^{(m)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Observe that in the graph $\mathcal{G}(N_h)$, node 1 is connected to node m and to node 1 itself, and each node $i = 2, \dots, m$ is connected to node $i-1$. From this structure, one sees that for any pair (i, j) there exists a path of length at most m connecting i to j. Further, for every $l \geq m$, there exists a path of length l between any two nodes, which implies that N_h is primitive, and $B(N_h^l) = U$ for any $h = 0, \dots, k$. Since $B(N_0) = B(N_h)$ for all $h = 1, \dots, k$, it follows that for $l \geq m$

$$\begin{aligned} B(M(k)^l) &= B\left((M_0 \cdots M_k)^l\right) = B\left(M_0 \cdots M_k M_0 \cdots M_k \cdots M_0 \cdots M_k\right) \\ &= B\left(B(M_0) \cdots B(M_k) B(M_0) \cdots B(M_k) \cdots B(M_0) \cdots B(M_k)\right) \\ &= B\left(B(N_0 + D_0) \cdots B(N_k + D_k) \cdots B(N_0 + D_0) \cdots B(N_k + D_k)\right) \\ &= B\left(B(N_0)^{l(k+1)} + A\right) \\ &= B\left(U + A\right), \end{aligned}$$

where A is a suitable non-negative matrix. Hence, $B(M(k)^l) = B(U) = U$, and the claim follows. □

Theorem 1.70. *Suppose that*

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = \left[\left(\overline{a_0^{(1)}, \dots, a_{t-1}^{(1)}}\right), \dots, \left(\overline{a_0^{(m)}, \dots, a_{t-1}^{(m)}}\right)\right].$$

Then, $\mathbb{Q}\left(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}\right)$ is an algebraic extension of $\mathbb{Q}$ of degree at most $m+1$.

Proof. Using (1.51) and (1.52), we get that

$$\alpha_0^{(i)} = \alpha_t^{(i)} \quad i = 1, \ldots, m.$$

By (1.56), we obtain then

$$M(t) \begin{bmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{bmatrix} = \left(\alpha_0^{(1)}\alpha_1^{(1)} \ldots \alpha_{t-1}^{(1)}\right) \begin{bmatrix} \alpha_0^{(1)} \\ \vdots \\ \alpha_0^{(m)} \\ 1 \end{bmatrix},$$

where $M(t) = M_0 M_1 \cdots M_{t-1}$. Hence, $\rho = \alpha_0^{(1)}\alpha_1^{(1)} \ldots \alpha_{t-1}^{(1)} > 0$ is an eigenvalue of $M(t)$.

Now, since $a_0^{(i)} = a_t^{(i)}$ for every $i = 1, \ldots, m$, the admissibility conditions (1.53) imply $a_0^{(i)} > 0$, hence $\alpha_0^{(i)} \geq 1$ for all i. Thus, we can apply Lemma 1.69 and Lemma 1.68 and obtain that $\operatorname{rank}(M(t) - \rho I) = m$. Hence, there exists a vector

$$\boldsymbol{w} = (w_1, \ldots, w_{m+1})$$

in the nullspace of $M(t) - \rho I$ with coordinates in the same field containing the entries of $M(t) - \rho I$, that is $w_i \in \mathbb{Q}(\rho)$. Since the eigenspace associated to ρ is one-dimensional, there exists a scalar s such that

$$(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}, 1) = s\boldsymbol{w}.$$

By looking at the last entry of the vectors, we have $s = 1/w_{m+1} \in \mathbb{Q}(\rho)$. Consequently,

$$\mathbb{Q}(\alpha_0^{(1)}, \ldots, \alpha_0^{(m)}) \subseteq \mathbb{Q}(\rho). \tag{1.58}$$

On the other hand, ρ is an eigenvalue of $M(t)$, hence it is a root of the characteristic polynomial of $M(t)$, which is a monic polynomial of degree $m+1$ with integer coefficients. This implies that ρ is an algebraic integer of degree at most $m+1$, and the theorem follows. □

The following corollary shows that the same conclusion holds in the ultimately periodic case.

Corollary 1.71. *Suppose that*

$$(\alpha_0^{(1)},\dots,\alpha_0^{(m)}) = \left[\left(a_0^{(1)},\dots,a_{k-1}^{(1)},\overline{a_k^{(1)},\dots,a_{k+t-1}^{(1)}}\right),\dots \right.$$
$$\left.\dots,\left(a_0^{(m)},\dots,a_{k-1}^{(m)},\overline{a_k^{(m)},\dots,a_{k+t-1}^{(m)}}\right)\right].$$

Then, $\mathbb{Q}\left(\alpha_0^{(1)},\dots,\alpha_0^{(m)}\right)$ *is an algebraic extension of* $\mathbb{Q}$ *of degree at most* $m+1$.

Proof. It is sufficient to observe that

$$(\alpha_0^{(1)},\dots,\alpha_0^{(m)}) = \left[\left(a_0^{(1)},\dots,a_{k-1}^{(1)},\alpha_k^{(1)}\right),\dots,\left(a_0^{(m)},\dots,a_{k-1}^{(m)},\alpha_k^{(m)}\right)\right],$$

where

$$(\alpha_k^{(1)},\dots,\alpha_k^{(m)}) = \left[\left(\overline{a_k^{(1)},\dots,a_{k+t-1}^{(1)}}\right),\dots,\left(\overline{a_k^{(m)},\dots,a_{k+t-1}^{(m)}}\right)\right].$$

Using (1.54), $\alpha_0^{(i)} \in \mathbb{Q}(\alpha_k^{(1)},\dots,\alpha_k^{(m)})$ for every $i = 1,\dots,m$, and the claim follows from Theorem 1.70. □

In general, the degree of $\alpha_0^{(i)}$ is not necessarily equal to $m+1$, since the characteristic polynomial of $M(t)$ in the proof of Theorem 1.70 needs not be irreducible. However, this does hold in the case $m = 2$.

Theorem 1.72. *Suppose that*

$$(\alpha_0^{(1)},\alpha_0^{(2)}) = \left[\left(a_0^{(1)},\dots,a_{k-1}^{(1)},\overline{a_k^{(1)},\dots,a_{k+t-1}^{(1)}}\right),\left(a_0^{(2)},\dots,a_{k-1}^{(2)},\overline{a_k^{(2)},\dots,a_{k+t-1}^{(2)}}\right)\right].$$

Then, $\mathbb{Q}\left(\alpha_0^{(1)},\alpha_0^{(2)}\right)$ *is an algebraic extension of* $\mathbb{Q}$ *of degree* 3.

Proof. By Theorem 1.70, we know that $d = \left[\mathbb{Q}(\alpha_0^{(1)},\alpha_0^{(2)}) : \mathbb{Q}\right] \leq 3$.

If $d = 1$, then $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ would be rational, but this cannot happen since both sequences of partial quotients are infinite (see Theorem 1.54).

Similarly, if $d = 2$, then either $\left[\mathbb{Q}(\alpha_0^{(1)},\alpha_0^{(2)}) : \mathbb{Q}(\alpha_0^{(1)})\right] = 1$ or $\left[\mathbb{Q}(\alpha_0^{(1)}) : \mathbb{Q}\right] = 1$. Both cases contradict again the infiniteness of the sequences of partial quotients. □

In fact, it is only the period of the MCF that determines the number field in which $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ lie, as shown in the following proposition.

Proposition 1.73. *Let*

$$(\alpha_0^{(1)},\alpha_0^{(2)}) = \left[\left(a_0^{(1)},\dots,a_{k-1}^{(1)},\overline{a_k^{(1)},\dots,a_{k+h-1}^{(1)}}\right),\left(a_0^{(2)},\dots,a_{k-1}^{(2)},\overline{a_k^{(2)},\dots,a_{k+h-1}^{(2)}}\right)\right],$$

and

$$(\bar{\alpha}_0^{(1)}, \bar{\alpha}_0^{(2)}) = \left[\left(c_0, \ldots, c_{k-1}, \overline{a_k^{(1)}, \ldots, a_{k+h-1}^{(1)}}\right), \left(d_0, \ldots, d_{k-1}, \overline{a_k^{(2)}, \ldots, a_{k+h-1}^{(2)}}\right)\right].$$

Then, $\alpha_0^{(1)}, \bar{\alpha}_0^{(1)}, \alpha_0^{(2)}, \bar{\alpha}_0^{(2)}$ *all belong to the same cubic number field.*

Proof. Consider

$$(\alpha_k^{(1)}, \alpha_k^{(2)}) = (\bar{\alpha}_k^{(1)}, \bar{\alpha}_k^{(2)}) = \left[\left(\overline{a_k^{(1)}, \ldots, a_{k+h-1}^{(1)}}\right), \left(\overline{a_k^{(2)}, \ldots, a_{k+h-1}^{(2)}}\right)\right].$$

By Theorem 1.72, $\alpha_k^{(1)}$ and $\alpha_k^{(2)}$ are cubic irrationals belonging to the same cubic field. Thus, recalling (1.54), the statement follows from

$$\alpha_0^{(1)} = \frac{\alpha_k^{(1)} A_{k-1}^{(1)} + \alpha_k^{(2)} A_{k-2}^{(1)} + A_{k-3}^{(1)}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}}, \quad \alpha_0^{(2)} = \frac{\alpha_k^{(1)} A_{k-1}^{(2)} + \alpha_k^{(2)} A_{k-2}^{(2)} + A_{k-3}^{(2)}}{\alpha_k^{(1)} C_{k-1} + \alpha_k^{(2)} C_{k-2} + C_{k-3}},$$

$$\bar{\alpha}_0^{(1)} = \frac{\alpha_k^{(1)} \bar{A}_{k-1}^{(1)} + \alpha_k^{(2)} \bar{A}_{k-2}^{(1)} + \bar{A}_{k-3}^{(1)}}{\alpha_k^{(1)} \bar{C}_{k-1} + \alpha_k^{(2)} \bar{C}_{k-2} + \bar{C}_{k-3}}, \quad \bar{\alpha}_0^{(2)} = \frac{\alpha_k^{(1)} \bar{A}_{k-1}^{(2)} + \alpha_k^{(2)} \bar{A}_{k-2}^{(2)} + \bar{A}_{k-3}^{(2)}}{\alpha_k^{(1)} \bar{C}_{k-1} + \alpha_k^{(2)} \bar{C}_{k-2} + \bar{C}_{k-3}},$$

where $\left(\frac{A_n^{(1)}}{C_n}, \frac{A_n^{(2)}}{C_n}\right)$ and $\left(\frac{\bar{A}_n^{(1)}}{\bar{C}_n}, \frac{\bar{A}_n^{(2)}}{\bar{C}_n}\right)$, for all $n \geq 0$, are the convergents of $(\alpha_0^{(1)}, \alpha_0^{(2)})$ and $(\bar{\alpha}_0^{(1)}, \bar{\alpha}_0^{(2)})$, respectively. □

2 Transcendence criteria for multidimensional continued fractions

Chapter Contents

This chapter presents several results obtained in joint work with Murru and Romeo [1]. These results concern transcendence criteria for MCFs. In particular, Section 2.3 uses Liouville's classical idea to construct transcendental numbers via MCFs whose partial quotients grow sufficiently rapidly. Section 2.4 extends Baker's construction for quasi-periodic continued fractions to the multidimensional setting. The first two sections, on the other hand, are devoted to presenting some preliminary new results on MCFs. The original contributions begin essentially with Lemma 2.13. The preceding results are either accompanied by explicit references or are expected to be known. In the latter case, however, they do not appear to be readily available in the literature, at least not in the form in which they are presented here.

Unless otherwise specified, from now on we shall consider only multidimensional continued fractions with infinite sequences of partial quotients.

2.1 Auxiliary results

In this section, we prove some results that will be used in the proofs of the main transcendence criteria.

First, we attach to a Jacobi–Perron MCF with partial quotients $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)}$, the auxiliary sequences

$$\begin{aligned} \tilde{A}_n^{(i)} &= A_n^{(i)} C_{n-1} - A_{n-1}^{(i)} C_n, \\ \tilde{\tilde{A}}_n^{(i)} &= A_n^{(i)} C_{n-2} - A_{n-2}^{(i)} C_n, \quad n \geq 0. \end{aligned}$$

If $m = 2$, the sequences $(\tilde{A}_n^{(i)})_{n \geq 0}$ enjoy particularly nice properties.

Lemma 2.1. *The sequences* $(\tilde{A}_n^{(1)})_{n\geq 0}$ *and* $(\tilde{A}_n^{(2)})_{n\geq 0}$ *satisfy*

$$\begin{aligned}\tilde{A}_n^{(1)} &= -a_n^{(2)}\tilde{A}_{n-1}^{(1)} - a_{n-1}^{(1)}\tilde{A}_{n-2}^{(1)} + \tilde{A}_{n-3}^{(1)},\\ \tilde{A}_n^{(2)} &= -a_n^{(2)}\tilde{A}_{n-1}^{(2)} - a_{n-1}^{(1)}\tilde{A}_{n-2}^{(2)} + \tilde{A}_{n-3}^{(2)},\end{aligned} \tag{2.1}$$

for all $n \geq 1$*, with initial conditions*

$$\begin{cases}\tilde{A}_{-2}^{(1)} = 0, \quad \tilde{A}_{-1}^{(1)} = 0, \quad \tilde{A}_0^{(1)} = -1,\\ \tilde{A}_{-2}^{(2)} = 1, \quad \tilde{A}_{-1}^{(2)} = 0, \quad \tilde{A}_0^{(2)} = 0.\end{cases}$$

Moreover,

$$|\tilde{A}_n^{(1)}| \leq C_n, \quad |\tilde{A}_n^{(2)}| \leq C_n, \quad |\tilde{\tilde{A}}_n^{(1)}| \leq C_n, \quad |\tilde{\tilde{A}}_n^{(2)}| \leq C_n,$$

for all $n \geq 0$*.*

Proof. See [78, Chapter 9]. □

Remark 2.2. The recurrent relation (2.1) shows, in particular, that $\tilde{A}_n^{(i)}$ is independent of $a_n^{(1)}$. In the m-dimensional case, a recursion for $\tilde{A}_n^{(i)}$ as simple as the one given in Lemma 2.1 could not be found. Nevertheless, it is still possible to prove that $\tilde{A}_n^{(i)}$ does not depend on $a_n^{(1)}$. Indeed, we have

$$\begin{aligned}\tilde{A}_n^{(i)} &= A_n^{(i)}C_{n-1} - A_{n-1}^{(i)}C_n = (a_n^{(1)}A_{n-1}^{(i)} + \ldots + a_n^{(m)}A_{n-m}^{(i)} + A_{n-m-1}^{(i)})C_{n-1}\\ &\quad - A_{n-1}^{(i)}(a_n^{(1)}C_{n-1} + \ldots + a_n^{(m)}C_{n-m} + C_{n-m-1}) =\\ &= a_n^{(2)}(A_{n-2}^{(i)}C_{n-1} - A_{n-1}^{(i)}C_{n-2}) + \ldots + a_n^{(m)}(A_{n-m}^{(i)}C_{n-1} - A_{n-1}^{(i)}C_{n-m})\\ &\quad + (A_{n-m-1}^{(i)}C_{n-1} - A_{n-1}^{(i)}C_{n-m-1}).\end{aligned}$$

But, by the recurrence relations of Lemma 1.66, the quantities $A_{n-2}^{(i)}, \ldots, A_{n-m-1}^{(i)}$ and $C_{n-1}, \ldots, C_{n-m-1}$ are independent of $a_n^{(1)}$, and hence so is $\tilde{A}_n^{(i)}$.

This observation will play a crucial role, as it will allow us to explicitly construct transcendental multidimensional continued fractions using Theorem 2.16 and Theorem 2.19 (see also Remark 2.17).

Lemma 2.1 yields the bound $\tilde{A}_n^{(i)} \leq C_n$ in the case $m = 2$. For completeness, we also show that $\tilde{A}_{n+1}^{(1)}$ can be bounded in terms of the denominator at the previous step C_n. This is established in the following proposition for Jacobi's MCFs, although a similar result can be obtained for $m \geq 3$.

Proposition 2.3. *Let* $[(a_0^{(1)}, a_1^{(1)}, \dots), (a_0^{(2)}, a_1^{(2)}, \dots)]$ *be such that* $a_{n+1}^{(1)} < C_n$ *for all* $n \in \mathbb{N}$. *Then,*

$$\tilde{A}_{n+1}^{(1)} < 3C_n^2, \qquad \tilde{A}_{n+1}^{(2)} < 3C_n^2,$$

for all $n \in \mathbb{N}$.

Proof. We know that, $\tilde{A}_{n+1}^{(1)} \leq C_{n+1}$. Hence,

$$\tilde{A}_{n+1}^{(1)} \leq C_{n+1} = a_{n+1}^{(1)} C_n + a_{n+1}^{(2)} C_{n-1} + C_{n-2} \leq a_{n+1}^{(1)}(C_n + C_{n-1} + C_{n-2}) < 3C_n^2,$$

since $a_{n+1}^{(2)} \leq a_{n+1}^{(1)} < C_n$. The same argument applies to $\tilde{A}_{n+1}^{(2)}$. □

We now seek to extend some elementary properties of convergents for classical continued fractions in the multidimensional setting. In particular, to prove the main theorems of Section 2.3, we need to derive an upper bound for the simultaneous approximation error $\left|\alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n}\right|$. The following lemma is merely a technical tool for the proof of Lemma 2.5, which indeed may be viewed as a multidimensional counterpart of Theorem 1.22, although it yields a weaker conclusion.

Lemma 2.4. *Let* $r_1, \dots, r_n$ *be pairwise distinct real numbers. Set*

$$m_n = \min(r_1, \dots, r_n), \quad M_n = \max(r_1, \dots, r_n).$$

If $x \in [m_n, M_n]$, *then there exists* $j \in \{1, \dots, n-1\}$ *such that*

$$x \in [\min(r_j, r_{j+1}), \max(r_j, r_{j+1})].$$

Proof. We argue by induction on n. The case $n = 2$ is trivial. Assume the claim holds for $n - 1$.

Let us add one more point r_n. Then, exactly one of the following occurs:

(i) $m_{n-1} = m_n$ and $M_{n-1} = M_n$, in which case the claim follows directly from the inductive hypothesis.

(ii) $m_n < m_{n-1}$ and $M_{n-1} = M_n$, so that $m_n = r_n$. Then, either $x \in [r_n, r_{n-1}]$, or $x \in [r_{n-1}, M_n] \subseteq [m_{n-1}, M_{n-1}]$, and the claim follows.

(iii) $m_{n-1} = m_n$ and $M_{n-1} < M_n$, so that $M_n = r_n$ and the argument is analogous to the one before. □

Lemma 2.5. *Given*

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), \dots, (a_0^{(m)}, a_1^{(m)}, \dots)],$$

then, for all $i = 1, \dots, m$,

$$\left| \alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n} \right| \leq \frac{|\tilde{A}_{n+1}^{(i)}|}{C_{n+1} C_n},$$

for infinitely many $n \in \mathbb{N}$.

Proof. Arguing as in the proof of Lemma 1.61 (for $m = 2$), one shows that, for all $i = 1, \dots, m$, setting

$$m_n^{(i)} = \min \left(\frac{A_n^{(i)}}{C_n}, \frac{A_{n+1}^{(i)}}{C_{n+1}}, \dots, \frac{A_{n+m}^{(i)}}{C_{n+m}} \right),$$
$$M_n^{(i)} = \max \left(\frac{A_n^{(i)}}{C_n}, \frac{A_{n+1}^{(i)}}{C_{n+1}}, \dots, \frac{A_{n+m}^{(i)}}{C_{n+m}} \right),$$

we have

$$m_n^{(i)} \leq \alpha_0^{(i)} \leq M_n^{(i)},$$

for all $n \in \mathbb{N}$ (see Theorem 1 of [12] for more details).

By Lemma 2.4, for each $n \in \mathbb{N}$ there exists $j \in \{0, \dots, m-1\}$ such that $\alpha_0^{(i)} \in \left[\frac{A_{n+j}^{(i)}}{C_{n+j}}, \frac{A_{n+j+1}^{(i)}}{C_{n+j+1}} \right]$ or $\alpha_0^{(i)} \in \left[\frac{A_{n+j+1}^{(i)}}{C_{n+j+1}}, \frac{A_{n+j}^{(i)}}{C_{n+j}} \right]$. Hence, we have

$$\left| \alpha_0^{(i)} - \frac{A_{n+j}^{(i)}}{C_{n+j}} \right| \leq \left| \frac{A_{n+j+1}^{(i)}}{C_{n+j+1}} - \frac{A_{n+j}^{(i)}}{C_{n+j}} \right| = \frac{|\tilde{A}_{n+j+1}^{(i)}|}{C_{n+j+1} C_{n+j}},$$

for infinitely many n. This in turn impies, by a suitable rescaling of the indexes, that

$$\left| \alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n} \right| \leq \left| \frac{A_{n+1}^{(i)}}{C_{n+1}} - \frac{A_n^{(i)}}{C_n} \right| = \frac{|\tilde{A}_{n+1}^{(i)}|}{C_{n+1} C_n},$$

for infinitely many n and for all $i = 1, \dots, m$. □

Remark 2.6. If $m = 2$, then by the same argument, *for every* n we obtain

$$\left| \alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n} \right| \leq \frac{|\tilde{A}_{n+1}^{(i)}|}{C_{n+1} C_n}, \quad \text{or} \quad \left| \alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n} \right| \leq \frac{|\tilde{A}_{n+2}^{(i)}|}{C_{n+2} C_{n+1}}.$$

Since $|\tilde{A}_{n+1}^{(i)}| < C_{n+1}$, $|\tilde{A}_{n+2}^{(i)}| < C_{n+2}$ and $C_{n+1} > C_n$, we can conclude that

$$\left| \alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n} \right| \leq \frac{1}{C_n}.$$

The results in Section 2.4 rely on Theorem 2.14. For the latter, in turn, we will

need to compare the numerators $A_n^{(i)}$ and the denominators C_n of the convergents of a MCF in order to obtain suitable bounds for certain quantities arising in the proof. A first attempt in this direction is to use the convergence properties of $A_n^{(i)}/C_n$. This is carried out in the following proposition.

Proposition 2.7. *Let $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) \in \mathbb{R}^m$ such that $0 < \alpha_0^{(i)} < 1$ for all $i = 1, \dots, m$. Then, there exists a constant $K \in \mathbb{N}$, depending only on $\alpha_0^{(1)}, \dots, \alpha_0^{(m)}$, such that $A_n^{(i)} \leq KC_n$ for all $n \geq 0$.*

Proof. For each $i = 1, \dots, m$, consider the sequence

$$M_k^{(i)} = \max\left(\frac{A_k^{(i)}}{C_k}, \frac{A_{k+1}^{(i)}}{C_{k+1}}, \dots, \frac{A_{k+m}^{(i)}}{C_{k+m}}\right).$$

From the proof of Theorem 1.59 (which easily extends to any $m > 2$), we know that$(M_k^{(i)})_{k\geq 0}$ is strictly decreasing and converges to $\alpha_0^{(i)} < 1$. Hence, there exists k_0 such that $A_k^{(i)} \leq C_k$ for all $k \geq k_0$. The claim follows. □

For the case $m = 2$, we can obtain a sharper bound, which is precisely the one that will be used in the proof of Theorem 2.14.

Lemma 2.8. *Let $(\alpha_0^{(1)}, \alpha_0^{(2)}) \in \mathbb{R}^2$ with $0 < \alpha_0^{(1)}, \alpha_0^{(2)} < 1$. Then,*

$$A_n^{(1)}, A_n^{(2)} \leq C_n,$$

for all $n \geq 0$.

Proof. We argue by induction. For $n = 0$, since $0 < \alpha_0^{(1)}, \alpha_0^{(2)} < 1$, we have $A_0^{(1)} = a_0^{(1)} = 0$, $A_0^{(2)} = a_0^{(2)} = 0$, and $C_0 = 1$, so the claim holds.

For $n = 1$, we have $A_1^{(1)} = a_1^{(2)}$, $A_1^{(2)} = 1$, and $C_1 = a_1^{(1)}$, hence again $A_1^{(i)} \leq C_1$.

For $n = 2$, we compute

$$A_2^{(1)} = a_2^{(1)}a_1^{(2)} + 1, \quad A_2^{(2)} = a_2^{(1)}, \quad C_2 = a_2^{(1)}a_1^{(1)} + a_2^{(2)}.$$

Clearly $A_2^{(2)} \leq C_2$, and $A_2^{(1)} \leq C_2$ follows from the admissibility condition (1.53): $a_2^{(2)} \geq 1$ whenever $a_1^{(1)} = a_1^{(2)}$.

Assume now the claim holds for $n - 1$, $n - 2$, and $n - 3$. Using the recurrence

relations,

$$\begin{aligned} A_n^{(1)} &= a_n^{(1)} A_{n-1}^{(1)} + a_n^{(2)} A_{n-2}^{(1)} + A_{n-3}^{(1)}, \\ A_n^{(2)} &= a_n^{(1)} A_{n-1}^{(2)} + a_n^{(2)} A_{n-2}^{(2)} + A_{n-3}^{(2)}, \\ C_n &= a_n^{(1)} C_{n-1} + a_n^{(2)} C_{n-2} + C_{n-3}, \end{aligned}$$

and comparing term by term, the conclusion follows by the inductive hypothesis. □

Following the same inductive proof of Lemma 2.8, it is possible to prove the following general result.

Corollary 2.9. *Let* $(\alpha_0^{(1)}, \alpha_0^{(2)}) \in \mathbb{R}^2$ *such that*

$$N < \alpha_0^{(1)} < N+1, \quad M < \alpha_0^{(2)} < M+1,$$

for some integers $N, M \geq 0$*. Then, for all* $n \geq 0$*,*

$$\begin{aligned} NC_n \leq A_n^{(1)} < (N+1)C_n, \\ MC_n \leq A_n^{(2)} < (M+1)C_n. \end{aligned}$$

In the proof of the main theorems in Section 2.4, we will also need a bound on the difference of two real numbers whose MCFs share a given number of initial partial quotients. The next two lemmas serve this purpose and can be seen as multidimensional analogues of Lemma 1.46, although we prove them only for $m = 2$, as they both relies on the inequalities of Lemma 2.1, that are specific for Jacobi's algorithm.

Lemma 2.10. *Given*

$$\begin{aligned} (\alpha_0^{(1)}, \alpha_0^{(2)}) &= [(a_0^{(1)}, a_1^{(1)}, \ldots), (a_0^{(2)}, a_1^{(2)}, \ldots)], \\ (\bar{\alpha}_0^{(1)}, \bar{\alpha}_0^{(2)}) &= [(\bar{a}_0^{(1)}, \bar{a}_1^{(1)}, \ldots), (\bar{a}_0^{(2)}, \bar{a}_1^{(2)}, \ldots)], \end{aligned}$$

with $a_i^{(1)} = \bar{a}_i^{(1)}$ *and* $a_i^{(2)} = \bar{a}_i^{(2)}$ *for all* $i = 0, \ldots, n$*, then*

$$|\alpha_0^{(1)} - \bar{\alpha}_0^{(1)}| < \frac{1}{C_{n-2}}, \quad |\alpha_0^{(2)} - \bar{\alpha}_0^{(2)}| < \frac{1}{C_{n-2}}.$$

Proof. Assume without loss of generality that $\bar{\alpha}_0^{(1)} \leq \alpha_0^{(1)}$. Since $\alpha_0^{(1)}$ and $\bar{\alpha}_0^{(1)}$ share up to the n-th convergent, for all $n \in \mathbb{N}$ we have

$$m_{n-2} < \bar{\alpha}_0^{(1)} \leq \alpha_0^{(1)} < M_{n-2}.$$

where m_{n-2} and M_{n-2} are as in the proof of Lemma 1.59. The claim follows by applying the inequalities of Lemma 2.1. The proof for $\alpha_0^{(2)}, \bar{\alpha}_0^{(2)}$ is analogous. □

The next lemma provides the bound that will actually be used in the proofs of Theorem 2.22 and Theorem 2.24.

Lemma 2.11. *Under the same assumptions as in Lemma 2.10, we have*

$$|\alpha_0^{(1)} - \bar{\alpha}_0^{(1)}| < \frac{2}{C_n}, \quad |\alpha_0^{(2)} - \bar{\alpha}_0^{(2)}| < \frac{2}{C_n}.$$

Proof. The claim follows by writing

$$|\alpha_0^{(1)} - \bar{\alpha}_0^{(1)}| < \left|\alpha_0^{(1)} - \frac{A_n^{(1)}}{C_n}\right| + \left|\bar{\alpha}_0^{(1)} - \frac{A_n^{(1)}}{C_n}\right| \leq \frac{2}{C_n},$$

where the last inequality follows from Theorem 2.6. The same argument applies to $\alpha_0^{(2)}, \bar{\alpha}_0^{(2)}$. □

Remark 2.12. Lemma 2.10 and Lemma 2.11 provide two different estimates for the difference of numbers sharing the first partial quotients of their MCF. In fact, although $\frac{2}{C_n}$ is often smaller than $\frac{1}{C_{n-2}}$, as $C_n = a_n^{(1)} C_{n-1} + a_n^{(2)} C_{n-2} + C_{n-3}$, it is not hard to see that this is not always the case.

The last ingredient needed for the proof of Theorem 2.22 and Theorem 2.24 is a lower bound for the denominator of the n-th convergent of Jacobi's MCF and, in the case of bounded partial quotients, also an upper bound. These bounds are provided in the following lemma.

Lemma 2.13. *Consider the bidimensional continued fraction*

$$[(a_0^{(1)}, a_1^{(1)}, \ldots), (a_0^{(2)}, a_1^{(2)}, \ldots)].$$

Then, $C_n > \psi^{n-2}$, where ψ is the unique real root of $f(x) = x^3 - x^2 - 1$.

Moreover, if $a_n^{(1)} \leq M$ for all n, then $C_n \leq \chi^n$, where χ is the positive real root of $g(x) = x^3 - Mx^2 - Mx - 1$.

Proof. We prove both statements by induction. Since $\psi \approx 1.46$ the claim holds for $n = 0, 1, 2$. For the inductive step,

$$C_{n+3} = a_{n+3}^{(1)} C_{n+2} + a_{n+3}^{(2)} C_{n+1} + C_n \geq C_{n+2} + C_n \geq \psi^{n-2}(\psi^2 + 1) = \psi^{n+1},$$

since $\psi^3 = \psi^2 + 1$.

For the upper bound, assume that $a_n^{(1)} \leq M$ for all n. In general, the polynomial g can have either one or three real roots. It has three real roots if and only if its discriminant is positive, i.e.

$$\Delta(g) = M^4 - 18M^2 - 27 > 0,$$

which happens, in particular, for all $M \geq 5$. It is possible to see that for $1 \leq M \leq 4$ the base case holds. For $M \geq 5$, by Descartes' rule of signs there is only a positive root χ, since the product of all roots is 1 and there are exactly two roots of same sign. Moreover, for the same reason, $\chi \geq M \geq 1$ as M is the trace of g. Therefore,

$$C_{-1} = 0 \leq \chi^{-1}, \quad C_0 = 1 = \chi^0, \quad C_1 = a_1 \leq M \leq \chi,$$

and the base of the induction is proved. For the inductive step observe that, for all n,

$$\begin{aligned} C_{n+2} &= a_{n+2}^{(1)} C_{n+1} + a_{n+2}^{(2)} C_n + C_{n-1} \leq M\chi^{n+1} + M\chi^n + \chi^{n-1} = \\ &= \chi^{n-1}(M\chi^2 + M\chi + 1) = \chi^{n+2}, \end{aligned}$$

as χ is a root of g, and the thesis is true. □

2.2 Bounds on the height of cubic irrationals

In this section, we establish an explicit bound for the height of cubic irrationals arising from periodic Jacobi MCFs. Our aim is to find an analogue of Lemma 1.47 to be used in the proofs of Theorem 2.22 and Theorem 2.24.

We first consider the case of cubic irrationals $(\alpha_0^{(1)}, \alpha_0^{(2)})$ such that $0 < \alpha_0^{(1)}, \alpha_0^{(2)} < 1$. The general case will then follow from Theorem 2.9 and is treated in Theorem 2.15.

Theorem 2.14. *Given*

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = \left[\left(a_0^{(1)}, \ldots, a_{k-1}^{(1)}, \overline{a_k^{(1)}, \ldots, a_{k+h-1}^{(1)}}\right), \left(a_0^{(2)}, \ldots, a_{k-1}^{(2)}, \overline{a_k^{(2)}, \ldots, a_{k+h-1}^{(2)}}\right)\right], \tag{2.2}$$

with $0 < \alpha_0^{(1)}, \alpha_0^{(2)} < 1$*, then* $\alpha_0^{(1)}$ *and* $\alpha_0^{(2)}$ *are cubic irrationals such that*

$$H(\alpha_0^{(1)}), H(\alpha_0^{(2)}) \leq 2592 C_{h+k-1}^9.$$

Proof. From (2.2), we have that $\alpha_k^{(1)} = \alpha_{k+h}^{(1)}$ and $\alpha_k^{(2)} = \alpha_{k+h}^{(2)}$, so that, by (1.55),

$$\begin{bmatrix} \alpha_k^{(1)} \\ \alpha_k^{(2)} \\ 1 \end{bmatrix} = \frac{1}{\alpha_1^{(1)} \dots \alpha_{k+h}^{(1)}} \begin{bmatrix} a_k^{(1)} & 1 & 0 \\ a_k^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{k+h-1}^{(1)} & 1 & 0 \\ a_{k+h-1}^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_k^{(1)} \\ \alpha_k^{(2)} \\ 1 \end{bmatrix}. \tag{2.3}$$

Moreover, applying (1.56) with $m = 2$ implies

$$\begin{bmatrix} \alpha_k^{(1)} \\ \alpha_k^{(2)} \\ 1 \end{bmatrix} = \alpha_1^{(1)} \alpha_2^{(1)} \dots \alpha_k^{(1)} \begin{bmatrix} a_{k-1}^{(1)} & 1 & 0 \\ a_{k-1}^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \cdots \begin{bmatrix} a_0^{(1)} & 1 & 0 \\ a_0^{(2)} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_0^{(1)} \\ \alpha_0^{(2)} \\ 1 \end{bmatrix}. \tag{2.4}$$

Following the notation introduced in (1.57), and substituting (2.4) into the right-hand side of (2.3), we obtain

$$\begin{aligned} \begin{bmatrix} \alpha_0^{(1)} \\ \alpha_0^{(2)} \\ 1 \end{bmatrix} &= \frac{1}{\alpha_{k+1}^{(1)} \dots \alpha_{k+h}^{(1)}} M_0 M_1 \dots M_{k+h-1} (M_0 \dots M_{k-1})^{-1} \begin{bmatrix} \alpha_0^{(1)} \\ \alpha_0^{(2)} \\ 1 \end{bmatrix} = \\ &= \frac{1}{\lambda'} \begin{bmatrix} A_{k+h-1}^{(1)} & A_{k+h-2}^{(1)} & A_{k+h-3}^{(1)} \\ A_{k+h-1}^{(2)} & A_{k+h-2}^{(2)} & A_{k+h-3}^{(2)} \\ C_{k+h-1} & C_{k+h-2} & C_{k+h-3} \end{bmatrix} \begin{bmatrix} A_{k-1}^{(1)} & A_{k-2}^{(1)} & A_{k-3}^{(1)} \\ A_{k-1}^{(2)} & A_{k-2}^{(2)} & A_{k-3}^{(2)} \\ C_{k-1} & C_{k-2} & C_{k-3} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_0^{(1)} \\ \alpha_0^{(2)} \\ 1 \end{bmatrix}, \end{aligned}$$

where $\lambda' = \alpha_{k+1}^{(1)} \dots \alpha_{k+h}^{(1)}$. With some computations, we have

$$\begin{bmatrix} A_{k-1}^{(1)} & A_{k-2}^{(1)} & A_{k-3}^{(1)} \\ A_{k-1}^{(2)} & A_{k-2}^{(2)} & A_{k-3}^{(2)} \\ C_{k-1} & C_{k-2} & C_{k-3} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{A}_{k-2}^{(2)} & -\tilde{A}_{k-2}^{(1)} & \tilde{U}_{k-2} \\ -\tilde{\tilde{A}}_{k-1}^{(2)} & \tilde{\tilde{A}}_{k-1}^{(1)} & -\tilde{\tilde{U}}_{k-1} \\ \tilde{A}_{k-1}^{(2)} & -\tilde{A}_{k-1}^{(1)} & \tilde{U}_{k-1} \end{bmatrix}.$$

where the latter sequences are defined recursively by

$$\begin{aligned} \tilde{U}_k &= A_k^{(1)} A_{k-1}^{(2)} - A_{k-1}^{(1)} A_k^{(2)}, \\ \tilde{\tilde{U}}_k &= A_k^{(1)} A_{k-2}^{(2)} - A_{k-2}^{(1)} A_k^{(2)}, \end{aligned}$$

for every $k \geq 0$. Thus, we derive the system

$$X_{1,1} \alpha_0^{(1)} + X_{1,2} \alpha_0^{(2)} + X_{1,3} = \lambda' \alpha_0^{(1)},$$

$$X_{2,1}\alpha_0^{(1)} + X_{2,2}\alpha_0^{(2)} + X_{2,3} = \lambda'\alpha_0^{(2)},$$
$$X_{3,1}\alpha_0^{(1)} + X_{3,2}\alpha_0^{(2)} + X_{3,3} = \lambda',$$

where

$$\begin{aligned}
X_{1,1} &= A^{(1)}_{k+h-1}\tilde{A}^{(2)}_{k-2} - A^{(1)}_{k+h-2}\tilde{\tilde{A}}^{(2)}_{k-1} + A^{(1)}_{k+h-3}\tilde{A}^{(2)}_{k-1},\\
X_{1,2} &= -A^{(1)}_{k+h-1}\tilde{A}^{(1)}_{k-2} + A^{(1)}_{k+h-2}\tilde{\tilde{A}}^{(1)}_{k-1} - A^{(1)}_{k+h-3}\tilde{A}^{(1)}_{k-1},\\
X_{1,3} &= A^{(1)}_{k+h-1}\tilde{U}_{k-2} - A^{(1)}_{k+h-2}\tilde{\tilde{U}}_{k-1} + A^{(1)}_{k+h-3}\tilde{U}_{k-1},\\
X_{2,1} &= A^{(2)}_{k+h-1}\tilde{A}^{(2)}_{k-2} - A^{(2)}_{k+h-2}\tilde{\tilde{A}}^{(2)}_{k-1} + A^{(2)}_{k+h-3}\tilde{A}^{(2)}_{k-1},\\
X_{2,2} &= -A^{(2)}_{k+h-1}\tilde{A}^{(1)}_{k-2} + A^{(2)}_{k+h-2}\tilde{\tilde{A}}^{(1)}_{k-1} - A^{(2)}_{k+h-3}\tilde{A}^{(1)}_{k-1},\\
X_{2,3} &= A^{(2)}_{k+h-1}\tilde{U}_{k-2} - A^{(2)}_{k+h-2}\tilde{\tilde{U}}_{k-1} + A^{(2)}_{k+h-3}\tilde{U}_{k-1},\\
X_{3,1} &= C_{k+h-1}\tilde{A}^{(2)}_{k-2} - C_{k+h-2}\tilde{\tilde{A}}^{(2)}_{k-1} + C_{k+h-3}\tilde{A}^{(2)}_{k-1},\\
X_{3,2} &= -C_{k+h-1}\tilde{A}^{(1)}_{k-2} + C_{k+h-2}\tilde{\tilde{A}}^{(1)}_{k-1} - C_{k+h-3}\tilde{A}^{(1)}_{k-1},\\
X_{3,3} &= C_{k+h-1}\tilde{U}_{k-2} - C_{k+h-2}\tilde{\tilde{U}}_{k-1} + C_{k+h-3}\tilde{U}_{k-1}.
\end{aligned}$$

Eliminating λ' gives

$$X_{1,1}\alpha_0^{(1)} + X_{1,2}\alpha_0^{(2)} + X_{1,3} = X_{3,1}\left(\alpha_0^{(1)}\right)^2 + X_{3,2}\alpha_0^{(1)}\alpha_0^{(2)} + X_{3,3}\alpha_0^{(1)}, \tag{2.5}$$
$$X_{2,1}\alpha_0^{(1)} + X_{2,2}\alpha_0^{(2)} + X_{2,3} = X_{3,1}\alpha_0^{(1)}\alpha_0^{(2)} + X_{3,2}\left(\alpha_0^{(2)}\right)^2 + X_{3,3}\alpha_0^{(2)}. \tag{2.6}$$

Since $\alpha_0^{(2)}$ is irrational, $X_{2,1} - X_{3,1}\alpha_0^{(2)} \neq 0$. Thus, from (2.6), we obtain

$$\alpha_0^{(1)} = \frac{X_{3,2}\left(\alpha_0^{(2)}\right)^2 + (X_{3,3} - X_{2,2})\alpha_0^{(2)} - X_{2,3}}{X_{2,1} - X_{3,1}\alpha_0^{(2)}}.$$

Substituting into (2.5), one gets

$$\begin{aligned}
&X_{3,1}\left(X_{3,2}\left(\alpha_0^{(2)}\right)^2 + (X_{3,3} - X_{2,2})\alpha_0^{(2)} - X_{2,3}\right)^2 - \left(X_{1,2}\alpha_0^{(2)} + X_{1,3}\right)\left(X_{2,1} - X_{3,1}\alpha_0^{(2)}\right)^2 +\\
&+ \left(X_{3,2}\alpha_0^{(2)} + X_{3,3} - X_{1,1}\right)\left(X_{3,2}\left(\alpha_0^{(2)}\right)^2 + (X_{3,3} - X_{2,2})\alpha_0^{(2)} - X_{2,3}\right)\left(X_{2,1} - X_{3,1}\alpha_0^{(2)}\right) = 0.
\end{aligned}$$

The two terms of degree 4 are $X_{3,1}X_{3,2}^2\left(\alpha_0^{(2)}\right)^4$ and $-X_{3,1}X_{3,2}^2\left(\alpha_0^{(2)}\right)^4$, therefore the left-hand side is a polynomial expression of the form

$$A\left(\alpha_0^{(2)}\right)^3 + B\left(\alpha_0^{(2)}\right)^2 + C\,\alpha_0^{(2)} + D = 0,$$

where

$$\begin{aligned}
A =& - X_{1,2}X_{3,1}^2 + X_{1,1}X_{3,1}X_{3,2} - X_{2,2}X_{3,1}X_{3,2} + X_{2,1}X_{3,2}^2,\\
B =& \, 2X_{1,2}X_{2,1}X_{3,1} - X_{1,1}X_{2,2}X_{3,1} + X_{2,2}X_{3,1}^2 - X_{1,3}X_{3,1}^2 - X_{1,1}X_{2,1}X_{3,2} +\\
& - X_{2,1}X_{2,2}X_{3,2} - X_{2,3}X_{3,1}X_{3,2} + X_{1,1}X_{3,1}X_{3,3} - X_{2,2}X_{3,1}X_{3,3} + 2X_{2,1}X_{3,2}X_{3,3},\\
C =& - X_{1,2}X_{2,1}^2 + X_{1,1}X_{2,1}X_{2,2} + 2X_{1,3}X_{2,1}X_{3,1} - X_{1,1}X_{2,3}X_{3,1} + 2X_{2,2}X_{2,3}X_{3,1} +\\
& - X_{2,1}X_{2,3}X_{3,2} - X_{1,1}X_{2,1}X_{3,3} - X_{2,1}X_{2,2}X_{3,3} - X_{2,3}X_{3,1}X_{3,3} + X_{2,1}X_{3,3}^2\\
D =& - X_{1,3}X_{2,1}^2 + X_{1,1}X_{2,1}X_{2,3} + X_{2,3}^2X_{3,1} - X_{2,1}X_{2,3}X_{3,3}.
\end{aligned}$$

By Theorem 1.72, this polynomial is in fact of degree 3 (that is, $A \neq 0$) and is irreducible. Notice also that h and k are constants, because they represent the period and the pre-period of the bidimensional continued fraction of $(\alpha_0^{(1)}, \alpha_0^{(2)})$. Hence, by Lemma 2.8, each $X_{i,j}$ satisfies

$$X_{i,j} \leq 6C_{h+k-1}^3. \tag{2.7}$$

Since $A_n^{(1)}, A_n^{(2)} \leq C_n$ for $0 < \alpha_0^{(1)}, \alpha_0^{(2)} < 1$ (see Lemma 2.8), this implies

$$H(\alpha_0^{(2)}) \leq 2592C_{h+k-1}^9.$$

Now we want to apply the same reasoning to $\alpha_0^{(1)}$, in order to find its minimal polynomial and a bound on its height. From (2.5), we obtain

$$\alpha_0^{(2)} = \frac{X_{3,1}\left(\alpha_0^{(1)}\right)^2 + (X_{3,3} - X_{1,1})\alpha_0^{(1)} - X_{1,3}}{X_{1,2} - X_{3,2}\alpha_0^{(1)}}.$$

Observe again that the denominator is nonzero as $\alpha_0^{(1)}$ is irrational. Substituting into (2.6) yields

$$\begin{aligned}
&X_{3,2}\left(X_{3,1}\left(\alpha_0^{(1)}\right)^2 + (X_{3,3} - X_{1,1})\alpha_0^{(1)} - X_{1,3}\right)^2 - \left(X_{2,1}\alpha_0^{(1)} + X_{2,3}\right)\left(X_{1,2} - X_{3,2}\alpha_0^{(1)}\right)^2\\
&+\left(X_{3,1}\alpha_0^{(1)} + X_{3,3} - X_{2,2}\right)\left(X_{3,1}\left(\alpha_0^{(1)}\right)^2 + (X_{3,3} - X_{1,1})\alpha_0^{(1)} - X_{1,3}\right)\left(X_{1,2} - X_{3,2}\alpha_0^{(1)}\right) = 0.
\end{aligned}$$

The two terms of degree 4 are $X_{3,2}X_{3,1}^2\left(\alpha_0^{(1)}\right)^4$ and $-X_{3,2}X_{3,1}^2\left(\alpha_0^{(1)}\right)^4$ and, reasoning as before, we can see that the left-hand side is again an irreducible polynomial of degree exactly 3 evaluted at $\alpha_0^{(1)}$. With some computations, the equation reduces to

$$A\left(\alpha_0^{(1)}\right)^3 + B\left(\alpha_0^{(1)}\right)^2 + C\,\alpha_0^{(1)} + D = 0,$$

where

$$\begin{aligned}
A &= X_{1,2}X_{3,1}^2 - X_{1,1}X_{3,1}X_{3,2} + X_{2,2}X_{3,1}X_{3,2} - X_{2,1}X_{3,2}^2,\\
B &= -X_{1,1}X_{1,2}X_{3,1} - X_{1,2}X_{2,2}X_{3,1} + X_{1,1}^2X_{3,2} + 2X_{1,2}X_{2,1}X_{3,2} - X_{1,1}X_{2,2}X_{3,2} +\\
&\quad - X_{1,3}X_{3,1}X_{3,2} - X_{2,3}X_{3,2}^2 + 2X_{1,2}X_{3,1}X_{3,3} - X_{1,1}X_{3,2}X_{3,3} + X_{2,2}X_{3,2}X_{3,3},\\
C &= -X_{1,2}^2X_{2,1} + X_{1,1}X_{1,2}X_{2,2} - X_{1,2}X_{1,3}X_{3,1} + 2X_{1,1}X_{1,3}X_{3,2} - X_{1,3}X_{2,2}X_{3,2} +\\
&\quad + 2X_{1,2}X_{2,3}X_{3,2} - X_{1,1}X_{1,2}X_{3,3} - X_{1,2}X_{2,2}X_{3,3} - X_{1,3}X_{3,2}X_{3,3} + X_{1,2}X_{3,3}^2,\\
D &= X_{1,2}X_{1,3}X_{2,2} - X_{1,2}^2X_{2,3} + X_{1,3}^2X_{3,2} - X_{1,2}X_{1,3}X_{3,3}.
\end{aligned}$$

Also in this case we use (2.7), and we obtain that

$$H(\alpha_0^{(1)}) \leq 2592C_{h+k-1}^9,$$

which concludes the proof. □

Theorem 2.15. *Given*

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = \left[\left(a_0^{(1)}, \dots, a_{k-1}^{(1)}, \overline{a_k^{(1)}, \dots, a_{k+h-1}^{(1)}}\right), \left(a_0^{(2)}, \dots, a_{k-1}^{(2)}, \overline{a_k^{(2)}, \dots, a_{k+h-1}^{(2)}}\right)\right],$$

with $0 < \alpha_0^{(1)} < N$, $0 < \alpha_0^{(2)} < M$ *for some positive integers* N, M, *then* $\alpha_0^{(1)}$ *and* $\alpha_0^{(2)}$ *are cubic irrationals such that*

$$H(\alpha_0^{(1)}), H(\alpha_0^{(2)}) \leq 2592N^5M^5C_{h+k-1}^9.$$

Proof. The argument is identical to that of the previous theorem, using Theorem 2.9 to bound the coefficients $X_{i,j}$ also in terms of N and M. □

2.3 Liouville-type multidimensional continued fractions

In this section we provide some infinite families of multidimensional continued fractions converging to transcendental numbers, using a method similar to Liouville's original approach.

Theorem 2.16. *Let*

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), (a_0^{(2)}, a_1^{(2)}, \dots)]$$

be a bidimensional continued fraction, and let $\delta > 0$.

If

$$a_n^{(1)} > \max\{|\tilde{A}_n^{(1)}|C_{n-1}^{\delta}, |\tilde{A}_n^{(2)}|C_{n-1}^{\delta}\}, \tag{2.8}$$

for all $n \in \mathbb{N}$*, then* $\alpha_0^{(1)}$ *and* $\alpha_0^{(2)}$ *are (rationally independent) transcendental numbers.*

Proof. By Lemma 2.5, for infinitely many $n \in \mathbb{N}$ we have

$$\left|\alpha_0^{(1)} - \frac{A_n^{(1)}}{C_n}\right| < \left|\frac{A_{n+1}^{(1)}}{C_{n+1}} - \frac{A_n^{(1)}}{C_n}\right| = \left|\frac{\tilde{A}_{n+1}^{(1)}}{C_{n+1}C_n}\right| < \frac{|\tilde{A}_{n+1}^{(1)}|}{a_{n+1}^{(1)}C_n^2} < \frac{1}{C_n^{2+\delta}},$$

where we used (2.8). By Roth's Theorem 1.27, $\alpha_0^{(1)}$ is transcendental.
The same argument applies to $\alpha_0^{(2)}$, yielding

$$\left|\alpha_0^{(2)} - \frac{A_n^{(2)}}{C_n}\right| < \left|\frac{A_{n+1}^{(2)}}{C_{n+1}} - \frac{A_n^{(2)}}{C_n}\right| = \left|\frac{\tilde{A}_{n+1}^{(2)}}{C_{n+1}C_n}\right| < \frac{|\tilde{A}_{n+1}^{(2)}|}{a_{n+1}^{(1)}C_n^2} < \frac{1}{C_n^{2+\delta}},$$

for infinitely many n, hence $\alpha_0^{(2)}$ is transcendental as well. Rational independence follows from Theorem 1.54. □

Remark 2.17. Condition (2.8) can be effectively used to construct pairs of transcendental numbers via Jacobi's algorithm. Indeed, $\tilde{A}_n^{(1)}$ and $\tilde{A}_n^{(2)}$ do not depend on $a_n^{(1)}$, but only on $a_n^{(2)}$ and previous partial quotients (see Remark 2.2). Therefore, for any arbitrary choice of the sequence of non-negative integers $(a_n^{(2)})_{n\geq 0}$, one obtains a different pair of transcendental numbers.

Example 2.18. For simplicity, let $\delta = 1$ and $a_n^{(2)} = 1$ for every $n \geq 0$. Setting $a_n^{(1)}$ to be the smallest positive integer satisfying (2.8) at each step, the first values that we obtain are displayed in Table 2.1.

n	$a_n^{(1)}$	C_n	$\tilde{A}_n^{(1)}$	$\tilde{A}_n^{(2)}$
0	1	1	−1	0
1	2	2	1	1
2	3	7	1	−1
3	36	255	−5	−2
4	9946	2536239	−30	39
5	126205788880	320088043783222582	49761	19852

Table 2.1: Values of $a_n^{(1)}$, C_n, $\tilde{A}_n^{(1)}$ and $\tilde{A}_n^{(2)}$ for $a_n^{(2)} = 1$ and $\delta = 1$.

Notice that the partial quotients $a_n^{(1)}$ grow extremely rapidly: already at step $n = 5$ one obtains a twelve-digit number. By Theorem 2.16, the pair $(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), (1, 1, \dots)]$ consists of rationally independent transcendental numbers.

Choosing instead $a_n^{(2)} = 2$ for every $n \geq 0$, the growth of $a_n^{(1)}$ is even faster, as shown in Table 2.2.

n	$a_n^{(1)}$	C_n	$\tilde{A}_n^{(1)}$	$\tilde{A}_n^{(2)}$
0	1	1	-1	0
1	3	3	2	1
2	7	23	-1	-2
3	300	6907	-13	-3
4	4192550	28957942899	328	607
5	1578280080147147208	457037444395302323670112898 29	54502493	12576434

Table 2.2: Values of $a_n^{(1)}$, C_n, $\tilde{A}_n^{(1)}$ and $\tilde{A}_n^{(2)}$ for $a_n^{(2)} = 2$ and $\delta = 1$.

In both cases, the pair $(\alpha_0^{(1)}, \alpha_0^{(2)})$ consists of rationally independent transcendental numbers by Theorem 2.16.

We now extend this construction to higher dimensions.

Theorem 2.19. *Let*

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = [(a_n^{(1)})_{n\geq 0}, \dots, (a_n^{(m)})_{n\geq 0}]$$

be a multidimensional continued fraction, and let $\delta > 0$. If

$$a_n^{(1)} > \max\{|\tilde{A}_n^{(1)}|C_{n-1}^{\delta}, |\tilde{A}_n^{(2)}|C_{n-1}^{\delta}, \dots, |\tilde{A}_n^{(m)}|C_{n-1}^{\delta}\}, \tag{2.9}$$

for all $n \in \mathbb{N}$, then all the numbers $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)})$ are transcendental.

Proof. The proof follows the same lines as that of Theorem 2.16. By Lemma 2.5, for infinitely many $n \in \mathbb{N}$,

$$\left|\alpha_0^{(i)} - \frac{A_n^{(i)}}{C_n}\right| < \left|\frac{A_{n+1}^{(i)}}{C_{n+1}} - \frac{A_n^{(i)}}{C_n}\right| = \left|\frac{\tilde{A}_{n+1}^{(i)}}{C_{n+1}C_n}\right| < \frac{|\tilde{A}_{n+1}^{(i)}|}{a_{n+1}^{(1)}C_n^2} < \frac{1}{C_n^{2+\delta}},$$

for each $i = 1, \dots, m$. Theorem 1.27 then implies that all $\alpha_0^{(i)}$ are transcendental. □

Thanks to Remark 2.2, it is again possible to define recursively the sequence $(a_n^{(1)})_{n\geq 0}$ so that (2.9) holds. Hence, one can construct multidimensional continued fractions converging to m-tuples of transcendental numbers.

For $m \geq 3$, it remains an interesting problem to determine conditions ensuring rational independence.

2.4 Quasi-periodic multidimensional continued fractions

We now study transcendence criteria for quasi-periodic multidimensional continued fractions.

Theorem 2.20. *Let* $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), \dots, (a_0^{(m)}, a_1^{(m)}, \dots)]$, *and let* $d \geq 1$. *If* $a_{n+1}^{(1)} < C_n^d$ *for all* $n \in \mathbb{N}$, *then there exists a constant* $K = K(d, m)$ *such that*

$$\log\log C_{n+1} < Kn, \tag{2.10}$$

for all $n \geq 0$, *where one can take*

$$K = \log(d+1) + \log\left(1 + \frac{1}{d}\right) + \log\log(m+1).$$

Proof. From the recurrence relation,

$$C_{n+1} = \sum_{j=1}^{m} a_{n+1}^{(j)} C_{n+1-j} + C_{n-m},$$

and using $a_{n+1}^{(j)} < C_n^d$ for all j, we obtain

$$C_{n+1} < (m+1) C_n^{d+1}.$$

Iterating this inequality yields, for all n,

$$\begin{aligned} C_{n+1} < (m+1) C_n^{d+1} < (m+1)^{1+(d+1)} C_{n-1}^{(d+1)^2} < \cdots < \\ < (m+1)^{1+(d+1)+\cdots+(d+1)^n} C_0^{(d+1)^{n+1}} = (m+1)^{\frac{(d+1)^{n+1}-1}{d}}, \end{aligned}$$

from which (2.10) follows. □

Definition 2.21. Let $(\lambda_k)_{k\geq 0}$, $(r_k)_{k\geq 0}$, $(n_k)_{k\geq 0}$ be sequences of positive integers with

$$n_{k+1} \geq n_k + r_k \lambda_k,$$

for all k. Let $(a_n^{(1)})_{n\geq 0}, \dots, (a_n^{(m)})_{n\geq 0}$ be m sequences of integers satisfying the admissibility conditions (1.53), and suppose that, for all $j = 1, \dots, m$,

$$a_{n+r_k}^{(j)} = a_n^{(j)}, \tag{2.11}$$

whenever $n_k \leq n \leq n_k + (\lambda_k - 1)r_k - 1$. Then, the multidimensional continued fraction $[(a_n^{(1)})_{n\geq 0}, \dots, (a_n^{(m)})_{n\geq 0}]$ is called *quasi-periodic*.

We remark that quasi-periodic sequences may, in principle, be periodic; this possibility is excluded in the following theorems, where "non-periodic continued fraction" means that at least one of the sequences is not periodic.

The following result may be regarded as a multidimensional analogue of Baker's Theorem 1.49. Its statement is similar in spirit: under suitable technical assumptions, if a pair of real numbers $(\alpha_0^{(1)}, \alpha_0^{(2)})$ has a quasi-periodic, non-periodic MCF whose partial quotients are grouped into blocks, each consisting of repeated copies of a finite sequence, and the numbers of repetitions grow sufficiently rapidly from block to block, then at least one of $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ is transcendental.

Theorem 2.22. *Let $(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), (a_0^{(2)}, a_1^{(2)}, \dots)]$ be a quasi-periodic non-periodic multidimensional continued fraction. Assume there exists $d \geq 1$ such that $a_{n+1}^{(1)} < C_n^d$ for all $k \in \mathbb{N}$, and suppose $r_k < cn_k$, where c is a constant. If*

$$\lim_{k\to+\infty} \frac{\log \lambda_k}{n_k} = +\infty, \tag{2.12}$$

then either both $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ are transcendental (and rationally independent), or one of them is cubic irrational and the other is transcendental.

Proof. The proof proceeds by contradiction, constructing approximations via periodic MCFs, obtained by suitably truncanting the MCF of $(\alpha_0^{(1)}, \alpha_0^{(2)})$, and applying Roth-type estimates, as in the one-dimensional Baker method.

Suppose that at least one of $\alpha_0^{(1)}$ or $\alpha_0^{(2)}$ is an algebraic number. Starting from the *quasi-periodic* MCF of $(\alpha^{(1)}, \alpha_0^{(2)})$, we construct a sequence of pairs $(\eta_k^{(1)}, \eta_k^{(2)})_{k\geq 0}$ in the following way:

$$(\eta_k^{(1)}, \eta_k^{(2)}) = \left[\left(a_0^{(1)}, \dots, a_{n_k-1}^{(1)}, \overline{a_{n_k}^{(1)}, \dots, a_{n_k+r_k-1}^{(1)}} \right), \right. \\ \left. \left(a_0^{(2)}, \dots, a_{n_k-1}^{(2)}, \overline{a_{n_k}^{(2)}, \dots, a_{n_k+r_k-1}^{(2)}} \right) \right], \tag{2.13}$$

for all $k \geq 0$. For simplicity, let us set, for all $k \geq 1$,

$$m_k = n_k + (\lambda_k - 1)r_k - 1.$$

Then, $\eta_k^{(1)}$ and $\eta_k^{(2)}$ are cubic irrationals for all k, that have in common with $\alpha^{(1)}$ and $\alpha^{(2)}$, respectively, the first m_k partial quotients. Assume first that $\alpha_0^{(i)} \neq \eta_k^{(i)}$, $i = 1, 2$, for infinitely many $k \geq 0$ in order to apply Theorem 1.42. Therefore, by Lemma 2.11, Theorem 2.15 and Theorem 1.42, choosing some $\varepsilon > 0$, one of the following chains of inequalities holds:

$$\frac{2}{C_{m_k}} > |\alpha_0^{(1)} - \eta_k^{(1)}| > \frac{c_1}{C_{n_k+r_k-1}^{9(6+\varepsilon)}},$$
$$\frac{2}{C_{m_k}} > |\alpha_0^{(2)} - \eta_k^{(2)}| > \frac{c_2}{C_{n_k+r_k-1}^{9(6+\varepsilon)}},$$

for some positive constants c_1, c_2. In either case, by using Lemma 2.13, we obtain

$$\psi^{m_k-2} < C_{m_k} < c_3 C_{n_k+r_k-1}^{9(6+\varepsilon)},$$

that is

$$\lambda_k < m_k - 2 < c_4 + c_5 \log(C_{n_k+r_k-1}) < c_6 \log(C_{n_k+r_k-1}),$$

for suitable constants $c_i > 0$. Hence

$$\log\log(C_{n_k+r_k-1}) > c_7 \log \lambda_k, \tag{2.14}$$

for some $c_7 > 0$. However, by Theorem 2.20, there exists $c_8 > 0$ such that

$$\log\log(C_{n_k+r_k-1}) < c_8(n_k + r_k - 1). \tag{2.15}$$

Combining (2.14) and (2.15), and since $r_k < cn_k$, we obtain for some constant $c_9 > 0$

$$\frac{\log \lambda_k}{n_k} < c_9,$$

which contradicts (2.12). Therefore, $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ are transcendental numbers and the rational independence follows from Theorem 1.54.

If $\alpha_0^{(1)} = \eta_k^{(1)}$ (or $\alpha_0^{(2)} = \eta_k^{(2)}$) for all but finitely many k, then we can conclude that $\alpha_0^{(1)}$ (or $\alpha_0^{(2)}$) is cubic irrational and $\alpha_0^{(2)}$ (or $\alpha_0^{(1)}$) is transcendental.

Notice that it is not possible to have $\alpha_0^{(1)} = \eta_k^{(1)}$ and $\alpha_0^{(2)} = \eta_k^{(2)}$ for some k, since the quasi-periodic continued fraction of $(\alpha_0^{(1)}, \alpha_0^{(2)})$ is not periodic. □

Example 2.23. We construct an explicit example of a quasi-periodic MCF satisfying the assumptions of Theorem 2.22. Let $(\lambda_k)_{k\geq 0}$ be the sequence of integers defined recursively by

$$\begin{aligned} &\lambda_0 = 1, \quad \lambda_k = k^{\lambda_0+\cdots+\lambda_{k-1}}, \\ &n_0 = 0, \quad n_k = \lambda_0 + \cdots + \lambda_{k-1}, \quad \text{for } k \geq 1, \end{aligned}$$

so that $\lambda_k = k^{n_k}$ and

$$\lim_{k\to+\infty} \frac{\log \lambda_k}{n_k} = \lim_{k\to+\infty} \log k = +\infty.$$

Set also $r_k = 1$ for $k \geq 0$, and let $(x_k^{(1)})_{k\geq 0}$ and $(x_k^{(2)})_{k\geq 0}$ be two sequences of integers such that

$$x_k^{(1)} \geq x_k^{(2)} > 1 \quad \text{for } k \geq 1. \tag{2.16}$$

We then define the sequences $\boldsymbol{a}^{(1)} = (a_n^{(1)})_{n\geq 0}$ and $\boldsymbol{a}^{(2)} = (a_n^{(2)})_{n\geq 0}$ by setting

$$a_0^{(i)} = x_0^{(i)}, \qquad a_n^{(i)} = x_k^{(i)} \quad \text{for } n_k \leq n \leq n_{k+1} - 1.$$

In other words, the sequences are given by

$$\begin{aligned} \boldsymbol{a}^{(1)} &= (x_0^{(1)},\ x_1^{(1)},\ \underbrace{x_2^{(1)},\dots,x_2^{(1)}}_{\lambda_2 \text{ times}},\ \underbrace{x_3^{(1)},\dots,x_3^{(1)}}_{\lambda_3 \text{ times}},\ \dots), \\ \boldsymbol{a}^{(2)} &= (x_0^{(2)},\ x_1^{(2)},\ \underbrace{x_2^{(2)},\dots,x_2^{(2)}}_{\lambda_2 \text{ times}},\ \underbrace{x_3^{(2)},\dots,x_3^{(2)}}_{\lambda_3 \text{ times}},\ \dots). \end{aligned}$$

The assumption (2.16) ensures that $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ satisfy the admissibility conditions in (1.39). Hence, they are the sequences of partial quotients of a quasi-periodic MCF, say

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}].$$

The last technical condition in the theorem is not restrictive. Indeed, we may choose $x_k^{(1)}$, and thus $a_n^{(1)}$, to be bounded, so that $a_{n+1}^{(1)} < C_n^d$ holds for some $d \geq 1$. For instance, we may take $x_k^{(1)} = x_k^{(2)} = 3$ if k is even, and $x_k^{(1)} = x_k^{(2)} = 2$ otherwise, that is,

$$\boldsymbol{a}^{(1)} = (3,\ 2,\ \underbrace{3,\dots,3}_{2^2 \text{ times}},\ \underbrace{2,2,\dots,2,2}_{3^{1+1+2^2} \text{ times}},\ \underbrace{3,3,3,\dots,3,3,3}_{4^{1+1+2^2+3^6} \text{ times}},\ \underbrace{2,2,2,2,\dots,2,2,2,2}_{5^{1+1+2^2+3^6+4^{735}} \text{ times}},\ \dots),$$

$$\boldsymbol{a}^{(2)} = (3,\ 2,\ \underbrace{3,\dots,3}_{2^2 \text{ times}},\ \underbrace{2,2,\dots,2,2}_{3^{1+1+2^2} \text{ times}},\ \underbrace{3,3,3,\dots,3,3,3}_{4^{1+1+2^2+3^6} \text{ times}},\ \underbrace{2,2,2,2,\dots,2,2,2,2}_{5^{1+1+2^2+3^6+4^{735}} \text{ times}},\ \dots).$$

By the previous theorem, at least one of $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ is transcendental.

When the partial quotients are bounded, the assumptions can be weakened, as the following theorem shows.

Theorem 2.24. *Let $(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), (a_0^{(2)}, a_1^{(2)}, \dots)]$ be a quasi-periodic non-periodic continued fraction. Assume there exist two positive integers M, N such that $a_k^{(1)}, a_k^{(2)} \leq M$ and $r_k \leq N$ for all $k \in \mathbb{N}$. Define the constant*

$$B = 54\frac{\log \chi}{\log \psi} - 1,$$

where χ and ψ are taken as in Lemma 2.13, i.e. the positive real roots of $x^3 - Mx^2 - Mx - 1$ and $x^3 - x^2 - 1$, respectively. If

$$\limsup_{k \to +\infty} \frac{\lambda_k}{n_k} > B, \tag{2.17}$$

then either both $\alpha_0^{(1)}$, $\alpha_0^{(2)}$ are transcendental (and rationally independent), or one of them is cubic irrational and the other is transcendental.

Proof. The argument refines that of Theorem 2.22, using explicit upper and lower bounds for C_n from Lemma 2.13.

Assume by contradiction that $\alpha_0^{(1)}$ or $\alpha_0^{(2)}$ is algebraic, and define $\eta_k^{(1)}$, $\eta_k^{(2)}$ as in (2.13). Suppose that $\alpha_0^{(1)} \neq \eta_k^{(1)}$ and $\alpha_0^{(2)} \neq \eta_k^{(2)}$ for infinitely many $k \geq 0$. Using the same argument as in the proof of Theorem 2.22, we can assert that, for all $\varepsilon > 0$, at least one of the following inequality holds:

$$\frac{2}{C_{m_k}} > |\alpha_0^{(1)} - \eta_k^{(1)}| > \frac{c_1}{(C_{n_k+r_k-1})^{9(6+\varepsilon)}},$$
$$\frac{2}{C_{m_k}} > |\alpha_0^{(2)} - \eta_k^{(2)}| > \frac{c_2}{(C_{n_k+r_k-1})^{9(6+\varepsilon)}},$$

for some positive constants c_1 and c_2, and $m_k = n_k + (\lambda_k - 1)r_k - 1$. In either case, we have

$$C_{m_k} < c_3(C_{n_k+r_k-1})^{9(6+\varepsilon)},$$

for a suitable positive constant c_2. From Lemma 2.13, we obtain

$$\psi^{m_k-2} < c_2\chi^{9(6+\varepsilon)(n_k+r_k-1)},$$

that is,

$$n_k + (\lambda_k - 1)r_k - 3 < \frac{\log c_2}{\log \psi} + 9(6+\varepsilon)(n_k + r_k - 1)\frac{\log \chi}{\log \psi}. \tag{2.18}$$

Since r_k is bounded and ε is constant, then by taking a proper constant c_3 we can rewrite (2.18) as

$$n_k + r_k\lambda_k < c_3 + 9(6+\varepsilon)n_k\frac{\log \chi}{\log \psi},$$

from which we get

$$\lambda_k < r_k\lambda_k < c_4 + n_k\left(9(6+\varepsilon)\frac{\log \chi}{\log \psi} - 1\right).$$

Hence,

$$\frac{\lambda_k}{n_k} < \frac{c_4}{n_k} + \left(54\frac{\log \chi}{\log \psi} - 1\right) + 9\varepsilon\frac{\log \chi}{\log \psi},$$

which contradicts (2.17), by taking a sufficiently small ε. Therefore, $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ are both transcendental and, again, the rational independence follows from Theorem 1.54.

If $\alpha_0^{(1)} = \eta_k^{(1)}$ (resp. $\alpha_0^{(2)} = \eta_k^{(2)}$) for all but finitely many k, then we can conclude that $\alpha_0^{(1)}$ (resp. $\alpha_0^{(2)}$) is a cubic irrational and $\alpha_0^{(2)}$ (resp. $\alpha_0^{(1)}$) is transcendental.

As in the previous proof, the non-periodicity of the continued fraction excludes the possibility that $\alpha_0^{(1)} = \eta_k^{(1)}$ and $\alpha_0^{(2)} = \eta_k^{(2)}$ simultaneously for some k. □

Example 2.25. Let $(r_k)_{k\geq 0}$, $(n_k)_{k\geq 0}$ be as in Example 2.23. Similarly, let

$$\boldsymbol{a}^{(1)} = (x_0^{(1)},\ x_1^{(1)},\ \underbrace{x_2^{(1)},\dots,x_2^{(1)}}_{\lambda_2 \text{ times}},\ \underbrace{x_3^{(1)},\dots,x_3^{(1)}}_{\lambda_3 \text{ times}},\ \dots),$$
$$\boldsymbol{a}^{(2)} = (x_0^{(2)},\ x_1^{(2)},\ \underbrace{x_2^{(2)},\dots,x_2^{(2)}}_{\lambda_2 \text{ times}},\ \underbrace{x_3^{(2)},\dots,x_3^{(2)}}_{\lambda_3 \text{ times}},\ \dots),$$

and

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}].$$

As before, setting $x_k^{(1)} = x_k^{(2)} = 3$ if k is even, and $x_k^{(1)} = x_k^{(2)} = 2$ otherwise, we have that the partial quotients of our MCF are bounded by $M = 3$. Then, by simple computations, we have

$$3 \leq \chi \leq 4, \quad 1.4 \leq \psi \leq 2, \quad B < 222.$$

In this case, Theorem 2.24 allows us to take blocks of partial quotients with considerably smaller numbers of repetitions. Namely, let us redefine the sequence $(\lambda_k)_{k\geq 0}$

as

$$\lambda_0 = 1, \quad \lambda_k = 222(\lambda_0 + \cdots + \lambda_{k-1}) = 222\, n_k \ \text{ for } k \geq 1.$$

Notice that this sequence is eventually bounded by that defined in Example 2.23. By construction, we have

$$\lim_{k\to+\infty} \frac{\lambda_k}{n_k} = \lim_{k\to+\infty} \frac{222\, n_k}{n_k} = 222 > B.$$

Hence, Theorem 2.24 implies that at least one of $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ is transcendental.

Remark 2.26. Using Theorem 1.43 instead of Theorem 1.42 in the proof of the previous theorems does not yield a significant improvement of the results. The only effect is a refinement of the constant B in Theorem 2.24. More precisely, one can take

$$B = 36\frac{\log \chi}{\log \psi} - 1,$$

since the lower bounds for the approximation improve to

$$|\alpha_0^{(1)} - \eta_k^{(1)}| > \frac{c_1}{(C_{n_k+r_k-1})^{9(4+\varepsilon)}},$$
$$|\alpha_0^{(2)} - \eta_k^{(2)}| > \frac{c_2}{(C_{n_k+r_k-1})^{9(4+\varepsilon)}}.$$

2.5 Conclusions and outlook

In this section we collect several open problems and directions for future research aimed at extending and strengthening the results obtained in this thesis.

In Theorems 2.22 and 2.24 the authors were not able to exclude the case in which one of $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ is a cubic irrational, except under the following conjecture.

Conjecture 2.27. *Let*

$$(\alpha_0^{(1)}, \alpha_0^{(2)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), (a_0^{(2)}, a_1^{(2)}, \dots)]$$

be a quasi-periodic continued fraction, and let $(\eta_k^{(1)}, \eta_k^{(2)})$ *be as in* (2.13)*. Then,*

$$\alpha_0^{(1)} \neq \eta_k^{(1)} \quad \textit{and} \quad \alpha_0^{(2)} \neq \eta_k^{(2)},$$

for infinitely many $k \geq 0$*.*

Note that the non-periodicity of the quasi-periodic MCF of $(\alpha_0^{(1)}, \alpha_0^{(2)})$ is not sufficient to exclude this phenomenon. In fact, since the periodicity problem for the

Jacobi–Perron algorithm remains open, it is possible that $\alpha_0^{(1)}$ or $\alpha_0^{(2)}$ is a cubic irrational while the associated MCF is not periodic.

A further obstruction to extending Theorems 2.22 and 2.24 to MCFs of dimension $m > 2$ is the lack of a higher-dimensional analogue of Theorem 2.14. One can formulate the following conjecture.

Conjecture 2.28. *Let* $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) \in \mathbb{R}^m$ *have the MCF*

$$\left[\left(a_0^{(1)}, \dots, a_{k-1}^{(1)}, \overline{a_k^{(1)}, \dots, a_{k+h-1}^{(1)}}\right), \dots, \left(a_0^{(m)}, \dots, a_{k-1}^{(m)}, \overline{a_k^{(m)}, \dots, a_{k+h-1}^{(m)}}\right)\right],$$

and assume $0 < \alpha_0^{(i)} < 1$ *for all* i. *Then, there exist constants* $L_1, L_2 \in \mathbb{Z}_{>0}$, *independent of the numbers* $\alpha_0^{(i)}$, *such that* $\alpha_0^{(1)}, \dots, \alpha_0^{(m)}$, *which are algebraic irrationals of degree at most* $m+1$, *satisfy*

$$H(\alpha_0^{(i)}) \le L_1 C_{h+k-1}^{L_2}, \qquad i = 1, \dots, m.$$

To approach this conjecture, one can proceed as in the proof of Theorem 2.14, which leads to a system of equations of the form

$$\begin{cases} \sum_{i=1}^m X_{1,i}\alpha_0^{(i)} + X_{1,m+1} = \lambda'\alpha_0^{(1)}, \\ \sum_{i=1}^m X_{2,i}\alpha_0^{(i)} + X_{2,m+1} = \lambda'\alpha_0^{(2)}, \\ \vdots \\ \sum_{i=1}^m X_{m,i}\alpha_0^{(i)} + X_{m,m+1} = \lambda'\alpha_0^{(m)}, \\ \sum_{i=1}^m X_{m+1,i}\alpha_0^{(i)} + X_{m+1,m+1} = \lambda', \end{cases}$$

where $\lambda' = \alpha_{k+1}^{(1)} \cdots \alpha_{k+h}^{(1)}$. Eliminating λ' yields

$$\begin{cases} \sum_{i=1}^m X_{1,i}\alpha_0^{(i)} + X_{1,m+1} = \alpha_0^{(1)} \cdot \left(\sum_{i=1}^m X_{m+1,i}\alpha_0^{(i)} + X_{m+1,m+1}\right), \\ \vdots \\ \sum_{i=1}^m X_{m,i}\alpha_0^{(i)} + X_{m,m+1} = \alpha_0^{(m)} \cdot \left(\sum_{i=1}^m X_{m+1,i}\alpha_0^{(i)} + X_{m+1,m+1}\right). \end{cases}$$

The main difficulty is that, unlike the case $m = 2$, it is not straightforward to extract explicit minimal polynomials for each $\alpha_0^{(i)}$ from this system. Nevertheless, elimination techniques show that there exist polynomials

$$P^{(i)} \in \mathbb{Q}[Y_{1,1}, \dots, Y_{m+1,m+1}, Z], \qquad i = 1, \dots, m,$$

whose coefficients depend only on m, such that

$$P^{(i)}(X_{1,1}, \dots, X_{m+1,m+1}, \alpha_0^{(i)}) = 0.$$

Thus, the conjecture would follow from a suitable generalization of Lemma 2.8. This is the key obstruction: Theorem 2.7 is not sufficient, since its constants depend on the specific m-tuple, whereas a uniform bound is required.

Another obstacle to extending our results to quasi-periodic MCFs in higher dimension is that an analogue of Lemma 2.1 generally fails for $m \geq 3$. In particular, one no longer has $|\tilde{A}_n^{(i)}| \leq C_n$ for all $n \in \mathbb{N}$.

In Theorem 2.16 and Theorem 2.19 (Liouville-type MCFs) and in Theorem 2.22 and Theorem 2.24 (quasi-periodic MCFs), we provided criteria ensuring that the associated m-tuples are composed of rationally independent transcendental numbers. It is natural to wonder if, under these conditions, the numbers are in fact algebraically independent over $\mathbb{Q}$. Establishing this, or identifying additional sufficient conditions, would be a significant improvement.

Beyond Liouville-type [56] and quasi-periodic continued fractions [11, 59], several transcendence criteria are known in the one-dimensional setting. Adamczewski and Bugeaud proved transcendence for *stammering* [3] and *palindromic* [5] continued fractions, and later, together with Davison [6], for broader classes. This line of research culminates in Bugeaud's result on automatic continued fractions [18]. Extending these notions and results to the multidimensional setting is a promising direction for future work.

Finally, Baker's results rely crucially on the Davenport–Roth bound (Theorem 1.29). Any refinement of this estimate could lead to sharper transcendence criteria. For instance, in [4], improvements are obtained via refined growth estimates for convergents, based on strengthened versions of (1.21) and applications of Schmidt's Subspace Theorem [76]. It would be interesting to investigate whether analogous refinements of Theorem 2.20 can be achieved in the multidimensional setting. A preliminary step in this direction is to better understand the role of the condition $a_{n+1}^{(1)} < C_n^d$: whether it is merely technical, or whether it already encodes significant information about the transcendence of multidimensional continued fractions.

Part II

On Fractional Parts of Powers of Pisot Numbers

3 On the number of residues of certain linear recurrences

Chapter Contents

The results presented in this chapter arise from a collaboration with Sanna and are published in [2]. We provide here a detailed exposition, culminating in a result on the distribution modulo 1 of certain powers of Pisot numbers.

Let $a_1, \dots, a_k$ be integers. A sequence of integers $\boldsymbol{x} = (x_n)_{n \geq 0}$ is called a *linear recurrence* with *characteristic polynomial*

$$f = X^k - a_1 X^{k-1} - a_2 X^{k-2} - \cdots - a_k \tag{3.1}$$

if it satisfies

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}, \tag{3.2}$$

for all integers $n \geq k$. The values $x_0, \dots, x_{k-1}$ are called the *initial values* of $\boldsymbol{x}$ and, together with f, uniquely determine the sequence via (3.2). We denote by $\mathcal{L}(f)$ the set of all integer sequences satisfying (3.2).

It is well known that every $\boldsymbol{x} \in \mathcal{L}(f)$ is ultimately periodic modulo any integer $m \geq 1$, and in fact purely periodic whenever $\gcd(m, a_k) = 1$. The arithmetic properties of linear recurrences modulo m have been extensively studied; in particular, one may ask which residue classes modulo m are attained by $\boldsymbol{x}$ and with what frequency [19,

32, 40, 61, 73, 84], as well as for which moduli m the sequence $\boldsymbol{x}$ contains a complete system of residues [10, 20, 49, 83].

For $\boldsymbol{x} \in \mathcal{L}(f)$ and $m \geq 1$, let $\tau(\boldsymbol{x}; m)$ denote the (minimal) period of $\boldsymbol{x}$ modulo m, i.e., the smallest integer $t \geq 1$ such that $x_{n+t} \equiv x_n \pmod{m}$ for all sufficiently large n. We also define

$$\rho(\boldsymbol{x}; m) = \Big|\{x_n \bmod m : n \geq 0\}\Big|,$$

and

$$\rho^*(\boldsymbol{x}; m) = \Big|\{x_n \bmod m : n \geq 0,\ x_n \not\equiv 0 \pmod{m}\}\Big|.$$

Set

$$\mathcal{R}(f) = \Big\{\rho(\boldsymbol{x}; m) : \boldsymbol{x} \in \mathcal{L}(f),\ m \in \mathbb{Z}^+\Big\},$$
$$\mathcal{R}^*(f) = \Big\{\rho^*(\boldsymbol{x}; m) : \boldsymbol{x} \in \mathcal{L}(f),\ m \in \mathbb{Z}^+\Big\}.$$

With this notation, Dubickas and Novikas [33] showed that

$$\mathcal{R}(X^2 - X - 1) = \mathbb{Z}^+ \quad \text{and} \quad \mathcal{R}^*(X^2 - X - 1) = \mathbb{Z}^+ \setminus \{1, 2, 3\}.$$

Sanna [71] studied the case where f is quadratic with roots α, β satisfying $\alpha\beta = \pm 1$ and such that α/β is not a root of unity. He proved, in particular, that $\mathcal{R}(f)$ contains all integers $n \geq 7$ with $n \neq 10$ and $4 \nmid n$. Moreover,

$$4D_0\,\mathbb{Z}^+ \subseteq \mathcal{R}(f) \quad \text{if } \alpha\beta = 1, \qquad 8D_0\,\mathbb{Z}^+ \subseteq \mathcal{R}(f) \quad \text{if } \alpha\beta = -1,$$

where D_0 denotes the squarefree part of the discriminant of f, under the assumptions $D_0 \equiv 1 \pmod{4}$ and $D_0 \geq 5$.

The main result in [2] is the following.

Theorem 3.1. *Let a_1 be a nonzero integer. Then, $\mathcal{R}(X^2 - a_1X - 1) = \mathbb{Z}^+$. Moreover, $\mathcal{R}^*(X^2 - a_1X - 1) = \mathbb{Z}^+$ if and only if $|a_1| \geq 2$. If $|a_1| = 1$, then $\mathcal{R}^*(X^2 - X - 1) = \mathbb{Z}^+ \setminus \{1, 2, 3\}$.*

The assumption that a_1 is nonzero is not restrictive, since one can easily check that $\mathcal{R}(X^2 - 1) = \{1, 2\}$. Indeed, every $\boldsymbol{x} \in \mathcal{L}(X^2 - 1)$ is of the form

$$\boldsymbol{x} = (x_0, x_1, x_0, x_1, x_0, x_1, \ldots),$$

so that $\rho(\boldsymbol{x}; m) = 2$ if $x_0 \not\equiv x_1 \pmod{m}$, and $\rho(\boldsymbol{x}; m) = 1$ otherwise.

It is worth highlighting how our approach differs from those of [33] and [71], since

both works are closely related to ours. Their core strategy is to find a suitable positive integer m, generally a prime p, and a sequence $\boldsymbol{x} \in \mathcal{L}(f)$ that, modulo m, becomes essentially a geometric progression whose ratio is a root of f modulo m. Consequently, the number of distinct residues $\rho(\boldsymbol{x}; m)$ is governed by the multiplicative order of that root modulo m, and producing a prescribed value n of ρ amounts to finding m and a root of f modulo m of order n.

Dubickas and Novikas [33] carried out this idea for the single polynomial $f = X^2 - X - 1$ (the case $a_1 = 1$, with our notation). For each target value of n, they construct a Fibonacci-type sequence with *ad hoc* initial values x_0, x_1, chosen so that the sequence reduces, modulo a suitable prime p, to a geometric progression of the desired order n. To select the prime p, the authors define an appropriate sequence of numbers $(L_n)_{n\geq 0}$, known as *Lucas numbers*, and take p to be a *primitive divisor* of such sequence (namely, p divides a term of the sequence and no term before it). Several (sometimes very large) primitive divisors of Lucas numbers are computed explicitly with the aid of a computer. When this strategy is not directly applicable, the problem is attacked by combining sequences with known number of residues modulo specific integers, so as to obtain a new sequence with n distinct residues modulo a suitable integer m. The resulting proof is split into numerous cases according to delicate divisibility properties of n (Tables 1, 5, 6, 9 of [33]), and both the choice of initial values and the choice of primes are tailored to the specific arithmetic of Fibonacci numbers: the arguments do not extend, even in principle, beyond this one polynomial.

Sanna [71] isolated and generalized the underlying mechanism to a wider family of quadratic polynomials f. His criterion for showing that $n \in \mathcal{R}(f)$ reduces the problem to the existence of a primitive divisor p of a suitable sequence attached to f (namely, a *Lehmer sequence*), satisfying the additional congruence $p \equiv 1 \pmod{n}$. This extra condition can be satisfied rather easily when $4 \nmid n$ (and $f(0) = -1$), but becomes a genuine obstruction when $4 \mid n$, where one needs a primitive divisor with even further properties – something guaranteed only under restrictive hypotheses on the discriminant of f. This is the source of the gaps left in [71]: all multiples of 4 (apart from those covered by such hypotheses) and finitely many small values are excluded.

Our argument removes this obstruction: rather than imposing additional conditions on the primitive divisor p, we work directly with the multiplicative order of the roots of f in the ring of integers of the splitting field of f (Lemma 3.16 and Lemma 3.18). This allows us to handle the case where $4 \mid n$ as well, by replacing Sanna's restrictive assumptions on p with the requirement that p is a *high* primitive

divisor (see Definition 3.11), which can be guaranteed in fuller generality.

We also remark that the proof of Theorem 3.1 is constructive: it provides an algorithm which, given an integer $a_1 \neq 0$ and an integer $n \geq 1$, produces an m and initial values x_0, x_1 such that the corresponding sequence $\boldsymbol{x} \in \mathcal{L}(X^2 - a_1X - 1)$ satisfies $\rho(\boldsymbol{x}; m) = n$, and, if $|a_1| \geq 2$ or $n \geq 4$, all the residues of $\boldsymbol{x}$ modulo m are nonzero.

The last part of this Chapter is devoted to a corollary of Theorem 3.1.

For every $t \in \mathbb{R}$, let $\lfloor t \rfloor$ be the *floor function* of t, that is, the greatest integer not exceeding t, and let $\{t\} = t - \lfloor t \rfloor$ be the *fractional part* of t. Let ϱ be the golden ratio. As a corollary of their aforementioned result, Dubickas and Novikas proved that for every $k \geq 2$ there exists a real number λ such that $\left(\{\lambda\varrho^n\}\right)_{n\geq 0}$ has exactly k limit points; whereas for no $\lambda \neq 0$ does the sequence $\left(\{\lambda\varrho^n\}\right)_{n\geq 0}$ have exactly one limit point.

From Theorem 3.1, we deduce the following result concerning the distribution modulo 1 of powers of certain Pisot numbers.

Corollary 3.2. *Let $a_1 \geq 1$ be an integer, and let $\alpha = \left(a_1 + \sqrt{a_1^2 + 4}\right)/2$. Then, for every integer $k \geq 2$, and for every integer $k \geq 1$ if $a_1 \geq 2$, there exists $\xi \in \mathbb{Q}(\alpha)$ such that the sequence $\left(\{\xi\alpha^n\}\right)_{n\geq 0}$ has exactly k limit points.*

3.1 Preliminary results

Hereafter, let a_1 be a nonzero integer, and cosndier the polynomial $f = X^2 - a_1X - 1$ with discriminant $D = a_1^2 + 4$. Since $a_1 \neq 0$, the polynomial f is irreducible over $\mathbb{Q}$, meaning D is not a perfect square. Thus, the splitting field $K = \mathbb{Q}\left(\sqrt{D}\right)$ is a quadratic number field. Denoting the roots of f in K by α and β, we observe that α and β are distinct real numbers such that $\alpha\beta = -1$, $\alpha + \beta = a_1$, $(\alpha - \beta)^2 = D$. Moreover, since the only roots of unity in a real number field are $-1, 1$, then α/β is not a root of unity, as otherwise we would have either $\alpha = 1 = -\beta$, or $\beta = 1 = -\alpha$. Both cases would contradict $\alpha + \beta = a_1 \neq 0$.

3.1.1 A preparatory lemma

We begin with the following lemma, which reduces the number of cases to be considered in the proof of Theorem 3.1.

Lemma 3.3. *Let $\boldsymbol{x} \in \mathcal{L}(f)$, let $m \geq 1$ be an integer, and let $t = \tau(\boldsymbol{x}; m)$. Then, there exists $\boldsymbol{y} \in \mathcal{L}(X^2 + a_1X - 1)$ such that $y_n \equiv x_{(-(n+1) \bmod t)} \pmod m$ for every integer*

$n \geq 0$, where $(-(n+1) \bmod t)$ is the unique integer $r \in [0,t)$ such that $r \equiv -(n+1) \pmod{t}$. In particular, we have that $\tau(\boldsymbol{x};m) = \tau(\boldsymbol{y};m)$, $\rho(\boldsymbol{x};m) = \rho(\boldsymbol{y};m)$, and that all residues of $\boldsymbol{x}$ modulo m are nonzero if and only if all residues of $\boldsymbol{y}$ modulo m are nonzero.

Proof. Let us extend $\boldsymbol{x}$ to negative indices by defining $x_n = -a_1 x_{n+1} + x_{n+2}$ for every integer $n \leq -1$. Then, we have that $x_n = -a_1 x_{n+1} + x_{n+2}$ and $x_{n+t} \equiv x_n \pmod{m}$ for every integer n. Let $\boldsymbol{y} \in \mathcal{L}(X^2 + a_1 X - 1)$ with $y_0 = x_{t-1}$ and $y_1 = x_{t-2}$. Since $y_{n+2} = -a_1 y_{n+1} + y_n$ for every integer $n \geq 0$, it follows easily by induction that $y_n = x_{t-(n+1)}$ for every integer $n \geq 0$. Hence, by the periodicity of $\boldsymbol{x}$, we get that $y_n \equiv x_{(-(n+1) \bmod t)} \pmod{m}$ for every integer $n \geq 0$, as desired. □

In light of Lemma 3.3, hereafter we assume that $a_1 \geq 1$.

Remark 3.4. Since Dubickas and Novikas already proved the case $a_1 = 1$ of Theorem 3.1 and Corollary 3.2, we could assume that $a_1 \geq 2$. However, we choose to include the case $a_1 = 1$ in our proofs in order to state some of the intermediary results with greater generality.

3.1.2 Lehmer sequences

Let γ, δ be complex numbers such that $\gamma\delta$ and $(\gamma+\delta)^2$ are nonzero coprime integers, and such that γ/δ is not a root of unity. The *Lehmer sequence* (ℓ_n) associated with γ and δ is defined by

$$\ell_n = \begin{cases} \dfrac{\gamma^n - \delta^n}{\gamma - \delta} & \text{if } n \text{ is odd,} \\ \dfrac{\gamma^n - \delta^n}{\gamma^2 - \delta^2} & \text{if } n \text{ is even,} \end{cases} \tag{3.3}$$

for all integers $n \geq 0$. The assumptions on γ and δ ensure that $\ell_n \in \mathbb{Z}$ for every n.

Definition 3.5. A prime number p is called a *primitive divisor* of ℓ_n if $p \mid \ell_n$ but

$$p \nmid (\gamma^2 - \delta^2)^2 \ell_1 \cdots \ell_{n-1}.$$

Since $\ell_1 = \ell_2 = 1$, primitive divisors can occur only for $n \geq 3$. See [14] for further details on the existence of primitive divisors. Let $\mathcal{P}(\ell_n)$ denote the set of all primitive divisors of ℓ_n.

We also need to introduce the sequence of *cyclotomic numbers* (ϕ_n) associated

with γ and δ. This sequence is defined by

$$\phi_n = \prod_{\substack{1\le k\le n\\ \gcd(n,k)=1}} \left(\gamma - \mathrm{e}^{\frac{2\pi \mathbf{i} k}{n}}\delta\right),$$

for all integers $n \geq 1$. It is known that $\phi_n \in \mathbb{Z}$ for every $n \geq 3$ (see [14, p. 84]). Recall that the n-th cyclotomic polynomial Φ_n is given by

$$\Phi_n(X) = \prod_{\substack{1\le k\le n\\ \gcd(n,k)=1}} \left(X - \mathrm{e}^{\frac{2\pi \mathbf{i} k}{n}}\right).$$

A direct computation shows that

$$\delta^{\varphi(n)}\Phi_n\left(\frac{\gamma}{\delta}\right) = \phi_n,$$

where φ is Euler's totient function. Moreover,

$$X^m - 1 = \prod_{d\mid m} \Phi_d(X),$$

and therefore

$$\gamma^m - \delta^m = \delta^m\left(\left(\frac{\gamma}{\delta}\right)^m - 1\right) = \delta^m \prod_{d\mid m} \Phi_d\left(\frac{\gamma}{\delta}\right) = \prod_{d\mid m} \delta^{\varphi(d)}\Phi_d\left(\frac{\gamma}{\delta}\right) = \prod_{d\mid m} \phi_d, \tag{3.4}$$

where we used the identity $m = \sum_{d\mid m} \varphi(d)$.

We now aim to establish a connection between cyclotomic numbers and primitive divisors of Lehmer sequences. To this end, we need the multiplicative form of the Möbius inversion formula, for which we also need to recall the *Möbius function* $\mu(n)$. If n be a positive integer with prime factorization $n = p_1^{e_1}\cdots p_{s_n}^{e_{s_n}}$, $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1,\\ 0 & \text{if } e_i \geq 2 \text{ for some } i,\\ (-1)^{s_n} & \text{if } e_i = 1 \text{ for all } i. \end{cases}$$

It is well known that, for every integer $n \geq 2$,

$$\sum_{d\mid n} \mu(d) = 0, \tag{3.5}$$

and that μ is multiplicative, i.e.,

$$\mu(n_1 n_2) = \mu(n_1)\mu(n_2) \quad \text{whenever } \gcd(n_1, n_2) = 1. \tag{3.6}$$

Lemma 3.6 (Möbius inversion formula)**.** *Let $f, g : \mathbb{N} \to \mathbb{C} \setminus \{0\}$. If*

$$g(m) = \prod_{d|m} f(d),$$

then

$$f(m) = \prod_{d|m} g(d)^{\mu(m/d)}.$$

Proof. This is a classical result; see, for example, [9, Chapter 2]. The lemma follows by translating the proof given there from additive to multiplicative notation. □

For every prime number p, let ν_p denote the p-adic valuation. For every integer $n \geq 4$, let $P_3(n)$ denote the greatest prime factor of $n/\gcd(n, 3)$.

Lemma 3.7. *Let $n \geq 5$ be an integer with $n \neq 6$. Then,*

$$|\phi_n| \Bigg(\prod_{p \in \mathcal{P}(\ell_n)} p^{\nu_p(\ell_n)} \Bigg)^{-1} \in \begin{cases} \{1, P_3(n)\} & \text{if } n \neq 12, \\ \{1, 2, 3, 6\} & \text{if } n = 12, \end{cases} \tag{3.7}$$

where the product runs over all the primitive divisors p of ℓ_n.

Proof. From (3.4) we have

$$\gamma^m - \delta^m = \prod_{d \,|\, m} \phi_d,$$

for every integer $m \geq 1$. From the definition (3.3), we have

$$(\gamma^d - \delta^d) = \begin{cases} \ell_d(\gamma - \delta) & \text{if } n \text{ is odd}, \\ \ell_d(\gamma - \delta)(\gamma + \delta) & \text{if } n \text{ is even}. \end{cases}$$

Therefore, applying Lemma 3.6 with $g(m) = \gamma^m - \delta^m$ and $f(d) = \phi_d$, we obtain

$$\phi_m = \prod_{d \,|\, m} \left(\gamma^d - \delta^d\right)^{\mu(m/d)} = \prod_{d \,|\, m} \ell_d^{\mu(m/d)} \prod_{d \,|\, m} (\gamma - \delta)^{\mu(m/d)} \prod_{\substack{d \,|\, m \\ d \text{ even}}} (\gamma + \delta)^{\mu(m/d)}.$$

for all integers $m \geq 3$.

Now,

$$\prod_{d\,|\,m}(\gamma-\delta)^{\mu(m/d)}=\prod_{d\,|\,m}(\gamma-\delta)^{\mu(d)}=1,$$

where the last inequality holds by (3.5). Moreover, using also (3.6) and writing $m=2^{v_2(m)}n$ with n odd, we get

$$\prod_{\substack{d\,|\,m\\ d\text{ even}}}(\gamma+\delta)^{\mu(d)}=\prod_{e=1}^{v_2(m)}\left(\prod_{d\,|\,n}(\gamma+\delta)^{\mu(n/d)}\right)^{\mu(2^{v_2(m)-e})}=1,$$

since

$$\prod_{d\,|\,n}(\gamma-\delta)^{\mu(n/d)}=\prod_{d\,|\,n}(\gamma-\delta)^{\mu(d)}=1,$$

where the last inequality holds again by (3.5). Hence,

$$\phi_m=\prod_{d\,|\,m}\left(\gamma^d-\delta^d\right)^{\mu(m/d)}=\prod_{d\,|\,m}\ell_d^{\mu(m/d)}. \tag{3.8}$$

Let p be a prime number. If p is a primitive divisor of ℓ_n, then $p\nmid\ell_k$ for every $k<n$, and therefore (3.8) yields

$$\nu_p(\phi_n)=\nu_p(\ell_n).$$

If p is not a primitive divisor of ℓ_n and $p\mid\phi_n$, then it is known that $\nu_p(\phi_n)=1$ and that either $p=P_3(n)$ (if $n\neq 12$), or $p\in\{2,3\}$ (if $n=12$); see the "only if" part of the proof of [14, Theorem 2.4], or see the paragraphs before, and the comment after, [85, Lemma 6]. □

We need a lower bound for the absolute values of cyclotomic numbers.

Lemma 3.8. *Suppose that γ,δ are real numbers with $\gamma\delta>0$. Then,*

$$|\phi_n|>|4(\gamma-\delta)^2\gamma\delta|^{\varphi(n)/4},$$

for every integer $n\geq 3$, where φ is Euler's totient function.

Proof. This result is due to Ward [91, p. 233]. □

Hereafter, we set $\gamma=\alpha$ and $\delta=-\beta$. Note that this choice does indeed satisfy the conditions of Lehmer sequence. Moreover, we have that $\gamma\delta=1$, $\gamma-\delta=a_1$, $(\gamma+\delta)^2=D$, and $(\gamma^2-\delta^2)^2=a_1^2D$.

Furthermore, the first values of (ℓ_n) are

$$\ell_1 = 1, \quad \ell_2 = 1, \quad \ell_3 = a_1^2 + 3, \quad \ell_4 = a_1^2 + 2,$$
$$\ell_5 = (a_1^2 + 1)(a_1^2 + 4) + 1, \quad \ell_6 = (a_1^2 + 1)\ell_3. \tag{3.9}$$

In particular, note that, since ℓ_3 or ℓ_4 is even, if $n \geq 5$ then every primitive divisor of ℓ_n (if it exists) is odd.

The problem of determining which terms of a Lehmer sequence have a primitive divisor has a very long history. The first complete classification was given by Bilu, Hanrot, and Voutier [14] (see also [71, Remark 3.1] for some missing values in [14, Table 4]). We make use of the following two results.

Lemma 3.9. *Let $n \geq 3$ be an integer. If $a_1 = 1$ and $n \notin \{3, 6, 10, 12\}$, or $a_1 = 3$ and $n \neq 3$, or $a_1 \notin \{1, 3\}$, then ℓ_n has at least one odd primitive divisor p.*

Proof. If $n \geq 13$, then it is known that ℓ_n has at least one odd primitive divisor [85, Lemma 8]. (In fact, this is true for all Lehmer sequences with real γ, δ.) Hereafter, assume that $n \leq 12$.

If $a_1 = 1$ and $n \notin \{3, 6, 10, 12\}$, then one can check that ℓ_n has an odd primitive divisor. If $a_1 = 3$ and $n \neq 3$, then one can check that ℓ_n has an odd primitive divisor.

Hereafter, assume that $a_1 \notin \{1, 3\}$. If $n \geq 5$ and $n \neq 6$, then by Lemma 3.8, recalling that $\gamma - \delta = a_1$, $\gamma\delta = 1$, we get that $|\phi_n| > 2^{\varphi(n)} > n$. In turn, by Lemma 3.7, this implies that ℓ_n has an odd primitive divisor, as otherwise (3.7) would give

$$n < |\phi_n| \leq P_3(n) \leq n,$$

which is a contradiction. Here we used the fact that, since $n \geq 5$ then ℓ_n does not admit even primitive divisors.

It remains to prove that if $n \in \{3, 4, 6\}$ then ℓ_n has an odd primitive divisor. Since $a_1 \notin \{1, 3\}$, by reasoning modulo 8 and modulo 9, it follows easily that $a_1^2 + 2$ is not a power of 2, and that neither $a_1^2 + 1$ nor $a_1^2 + 3$ is equal to $2^s 3^t$ for some integers $s, t \geq 0$. Hence, there exists a prime number $p \geq 5$ dividing $a_1^2 + 1$. We claim that p is an odd primitive divisor of ℓ_6. In fact, since $p \geq 5$, by (3.9) and recalling that $D = a_1^2 + 4$, we get that $p \mid \ell_6$ and $p \nmid a_1 D \ell_1 \cdots \ell_5$. The proof that ℓ_3 and ℓ_4 have an odd primitive divisor proceeds similarly. □

Remark 3.10. Lemma 3.9 is optimal, that is, one can verify that if $a_1 = 1$ and $n \in \{3, 6, 10, 12\}$, or if $a_1 = 3$ and $n = 3$, then ℓ_n does not have an odd primitive divisor.

Definition 3.11. For every integer $n \geq 1$, we say that a prime number p is a *high primitive divisor* of ℓ_n if p is an odd primitive divisor of ℓ_n such that $p^{\nu_p(\ell_n)} > n$.

Lemma 3.12. *Suppose that $n \geq 5$ and $n \neq 6$. If $(2a_1)^{\varphi(n)/2} > n^2$, then ℓ_n has a high primitive divisor.*

Proof. Recalling that $\gamma - \delta = a_1$ and $\gamma\delta = 1$, by Lemma 3.8 we have that $|\phi_n| > (2a_1)^{\varphi(n)/2}$. Hence, by Lemma 3.7, we get that $(2a_1)^{\varphi(n)/2} > n^2$ implies that

$$\prod_{q \in \mathcal{P}(\ell_n)} q^{\nu_q(\ell_n)} > n. \tag{3.10}$$

In particular, from (3.10) it follows that ℓ_n has at least one primitive divisor. Let p be the greatest primitive divisor of ℓ_n. It is known that each primitive divisor q of ℓ_n satisfies $q \equiv \pm 1 \pmod{n}$ [85, p. 427]. Hence, either $p \geq n+1$, or $p = n-1$ and p is the unique primitive divisor of ℓ_n. In both cases, taking into account (3.10), we get that $p^{\nu_p(\ell_n)} > n$. Hence, we have that p is a high primitive divisor of ℓ_n (note that p is odd since $n \geq 5$). □

Lemma 3.13. *Let $n \geq 3$ be an integer. If $a_1 = 1$ and $n \notin \{3, 4, 6, 8, 10, 12, 14, 18, 24\}$, or $a_1 = 2$ and $n \notin \{4, 6, 12\}$, or $a_1 = 3$ and $n \notin \{3, 6\}$, or $a_1 \geq 4$, then ℓ_n has at least one high primitive divisor.*

Proof. If $n \geq 2 \cdot 10^9$, then $\varphi(n) > n/\log n$ [91, Lemma 4.1]. Hence, we get that

$$(2a_1)^{\varphi(n)/2} > 2^{n/(2\log n)} > n^2.$$

If $180 \leq n < 2 \cdot 10^9$, then $\varphi(n) > n/6$ [91, Lemma 4.2]. Hence, we get that

$$(2a_1)^{\varphi(n)/2} > 2^{n/12} > n^2.$$

If $5 \leq n \leq 179$ with $n \neq 6$, and $|a_1| \geq 7$, then

$$(2a_1)^{\varphi(n)/2} \geq 14^{\varphi(n)/2} > n^2.$$

Hence, if $n \geq 180$, or $5 \leq n \leq 179$ with $n \neq 6$ and $|a_1| \geq 7$, by Theorem 3.12, ℓ_n has a high primitive divisor.

If $5 \leq n \leq 179$ with $n \neq 6$, and $a_1 \leq 6$, then with the aid of a computer one can check that ℓ_n has a high primitive divisor, unless $a_1 = 1$ and $n \in \{8, 10, 12, 14, 18, 24\}$, or $a_1 = 2$ and $n = 12$. Note that this verification is done more efficiently by factorizing ℓ_n only in the cases in which the inequality $(2a_1)^{\varphi(n)/2} > n^2$ does not hold.

If $a_1 = 2$ or $a_1 = 3$, then ℓ_3 or ℓ_4 has a high primitive divisor, respectively. Hereafter, assume that $n \in \{3, 4, 6\}$ and $a_1 \geq 4$. It remains to prove that ℓ_n has a high primitive divisor. This is done similarly to the end of the proof of Lemma 3.9. Since $a_1 \geq 4$, by reasoning modulo 8 and modulo 9, it follows easily that $a_1^2 + 2$ is not equal to 2^s or $2^s 3$, for some integer $s \geq 0$, and that neither $a_1^2 + 1$ nor $a_1^2 + 3$ is equal to $2^s 3^t$ or $2^s 3^t 5$, for some integers $s, t \geq 0$. Hence, there exists a prime number $p \geq 5$ dividing $a_1^2 + 1$ and such that $p^{\nu_p(a_1^2+1)} > 6$. We claim that p is a high primitive divisor of ℓ_6. In fact, since $p \geq 5$, by (3.9) and recalling that $D = a_1^2 + 4$, we get that $p \mid \ell_6$, $p \nmid a_1 D \ell_1 \cdots \ell_5$, and $p^{\nu_p(\ell_6)} > 6$. The proof that ℓ_3 and ℓ_4 have a high primitive divisor proceeds similarly. □

Remark 3.14. Lemma 3.13 is optimal, that is, one can verify that if $a_1 = 1$ and $n \in \{3, 4, 6, 8, 10, 12, 14, 18, 24\}$, or $a_1 = 2$ and $n \in \{4, 6, 12\}$, or $a_1 = 3$ and $n \in \{3, 6\}$, then ℓ_n has no high primitive divisor.

For every odd prime number p, let $\left(\frac{D}{p}\right)$ denote the *Legendre symbol*. Observe that $\left(\frac{D}{p}\right) = 1$ if and only if f splits modulo p.

Lemma 3.15. *Let $n \geq 3$ be an odd integer and let p be an odd prime factor of ℓ_n. Then, we have that $\left(\frac{D}{p}\right) = 1$.*

Proof. Define $v_n = (\gamma^n + \delta^n)/(\gamma + \delta) = (\alpha^n - \beta^n)/(\alpha - \beta)$. Then, since n is odd, we have that v_n is a symmetric expression of the algebraic integers α, β. Hence, we get that $v_n \in \mathbb{Q}$. On the other hand, v_n is an algebraic integer. Indeed, let $T(X) = (X^n - 1)/(X - 1) \in \mathbb{Z}[X]$, then

$$v_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \beta^{n-1} \frac{(\alpha/\beta)^n - 1}{(\alpha/\beta) - 1} = \beta^{n-1} T(\alpha/\beta) = S(\alpha, \beta),$$

where S is a polynomial in two variables with integer coefficients. Consequently, $v_n \in \mathbb{Z}$. Furthermore, recalling that $a_1 = \alpha + \beta = \gamma - \delta$ and $D = (\alpha + \beta)^2 + 4 = (\alpha + \beta)^2 - 4\alpha\beta = (\alpha - \beta)^2 = (\gamma + \delta)^2$, we obtain the following identity

$$D v_n^2 - a_1^2 \ell_n^2 = (\gamma + \delta)^2 v_n^2 - (\gamma - \delta)^2 \ell_n^2 = (\gamma^n + \delta^n)^2 - (\gamma^n - \delta^n)^2 = 4(\gamma\delta)^n = 4.$$

From this and by the fact that $p \mid \ell_n$, we get that $D v_n^2 \equiv 2^2 \pmod{p}$. Since p is odd, it follows that $p \nmid v_n$ and so $D \equiv (2v_n^{-1})^2 \pmod{p}$. Thus $\left(\frac{D}{p}\right) = 1$, as desired. □

3.1.3 Multiplicative orders

Let $\mathcal{O}_K$ be the ring of integers of K. For every ideal $\mathfrak{i}$ of $\mathcal{O}_K$ and for every $\theta \in \mathcal{O}_K$, let $\mathrm{ord}_{\mathfrak{i}}(\theta)$ denote the multiplicative order of θ modulo $\mathfrak{i}$ (if it exists).

Lemma 3.16. *Let $n, v \geq 1$ be integers, let p be a prime number not dividing $a_1 D$, and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p. Then, $p^v \mid \ell_n$ if and only if $\mathrm{ord}_{\mathfrak{p}^v}(\alpha^2) \mid n$. In particular, we have that p is a primitive divisor of ℓ_n if and only if $n = \mathrm{ord}_{\mathfrak{p}}(\alpha^2)$.*

Proof. Since $p \nmid a_1 D$, we have that $\alpha + \beta$ and $\alpha - \beta$ are invertible modulo $\mathfrak{p}^v$. Hence, using (3.3), we get that $p^v \mid \ell_n$ is equivalent to $\alpha^n \equiv (-\beta)^n \pmod{\mathfrak{p}^v}$, which in turn is equivalent to $(\alpha^2)^n \equiv 1 \pmod{\mathfrak{p}^v}$, since $\alpha/(-\beta) = \alpha^2$. Thus the claim follows. □

Lemma 3.17. *Let p be an odd prime number, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p, and let $v \geq 1$ be an integer. Then, the equation $X^2 \equiv 1 \pmod{\mathfrak{p}^v}$ has exactly two solutions modulo $\mathfrak{p}^v$, namely 1 and -1.*

Proof. The equation $X^2 \equiv 1 \pmod{\mathfrak{p}^v}$ is equivalent to $\nu_{\mathfrak{p}}(X-1) + \nu_{\mathfrak{p}}(X+1) \geq v$, where $\nu_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation over $\mathcal{O}_K$. Moreover, we have that $\nu_{\mathfrak{p}}(X-1)$ and $\nu_{\mathfrak{p}}(X+1)$ cannot both be positive, since $2 \notin \mathfrak{p}$. Hence, either $X \equiv 1 \pmod{\mathfrak{p}^v}$ or $X \equiv -1 \pmod{\mathfrak{p}^v}$. □

Lemma 3.18. *Let p be an odd prime number, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p, and let $v \geq 1$ be an integer. Set $a = \mathrm{ord}_{\mathfrak{p}^v}(\alpha)$, $b = \mathrm{ord}_{\mathfrak{p}^v}(\beta)$, and $c = \mathrm{ord}_{\mathfrak{p}^v}(\alpha^2)$. Then, the following statements hold.*

(i) If c is odd then $a = c$ and $b = 2c$, or $a = 2c$ and $b = c$.

(ii) If c is even then $a = b = 2c$.

(iii) $\mathrm{lcm}(a, b) = 2c$.

Proof. Hereafter, all congruences are modulo $\mathfrak{p}^v$. Since $\alpha^{2c} \equiv 1$, we have that $a \mid 2c$ and, by Lemma 3.17, that $\alpha^c \equiv \pm 1$. Moreover, from $\alpha^a \equiv 1$, we have that $\alpha^{2a} \equiv 1$, and so $c \mid a$. Hence, we get that $a = c$ if $\alpha^c \equiv 1$, and $a = 2c$ if $\alpha^c \equiv -1$.

Using the fact that $\beta^2 = \alpha^{-2}$, we get that $c = \mathrm{ord}_{\mathfrak{p}^v}(\beta^2)$. Hence, by a reasoning similar to the previous one, we have that $\beta^c \equiv \pm 1$, while $b = c$ if $\beta^c \equiv 1$, and $b = 2c$ if $\beta^c \equiv -1$.

Suppose that c is odd. Since $\alpha\beta = -1$, we have that $\alpha^c \beta^c \equiv (-1)^c \equiv -1$. Hence, either $\alpha^c \equiv 1$ and $\beta^c \equiv -1$, or $\alpha^c \equiv -1$ and $\beta^c \equiv 1$. By the previous considerations, it follows that either $a = c$ and $b = 2c$, or $a = 2c$ and $b = c$. This proves (i).

Suppose that c is even. Then, by the minimality of c, we get that $\alpha^c \equiv \beta^c \equiv -1$. Hence, by the previous considerations, we have that $a = b = 2c$. This proves (ii).

Finally, we obtain (iii) as a direct consequence of (i) and (ii). □

3.1.4 Second-order linear recurrences

In this section, we collect some results on linear recurrences in $\mathcal{L}(f)$.

Lemma 3.19. *Let $\boldsymbol{x} \in \mathcal{L}(f)$. Then, we have that*

$$x_n = \frac{(x_1 - \beta x_0)\alpha^n - (x_1 - \alpha x_0)\beta^n}{\alpha - \beta},$$

for every integer $n \geq 0$.

Proof. This is a special case of the general expression of linear recurrences as generalized power sums [36, Sec. 1.1.6] and can be easily proven by induction. □

Lemma 3.20. *Let $\boldsymbol{x} \in \mathcal{L}(f)$, let p be a prime number not dividing $2(x_1^2 - a_1 x_1 x_0 - x_0^2)D$, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p, and let $v \geq 1$ be an integer. Then, we have that $\tau(\boldsymbol{x}; p^v) = 2\,\mathrm{ord}_{\mathfrak{p}^v}(\alpha^2)$.*

Proof. The claim can be derived from more general results on the period of linear recurrences modulo integers [36, Sec. 3.1], but we provide a short proof for completeness.

Define the matrix

$$M = \begin{pmatrix} (x_1 - \beta x_0) & -(x_1 - \alpha x_0) \\ (x_1 - \beta x_0)\alpha & -(x_1 - \alpha x_0)\beta \end{pmatrix},$$

and note that

$$\det(M) = (x_1^2 - a_1 x_1 x_0 - x_0^2)(\alpha - \beta).$$

Since $p \nmid (x_1^2 - a_1 x_1 x_0 - x_0^2)D$, we get that $\alpha - \beta$ and M are invertible modulo $\mathfrak{p}^v$. Hence, by Lemma 3.19, for every integer $n \geq 1$, we have that

$$\begin{cases} x_n \equiv x_0 \\ x_{n+1} \equiv x_1 \end{cases} \pmod{p^v} \iff M \begin{pmatrix} \alpha^n \\ \beta^n \end{pmatrix} \equiv (\alpha - \beta) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \pmod{\mathfrak{p}^v}$$

$$\iff \begin{pmatrix} \alpha^n \\ \beta^n \end{pmatrix} \equiv M^{-1}(\alpha - \beta) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \pmod{\mathfrak{p}^v} \iff \begin{pmatrix} \alpha^n \\ \beta^n \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod{\mathfrak{p}^v}.$$

Therefore, we get that $\tau(\boldsymbol{x}; p^v) = \mathrm{lcm}\Big(\mathrm{ord}_{\mathfrak{p}^v}(\alpha), \mathrm{ord}_{\mathfrak{p}^v}(\beta)\Big)$. Then, since p is odd, the claim follows from Theorem 3.18(iii). □

Lemma 3.21. *Let $n \geq 3$ be an integer. Suppose that p is an odd primitive divisor of ℓ_n such that $\left(\frac{D}{p}\right) = 1$. Then, there exists $\boldsymbol{x} \in \mathcal{L}(f)$ such that $\tau(\boldsymbol{x}; p) = \rho(\boldsymbol{x}; p) = 2n$*

and all the residues of $\boldsymbol{x}$ modulo p are nonzero. Moreover, if n is odd, then there exists $\boldsymbol{y} \in \mathcal{L}(f)$ such that $\tau(\boldsymbol{y}; p) = \rho(\boldsymbol{y}; p) = n$ and all the residues of $\boldsymbol{y}$ modulo p are nonzero.

Proof. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p. Since p is a primitive divisor of ℓ_n, by Lemma 3.16 we have that $\mathrm{ord}_{\mathfrak{p}}(\alpha^2) = n$. Since p is odd and $\left(\frac{D}{p}\right) = 1$, we have that f splits modulo p, that is, there exist integers a, b such that $a \equiv \alpha \pmod{\mathfrak{p}}$ and $b \equiv \beta \pmod{\mathfrak{p}}$. In particular, we have that $\mathrm{ord}_p(a) = \mathrm{ord}_{\mathfrak{p}}(\alpha)$ and $\mathrm{ord}_p(b) = \mathrm{ord}_{\mathfrak{p}}(\beta)$, where ord_p denotes the multiplicative order modulo p. By Lemma 3.18(i)–(ii), we get that $\mathrm{ord}_p(a) = 2n$ or $\mathrm{ord}_p(b) = 2n$. By swapping a and b, we can assume that $\mathrm{ord}_p(a) = 2n$. Let $\boldsymbol{x} \in \mathcal{L}(f)$ such that $x_0 = 1$ and $x_1 = a$. Since $f(a) \equiv 0 \pmod{p}$, it follows easily by induction that $x_n \equiv a^n \pmod{p}$ for every integer $n \geq 0$. Hence, we get that $\tau(\boldsymbol{x}; p) = \rho(\boldsymbol{x}; p) = 2n$ and all the residues of $\boldsymbol{x}$ modulo p are nonzero, as desired.

If n is odd, then by Lemma 3.18(i), we have that $\mathrm{ord}_p(b) = n$. Let $\boldsymbol{y} \in \mathcal{L}(f)$ such that $y_0 = 1$ and $y_1 = b$. Then, reasoning as for $\boldsymbol{x}$, we get that $\tau(\boldsymbol{y}; p) = \rho(\boldsymbol{y}; p) = n$ and all the residues of $\boldsymbol{y}$ modulo p are nonzero, as desired. □

Lemma 3.22. *Let $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 = 1$, let p be a prime number not dividing D, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p, let $v \geq 1$ be an integer, and let $s \geq 0$ be an integer. Then, we have that $x_s \equiv 0 \pmod{p^v}$ if and only if*

$$x_1 \equiv \frac{\beta - \alpha y_s}{1 - y_s} \pmod{\mathfrak{p}^v}, \tag{3.11}$$

and $1 - y_s$ is invertible modulo $\mathfrak{p}^v$, where $y_s = (-\alpha^2)^{-s}$.

Proof. Since $p \nmid D$, we get that $\alpha - \beta$ is invertible modulo $\mathfrak{p}^v$. Hence, employing Lemma 3.19 and the fact that $\beta = -\alpha^{-1}$, it follows easily that $x_s \equiv 0 \pmod{p^v}$ is equivalent to

$$x_1(1 - y_s) \equiv \beta - \alpha y_s \pmod{\mathfrak{p}^v}. \tag{3.12}$$

We claim that if (3.12) holds then $1 - y_s$ is invertible modulo $\mathfrak{p}^v$. Indeed, if $1 - y_s$ is not invertible modulo $\mathfrak{p}^v$, then $y_s \equiv 1 \pmod{\mathfrak{p}}$, and so (3.12) implies that $\alpha - \beta \equiv 0 \pmod{\mathfrak{p}}$, which is a contradiction, since $p \nmid D$. Hence, we have that $1 - y_s$ is invertible modulo $\mathfrak{p}^v$. Consequently, we get that (3.12) is equivalent to (3.11) and $1 - y_s$ being invertible modulo $\mathfrak{p}^v$, as desired. □

Lemma 3.23. *Let $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 = 1$, let p be a prime number not dividing D, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p, let $v \geq 1$ be an integer, let $n = \mathrm{ord}_{\mathfrak{p}}(\alpha^2)$,*

and let s, t be integers such that $0 \leq s < t < 2n$ and $s \equiv t \pmod 2$. Then, we have that $x_s \equiv x_t \pmod{p^v}$ if and only if

$$x_1 \equiv \frac{\beta - \alpha z_{s,t}}{1 - z_{s,t}} \pmod{\mathfrak{p}^v}, \tag{3.13}$$

and $1 - z_{s,t}$ is invertible modulo $\mathfrak{p}^v$, where $z_{s,t} = (-1)^{t+1}\alpha^{-(s+t)}$.

Proof. Since $p \nmid D$, we get that $\alpha - \beta$ is invertible modulo $\mathfrak{p}^v$. Hence, employing Lemma 3.19, we have that $x_s \equiv x_t \pmod{p^v}$ is equivalent to

$$(x_1 - \beta)\alpha^s - (x_1 - \alpha)\beta^s \equiv (x_1 - \beta)\alpha^t - (x_1 - \alpha)\beta^t \pmod{\mathfrak{p}^v}. \tag{3.14}$$

Since $0 \leq s < t < 2n$, $s \equiv t \pmod 2$, and $n = \operatorname{ord}_{\mathfrak{p}}(\alpha^2)$, we have that $1 - \alpha^{t-s}$ is invertible modulo $\mathfrak{p}^v$. Hence, it follows that (3.14) is equivalent to

$$x_1 - \beta \equiv \frac{\beta^t(\beta^{s-t} - 1)}{\alpha^s(1 - \alpha^{t-s})}(x_1 - \alpha) \pmod{\mathfrak{p}^v}. \tag{3.15}$$

Since $\beta = -\alpha^{-1}$ and $s \equiv t \pmod 2$, we get that

$$\frac{\beta^t(\beta^{s-t} - 1)}{\alpha^s(1 - \alpha^{t-s})} = (-1)^{t+1}\alpha^{-(s+t)} = z_{s,t}.$$

Therefore, we have that (3.15) is equivalent to

$$x_1(1 - z_{s,t}) \equiv \beta - \alpha z_{s,t} \pmod{\mathfrak{p}^v}. \tag{3.16}$$

We claim that if (3.16) holds then $1 - z_{s,t}$ is invertible modulo $\mathfrak{p}^v$. Indeed, if $1 - z_{s,t}$ is not invertible modulo $\mathfrak{p}^v$, then $z_{s,t} \equiv 1 \pmod{\mathfrak{p}}$, and so (3.16) implies that $\alpha - \beta \equiv 0 \pmod{\mathfrak{p}}$, which is a contradiction, since $p \nmid D$. Hence, we have that $1 - z_{s,t}$ is invertible modulo $\mathfrak{p}^v$. Consequently, we get that (3.16) is equivalent to (3.13) and $1 - z_{s,t}$ being invertible modulo $\mathfrak{p}^v$, as desired. □

Lemma 3.24. *Let $n \geq 4$ be an even integer. Suppose that p is a high primitive divisor of ℓ_n such that $\left(\frac{D}{p}\right) = -1$, and set $v = \nu_p(\ell_n)$. Then, there exists $\boldsymbol{x} \in \mathcal{L}(f)$ such that $\tau(\boldsymbol{x}; p^v) = 2n$, $x_s \not\equiv x_t \pmod{p^v}$ for all integers s, t with $0 \leq s < t < 2n$ and $s \equiv t \pmod 2$, and all the residues of $\boldsymbol{x}$ modulo p^v are nonzero.*

Proof. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p. Since p is a primitive divisor of ℓ_n and $p^v \mid \ell_n$, from Lemma 3.16 we have that $n = \operatorname{ord}_{\mathfrak{p}}(\alpha^2) = \operatorname{ord}_{\mathfrak{p}^v}(\alpha^2)$.

We claim that there exists an integer c that satisfies

$$c \not\equiv \frac{\beta - \alpha^{2k+1}}{1 - \alpha^{2k}} \pmod{\mathfrak{p}^v}, \tag{3.17}$$

for every integer $k \geq 0$ such that $1 - \alpha^{2k}$ is invertible modulo $\mathfrak{p}^v$. Indeed, we have p^v choices for c modulo $\mathfrak{p}^v$, while the right-hand side of (3.17) takes at most $n = \mathrm{ord}_{\mathfrak{p}^v}(\alpha^2)$ values modulo $\mathfrak{p}^v$. Since p is a high primitive divisor of ℓ_n, we have that $p^v > n$ and so it is possible to choose a value of c with the desired property.

Let $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 = 1$ and $x_1 = c$. Since $\left(\frac{D}{p}\right) = -1$, we have that $p \nmid D$ and that f has no roots modulo p. Hence, also since p is odd, we get that $p \nmid 2(x_1^2 - a_1 x_1 - 1)D$. Consequently, by Lemma 3.20, we get that $\tau(\boldsymbol{x}; p^v) = 2n$, as desired.

Since $n = \mathrm{ord}_{\mathfrak{p}^v}(\alpha^2)$, we have that $\alpha^{2n} \equiv 1 \pmod{\mathfrak{p}^v}$. Consequently, by Lemma 3.17, we get that $\alpha^n \equiv \pm 1 \pmod{\mathfrak{p}^v}$. In fact, since n is even and $n = \mathrm{ord}_{\mathfrak{p}^v}(\alpha^2)$, it follows that $\alpha^n \equiv -1 \pmod{\mathfrak{p}^v}$. Hence, we have that -1 is equal to a power of α^2 modulo $\mathfrak{p}^v$.

Suppose that there exist integers s, t such that $0 \leq s < t < 2n$, $s \equiv t \pmod 2$, and $x_s \equiv x_t \pmod{p^v}$. By Lemma 3.23, we get that

$$c \equiv x_1 \equiv \frac{\beta - \alpha z_{s,t}}{1 - z_{s,t}} \pmod{\mathfrak{p}^v}, \tag{3.18}$$

where $z_{s,t} = (-1)^{t+1}\alpha^{-(s+t)}$ is equal to a power of α^2 modulo $\mathfrak{p}^v$. But then (3.17) and (3.18) are in contradiction. Therefore, it follows that $x_s \not\equiv x_t \pmod{p^v}$ for all integers s, t with $0 \leq s < t < 2n$ and $s \equiv t \pmod 2$.

Suppose that $x_s \equiv 0 \pmod{p^v}$ for some integer $s \geq 0$. Then, by Lemma 3.22, we have that

$$c \equiv x_1 \equiv \frac{\beta - \alpha y_s}{1 - y_s} \pmod{\mathfrak{p}^v}, \tag{3.19}$$

where $y_s = (-\alpha^2)^{-s}$ is equal to a power of α^2 modulo $\mathfrak{p}^v$. But then (3.17) and (3.19) are in contradiction.

Therefore, it follows that $x_s \not\equiv 0 \pmod{p^v}$ for all integers $s \geq 0$. □

For every $\boldsymbol{x} \in \mathcal{L}(f)$ and for all integers $m, d \geq 1$ and r, let us define

$$\rho(\boldsymbol{x}; m, r, d) = \left|\left\{x_n \bmod m : n \geq 0,\ n \equiv r \pmod d\right\}\right|.$$

We need the following lemma.

Lemma 3.25. *Let $m_1, m_2 \geq 1$ be coprime integers, let $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in \mathcal{L}(f)$, and set $\tau_i = \tau(\boldsymbol{x}^{(i)}; m_i)$ for each $i \in \{1, 2\}$. If $\rho(\boldsymbol{x}^{(2)}; m_2) = \tau_2$, then there exists $\boldsymbol{x} \in \mathcal{L}(f)$ such that $\boldsymbol{x} \equiv \boldsymbol{x}^{(i)} \pmod{m_i}$ for each $i \in \{1, 2\}$, and*

$$\rho(\boldsymbol{x}; m_1 m_2) = \frac{\tau_2}{d} \sum_{r=0}^{d-1} \rho(\boldsymbol{x}^{(1)}; m_1, r, d),$$

where $d = \gcd(\tau_1, \tau_2)$.

Proof. Since m_1, m_2 are coprime, we can pick $\boldsymbol{x} \in \mathcal{L}(f)$ such that $x_j \equiv x_j^{(i)} \pmod{m_i}$ for each $i \in \{1, 2\}$ and $j \in \{0, 1\}$. Then, it follows easily by induction that $x_j \equiv x_j^{(i)} \pmod{m_i}$ for each $i \in \{1, 2\}$ and for every integer $j \geq 0$. At this point, the rest of the claim is a consequence of [33, Lemma 6]. □

3.2 Proof of the main theorem

Here is presented the proof of Theorem 3.1. The proof realizes every $n \in \mathbb{Z}^+$ as $\rho(\boldsymbol{x}; m)$ for a suitable $\boldsymbol{x} \in \mathcal{L}(f)$ and modulus m, by exploiting primitive divisors of the Lehmer sequence $(\ell_n)_{n \geq 0}$ attached to α, β. The argument considers essentially three cases and is organized as follows.

The case $n = 4$. This case is treated separately in Lemma 3.26 via explicit choices of x_0, x_1, m, depending on the parity and size of a_1, that yield $4 \in \mathcal{R}(f)$.

The cases n and $2n$ for odd n. If $n \geq 1$ is odd, Lemma 3.9, Lemma 3.15 and Lemma 3.21 allow one to produce a sequence $\boldsymbol{x}$ with $\rho(\boldsymbol{x}; p) = n$ and $\rho(\boldsymbol{x}; p) = 2n$, where p is an odd primitive divisor of ℓ_n. The cases $a_1 \in \{1, 3\}$ with $n = 3$ are not directly covered by the lemmas and are checked directly via Table 3.1. This yields $\{n, 2n\} \subseteq \mathcal{R}(f)$ for every odd n.

The case $2n$ for even n. Since the case 4 is treated separately, we suppose $n \geq 4$. Then, by Lemma 3.13, ℓ_n has a *high* primitive divisor p (apart from finitely many exceptions, which are covered by Table 3.1). We distinguish two more cases, according to the value of $\left(\frac{D}{p}\right)$. If $\left(\frac{D}{p}\right) = 1$, Lemma 3.21 is the only thing that we need. Instead, if $\left(\frac{D}{p}\right) = -1$, by Lemma 3.24 we construct, modulo $m_1 = p^{\nu_p(\ell_n)}$, a sequence $\boldsymbol{x}^{(1)}$ of period $2n$ whose values are evenly distributed among residue classes mod 2 (and mod 4). This is then combined, via Lemma 3.25, with a short auxiliary sequence $\boldsymbol{x}^{(2)}$ (taken from Table 3.1 if $a_1 = 1$, or trivially constructed mod a_1 if $a_1 \geq 2$), yielding a new sequence $\boldsymbol{x}$ with $\rho(\boldsymbol{x}; m_1 m_2) = 2n$, for a suitable choice of m_2. Hence $2n \in \mathcal{R}(f)$ for every even n.

Throughout, the constructions are arranged so that all residues attained are

nonzero, which gives the characterization of $\mathcal{R}^*(f)$; the only exceptions that occur are excluded by the result of Dubickas and Novikas [33].

3.2.1 The case $n = 4$

Lemma 3.26. *There exist $\boldsymbol{x} \in \mathcal{L}(f)$ and $m \in \mathbb{Z}^+$ such that $\rho(\boldsymbol{x}; m) = 4$ and all the residues of $\boldsymbol{x}$ modulo m are nonzero. In particular, we have that $4 \in \mathcal{R}(f)$.*

Proof. If $a_1 = 1$ or $a_1 = 2$, then the claim follows from rows 2 or 13 of Table 3.1, respectively.

If a_1 is odd and $a_1 \geq 3$, then pick $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 = 1$, $x_1 = 2$, and let $m = 2a_1$. Since $2a_1 \equiv 0 \pmod{m}$ and $a_1^2 \equiv a_1 \pmod{m}$, we get that the first terms of $\boldsymbol{x}$ are congruent to

$$1,\ 2,\ 1,\ a_1 + 2,\ a_1 + 1,\ a_1 + 2,\ 1,\ 2,\ \ldots \pmod{m}.$$

Noting that $m \geq 6$ and $a_1 \not\equiv -2, -1, 0, 1 \pmod{m}$, it follows easily that $\tau(\boldsymbol{x}; m) = 6$, $\rho(\boldsymbol{x}; m) = 4$ and that all the residues of $\boldsymbol{x}$ modulo m are nonzero.

If a_1 is even and $a_1 \geq 4$, then pick $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 = 1$, $x_1 = 3$, and let $m = 2a_1$. Since $a_1^2 \equiv 2a_1 \equiv 0 \pmod{m}$, we get that the first terms of $\boldsymbol{x}$ are congruent to

$$1,\ 3,\ a_1 + 1,\ a_1 + 3,\ 1,\ 3,\ \ldots \pmod{m}.$$

Noting that $m \geq 8$ and $a_1 \not\equiv -3, -2, -1, 0, 2 \pmod{m}$, it follows easily that $\tau(\boldsymbol{x}; m) = \rho(\boldsymbol{x}; m) = 4$ and that all the residues of $\boldsymbol{x}$ modulo m are nonzero. □

3.2.2 The cases n and $2n$ for odd n

Now, we prove that $\{n, 2n\} \subseteq \mathcal{R}(f)$ for every odd integer $n \geq 1$. We have that $1 \in \mathcal{R}(f)$, since $\rho(\boldsymbol{x}; 1) = 1$ for every $\boldsymbol{x} \in \mathcal{L}(f)$; and $2 \in \mathcal{R}(f)$, since $\rho(\boldsymbol{x}; 2) = 2$ for every $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 \not\equiv x_1 \pmod{2}$. If $a_1 = 1$, or $a_1 = 3$, then $\{3, 6\} \subseteq \mathcal{R}(f)$ by rows 1 and 3, or 17 and 18, of Table 3.1, respectively. Hence, assume that $n \geq 3$ is an odd integer, and $a_1 \notin \{1, 3\}$ or $n \neq 3$. By Lemma 3.9, we have that ℓ_n has an odd primitive divisor p. Moreover, since n is odd, from Lemma 3.15 it follows that $\left(\frac{D}{p}\right) = 1$. Therefore, from Lemma 3.21 we obtain that $\{n, 2n\} \subseteq \mathcal{R}(f)$.

row n.	a_1	x_0	x_1	m	$\tau(\boldsymbol{x};m)$	$\rho(\boldsymbol{x};m)$	
1	1	0	1	3	8	3	
2	1	1	3	5	4	4	$*$
3	1	1	3	8	12	6	$*$
4	1	1	3	10	12	8	$*$
5	1	1	3	13	28	12	$*$
6	1	1	3	17	36	16	$*$
7	1	1	3	28	48	20	$*$
8	1	1	3	26	84	24	$*$
9	1	1	3	56	48	28	$*$
10	1	1	3	52	84	36	$*$
11	1	1	3	78	168	48	$*$
12	2	1	1	4	4	2	$*$
13	2	1	1	5	12	4	$*$
14	2	1	1	28	12	8	$*$
15	2	1	1	13	28	12	$*$
16	2	1	1	39	56	24	$*$
17	3	1	1	9	6	3	$*$
18	3	1	1	8	12	6	$*$
19	3	1	1	17	16	12	$*$

Table 3.1: Values of $\tau(\boldsymbol{x};m)$ and $\rho(\boldsymbol{x};m)$ for $\boldsymbol{x} \in \mathcal{L}(X^2 - a_1X - 1)$ with initial values x_0, x_1. The symbol $*$ means that all the residues of $\boldsymbol{x}$ modulo m are nonzero.

3.2.3 The case $2n$ for even n

It remains to prove that $2n \in \mathcal{R}(f)$ for every even integer $n \geq 2$. By Lemma 3.26, we have that $4 \in \mathcal{R}(f)$. Hence, we can assume that $n \geq 4$.

If $a_1 = 1$ and $n \in \{4, 6, 8, 10, 12, 14, 18, 24\}$, or $a_1 = 2$ and $n \in \{4, 6, 12\}$, or $a_1 = 3$ and $n = 6$, then $2n \in \mathcal{R}(f)$ by rows 4–11, 14–16, or 19 of Table 3.1, respectively.

Hence, assume that $n \geq 4$ is an even integer, and that $a_1 = 1$ and $n \notin \{4, 6, 8, 10, 12, 14, 18, 24\}$, or $a_1 = 2$ and $n \notin \{4, 6, 12\}$, or $a_1 = 3$ and $n \neq 6$, or $a_1 \geq 4$. Then, by Lemma 3.13, we have that ℓ_n has a high primitive divisor p.

If $\left(\frac{D}{p}\right) = 1$, then Lemma 3.21 implies that $2n \in \mathcal{R}(f)$, as desired. Hence, suppose that $\left(\frac{D}{p}\right) = -1$. Thanks to Lemma 3.24, there exists $\boldsymbol{x}^{(1)} \in \mathcal{L}(f)$ such that $\tau(\boldsymbol{x}^{(1)}; m_1) = 2n$ and $\rho(\boldsymbol{x}^{(1)}; m_1, r, 2) = n$ for each $r \in \{0, 1\}$, where

$m_1 = p^{\nu_p(\ell_n)}$. Note that, since n is even, we also get that $\rho(\boldsymbol{x}^{(1)}; m_1, r, 4) = n/2$ for each $r \in \{0, 1, 2, 3\}$. We define $\boldsymbol{x}^{(2)} \in \mathcal{L}(f)$ and $m_2 \in \mathbb{Z}^+$ as follows. If $a_1 = 1$ then we pick $\boldsymbol{x}^{(2)}$ and m_2 from rows 2 of Table 3.1. If $a_1 \geq 2$ then we let $x_0^{(2)} = 0$, $x_1^{(2)} = 1$, and put $m_2 = a_1$. It follows easily that $\tau(\boldsymbol{x}^{(2)}; m_2) = \rho(\boldsymbol{x}^{(2)}; m_2) \in \{2, 4\}$. Since p is a primitive divisor of ℓ_n, we have that $p \nmid a_1 D$. Hence, we get that m_1, m_2 are coprime integers. (Note that $m_2 = D = 5$ if $a_1 = 1$.) Set $\tau_i = \tau(\boldsymbol{x}^{(i)}; m_i)$ for each $i \in \{1, 2\}$, and let $d = \gcd(\tau_1, \tau_2)$. Note that, since n is even, we have that $d = \tau_2 \in \{2, 4\}$. Therefore, applying Lemma 3.25 we get that there exists $\boldsymbol{x} \in \mathcal{L}(f)$ such that

$$\rho(\boldsymbol{x}; m_1 m_2) = \frac{\tau_2}{d} \sum_{r=0}^{d-1} \rho(\boldsymbol{x}^{(1)}; m_1, r, d) = 1 \cdot \sum_{r=0}^{d-1} \frac{2n}{d} = 2n,$$

so that $2n \in \mathcal{R}(f)$, as desired. The proof that $\mathcal{R}(f) = \mathbb{Z}^+$ is complete.

At this point, note that for each integer $n \geq 4$, and for each integer $n \geq 3$ if $a_1 \geq 2$, the sequence $\boldsymbol{x} \in \mathcal{L}(f)$ and the modulo m that we constructed so that $\rho(\boldsymbol{x}; m) = n$ have the additional property that all the residues of $\boldsymbol{x}$ modulo m are nonzero. This follows from Table 3.1, Lemma 3.21, Lemma 3.26, and Lemma 3.24–Lemma 3.25 (note that $\boldsymbol{x}$ has no zero modulo $m_1 m_2$ since $\boldsymbol{x}^{(1)}$ has no zero modulo m_1). Moreover, if $a_1 \geq 2$, then picking $\boldsymbol{x} \in \mathcal{L}(f)$ with $x_0 = 1$, $x_1 = 1$, and $m = a_1$, we get that $\rho(\boldsymbol{x}; m) = 1$ and all the residues of $\boldsymbol{x}$ modulo m are nonzero. If $a_1 = 2$ or $a_1 \geq 3$, then picking $\boldsymbol{x}$ and m as in row 12 of Table 3.1, or taking $x_0 = 1$, $x_1 = 2$, and $m = a_1$, we get that $\rho(\boldsymbol{x}; m) = 2$ and all the residues of $\boldsymbol{x}$ modulo m are nonzero.

It remains to prove that if $a_1 = 1$ and $n \in \{1, 2, 3\}$ then there exist no $\boldsymbol{x} \in \mathcal{L}(f)$ and $m \in \mathbb{Z}^+$ such that $\rho(\boldsymbol{x}; m) = n$ and all the residues of $\boldsymbol{x}$ modulo m are nonzero. This is done in the last paragraph of [33, p. 122].

The proof is complete.

3.3 Proof of the corollary

We present here the proof of Theorem 3.2.

If $a_1 = 1$ then the result is in fact [33, Corollary 2]. Hence, assume that $a_1 \geq 2$.

We order α and β so that $\alpha = \left(a_1 + \sqrt{D}\right)/2$ and $\beta = \left(a_1 - \sqrt{D}\right)/2$. In particular, note that $\alpha > 1$ and $-1 < \beta < 0$.

Let $k \geq 1$ be an integer. By Theorem 3.1, there exist $\boldsymbol{x} \in \mathcal{L}(f)$ and $m \in \mathbb{Z}^+$ such that $\rho(\boldsymbol{x}; m) = k$ and $x_n \not\equiv 0 \pmod{m}$ for every integer $n \geq 0$. Moreover, by

Lemma 3.19, we have that $x_n = c_1\alpha^n - c_2\beta^n$ for all integers $n \geq 0$, where

$$c_1 = \frac{x_1 - \beta x_0}{\alpha - \beta} \quad \text{and} \quad c_2 = \frac{x_1 - \alpha x_0}{\alpha - \beta}.$$

By eventually replacing $\boldsymbol{x}$ with $-\boldsymbol{x} = (-x_n)_{n\geq 0}$, we can assume that $c_1 > 0$. Furthermore, note that $c_2 \neq 0$. Let $\xi = c_1/m$, so that $\xi \in \mathbb{Q}(\alpha)$. Then, we have that

$$\xi\alpha^n - \frac{x_n}{m} = \frac{c_2\beta^n}{m} \to 0, \tag{3.20}$$

as $n \to +\infty$, since $|\beta| < 1$. Let $r_1, \dots, r_k \in \mathbb{Z}$, with $0 < r_1 < \cdots < r_k < m$, be the residues of $\boldsymbol{x}$ modulo m. From (3.20) it follows easily that the set of limit points of $(\{\xi\alpha^n\})_{n\geq 0}$ is equal to $\{r_1/m, \dots, r_k/m\}$. Hence, we get that $(\{\xi\alpha^n\})_{n\geq 0}$ has exactly k limit points.

The proof is complete.

3.4 Conclusions and outlook

Given a general monic polynomial $f \in \mathbb{Z}[X]$, a natural question arising from the preceding discussion is whether one can determine explicitly the sets $\mathcal{R}(f)$ and $\mathcal{R}^*(f)$. As already observed in [33], this problem appears to be highly nontrivial in full generality. It is therefore reasonable to approach it through suitable restrictions or partial generalizations.

A first natural simplification is to restrict attention to irreducible polynomials. Indeed, for a reducible polynomial f one can relate the structure of $\mathcal{L}(f)$ to that of its factors. To make this precise, for any polynomial $f = -a_0X^k - a_1X^{k-1} - \cdots - a_k$ consider the operator

$$\begin{aligned} \Delta_f : \mathbb{Z}^{\mathbb{N}} &\to \mathbb{Z}^{\mathbb{N}} \\ \boldsymbol{x} &\mapsto \Delta_f(\boldsymbol{x}), \end{aligned}$$

defined by

$$\left(\Delta_f(\boldsymbol{x})\right)_n = -a_0x_{n+k} - a_1x_{n+k-1} - \cdots - a_kx_n,$$

for every $n \geq 0$. It is immediate to verify that, if $a_0 = -1$, then $\boldsymbol{x} \in \mathcal{L}(f)$ if and only if $\Delta_f(\boldsymbol{x}) = \mathbf{0}$, where $\mathbf{0} = (0, 0, 0, \dots)$.

Proposition 3.27. *Let f and h be two monic polynomials with integer coefficients such that $h \mid f$. Then, $\mathcal{L}(h) \subseteq \mathcal{L}(f)$.*

Proof. Let $f = g \cdot h$, and write

$$\begin{aligned} g &= -b_0X^s - b_1X^{s-1} - \cdots - b_s, \\ h &= -c_0X^t - c_1X^{t-1} - \cdots - c_t, \\ f &= -a_0X^{s+t} - a_1X^{s+t-1} - \cdots - a_{s+t}, \end{aligned}$$

where

$$a_v = -\sum_{i+j=v} b_i c_j,$$

for every $v = 0, \ldots, s+t$ and $a_0 = b_0 = c_0 = -1$. We claim that $\Delta_f = \Delta_g \circ \Delta_h$. Indeed, let $\boldsymbol{x} \in \mathbb{Z}^{\mathbb{N}}$, and set $\boldsymbol{y} = \Delta_h(\boldsymbol{x})$. Then, $(\Delta_g(\boldsymbol{y}))_n = -b_0y_{n+s} - b_1y_{n+s-1} - \cdots - b_sy_n$, hence

$$\begin{aligned} (\Delta_g(\boldsymbol{y}))_n &= -\sum_{i=0}^{s} b_i y_{n+s-i} \\ &= -\sum_{i=0}^{s} b_i \left(-\sum_{j=0}^{t} c_j x_{n+s+t-i-j} \right) \\ &= \sum_{i=0}^{s}\sum_{j=0}^{t} b_i c_j x_{n+s+t-i-j} \\ &= \sum_{v=0}^{s+t}\sum_{i+j=v} b_i c_j x_{n+s+t-v} \\ &= -\sum_{v=0}^{s+t} a_v x_{n+s+t-v} = (\Delta_f(\boldsymbol{x}))_n, \end{aligned}$$

for every $n \geq 0$. Hence, if $\boldsymbol{x} \in \mathcal{L}(h)$, then $\Delta_h(\boldsymbol{x}) = \mathbf{0}$ and therefore $\Delta_f(\boldsymbol{x}) = \mathbf{0}$, which proves the claim. □

As a direct consequence, we obtain the following partial extension of Theorem 3.1.

Corollary 3.28. *Let a_1 be a non-zero integer, and let f be a multiple of $X^2 - a_1X - 1$. Then, $\mathcal{R}(f) = \mathbb{Z}^+$. Moreover, $\mathcal{R}^*(f) \supseteq \mathbb{Z}^+ \setminus \{1, 2, 3\}$ if $|a_1| = 1$, and $\mathcal{R}^*(f) = \mathbb{Z}^+$ if $|a_1| \geq 2$.*

However, even in this relatively simple setting, some questions remain open. For instance, in the case $|a_1| = 1$, it would be interesting to determine whether the values $1, 2, 3$ can occur in $\mathcal{R}^*(f)$ when f has additional factors, and more generally how these factors influence the distribution of nonzero residues.

Focusing on irreducible polynomials, a natural next step is to investigate other families of quadratic polynomials, such as $f = X^2 - a_1X + 1$, or more generally $f =$

$X^2 - a_1X - a_2$. At first glance, one might conjecture that $\mathcal{R}(f) = \mathbb{Z}^+$ holds in great generality. However, this expectation must be tempered by simple counterexamples. For instance, if $f = X^2+X+1$, which coincides with the third cyclotomic polynomial Φ_3, then every sequence $\boldsymbol{x} \in \mathcal{L}(f)$ is purely periodic of period 3 (see, e.g., [72, Theorem 3.1]). Consequently, $\mathcal{R}(f) \subseteq \{1, 2, 3\}$. In fact, in this case, we have $\mathcal{R}(f) = \mathcal{R}^*(f) = \{1, 2, 3\}$. Indeed, if $x_0 = x_1 = 1$, then $\rho(\boldsymbol{x}; 3) = \rho^*(\boldsymbol{x}; 3) = 1$ and $\rho(\boldsymbol{x}; 4) = \rho^*(\boldsymbol{x}; 4) = 2$; if $x_0 = 1$ and $x_1 = 2$, then $\rho(\boldsymbol{x}; 6) = \rho^*(\boldsymbol{x}; 6) = 3$.

In this direction, it would be of particular interest to determine whether, for every $n \geq 3$, one has

$$\mathcal{R}(\Phi_n) = \{1, 2, \ldots, n\},$$

or to characterize precisely those cyclotomic polynomials for which such a property holds.

Another possible direction is the study of higher-degree polynomials with simple structure, such as the Tribonacci polynomial $f = X^3 - X^2 - X - 1$. It would be interesting to understand to what extent the techniques developed in this work can be adapted to such settings, and whether analogous results can still be obtained under suitable assumptions.

An important motivation for these investigations comes from the study of the distribution modulo 1 of sequences of the form $(\xi\alpha^n)_{n\geq 0}$, where α is a root of f. The results presented in this chapter show that the structure of the sets $\mathcal{R}(f)$ and $\mathcal{R}^*(f)$ is closely related to the possible distribution patterns of such sequences. A deeper understanding of these sets could therefore lead to further progress in the study of the distribution of powers of algebraic numbers, especially in the case of Pisot numbers.

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