Exploring linear recurrent sequences: Theory and Applications

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The aim of this dissertation is to study linear recurrent sequences of any degree within the context of algebraic structures, as well as to examine certain aspects of second-order recurrences. Specifically, the study focuses on the Lucas sequence, it introduces a new definition of Lucas atoms, and it investigates the Fibonacci sequence. Additionally, the research presents results regarding the algebraic independence of certain Hurwitz-type continued fractions and their convergents.

The first part of this manuscript is about linear recurrent sequences. Let R be an associative, commutative ring having characteristic zero and unity, and let $\mathcal{S}(R)$ be the set of sequences of elements belonging to the ring R and $\mathcal{W}(R) \subset \mathcal{S}(R)$ the set of linear recurrent sequences. Both sets can be equipped with several operations giving them interesting algebraic structures. In particular, if R is a field, it is immediate to see that the element-wise sum or product (also called Hadamard product) of two linear recurrent sequences is still a linear recurrent sequence. Cerruti and Vaccarino proved this in the general case where R is a ring, showing that $\mathcal{W}(R)$ is an $R-$algebra and also giving explicitly the characteristic polynomial of the element-wise sum and Hadamard product of two linear recurrent sequences [6]. In the same manner, $\mathcal{W}(R)$ equipped with the element-wise sum and the convolution product (or Cauchy product) has been deeply studied. In particular, $\mathcal{W}(R)$ is still an $R-$algebra and the characteristic polynomial of the convolution product between two linear recurrent sequences can be explicitly found [6]. The convolution product of linear recurrent sequences has been explored also from a combinatorial point of view [1] and over finite fields [11]. Another important operation between sequences is the binomial convolution (called also Hurwitz product). In [12], Keigher introduced in a systematic way the Hurwitz series ring. This has also been explored by other several authors [4, 5, 13]. However, there are few results when focusing on linear recurrent sequences [14]. We expose our results about the algebraic structure of linear recurrent sequences considering in particular the Hurwitz product and the Newton product. Moreover, we study whether isomorphisms exist between these structures.

The Lucas sequence is a specific second-order linear recurrence sequence from which Lucas polynomials are defined. Lucas atoms have been introduced by Sagan and Tirrell [17], we introduce them in a new and different perspective, providing straightforward proofs for their main properties. Specifically, we revisit some of the main properties of Lucas atoms, obtaining them with elementary proofs. The p-adic valuations of integer sequences is a well studied topic, in particular the case of Lucas sequences has been deepened by several authors (see, e.g., [3, 15, 18, 21]). We present our results about the p-adic valuations of Lucas atoms. In [17], the authors dealt with Lucas atoms and some divisibility properties by $p = 2, 3$. They left open, addressing it as a hard problem, the extension of these results to arbitrary primes. We solve this problem and we completely characterize the p-adic valuations of Lucas atoms. Finally, we exploit the results on the p-adic valuations of Lucas atoms to prove that the sequence of Lucas atoms is not holonomic, i.e., it does not satisfy any recurrence relation, also considering coefficients being polynomials, contrarily to the Lucas sequence.

A particular case of Lucas sequence is the Fibonacci sequence whose elements are called Fibonacci numbers. The theorem of Zeckendorf asserts that any positive integer can be expressed in a unique manner as the sum of one or more distinct non-consecutive Fibonacci numbers [22]. This kind of representation of integer sequences has been studied in several works. In particular, the Zeckendorf representation of numbers of the form f_{kn}/f_n, f_n^2/d and L_n^2/d, where L_n are the Lucas numbers and d is a Lucas or Fibonacci number, have been studied by Filipponi and Freitag [7, 9]. Whereas, the Zeckendorf representation of numbers of the form mf_n have been analyzed by Filipponi, Hart, and Sanchis [8, 10]. Filipponi determined the Zeckendorf representation of mf_nf_{n+k} and mL_nL_{n+k} for $m \in \{1,2,3,4\}$, see e.g. [8]. For all integers a and $m \geq 1$ with $\gcd(a,m) = 1$, let $(a^{-1} \bmod m)$ denote the least positive multiplicative inverse of a modulo m, that is, the unique $b \in \{1,\ldots,m\}$ such that $ab \equiv 1 \pmod{m}$. In [16], Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of $(2^{-1} \bmod f_n)$, for every positive integer n that is not divisible by 3. We extend their result by determining the Zeckendorf representation of the multiplicative inverse of a modulo f_n, for every fixed integer $a \geq 3$ and every positive integer n with $\gcd(a, f_n) = 1$.

The last part of this dissertation focuses on the area of algebraic independence. The transcendence of π and that of e has been known since the end of the 19th century, but the question of the algebraic independence of these two numbers over $\mathbb{Q}$ has still not been answered. It concerns the exclusion of the existence of a non-identical vanishing polynomial $P(X,Y)$ with rational coefficients such that $P(\pi, e) = 0$. The theorem of Lindemann - Weierstrass (1885), from which the transcendence of π and of e can be derived, is the beginning of a general theory on algebraic independence of complex numbers over $\mathbb{Q}$. In one of its equivalent formulations this theorem states that in the case of the *linear* independence of algebraic numbers $\alpha_1,\ldots,\alpha_n$ over $\mathbb{Q}$, the numbers $e^{\alpha_1},\ldots,e^{\alpha_n}$ are *algebraically* independent over $\mathbb{Q}$ [19]. An additional significant achievement is the theorem of Gelfond-Schneider that states the transcendence of α^β when α and β are algebraic over $\mathbb{Q}$, assuming that $\alpha \neq 0,1$ and $\beta \notin \mathbb{Q}$ [19]. Another important result is Baker's Theorem on linear forms of logarithms that states that given $\alpha_1,\ldots,\alpha_n$ algebraic numbers different from zero such that $\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over the rational numbers, then the numbers $1,\log\alpha_1,\ldots,\log\alpha_n$ are linearly independent over the field of all algebraic numbers [2]. In terms of algebraic independence of continued fractions, Tanaka [20] gave a necessary and sufficient condition for the values of $\Theta(x,a,q)$ to be algebraically independent, where $\Theta(x,a,q)$ is a sort of q-hypergeometric series. In particular, he showed under which conditions the values of the continued fractions obtained when $x = a$, namely $\Theta(a,q)$, are algebraically dependent. The last chapter of this manuscript contains results on algebraic independence or dependence of number sets. There will be presented a criterion and its variants such that, starting from a set of known algebraic independent numbers we get a new one, where the numbers in both sets satisfy a system of polynomial equations. Moreover, this criterion will be applied to prove the algebraic independence of certain Hurwitz-type continued fractions and their convergents.

References

[1] M. Abrate, S. Barbero, U. Cerruti, and N. Murru. Colored compositions, invert operator and elegant compositions with the “black tie”. *Discrete Mathematics*, 335:1–7, 2014.

[2] A. Baker. *Transcendental number theory*. Cambridge university press, 2022.

[3] C. Ballot. The p-adic valuation of Lucas sequences when p is a special prime, Fibonacci quart. 57 (2019), no. 3, 265–275. *Fibonacci Quart*, 57(3):265–275, 2019.

[4] S. Barbero, U. Cerruti, and N. Murru. Some combinatorial properties of the Hurwitz series ring. *Ricerche di Matematica*, 67:491–507, 2018.

[5] A. Benhissi. Ideal structure of Hurwitz series rings. *Beiträge zur Algebra und Geometrie*, 48(1):251–256, 2007.

[6] U. Cerruti and F. Vaccarino. R-algebras of linear recurrent sequences. *Journal of Algebra*, 175(1):332–338, 1995.

[7] P. Filipponi and H. T. Freitag. The Zeckendorf representation of {F kn/F n}. In *Applications of Fibonacci Numbers: Volume 5 Proceedings of ‘The Fifth International Conference on Fibonacci Numbers and Their Applications’, The University of St. Andrews, Scotland, July 20–July 24, 1992*, pages 217–219. Springer, 1993.

[8] P. Filipponi and E. L. Hart. The Zeckendorf decomposition of certain Fibonacci-Lucas products. *Fibonacci Quarterly*, 36:240–247, 1998.

[9] H. T. Freitag and P. Filipponi. On the f-representation of integral sequences [fn2-d] and [ln2-d] where d is either a fibonacci or a lucas number. *Fibonacci Quarterly*, 27(3):276–282, 1989.

[10] E. Hart and L. Sanchis. On the occurrence of f_n in the Zeckendorf decomposition of nf_n. *Fibonacci Quart*, 37:21–33, 1999.

[11] P. Haukkanen. On a convolution of linear recurring sequences over finite fields. *J. Algebra*, 149(1):179–182, 1992.

[12] W. F. Keigher. On the ring of Hurwitz series. *Communications in Algebra*, 25(6):1845–1859, 1997.

[13] W. F. Keigher and F. L. Pritchard. Hurwitz series as formal functions. *Journal of Pure and Applied Algebra*, 146(3):291–304, 2000.

[14] V. L. Kurakin. Convolution of linear recurrent sequences. *Russian Mathematical Surveys*, 48(4):249, 1993.

[15] T. Lengyel. The order of the Fibonacci and Lucas numbers. *Fibonacci Quart*, 33(3):234–239, 1995.

[16] B. Prempreesuk, P. Noppakaew, and P. Pongsriiam. Zeckendorf representation and multiplicative inverse of f_m mod f_n. *Int. J. Math. Comput. Sci*, 15(1):17–25, 2020.

[17] B. E. Sagan and J. Tirrell. Lucas atoms. *Advances in Mathematics*, 374:107387, 2020.

[18] C. Sanna et al. The p-adic valuation of Lucas sequences. *Fibonacci Quart*, 54(2):118–124, 2016.

[19] A. B. Shidlovskii. *Transcendental numbers*, volume 12. Walter de Gruyter, 2011.

[20] T.-a. Tanaka. Conditions for the algebraic independence of certain series involving continued fractions and generated by linear recurrences. *Journal of Number Theory*, 129(12):3081–3093, 2009.

[21] M. Ward. The linear p-adic recurrence of order two. *Illinois Journal of Mathematics*, 6(1):40–52, 1962.

[22] É. Zeckendorf. Representations des nombres naturels par une somme de nombres de Fibonacci on de nombres de lucas. *Bulletin de La Society Royale des Sciences de Liege*, pages 179–182, 1972.