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# Acceleration of the Surface Test Integral Using Vertex Functions

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***Abstract*—In recent years, many papers have reported on the efficient and accurate evaluation of the double surface integrals that arise in the Method of Moments. Most have focused on the careful evaluation of the inner integral and assumed that the outer integral is sufficiently smooth to be easily evaluated numerically. More recently, several papers have appeared where the double integral is treated as a whole using the divergence theorem. These papers show promising results, though their implementation may imply changes to the integration paradigm for the associated codes. Here, instead, we investigate a technique that improves the numerical evaluation of the test integral without affecting the treatment of the source integral. From the integrand of the outer integral, we subtract pairs of quasi-static, so-called vertex functions defined on the source triangle. The approach is compared to standard Gauss-triangle schemes to demonstrate its effectiveness.**

## I. Introduction

The accurate solution of direct or inverse electromagnetics problems using surface integral equation formulations requires cost-effective and accurate numerical evaluation of double surface reaction integrals. Considerable literature exists on the accurate evaluation of the *source* integral [1]–[7] and it is sometimes claimed (but more often, simply implied) that the smoothing provided by the source integral renders the test integral easy to integrate numerically. This assertion does not hold entirely, however, especially if the source and test domains share points in common, as shown in [8] in the case of the magnetic field integral equation (MFIE). The most common approaches for dealing with the source integral are the *singularity subtraction* or *singularity cancellation* methods. For singularity subtraction [1]–[3], a simplified asymptotic form of the integrand is first identified and subtracted from integrand. The resulting difference integrand should be less singular than the original, and the subtracted term should be analytically integrable (or at least easily evaluated numerically); adding the analytical integral to the difference integral restores the original value of the integral. On the other hand, for singularity cancellation [3]–[7], variable transforms are chosen whose Jacobian cancels or regularizes any singularities. More recently, several papers have considered the possibility of treating the double surface integral as a whole [9]–[12], effectively applying the divergence theorem twice. These approaches demonstrate good accuracy but their implementation is non-trivial and may require extensive modification of an existing code.

Here, we focus on the numerical evaluation of the outer test integral and for this we adapt the well-known singularity extraction scheme usually applied to the source integral. That is, we subtract from the integrand of the outer integral (i.e., the source integral, as evaluated by any existing method), the static vertex function pairs as described by the authors in [13], and which sufficiently smooth the integrand to allow us to evaluate the outer integral by a Gauss integration scheme. To reconstruct the original integral, we must add back the integral of the vertex function previously subtracted. To evaluate these integrals we propose a radial-angular scheme for each vertex function, allowing us to evaluate the integrals of the vertex functions very efficiently, since, as can be seen in [13], the vertex functions have a linear radial dependence.

## II. Formulation

The evaluation of the electromagnetic interaction between a pair of triangles in the Method of Moments (MoM) leads to the evaluation of the double surface integral

$$\int_S \int_{S'} F(\mathbf{r},\mathbf{r}')\, dS'\, dS, \qquad (1)$$

where typically $F(\mathbf{r},\mathbf{r}')$takes the form

$$F(\mathbf{r},\mathbf{r}') = t(\mathbf{r}) g(\mathbf{r},\mathbf{r}')\, b(\mathbf{r}'), \qquad (2)$$

and where $t(\mathbf{r})$is either a scalar or a vector component of a testing function, $b(\mathbf{r}')$is similarly defined for a basis function, and $g(\mathbf{r},\mathbf{r}')$ is either a scalar or a scalar component of a vector or dyadic Green's function, with a $\mathcal{O}(|\mathbf{r}-\mathbf{r}'|^{-1})$ or $\mathcal{O}(\nabla|\mathbf{r}-\mathbf{r}'|^{-1})$ singularity.

We first assume the inner integral in (1) evaluates to $I_{S'}(\mathbf{r})$ by any of the procedures described in [1]–[7], and write (1) as

$$I = \int_S I_{S'}\, dS. \qquad (3)$$

The integrand of (3) can exhibit, e.g. in the case of a source and test cell with an edge in common, a non-smooth behavior that makes standard numerical evaluation inefficient, as seen in Fig. 1. To improve the behavior we regularize the integrand by subtracting its static part in the form of three pairs of so-called vertex functions defined on the source triangle. These vertex functions are element-independent functions that contain any (possibly) singular behavior of potentials (or their gradients) near a given vertex or edge for constant or linear sources on a planar element. Thus the integral (3) can be written as

$$I = I_d + \int_S I_{V_1}\, dS + \int_S I_{V_2}\, dS + \int_S I_{V_3}\, dS, \qquad (4)$$

where $I_d$ is the difference integral defined as

$$I_d = \int_S (I_{S'} - I_{V_1} - I_{V_2} - I_{V_3})\, dS, \qquad (5)$$

and where $I_{V_i},\ i = 1, 2, 3$ are pairs of the vertex functions as defined in [13]. The integral of vertex function pairs, i.e.,

$$\int_S I_{V_i} dS, \quad i = 1, 2, 3, \qquad (6)$$

can be evaluated easily using a radial-angular scheme projecting from each of the vertices of the source triangle.

## III. Preliminary Numerical Results

To demonstrate the accuracy of the proposed scheme, we analyze the convergence behavior of the test scalar integral (3) and the difference integral (5). We consider a pair of triangles with a common edge. For both cases the standard Gauss-triangle (GT) quadrature scheme [14]. The reference for each of the plot is evaluated with the highest number of points we have available for this scheme (166 points). The source integral is evaluated using the Radial-Angular transformation and GT quadrature scheme with a number of points able to provide machine precision [3].

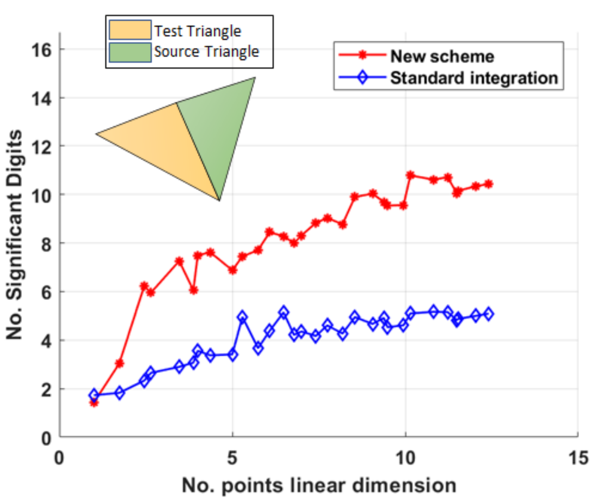


Fig. 1. Near-field convergence of test integrals. Inset: Orientation of a pair of triangle elements in space.

Figure 1 shows the convergence of the test integrals, that is, the behavior of the integrals with increasing number of surface sample points for the boundary integrals, comparing the proposed approach (red line) and the standard integration scheme, i.e. applying the Gauss-Triangle quadrature scheme (blue line). It is evident that the proposed method shows better convergence, reaching more than 10 significant digits using about 10 points per linear dimension.

## IV. Conclusions and Perspectives

Preliminary results show good accuracy and efficiency of the method.The next step of this research activity is to examine the possibility of optimizing the radial-angular scheme for the vertex functions to further smooth the resulting integrands and hence accelerate their convergence.

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