Peaked and Low Action Solutions of NLS Equations on Graphs with Terminal Edges

PEAKED AND LOW ACTION SOLUTIONS OF NLS EQUATIONS ON GRAPHS WITH TERMINAL EDGES

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Abstract. We consider the nonlinear Schrödinger equation with focusing power–type nonlinearity on compact graphs with at least one terminal edge, i.e. an edge ending with a vertex of degree 1. On the one hand, we introduce the associated action functional and we provide a profile description of positive low action solutions at large frequencies, showing that they concentrate on one terminal edge, where they coincide with suitable rescaling of the unique solution to the corresponding problem on the half line. On the other hand, a Ljapunov–Schmidt reduction procedure is performed to construct one–peaked and multipeaked positive solutions with sufficiently large frequency, exploiting the presence of one or more terminal edges.

Key words. quantum graphs, nonlinear Schrödinger, least action, terminal edges, Ljapunov–Schmidt reduction, peaked solutions

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1. Introduction.

A connected metric graph (or network) $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a locally one–dimensional structure built of several intervals, the *edges* $e \in E(\mathcal{G})$, glued together at some of their endpoints, the *vertices* $\mathrm{v} \in V(\mathcal{G})$. The specific way in which the edges are joined determines the topology of the graph. Each edge $e \in E(\mathcal{G})$ is identified either with a bounded interval $I_e = [0, \ell_e]$, $\ell_e > 0$, or with a (copy of a) half–line. Functions $u = (u_e)_{e \in E(\mathcal{G})}$ supported on $\mathcal{G}$ are defined by their restrictions u_e to the edges of the graph, and the functional spaces $L^p(\mathcal{G})$, $H^1(\mathcal{G})$ etc. are defined in the usual way (for a wider discussion of definitions and notations on metric graphs, see for instance the monograph [12]). When a differential operator acting on functions supported on the graph is defined, we also speak of *quantum graphs*.

The birth of quantum graphs can be traced back to the first half of the Fifties of the last century [40], when the spectral analysis of Schrödinger operators on a network modelling molecular bonds has been proposed to investigate the behaviour of valence electrons in a naphthalene molecule. Since then, graphs have been assumed to provide a meaningful tool to model the dynamics of systems confined to ramified domains.

Despite the fact that, in general, to rigorously justify the graph approximation is still an open problem (see for instance [26, 34] as well as [17, 24] and references therein), the last decades have been witnessing a renewed interest in the theory of quantum graphs, mainly driven by a wide variety of applications, e.g. quantum wires, Josephson junctions, propagations of signals, nonlinear optics and so on. Linear models have been proposed and extensively studied through the years, and the research in this setting continues to be significantly active (see the milestone paper [31], as well as [12] for a thorough presentation of the subject and a detailed bibliography, and for instance [10, 11, 13, 23, 30] and references therein for some recent developments).

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To date, a huge and constantly growing amount of works are available on the analysis of nonlinear evolutions on metric graphs too. Among these, physical applications such as for instance the theory of Bose–Einstein condensates contributed to gather a prominent focus on *nonlinear Schrödinger* (NLS) equations as

$$-i\partial_t\psi(x,t) = \Delta_x\psi(x,t) + |\psi(x,t)|^{p-1}\psi(x,t)\,. \tag{1.1}$$

Particularly, many efforts have been profused in the analysis of *standing waves* of (1.1), i.e. solutions of the form $\psi(x,t) = e^{i\lambda t}u(x)$, for suitable $\lambda \in \mathbb{R}$ and u solving the associated stationary equation

$$-u'' + \lambda u = |u|^{p-1}u\,. \tag{1.2}$$

First investigations have been developed on specific examples of graphs with half–lines, such as star graphs (see for instance [1, 2, 36]) and the tadpole graph [37]. Later, the problem has been addressed on general non–compact graphs with half–lines, for which a quite well–established theory of existence of standing waves is nowadays available (see the series of works [5, 6, 7] for the case of the nonlinearity extended to the whole graph, and [21, 22, 41, 42, 43] for the counterpart with nonlinearities restricted to the compact core). Broadening the discussion, several results have been accomplished also on compact graphs [18, 19, 33] and periodic graphs [3, 4, 20, 38, 39]. Furthermore, similar investigations have been recently initiated on different families of nonlinear equations too, i.e. nonlinear KdV equation, [35], and nonlinear Dirac equation [15, 16].

As it will play a crucial role in the subsequent discussion, let us also recall that existence and uniqueness of solutions of (1.2) on the real line $\mathbb{R}$ is well–known since decades (see for instance [32]). In particular, the unique -up to translations- positive solution in $H^1(\mathbb{R})$ of

$$-U'' + U = U^p \quad \text{ in } \mathbb{R} \tag{1.3}$$

is given by

$$U(x) = \left(\frac{p+1}{2}\right)^{\frac{2}{p-1}}\left[\cosh\left(\frac{p-1}{2}x\right)\right]^{-\frac{2}{p-1}}\,. \tag{1.4}$$

Similarly, uniqueness of positive H^1 solutions of (1.3) holds true also when the equation is set in $\mathbb{R}^+$. In this case, any solution in $H^1(\mathbb{R}^+)$ is given by a suitable translation $U(x-x_0)\chi_{[0,+\infty)}$, for some $x_0 \in \mathbb{R}$.

From the standpoint of Critical Point Theory, solutions of (1.2) can be identified at least in two different ways. On the one hand, one can search for critical points of the *energy functional* $E : H^1(\mathcal{G}) \to \mathbb{R}$

$$E(u,\mathcal{G}) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p+1}\int_{\mathcal{G}} |u|^{p+1}\,dx$$

in the constrained space of functions $u \in H^1(\mathcal{G})$ with *prescribed mass* ρ^2, that is

$$\int_{\mathcal{G}} |u|^2\,dx = \rho^2\,.$$

This is for instance the general framework of [5, 6, 7] and related works, where it has been shown that the problem is sensitive both to topological and metric properties of the graph.

On the other hand, given $\lambda > 0$, one can look for unconstrained critical points of the *action functional* $I : H^1(\mathcal{G}) \to \mathbb{R}$

$$I(u, \mathcal{G}) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p+1}\int_{\mathcal{G}} |u|^{p+1}\,dx + \frac{\lambda}{2}\int_{\mathcal{G}} |u|^2\,dx\,. \tag{1.5}$$

This approach has been exploited in [38] in the case of periodic graphs, and in [27, 28, 29] on star–graphs. Precisely, in [38], minimization on a generalized Nehari manifold is performed to show existence of least action solutions, whereas in [27, 28, 29] the focus is set on stability properties of specific critical points of the functional.

Standard variational arguments show that both critical points of the energy with fixed mass and unconstrained critical points of the action are solutions of the problem

$$\begin{cases} -u'' + \lambda u = |u|^{p-1}u & \text{on every edge of } \mathcal{G} \\ \sum_{e \succ \mathrm{v}} \frac{du_e}{dx}(\mathrm{v}) = 0 & \forall \mathrm{v} \in V\,, \end{cases} \tag{1.6}$$

that is they are solutions of the NLS equation (1.2) on every edge of the graph and they satisfy the homogeneous Kirchhoff condition at every vertex. Here, by $e \succ \mathrm{v}$ we mean that the edge e is incident at the vertex v, and we use the convention

$$\frac{du_e}{dx}(\mathrm{v}) = u'(0) \quad \text{or} \quad \frac{du_e}{dx}(\mathrm{v}) = -u'(l_e)$$

according to whether the x coordinate on e is equal to 0 or l_e at v.

Our work here fits in the investigation of the action functional (1.5). In what follows, we restrict our attention to compact graphs with at least one terminal edge, i.e. an edge ending with a vertex of degree 1 (recall that the degree of a graph is the total number of edges incident at it). For this class of graphs, we are interested in positive solutions of problem (1.6) for $p > 1$ and $\lambda > 0$. Our aim is twofold.

On the one side, it is easy to show that a solution of (1.2) can always be found minimizing the action on a suitable Nehari manifold. Hence, we concentrate on low action positive solutions and, given λ large enough, we provide a profile description for such states.

To this end, for every $\lambda > 0$, let us introduce the renormalized action functional $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$

$$J_\lambda(u) := \lambda^{\frac{1}{2} - \frac{p+1}{p-1}} \int_{\mathcal{G}} \frac{(u')^2}{2} + \frac{\lambda u^2}{2} - \frac{(u^+)^{p+1}}{p+1}\,dx\,, \tag{1.7}$$

where $u^+ := \max\{u, 0\}$ denotes the positive part of $u \in H^1(\mathcal{G})$, and consider the associated Nehari manifold

$$\begin{aligned} \mathcal{N}_\lambda :&= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \ : \ J'_\lambda(u)[u] = 0\right\} \\ &= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \ : \ \|u\|_\lambda^2 = |u|_{p+1}^{p+1}\right\}\,. \end{aligned} \tag{1.8}$$

It is standard to prove that $\mathcal{N}_\lambda$ is a natural constraint (see also Remark 1 in Section 2 below). As the arguments developed in Section 3 will display clearly, the scaling of λ in (1.7) is the natural one that allows the functional J_λ to exhibit a non–trivial limit as $\lambda \to +\infty$.

The first of our main results here shows that low action solutions have a unique peak at a vertex of degree 1, they are similar to a suitable rescaling of U on the corresponding terminal edge and negligible in L^∞ norm on the rest of the graph. This is stated in the next theorem.

THEOREM 1.1. *Let $\mathcal{G}$ be a compact graph with at least one terminal edge and $p > 1$. Let $\lambda_n \to \infty$ and let, for any n, u_n be a positive solution of (1.6) with $J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) \to m_\infty$, where*

$$m_\infty := \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|U\|^2_{H^1(\mathbb{R})}. \tag{1.9}$$

Then, up to subsequences, u_n has a unique maximum point located at a terminal vertex V. *Moreover, denoting by $I = [0, l]$ the terminal edge where u_n attains its maximum (with the convention that the vertex* V *with degree 1 coincides with* 0*) we have that, while $n \to 0$*

1. $u_n(0) \to +\infty$.
2. $\lambda_n^{\frac{1}{1-p}} u_n\left(\frac{x}{\sqrt{\lambda_n}}\right)\chi_l\left(\frac{x}{\sqrt{\lambda_n}}\right) \to U(x)$ *weakly in $H^1(\mathbb{R}^+)$ and strongly in $C^0(\mathbb{R}^+)$, in $C^2_{loc}(\mathbb{R}^+)$ and in $L^t_{loc}(\mathbb{R}^+)$ for all $t \geq 2$. Here χ_l is a cut off function.*
3. $\lambda_n^{\frac{1}{1-p}}\|u_n(x) - \lambda_n^{\frac{1}{p-1}}U(x\sqrt{\lambda_n})\|_{C^0([0,l/2])} \to 0$
4. *For every $l_1 \in (0, l)$ and every $0 < l_1 < x \leq l$, there exist two constants $c_1, c_2 > 0$, depending on l_1 but independent from n, such that*

$$u_n(x) \leq c_1\lambda_n^{\frac{1}{p-1}} e^{-c_2\sqrt{\lambda_n}x} \ \text{ on } [l_1, l] \subset I,$$
$$\|u_n\|_{L^\infty(\mathcal{G}\setminus I)} \leq c_1\lambda_n^{\frac{1}{p-1}} e^{-c_2\sqrt{\lambda_n}l}.$$

Some remarks are in order. Firstly, we point out that the assumption $J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) \to m_\infty$ is consistent, as the sets of solutions u_n fulfilling it is actually not empty (see Section 2 and Corollary 2.1). Secondly, we highlight the fact that the presence of a terminal edge is crucial in the proof of Theorem 1.1, as it allows to locate precisely the point where low action solutions attain their maximum value. If one were interested in extending the above results to graphs without terminal edges, aiming to prove that concentration occurs in the internal of some given edge, then the first problem in adapting our argument would be to keep track of the exact location of maximum points along the sequence of solutions under exam.

On the other side, and reversing the perspective, whenever $\mathcal{G}$ has at least a terminal edge and again for large λ, it is possible to construct one–peaked and multipeaked positive solutions to (1.2), i.e. solutions with one or more maximum points at the vertices of degree 1, respectively, and negligibly small on the rest of the graph. Such solutions are obtained using the function U in (1.4) as a model and exploiting a Ljapunov–Schmidt reduction procedure. These results are stated in the next two theorems.

THEOREM 1.2. *Let $\mathcal{G}$ be a compact graph with a vertex v_1 with degree 1 and $p > 1$. Denote by $I_1 = [0, l_1]$ the terminal edge ending at v_1, with the convention that v_1 coincides with 0. Then, provided λ is sufficiently large, there exists a solution u_λ of (1.6) with a single peak at v_1, i.e. u_λ of the form*

$$u_\lambda := W_\lambda + \phi,$$

with

$$W_\lambda(x) = \chi(x)U_\lambda(x)$$

where χ is a smooth cut–off function supported on $[0, l] \subset I_1$, for some $l < l_1$, and

$$U_\lambda(x) = \begin{cases} \lambda^{\frac{1}{p-1}}U(\lambda x) & \text{on } I_1 \\ 0 & \text{on } \mathcal{G}\setminus I_1, \end{cases}$$

U being as in (1.4), and

$$\|\phi\|_\lambda = O(\lambda^{-\alpha})$$

for every $\alpha > 0$. Furthermore,

$$\rho^2 := |u_\lambda|^2_{L^2(\mathcal{G})} = C\lambda^{\frac{5-p}{2(p-1)}}\left(|U|^2_{L^2(\mathbb{R}^+)} + o(1)\right). \tag{1.10}$$

THEOREM 1.3. *Let $\mathcal{G}$ be a compact graph with $m \geq 1$ vertices with degree 1 and $p > 1$. Choose $\mathrm{v}_1, \dots, \mathrm{v}_k$ vertices of degree 1 with $1 \leq k \leq m$. Let also $I_i = [0, l_i]$ denote the terminal edge ending at v_i, with the convention that v_i coincides with 0. Then, provided λ is sufficiently large, there exists a k-peaked solution u_λ of (1.6) with a single peak at every vertex v_i, $i = 1, \dots, k$, i.e. u_λ of the form*

$$u_\lambda = W_\lambda + \phi,$$

with

$$W_\lambda(x) = \sum_{i=1}^{k} \chi_i(x) U_{\lambda,i}(x)$$

where χ_i is a smooth cut–off function supported on $[0, l] \subset I_i$, for some $l < \min_{1 \leq i \leq k} l_i$, and

$$U_{\lambda,i}(x) = \begin{cases} \lambda^{\frac{1}{p-1}} U(\lambda x) & \text{on } I_i \\ 0 & \text{on } \mathcal{G} \setminus I_i, \end{cases}$$

U being as in (1.4), and

$$\|\phi\|_\lambda = O(\lambda^{-\alpha})$$

for every $\alpha > 0$. Furthermore,

$$\rho^2 := |u|^2_{L^2(\mathcal{G})} = C\lambda^{\frac{5-p}{2(p-1)}}\left(k|U|^2_{L^2(\mathbb{R}^+)} + o(1)\right). \tag{1.11}$$

To comment on these results, let us first notice that the procedure leading to the proof of Theorems 1.2–1.3 is possible for every $p > 1$. Moreover, the dependence on λ of the mass ρ^2 of the solutions we construct is given explicitly by (1.10)–(1.11). Particularly, note that if $p \in (1, 5)$, i.e. p is in the so–called L^2–subcritical regime, then the above solutions share large masses, whereas they have vanishing mass in the limit $\lambda \to +\infty$ if we consider L^2–supercritical powers $p \in (5, +\infty)$. At the L^2–critical power $p = 5$, all functions identified in Theorems 1.2–1.3 share the same mass.

A further observation about one-peaked solutions is possible. Given a sequence $\lambda_n \to +\infty$, let u_{λ_n} be the corresponding one-peaked solution obtained by Theorem 1.2. The sequence $\{u_{\lambda_n}\}_n$ fulfills the hypotheses of Theorem 1.1, so it inherits all the properties given by Theorem 1.1: the exact location of the unique maximum point, the decreasing monotonicity, the decay rate and so on.

Similarly to Theorem 1.1, we stress the fact that terminal edges play a key role in the Ljapunov–Schmidt scheme of Theorems 1.2–1.3. If such an assumption is removed, then it is not clear to us what suitable model function should be considered, so to possibly generalize our arguments to build peaked solutions with maximum points inside any edge not incident to vertices of degree 1.

To conclude this introduction, we note that all the results derived in this paper contribute to clarify the deep dependence of these problems on the topology of the underlying graphs. This is actually a common feature when dealing with graphs (just to name an example in the framework of compact graphs, the role of terminal edges in existence issues for the mass–constrained case has been pointed out in [19]). On the contrary, it remains an open problem to understand whether the methods we exploited here could be of some help in investigating the role of the metric, i.e. helping for instance to understand where low action solutions concentrate in the presence of multiple terminal edges of different lengths. Also, it is unclear whether a profile description analogous to the one in Theorem 1.1 can be given when minimizing the energy functional under a mass constraint. With respect to these questions, it might be worth briefly comparing our results with the ones in [14] and in [8].

Between the submission and the first revision of this paper, we came to know the preprint version of [14], where existence and properties of stationary states localized on single edges are investigated for the cubic NLS, i.e. $p = 3$ in (1.2). Precisely, given any edge e in the graph and in the limit of large λ, Theorem 1.1 in [14] proves, both for compact graphs and non–compact graphs with a finite number of half–lines, the existence of a positive stationary state Ψ realizing its maximum at one point of e and being monotonic from this maximum point to the endpoints of the edge. Furthermore, concentration of this solution is proved in the L^2 norm, according to the following estimate (which directly follows by formula (1.6) in [14])

$$\frac{\|\Psi\|_{L^2(\mathcal{G}\setminus e)}}{\|\Psi\|_{L^2(\mathcal{G})}} \le Ce^{-2\sqrt{\lambda}\ell} \tag{1.12}$$

(here $\ell := |e|$ denotes the length of the edge e). A quite remarkable aspect of the analysis in [14] is that it addresses the problem of identifying, among these localized states, which is the one that minimizes the energy at prescribed mass. A selection principle is stated in Theorem 1.2 in [14] for compact graphs in the regime of large mass. In particular, it is shown that, in the presence of vertices with degree 1, ground states of the energy will concentrate on the longest terminal edge. The methods developed in [14] are quite different from the one our analysis in this paper is built upon. Indeed, the authors of that paper consider suitable nonlinear generalizations of Dirichlet–to–Neumann maps and they take advantage of elliptic functions available when dealing with the nonlinearity power $p = 3$.

Note that, if u_n is the low action solution of our Theorem 1.1 above, then a slightly sharper version of (1.12) holds. Indeed, by Theorem 1.1, statement 2., it follows that $\|u_n\|_{L^2(\mathcal{G})} \sim \lambda_n^{\frac{p+1}{2(p-1)}}\|U\|_{L^2(\mathbb{R}^+)}$ as $\lambda_n \to +\infty$, so that applying (1.12) to u_n would read (recalling that $p = 3$)

$$\|u_n\|_{L^2(\mathcal{G}\setminus I)} \le C\lambda_n e^{-2\sqrt{\lambda_n}\ell}\,. \tag{1.13}$$

Conversely, making use of statement 4. in Theorem 1.1, we obtain for every $p > 1$

$$\|u_n\|_{L^2(\mathcal{G}\setminus I)} \le \|u_n\|_{L^\infty(\mathcal{G}\setminus I)}\sqrt{|\mathcal{G}\setminus I|} \le C\lambda_n^{\frac{1}{p-1}}e^{-c_2\sqrt{\lambda_n}\ell}$$

which improves (1.13) since $\frac{1}{p-1} = \frac{1}{2}$ when $p = 3$.

Finally, in [8], working within the context of mass–constrained critical points of the energy on non–compact graphs with a finite number of half–lines, the authors prove the existence of solutions attaining their maximum only inside a given edge, provided the mass is sufficiently large. Solutions of this fashion are constructed for every edge in the graph, regardless of it being terminal or not. Moreover, this result is achieved for every L^2–subcritical nonlinearity $p \in (1, 5)$. The analysis therein is of variational nature, based on the discussion of the doubly–constrained minimization problem of minimizing the energy among functions with prescribed mass and attaining their L^∞ norm on a given edge.

The remainder of the paper is organised as follows. Section 2 is devoted to the profile description of low action solutions, developing the proof of Theorem 1.1, whereas Section 3 carries on the construction of peaked solutions as in Theorems 1.2–1.3.

1.1. Notation. Hereafter we will use the following recurrent notations.

- $B_{P,r} = B(P,r)$ is the ball centered at P with radius r. We use the same notation either if $B_{P,r} \subset \mathbb{R}$ or $B_{P,r} \subset \mathbb{R}^+$. In the last case, if $0 \le P < r$ we intend $B_{P,r} = \{0 \le x < P + r\}$. Finally, $B_r := B(0,r)$.
- χ_ρ is a smooth cut–off function such that $\chi_\rho = 1$ when $x \in B_{\rho/2}$ and $\chi_\rho = 0$ outside a ball of radius ρ. When no ambiguity is possible we will omit the subscript ρ.
- $\chi_{[0,+\infty)}$ is the characteristic function of $[0,+\infty)$.
- With abuse of notation we often identify an edge $I \in \mathcal{G}$ with $[0,l]$, l being the lenght of the edge. When the edge is a terminal one, the vertex v of degree 1 will be identified with 0.
- Given a vertex $\mathrm{v} \in \mathcal{G}$ we will suppose w.l.o.g. that the degree of that vertex is either 1 or strictly larger than 2. In fact, degree 2 vertices are indistinguishable from internal points.
- Since throughout the paper we always consider $\lambda > 0$, we endow $H^1(\mathcal{G})$ with the following equivalent scalar product

$$\langle u, v\rangle_\lambda = \int_{\mathcal{G}} u'(x)v'(x)dx + \lambda \int_{\mathcal{G}} u(x)v(x)dx.$$

From now on, unless otherwise specified, we will always consider this product (and its related norm $\|\cdot\|_\lambda$) as the scalar product (and the norm) on $H^1(\mathcal{G})$.

2. Profile of low action solution. As anticipated in the Introduction, for any $\lambda > 0$ a positive solution of (1.6) can be obtained as a critical point of the action functional J_λ defined in (1.7) constrained to the Nehari manifold (1.8) (a standard reference on minimization on Nehari manifolds is for instance [9]).

Remark 1. *Note that any critical point of J_λ is a solution of*

$$\begin{cases} -u'' + \lambda u = (u^+)^p & \text{on every edge of } \mathcal{G} \\ \sum_{e \succ \mathrm{v}} \frac{du_e}{dx}(\mathrm{v}) = 0 & \forall \mathrm{v} \in V\,. \end{cases} \tag{2.1}$$

Clearly, any positive solution of (1.6) is also a solution of (2.1). Moreover, one can prove also that any nontrivial solution of (2.1) is a positive solution of (1.6), so that any nontrivial critical point of J_λ is a positive solution of (1.6). To see this, it is sufficient to show that any solution $\bar{u} \not\equiv 0$ of (2.1) is strictly positive. Since $\mathcal{G}$ is compact, $\bar{u}$ has a minimum point $P \in \mathcal{G}$. By contradiction, let us suppose that $\bar{u}(P) \le 0$. If P lies in the interior of some edge $e \in E(\mathcal{G})$, then

$$0 \le \bar{u}''(P) = \lambda\bar{u} - (\bar{u}^+)^p = \lambda\bar{u} \le 0,$$

so $\bar{u}''(P) = \bar{u}'(P) = \bar{u}(P) = 0$ and by uniqueness of solutions to the Cauchy problem associated to (1.2), it follows $\bar{u} \equiv 0$ on the whole edge. Then Kirchhoff condition implies that all edges incident at the vertices of e realize $\bar{u}' = 0$ at those vertices.

Iterating the argument thus leads to $\bar{u} \equiv 0$ on $\mathcal{G}$, which is a contradiction. On the other hand, if P coincides with a terminal vertex, then $\bar{u}'(P) = 0$ by Kirchhoff condition. Therefore, either $\bar{u}(P) = 0$, which then implies $\bar{u} \equiv 0$ on $\mathcal{G}$ as above, or $\bar{u}(P) < 0$ and $\bar{u}''(P) > 0$, and P cannot be a minimum point. In both cases, we get again a contradiction. Finally, if P coincides with a vertex of degree greater than or equal to 3, then Kirchhoff condition implies that $u' = 0$ at this vertex along every edge incident at it. As this entails again $\bar{u} \equiv 0$ on $\mathcal{G}$, we conclude.

It is standard to prove that $\mathcal{N}_\lambda$ is a C^1 manifold and that the Palais-Smale condition holds on $\mathcal{N}_\lambda$. Moreover, by (1.8) we have that

$$J_\lambda|_{\mathcal{N}_\lambda}(u) = \lambda^{\frac{1}{2}-\frac{p+1}{p-1}} \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u\|_\lambda^2.$$

The Nehari manifold is not empty as problem (1.6) always admits a constant solution. Also, any solution u_λ that we will find in Section 3 belongs to $\mathcal{N}_\lambda$.

One can easily prove that $\inf_{\mathcal{N}_\lambda} J_\lambda > 0$ and, since Palais-Smale holds, that a non trivial minimizer exists. We set

$$m_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda|_{\mathcal{N}_\lambda}(u) > 0.$$

The one peaked solution of Section 3 allows also to estimate m_λ in term of the $H^1(\mathbb{R})$ norm of the function U defined in (1.4). Let us take u_λ a one-peaked solution given by Theorem 1.2. Let $I_1 = [0, l_1]$ be the terminal edge where the peak is located, and suppose that the terminal vertex is in $x = 0$. We know that

$$u_\lambda = W_\lambda(x) + \phi$$

where $W_\lambda(x) = \chi(x) U_\lambda(x)$, $\chi = 1$ if $x \in I_1$ and $0 \le x \le \delta$, $\chi = 0$ if $x \in I_1$ and $2\delta \le x \le l_1$ for some fixed δ and

$$U_\lambda(x) = \begin{cases} \lambda^{\frac{1}{p-1}} U(x\sqrt{\lambda}) & \text{on } I_1 \\ 0 & \text{elsewhere} \end{cases}.$$

Immediately we have, for λ large,

$$\|U_\lambda\|_\lambda^2 = \|U\|_{H^1(\mathbb{R}^+)}^2 + o(1) = \frac{1}{2}\|U\|_{H^1(\mathbb{R})}^2 + o(1).$$

In addition $\|\phi\|_\lambda \le \lambda^{-\alpha}$ for any positive α, thus we compute

$$\begin{aligned} J_\lambda|_{\mathcal{N}_\lambda}(u_\lambda) =& \lambda^{\frac{1}{2}-\frac{p+1}{p-1}} \left[\left(\frac{1}{2} - \frac{1}{p+1}\right) \|U_\lambda\|_\lambda^2\right] + o(1) \\ =& \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right) \|U\|_{H^1(\mathbb{R})}^2 + o(1) \end{aligned}$$

and we obtain

$$0 \le \limsup_{\lambda \to \infty} m_\lambda \le m_\infty, \tag{2.2}$$

where m_∞ is defined in (1.9). This proves, also, that it is possible to find a sequence $\{u_n\}_n$ fulfilling the hypothesis of Theorem 1.1. We are able, by proving this theorem, to give an asymptotic profile description for a positive low action solution of problem (1.6).

Proof of Theorem 1.1. The proof is divided in several steps.

Step 1: For n large u_n is not constant.

Indeed, if $u_n \equiv C$, then, by (1.6) necessarily $C = \sqrt[p-1]{\lambda_n}$. Then

$$\begin{aligned} J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) &= \lambda_n^{\frac{1}{2}-\frac{p+1}{p-1}}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_n\|_\lambda^2\right] = \lambda_n^{\frac{1}{2}-\frac{p+1}{p-1}}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)\lambda_n^{\frac{p+1}{p-1}}|\mathcal{G}|\right] \\ &= \lambda_n^{\frac{1}{2}}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)|\mathcal{G}|\right] \to \infty \text{ for } \lambda_n \to \infty, \end{aligned}$$

where $|\mathcal{G}|$ is the total length of the graph. This contradicts $J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) \to m_\infty$.

Step 2: u_n has a maximum point P_n. Moreover, $u_n(P_n) \geq \sqrt[p-1]{\lambda_n}$.

First, by standard regularity theory, we have that u_n is a regular solution, that is, for any edge $I \subset \mathcal{G}$, $u_n|_I \in C^2(\bar{I})$. Since u_n is not constant, and the graph is compact, u_n has a global maximum point $P_n \in \mathcal{G}$.

Now, if P_n is in the interior of some edge I, we have that $u_n'(P_n) = 0$ and $u_n''(P_n) \leq 0$. Thus, by (1.6) we get $\lambda u_n(P_n) - u_n^p(P_n) = u_n''(P_n) \leq 0$, so $u_n(P_n) \geq \sqrt[p-1]{\lambda_n}$.

If P_n is attained on a terminal vertex, again we have $u_n'(P_n) = 0$ by Kirchhoff condition, so necessarily we have $u_n''(P_n) \leq 0$. Thus again $u_n(P_n) \geq \sqrt[p-1]{\lambda_n}$.

Finally suppose that P_n is attained at a vertex of degree greater than 1. Since P_n is a maximum point, $\frac{d(u_n)_e}{dx}(P_n) \leq 0$ on any edge e that leaves the vertex. Since, by (1.6), $\sum_{e \succ P_n} \frac{d(u_n)_e}{dx}(P_n) = 0$, we have $\frac{d(u_n)_e}{dx}(P_n) = 0$. At this point there exists at least an edge $e \succ P_n$ for which $(u_n)_e''(P_n) \leq 0$ and we conclude as before.

Step 3: There exists a vertex $\mathrm{v} \in \mathcal{G}$ such that, up to subsequences, $d(P_n, \mathrm{v}) \to 0$ while $n \to \infty$.

Suppose, by contradiction, that $\lim_n \inf_{\mathrm{v} \in V(\mathcal{G})} d(P_n, \mathrm{v}) = \delta > 0$. Up to subsequences, we can suppose that $P_n \in I$ for all n and we can identify $I = [0, l]$ so that v coincides with 0, where the vertex v verifies $\inf_{\mathrm{w} \in V(\mathcal{G})} d(P_n, \mathrm{w}) = d(P_n, \mathrm{v})$ for every n. Thus we define

$$v_n(x) := \lambda_n^{\frac{1}{1-p}} u_n\left(\frac{x}{\sqrt{\lambda_n}} + P_n\right)\chi_\delta\left(\frac{x}{\sqrt{\lambda_n}} + P_n\right) \text{ for } |x/\sqrt{\lambda_n}| \leq \delta.$$

The function v_n belongs to $H^1(\mathbb{R})$, moreover

$$\begin{aligned} \|v_n\|_{H^1(\mathbb{R})}^2 &\leq C\lambda_n^{\frac{2}{1-p}} \int_{B_{\delta\sqrt{\lambda_n}}} \left[\frac{d}{dx}u_n\left(\frac{x}{\sqrt{\lambda_n}} + P_n\right)\right]^2 + \left[u_n\left(\frac{x}{\sqrt{\lambda_n}} + P_n\right)\right]^2 dx \\ &= C\lambda_n^{\frac{2}{1-p}} \int_{B_{\delta\sqrt{\lambda_n}}} \frac{1}{\lambda_n}\left(u_n'\right)^2\left(\frac{x}{\sqrt{\lambda_n}} + P_n\right) + u_n^2\left(\frac{x}{\sqrt{\lambda_n}} + P_n\right) dx \\ &= C\lambda_n^{\frac{p+1}{1-p}}\sqrt{\lambda_n} \int_{B_{P_n,\delta}} \left(u_n'\right)^2(x) + \lambda_n u_n^2(x)\, dx \\ &\leq C\lambda_n^{\frac{p+1}{1-p}}\sqrt{\lambda_n} \int_I \left(u_n'\right)^2(x + P_n) + \lambda_n u_n^2(x + P_n)\, dx \\ &\leq C\lambda_n^{\frac{p+1}{1-p}}\sqrt{\lambda_n}\|u_n\|_\lambda^2 \leq C\left(\frac{p-1}{2(p+1)}\right) J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) \leq Cm_\infty. \end{aligned}$$

So $\{v_n\}_n$ is bounded in $H^1(\mathbb{R})$, hence there exists $v \in H^1(\mathbb{R})$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R})$ and $v_n \to v$ strongly in $L^t_{\mathrm{loc}}(\mathbb{R})$ for any $t \geq 2$ and in $C^0_{\mathrm{loc}}(\mathbb{R})$. We want to prove that v is a nontrivial solution of (1.3).

Take $\varphi \in C_0^\infty(\mathbb{R})$. For n large we have that the support $\mathrm{spt}(\varphi)$ of φ is contained in $B_{\frac{\delta}{2}\sqrt{\lambda_n}}$. We define a sequence of functions $\{\varphi_n\}_n \in H^1(\mathcal{G})$ (for n large) as

$$\varphi_n(x) = \begin{cases} \lambda_n^{\frac{1}{p-1}} \varphi\left(\sqrt{\lambda_n}(x-P_n)\right) & \text{on } I \\ 0 & \text{elsewhere}. \end{cases}$$

Since u_n is a solution of (1.6) we have

$$\begin{aligned}
0 =& J'_{\lambda_n}(u_n)[\varphi_n] = \int_I u'_n\varphi'_n + \lambda_n u_n\varphi_n - u_n^p\varphi_n dx \\
=& \lambda_n^{\frac{2}{p-1}} \int_I \frac{d}{dx} v_n\left(\sqrt{\lambda_n}(x-P_n)\right) \frac{d}{dx}\varphi\left(\sqrt{\lambda_n}(x-P_n)\right) dx \\
&+ \lambda_n^{\frac{2}{p-1}}\lambda_n \int_I v_n\left(\sqrt{\lambda_n}(x-P_n)\right)\varphi\left(\sqrt{\lambda_n}(x-P_n)\right) dx \\
&- \lambda_n^{\frac{p+1}{p-1}} \int_I v_n\left(\sqrt{\lambda_n}(x-P_n)\right)\varphi\left(\sqrt{\lambda_n}(x-P_n)\right) dx \\
=& \lambda_n^{\frac{p+1}{p-1}-\frac{1}{2}} \int_{\mathbb{R}} v'_n\varphi' + v_n\varphi - v_n^p\varphi dx,
\end{aligned}$$

so by weak convergence on $H^1(\mathbb{R})$

$$\int_{\mathbb{R}} v'\varphi' + v\varphi - v^p\varphi = 0 \text{ for any } \varphi \in C_0^\infty(\mathbb{R}).$$

Since, by Step 2, $u_n(P_n) \geq \lambda_n^{\frac{1}{p-1}}$ then $v_n(0) = \lambda_n^{\frac{1}{1-p}} u_n(P_n) \geq 1$, so by L^t_{loc} convergence we can prove that $v \neq 0$. Thus, by uniqueness of solutions of (1.3) we have that $v = U$. This leads to a contradiction. In fact, there exists $R > 0$ such that

$$|U|^{p+1}_{L^{p+1}(B_R)} > \frac{3}{4}|U|^{p+1}_{L^{p+1}(\mathbb{R})}$$

and, since $v_n \to v = U$ in L^{p+1}_{loc} there exists $n_0 > 1$ such that

$$|v_n|^{p+1}_{L^{p+1}(B_R)} > \frac{3}{4}|U|^{p+1}_{L^{p+1}(\mathbb{R})} \text{ for } n > n_0. \tag{2.3}$$

On the other hand, there exists $n_1 > 1$ such that, for $n > n_1$ it holds $R/\sqrt{\lambda_n} < \delta/2$, so that if $|x| \leq R$, then $x/\sqrt{\lambda_n} + P_n \in B_{P_n,\frac{\delta}{2}}$ and $\chi(x) \equiv 1$. So, for n large we have

$$\begin{aligned}
|v_n|^{p+1}_{L^{p+1}(B_R)} &\leq \lambda_n^{-\frac{p+1}{p-1}} \int_{B_R} |u_n|^{p+1}(x/\sqrt{\lambda_n} + P_n) dx \leq \lambda_n^{\frac{1}{2}-\frac{p+1}{p-1}} \int_{B_{P_n,\delta}} |u_n|^{p+1} dx \\
&\leq \lambda_n^{\frac{1}{2}-\frac{p+1}{p-1}} |u_n|^{p+1}_{L^{p+1}(\mathcal{G})}.
\end{aligned}$$

So

$$\begin{aligned}
J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) &= \lambda_n^{\frac{1}{2}-\frac{p+1}{p-1}} \left[\left(\frac{1}{2} - \frac{1}{p+1}\right)|u_n|^{p+1}_{L^{p+1}(\mathcal{G})}\right] \geq \left[\left(\frac{1}{2} - \frac{1}{p+1}\right)|v_n|^{p+1}_{L^{p+1}(B_R)}\right] \\
&> \frac{3}{4}\left(\frac{1}{2} - \frac{1}{p+1}\right)|U|^{p+1}_{L^{p+1}(\mathbb{R})} = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|U\|^2_{H^1(\mathbb{R})} = \frac{3}{2}m_\infty \qquad (2.4)
\end{aligned}$$

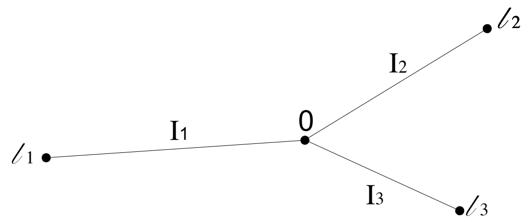

Fig. 2.1. *example of a labelling of the edges entering a vertex* V *as in Step 5 of the proof of Theorem 1.1.*

that contradicts our assumption, thus implying $\lim_n \inf_{\mathrm{W}\in V(\mathcal{G})} d(P_n, \mathrm{W}) = 0$.

Step 4: Given V *as in the previous step, we have* $\lim_n d(P_n, \mathrm{V})\sqrt{\lambda_n} = 0$.

Suppose, by contradiction, that $\lim_n d(P_n, \mathrm{V})\sqrt{\lambda_n} = \delta > 0$. Define

$$w_n(x) := \lambda_n^{\frac{1}{1-p}} u_n\left(\frac{x}{\sqrt{\lambda_n}}\right)\chi_l\left(\frac{x}{\sqrt{\lambda_n}}\right) \text{ for } 0 \le x/\sqrt{\lambda_n} \le l. \tag{2.5}$$

The function w_n belongs to $H^1(\mathbb{R}^+)$, and, in analogy with Step 3, we can prove that $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^+)$ and $w_n \to w$ strongly in $L^t_{\mathrm{loc}}(\mathbb{R}^+)$ for any $t \ge 2$ and in $C^0_{\mathrm{loc}}(\mathbb{R}^+)$. Given $\varphi \in C_0^\infty((0, +\infty))$, for n large we have that the support $\mathrm{spt}(\varphi)$ of φ is contained in $B_{\frac{l}{2}\sqrt{\lambda_n}}$ and we can prove, as before, that w is a nontrivial positive solution of (1.3) on $\mathbb{R}^+$, although we do not know its value at the origin. By uniqueness of solutions of (1.3) on $\mathbb{R}^+$, we have that $w = U(x - x_0)\chi_{[0,\infty)}$ for some suitable $x_0 \in \mathbb{R}$. Since for the maximum point of u_n it holds $P_n\sqrt{\lambda_n} \ge \delta/2 > 0$, we have that w has a maximum point in $(0, +\infty)$, so $x_0 > 0$. At this point we can prove, similarly to Step 3, that there exists $K > 1$ such that

$$J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) > K m_\infty$$

which contradicts our hypothesis.

Step 5: V *coincides with an extremal vertex.*

Suppose, by contradiction that V is a vertex with degree $k \ge 3$.

To simplify the notation, let $I_1 = [0, l_1], \ldots, I_k = [0, l_k]$ the edges that intersect in V and let us suppose that for any I_j, coordinates x_j are defined on I_j so that V coincides with $x_j = 0$, as shown in Figure 2.1. Suppose, also, that $P_n \in I_1$.

Choose $\rho < \min_k l_k$ and define, for $j = 1, \ldots, k$, $u_n^j := u_n|_{I_j}$ and

$$v_n^j(x) := \lambda_n^{\frac{1}{1-p}} u_n^j\left(\frac{x}{\sqrt{\lambda_n}}\right)\chi_\rho\left(\frac{x}{\sqrt{\lambda_n}}\right) \text{ for } 0 \le x/\sqrt{\lambda_n} \le \rho.$$

As before, for any j, $\{v_n^j\}_n$ is bounded in $H^1(\mathbb{R}^+)$, and converges to some v^j weakly in $H^1(\mathbb{R}^+)$ and strongly in $L^t_{\mathrm{loc}}(\mathbb{R}^+)$ for any $t \ge 2$ and in $C^0_{\mathrm{loc}}(\mathbb{R}^+)$.

Given any $R > 0$, there exists n sufficiently large such that $R < \rho\sqrt{\lambda_n}/2$, so on $[0, R]$ we have that $v_n^j(x) \equiv \lambda_n^{\frac{1}{1-p}} u_n^j\left(\frac{x}{\sqrt{\lambda_n}}\right)$. Now, since u_n solves (1.6), we have that

$$(v_n^j)'' = v_n^j - (v_n^j)^p \text{ on } [0, R]$$

and, since $v_n^j \to v^j$ in $C^0([0,R])$, and by the arbitrariness of R we have that $v_n^j \to v^j$ in $C^0_{\mathrm{loc}}(\mathbb{R}^+)$ for all j.

Finally v^j is a nontrivial positive solution of (1.3) on $\mathbb{R}^+$, so

$$v^j(x) = U(x - x_j)\chi_{[0,+\infty)} \text{ for some } x_j \in \mathbb{R}.$$

We can prove that $x_j = 0$ for all j. In fact, we have that P_n is a maximum point for u_n, so $P_n\sqrt{\lambda_n}$ is a maximum point for v_n^1, so $(v_n^1)'(P_n\sqrt{\lambda_n}) = 0$. Since, by Step 4, $P_n\sqrt{\lambda_n} \to 0$, we have that $(v^1)'(0) = 0$ for C^2 convergence. Thus $x_1 = 0$. Moreover $u_n^j(0) = u_n^1(0)$ for any j by continuity of u_n. Then also $v_n^j(0) = v_n^1(0)$ and, passing to the limit in n, also that $v^j(0) = v^1(0)$ for any j. Thus $x_j = 0$ for all j, since U has a unique maximum. At this point, note that, adapting the argument in Step 3, one obtains that, for every $j = 1, \dots, k$ and for sufficiently large n

$$\|v_n^j\|^{p+1}_{L^{p+1}(B_R)} > \frac{3}{4}\|U\|^{p+1}_{L^{p+1}(\mathbb{R}^+)} \,,$$

where B_R is a suitable neighborhood of the origin in $\mathbb{R}^+$. Since the previous estimate is the analogue of (2.3), proceeding as in (2.4) leads to

$$\begin{aligned} J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) \geq & \frac{3k}{4}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|U\|^2_{H^1(\mathbb{R}^+)} \\ = & \frac{3k}{8}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|U\|^2_{H^1(\mathbb{R})} = \frac{3k}{4}m_\infty > m_\infty \end{aligned}$$

being $k \geq 3$, i.e. a contradiction.

Step 6: u_n has a unique maximum. Moreover, this maximum coincides with V.

By contradiction, suppose that u_n has another maximum point $Q_n \neq P_n$. By the previous step, up to subsequences, it is possible to prove that there exists a terminal vertex W in $\mathcal{G}$ such that $\lim_n d(Q_n, \mathrm{W})\sqrt{\lambda_n} = 0$. Moreover, one can check that W must coincide with V, otherwise $J_{\lambda_n}|_{\mathcal{N}} > \frac{3}{2}m_\infty$.

Thus $\lim_n d(Q_n, \mathrm{V})\sqrt{\lambda_n} = 0$. At this point, let w_n be as in (2.5) in Step 4

$$w_n(x) = \lambda_n^{\frac{1}{1-p}} u_n\left(\frac{x}{\sqrt{\lambda_n}}\right)\chi_l\left(\frac{x}{\sqrt{\lambda_n}}\right) \text{ for } 0 \leq x/\sqrt{\lambda_n} \leq l$$

and, setting $p_n = P_n\sqrt{\lambda_n}$, $q_n = Q_n\sqrt{\lambda_n}$, we have

$$p_n, q_n \to 0 \text{ as } n \to \infty, \text{ and } w_n'(p_n) = w_n'(q_n) = 0. \tag{2.6}$$

By the previous steps it holds $w_n(x) \to U(x)\chi_{[0,+\infty)}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^+)$, thus implying $w''(0) < 0$. On the other hand, in light of (2.6) we have $w''(0) = 0$ which gives us a contradiction.

We can prove that $P_n \equiv \mathrm{V}$ exactly with the same argument, using the fact that $u_n'(\mathrm{V}) = 0$ since u_n solves (1.6).

Step 7: $w_n(x) \to U(x)\chi_{[0,+\infty)}$ in $C^0(\mathbb{R}^+)$.

With the same argument of Step 6, we can prove also that u_n cannot have any local maximum point other than V. So we get that u_n is monotone on the graph and consequently that w_n is monotone on $[P_n\sqrt{\lambda_n}, +\infty)$. Now, given $\varepsilon > 0$ there exists an $R = R(\varepsilon)$ such that $U(R) \leq \varepsilon/4$. Moreover, there exists $\bar{n} = \bar{n}(R)$ such that, for $n > \bar{n}$, $\|w_n - U\|_{C^0([0,R])} \leq \varepsilon/4$. So

$$\begin{aligned} \|w_n - U\|_{C^0(\mathbb{R}^+)} &\leq \|w_n - U\|_{C^0([0,R])} + \|w_n\|_{C^0([R,+\infty))} + \|U\|_{C^0([R,+\infty))} \\ &\leq \|w_n - U\|_{C^0([0,R])} + w_n(R) + U(R) \\ + &\leq \|w_n - U\|_{C^0([0,R])} + |w_n(R) - U(R)| + 2U(R) \\ &\leq 2\|w_n - U\|_{C^0([0,R])} + 2U(R) \leq \varepsilon. \end{aligned} \tag{2.7}$$

Step 8: Proof of Claims 1–2–3.

The proof of Claims 1 and 2 of the Theorem is a direct consequence of the previous steps. Moreover by Step 7

$$\left\| \lambda_n^{\frac{1}{1-p}} \, u_n|_I \left(\frac{x}{\sqrt{\lambda_n}} \right) - U(x) \right\|_{C^0([0, l\sqrt{\lambda_n}/2])} \to 0$$

and by a change of variable we obtain Claim 3.

Step 9: proof of Claim 4.

Let $l_1 \in (0, l)$ be given. First, we can repeat the argument of the previous steps to prove that u_n has no local maximum point except for the extremal vertex. Therefore, u_n is strictly decreasing on any edge of the graph.

Given again as in (2.5)

$$w_n(x) := \lambda_n^{\frac{1}{1-p}} u_n \left(\frac{x}{\sqrt{\lambda_n}} \right) \chi_l \left(\frac{x}{\sqrt{\lambda_n}} \right) \text{ for } x \geq 0,$$

by Step 7 we have $w_n(x) \to U(x)\chi_{[0,+\infty)}$ in $C^2_{\text{loc}}(\mathbb{R}^+)$ and, by definition, there exists a constant C_0 for which

$$U \leq C_0 e^{-x} \text{ for } x > 0.$$

Now, fix $0 < \varepsilon < 1/4$ and choose $R = 2\log(C_0/\varepsilon)$. Then there exists $\bar{n} = \bar{n}(\varepsilon)$ such that

$$\|w_n - U\|_{C^2[0,R]} \leq \varepsilon \text{ for } n \geq \bar{n}.$$

We have that

$$w_n(x) \leq 2\varepsilon \text{ on } R/2 \leq x \leq R, \tag{2.8}$$

indeed

$$w_n(x) \leq U(x) + \varepsilon \leq C_0 e^{-R/2} + \varepsilon \leq 2\varepsilon.$$

Now (2.8) implies, by rescaling, that

$$u_n \left(\frac{x}{\sqrt{\lambda_n}} \right) \chi_l \left(\frac{x}{\sqrt{\lambda_n}} \right) \leq 2\lambda_n^{\frac{1}{p-1}} \varepsilon \text{ on } R/2 \leq x \leq R,$$

so that, since $\frac{R}{\sqrt{\lambda_n}} \leq \frac{l}{2}$ for n large, we have

$$u_n(y) \leq 2\lambda_n^{\frac{1}{p-1}} \varepsilon \text{ on } \frac{R}{2\sqrt{\lambda_n}} \leq y \leq \frac{R}{\sqrt{\lambda_n}},$$

and, u_n being strictly decreasing,

$$u_n(y) \leq 2\lambda_n^{\frac{1}{p-1}} \varepsilon \text{ on } \frac{R}{2\sqrt{\lambda_n}} \leq y \leq l. \tag{2.9}$$

Now, u_n solves

$$u_n'' - \left(\lambda_n - u_n^{p-1}\right) u_n = 0 \text{ on } 0 \le y \le l$$

and, by (2.9) and since $\varepsilon \le 1/4$, there exists $a > 0$ independent from n such that

$$\lambda_n - u_n^{p-1} \ge \lambda_n \left(1 - (2\varepsilon)^{p-1}\right) \ge a\lambda_n \text{ on } \frac{R}{2\sqrt{\lambda_n}} \le y \le l.$$

Since it is well–known (see Lemma 2.4 of [25]) that, whenever

$$u'' - \lambda_n q(x) u = 0 \text{ on } 0 < l_1 \le x \le l, \qquad q \ge a,$$

there exist two constants $c_1, c_2 > 0$, independent of λ_n, such that

$$u(x) \le c_1 \lambda_n^{\frac{1}{p-1}} e^{-c_2\sqrt{\lambda_n} x}$$

for every $l_1 \le x \le l$, we conclude. ∎

Corollary 2.1. *We have*

$$\lim_{\lambda \to \infty} m_\lambda = m_\infty.$$

Proof. By (2.2) we have $\lim_{\lambda\to\infty} m_\lambda \le m_\infty$. To prove the reverse inequality, assume by contradiction that there exists a sequence $\{u_n\}_n$ of solutions with $\lim_n J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) < m_\infty$, and, as in the proof of the previous theorem, we have

$$w_n \to U \text{ in } L^{p+1}_{\mathrm{loc}}(\mathbb{R}^+),$$

w_n being given by (2.5). Now, for any η, there exists an $R = R(\eta) > 0$ such that

$$|U|^{p+1}_{L^{p+1}([0,R])} > (1-\eta)|U|^{p+1}_{L^{p+1}(\mathbb{R}^+)}$$

and, since $w_n \to U$ in $L^{p+1}([0,R])$ there exists $n_0 > 1$ such that

$$|w_n|^{p+1}_{L^{p+1}([0,R])} > (1-2\eta)|U|^{p+1}_{L^{p+1}(\mathbb{R}^+)} \text{ for } n > n_0.$$

At this point we can proceed similarly to (2.4), obtaining

$$J_{\lambda_n}|_{\mathcal{N}_{\lambda_n}}(u_n) \ge (1-2\eta) m_\infty.$$

The arbitrariness of η provides the contradiction. ∎

3. Construction of peaked solutions. In order to perform the finite dimensional reduction, we have to linearize Problem (1.3) around the solution U and to study the null space of the linearized problem, that is the set of solutions to the Neumann boundary value problem

$$\begin{cases} -\psi'' + \psi = pU^{p-1}\psi & \text{in } \mathbb{R}^+ \\ \psi'(0) = 0. \end{cases} \tag{3.1}$$

While the equation $-\psi'' + \psi = pU^{p-1}\psi$ in $\mathbb{R}$ has a one-dimensional space of solutions generated by $Z(t) = U'(t)$, it is easy to show that problem (3.1) has only the trivial solution, due to the boundary condition.

For a given compact graph $\mathcal{G}$, we then consider the compact immersion

$$i_\lambda : \left(H^1(\mathcal{G}), \langle , \rangle_\lambda\right) \to \left(L^2(\mathcal{G}), \langle , \rangle_{L^2}\right)$$

and define its adjoint map

$$i_\lambda^* : \left(L^2(\mathcal{G}), \langle , \rangle_{L^2}\right) \to \left(H^1(\mathcal{G}), \langle , \rangle_\lambda\right)$$

such that

$$\langle i_\lambda^*(f), v\rangle_\lambda = \langle f, v\rangle_{L^2} \text{ for all } v \in H^1(\mathcal{G})\,,$$

or equivalently

$$u = i_\lambda^*(f) \Leftrightarrow u \text{ solves } \begin{cases} -u'' + \lambda u = f & \text{in } \mathcal{G} \\ \sum_{e \succ \mathrm{v}} \frac{du_e}{dx}(\mathrm{v}) = 0 & \forall \mathrm{v} \in V\,. \end{cases}$$

3.1. One peaked solutions. We construct now a model profile for a solution which has a peak on the extremal vertex v_1 (the vertex of degree 1) of the first edge $I_1 = [0, l_1]$. We suppose, without loss of generality that v_1 corresponds to the coordinate $x = 0$. We define

$$U_\lambda(x) = \begin{cases} \lambda^{\frac{1}{p-1}} U\left(\sqrt{\lambda}x\right) & \text{on } [0, l_1] \\ 0 & \text{elsewhere} \end{cases}$$

and, given a cut off function $\chi := \chi_l(x)$, with $l < l_1$, we define

$$W_\lambda(x) = \chi(x) U_\lambda(x) \tag{3.2}$$

and we search a solution of (1.6) of the form $u = W_\lambda(x) + \phi$, ϕ being a small error in $H^1(\mathcal{G})$. To improve the readability of the paper, herafter we denote

$$f(s) := (s^+)^p,$$

so a solution of (1.6) can be written as

$$W_\lambda + \phi = i_\lambda^*(f(W_\lambda + \phi)). \tag{3.3}$$

We define a linear operator

$$\begin{gathered} \mathcal{L}_\lambda : H^1(\mathcal{G}) \to H^1(\mathcal{G}) \\ \mathcal{L}_\lambda(\phi) = \phi - i_\lambda^*(f'(W_\lambda)\phi) \end{gathered}$$

and we recast equation (3.3) as

$$\mathcal{L}_\lambda(\phi) = N_\lambda(\phi) + R_\lambda$$

where

$$\begin{gathered} N_\lambda(\phi) := i_\lambda^* \left[f(W_\lambda + \phi) - f(W_\lambda) - f'(W_\lambda)\phi\right] \\ R_\lambda := i_\lambda^*(f(W_\lambda)) - W_\lambda. \end{gathered}$$

The following result implies the invertibility of $\mathcal{L}_\lambda$ for λ sufficiently large.

Lemma 3.1. *There exists $\lambda_0, c > 0$ such that $\forall \lambda > \lambda_0$, $\forall \phi \in H^1(\mathcal{G})$ it holds*

$$\|\mathcal{L}_\lambda(\phi)\|_\lambda \geq c\|\phi\|_\lambda$$

Proof. We proceed by contradiction, assuming that there exist a sequence $\lambda_n \to \infty$ and a sequence of functions $\phi_n \in H^1(\mathcal{G})$ such that $\|\phi_n\|_\lambda = 1$ and

$$\|\mathcal{L}_{\lambda_n}(\phi_n)\|_{\lambda_n} \to 0.$$

By definition of $\mathcal{L}_\lambda$ we have

$$\phi_n - \mathcal{L}_{\lambda_n}(\phi_n) = i^*_{\lambda_n}(f'(W_{\lambda_n})\phi_n)$$

that is

$$\begin{cases} -(\phi_n - \mathcal{L}_{\lambda_n}(\phi_n))'' + \lambda_n(\phi_n - \mathcal{L}_{\lambda_n}(\phi_n)) = f'(W_{\lambda_n})\phi_n & \text{on } \mathcal{G} \\ \sum_{e \succ \mathrm{v}} \frac{d(\phi_n - \mathcal{L}_{\lambda_n}(\phi_n))_e}{dx}(\mathrm{v}) = 0 & \forall \mathrm{v} \in V \end{cases}$$

and, setting $z_n := \phi_n - \mathcal{L}_{\lambda_n}(\phi_n)$, and $h_n := \mathcal{L}_{\lambda_n}(\phi_n)$ we get

$$\begin{cases} -z_n'' + \lambda_n z_n = f'(W_{\lambda_n})z_n + f'(W_{\lambda_n})h_n & \text{on } \mathcal{G} \\ \sum_{e \succ \mathrm{v}} \frac{dz_e}{dx}(\mathrm{v}) = 0 & \forall \mathrm{v} \in V \end{cases}. \tag{3.4}$$

Also, we have

$$\|z_n\|^2_{\lambda_n} = \|\phi_n\|^2_{\lambda_n} + \|\mathcal{L}_{\lambda_n}(\phi_n)\|^2_{\lambda_n} - 2\langle \phi_n, \mathcal{L}_{\lambda_n}(\phi_n)\rangle_{\lambda_n} \to 1 \tag{3.5}$$

and, on the other hand,

$$\begin{aligned} \|z_n\|^2_{\lambda_n} &= \int_{\mathcal{G}} (z_n')^2\, dx + \lambda_n \int_{\mathcal{G}} (z_n)^2\, dx \\ &= \int_{\mathcal{G}} (-z_n'' + \lambda_n z_n)\, z_n dx + \sum_{v \in V} \sum_{e \succ \mathrm{v}} z_n'(\mathrm{v}) z_n(\mathrm{v}). \end{aligned}$$

In light of (3.4) we have that $\sum_{e \succ \mathrm{v}} z_n'(\mathrm{v})z_n(\mathrm{v}) = 0$ for all $\mathrm{v} \in V$, and, since $W_{\lambda_n} = 0$ outside the first edge I_1, also that $-z_n'' + \lambda_n z_n = 0$ on I_e, $e \neq 1$. Thus

$$\|z_n\|^2_{\lambda_n} = \int_{I_1} (-z_n'' + \lambda_n z_n)\, z_n dx = \int_{I_1} f'(W_{\lambda_n})z_n^2 + f'(W_{\lambda_n})\mathcal{L}_{\lambda_n}(\phi_n) z_n dx,$$

and, since $\mathcal{L}_{\lambda_n}(\phi_n) \to 0$ in $H^1(\mathcal{G})$ and by (3.5), we have

$$\int_{I_1} f'(W_{\lambda_n}) z_n^2 \to 1 \text{ while } n \to \infty. \tag{3.6}$$

On the edge I_1 we consider the rescaling $s = x\sqrt{\lambda_n}$ and we set

$$\tilde{z}_n(s) = \lambda_n^{1/4} z_n\left(\frac{s}{\sqrt{\lambda_n}}\right) \text{ for } s \in [0, l_1\sqrt{\lambda_n}].$$

Of course

$$\tilde{z}_n'(s) = \lambda_n^{-\frac{1}{4}} z_n'\left(\frac{s}{\sqrt{\lambda_n}}\right) \text{ and } \tilde{z}_n''(s) = \lambda_n^{-\frac{3}{4}} z_n''\left(\frac{s}{\sqrt{\lambda_n}}\right)$$

and, recalling the definition (3.2) of W_λ, and (3.4),

$$-\tilde{z}_n''(s) + \tilde{z}_n'(s) = p\chi^{p-1}\left(\frac{s}{\sqrt{\lambda_n}}\right) U^{p-1}(s)\left[\tilde{z}_n(s) + \tilde{h}_n(s)\right] \text{ for } s \in [0, l_1\sqrt{\lambda_n}]$$

where $\tilde h_n(s) := \lambda_n^{\frac14} h_n\left(\frac{s}{\sqrt{\lambda_n}}\right)$. Moreover it holds, for some constant $C>0$,

$$\|\tilde z_n\|_{H^1([0,l_1\sqrt{\lambda_n}])} \le C, \tag{3.7}$$

since

$$\int_0^{l_1\sqrt{\lambda_n}} |\tilde z_n'(s)|^2 + \tilde z_n^2(s)ds = \int_0^{l_1} |z_n'(x)|^2 + \lambda_n z_n^2(x)dx \le \|z_n\|^2_{\lambda_n}$$

which is bounded by (3.5). Analogously

$$\|\tilde h_n\|_{H^1([0,l_1\sqrt{\lambda_n}])} \le \|h_n\|_{\lambda_n} \to 0.$$

By (3.7) we have that there exists a function $\tilde z$ defined on $\mathbb{R}^+$ such that, for every fixed $T>0$,

$$\begin{aligned}
&\tilde z_n \to \tilde z \text{ a.e. in } \mathbb{R}^+\\
&\tilde z_n \to \tilde z \text{ in } L^p([0,T]) \text{ for all } q>1\\
&\tilde z_n \rightharpoonup \tilde z \text{ weakly in } H^1([0,T]).
\end{aligned}$$

We can show, indeed, that $\tilde z \in H^1([0,T])$. Consider

$$\zeta_n = \tilde z_n \chi\left(\frac{s}{\sqrt{\lambda_n}}\right).$$

Since $\lambda_n \to \infty$ we have that $\|\zeta_n\|_{H^1(\mathbb{R}^+)} \le C\|\tilde z_n\|_{H^1([0,l_1\sqrt{\lambda_n}])} \le C$, thus ζ_n admits a weak limit in $H^1(\mathbb{R}^+)$. Also, $\zeta_n = \tilde z_n$ on $[0,\delta\sqrt{\lambda_n}]$, so $\zeta_n \rightharpoonup \tilde z$ weakly in $H^1(\mathbb{R}^+)$ and $\tilde z \in H^1(\mathbb{R}^+)$.

Now, take a function $\varphi \in C^\infty(\mathbb{R}^+)$ and let $T>0$ be such that the support of φ is contained in $[0,T]$, so that

$$\begin{aligned}
\int_{[0,T]} &(-\tilde z_n''(s) \ +\tilde z_n'(s))\,\varphi(s)ds\\
&= \int_{[0,T]} p\left(\chi^{p-1}\left(\frac{s}{\sqrt{\lambda_n}}\right) U^{p-1}(s)\left[\tilde z_n(s) + \tilde h_n(s)\right]\right)\varphi(s)ds\\
&= \int_{[0,T]} pU^{p-1}(s)\tilde z_n(s)\varphi(s)ds + o(1)\,.
\end{aligned}$$

Integrating by parts the first term and passing to the limit we have that

$$\int_{\mathbb{R}^+} \tilde z'(s)\varphi(s) + \tilde z'(s)\varphi(s)ds = \int_{\mathbb{R}^+} pU^{p-1}(s)\tilde z(s)\varphi(s)ds.$$

Since φ is arbitrary, we have that $\tilde z$ is a solution of (3.1), so $\tilde z \equiv 0$. Moreover, extending by zero $\tilde z_n$ to the whole half line, we have $\tilde z_n \rightharpoonup 0$ in $L^2(\mathbb{R}^+)$, thus

$$p\int_0^{l_1\sqrt{\lambda_n}} U^{p-1}(s)\tilde z_n^2(s)ds = p\int_{\mathbb{R}^+} U^{p-1}(s)\tilde z_n^2(s)ds \to 0.$$

This leads to a contradiction in light of (3.6), since

$$\begin{aligned}
p\int_0^{l_1\sqrt{\lambda_n}} U^{p-1}(s)\tilde z_n^2(s)ds &\ge p\int_0^{l_1\sqrt{\lambda_n}} \chi^{p-1}\left(\frac{s}{\sqrt{\lambda_n}}\right) U^{p-1}(s)\tilde z_n^2(s)ds\\
&= \int_0^{l_1} f'(W_{\lambda_n}) z_n^2 dx \to 1.
\end{aligned}$$

This concludes the proof. ◻

Proposition 3.2. *We have* $\|R\|_\lambda \le \lambda^{-\alpha}$ *for any* $\alpha > 0$.

Proof. Take $V = i^*_\lambda(f(W_\lambda))$. Then we have, by direct computation, that

$$-(V - W_\lambda)''(x) + \lambda(V - W_\lambda)(x) = (\chi^p - \chi)(x)\lambda^{\frac{p}{p-1}} U^p(x\sqrt{\lambda}) \\ - \lambda^{\frac{1}{p-1}}\chi''(x)U(x\sqrt{\lambda}) - 2\lambda^{\frac{1}{p-1}}\sqrt{\lambda}\chi'(x)U'(x\sqrt{\lambda}) \tag{3.8}$$

and $V'(0) = 0$. Thus, multiplying (3.8) by $V - W_\lambda$, and integrating by parts we have

$$\begin{aligned}\|R\|_\lambda = \|V - W_\lambda\|_\lambda &\le C\lambda^{\frac{p}{p-1}} |(\chi^p - \chi)(x)U^p(x\sqrt{\lambda})|_{L^2([0,l_1])} \\ &+ C\lambda^{\frac{1}{p-1}} |\chi''(x)U(x\sqrt{\lambda})|_{L^2([0,l_1])} + C\lambda^{\frac{1}{p-1}}\sqrt{\lambda}|\chi'(x)U'(x\sqrt{\lambda})|_{L^2([0,l_1])} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By a change of variables, and since $U(x)$ decays exponentially in x, we have

$$\begin{aligned}I_1^2 &\le C\lambda^{\frac{2p}{p-1}} \int_\delta^{2\delta} U^{2p}(x\sqrt{\lambda})dx = C\lambda^{\frac{2p}{p-1}-\frac{1}{2}} \int_{\delta\sqrt{\lambda}}^{2\delta\sqrt{\lambda}} U^{2p}(s)ds \\ &\le C\lambda^{\frac{3p+1}{2(p-1)}} \int_{\delta\sqrt{\lambda}}^{2\delta\sqrt{\lambda}} e^{-2ps}ds \le C\lambda^{\frac{3p+1}{2(p-1)}} \left[-\frac{e^{-2ps}}{2p}\right]_{\delta\sqrt{\lambda}}^{2\delta\sqrt{\lambda}} \le C\lambda^{\frac{3p+1}{2(p-1)}} e^{-2p\delta\sqrt{\lambda}}.\end{aligned}$$

In the same way we can proceed for I_2 and I_3, obtaining the claim. ∎

Proof of Theorem 1.2. We look for a solution of (3.3) in the form $W_\lambda + \phi$, where W_λ is defined in (3.2). This corresponds to find a fixed point of the map

$$\begin{aligned}T_\lambda :&H^1(\mathcal{G}) \to H^1(\mathcal{G}) \\ T_\lambda(\phi) :=&\mathcal{L}_\lambda^{-1}\left(N_\lambda(\phi) + R_\lambda\right).\end{aligned}$$

We prove that T is a contraction on $\left\{\phi \in H^1(\mathcal{G}),\ \|\phi\|_\lambda \le c\lambda^{-\alpha}\right\}$ for some positive α, c. By Lemma 3.1, there exists $c > 0$ such that

$$\begin{aligned}\|T_\lambda(\phi)\|_\lambda &\le c\left(\|N_\lambda(\phi)\|_\lambda + \|R_\lambda\|_\lambda\right) \\ \|T_\lambda(\phi_1) - T_\lambda(\phi_2)\|_\lambda &\le c\left(\|N_\lambda(\phi_1) - N_\lambda(\phi_2)\|_\lambda\right).\end{aligned}$$

By the mean value theorem and by the properties of i^*_λ there exists $0 < \theta(x) < 1$ such that

$$\|N_\lambda(\phi_1) - N_\lambda(\phi_2)\|^2_\lambda \le c\int_{\mathcal{G}} \left[(W_\lambda + \phi_2 + \theta(\phi_1 - \phi_2))^{p-1} - (W_\lambda)^{p-1}\right]^2 (\phi_1 - \phi_2)^2\, dx,$$

so, if $\|\phi_i\|_\lambda$ is small enough, then also $|\phi_i|_{L^2(\mathcal{G})}$ is small and we can find a constant $0 < K < 1$ such that

$$\|N_\lambda(\phi_1) - N_\lambda(\phi_2)\|_\lambda \le K\|\phi_1 - \phi_2\|_\lambda.$$

In a similar way we can prove that, if $\|\phi\|_\lambda$ is small enough, by Proposition 3.2

$$\|T_\lambda(\phi)\|_\lambda \le c\left(\|N_\lambda(\phi)\|_\lambda + \|R_\lambda\|_\lambda\right) \le c\left(\|\phi\|^2_\lambda + \lambda^{-\alpha}\right).$$

Then there exists $c > 0$ such that T_λ maps a ball of center 0 and radius $c\lambda^{-\alpha}$ in $H^1(\mathcal{G})$ into itself and it is a contraction. So there exists a fixed point ϕ_λ with norm $\|\phi_\lambda\|_\lambda = O(\lambda^{-\alpha})$.

At this point we proved that (1.6) has a one-peaked solution $u = W_\lambda + \phi$, with $\|\phi_\lambda\|_\lambda = O(\lambda^{-\alpha})$. To conclude the proof we compute the L^2 norm of the solution, that is

$$
\begin{aligned}
|u|^2_{L^2(\mathcal{G})} &= C|W_\lambda|^2_{L^2(\mathcal{G})} + l.o.t. = C\int_0^{l_1} U^2_\lambda(x)\chi^2(x)dx + l.o.t.\\
&= C\lambda^{\frac{5-p}{2(p-1)}}\left(|U|^2_{L^2(\mathbb{R}^+)} + o(1)\right).
\end{aligned}
$$

which concludes the proof. □

3.2. Multipeaked solutions. Let us now consider a graph $\mathcal{G}$ which has at least k vertices $\mathrm{v}_1 \dots, \mathrm{v}_k$ of degree 1, and we construct a solution of (1.6) which has a positive peak on any vertex v_i, $i = 1, \dots, k$. Without loss of generality we suppose that each vertex v_i, $i = 1, \dots, k$ lies on of the edge $I_i = [0, l_i]$ and that v_i corresponds to the coordinate $x = 0$.

The strategy of the proof is similar to the previous one, so we only underline the differences. We define

$$
W_\lambda(x) = \sum_{i=1}^{k} \chi_i(x) U_{\lambda,i}(x) \tag{3.9}
$$

where

$$
U_{\lambda,i}(x) = \begin{cases} \lambda^{\frac{1}{p-1}} U(x\sqrt{\lambda}) & \text{on } [0, l_i] \\ 0 & \text{elsewhere} \end{cases}
$$

and, $\chi_i := \chi_{\delta,i}(x)$ is a cut off function which is 1 on $[0, \delta/2] \subset [0, l_i]$ and 0 on $[\delta, l_i]$ and on every other edge I_j, $j \neq i$. Here $\delta < \min_i l_i$.

It is clear that $W_\lambda(x) \in H^1(\mathcal{G})$. As before, we look for a solution of (1.6) of the form $u = W_\lambda(x) + \phi$, ϕ being a small error in $H^1(\mathcal{G})$. We can prove the invertibility of the operator $\mathcal{L}_\lambda$ as follows.

Lemma 3.3. *There exist $\lambda_0, c > 0$ such that $\forall \lambda > \lambda_0$, $\forall \phi \in H^1(\mathcal{G})$ it holds*

$$
\|\mathcal{L}_\lambda(\phi)\|_\lambda \geq c\|\phi\|_\lambda \,.
$$

Proof. As before, we proceed by contradiction, assuming that there exist a sequence $\lambda_n \to \infty$ and a sequence of functions $\phi_n \in H^1(\mathcal{G})$ such that $\|\phi_n\|_\lambda = 1$ and $\|\mathcal{L}_{\lambda_n}(\phi_n)\|_{\lambda_n} \to 0$.

Setting $z_n := \phi_n - \mathcal{L}_{\lambda_n}(\phi_n)$ and $h_n := \mathcal{L}_{\lambda_n}(\phi_n)$, we can prove as in Lemma 3.1 that z_n solves equation (3.4) and that $\|z_n\|^2_{\lambda_n} \to 1$ as $n \to \infty$. Since $W_{\lambda_n} = 0$ outside the first k edges $I_1, \dots, I_k$, we have

$$
\|z_n\|^2_{\lambda_n} = \sum_{i=1}^{k} \int_{I_i} \left(-z_n'' + \lambda_n z_n\right) z_n dx = \sum_{i=1}^{k} \int_{I_i} f'(W_{\lambda_n}) z_n^2 dx + o(1). \tag{3.10}
$$

This means that there is at least one edge $I_{\bar{i}}$ such that

$$
\int_{I_{\bar{i}}} f'(W_{\lambda_n}) z_n^2 dx \not\to 0 \,. \tag{3.11}
$$

Letting now $z_{n,\bar{i}} = z_n|_{I_{\bar{i}}}$, we can define the functions

$$
\tilde{z}_n(s) = \lambda_n^{1/4} z_{n,\bar{i}}\left(\frac{s}{\sqrt{\lambda_n}}\right) \text{ for } s \in [0, l_{\bar{i}}\sqrt{\lambda_n}]
$$

and we can repeat the argument of Lemma 3.1 to prove that $\tilde{z}_n \rightharpoonup 0$ in $L^2(\mathbb{R}^+)$ as $n \to \infty$. This contradicts (3.11). ∎

PROPOSITION 3.4. *We have $\|R\|_\lambda \leq \lambda^{-\alpha}$ for any $\alpha > 0$.*

Proof. As in Proposition 3.2, we take $V = i^*_\lambda(f(W_\lambda))$, where W_λ is defined in (3.9). Then we find that $V - W_\lambda$ solves the following differential equation

$$
\begin{aligned}
(3.12)\qquad -(V-W_\lambda)''(x) + \lambda(V-W_\lambda)(x) &= \sum_{i=1}^{k}(\chi_i^p - \chi_i)(x)\lambda^{\frac{p}{p-1}}U^p(x\sqrt{\lambda}) \\
&\quad - \lambda^{\frac{1}{p-1}}\sum_{i=1}^{k}\chi_i''(x)U(x\sqrt{\lambda}) - 2\lambda^{\frac{1}{p-1}}\sqrt{\lambda}\sum_{i=1}^{k}\chi_i'(x)U'(x\sqrt{\lambda})
\end{aligned}
$$

which leads to the same conclusion of Proposition 3.2. ∎

Proof of Theorem 1.3. The proof of this theorem is verbatim the proof of Theorem 1.2. ∎

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