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**RESEARCH ARTICLE**

# Rank two bundles on $\mathbf{P}^n$ with isolated cohomology

**F. Malaspina<sup>1</sup> | A. P. Rao<sup>2</sup>**
<sup>1</sup>Politecnico di Torino, Torino, Italy

<sup>2</sup>University of Missouri-St.Louis, St.Louis, Missouri, USA

**Correspondence**

 F. Malaspina, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.  
Email: [francesco.malaspina@polito.it](mailto:francesco.malaspina@polito.it)
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**Abstract**

The purpose of this paper is to study minimal monads associated to a rank two vector bundle  $\mathcal{E}$  on  $\mathbf{P}^n$ . In particular, we study situations where  $\mathcal{E}$  has  $H_*^i(\mathcal{E}) = 0$  for  $1 < i < n - 1$ , except for one pair of values  $(k, n - k)$ . We show that on  $\mathbf{P}^8$ , if  $H_*^3(\mathcal{E}) = H_*^4(\mathcal{E}) = 0$ , then  $\mathcal{E}$  must be decomposable. More generally, we show that for  $n \geq 4k$ , there is no indecomposable bundle  $\mathcal{E}$  for which all intermediate cohomology modules except for  $H_*^1, H_*^k, H_*^{n-k}, H_*^{n-1}$  are zero.

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## 1 | INTRODUCTION

It has been difficult to disprove the existence of an indecomposable rank two bundle  $\mathcal{E}$  on  $\mathbf{P}^n$  for large  $n$ . Most known results have been obtained by imposing other conditions on  $\mathcal{E}$  to show that  $\mathcal{E}$  cannot exist or must be split. For example, the so-called Babylonian condition which requires  $\mathcal{E}$  to be extendable to  $\mathbf{P}^{n+m}$  for every  $m$  has been studied by a number of people including Barth and van de Ven [2] and Coanda and Trautmann [3]. Numerical criteria that force splitting are found again in Barth and van de Ven, where for a normalized rank two bundle with second Chern class  $a$  and with splitting type  $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(b)$  on the general line  $l$ , a function  $f(a, b)$  is found such that if  $n > f(a, b)$ , then a bundle on  $\mathbf{P}^n$  with these invariants must be split.

Cohomological criteria for forcing the splitting of  $\mathcal{E}$  start with Horrocks [7]. If  $S$  is the polynomial ring corresponding to  $\mathbf{P}^n$ , then  $H_*^i(\mathcal{E})$  (defined as  $\bigoplus_{\nu} H^i(\mathbf{P}^n, \mathcal{E}(\nu))$ ) is an  $S$ -module. The intermediate cohomology modules  $H_*^i(\mathcal{E})$ ,  $1 \leq i \leq n - 1$  are all graded modules of finite length and there is a strong relationship between  $\mathcal{E}$  and its intermediate cohomology modules. He shows that if  $H_*^i(\mathcal{E}) = 0$  for all  $i$  with  $1 \leq i \leq n - 1$ , then  $\mathcal{E}$  is split. Moreover, Horrocks in [7] established

that a vector bundle on  $\mathbf{P}^n$  is determined up to isomorphism and up to a sum of line bundles (i.e., up to stable equivalence) by its collection of intermediate cohomology modules and also a certain collection of extension classes involving these modules. This correspondence has been generalized to any Arithmetically Cohen-Macaulay (ACM) varieties in [12]. The Syzygy Theorem ([5, 6]) shows that for a rank two bundle  $\mathcal{E}$ , it is enough to know that  $H_*^1(\mathcal{E}) = 0$  to force splitting. In [14], it is shown that for an indecomposable rank two bundle on  $\mathbf{P}^n$ , in addition to  $H_*^1(\mathcal{E})$  and  $H_*^{n-1}(\mathcal{E})$  being nonzero, some intermediate cohomology module  $H_*^k(\mathcal{E})$  ( $1 < k < n - 1$ ) (and hence also  $H_*^{n-k}(\mathcal{E})$ ) must be nonzero. Various calculations in [13] and [14] show that there are limitations on the module structure of  $H_*^1(\mathcal{E})$  and  $H_*^2(\mathcal{E})$  for some values of  $n$ .

In this paper, we study situations where a rank two bundle  $\mathcal{E}$  on  $\mathbf{P}^n$  has  $H_*^i(\mathcal{E}) = 0$  for  $1 < i < n - 1$ , except for one pair of values  $(k, n - k)$ . We describe the minimal monads associated to  $\mathcal{E}$ . We show that on  $\mathbf{P}^8$ , if  $H_*^3(\mathcal{E}) = H_*^4(\mathcal{E}) = 0$ , then  $\mathcal{E}$  must be decomposable. More generally, we show that for  $n \geq 4k$ , there is no indecomposable bundle  $\mathcal{E}$  for which all intermediate cohomology modules except for  $H_*^1, H_*^k, H_*^{n-k}, H_*^{n-1}$  are zero. The proof utilizes the space between  $k$  and  $n - k$  when  $n \geq 4k$  for making cohomological computations.

## 2 | MONADS FOR RANK TWO VECTOR BUNDLES ON $\mathbf{P}^n$

Let  $\mathcal{E}$  be an indecomposable rank two vector bundle on  $\mathbf{P}^n$  over an algebraically closed field of characteristic different from two. If  $S$  is the polynomial ring on  $n + 1$  variables, let  $N_i = H_*^i(\mathcal{E}) = \bigoplus_{\nu} H^i(\mathcal{E}(\nu))$  be the finite length graded  $S$ -module over  $S$ , for  $1 \leq i \leq n - 1$ . By the Syzygy Theorem, both  $N_1$  and  $N_{n-1}$  are nonzero modules. Horrocks ([8]) gives a brief description of the construction of a minimal monad for a bundle  $\mathcal{E}$  of any rank on  $\mathbf{P}^n$  by “killing both  $H_*^1$  and  $H_*^{n-1}$ .” Barth and Hulek ([1]) use this idea to construct (with more detail)  $l$ - $m$  minimal monads for a bundle  $\mathcal{E}$  on  $\mathbf{P}^n$ , where only  $H_{\geq l}^1(\mathcal{E})$  and  $H_{< -m-n}^{n-1}(\mathcal{E})$  are killed. Horrocks’ construction, which we use below, is obtained when both  $l$  and  $m$  are very negative.

The monad is a complex

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \rightarrow 0,$$

where  $\mathcal{P}$  is a bundle with  $H_*^i(\mathcal{P}) = 0$  for  $i = 1$  and  $i = n - 1$ , and where  $\mathcal{A}, \mathcal{B}$  are free bundles. Let  $\mathcal{G}$  be kernel  $\beta$ . We have two sequences

$$\begin{aligned} 0 \rightarrow \mathcal{G} \rightarrow \mathcal{P} \rightarrow \mathcal{B} \rightarrow 0, \\ 0 \rightarrow \mathcal{A} \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow 0, \end{aligned} \tag{1}$$

from which we see that  $H_*^i(\mathcal{E}) = H_*^i(\mathcal{G})$  for  $1 \leq i \leq n - 2$ , while  $H_*^{n-1}(\mathcal{G}) = 0$ , and  $H_*^i(\mathcal{E}) = H_*^i(\mathcal{P})$  for  $2 \leq i \leq n - 2$ , while  $H_*^1(\mathcal{P}) = H_*^{n-1}(\mathcal{P}) = 0$ .

The minimality of the complex means that the rank of  $\mathcal{B}$  equals the number of generators of  $H_*^1(\mathcal{G}) = N_1$  and the rank of  $\mathcal{A}^\vee$  equals the number of generators of  $H_*^1(\mathcal{E}^\vee)$ . When the rank of the bundle  $\mathcal{E}$  equals 2, we find that  $\mathcal{A}$  and  $\mathcal{B}$  have the same rank.

The two sequences give rise to

$$\begin{aligned} 0 \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{P} \rightarrow \mathcal{B} \otimes \mathcal{P} \rightarrow S^2 \mathcal{B} \rightarrow 0, \\ 0 \rightarrow S^2 \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{E} \rightarrow 0. \end{aligned} \tag{2}$$

(Since the characteristic of the field is different from 2, we can assume that  $S^2$  commutes with duals.)

**Lemma 2.1.** *If  $H_*^2(\mathcal{E}) = 0$ , then  $H_*^1(\wedge^2\mathcal{P})$  and  $H_*^{n-1}(\wedge^2\mathcal{P})$  are nonzero. If  $H_*^l(\mathcal{E}) = 0$  for some  $l$ , with  $2 \leq l \leq n-2$ , then  $H_*^l(\wedge^2\mathcal{P}) = 0$ .*

*Proof.* See [11, Theorem 2.2] for the first part. Next, suppose  $H_*^l(\mathcal{E}) = 0$  for some  $l$ , with  $2 \leq l \leq n-2$ . So,  $N_l = N_{n-l} = 0$  by Serre duality. In particular,  $\mathcal{G}$  and  $\mathcal{P}$  have  $H_*^l = 0$  as well. It follows from Equation (2), that  $H_*^l(\wedge^2\mathcal{G}) = 0$  and hence  $H_*^l(\wedge^2\mathcal{P}) = 0$ .  $\square$

**Lemma 2.2.** *Let  $2 \leq t \leq n-2$ . Let  $A = H_*^0(\mathcal{A})$ ,  $B = H_*^0(\mathcal{B})$ . There is an exact sequence*

$$A \otimes N_t \rightarrow H_*^t(\wedge^2\mathcal{P}) \rightarrow B \otimes N_t,$$

*which is injective on the left if  $t \geq 3$  and  $N_{t-1} = 0$ , and is surjective on the right if  $t \leq n-3$  and  $N_{t+1} = 0$ .*

*Proof.* Break up the first sequence in 2 as  $0 \rightarrow \wedge^2\mathcal{G} \rightarrow \wedge^2\mathcal{P} \rightarrow \mathcal{D} \rightarrow 0$ ,  $0 \rightarrow \mathcal{D} \rightarrow B \otimes \mathcal{P} \rightarrow S^2\mathcal{B} \rightarrow 0$ . We get long exact sequences

$$H_*^{t-1}(\mathcal{D}) \rightarrow H_*^t(\wedge^2\mathcal{G}) \rightarrow H_*^t(\wedge^2\mathcal{P}) \rightarrow H_*^t(\mathcal{D}) \rightarrow H_*^{t+1}(\wedge^2\mathcal{G}),$$

where  $H_*^t(\mathcal{D}) \cong B \otimes N_t$  (always) and  $H_*^{t-1}(\mathcal{D}) \cong B \otimes N_{t-1}$  provided  $t \geq 3$ . Likewise break up the second sequence as  $0 \rightarrow S^2\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{G} \rightarrow \mathcal{C} \rightarrow 0$ ,  $0 \rightarrow \mathcal{C} \rightarrow \wedge^2\mathcal{G} \rightarrow \wedge^2\mathcal{E} \rightarrow 0$ . We see that  $H_*^i(\wedge^2\mathcal{G}) \cong H_*^i(\mathcal{C})$  for  $i = t, t+1$ ,  $H_*^t(\mathcal{C}) \cong A \otimes N_t$  and when  $t \leq n-3$ ,  $H_*^{t+1}(\mathcal{C}) \cong A \otimes N_{t+1}$ .  $\square$

Moreover, when  $H_*^l(\mathcal{E}) = 0$  for some  $l$ , with  $2 \leq l \leq n-3$ , from Equation (2) since  $H_*^{l+1}(S^2\mathcal{A}) = H_*^{l+2}(\mathcal{A} \otimes \mathcal{G}) = H_*^l(\wedge^2\mathcal{E}) = H_*^{l+1}(\wedge^2\mathcal{E}) = 0$ , we get  $H_*^{l+1}(\wedge^2\mathcal{G}) \cong H_*^{l+1}(\mathcal{A} \otimes \mathcal{G})$ . Since  $H_*^{l-1}(S^2\mathcal{B}) = H_*^l(B \otimes \mathcal{P}) = 0$ ,  $H_*^{l+1}(\wedge^2\mathcal{G}) \hookrightarrow H_*^{l+1}(\wedge^2\mathcal{P})$ . From

$$0 \rightarrow \mathcal{D} \rightarrow B \otimes \mathcal{P} \rightarrow S^2\mathcal{B} \rightarrow 0,$$

we obtain  $H_*^{l+1}(B \otimes \mathcal{P}) \cong H_*^{l+1}(\mathcal{D})$  giving an exact sequence

$$0 \rightarrow A \otimes N_{l+1} \rightarrow H_*^{l+1}(\wedge^2\mathcal{P}) \rightarrow B \otimes N_{l+1}, \quad (3)$$

where  $A = H_*^0(\mathcal{A})$ ,  $B = H_*^0(\mathcal{B})$ . Notice that the sequence is exact on the right if  $N_{l+2} = 0$ .

Likewise, if  $H_*^l(\mathcal{E}) = 0$  for some  $l$ , with  $3 \leq l \leq n-2$ , we get the exact sequence

$$A \otimes N_{l-1} \rightarrow H_*^{l-1}(\wedge^2\mathcal{P}) \rightarrow B \otimes N_{l-1} \rightarrow 0. \quad (4)$$

Notice that the sequence is exact on the right if  $N_{l-2} = 0$ .

The following proposition is a typical one that shows that a minimal monad for a rank two bundle is built very minimally out of the cohomological data for  $\mathcal{E}$ . Other examples of such a result can be found in [15] and [13]. Decker ([4]) has conjectured such a minimality for rank two bundles on  $\mathbf{P}^4$ .

**Proposition 2.3.** *Suppose that  $\mathcal{E}$  is a nonsplit rank two bundle on  $\mathbf{P}^n$  ( $n \geq 6$ ), with  $H_*^l(\mathcal{E}) = 0$  for some  $l$  with  $2 \leq l \leq n - 2$ . Then in the minimal monad for  $\mathcal{E}$ , the bundle  $\mathcal{P}$  has no line bundle summands.*

*Proof.* Note that the statement is vacuous for  $n = 4, 5$ , since  $\mathcal{E}$  will be split by [14]. So, assume that  $n \geq 6$  and that  $\mathcal{E}$  satisfies  $H_*^l(\mathcal{E}) = 0$  for some  $2 \leq l \leq n - 2$ . By [14], there must also be a  $j$  such that  $H_*^j(\mathcal{E}) \neq 0$  for some  $2 \leq j \leq n - 2$ .

We may choose  $l$  to be the lowest value with  $H_*^l(\mathcal{E}) = 0$  and let us suppose that  $l \geq 3$ . Then  $H_*^{l-1}(\mathcal{E}) = N_{l-1} \neq 0$ . Consider the exact sequence using Lemma 2.2 (with  $t = l - 1$ )

$$A \otimes N_{l-1} \rightarrow H_*^{l-1}(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_{l-1} \rightarrow 0.$$

Now if  $\mathcal{P} \cong \mathcal{Q} \oplus \mathcal{O}_{\mathbf{P}}(a)$ , then  $H_*^{l-1}(\wedge^2 \mathcal{P}) \cong H_*^{l-1}(\wedge^2 \mathcal{Q}) \oplus [S(a) \otimes N_{l-1}]$ , where  $N_{l-1} \neq 0$ . The map  $S(a) \otimes N_{l-1} \rightarrow B \otimes N_{l-1}$  in the sequence is induced by the map  $\mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}_k \xrightarrow{\beta_2 \otimes I} B \otimes \mathcal{P}_k$ , where  $\beta = [\beta_1, \beta_2]$  in the monad for  $\mathcal{E}$ .

The map  $A \otimes N_{l-1} \rightarrow S(a) \otimes N_{l-1}$  is induced by the map  $\mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{G} \hookrightarrow \wedge^2 \mathcal{P} \rightarrow \mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}$ , hence by  $\mathcal{A} \otimes \mathcal{P} \xrightarrow{\alpha_2 \otimes I} \mathcal{L} \otimes \mathcal{P}$  if  $\alpha = [\alpha_1, \alpha_2]^T$  in the monad.

The sequence above now reads

$$A \otimes N_{l-1} \xrightarrow{\begin{bmatrix} * \\ \alpha_2 \otimes I \end{bmatrix}} H_*^{l-1}(\wedge^2 \mathcal{Q}) \oplus [S(a) \otimes N_{l-1}] \xrightarrow{[* , \beta_2 \otimes I]} B \otimes N_{l-1} \rightarrow 0.$$

If we tensor the sequence by the quotient  $k = S/(X_0, \dots, X_{n+1})$ , since the matrix  $\beta_2$  is a minimal matrix,  $(\beta_2 \otimes I) \otimes k = 0$ , hence  $[S(a) \otimes N_{l-1} \otimes k]$  is inside the kernel of  $[* , \beta_2 \otimes I] \otimes k$ . By exactness,  $S(a) \otimes N_{l-1} \otimes k$  is inside the image of  $(\alpha_2 \otimes I) \otimes k$ , which is not possible since  $\alpha_2$  is also a minimal matrix.

It remains to study the case where  $l = 2$ . There is a value  $l'$  between 3 and  $n - 3$  for which  $H_*^{l'}(\mathcal{E}) = N_{l'} \neq 0$  and  $H_*^{l'+1}(\mathcal{E}) = 0$ . We now have an exact sequence of nonzero  $S$ -modules

$$A \otimes N_{l'} \rightarrow H_*^{l'}(\wedge^2 \mathcal{P}) \rightarrow B \otimes N_{l'} \rightarrow 0,$$

and we repeat the earlier argument to get a contradiction. □

**Definition 2.4.** A rank two bundle  $\mathcal{E}$  on  $\mathbf{P}^n$ ,  $n \geq 6$ , will be said to have isolated cohomology of type  $(n, k)$  if there exists an integer  $k$ ,  $1 < k \leq \frac{n}{2}$ , with  $H_*^k(\mathcal{E})$  and  $H_*^{n-k}(\mathcal{E})$  nonzero modules, and  $H_*^i(\mathcal{E}) = 0$  for  $i \neq 1, k, n - k, n - 1$ .

*Remark 2.5.* By Lemma 2.1, we get that if  $\mathcal{E}$  has isolated cohomology of type  $(n, k)$ , then  $H_*^i(\wedge^2 \mathcal{P}) = 0$  for  $i \neq 1, k, n - k, n - 1$ .

A special case in the definition is when the middle cohomology is not zero, that is, of type  $(n, k)$ , where  $n$  is even, equal to  $2k$ , and the only nonzero cohomology modules are  $H_*^1(\mathcal{E}), H_*^k(\mathcal{E}), H_*^{n-1}(\mathcal{E})$ .

Note that the conditions that  $H_*^1(\mathcal{E}), H_*^{n-1}(\mathcal{E})$  are both nonzero for an indecomposable rank two bundle follow from the Syzygy Theorem. In [14], it is proved that for an indecomposable rank two bundle on  $\mathbf{P}^n$ ,  $n \geq 4$ , at least one cohomology module  $H_*^l(\mathcal{E})$  must be nonzero with  $1 < l < n - 1$ . The reason  $n$  is chosen to be  $\geq 6$  in the definition is that first, the definition is vacuous for  $n = 2, 3$  and second, for  $n = 4, 5$ ,  $k$  must be 2, and the definition made is always satisfied by any possible indecomposable rank two bundle on  $\mathbf{P}^4$  or  $\mathbf{P}^5$ , and hence imposes no restrictions.

Let  $\mathcal{P}_k(N)$  be the  $k$ th syzygy bundle of the finite length module  $N$ . By this, we mean that in a minimal free resolution for  $N$  over the polynomial ring  $S$ :

$$0 \rightarrow L_{n+1} \xrightarrow{f_{n+1}} L_n \rightarrow \dots \rightarrow L_{k+1} \xrightarrow{f_{k+1}} L_k \rightarrow \dots \rightarrow L_1 \xrightarrow{f_1} L_0 \rightarrow N \rightarrow 0.$$

$P_k(N)$  will denote the image of  $f_{k+1}$  and  $\mathcal{P}_k(N)$  will denote the sheafification of  $P_k(N)$ . Hence,  $H_*^k(\mathcal{P}_k(N)) = N$ , with  $H_*^i(\mathcal{P}_k(N)) = 0$  when  $i \neq 0, k, n$ . According to [7], if  $\mathcal{P}$  is any bundle on  $\mathbf{P}^n$  with the property that  $H_*^k(\mathcal{P}) = N$  and  $H_*^i(\mathcal{P}) = 0$  when  $i \neq 0, k, n$ , then  $\mathcal{P} \cong \mathcal{P}_k(N) \oplus \mathcal{F}$  where  $\mathcal{F}$  is a direct sum of line bundles.

**Lemma 2.6.** *Let  $\mathcal{P}$  be a vector bundle on  $\mathbf{P}^n$  with nonzero cohomology modules  $H_*^k(\mathcal{P}) = N$ ,  $H_*^l(\mathcal{P}) = M$  for  $1 \leq k < l \leq n - 1$ , and with  $H_*^i(\mathcal{P}) = 0$  when  $i \neq 0, k, l, n$ . Then there is an exact sequence*

$$0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_l(M) \rightarrow 0,$$

where  $\mathcal{F}$  is some free bundle.

*Proof.* This too follows from [7]. Letting  $P$  denote  $H_*^0(\mathcal{P})$ , form an exact sequence (by partially resolving  $P^\vee$ )

$$0 \rightarrow P \rightarrow L_k \rightarrow L_{k-1} \rightarrow \dots \rightarrow L_1 \rightarrow A \rightarrow N \rightarrow 0,$$

where  $A$  is not a free module. Compare this with a truncated minimal free resolution of  $N$ :

$$0 \rightarrow P_k(N) \rightarrow L'_k \rightarrow L'_{k-1} \rightarrow \dots \rightarrow L'_1 \rightarrow L'_0 \rightarrow N \rightarrow 0.$$

The induced map  $P_k(N) \rightarrow P$  gives a map  $\mathcal{P}_k(N) \rightarrow \mathcal{P}$  that is an isomorphism at the cohomology level  $H_*^k$ . Minimally add a free module  $F$  to  $P$  to force a surjection  $P^\vee \oplus F^\vee \rightarrow P_k(N)^\vee$ . This gives an inclusion of bundles  $\mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus F$  whose cokernel is  $\mathcal{P}_l(M) \oplus \mathcal{F}'$  where  $\mathcal{F}'$  is a free bundle (since it has only  $H_*^l$  intermediate cohomology). We notice that both for  $k = 1$  and for  $k > 1$ , the map  $H_*^1(\mathcal{P}_k(N)) \rightarrow H_*^1(\mathcal{P} \oplus F)$  is an isomorphism, so we get a surjection from  $H_*^0(\mathcal{P} \oplus F)$  to  $H_*^0(\mathcal{P}_l(M) \oplus \mathcal{F}')$ . By the minimality of  $F$ , we may conclude that  $\mathcal{F}' = 0$  □

Summarizing this below, we get the following.

**Proposition 2.7.** *Let  $\mathcal{E}$  be a rank two bundle on  $\mathbf{P}^n$ ,  $n \geq 6$  with isolated cohomology of type  $(n, k)$  with  $H_*^k(\mathcal{E}) = N$ , for some  $k$  strictly between 1 and  $\frac{n}{2}$ . Then  $\mathcal{E}$  has the monad*

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \rightarrow 0,$$

where

- $\mathcal{P}$  satisfies an exact sequence  $0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \rightarrow 0$ , where  $\mathcal{F}$  is some free bundle,  $M = H_*^{n-k}(\mathcal{E})$  (which can be identified with  $N^\vee$  up to twist).
- $H_*^i(\wedge^2 \mathcal{P}) = 0$  for  $i \neq 1, k, n - k, n - 1$ .
- $H_*^1(\wedge^2 \mathcal{P})$  and  $H_*^{n-1}(\wedge^2 \mathcal{P})$  are nonzero if  $k \neq 2$ .

In the case left out in the above proposition, where  $\mathcal{E}$  has isolated middle cohomology with  $n = 2k$  and with  $H_*^k(\mathcal{E}) = N \neq 0$  equal to the only nonzero cohomology module in the range  $1 < i < n - 1$ , the monad for  $\mathcal{E}$  has the form

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{B} \rightarrow 0.$$

Also, there is a short exact sequence

$$0 \rightarrow A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k(N)) \rightarrow B \otimes N \rightarrow 0.$$

Thus,

**Proposition 2.8.** *Let  $\mathcal{E}$  be a rank two bundle on  $\mathbf{P}^n$ ,  $n = 2k, n \geq 6$ , with  $H_*^k(\mathcal{E}) = N, H_*^i(\mathcal{E}) = 0, i \neq 1, k, n$ . Let  $\mathcal{P}_k$  be the  $k$ th syzygy bundle of  $N$  where  $\mathcal{P}_k$  is the sheafification of  $P_k$  with  $P_k = \text{Image of } (f_{k+1} : L_{k+1} \rightarrow L_k)$  in a minimal free resolution of  $N$ . Then  $\mathcal{E}$  has the monad*

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{P}_k \xrightarrow{\beta} \mathcal{B} \rightarrow 0,$$

where  $\mathcal{A}, \mathcal{B}$  are sheafifications of free summands  $A, B$  of  $L_{k+1}$  and  $L_k$ , respectively, and where  $\alpha, \beta$  are induced by  $f_{k+1}$ . Furthermore,

- $H_*^i(\wedge^2 \mathcal{P}_k) = 0$  for  $i \neq 1, k, n - 1$ ,
- the induced sequence  $0 \rightarrow A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k) \rightarrow B \otimes N \rightarrow 0$  is exact,
- $H_*^1(\wedge^2 \mathcal{P}_k)$  and  $H_*^{n-1}(\wedge^2 \mathcal{P}_k)$  are nonzero.

*Proof.* The only item to verify is that  $\mathcal{A}, \mathcal{B}$  are sheafifications of free summands  $A, B$  of  $L_{k+1}$  and  $L_k$ , respectively, and that  $\alpha, \beta$  are induced by  $f_{k+1}$ . Since  $L_{k+1} \rightarrow P_k$  is surjective,  $\alpha : A \rightarrow P_k$  factors through  $\tilde{\alpha} : A \rightarrow L_{k+1}$ . Likewise, since  $L_k^\vee \rightarrow P_k^\vee$  is surjective,  $\beta^\vee : B^\vee \rightarrow P_k^\vee$  factors through  $\tilde{\beta}^\vee : B^\vee \rightarrow L_k^\vee$ . It remains to show that the matrices  $\tilde{\alpha}, \tilde{\beta}$  have full rank when tensored by  $k$ .

The map  $H_*^k(\wedge^2 \mathcal{P}_k) \rightarrow B \otimes N \rightarrow 0$  in the short sequence above is obtained from  $\wedge^2 \mathcal{P}_k \rightarrow \mathcal{B} \otimes \mathcal{P}_k$  where  $p \wedge q$  maps to  $\beta(p) \otimes q - \beta(q) \otimes p$ . This factors through  $\mathcal{L}_k \otimes \mathcal{P}_k$  via the lift  $\tilde{\beta}$ . In particular, the map  $L_k \otimes N \rightarrow B \otimes N$ , given by  $\tilde{\beta} \otimes I$ , is onto. Hence so is  $(\tilde{\beta} \otimes k) \otimes I$ , a map of vector spaces. Hence, the matrix  $\tilde{\beta} \otimes k$  has rank equal to the rank of  $B$ . So,  $B$  is a direct summand of  $L_k$ .

The map  $0 \rightarrow A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k)$  is obtained from  $H_*^k(\mathcal{A} \otimes \mathcal{G}) \cong H_*^k(\wedge^2 \mathcal{G}) \hookrightarrow H_*^k(\wedge^2 \mathcal{P}_k)$ , which, in turn, is obtained from  $\mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{G} \hookrightarrow \wedge^2 \mathcal{P}_k$ , where  $a \otimes g$  maps to  $\alpha(a) \wedge g$  in  $\wedge^2 \mathcal{P}_k$ . This map  $\mathcal{A} \otimes \mathcal{G} \rightarrow \wedge^2 \mathcal{P}_k$  factors through  $\mathcal{L}_{k+1} \otimes \mathcal{G}$ , via the lift  $\tilde{\alpha}$ .

It follows that the injection  $A \otimes N \rightarrow H_*^k(\wedge^2 \mathcal{P}_k)$  factors through  $A \otimes N \rightarrow L_{k+1} \otimes N$ , by the map  $\tilde{\alpha} \otimes I$ . This must also be injective. Choose a socle element  $n$  in  $N$  (an element that is annihilated by all linear forms in  $S$ ). The submodule generated by  $n, \langle n \rangle$ , is a one-dimensional vector space and  $A \otimes \langle n \rangle$  is mapped injectively by  $\tilde{\alpha} \otimes I$  to  $L_{k+1} \otimes N$ . Since the image of  $\tilde{\alpha} \otimes I$  on  $A \otimes \langle n \rangle$  is the same as the image of  $(\tilde{\alpha} \otimes k) \otimes I$  on  $(A \otimes k) \otimes \langle n \rangle$ , it follows that the rank of the matrix  $\tilde{\alpha} \otimes k$  has rank equal to the rank of  $A$ . Thus,  $A$  is a direct summand of  $L_{k+1}$ .  $\square$

We now review a result of Jyotilingam [9] about cohomology modules of tensor products, applying it to the special case of syzygy bundles for our purposes. In the theorem below,  $N$  and  $M$  will be graded finite length  $S$ -modules where  $S = k[X_0, X_1, \dots, X_n]$  corresponding to  $\mathbf{P}^n$ .  $\mathcal{P}_k(N)$  and  $\mathcal{Q}_i(M)$  will indicate syzygy bundles obtained from minimal free resolutions of  $N$  and  $M$ . Note that in the minimal free resolution,

$$0 \rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow N \rightarrow 0,$$

when we tensor by  $M$ , the map  $L_{n+1} \otimes M \rightarrow L_n \otimes M$ , cannot be injective since  $M$  has finite length, hence  $\text{Tor}_{n+1}^S(N, M) \neq 0$ , and by Lichtenbaum’s theorem [10]  $\text{Tor}_i^S(N, M) \neq 0$  for all  $i \leq n + 1$ .

**Theorem 2.9.** *Let  $N$  be a finite  $S$ -module and let  $\mathcal{P}_k$  be its  $k$ th syzygy bundle on  $\mathbf{P}^n$ , with  $k \geq 1$ . Let  $\mathcal{Q}$  be a bundle on  $\mathbf{P}^n$  with  $H_*^l(\mathcal{Q}) = M \neq 0$ , with  $k \leq l \leq n - 2$ , and with  $H_*^i(\mathcal{Q}) = 0$  for  $i = l - 1, l - 2, \dots, l - k + 2$ . Then  $H_*^{l+1}(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$ .*

*Proof.* The cases  $k = 1$  and  $k = 2$  require no conditions on  $H_*^{l-1}(\mathcal{Q})$ . When  $k = 1$ , we get the sequence  $H_*^l(\mathcal{L}_1 \otimes \mathcal{Q}) \rightarrow H_*^l(\mathcal{L}_0 \otimes \mathcal{Q}) \rightarrow H_*^{l+1}(\mathcal{P}_1 \otimes \mathcal{Q}) \rightarrow 0$  and the map  $L_1 \otimes M \rightarrow L_0 \otimes M$  can never be surjective. When  $k > 1$ , consider the diagram obtained from the sequences  $0 \rightarrow \mathcal{P}_i \otimes \mathcal{Q} \rightarrow \mathcal{L}_i \otimes \mathcal{Q} \rightarrow \mathcal{P}_{i-1} \otimes \mathcal{Q} \rightarrow 0$ ,  $i = k, k - 1, k - 2$  (with  $\mathcal{P}_j = 0$  if  $j < 0$  and  $\mathcal{P}_0 = \mathcal{L}_0$ ):

$$\begin{array}{ccccc} L_k \otimes M & = & L_k \otimes M & H_*^{l-1}(\mathcal{P}_{k-3} \otimes \mathcal{Q}) & \\ \downarrow & & \downarrow \gamma & \downarrow & \\ H_*^l(\mathcal{P}_{k-1} \otimes \mathcal{Q}) & \xrightarrow{\alpha} & L_{k-1} \otimes M & \xrightarrow{\beta} & H_*^l(\mathcal{P}_{k-2} \otimes \mathcal{Q}) \\ \downarrow \mu & & \downarrow \delta & \downarrow & \\ H_*^{l+1}(\mathcal{P}_k \otimes \mathcal{Q}) & L_{k-2} \otimes M = & L_{k-2} \otimes M & & \end{array}$$

The vanishing conditions on  $H_*^i(\mathcal{Q})$  show that  $H_*^{l-1}(\mathcal{P}_{k-3} \otimes \mathcal{Q}) = H_*^{l-2}(\mathcal{P}_{k-4} \otimes \mathcal{Q}) = \dots = H_*^{l-k+2}(\mathcal{L}_0 \otimes \mathcal{Q}) = 0$ . So,  $\ker \delta = \text{im } \alpha$  and the diagram induces a surjection  $\text{im } \mu \rightarrow \text{Tor}_{k-1}^S(N, M)$ . By Lichtenbaum’s theorem,  $H_*^{l+1}(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$ .  $\square$

### 3 | ISOLATED COHOMOLOGY OF TYPE $(n, k)$ , WITH $n \geq 4k$

In this section, we will prove that there are no indecomposable rank two bundles on  $\mathbf{P}^n$  with isolated cohomology of type  $(n, k)$ , where  $n \geq 4k$ . We study the sequence  $0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \rightarrow 0$  of Proposition 2.7. We will need to pay special attention to the case where  $N$  is a cyclic module. Hence the following lemma.

**Lemma 3.1.** *Let  $N$  be a graded cyclic  $S$ -module. For the corresponding syzygy bundle  $\mathcal{P}_2(N)$  on  $\mathbf{P}^n$ ,  $H_*^3(S^2\mathcal{P}_2(N)) = 0$  and  $H_*^3(\wedge^2\mathcal{P}_2(N)) \neq 0$ .*

*Proof.* From the sequence  $0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{L}_2 \rightarrow \mathcal{P}_1 \rightarrow 0$  obtained from a minimal resolution of  $N$ , it suffices to show that the map  $H_*^1(\mathcal{L}_2 \otimes \mathcal{P}_1) \rightarrow H_*^1(\wedge^2\mathcal{P}_1)$  is surjective to prove that



$H_*^3(S^2\mathcal{P}_2(N)) = 0$ . This map can be studied using the natural commuting diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{P}_1 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{L}_1 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{L}_0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \wedge^2 \mathcal{P}_1 & \rightarrow & \wedge^2 \mathcal{L}_1 & \rightarrow & \mathcal{L}_1 \otimes \mathcal{L}_0 \rightarrow S^2 \mathcal{L}_0 \end{array}$$

It simplifies when  $\mathcal{L}_0$  has rank one, where without loss of generality, we can take  $\mathcal{L}_0$  to be  $\mathcal{O}_{\mathbb{P}^n}$ , yielding

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{P}_1 & \rightarrow & \mathcal{L}_2 \otimes \mathcal{L}_1 & \rightarrow & \mathcal{L}_2 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \wedge^2 \mathcal{P}_1 & \rightarrow & \wedge^2 \mathcal{L}_1 & \rightarrow & \mathcal{P}_1 \rightarrow 0 \end{array}$$

Since  $\mathcal{L}_2$  surjects onto the global sections of  $\mathcal{P}_1$ , it follows from the diagram of long exact sequences of cohomology modules that  $H_*^1(\mathcal{L}_2 \otimes \mathcal{P}_1) \rightarrow H_*^1(\wedge^2 \mathcal{P}_1)$  is onto.

For the second part, we will show that  $H_*^3(\mathcal{P}_2 \otimes \mathcal{P}_2) \neq 0$ . (This argument will be repeated later in a slightly different setting.) With  $H_*^3(S^2\mathcal{P}_2) = 0$ , since  $H_*^3(\mathcal{P}_2 \otimes \mathcal{P}_2) = H_*^3(S^2\mathcal{P}_2) \oplus H_*^3(\wedge^2 \mathcal{P}_2)$ , the conclusion of the lemma follows.

Consider  $0 \rightarrow \mathcal{P}_2 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_2 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_1 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_0 \otimes \mathcal{P}_2 \rightarrow 0$ . From  $0 \rightarrow \mathcal{P}_1 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_1 \otimes \mathcal{P}_2 \rightarrow \mathcal{L}_0 \otimes \mathcal{P}_2 \rightarrow 0$ , we get

$$H_*^2(\mathcal{P}_1 \otimes \mathcal{P}_2) = \ker(L_1 \otimes N \rightarrow L_0 \otimes N) = L_1 \otimes N$$

since  $N$  is cyclic. Hence, we get

$$H_*^3(\mathcal{P}_2 \otimes \mathcal{P}_2) = \text{coker}(L_2 \otimes N \rightarrow L_1 \otimes N),$$

which is clearly nonzero. □

**Proposition 3.2.** *Suppose that  $\mathcal{E}$  on  $\mathbb{P}^n$  is a rank two bundle of type  $(n, k)$  with  $n \geq 7$ ,  $k$  strictly less than  $\frac{n}{2}$ . Then the sequence  $0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_1(M) \rightarrow 0$ , in Proposition 2.7, is not-split.*

*Proof.* Suppose  $\mathcal{P} \oplus \mathcal{F} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$ . Neither  $\mathcal{P}_k(N)$  nor  $\mathcal{P}_{n-k}(M)$  has any line bundle summands, hence  $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$ . So,  $\wedge^2 \mathcal{P}$  has summands  $\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M)$  and  $\wedge^2 \mathcal{P}_k(N)$ . If  $k > 2$ , then using Proposition 2.9,  $H_*^{n-k+1}(\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M))$  is nonzero which contradicts the requirement in Proposition 2.7 that  $H_*^{n-k+1}(\wedge^2 \mathcal{P}) = 0$ .

If  $k = 2$ , there are two cases: if  $N$  is cyclic, then  $H_*^3(\wedge^2 \mathcal{P}_2(N)) \neq 0$  by Lemma 3.1, which contradicts Proposition 2.7 since  $n - k > 3$  when  $n \geq 6$ .

If  $N$  is noncyclic, then from the sequences  $0 \rightarrow \mathcal{P}_2(N) \rightarrow \mathcal{L}_2 \rightarrow \mathcal{P}_1(N) \rightarrow 0$  and  $0 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow 0$ , we get  $H_*^4(\wedge^2 \mathcal{P}_2(N)) \neq 0$ . This a contradiction to Proposition 2.7 when  $n \geq 7$ . □

*Remark 3.3.* The case  $n = 6, k = 2$  is not answered above. A weaker argument can be made here that even though  $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$ ,  $N$  itself is neither cyclic nor a direct sum of submodules  $N_1 \oplus N_2$ .

**Theorem 3.4.** *Let  $\mathcal{E}$  be a rank two vector bundle on  $\mathbf{P}^8$  with  $H_*^3(\mathcal{E}) = H_*^4(\mathcal{E}) = 0$ , then  $\mathcal{E}$  splits.*

*Proof.* Let  $N = H_*^2(\mathcal{E})$  and  $M = H_*^6(\mathcal{E})$ . Both are nonzero unless  $\mathcal{E}$  splits. By Proposition 3.2 (with  $k = 2$ ), we know that the sequence below is nonsplit.

$$0 \rightarrow \mathcal{P}_2(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_6(M) \rightarrow 0. \quad (5)$$

The proof will analyze the consequences of the two sequences below obtained from sequence.

$$0 \rightarrow S^2\mathcal{P}_2(N) \rightarrow \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \wedge^2\mathcal{P}_6(M) \rightarrow 0, \quad (6)$$

$$0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_6(M) \rightarrow 0. \quad (7)$$

**Case 1** If  $N$  is cyclic, we look at the sequence (6).

It breaks into

$$\begin{aligned} 0 \rightarrow S^2\mathcal{P}_2(N) \rightarrow \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow \mathcal{D} \rightarrow 0, \\ 0 \rightarrow \mathcal{D} \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \wedge^2\mathcal{P}_6(M) \rightarrow 0, \end{aligned} \quad (8)$$

$H_*^3(\mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}]) \neq 0$  by the same argument in the second part of the proof of Lemma 3.1, and by the same lemma,  $H_*^3(S^2\mathcal{P}_2(N)) = 0$ . Hence,  $H_*^3(\mathcal{D}) \neq 0$  from the first sequence in (8).

In the second sequence in (8),  $H_*^3(\mathcal{P}) = 0$ . Hence so is  $H_*^3(\wedge^2\mathcal{P})$ . Finally,  $\mathcal{P}_6(M)$  fits into a sequence with free bundles

$$0 \rightarrow \mathcal{L}'_9 \rightarrow \mathcal{L}'_8 \rightarrow \mathcal{L}'_7 \rightarrow \mathcal{P}_6 \rightarrow 0.$$

This yields two exact sequences

$$\begin{aligned} 0 \rightarrow S^2\mathcal{P}_7 \rightarrow S^2\mathcal{L}'_7 \rightarrow \mathcal{L}'_7 \otimes \mathcal{P}_6 \rightarrow \wedge^2\mathcal{P}_6 \rightarrow 0, \\ 0 \rightarrow \wedge^2\mathcal{L}'_9 \rightarrow \wedge^2\mathcal{L}'_8 \rightarrow \mathcal{L}'_8 \otimes \mathcal{P}_7 \rightarrow S^2\mathcal{P}_7 \rightarrow 0. \end{aligned} \quad (9)$$

From these, we can chase down  $H_*^2(\wedge^2\mathcal{P}_6)$  to be equal to zero since  $H_*^2(\mathcal{P}_6) = 0$ ,  $H_*^4(\mathcal{P}_7) = 0$ ,  $H_*^6(\wedge^2\mathcal{L}'_9) = 0$ . Hence,  $H_*^3(\mathcal{D})$  is both zero and nonzero, a contradiction.

**Case 2** If  $N$  is noncyclic, we look at the sequence (7)

$$0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_6(M) \rightarrow 0.$$

It breaks into

$$\begin{aligned} 0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{D} \rightarrow 0, \\ 0 \rightarrow \mathcal{D} \rightarrow \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_6(M) \rightarrow 0. \end{aligned} \quad (10)$$

From

$$\begin{aligned} 0 \rightarrow S^2\mathcal{P}_1(N) \rightarrow S^2\mathcal{L}_1 \rightarrow \mathcal{L}_1 \otimes \mathcal{L}_0 \rightarrow \wedge^2\mathcal{L}_0 \rightarrow 0, \\ 0 \rightarrow \wedge^2\mathcal{P}_2(N) \rightarrow \wedge^2\mathcal{L}_2 \rightarrow \mathcal{L}_2 \otimes \mathcal{L}_1 \rightarrow S^2\mathcal{P}_1(N) \rightarrow 0, \end{aligned}$$

we get  $H_*^2(S^2\mathcal{P}_1(N)) \neq 0$  and  $H_*^4(\wedge^2\mathcal{P}_2(N)) \neq 0$ . Since  $H_*^4(\mathcal{P})$  and  $H_*^4(\wedge^2\mathcal{P})$  are zero, we obtain  $H_*^3(\mathcal{D}) \neq 0$ .

Again, in the second sequence in (10),  $H_*^3(\mathcal{P}_6(M) \otimes \mathcal{F}) = 0$  and  $H_*^3(\mathcal{P}_6(M) \otimes \mathcal{P})$  can be studied using a resolution for  $\mathcal{P}_6(M)$  and tensoring with  $\mathcal{P}$ .

$$0 \rightarrow \mathcal{L}'_9 \otimes \mathcal{P} \rightarrow \mathcal{L}'_8 \otimes \mathcal{P} \rightarrow \mathcal{L}'_7 \otimes \mathcal{P} \rightarrow \mathcal{P}_6(M) \otimes \mathcal{P} \rightarrow 0.$$

Then  $H_*^3(\mathcal{P}_6(M) \otimes \mathcal{P}) = 0$  since  $H_*^3(\mathcal{P}), H_*^4(\mathcal{P}), H_*^5(\mathcal{P})$  are all zero.

We compute  $H_*^2(S^2\mathcal{P}_6(M))$ , breaking up the resolution of  $\mathcal{P}_6$  (suppressing the letter  $M$ ) into short exact sequences:

$$\begin{aligned} 0 \rightarrow \wedge^2\mathcal{P}_7 \rightarrow \wedge^2\mathcal{L}'_7 \rightarrow \mathcal{L}'_7 \otimes \mathcal{P}_6 \rightarrow S^2\mathcal{P}_6 \rightarrow 0, \\ 0 \rightarrow S^2\mathcal{L}'_9 \rightarrow S^2\mathcal{L}'_8 \rightarrow \mathcal{L}'_8 \otimes \mathcal{P}_7 \rightarrow \wedge^2\mathcal{P}_7 \rightarrow 0. \end{aligned} \tag{11}$$

$H_*^2(S^2\mathcal{P}_6(M))$  will vanish since  $H_*^2(\mathcal{P}_6), H_*^4(\mathcal{P}_7)$  and  $H_*^6(S^2\mathcal{L}'_9)$  are all zero. □

**Corollary 3.5.** *Let  $n \geq 8$ . Let  $\mathcal{E}$  be a rank two vector bundle on  $\mathbf{P}^n$  with  $H_*^i(\mathcal{E}) = 0$  for  $i = 3, \dots, n - 3$ . Then  $\mathcal{E}$  splits.*

*Proof.* Use induction on  $n$ . The case  $n = 8$  is proved in the above theorem. Assume the result for  $n - 1$ . Let  $\mathcal{E}$  be a rank two vector bundle on  $\mathbf{P}^n$  with  $H_*^i(\mathcal{E}) = 0$  for  $i = 3, \dots, n - 3$ . For a hyperplane  $H$ , by the restriction sequence in cohomology,

$$H_*^i(\mathcal{E}) \rightarrow H_*^i(\mathcal{E}_H) \rightarrow H_*^{i+1}(\mathcal{E}(-1)),$$

we get that  $H_*^i(\mathcal{E}_H) = 0$  for  $i = 3, \dots, n - 4$  on  $\mathbf{P}^{n-1}$ . So,  $\mathcal{E}_H$  splits and hence also  $\mathcal{E}$ . □

The theorem above can be generalized to arbitrary  $k$  using the similar calculations.

**Theorem 3.6.** *Let  $n \geq 4k$ , with  $k > 1$ . Then there cannot exist a rank two bundle  $\mathcal{E}$  on  $\mathbf{P}^n$ , for which the only nonzero intermediate cohomology modules are  $H_*^1(\mathcal{E}), H_*^k(\mathcal{E}) = N, H_*^{n-k}(\mathcal{E}) = M$ , and  $H_*^{n-1}(\mathcal{E})$ .*

*Proof.* The case  $k = 2$  was done in the corollary above. So, we assume that  $k > 2$ . The proof will analyze the consequences of the sequence

$$0 \rightarrow \mathcal{P}_k(N) \rightarrow \mathcal{P} \oplus \mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \rightarrow 0, \tag{12}$$

which is nonsplit by Proposition 3.2. We get the collateral sequence:

$$0 \rightarrow \wedge^2\mathcal{P}_k(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_{n-k}(M) \rightarrow 0. \tag{13}$$

We will prove it using several cases.

**Case 1** The case where  $N$  is cyclic,  $k$  is even and  $> 2$ .

We look at the sequence (13) which breaks into

$$\begin{aligned} 0 \rightarrow \wedge^2\mathcal{P}_k(N) \rightarrow \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \rightarrow \mathcal{D} \rightarrow 0, \\ 0 \rightarrow \mathcal{D} \rightarrow \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \rightarrow S^2\mathcal{P}_{n-k}(M) \rightarrow 0, \end{aligned} \tag{14}$$



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