On the Separability of Functions and Games

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***Abstract*—We study the notion of (additive) separability of a function of several variables with respect to a hypergraph (H-graph). We prove the existence of a unique minimal H-graph with respect to which a function is separable and show that the corresponding minimal decomposition of the function can be obtained through a recursive algorithm. We then focus on (strategic form) games and propose a concept of separability for a game with respect to a forward directed hypergraph (FDH-graph). This notion refines and generalizes that of graphical game and is invariant with respect to strategic equivalence. We show that every game is separable with respect to a minimal FDH-graph. Moreover, for exact potential games, such minimal FDH-graph reduces to the minimal H-graph of the potential function. Our results imply and refine known results on graphical potential games and yield a new proof of the celebrated Hammersely-Clifford Theorem on Markov Random Fields.**

Index terms: Separable Functions, Network Games, Hypergraphical Games, Potential Games, Hammersley-Clifford Theorem

I. Introduction

In many fields of data science, statistical physics, network economics, optimization, and control, we deal with functions defined on highly dimensional configuration spaces (i.e., large product spaces) representing the global state of a multi-agent system. It is often the case that such functions decompose as sums of simpler local functions each only depending on a relatively small subset of variables. We refer to functions exhibiting this feature generically as separable functions.[1] Instances are the target functions of many problems in combinatorial optimization (e.g., graph coloring), the $\log$ of the distribution functions of Markov Random Fields or of the Hamiltonian of an interacting particle system (e.g., Ising model).

An important area where separable functions naturally arise is that of network games, which have recently emerged as a unified framework for modeling interactions in many social and economic settings [1]–[5]. In such models, players are each affected by the actions of a typically small subset of the other players (e.g., friends, colleagues, and peers in a social network, or costumers and competitors in an economic network) and their utility function can be typically represented as the superimposition of various terms each depending on the actions of a specific subset of players only. The most common formalization is through the notion of graphical game [6] (see also [7, Chapter 7]), where players are identified with the nodes of a graph and each player's utility depends only on her own action and the actions of players that are neighbors in such graph. Another instance is represented by polymatrix games [8], where each player perceives a utility as if she was simultaneously playing a two-player game with possibly every other player and playing in all games the same action: in this case, the utility of a player is expressed as the sum of all these pairwise contributions. Hypergraphical games [9] encompass both these classes. Hyperlinks determine the group of players participating in local games and the utility of a player is expressed as the sum of utilities of all local games in which she is involved. In the case when hyperlinks are all of cardinality two so that the hypergraph reduces to a graph, hypergraphical games reduce to polymatrix games. We refer to such games as pairwise separable: examples include network coordination or anti-coordination games. Hypergraphical versions have been studied in [10], under the name of synchronization games.

A key observation is that the representation of a separable function in terms of local expressions is typically not unique. Expressions can be reassembled in various ways and may lead to substantially different decompositions. For many computational techniques involving separable functions the size of the subsets of variables that show up in the local functions is a crucial complexity parameter. This is the case, e.g., for distributed inference and learning algorithms for Markov Random Fields, such as Belief Propagation and Iterative Proportional Fitting [11]. Similar considerations apply to many algorithms proposed to compute Nash and correlated equilibria in graphical games [6], [12], polymatrix games [13], or hypergraphical games [9] [10]. This indicates that finding parsimonious representations of separable functions is a crucial problem. To the best of our knowledge, the problem of existence, uniqueness, and characterization of a minimal decomposition of a separable function has not been addressed in the literature.

As a first contribution, we prove that every function defined on a product space admits a unique *minimal* decomposition. We also propose a constructive way to find such a minimal decomposition of a separable function. We then consider games up to strategic equivalence, meaning that we are only concerned with variations of the utility of a player when she modifies her action rather than their absolute values: in fact, most classical game-theoretic concepts such as domination, Nash equilibrium, correlated equilibrium, best-response dynamics or the logit dynamics are invariant with respect to strategic equivalence. While this brings in even more freedom in decomposing the utility functions, we show that a minimal

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[1]Notice that in some of the literature (particularly in econometrics) the term *additively separable* function is used in a stricter sense to indicate a function of n variables that can be split into the sum of n functions each depending on just a single variable.

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3314552

decomposition exists also in this setup, that it is unique, and that it can be found applying the techniques developed for single functions. Finally, for the class of exact potential games, we establish an direct correspondence between the separability property of a game and the separability of the potential function. As a corollary, we generalize a result on graphical games [14], as well as provide an alternative proof of the Hammersley-Clifford Theorem for Markov Random Fields.

The rest of the paper is organized as follows. Section II introduces some basic notions on hypergraphs and the formal definition of separability of a function with respect to a hypergraph. The main result of this section is Theorem 1 that proves, for every function defined on a product space, the existence of a minimal hypergraph with respect to which it is separable. The following Section III is devoted to defining a constructive way to compute such minimal decomposition. This is based on two factors: a technique that reformulates separability in terms of difference equations and leads to the identification of the minimal hypergraph, and a projection technique that then explicitly constructs the decomposition. In Section IV, we extend the concept of separability to games defined up to strategic equivalence. The natural graph-theoretic object here is a directed version of a hypergraph, which is called forward directed hypergraph (FDH-graph). The main result of this section is Theorem 2 that proves the existence of a minimal FDH-graph with respect to which a game is separable. Section V is devoted to potential games and it contains Theorem 3 that describes the relation between the minimal FDH-graph associated with the game and the minimal H-graph capturing the separability structure of the potential function. We then show how this result implies results on graphical potential games and on Markov Random Fields. The paper ends with a conclusive Section VI.

II. Minimal separability of functions

In this section, we first introduce the notion of separability of a function of several variables with respect to a hypergraph. We then prove existence and uniqueness of a minimal hypergraph with respect to which a function is separable.

A. Hypergraphs and separable functions

A *hypergraph* (shortly, a *H-graph*) is the pair $\mathcal{H}=(\mathcal{V},\mathcal{L})$ of a finite node set $\mathcal{V}$ and of a set $\mathcal{L}$ of hyperlinks, each of which is a nonempty subset of nodes [15]. A H-graph $\mathcal{H}=(\mathcal{V},\mathcal{L})$ is called *simple* if no hyperlink $\mathcal{J}$ in $\mathcal{L}$ is strictly contained in another hyperlink $\mathcal{K}$ in $\mathcal{L}$. The *simple closure* of a H-graph $\mathcal{H}=(\mathcal{V},\mathcal{L})$ is the simple H-graph $\overline{\mathcal{H}}=(\mathcal{V},\overline{\mathcal{L}})$ with set of hyperlinks $\overline{\mathcal{L}}=\{\mathcal{J}\in\mathcal{L} : \nexists\mathcal{K}\in\mathcal{L} \text{ s.t. } \mathcal{K}\supsetneq\mathcal{J}\}$.

Throughout the paper, we shall work with a fixed nonempty finite set $\mathcal{V}$ and, for every i in $\mathcal{V}$, a set of states $\mathcal{A}_i$. We then refer to the product set $\mathcal{X}=\prod_{i\in\mathcal{V}}\mathcal{A}_i$ as the *configuration space*. For a subset $\mathcal{J}\subset\mathcal{V}$, we put $\mathcal{X}_\mathcal{J}=\prod_{i\in\mathcal{J}}\mathcal{A}_i$. The restriction of a configuration x in $\mathcal{X}$ to a subset $\mathcal{J}\subset\mathcal{V}$ is denoted by $x_\mathcal{J}$ in $\mathcal{X}_\mathcal{J}$. Following a standard convention in game theory, we use the notation $-i=\mathcal{V}\setminus\{i\}$ and correspondingly write $\mathcal{X}_{-i}$ for $\mathcal{X}_{\mathcal{V}\setminus\{i\}}$ and x_{-i} for $x_{\mathcal{V}\setminus\{i\}}$. Moreover, given a function $f:\mathcal{X}\to\mathbb{R}$ we sometimes use the notation $f(x_i,x_{-i})$ for its value $f(x)$ in a configuration x.

Definition 1: A function[2] $f:\mathcal{X}\to\mathbb{R}$ is $\mathcal{H}$*-separable*, where $\mathcal{H}=(\mathcal{V},\mathcal{L})$ is a H-graph, if there exist functions $f_\mathcal{J}:\mathcal{X}_\mathcal{J}\to\mathbb{R}$, for $\mathcal{J}$ in $\mathcal{L}$, such that

$$f(x)=\sum_{\mathcal{J}\in\mathcal{L}}f_\mathcal{J}(x_\mathcal{J})\,,\qquad\forall x\in\mathcal{X}\,. \tag{1}$$

Separability of a function $f:\mathcal{X}\to\mathbb{R}$ with respect to a H-graph $\mathcal{H}$ thus consists in the possibility of representing $f(x)$ as a sum of functions each depending exclusively on the variables $x_\mathcal{J}$ associated to a hyperlink $\mathcal{J}$ of $\mathcal{H}$. Clearly, a function f is $\mathcal{H}$-separable if and only if it is $\overline{\mathcal{H}}$-separable. We shall refer to (1) as a *decomposition* of f with respect to $\mathcal{H}$.

Remark 1: Notice that Definition 1 does not require hyperlinks in $\mathcal{L}$ to be disjoint, i.e., we do not require that $\mathcal{J}\cap\mathcal{K}=\emptyset$ for $\mathcal{J}\neq\mathcal{K}$ in $\mathcal{L}$, as is done on some of the existing literature.

Example 1: Consider a finite-valued random vector $X=(X_i)_{i\in\mathcal{V}}$, with positive probability distribution

$$\mathbb{P}(X=x)\propto e^{f(x)}\,,\qquad\forall x\in\mathcal{X}\,.$$

Let $\mathcal{H}=(\mathcal{V},\mathcal{L})$ be a H-graph. Then, the function f is $\mathcal{H}$-separable if and only if the distribution of X factorizes as

$$\mathbb{P}(X=x)=\frac{1}{Z(f)}\prod_{\mathcal{J}\in\mathcal{L}}e^{f_\mathcal{J}(x_\mathcal{J})}\,,\qquad\forall x\in\mathcal{X}\,, \tag{2}$$

where $Z(f)=\sum_{x\in\mathcal{X}}e^{f(x)}$. In the special case when $\mathcal{H}$ has disjoint hyperlinks, the factorization above reduces to

$$\mathbb{P}(X=x)=\prod_{\mathcal{J}\in\mathcal{L}}\mathbb{P}(X_\mathcal{J}=x_\mathcal{J})\,,$$

i.e., mutual independence of the subvectors $X_\mathcal{J}=(X_i)_{i\in\mathcal{J}}$, for $\mathcal{J}$ in $\mathcal{L}$.

Definition 2: Let $\mathcal{H}_1=(\mathcal{V},\mathcal{L}_1)$ and $\mathcal{H}_2=(\mathcal{V},\mathcal{L}_2)$ be two H-graphs with the same node set. Then:

(i) $\mathcal{H}_1$ is finer that $\mathcal{H}_2$ (shortly, $\mathcal{H}_1\preceq\mathcal{H}_2$) if, for every $\mathcal{J}_1$ in $\mathcal{L}_1$, there exists $\mathcal{J}_2$ in $\mathcal{L}_2$ such that $\mathcal{J}_1\subseteq\mathcal{J}_2$;

(ii) $\mathcal{H}_1$ is *strictly finer* than $\mathcal{H}_2$ (shortly, $\mathcal{H}_1\prec\mathcal{H}_2$) if $\mathcal{H}_1\preceq\mathcal{H}_2$ but $\mathcal{H}_2\not\preceq\mathcal{H}_1$;

(iii) the *intersection* of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the H-graph $\mathcal{H}_1\sqcap\mathcal{H}_2=(\mathcal{V},\mathcal{L})$ with

$$\mathcal{L}=\{\mathcal{J}=\mathcal{J}_1\cap\mathcal{J}_2\,:\;\mathcal{J}_1\in\mathcal{L}_1,\,\mathcal{J}_2\in\mathcal{L}_2\}\,;$$

(iv) the *union* of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the H-graph $\mathcal{H}_1\sqcup\mathcal{H}_2=(\mathcal{V},\mathcal{L}_1\cup\mathcal{L}_2)$.

Remark 2: Both $\preceq$ and $\prec$ are transitive relations and $\mathcal{H}\preceq\overline{\mathcal{H}}\preceq\mathcal{H}$ for every H-graph $\mathcal{H}$. Moreover, for two H-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$, we have that

$$\mathcal{H}_1\preceq\mathcal{H}_2\;\Leftrightarrow\;\bar{\mathcal{H}}_1\preceq\bar{\mathcal{H}}_2\,,\qquad\mathcal{H}_1\prec\mathcal{H}_2\;\Leftrightarrow\;\bar{\mathcal{H}}_1\prec\bar{\mathcal{H}}_2\,; \tag{3}$$

$$\mathcal{H}_1,\mathcal{H}_2\text{ simple and }\mathcal{H}_1\preceq\mathcal{H}_2\preceq\mathcal{H}_1\quad\Rightarrow\quad\mathcal{H}_1=\mathcal{H}_2\,. \tag{4}$$

[2]Every statement and reasoning in this subsection continues to hold true for function $f:\mathcal{X}\to\mathcal{Z}$, where $\mathcal{Z}$ is an arbitrary Abelian group.

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3314552

B. Minimal separability of functions

Notice that every function $f : \mathcal{X} \to \mathbb{R}$ is $\mathcal{H}$-separable with respect to the trivial H-graph $\mathcal{H} = (\mathcal{V}, \{\mathcal{V}\})$ having an unique hyperlink consisting of all nodes. Moreover, given two H-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \preceq \mathcal{H}_2$, we have that if f is $\mathcal{H}_1$-separable, then it is also $\mathcal{H}_2$-separable. In the rest of this section, we shall prove that, for every function $f : \mathcal{X} \to \mathbb{R}$ there exists a H-graph $\mathcal{H}_f$ that is the minimal one (with respect to the transitive relation $\preceq$) among all those H-graphs $\mathcal{H}$ such that f is $\mathcal{H}$-separable.

We start by proving the following fundamental technical result that will be instrumental to our future derivations.

Proposition 1: Let a function $f : \mathcal{X} \to \mathbb{R}$ be both $\mathcal{H}_1$-separable and $\mathcal{H}_2$-separable for two H-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$. Then, f is also $\mathcal{H}$-separable, where $\mathcal{H} = \mathcal{H}_1 \sqcap \mathcal{H}_2$.

Proof: Let Σ_f be the family of all H-graphs $\mathcal{H}$ such that f is $\mathcal{H}$-separable and, for $i = 1, 2$, let $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ in Σ_f be an H-graph such that f is $\mathcal{H}_i$-separable. We can then write

$$f(x) = \sum_{\mathcal{J} \in \mathcal{L}_1} g^{(1)}_{\mathcal{J}}(x_{\mathcal{J}}) = \sum_{\mathcal{K} \in \mathcal{L}_2} g^{(2)}_{\mathcal{K}}(x_{\mathcal{K}}) \,, \qquad (5)$$

for every x in $\mathcal{X}$. Then, for every $\mathcal{J}$ in $\mathcal{L}_1$, we have that

$$g^{(1)}_{\mathcal{J}}(x_{\mathcal{J}}) = \sum_{\mathcal{K} \in \mathcal{L}_2} g^{(2)}_{\mathcal{K}}(x_{\mathcal{K}}) - \sum_{\mathcal{I} \in \mathcal{L}_1 \setminus \{\mathcal{J}\}} g^{(1)}_{\mathcal{I}}(x_{\mathcal{I}}) \,. \qquad (6)$$

Now, observe that, since the lefthand side of (6) is independent from $x_{\mathcal{V} \setminus \mathcal{J}}$, so is its righthand side. Therefore, we may rewrite (6) as

$$g^{(1)}_{\mathcal{J}}(x_{\mathcal{J}}) = \sum_{\mathcal{K} \in \mathcal{L}_2} h^{(2)}_{\mathcal{K} \cap \mathcal{J}}(x_{\mathcal{K} \cap \mathcal{J}}) - \sum_{\substack{\mathcal{I} \in \mathcal{L}_1: \\ \mathcal{I} \neq \mathcal{J}}} h^{(1)}_{\mathcal{I} \cap \mathcal{J}}(x_{\mathcal{I} \cap \mathcal{J}}) \,, \qquad (7)$$

where, for $i = 1, 2$ and an arbitrarily chosen y in $\mathcal{X}$,

$$h^{(i)}_{\mathcal{K} \cap \mathcal{J}}(x_{\mathcal{K} \cap \mathcal{J}}) = g^{(i)}_{\mathcal{K}}(x_{\mathcal{K} \cap \mathcal{J}}, y_{\mathcal{K} \setminus \mathcal{J}}) \,, \qquad (8)$$

for every $\mathcal{K}$ in $\mathcal{L}_1 \cup \mathcal{L}_2$ and x in $\mathcal{X}$. It then follows from (5), (7) and (8) that

$$f(x) = \sum_{\mathcal{J} \in \mathcal{L}_1} \sum_{\mathcal{K} \in \mathcal{L}_2} h^{(2)}_{\mathcal{K} \cap \mathcal{J}}(x_{\mathcal{K} \cap \mathcal{J}}) - \sum_{\mathcal{J} \in \mathcal{L}_1} \sum_{\substack{\mathcal{I} \in \mathcal{L}_1: \\ \mathcal{I} \neq \mathcal{J}}} h^{(1)}_{\mathcal{I} \cap \mathcal{J}}(x_{\mathcal{I} \cap \mathcal{J}}) \,. \qquad (9)$$

Observe that (9) is not yet the desired separability decomposition because of the presence of the second term in its righthand side. However, a suitable iterative application of (9) allows us to prove the claim. To formally see this, it is convenient to first introduce the following definition. Given a H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, let the H-graphs $\sqcap^k \mathcal{H} = (\mathcal{V}, \mathcal{L}^k)$ be defined by

$$\mathcal{L}^k = \{\mathcal{J}_1 \cap \cdots \cap \mathcal{J}_k \mid \mathcal{J}_s \in \mathcal{L} \,, \mathcal{J}_s \neq \mathcal{J}_t \; \forall s \neq t\} \,, \qquad (10)$$

and notice that $\sqcap^2(\sqcap^k \mathcal{H}) \preceq \sqcap^{k+1} \mathcal{H}$. We can now interpret (9) as saying that

$$(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^2 \mathcal{H}_1) \in \Sigma_f \,. \qquad (11)$$

We now prove by induction that, for every $k \geq 2$,

$$(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1) \in \Sigma_f \,. \qquad (12)$$

Indeed, assume that (12) holds true for a certain k and let us prove it for $k + 1$. Considering that (11) is true for any pair of H-graphs $\mathcal{H}_1, \mathcal{H}_2$ in Σ_f, if we apply it replacing $\mathcal{H}_1$ with $(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)$, we obtain that

$$(((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)) \sqcap \mathcal{H}_2) \sqcup (\sqcap^2((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1))) \in \Sigma_f \,. \qquad (13)$$

Notice now that

$$((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)) \sqcap \mathcal{H}_2 \preceq \mathcal{H}_1 \sqcap \mathcal{H}_2 \,, \qquad (14)$$

and

$$\begin{aligned} \sqcap^2((\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^k \mathcal{H}_1)) &\preceq I(\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^2(\sqcap^k \mathcal{H}_1)) \\ &\preceq (\mathcal{H}_1 \sqcap \mathcal{H}_2) \sqcup (\sqcap^{k+1} \mathcal{H}_1) \,. \end{aligned} \qquad (15)$$

Relations (13), (14), and (15) imply (12) for $k + 1$. Therefore, (12) holds true for every value of k. Finally, notice that, for $k > |\mathcal{L}|$, $\sqcap^k \mathcal{H}$ is the H-graph with an empty set of hyperlinks. This proves that $\mathcal{H}_1 \sqcap \mathcal{H}_2 \in \Sigma_f$. ∎

Proposition 1 implies that there exists a finest simple H-graph with respect to which f is separable, as stated below.

Theorem 1: Given a function $f : \mathcal{X} \to \mathbb{R}$, there exists a unique simple H-graph $\mathcal{H}_f$ such that:

- f is $\mathcal{H}_f$-separable;
- $\mathcal{H}_f \preceq \mathcal{H}$ for every H-graph $\mathcal{H}$ for which f is $\mathcal{H}$-separable.

Proof: Consider the $\sqcap$-intersection $\sqcap\mathcal{H}$ of all H-graphs $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ such that f is $\mathcal{H}$-separable and then take the simple closure $\mathcal{H}_f = \overline{\sqcap\mathcal{H}}$. By Lemma 1, f is $\mathcal{H}_f$-separable. Moreover, if f is $\mathcal{H}'$-separable for some H-graph $\mathcal{H}'$, necessarily $\sqcap\mathcal{H} \preceq \mathcal{H}'$ and thus also $\mathcal{H}_f \preceq \mathcal{H}'$. Uniqueness follows from the fact that if $\mathcal{H}'$ is another simple H-graph for which the two properties above are true, necessarily, $\mathcal{H}_f \preceq \mathcal{H}' \preceq \mathcal{H}_f$. As both $\mathcal{H}_f$ and $\mathcal{H}'$ are simple H-graphs, this and (4) yield that $\mathcal{H}_f = \mathcal{H}'$. ∎

The H-graph $\mathcal{H}_f$ whose existence and properties are determined by Theorem 1 is called the *minimal H-graph* for f.

III. Minimal representations

While the results of Section II guarantee existence of a minimal H-graph for a function f, they are not constructive. In this section, we show how minimal representations can be indeed computed. We first focus on how to construct the minimal H-graph of a function. Then, we show how to compute the minimal decomposition by meas of orthogonal projections.

A. Construction of the minimal H-graph

To start with, we introduce some useful background on difference operators. For i in $\mathcal{V}$ and y_i in $\mathcal{A}_i$, define the linear operator $D_i^{(y_i)} : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ as

$$(D_i^{(y_i)} f)(x) = f(x) - f(y_i, x_{-i}) \,, \qquad \forall x \in \mathcal{X} \,.$$

Notice that for every i and j in $\mathcal{V}$, y_i in $\mathcal{A}_i$, and y_j in $\mathcal{A}_j$, the operators $D_i^{(y_i)}$ and $D_j^{(y_j)}$ commute. Therefore, for a nonempty subset $\mathcal{J} \subseteq \mathcal{V}$ and $y_\mathcal{J}$ in $\mathcal{X}_\mathcal{J}$, we can define the operator $D_\mathcal{J}^{(y_\mathcal{J})}$ on $\mathbb{R}^\mathcal{X}$ as the composition

$$D_\mathcal{J}^{(y_\mathcal{J})} = D_{j_1}^{(y_\mathcal{J})_{j_1}} \cdot \ldots \cdot D_{j_s}^{(y_\mathcal{J})_{j_s}} . \tag{16}$$

where $(j_1, j_2, \ldots, j_s)$ is an arbitrary ordering of $\mathcal{J}$. Finally, we define the operator $D_\mathcal{J} : \mathbb{R}^\mathcal{X} \to \mathbb{R}^{\mathcal{X}\times\mathcal{X}_\mathcal{J}}$ as

$$(D_\mathcal{J} f)(x, y_\mathcal{J}) := \left(D_\mathcal{J}^{(y_\mathcal{J})} f\right)(x) . \tag{17}$$

For i in $\mathcal{V}$ we denote $D_i := D_{\{i\}}$. When $\mathcal{J} = \mathcal{V}$ we often drop the index and simply write Df for $D_\mathcal{V} f$. E.g., when $\mathcal{V} = \{1, 2\}$, we get the formula

$$Df(x, y) = f(x_1, x_2) - f(x_1, y_2) - f(y_1, x_2) + f(y_1, y_2) .$$

Remark 3: In general, the computation of Df may prove computationally heavy as the number of variables $n = |\mathcal{V}|$ grows large. However, such complexity con be significantly reduced if the function f has symmetries. This is especially the case when f is permutation invariant. E.g., if $\mathcal{A}_i = \{0, 1\}$ for every i, so that $\mathcal{X} = \{0, 1\}^\mathcal{V}$, and $f : \mathcal{X} \to \mathbb{R}$ is such that

$$f(x) = g\Big(\sum_{i\in\mathcal{V}} x_i\Big) , \tag{18}$$

for a function $g : \{0, 1, \ldots, n\} \to \mathbb{R}$, then a direct recursive argument, leads to the following version of the inclusion-exclusion principle:

$$Df(0, \mathbb{1}) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} g(k) . \tag{19}$$

We now present two preparatory results.

Lemma 1: Let a function $f : \mathcal{X} \to \mathbb{R}$ admit the decomposition

$$f(x) = \sum_{\mathcal{J}\in\mathcal{L}} f_\mathcal{J}(x_\mathcal{J})$$

with respect to some H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$. Then,

(i) $D_{\mathcal{J}_0} f = 0$ for every $\mathcal{J}_0 \subseteq \mathcal{V}$ such that $\mathcal{J}_0 \nsubseteq \mathcal{J}$ for all $\mathcal{J}$ in $\mathcal{L}$;

(ii) if $\mathcal{H}$ is simple, then

$$D_\mathcal{J} f(x, y) = D_\mathcal{J} f_\mathcal{J}(x_\mathcal{J}, y_\mathcal{J}) , \qquad \forall \mathcal{J} \in \mathcal{L} . \tag{20}$$

Proof: Define the immersion operator $\iota_\mathcal{J} : \mathbb{R}^{\mathcal{X}_\mathcal{J}} \to \mathbb{R}^\mathcal{X}$ as $(\iota_\mathcal{J} g)(x) = g(x_\mathcal{J})$, and simply write $D_i f_\mathcal{J}$ for $D_i \iota_\mathcal{J} f_\mathcal{J}$. Then, notice that $D_i f_\mathcal{J} = 0$ for $\mathcal{J}$ in $\mathcal{L}$ and i in $\mathcal{V} \setminus \mathcal{J}$. This yields (i). If $\mathcal{H}$ is simple, then every $\mathcal{J}$ in $\mathcal{L}$ is such that $\mathcal{J} \nsubseteq \mathcal{K}$ for all $\mathcal{K} \neq \mathcal{J}$ in $\mathcal{L}$. Hence, $D_\mathcal{J} f_\mathcal{K} = 0$ for all $\mathcal{K} \neq \mathcal{J}$ and (20) follows. ■

Lemma 2: For a function $f : \mathcal{X} \to \mathbb{R}$, the following conditions are equivalent:

(a) $Df = 0$;

(b) there exist functions $f_{-i} : \mathcal{X}_{-i} \to \mathbb{R}$ for i in $\mathcal{V}$ such that

$$f(x) = \sum_{i\in\mathcal{V}} f_{-i}(x_{-i}) . \tag{21}$$

Proof: We prove more general statement (a')⇔(b'), of which (a)⇔(b) is a special case. Let $\mathcal{J} \subset \mathcal{V}$, $\mathcal{J} \neq \emptyset$. The following conditions are equivalent:

(a') $D_\mathcal{J} f = 0$;

(b') there exist functions $f_{-i} : \mathcal{X}_{-i} \to \mathbb{R}$ for i in $\mathcal{J}$ such that

$$f(x) = \sum_{i\in\mathcal{J}} f_{-i}(x_{-i}) \tag{22}$$

(a')⇒(b') We prove it by induction on $n = |\mathcal{J}|$. If $n = 1$, $\mathcal{J} = \{k\}$ for some $k \in \mathcal{V}$. We fix an arbitrary action $\bar{x}_k \in \mathcal{A}_k$ and write

$$0 = D_k^{(\bar{x}_k)} f(x) = f(x) - f(x_{-k}, \bar{x}_k) = 0 .$$

If we define $f_{-k}(x_{-k}) = f(x_{-k}, \bar{x}_k)$ we have the thesis. We now assume the result to be true for some $n \geq 1$ and we prove it for $n + 1$. We take any $k \in \mathcal{J}$ and fix an arbitrary action $\bar{x}_k \in \mathcal{A}_k$. We define $g = D_k^{(\bar{x}_k)} f$. By assumption, we have that $D_{\mathcal{J}\setminus\{k\}} g = 0$ and $|\mathcal{J} \setminus \{k\}| = |\mathcal{J}| - 1 = n$. Then, by the inductive hypothesis, we can decompose g as

$$g(x) = \sum_{i\in\mathcal{J}\setminus\{k\}} g_{-i}(x_{-i}) = f(x) - f(x_{-k}, \bar{x}_k).$$

By defining

$$f_{-i}(x_{-i}) = \begin{cases} g_{-i}(x_{-i}) \text{ if } i \neq k \\ f(x_{-k}, \bar{x}_k) \text{ if } i = k , \end{cases}$$

we obtain the claim, as we can express

$$f(x) = \sum_{i\in\mathcal{J}} f_{-i}(x_{-i}) .$$

(b')⇒(a') is a direct consequence of the fact that the difference operators commute and that $D_i f_{-i} = 0$ for every i in $\mathcal{V}$. ■

The following result is at the basis of our construction of minimal decompositions.

Proposition 2: For every function $f : \mathcal{X} \to \mathbb{R}$ there exists a simple H-graph $\tilde{\mathcal{H}}_f = (\mathcal{V}, \tilde{\mathcal{L}}_f)$ such that $D_\mathcal{J} f \neq 0$ for every $\mathcal{J}$ in $\tilde{\mathcal{L}}_f$ and f is $\tilde{\mathcal{H}}_f$-separable. Moreover, such H-graph is unique and coincides with the minimal H-graph of f, i.e., $\tilde{\mathcal{H}}_f = \mathcal{H}_f$.

Proof: Consider a simple H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ such that f is $\mathcal{H}$-separable:

$$f(x) = \sum_{\mathcal{J}\in\mathcal{L}} f_\mathcal{J}(x_\mathcal{J}) , \tag{23}$$

for some functions $f_\mathcal{J} : \mathcal{X}_J \to \mathbb{R}$. Suppose there exists $\mathcal{J}_0$ in $\mathcal{L}$ such that $D_{\mathcal{J}_0} f = 0$. Using (1) we deduce that

$$D_{\mathcal{J}_0} f_{\mathcal{J}_0}(x_{\mathcal{J}_0}, y_{\mathcal{J}_0}) = D_{\mathcal{J}_0} f(x, y) = 0 .$$

We can now apply (2) to $f_{\mathcal{J}_0}$ and conclude that

$$f_{\mathcal{J}_0}(x_{\mathcal{J}_0}) = \sum_{i\in\mathcal{J}_0} f_{\mathcal{J}_0\setminus i}(x_{\mathcal{J}_0\setminus i}) , \tag{24}$$

for some functions $f_{\mathcal{J}_0\setminus i} : \mathcal{X}_{\mathcal{J}_0\setminus i} \to \mathbb{R}$. Inserting (24) inside (23) we obtain another H-graph $\mathcal{H}' = (\mathcal{V}, \mathcal{L}')$ with

$$\mathcal{L}' = (\mathcal{L} \setminus \{\mathcal{J}_0\}) \bigcup_{i\in\mathcal{J}_0} \{\mathcal{J}_0 \setminus i\} ,$$

with respect to which f is also separable. Notice that, by construction, $\mathcal{H}' \preceq \mathcal{H}$ while $\mathcal{H} \npreceq \mathcal{H}'$, as $\mathcal{J}_0$ is not contained

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3314552

in any hyperlink of $\mathcal{H}'$. Hence, $\mathcal{H}' \prec \mathcal{H}$ and, by (3), also $\bar{\mathcal{H}}' \prec \bar{\mathcal{H}} = \mathcal{H}$. Hence, starting from $\mathcal{H}$, we have found a strictly finer simple H-graph with respect to which f is decomposable. As there are finitely many H-graphs with node set $\mathcal{V}$, this implies that there must exist a simple H-graph $\tilde{\mathcal{H}}_f = (\mathcal{V}, \tilde{\mathcal{L}}_f)$ such that $D_{\mathcal{J}} f \neq 0$ for all $\mathcal{J}$ in $\bar{\mathcal{L}}$ and f is $\tilde{\mathcal{H}}_f$-separable.

We now prove the remaining part of the statement. Consider the minimal H-graph $\mathcal{H}_f = (\mathcal{V}, \mathcal{L}_f)$ of f and the corresponding decomposition

$$f(x) = \sum_{\mathcal{J} \in \mathcal{L}_f} f_{\mathcal{J}}(x_{\mathcal{J}})$$

We prove that $\tilde{\mathcal{H}}_f \preceq \mathcal{H}_f$. If not, there would exist $\mathcal{J}_o$ in $\tilde{\mathcal{L}}_f$ such that $\mathcal{J}_o \not\subseteq \mathcal{J}$ for every $\mathcal{J}$ in $\mathcal{L}_f$. By Lemma 1(i), we obtain that $D_{\mathcal{J}_o} f = 0$. This contradicts the fact that $D_{\mathcal{J}_o} f \neq 0$, by the way $\tilde{\mathcal{H}}_f$ was constructed. Hence, $\tilde{\mathcal{H}}_f \preceq \mathcal{H}_f$. Since $\mathcal{H}_f$ is the minimal H-graph of f, we have that $\mathcal{H}_f \preceq \tilde{\mathcal{H}}_f$, so that property (4) implies the claim. ■

The proof of Proposition 2 suggests an algorithm to find the minimal H-graph for every function f:

Algorithm 1:

0. *Start from any simple H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ for which f is $\mathcal{H}$-separable*
1. *If there exists $\mathcal{J}_0$ in $\mathcal{L}$ such that $D_{\mathcal{J}_o} f = 0$, then*
   - $\mathcal{L} \leftarrow \overline{(\mathcal{L} \setminus \{\mathcal{J}_0\}) \cup \bigcup_{i \in \mathcal{J}_0} \{\mathcal{J}_0 \setminus i\}}$
   - *Go to 1.*
2. *Otherwise $\mathcal{H}_f = \mathcal{H}$*

Notice that our procedure determines the minimal H-graph of a function, but does not yield an explicit decomposition of the function with respect to this H-graph. This can be achieved by a projection technique that is illustrated in the next subsection.

B. Minimal decompositions of square integrable functions

Minimal decompositions can be explicitly computed by means of orthogonal projections. In the case when the configuration space $\mathcal{X}$ is not finite, the price to pay to use this approach is a restriction of the original space of functions $\mathbb{R}^{\mathcal{X}}$. Specifically, we equip each state set $\mathcal{A}_i$ with a σ-algebra $\mathcal{B}_i$ and a probability measure μ_i on $\mathcal{B}_i$. Then, we consider the usual product probability space $(\mathcal{X}, \mathcal{B}, \mu)$ where

$$\mathcal{B} = \bigotimes_{i \in \mathcal{V}} \mathcal{B}_i, \qquad \mu = \prod_{i \in \mathcal{V}} \mu_i \,.$$

From now on, we consider functions in the Hilbert space $\mathcal{S} = L^2(\mathcal{X}, \mathcal{B}, \mu)$ equipped with the usual inner product. For a subset $\mathcal{J} \subseteq \mathcal{V}$, let

$$\mathcal{B}_{\mathcal{J}} = \bigotimes_{i \in \mathcal{J}} \mathcal{B}_i \bigotimes_{i \in \mathcal{V} \setminus \mathcal{J}} \{\emptyset, \mathcal{A}_i\} \,,$$

be the sub-σ-algebra of cylinders in $\mathcal{B}$. The subset of functions $\mathcal{S}_{\mathcal{J}} = L^2(\mathcal{X}, \mathcal{B}_{\mathcal{J}}, \mu)$ coincides with those functions in $\mathcal{S}$ that only depend on the variables in $\mathcal{J}$: it is a closed subspace of $\mathcal{S}$. We denote by $P_{\mathcal{J}}$ the orthogonal projection onto $\mathcal{S}_{\mathcal{J}}$. It is a well known fact that such operator coincides with the conditional expectation given the σ-algebra $\mathcal{B}_{\mathcal{J}}$ and takes the following explicit form:

$$(P_{\mathcal{J}} f)(x) = \int_{\mathcal{X}_{\mathcal{J}^c}} f(x_{\mathcal{J}}, y) \mathrm{d}\mu_{\mathcal{J}^c}(y) \,, \qquad \forall f \in \mathcal{S} \,, \tag{25}$$

where

$$\mu_{\mathcal{J}^c} = \prod_{i \in \mathcal{V} \setminus \mathcal{J}} \mu_i$$

denotes the product probability measure on $\mathcal{X}_{\mathcal{J}^c}$.

Remark 4: In the special case when $\mathcal{A}_i$ is finite for every i in $\mathcal{V}$, we can chose μ_i as the uniform probability measure on $\mathcal{A}_i$. In this particular case, we have that $\mathcal{S} = \mathbb{R}^{\mathcal{X}}$ and formula (25) reduces to

$$(P_{\mathcal{J}} f)(x) = \frac{1}{|\mathcal{X}_{\mathcal{J}^c}|} \sum_{y \in \mathcal{X}_{\mathcal{J}^c}} f(x_{\mathcal{J}}, y) \,. \tag{26}$$

Given an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, the $\mathcal{H}$-separable functions inside $\mathcal{S}$ form a linear subspace $\mathcal{S}_{\mathcal{H}}$ that can be represented as the sum of subspaces

$$\mathcal{S}_{\mathcal{H}} = \sum_{\mathcal{J} \in \mathcal{L}} \mathcal{S}_{\mathcal{J}} \,.$$

We denote by $P_{\mathcal{H}}$ the orthogonal projector onto $\mathcal{S}_{\mathcal{H}}$ with respect to the inner product in $\mathcal{S}$. Since $\mathcal{S}_{\mathcal{H}}$ is the finite sum of the (possibly intersecting) subspaces $\mathcal{S}_{\mathcal{J}}$ as $\mathcal{J}$ varies in $\mathcal{L}$ and since the individual projectors $P_{\mathcal{J}}$ commute (as is evident from (25)), the orthogonal projector $P_{\mathcal{H}}$ can be expressed as an alternating sum of projectors $P_{\mathcal{J}}$. Precisely, we have the following result.

Proposition 3: Given an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ and a function f in $\mathcal{S}$,

$$f_{\mathcal{H}} = P_{\mathcal{H}} f = \sum_{\substack{\mathcal{K} \subset \mathcal{L} \\ \mathcal{K} \neq \emptyset}} (-1)^{|\mathcal{K}|+1} P_{\cap \mathcal{K}} f. \tag{27}$$

where we are indicating by $\cap \mathcal{K}$ the intersection of all subsets contained in $\mathcal{K}$.

Proof: See Appendix A. ■

Formula (27) allows one to construct the best $\mathcal{H}$-separable approximation, in the L^2 metric, of any given function f in $\mathcal{S}$. In situations where it is a priori known that f is $\mathcal{H}$-separable, this formula provides a decomposition of f with respect to $\mathcal{H}$. In particular, this fact can be used to complement our Algorithm 1 to find minimal decompositions. If the minimal H-graph $\mathcal{H}_f = (\mathcal{V}, \mathcal{L}_f)$ of a function f in $\mathcal{S}$ is known, we can build a decomposition of f with respect to $\mathcal{H}_f$ simply applying the above projection:

$$f = P_{\mathcal{H}_f} f = \sum_{\substack{\mathcal{K} \subset \mathcal{L} \\ \mathcal{K} \neq \emptyset}} (-1)^{|\mathcal{K}|+1} P_{\cap \mathcal{K}} f \tag{28}$$

computing the terms $P_{\cap \mathcal{K}} f$ using formula (25).

Example 2: Let $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{A}_i = \{0, 1\}$ for i in $\mathcal{V}$, and consider the permutation-invariant function

$$\begin{aligned} f(x_1, x_2, x_3) = \log\big(&1 + 2(x_1 + x_2 + x_3) + x_1^2 + x_2^2 + x_3^2 \\ &+ 3(x_1 x_2 + x x_3 + x_2 x_3) + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 \\ &+ x_3 x_1^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3\big) \,. \end{aligned}$$

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3314552

First, one can verify that $Df = 0$, so that f admits a non trivial decomposition. Applying Algorithm 1, we then set

$$\mathcal{L} \leftarrow \{\mathcal{V}\} \setminus \{\mathcal{V}\} \cup \{\mathcal{V} \setminus \{i\} : i \in \mathcal{V}\} = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}.$$

To rule out the possibility that f admits an even finer separation, we compute $D_{\mathcal{J}} f$ for $\mathcal{J}$ in $\mathcal{L}$. E.g., for $\mathcal{J} = \{2,3\}$ we get

$$\begin{aligned} 0 \;&\neq\; \log 3/4 \\ &= f(0,0,0) - f(0,0,1) - f(0,1,0) + f(0,1,1) \\ &= D_{\{2,3\}} f((0,0,0),(1,1)), \end{aligned}$$

and similarly for the other hyperlinks due to the permutation invariance of f. It then follows that the minimal H-graph of f is indeed $\mathcal{H}_f = (\mathcal{V}, \mathcal{L})$ where $\mathcal{L} = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$.

We then compute the various projections $P_{\cap\mathcal{K}} f$ for $\mathcal{K} \subset \mathcal{L}$, $\mathcal{K} \neq \emptyset$. For instance, we get

$$\begin{aligned} P_{\{1,2\}} f(x) =\; & \frac{1}{2}\left(f(x_1,x_2,0) + f(x,x_2,1)\right) \\ =\; & \log(x_1+x_2+1) + \frac{1}{2}\log\left((x_1+1)(x_1+2)\right) \\ & + \frac{1}{2}\log\left((x_2+1)(x_2+2)\right), \end{aligned}$$

$$\begin{aligned} P_{\{1\}} f(x) &= \frac{1}{4}\left(f(x,0,0) + f(x,0,1) + f(x,1,0) + f(x,1,1)\right) \\ &= \log\left((x_1+1)(x_2+1)\right) + \frac{1}{4}\log(12), \end{aligned}$$

while $P_{\{1,3\}} f$, $P_{\{2,3\}} f$, $P_{\{2\}} f$, and $P_{\{3\}} f$ can be obtained by symmetry. Finally, (28) gives the minimal decomposition

$$\begin{aligned} f(x) &= \sum_{\substack{\mathcal{K} \subset \mathcal{L} \\ \mathcal{K} \neq \emptyset}} (-1)^{|\mathcal{K}|+1} P_{\cap\mathcal{K}} f(x) \\ &= \log(x_1+x_2+1) + \log(x_2+x_3+1) + \log(x_1+x_3+1). \end{aligned}$$

IV. On the separability of games

In this section, we introduce the concept of separable game and we apply the results of the previous sections to derive minimal decompositions.

A. Separable games

We consider strategic form games with finite nonempty player set $\mathcal{V}$ and a nonempty action set $\mathcal{A}_i$ for each player i in $\mathcal{V}$. Recall the standing notation $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$. We shall refer to two configurations x and y in $\mathcal{X}$ as i-comparable and write $x \sim_i y$ when $x_{-i} = y_{-i}$, i.e., when x and y coincide except for possibly in their i-th entry.

Assume we have equipped $\mathcal{X}$ with a product σ-algebra $\mathcal{B}$ and product probability measure μ as described in (III-B) and consider the space $\mathcal{S} = L^2(\mathcal{X}, \mathcal{B}, \mu)$ of square-integrable functions on the product probability space $(\mathcal{X}, \mathcal{B}, \mu)$. We let each player i in $\mathcal{V}$ be equipped with a utility function $u_i : \mathcal{X} \to \mathbb{R}$ in $\mathcal{S}$. We shall identify a game with player set $\mathcal{V}$ and strategy profile space $\mathcal{X}$ with the vector u assembling all the players' utilities. Notice that, in this way, the set of all games with player set $\mathcal{V}$ and strategy profile space $\mathcal{X}$, to be denoted as $\mathcal{U}$, is isomorphic to the Hilbert space $\mathcal{S}^{\mathcal{V}}$.

A game u is referred to as *non-strategic* if the utility of each player i in $\mathcal{V}$ does not depend on her own action, i.e., if

$$u_i(x) = u_i(y), \qquad \forall x, y \in \mathcal{X} \text{ s.t. } y \sim_i x. \tag{29}$$

Two games u and $\tilde{u}$ are referred to as *strategically equivalent* if their difference is a non-strategic game. Strategic equivalence is an equivalence relation on $\mathcal{U}$. In the rest of this paper, we will focus on properties of a game that are invariant with respect to strategic equivalence.

To describe the separability property of a game, we introduce a new type of H-graph. A *forward directed hypergraph* (*FDH-graph*) [16] is a pair $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ consisting of a finite node set $\mathcal{V}$ and of a finite hyperlink set $\mathcal{D}$, where each hyperlink $d = (i, \mathcal{J})$ in $\mathcal{D}$ is an ordered pair of a node i in $\mathcal{V}$ (to be referred to as its tail node) and a nonempty subset of head nodes $\mathcal{J} \subseteq \mathcal{V} \setminus \{i\}$ (to be referred to as the hyperlink's head set).

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of links, can naturally be seen as a FDH-graph on $\mathcal{V}$ whose hyperlinks are the original links in the graph (i,j) interpreted as $(i, \{j\})$; with slight abuse of notation in the following we will identify such FDH-graph with the original graph $\mathcal{G}$. There is also a different FDH-graph that can be naturally associated to a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. This is the FDH-graph $\mathcal{F}^{\mathcal{G}} = (\mathcal{V}, \mathcal{D}^{\mathcal{G}})$ where $\mathcal{D}^{\mathcal{G}}$ consists of the hyperlinks $(i, \mathcal{N}_i)$ for i in $\mathcal{V}$, where $\mathcal{N}_i$ indicates the out-neighborhood of node i.

The following definition models the way the utility of a player in a multi-player game depends on the actions of the remaining players and incorporates the notion of strategic equivalence.

Definition 3: Given a FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, a game u in $\mathcal{U}$ is *$\mathcal{F}$-separable* if the utility of each player i in $\mathcal{V}$ can be decomposed as

$$u_i(x) = \sum_{(i,\mathcal{J}) \in \mathcal{D}} u_i^{\mathcal{J}}(x_i, x_{\mathcal{J}}) + n_i(x_{-i}), \tag{30}$$

where $u_i^{\mathcal{J}} : \mathcal{A}_i \times \prod_{j \in \mathcal{J}} \mathcal{A}_j \to \mathbb{R}$ are functions that depend on the actions of player i and of players in the subset $\mathcal{J}$ of head nodes of hyperlink $(i, \mathcal{J})$ only, while $n_i : \mathcal{X}_{-i} \to \mathbb{R}$ is a non-strategic component that does not depend on the action of player i.

Definition 3 captures not only locality of the relative influences among players in the game, but also the fact that players may have separate interactions with different groups of other players. Up to a non-strategic component, this grouping of the player set is modeled as a FDH-graph with node set coinciding with the player set $\mathcal{V}$ and where each group jointly influencing player i corresponds to a directed hyperlink with tail node i. This notion of separability, albeit based on an undirected H-graph and not considering strategic equivalence, was originally introduced in [9].

A special case is when the FDH-graph is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A game u in $\mathcal{U}$ that is $\mathcal{G}$-separable admits a representation of the following type:

$$u_i(x) = \sum_{j \in \mathcal{N}_i} u_{ij}(x_i, x_j) + n_i(x_{-i}) \qquad \forall x \in \mathcal{X}, \tag{31}$$

where $u_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathbb{R}$ for (i,j) in $\mathcal{E}$. We refer to such games as *pairwise-separable*. Apart for the possible presence of nonstrategic terms, these games are instances of polymatrix games. In the special case when $\mathcal{G}$ is undirected, such games can be interpreted as if each pair of players i, j connected by a link were involved in a two-player game having utility functions, respectively, $u_{ij}(x_i, x_j)$ and $u_{ji}(x_j, x_i)$. Each player i in $\mathcal{V}$ chooses the same action x_i in $\mathcal{A}_i$ for all the two-player games she is engaged in and gets a utility that is the aggregate of the utilities of all such games.

The following is a popular example of a pairwise-separable game.

Example 3 (Network coordination games): For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a *network coordination game* on $\mathcal{G}$ is a game u where every player i in $\mathcal{V}$ has binary action set $\mathcal{A}_i = \{0,1\}$ and utility function

$$u_i(x) = \sum_{j \in \mathcal{N}_i} \zeta(x_i, x_j)\,, \tag{32}$$

where $\zeta(x_i, x_j) = \zeta(x_j, x_i)$ is a symmetric function such that $\zeta(0,0) \geq \zeta(0,1) = \zeta(1,0)$ and $\zeta(1,1) \geq \zeta(0,1) = \zeta(1,0)$. Clearly, every network coordination game with utilities (32) is a pairwise-separable game on $\mathcal{G}$.

The concept of separable game encompasses that of graphical game. Given a graph $\mathcal{G}$, games that are separable with respect to the FDH-graph $\mathcal{F}^{\mathcal{G}}$ are called graphical with respect to $\mathcal{G}$, or simply $\mathcal{G}$-games. Pairwise-separable games with respect to $\mathcal{G}$ are an example of such games. An example of a graphical game that is not pairwise-separable is reported below.

Example 4 (Best-shot public good game): Consider a graph $\mathcal{G}$ and the game where every player i in $\mathcal{V}$ has binary action set $\mathcal{A}_i = \{0,1\}$ and utility:

$$u_i(x) = \begin{cases} 1-c & \text{if} \quad x_i = 1 \\ 1 & \text{if} \quad x_i = 0 \text{ and } x_j = 1 \text{ for some } j \in \mathcal{N}_i \\ 0 & \text{if} \quad x_i = 0 \text{ and } x_j = 0 \text{ for every } j \in \mathcal{N}_i\,. \end{cases} \tag{33}$$

The game u constructed in this way is an instance the so called "public good games" [4]. It models a more complex behaviour for the population $\mathcal{V}$ compared to simple coordination: players benefit form acquiring some good, represented by taking action 1 and which is public in the sense that it can be lent from one player to another. Taking action 1 has a cost c, so players would prefer that one of their neighbors takes that action, but taking the action and paying the cost is still the best choice if no one of their neighbors does. The best-shot public good game is a graphical game on $\mathcal{G}$ but it is not pairwise-separable.

B. Normalized games and minimal representations

As for functions, also for games we can define the concept of minimal representations. The idea is to reformulate separability for a game as separability of the single utility functions.

To this aim, it is useful to introduce the 'local' H-graphs associated to an FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$

$$\mathcal{H}_i^{\mathcal{F}} = (\mathcal{V}, \mathcal{L}_i^{\mathcal{F}}), \quad \mathcal{L}_i^{\mathcal{F}} = \{\{i\} \cup \mathcal{J} : (i, \mathcal{J}) \in \mathcal{D}\}, \quad i \in \mathcal{V}\,. \tag{34}$$

If the terms $n_i(x_{-i})$ in expression (30) were not present, then, effectively, the $\mathcal{F}$-separability of u would be equivalent to each utility function u_i to be $\mathcal{H}_i^{\mathcal{F}}$-separable in the sense of Definition 1. To maintain this equivalence in presence of the non-strategic terms $n_i(x_{-i})$, we would need to augment each local H-graph $\mathcal{H}_i^{\mathcal{F}}$ with the hyperlink $(i, \mathcal{V}_{-i})$. We take a slightly different road and prove that, for every game u, there always exists a game strategically equivalent to u for which a decomposition like (30) holds without the presence of the non-strategic terms $n_i(x_{-i})$.

The space $\mathcal{U} = \mathcal{S}^{\mathcal{V}}$ of all games is a natural Hilbert space with respect to the inner product[3]

$$\langle u, v \rangle_{\mathcal{U}} = \sum_{i \in \mathcal{V}} \int_{\mathcal{X}} u_i(x) v_i(x) \mathrm{d}\mu(x)\,. \tag{35}$$

We indicate with $\mathcal{U}_\bullet$ the subspace of non strategic games. Notice that u belongs to $\mathcal{U}_\bullet$ if and only if u_i belongs to $\mathcal{S}_{-i}$ for every i in $\mathcal{V}$. Consequently, the orthogonal space of $\mathcal{U}_\bullet$ is described by

$$\begin{aligned} u \in \mathcal{U}_\bullet^\perp \quad &\Leftrightarrow \quad u_i \in \mathcal{S}_{-i}^\perp \; \forall i \in \mathcal{V} \\ &\Leftrightarrow \quad \int_{\mathcal{X}_i} u_i(y_i, x_{-i}) \mathrm{d}\mu_i(y_i) = 0, \quad \forall x \in \mathcal{X}, \forall i \in \mathcal{V}\,. \end{aligned} \tag{36}$$

Games in $\bar{\mathcal{U}} = \mathcal{U}_\bullet^\perp$ are called *normalized* [17]. In every class of strategically equivalent games there exists a unique normalized game $\bar{u}$ in $\bar{\mathcal{U}}$. As u belongs to $\bar{\mathcal{U}}$ if and only if u_i belongs to $\mathcal{S}_{-i}^\perp$ for all i, we can use formula (25) to represent the orthogonal projection onto $\bar{\mathcal{U}}$. Precisely, for every finite game u, the normalized strategically equivalent game $\bar{u}$ can be obtained as

$$\bar{u}_i(x) = u_i(x) - (P_{-i} u_i)(x) = u_i(x) - \int_{\mathcal{X}_i} u_i(y_i, x_{-i}) \mathrm{d}\mu_i(y_i)\,, \tag{37}$$

for all x in $\mathcal{X}$ and i in $\mathcal{V}$.

It turns out that studying the separability of a game u is equivalent to studying that of its normalized projection $\bar{u}$. Indeed, we have the following result.

Proposition 4: For every game u and FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ the following conditions are equivalent:

(i) u is $\mathcal{F}$-separable;
(ii) $\bar{u}$ is $\mathcal{F}$-separable;
(iii) $\bar{u}_i$ is $\mathcal{H}_i^{\mathcal{F}}$-separable for every i in $\mathcal{V}$.

Moreover, if any of the above is satisfied, $\bar{u}$ admits a representation of type

$$\bar{u}_i(x) = \sum_{(i,\mathcal{J}) \in \mathcal{D}} \bar{u}_i^{\mathcal{J}}(x_i, x_{\mathcal{J}})\,, \tag{38}$$

where $\bar{u}_i^{\mathcal{J}}$ are normalized in the sense that

$$\int_{y_i \in \mathcal{X}_i} \bar{u}_i^{\mathcal{J}}(y_i, x_{\mathcal{J}}) \mathrm{d}\mu_i(y_i) = 0\,, \qquad \forall x_{\mathcal{J}} \in \mathcal{X}_{\mathcal{J}}\,.$$

Proof: Conditions (i) and (ii) are equivalent by the definition of separability that is invariant over strategically equivalent games. Suppose now that (i) holds and that u admits

[3] We acknowledge that the same inner product (35) has been used to analyze different decompositions of games [17], [18].

a representation as in (30). By applying the normalization operator, we obtain for $\bar{u}$ a representation as (38). Finally, notice that (38) is equivalent to requiring that each $\bar{u}_i$ is $\mathcal{H}_i^{\mathcal{F}}$-separable. ■

Two remarks are in order concerning the previous result. First, Proposition 4 establishes that for normalized games, separability can be expressed in the stronger form (38). Second, it reduces the separability of a game to the separability of the single utility functions. This paves the way to apply the results of Section II that yield the existence of minimal separable representations and algorithms to construct it.

We first notice that using local H-graphs, we can extend to FDH-graphs most of the concepts that we introduced for H-graphs. In particular, given two FDH-graphs $\mathcal{F}_k = (\mathcal{V}, \mathcal{D}_k)$ for $k = 1, 2$, we write $\mathcal{F}_1 \preceq \mathcal{F}_2$ if $\mathcal{H}_i^{\mathcal{F}_1} \preceq \mathcal{H}_i^{\mathcal{F}_2}$ for every i in $\mathcal{V}$. We say that an FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ is simple if $\mathcal{H}_i^{\mathcal{F}}$ is simple for every i in $\mathcal{V}$. The simple closure is similarly obtained closing each single local H-graph:

$$\bar{\mathcal{F}} = (\mathcal{V}, \bar{\mathcal{D}}), \qquad \bar{\mathcal{D}} = \{(i, \mathcal{J}) \,|\, i \in \mathcal{V},\ \mathcal{J} \cup \{i\} \in \mathcal{L}_i^{\bar{\mathcal{F}}}\}\,.$$

We have the following result.

Theorem 2: Given a game u in $\mathcal{U}$, there exists a unique simple FDH-graph $\mathcal{F}_u$ such that:

- u is $\mathcal{F}_u$-separable;
- $\mathcal{F}_u \preceq \mathcal{F}$ for every FDH-graph $\mathcal{F}$ for which u is $\mathcal{F}$-separable.

Proof: Proposition 4 implies that it is sufficient to prove the statement for normalized games. In this case, by virtue of Theorem 1, the result follows if we define $\mathcal{F}_u = (\mathcal{V}, \mathcal{D}_u)$ where $\mathcal{D}_u = \{(i, \mathcal{J}) \,|\, i \in \mathcal{V},\ \mathcal{J} \cup \{i\} \in \mathcal{L}_{u_i}\}$. ■

For pairwise-separable games, the definition of minimality implies the following.

Corollary 1: The minimal FDH-graph of a pairwise-separable game u in $\mathcal{U}$ is a graph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E})$.

The following examples show how to check for minimality of a given separable representation and how to obtain the minimal representation for specific games.

Example 5 (Best-shot public good game cont.): Consider the best-shot public good game u with respect to a graph $\mathcal{G}$, as illustrated in Example 4. The minimal FDH-graph for u is $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ where

$$\mathcal{D} = \{(i, \mathcal{N}_i),\ i \in \mathcal{V}\}\,. \tag{39}$$

Since u is $\mathcal{G}$-graphical, it follows that it is $\mathcal{F}$-separable. To prove the minimality of $\mathcal{F}$ we exploit Proposition 4. Precisely, we show that $D_{\mathcal{N}_i^\bullet} \bar{u}_i(0, \mathbb{1}) = D_{\mathcal{N}_i^\bullet} u_i(0, \mathbb{1}) \neq 0$ for all i, where the first equality follows from the fact that $D_i(u_i - \bar{u}_i) = D_i n_i = 0$ since the non-strategic term n_i does not depend on i. To this aim, we first denote by $f_i^a : \{0,1\}^{\mathcal{N}_i} \to \mathbb{R}$ defined by $f_i^a(x) = u_i(a, x)$ the utility of player i in $\mathcal{V}$ when she plays a and her neighbors play x. Notice that

$$f_i^0(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad f_i^1(x) = 1 - c\,, \qquad \forall x \in \mathcal{X}\,.$$

This shows, in particular, that the two functions only depend on the cardinality of players in $\mathcal{N}_i$ that are playing 1. We can thus apply formula (19) in Remark 3 and we obtain

$$D_{\mathcal{N}_i} f_i^0(0, \mathbb{1}) = \sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k} = (1-1)^m - 1 = -1\,,$$

$$D_{\mathcal{N}_i} f_i^1(0, \mathbb{1}) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (1 - c) = 0\,.$$

Therefore

$$D_{\mathcal{N}_i^\bullet} u_i(0, \mathbb{1}) = D_{\mathcal{N}_i} f_i^1(0, \mathbb{1}) - D_{\mathcal{N}_i} f_i^0(0, \mathbb{1}) = -1 \neq 0\,.$$

This shows that all utility functions do not admit a finer decomposition. As a result, the FDH-graph described by (39) is the minimal one for u.

Example 6 (Two-level coordination game): We consider the following variation of the network coordination game presented in Example 3. We fix a set of players $\mathcal{V}$ and the same action set for all players: $\mathcal{A} = \mathcal{A}_i = \{0, 1\}$ for all i in $\mathcal{V}$. For every pair of players i, j we consider functions $u_{ij} : \mathcal{A}^2 \to \mathbb{R}$ defined as the pairwise utility function ζ of the network coordination game (32). We now consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and, for every i in $\mathcal{V}$, functions $\tilde{u}_i : \prod_{j \in \mathcal{N}_i^\bullet} \mathcal{A}_j \to \mathbb{R}$ given by

$$\tilde{u}_i(x) = \begin{cases} L & \text{if } x_i = x_k\ \forall k \in \mathcal{N}_i \\ 0 & \text{otherwise,} \end{cases}$$

where $L > 0$. We finally define the utility of player i as:

$$u_i(x) = \sum_{j \neq i} u_{ij}(x_i, x_j) + \tilde{u}_i(x_{\mathcal{N}_i^\bullet})$$

The interpretation is the following: each player has a benefit that is in part linearly proportional to the number of individuals playing the same action and, additionally, it has an extra value L if the player's action is in complete agreement with her neighbors. This type of utility function models, for example, the situation where players' actions represent the acquisition of a new technology and the benefit to a player comes from two channels: the range of diffusion of the technology in the whole population and the opportunity to use such technology with her strict collaborators. By construction, u is separable with respect to the FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$ where

$$\mathcal{D} = \{(i, \{j\})\,,\ j \neq i\} \cup \{(i, \mathcal{N}_i),\ i \in \mathcal{V}\}\,.$$

However, $\mathcal{F}$ is not simple and its simple closure is given by $\overline{\mathcal{F}} = (\mathcal{V}, \overline{\mathcal{D}})$ where

$$\overline{\mathcal{D}} = \{(i, \{j\})\,,\ j \notin \mathcal{N}_i^\bullet\} \cup \{(i, \mathcal{N}_i),\ i \in \mathcal{V}\}\,.$$

The FDH $\overline{\mathcal{F}}$ is minimal for u as can be checked analogously to what was done in Example 5.

V. Separable potential games

In this section, we focus on potential games and study how their separability properties are intertwined with the separability of the corresponding potential functions. This is the content of our next result Theorem 3. We then derive, as a corollary, results on graphical potential games first appeared in

This article has been accepted for publication in IEEE Transactions on Control of Network Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TCNS.2023.3314552

[14] and we provide an alternative proof of the Hammersley-Clifford theorem for Markov Random Fields.

A game u in $\mathcal{U}$ is referred to as *(exact) potential* [19] if it is strategically equivalent to a game where all utility functions are identical (identical interest game), namely if there exists a function $\phi : \mathcal{X} \to \mathbb{R}$ —called potential— such that

$$u_i(x) = \phi(x) + n_i(x_{-i}) , \qquad \forall x \in \mathcal{X} ,$$

for every i in $\mathcal{V}$, where $n_i(x_{-i})$ is a non-strategic component.

In order to connect the separability of ϕ to the separability of the game, we define an operation that associates an FDH-graph to every H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$ by putting

$$\mathcal{F}^{\mathcal{H}} = (\mathcal{V}, \mathcal{D}^{\mathcal{H}}), \qquad \mathcal{D}^{\mathcal{H}} = \{(i, \mathcal{J}) \,|\, i \in \mathcal{V}\backslash\mathcal{J},\ \{i\}\cup\mathcal{J} \in \mathcal{L}\} . \tag{40}$$

We have the following result.

Theorem 3: Let u in $\mathcal{U}$ be a potential game with potential function ϕ. Then, the minimal FDH-graph of u is the FDH-graph associated to the minimal H-graph of ϕ, i.e.,

$$\mathcal{F}_u = \mathcal{F}^{\mathcal{H}_\phi} . \tag{41}$$

Proof: For the sake of simplicity of notation, we put $\mathcal{F}^\bullet = \mathcal{F}^{\mathcal{H}_\phi}$. Since ϕ is $\mathcal{H}_\phi$-separable, we have that the game $\phi\mathbb{1}$ consisting of all utilities equal to ϕ is $\mathcal{F}^\bullet$-separable by construction. Since separability is by definition invariant over strategically equivalence classes, also u is $\mathcal{F}^\bullet$-separable.

It remains to be proven that if u is $\mathcal{F}$-separable, for some FDH-graph $\mathcal{F} = (\mathcal{V}, \mathcal{D})$, then, $\mathcal{F}^\bullet \preceq \mathcal{F}$. Notice that the corresponding local H-graphs (34) of $\mathcal{F}^\bullet$ are given by

$$\mathcal{H}_i^\bullet = (\mathcal{V}, \mathcal{L}_i^\bullet) , \qquad \mathcal{L}_i^\bullet = \{\mathcal{K} \in \mathcal{L}_\phi \mid i \in \mathcal{K}\} , \qquad i \in \mathcal{V} .$$

We denote by $\mathcal{H}_i = (\mathcal{V}, \mathcal{L}_i)$ the local H-graphs associated with $\mathcal{F}$ and by $\mathcal{H}_i^+ = (\mathcal{V}, \mathcal{L}_i \cup \{\mathcal{V} \setminus \{i\}\}))$ the ones augmented of the hyperlinks relative to nonstrategic terms. Since also $\phi\mathbb{1}$ is $\mathcal{F}$-separable, by definition of game separability, we have that ϕ is $\mathcal{H}_i^+$-separable for every i in $\mathcal{V}$. This implies that $\mathcal{H}_\phi \preceq \mathcal{H}_i^+$ for every i in $\mathcal{V}$. Since all hyperlinks of $\mathcal{H}_i^\bullet$ are hyperlinks of $\mathcal{H}_\phi$ containing i by construction, they cannot be contained inside $\mathcal{V} \setminus \{i\}$ and thus, necessarily, $\mathcal{H}_i^\bullet \preceq \mathcal{H}_i$ for every i in $\mathcal{V}$. By definition, this says that $\mathcal{F}^\bullet \preceq \mathcal{F}$. ■

The above result implies the following relation between the minimal graph of a potential game and the separability of the potential function. Given an undirected graph $\mathcal{G}$ we denote by $\mathcal{C}l(\mathcal{G})$ the set of maximal cliques in $\mathcal{G}$, and let $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l} = (\mathcal{V}, \mathcal{C}l(\mathcal{G}))$ be the cliques H-graph of $\mathcal{G}$.

Corollary 2: Let u in $\mathcal{U}$ be a potential game with potential function ϕ. Then, u is a $\mathcal{G}$-game for an undirected graph $\mathcal{G}$ if and only if its potential function ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable.

Proof: Consider the minimal FDH-graph $\mathcal{F}_u = (\mathcal{V}, \mathcal{D}_u)$ of u and the minimal H-graph $\mathcal{H}_\phi = (\mathcal{V}, \mathcal{L}_\phi)$ of ϕ. Recall that they are related through formula (41).

Suppose that u is a $\mathcal{G}$-game for some undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $\mathcal{K}$ in $\mathcal{L}_\phi$. Then, for every i in $\mathcal{K}$, (41) implies that $(i, \mathcal{K} \setminus \{i\})$ belongs to $\mathcal{D}_u$. Since $\mathcal{F}_u \preceq \mathcal{F}^{\mathcal{G}}$, it must hold that $\mathcal{K} \setminus \{i\} \subseteq \mathcal{N}_i$. As this is true for every i in $\mathcal{K}$, we have that $\mathcal{K}$ is a clique in $\mathcal{G}$. Therefore, $\mathcal{H}_\phi \preceq \mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$, thus showing that ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable.

Conversely, if ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable, then necessarily $\mathcal{H}_\phi \preceq \mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$, so that every undirected hyperlink in $\mathcal{L}_\phi$ is contained in a clique of $\mathcal{G}$. From formula (41), we obtain that every hyperlink $(i, \mathcal{J})$ in $\mathcal{D}_u$ is such that $\{i\} \cup \mathcal{J} \subseteq \mathcal{K}$ for some clique $\mathcal{K}$ of $\mathcal{G}$. This implies in particular that u is a $\mathcal{G}$-game. ■

Remark 5: The second part of Corollary 2 is equivalent to Theorems 4.2 and 4.4 in [14]. In that paper, the authors prove their results relying on the Hammersley-Clifford theorem, while our proofs are instead self-contained.

In fact, we wish to emphasize that Theorem 3 is more informative than Corollary 2. Indeed, the latter does not relate the minimal separability of a potential game with that of its corresponding potential function. This is evident, e.g., in the special case of a potential game u that is pairwise separable with respect to an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. In this case, (41) implies that the potential function ϕ is separable with respect to the H-graph $\mathcal{H}$ coinciding with $\mathcal{G}$, i.e., that it can be decomposed in a pairwise fashion

$$\phi(x) = \sum_{(i,j)\in\mathcal{E}} \phi_{ij}(x_i, x_j) ,$$

for some symmetric functions $\phi_{ij}(x_i, x_j) = \phi_{ji}(x_j, x_i)$. This is in general a much finer decomposition than the one on the maximal cliques of $\mathcal{G}$.

Example 7 (Synchronization games): Synchronization games are a family of separable games [10] that model high-order coordination behavior of players. They are an extension of network coordination games on graphs presented in Example 3 to the setting of H-graphs: players, corresponding to nodes of an H-graph, aim at simultaneously coordinating on some action with multiple *groups* of players, represented by hyperlinks. When all members of an hyperlink choose the same action, each player receives a positive payoff, which is additively combined with the ones deriving from all hyperlinks the player participates in. Formally, given an H-graph $\mathcal{H} = (\mathcal{V}, \mathcal{L})$, an action space $\mathcal{A}$ and a weight function w that associates to any action a in $\mathcal{A}$ and hyperlink $\mathcal{K}$ in $\mathcal{L}$ a positive integer weight $w(a, \mathcal{K}) > 0$, the synchronization game on $\mathcal{H}$ with player set $\mathcal{V}$ is defined by the utility functions:

$$u_i(x) = \sum_{\substack{\mathcal{K}\in\mathcal{L} \\ i\in\mathcal{K}}} \bar{w}(x, \mathcal{K}), \qquad \forall i \in \mathcal{V}, \forall x \in \mathcal{X} = \mathcal{A}^{\mathcal{V}} , \tag{42}$$

where

$$\bar{w}(x, \mathcal{K}) = \begin{cases} 0 & \text{if } \exists j, k \in \mathcal{K} : x_j \neq x_k \\ w(x_j, \mathcal{K}) & \text{for any } j \in \mathcal{K}\text{, otherwise.} \end{cases}$$

An instance of synchronization game is the two-level coordination game introduced in Example 6. As shown in [10, Lemma 6], every synchronization game on an H-graph is a potential game with potential function

$$\phi(x) = \sum_{\mathcal{K}\in\mathcal{L}} \bar{w}(x, \mathcal{K}) , \tag{43}$$

whose minimal H-graph is $\mathcal{H}_\phi = \bar{\mathcal{H}}$. It can be directly verified from the definition that, according to Theorem 3, a synchronization game on a H-graph $\mathcal{H}$ is minimally separable with respect to the FDH-graph $\mathcal{F}_u = \mathcal{F}^{\bar{\mathcal{H}}}$.

We conclude this section by showing how the celebrated Hammersley-Clifford Theorem on the structure of Markov Random Fields can be deduced from Corollary 2. Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a vector of finite-valued random variables $X = (X_i)_{i \in \mathcal{V}}$ indexed by the nodes of $\mathcal{G}$. Denote by $\mathcal{A}_i$ the set where the random variable X_i takes its values. Put $\mathcal{X} = \prod_i \mathcal{A}_i$. For every subset $\mathcal{W} \subseteq \mathcal{V}$, let $X_{\mathcal{W}}$ denote the subvector of X consisting of the random variables X_i with i in $\mathcal{W}$. We shall refer to the random vector X as positive if its probability distribution is equivalent to the product of the marginals, namely, if $\mathbb{P}(X = x) > 0$ whenever $\mathbb{P}(X_i = x_i) > 0$ for every i in $\mathcal{V}$. We shall refer to the random vector X as a Markov Random Field on $\mathcal{G}$ if, for every node i in $\mathcal{V}$, the random variables X_i and $X_{\mathcal{V} \setminus \mathcal{N}_i^\bullet}$ are conditionally independent given $X_{\mathcal{N}_i}$.[4]

Theorem 4: Let X be a positive Markov Random Field on an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then, its probability distribution admits the following decomposition:

$$\mathbb{P}(X = x) = \prod_{\mathcal{C} \in \mathcal{C}l(\mathcal{G})} \zeta_{\mathcal{C}}(x_{\mathcal{C}}), \quad \forall x \in \mathcal{X}\,, \tag{44}$$

where $\mathcal{C}l(\mathcal{G})$ is the family of maximal cliques of the graph $\mathcal{G}$.

Proof: Without loss of generality we can assume that $\mathbb{P}(X_i = x_i) > 0$ for every i in $\mathcal{V}$ and x_i in $\mathcal{A}_i$ so that, by the positivity assumption we have that $\mathbb{P}(X = x) > 0$ for every x in $\mathcal{X}$. Let

$$\phi(x) = \log \mathbb{P}(X = x)\,, \qquad \forall x \in \mathcal{X}\,. \tag{45}$$

Then, consider the identical interest game $u = u^\phi$ in $\mathcal{U}$ with utility functions $u_i(x) = \phi(x)$ for every i in $\mathcal{V}$. Clearly, u is a potential game with potential function ϕ. We shall now prove that u is $\mathcal{G}$-graphical. Indeed, conditional independence implies that

$$\begin{aligned}
\mathbb{P}(X = x) &= \mathbb{P}(X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i} = x_{\mathcal{V}\setminus\mathcal{N}_i} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \\
&= \mathbb{P}(X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \mathbb{P}(X_i = x_i \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \cdot \\
&\quad \cdot \mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} = x_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i}) \\
&= \mathbb{P}(X_{\mathcal{N}_i^\bullet} = x_{\mathcal{N}_i^\bullet}) \mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} = x_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i})\,,
\end{aligned}$$

for every i in $\mathcal{V}$. We can then write

$$u_i(x) = u_i^{\mathcal{N}_i}(x_i, x_{\mathcal{N}_i}) + n_i(x_{-i})\,,$$

where

$$u_i^{\mathcal{N}_i}(x_i, x_{\mathcal{N}_i}) = u_i^{\mathcal{N}_i}(x_{\mathcal{N}_i^\bullet}) = \log \mathbb{P}(X_{\mathcal{N}_i^\bullet} = x_{\mathcal{N}_i^\bullet})$$

only depends on the actions played by player i and her neighbors in $\mathcal{N}_i$, while

$$n_i(x_{-i}) = \log \mathbb{P}(X_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} = x_{\mathcal{V}\setminus\mathcal{N}_i^\bullet} \,|\, X_{\mathcal{N}_i} = x_{\mathcal{N}_i})$$

is a non strategic term. Thus u is $\mathcal{G}$-graphical so that Corollary 2 implies that its potential function ϕ is $\mathcal{H}_{\mathcal{G}}^{\mathcal{C}l}$-separable. Together with (45), this yields the claim. ■

[4]In the literature on probabilistic graphical models, this is referred to as the *local* Markov property [20, Ch. 3.1], which is known to be implied by the so-called *global* Markov property and in turn to imply the so-called *pairwise* Markov property [20, Proposition 3.4]. The three Markov properties are in fact known to be all equivalent to one another for positive random vectors [20, Theorem 3.7].

VI. Conclusion

In this paper, we have introduced the notion of separability with respect to a H-graph for functions defined on product spaces. We have proven, for every function, the existence of a minimal H-graph yielding the most parsimonious decomposition for that function. We have proposed a constructive procedure for computing the minimal H-graph of a function, based on a characterization of it in terms of difference operators. Furthermore, we have described how to explicitly obtain the corresponding minimal decomposition by means of projection operators onto spaces of separable functions.

We have then applied our results to games, encompassing and refining the notion of graphical game. Our proposed definition of separability with respect to a FDH-graph characterizes, up to a nonstrategic component, the way utility functions can be minimally split as the sum of functions depending on subgroups of players. By expressing separability of games in terms of separability of their utility functions, we have derived an algorithm to compute minimal separable representations of games. In the special case of potential games, we have shown that the separability of a game is intimately connected to the separability of the corresponding potential function. This result generalizes and refines one recently proved in [14] and, interestingly, it implies the celebrated Hammersley-Clifford Theorem for Markov Random Fields.

The potential implications of these results should be further investigated in a variety of directions including: (i) studying how information on the minimal separating FDH-graph of a game can be used to reduce the complexity of learning algorithms; and (ii) understanding the implications of separability in the behavior of evolutionary and learning dynamics.

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APPENDIX A
SUM OF PROJECTORS

We start with a general fact of linear algebra. Let V be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$. Given a closed subspace H of V, we denote by $P_H : V \to V$ the orthogonal projection operator onto H. Two subspaces H and K of V are called *perpendicular* if $P_H P_K = P_K P_H$.

Lemma 3: Let $\mathcal{F}$ be a finite family of closed subspaces of V that are pairwise perpendicular. Given $S \subseteq \mathcal{F}$, we denote by $\sum \mathcal{S}$ and by $\cap\mathcal{S}$, respectively the sum and the intersection of the subspaces in $\mathcal{S}$. Then,

$$P_{\cap\mathcal{F}} = \prod_{H\in\mathcal{F}} P_H, \qquad P_{\sum\mathcal{F}} = \sum_{\substack{\mathcal{K}\subset\mathcal{F}\\ \mathcal{K}\neq\emptyset}} (-1)^{|\mathcal{K}|+1} P_{\cap\mathcal{K}}\,. \qquad (46)$$

Proof: We first prove that if H and K are two perpendicular subspaces, then $P_H P_K = P_{H\cap K}$. Indeed notice that $T = P_H P_K$ is an orthogonal projector as

$$T^2 = P_H P_K P_H P_K = P_H P_H P_K P_K = T,$$

$$T^* = P_K^* P_H^* = P_K P_H = P_H P_K = T\,.$$

Since $T(V) \subseteq H \cap K$ and $T_{|H\cap K}$ coincides with the identity, we conclude that $T = P_{H\cap K}$. A direct inductive argument now proves the first identity in (46).

To prove the second identity in (46), first notice that

$$\mathcal{F}^\perp = \{H^\perp \mid H \in \mathcal{F}\}$$

is a family of pairwise perpendicular subspaces as the corresponding orthogonal projectors are $P_{H^\perp} = I - P_H$. Using standard orthogonality properties and the first relation in (46), we can now compute as follows

$$P_{\sum\mathcal{F}} = I - P_{\cap\mathcal{F}^\perp} = I - \prod_{H\in\mathcal{F}} P_{H^\perp} = I - \prod_{H\in\mathcal{F}} (I - P_H)\,,$$

from which the second relation in (46) directly follows. ■

Proof of Proposition 3 We first prove that $\{\mathcal{S}_\mathcal{J} \mid \mathcal{J} \in \mathcal{L}\}$ is a perpendicular family of subspaces. Indeed, taken $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{V}$ the composed application $P_{\mathcal{J}_1} P_{\mathcal{J}_2}$ can be described as follows. Imagine configuration vectors split into four parts, corresponding to the four subsets of labels:

$$\mathcal{I} = \mathcal{J}_1 \cap \mathcal{J}_2\,, \quad \bar{\mathcal{J}}_1 = \mathcal{J}_1 \setminus \mathcal{J}\,, \bar{\mathcal{J}}_2 = \mathcal{J}_2 \setminus \mathcal{I}\,, \quad \mathcal{K} = \mathcal{V} \setminus \mathcal{I}\,.$$

Then,

$$P_{\mathcal{J}_1} P_{\mathcal{J}_2} f(x) =$$
$$\int_{\mathcal{X}_{\bar{\mathcal{J}}_1}} \int_{\mathcal{X}_{\bar{\mathcal{J}}_2}} \int_{\mathcal{X}_\mathcal{K}} f(x_\mathcal{I}, y_1, y_2, z) \mathrm{d}\mu_{\bar{\mathcal{J}}_1}(y_1) \mathrm{d}\mu_{\bar{\mathcal{J}}_2}(y_2) \mathrm{d}\mu_\mathcal{K}(z)\,.$$

The form of the right hand side implies that $P_{\mathcal{J}_1} P_{\mathcal{J}_2} = P_{\mathcal{J}_2} P_{\mathcal{J}_1}$. By definition, $\mathcal{S}_\mathcal{H} = \sum_{\mathcal{J}\in\mathcal{L}} \mathcal{S}_\mathcal{J}$, so that (27) is a direct consequence of Lemma 3. ■

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