An extension theorem from connected sets and homogenization of non-local functionals

An extension theorem from connected sets and homogenization of non-local functionals

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Abstract. We study the asymptotic behaviour of convolution-type functionals defined on general periodic domains by proving an extension theorem.

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1 Introduction

In this paper we consider energies of convolution-type whose prototypes are functionals of the form

$$\frac{1}{\varepsilon^{d+p}}\int_{\Omega\times\Omega} a\left(\frac{y-x}{\varepsilon}\right)|u(y)-u(x)|^p dx\, dy, \tag{1}$$

where a is a non-negative convolution kernel, $p\in(1,+\infty)$, ε is a scaling parameter and Ω is a Lipschitz domain in $\mathbb{R}^d$. The kernel $a:\mathbb{R}^d\to[0,+\infty[$, describing the strength of the interaction at a given distance, satisfies

$$\int_{\mathbb{R}^d} a(\xi)(1+|\xi|^p)\, d\xi<+\infty, \tag{2}$$

and

$$a(\xi)\geq c>0, \qquad \text{if} \quad |\xi|\leq r_0, \tag{3}$$

for some $r_0>0$ and $c>0$.

Functionals of this form have been used as an approximation of the L^p-norm of the gradient as $\varepsilon\to 0$ and as such give an alternative way of defining Sobolev spaces (see *e.g.* [2, 10]). In the case $p=2$ perturbations of such energies (1) arise from models in population dynamics where the macroscopic properties are reduced to studying the

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evolution of the first-correlation function describing the population density u in the system [11], and recently they have also been used in problems in Data Science [12]. Furthermore discrete versions of such energies have been extensively studied in a general setting (see *e.g.* [3, 5] and related works).

A rather complete analysis of perturbations of functionals (1), more precisely, of functionals that are dominated from below and above by functionals of type (1), is presented in [4]. In this paper we consider another type of perturbation of (1) in the framework of the so-called *perforated domains*, that cannot be reduced to the analysis in [4] since it is 'degenerate' on the complement of a periodic connected set.

In our analysis we consider a typical situation arising in the study of inhomogeneous media with a periodic microstructure, when one sets the model in a domain obtained by removing inclusions representing sites with which the system does not interact. Usually, such a periodically perforated domain is obtained by intersecting Ω with a periodic open subset $E_\delta = \delta E$ of $\mathbb{R}^d$, where E is a periodic set with Lipschitz boundary and δ is the (small) period of the microstructure. In the setting of energies (1) the relevant scale of the period δ is of order ε. Indeed, in the other cases we have a multi-scale problem that can be decomposed into two separate limit analyses that fall within known results corresponding to letting first $\delta \to 0$ and then $\varepsilon \to 0$, or the converse (see [8]). Hence, we will consider energies whose prototypes are of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+p}} \int_{(\Omega\cap\varepsilon E)\times(\Omega\cap\varepsilon E)} a\left(\frac{y-x}{\varepsilon}\right) |u(x)-u(y)|^p dy\, dx, \tag{4}$$

where Ω is a fixed domain in $\mathbb{R}^d$.

In order to study the asymptotic analysis of such energies, it is necessary to prove that sequences with equi-bounded energy (and equi-bounded L^p-norm) are precompact. For the analog energy on Sobolev spaces

$$F_\varepsilon^{\mathrm{Sob}}(u) = \int_{\Omega\cap\varepsilon E} |\nabla u|^p dy\, dx.$$

this has been done in [1] through the construction of suitable extension operators $T_\varepsilon : L^p(\Omega\cap\varepsilon E) \to L^p(\Omega)$ which, for each Ω' compactly contained in Ω, provide an embedding of $W^{1,p}(\Omega')$ in $W^{1,p}(\Omega\cap\varepsilon E)$ uniformly for ε small enough (below a threshold explicitly depending on the distance between Ω' and $\partial\Omega$). The compact embedding of $W^{1,p}(\Omega')$ in $L^p(\Omega')$ then provides the desired compactness property. In our case, since the energies are non-local, a more complex statement is necessary. After noting that by condition (3) it is sufficient to prove compactness when a is the characteristic function of a ball centered in 0 and given radius r_0, we prove the existence of extension operators $T_\varepsilon : L^p(\Omega\cap\varepsilon E) \to L^p(\Omega)$ with the property that R and C exists such that for each Ω' compactly contained in Ω,

$$\begin{aligned}&\int_{\Omega'\times\Omega'} \chi_{B_R}\left(\frac{y-x}{\varepsilon}\right) |T_\varepsilon u(x)-T_\varepsilon u(y)|^p dy\, dx\\ &\leq C \int_{(\Omega\cap\varepsilon E)\times(\Omega\cap\varepsilon E)} \chi_{B_{r_0}}\left(\frac{y-x}{\varepsilon}\right) |u(x)-u(y)|^p dy\, dx,\end{aligned} \tag{5}$$

for ε small enough, with C and R independent of ε (here B_ρ denotes the ball of centre 0 and radius ρ and χ_A is the characteristic function of the set A). The precise statement of this result is given in Theorem 2.2. It provides a uniform bound for energies of the type (1) on Ω' in terms of energies (4), which in turn allows to apply the compactness results in [4] (see Section 2.2). Moreover, the asymptotic analysis of functionals (1) ensure that limits of functions with equibounded energies are in $W^{1,p}(\Omega')$ with a uniform bound and hence they belong to $W^{1,p}(\Omega)$.

The case $p = 2$ in (4) and with compact perforations; *i.e.*, with E of the form $E = \mathbb{R}^d \setminus (K_0 + \mathbb{Z}^d)$, where K_0 is a compact subset of $\mathbb{R}^d$ with Lipschitz boundary such that $(K_0 + i) \cap (K_0 + j) = \emptyset$ if $i, j \in \mathbb{Z}^d$ and $i \neq j$, has been studied in [8], together with some variants that allow to consider random perforations [9]. The main feature of our paper is the proof of the extension theorem under the only assumption that the periodic set E is connected and with Lipschitz boundary, and holds for any $p > 1$. The construction of T_ε is inspired by the arguments of [1], consisting in proving a local extension result on cubes and then using a periodic partition of the unity. The non-locality of the energies adds further technical difficulties to the possible non-connectedness or non-regularity of the restriction of E to cubes, already present in the case of Sobolev functions, and forces the introduction of the radius of interaction R in inequality (5).

As an application, we study the asymptotic behaviour of energies of the form

$$H_\varepsilon(u) = \frac{1}{\varepsilon^d} \int_{(\Omega\cap\varepsilon E)^2} h\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{u(y) - u(x)}{\varepsilon}\right) dx\, dy,$$

with $u \in L^p(\Omega; \mathbb{R}^m)$, upon some structure hypotheses on h as those considered in [4], that allow H_ε to be compared with F_ε. In Section 3 we obtain a homogenization theorem for H_ε as $\varepsilon \to 0$ proving that the Γ-limit of H_ε is defined on $W^{1,p}(\Omega; \mathbb{R}^m)$ and has a standard local form

$$\int_\Omega h_{\rm hom}(Du)dx,$$

with $h_{\rm hom}$ characterized by non-local homogenization formulas and of p-growth by (2) and (3). The proof is obtained by a perturbation argument that allows to use homogenization theorems proved in [4] for the corresponding energies defined on 'solid' domains, applied to functionals of the form $H_\varepsilon + \delta F_\varepsilon$. The Extension Theorem provides uniform estimates that allow to invert the passage to the limit as $\varepsilon \to 0$ and $\delta \to 0$. We note that a discrete analog of this result can be found in [6], where the discrete setting allows easier extension results from the discrete version of a perforated domain.

Before stating and proving the main result we gather some of the notation used in the following.

Notation

- $Q = (0, 1)^d$ denotes the unit cube in $\mathbb{R}^d$.

- χ_A denotes the characteristic function of the set A.
- $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$.
- $\mathbf{M}^{m\times d}$ is the space of $m \times d$ real matrices.
- if $\Xi \in \mathbf{M}^{m\times d}$ and $x \in \mathbb{R}^d$ then $\Xi x \in \mathbb{R}^m$ is defined by the usual row-by-column product.
- For any open set $\Omega \subset \mathbb{R}^d$ and for any $\lambda > 0$, $\lambda\Omega$ denotes the λ-homothetic set

$$\lambda\Omega := \{\lambda x : x \in \Omega\},$$

and $\Omega(\lambda)$ is the retracted set

$$\Omega(\lambda) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \lambda\}. \tag{6}$$

- For $R > 0$, D_R denotes the set of points in $\mathbb{R}^d \times \mathbb{R}^d$ whose distance is less than R; *i.e.*,

$$D_R := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq R\}.$$

- Given an open set with finite Lebesgue measure $|A| < \infty$, the mean value of u over A is given by

$$u_A = \frac{1}{|A|} \int_A u(x)\, dx. \tag{7}$$

- We say that a set $E \subset \mathbb{R}^d$ is periodic (more precisely, Q-periodic) if $E + e_i = E$ for every $i = 1, 2, \cdots, d$ where $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$.

2 The extension theorem

In this section, we prove the existence of an extension operator for non-local functionals defined on general connected domains. The main result of the paper is Theorem 2.2, from which we deduce a compactness result in Section 2.2. Before stating it, we recall the definition of a set with Lipschitz boundary.

Definition 2.1. *An open set $E \subset \mathbb{R}^n$ has* Lipschitz boundary *at $x \in \partial E$ if ∂E is locally the graph of a Lipschitz function, in the sense that there exist a coordinate system $(y_1, \ldots, y_d)$, a Lipschitz function Φ of $d-1$ variables, and an open rectangle U_x in the y-coordinates, centred at x, such that $E \cap U_x = \{y : y_n < \Phi(y_1, \ldots, y_{d-1})\}$ and that ∂E splits U_x into two connected sets, $E \cap U_x$ and $U_x \setminus \overline{E}$. If this property holds for every $x \in \partial E$ with the same Lipschitz constant, we say that E has* Lipschitz boundary.

Theorem 2.2. *Let E be a periodic open subset of $\mathbb{R}^d$ with Lipschitz boundary and let Ω be a bounded open subset of $\mathbb{R}^d$. Then, there exist $R = R(E) > 0$ and $k_0 > 0$ such that for all $\varepsilon > 0$ there exists a linear and continuous extension operator $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \to L^p(\Omega)$ such that for all $r > 0$ and for all $u \in L^p(\Omega \cap \varepsilon E)$,*

$$T_\varepsilon(u) = u \quad \text{a.e. in} \quad \Omega \cap \varepsilon E, \tag{8}$$

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon(u)|^p \, dx \leq c_1 \int_{\Omega \cap \varepsilon E} |u|^p \, dx, \tag{9}$$

$$\int_{(\Omega(\varepsilon k_0))^2 \cap D_{\varepsilon R}} |T_\varepsilon(u)(x) - T_\varepsilon(u)(y)|^p \, dx \, dy \leq c_2(r) \int_{(\Omega \cap \varepsilon E)^2 \cap D_{\varepsilon r}} |u(x) - u(y)|^p \, dx \, dy, \tag{10}$$

where we use notation (6). The positive constants c_1 and c_2 depend on E and d and, in addition, c_2 depends also on r, but both are independent of ε.

The proof, which will be given in the next subsection, is quite technical and it is split into several lemmas.

2.1 Technical lemmas and proof of the main result

In order to give an idea of the construction of the extension operator, we assume that $E \cap 2Q$ is connected and has Lipschitz boundary. Under these assumptions, there exists a linear and continuous operator $\Phi : L^p(E \cap 2Q) \to L^p(2Q)$ satisfying, in particular, an estimate analogous to (10) (see Lemma 2.5). Then, we consider the family Φ^α of the extension operator obtained by traslating Φ by an integer vector $\alpha \in \mathbb{Z}^d$. Finally, thanks to a periodic partition of unity, the construction of a global extension operator is achieved glueing together Φ^α (see Lemma 2.7). Now, the assumptions that $E \cap 2Q$ is connected and has Lipschitz boundary in general are not satisfied (unless the complement of E is a disjoint union of compact sets, which is the case studied in [8]), so that the first step consists to overcome the lack of connectedness of $E \cap 2Q$ and the regularity of its boundary. To this end, we state a slightly modified version of [1, Lemma 2.3], which is a key tool for the construction of the extension operator. The proof remains analogous to that of [1, Lemma 2.3] and is not repeated here.

Lemma 2.3. *Let E be a connected open subset of $\mathbb{R}^d$ with Lipschitz boundary. Then, there exists $k \in \mathbb{N}$, $k \geq 4$, such that $3Q \cap E$ is contained in a single connected component C of $kQ \cap E$. Moreover, C has Lipschitz boundary at each point of $\partial C \cap 3\overline{Q}$.*

We denote henceforth by $\tilde{C}$ the positive constant given by $\tilde{C} := 2\sqrt{d}k$, where k is defined as Lemma 2.3.

The next lemma is an easy consequence of the Hölder inequality.

Lemma 2.4. *Let A be an open subset of $\mathbb{R}^d$. Assume that A has finite and positive Lebesgue measure $|A| < \infty$. Then, for every $u \in L^p(A)$, with $1 < p < \infty$,*

$$\int_A |u_A - u(x)|^p dx \leq \frac{1}{|A|} \int_{A \times A} |u(x) - u(y)|^p dx dy. \tag{11}$$

Proof. Denote by p' the conjugate exponent of p. Thanks to Hölder's inequality, we deduce

$$\begin{aligned}\int_A |u_A - u(x)|^p dx &= \frac{1}{|A|^p}\int_A \left|\int_A (u(y)-u(x))dy\right|^p dx \\ &\le \frac{|A|^{p/p'}}{|A|^p}\int_A\int_A |u(y)-u(x)|^p dydx \\ &= \frac{1}{|A|}\int_{A\times A} |u(y)-u(x)|^p dxdy,\end{aligned}$$

which concludes the proof. □

The next lemma shows the existence of an extension operator Φ on general sets of $\mathbb{R}^d$. It is an adaptation of [1, Lemma 2.6].

Lemma 2.5. *Let B, ω, ω' be bounded open subsets of $\mathbb{R}^d$. Assume that ∂B is Lipschitz-continuous at each point of $\partial B \cap \overline{\omega}$ and $\omega' \subset\subset \omega$. Then, there exist a positive real number $R > 0$ and a linear and continuous extension operator $\Phi : L^p(B) \to L^p(\omega')$ such that, for all $u \in L^p(B)$,*

$$\Phi(u) = u \qquad a.e.\ in\ \ B \cap \omega', \tag{12}$$

$$\int_{\omega'} |\Phi(u)|^p \, dx \le c_1 \int_{B\cap\omega} |u|^p \, dx, \tag{13}$$

$$\int_{(\omega'\times\omega')\cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p \, dx\, dy \le c_2 \int_{(B\cap\omega)^2} |u(x)-u(y)|^p \, dx\, dy, \tag{14}$$

where c_1 and c_2 are positive constant depending only on B, ω', ω and p.

Proof. Since ∂B has Lipschitz boundary at each point of $\partial B \cap \overline{\omega}$, there exist a neighbourhood U of $\partial B \cap \overline{\omega}$ and a bi-lipschitz map $\mathcal{R} : U \cap B \to U \setminus B$ such that, for any $x_1, x_2 \in U \cap B$,

$$\frac{1}{2}|\mathcal{R}(x_1) - \mathcal{R}(x_2)| \le |x_1 - x_2| \le 2|\mathcal{R}(x_1) - \mathcal{R}(x_2)|.$$

For fixed $t > 0$ chosen below, we consider the set

$$A_t := \{x \in \omega \setminus B : \mathrm{dist}(x, \partial B) < t\}. \tag{15}$$

We may fix $t > 0$ small enough such that

$$A_t \cap \omega' \subset U \setminus B \qquad \text{and} \qquad \mathcal{R}^{-1}(A_t \cap \omega') \subset B \cap \omega. \tag{16}$$

Let φ be a C^∞ function such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $\overline{B}$ and $\varphi \equiv 0$ in $\{x \in \mathbb{R}^d \setminus B : \mathrm{dist}(x, \partial B) \ge t\}$. We define the operator $\Phi : L^p(B) \to L^p(\omega')$ as follows

$$\Phi(u)(x) := \begin{cases} u(x), & x \in B \cap \omega', \\ \varphi(x)u(\mathcal{R}^{-1}(x)) + (1-\varphi(x))u_{B\cap\omega}, & x \in A_t \cap \omega', \\ u_{B\cap\omega}, & x \in \omega' \setminus A_t, \end{cases} \tag{17}$$

where $u_{B\cap\omega}$ denotes the mean value of the function u over $B\cap\omega$ (see (7)). It follows that $\Phi(u)\in L^p(\omega')$ and $\Phi(u)=u$ a.e. in $B\cap\omega'$; *i.e.*, condition (12) is satisfied.

We now show condition (13). To this end, note that ω' can be written as

$$\omega' = (B\cap\omega')\cup(A_t\cap\omega')\cup(\omega'\setminus A_t).$$

This, combined with the Jensen inequality and the definition (17) of Φ, yields

$$\begin{aligned}\int_{\omega'}|\Phi(u)(x)|^p dx &= \int_{B\cap\omega'}|\Phi(u)(x)|^p dx + \int_{A_t\cap\omega'}|\Phi(u)(x)|^p dx + \int_{\omega'\setminus A_t}|\Phi(u)(x)|^p dx\\ &= \int_{B\cap\omega'}|u(x)|^p dx + \int_{A_t\cap\omega'}|\varphi(x)u(\mathcal{R}^{-1}(x)) + (1-\varphi(x))u_{B\cap\omega}|^p d\xi\\ &\quad + |\omega'\setminus A_t||u_{B\cap\omega}|^p\\ &\le \int_{B\cap\omega'}|u(x)|^p dx + 2^{p-1}\int_{A_t\cap\omega'}|u(\mathcal{R}^{-1}(x))|^p dx\\ &\quad + |u_{B\cap\omega}|^p(2^{p-1}|\omega'\cap A_t| + |\omega'\setminus A_t|).\end{aligned} \tag{18}$$

Since $\mathcal{R}$ is a bi-Lipschitz map, the Jacobian $\left|\frac{\partial\mathcal{R}}{\partial x}(x)\right|$ is a bounded function; *i.e.*, there exists a positive constant $c_{\mathcal{R}}$ such that

$$\left|\frac{\partial\mathcal{R}}{\partial x}(x)\right| \le c_{\mathcal{R}}, \tag{19}$$

so that, thanks to the change of variables $x' = \mathcal{R}^{-1}(x)$ and properties (16), we have

$$\int_{A_t\cap\omega'}|u(\mathcal{R}^{-1}(x))|^p dx \le c_{\mathcal{R}}\int_{B\cap\omega}|u(x')|^p dx'.$$

This, along with (18), implies that

$$\begin{aligned}\int_{\omega'}|\Phi(u)(x)|^p dx &\le (c_{\mathcal{R}}2^{p-1}+1)\int_{B\cap\omega'}|u(x)|^p dx + |u_{B\cap\omega}|^p(2^{p-1}|\omega'\cap A_t| + |\omega'\setminus A_t|)\\ &\le c_1\int_{B\cap\omega}|u(x)|^p dx,\end{aligned}$$

where c_1 denotes a positive constant depending only on p,ω',B and $\mathcal{R}$. Hence, condition (13) is proven.

To conclude the proof, it remains to check condition (14). Fix $R<t$. For $(x,y)\in(\omega'\times\omega')\cap D_R$, it is enough to estimate the integral in the left-hand side of (14) by

examining separately the sets

$$\begin{aligned}
S_1 &= ((B \cap \omega') \times (B \cap \omega')) \cap D_R, \\
S_2 &= ((B \cap \omega') \times (A_t \cap \omega')) \cap D_R, \\
S_2' &= ((A_t \cap \omega') \times (B \cap \omega') \cap D_R, \\
S_3 &= ((A_t \cap \omega') \times (A_t \cap \omega')) \cap D_R, \\
S_4 &= ((A_t \cap \omega') \times (\omega' \setminus A_t)) \cap D_R, \\
S_4' &= ((\omega' \setminus A_t) \times (A_t \cap \omega')) \cap D_R, \\
S_5 &= ((\omega' \setminus A_t) \times (\omega' \setminus A_t)) \cap D_R.
\end{aligned}$$

Note that the other cases do not occur since the distance between the points is grater than R. Indeed, take, for example, $(x, y) \in (B \cap \omega') \times (A_t \setminus \omega')$. Due to definition of A_t and since $R < t$, the distance $|x - y|$ is greater than R.
Now, we evaluate the left-hand side of (14) on the set S_i defined above. In view of the definition (17) of Φ, we have

$$\begin{aligned}
\int_{S_1} |\Phi(u)(x) - \Phi(u)(y)|^p \, dxdy &= \int_{S_1} |u(x) - u(y)|^p \, dxdy \\
&\leq \int_{(B\cap\omega)^2} |u(x) - u(x)|^p \, dxdy.
\end{aligned}$$

Here, we used the fact that $S_1 \subset (B \cap \omega')^2 \subset (B \cap \omega)^2$.
Due to definition (17) of Φ, an application of Jensen's inequality yields

$$\begin{aligned}
\int_{S_2} |\Phi(u)(x) - \Phi(u)(y)|^p dxdy &= \int_{S_2} |u(x) - u(\mathcal{R}^{-1}(y)) + (1 - \varphi(y))(u(\mathcal{R}^{-1}(y)) - u_{B\cap\omega})|^p dxdy \\
&\leq 2^{p-1} \int_{S_2} |u(x) - u(\mathcal{R}^{-1}(y))|^p \, dxdy \\
&\quad + 2^{p-1} \int_{S_2} |1 - \varphi(y)|^p \left|u(\mathcal{R}^{-1}(y)) - u_{B\cap\omega}\right|^p \, dxdy \qquad (20)
\end{aligned}$$

Using the change of variables $y' = \mathcal{R}^{-1}(y)$ and properties (16) and (19), the first integral in the left-hand side of (20) can be estimates as

$$\begin{aligned}
\int_{S_2} |u(x) - u(\mathcal{R}^{-1}(y))|^p \, dxdy &\leq \int_{B\cap\omega'} \left(\int_{A_t\cap\omega'} |u(x) - u(\mathcal{R}^{-1}(y))|^p dy \right) dx \\
&\leq c_{\mathcal{R}} \int_{(B\cap\omega')^2} |u(x) - u(y')|^p \, dxdy' \\
&\leq c_{\mathcal{R}} \int_{(B\cap\omega)^2} |u(x) - u(y)|^p \, dxdy. \qquad (21)
\end{aligned}$$

By applying Lemma 2.4 and taking into account condition (19), the second integral in the right-hand side of (20) can be estimated as

$$\begin{aligned}\int_{S_2} |1-\varphi(y)|^p \left|u(\mathcal{R}^{-1}(y)) - u_{B\cap\omega}\right|^p \,dxdy &\leq |B\cap\omega'| \int_{A_t\cap\omega'} \left|u(\mathcal{R}^{-1}(y)) - u_{B\cap\omega}\right|^p \,dy \\ &\leq c_{\mathcal{R}} |B\cap\omega'| \int_{B\cap\omega} |u(y') - u_{B\cap\omega}|^p \,dy' \\ &\leq c_{\mathcal{R}} \frac{|B\cap\omega'|}{|B\cap\omega|} \int_{(B\cap\omega)^2} |u(x) - u(y)|^p \,dxdy.\end{aligned}$$

Combined with (20) and (21), this implies

$$\int_{S_2} |\Phi u(x) - \Phi u(y)|^p dxdy \leq c \int_{(B\cap\omega)^2} |u(x) - u(y)|^p dxdy,$$

where c is a positive constant depending on p, B, ω, ω' and $\mathcal{R}$. Similarly, we have that

$$\int_{S_2'} |\Phi u(x) - \Phi u(y)|^p dxdy \leq c \int_{(B\cap\omega)^2} |u(x) - u(y)|^p dxdy.$$

Now, consider $(x, y) \in S_3$. From the definition (17) of Φ, we have

$$\Phi u(x) - \Phi u(y) = F_1(x, y) + F_2(x, y), \tag{22}$$

where $F_1(x, y)$ and $F_2(x, y)$ are given by

$$\begin{aligned} F_1(x, x) &:= (u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega})(\varphi(x) - \varphi(y)), \\ F_2(x, y) &:= \varphi(y) \left(u(\mathcal{R}^{-1}(x)) - u(\mathcal{R}^{-1}(y))\right). \end{aligned}$$

Thanks to Lemma 2.4 and due to properties (16) and the estimate $|\varphi(x) - \varphi(y)| \leq 2$, we deduce that

$$\begin{aligned}\int_{S_3} |F_1(x, y)|^p \,dxdy &\leq 2^p \int_{(A_t\cap\omega')^2} \left|u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega}\right|^p \,dxdy \\ &= 2^p |A_t\cap\omega'| \int_{(A_t\cap\omega')} \left|u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega}\right|^p \,dx \\ &\leq 2^p |A_t\cap\omega'| c_{\mathcal{R}} \int_{B\cap\omega} |u(x') - u_{B\cap\omega}|^p \,dx' \\ &\leq 2^p c_{\mathcal{R}} \frac{|A_t\cap\omega'|}{|B\cap\omega|} \int_{(B\cap\omega)^2} |u(x') - u(y)|^p \,dx'dy. \end{aligned} \tag{23}$$

On the other hand, using the changes of variables $x' = \mathcal{R}^{-1}(x)$ and $y' = \mathcal{R}^{-1}(y)$, we get

$$\begin{aligned}\int_{S_3} |F_2(x, y)|^p \,dxdy &\leq \int_{(A_t\cap\omega')^2} \left|u(\mathcal{R}^{-1}(x)) - u(\mathcal{R}^{-1}(y))\right|^p \,dxdy \\ &\leq c_{\mathcal{R}}^2 \int_{(B\cap\omega)^2} |u(x') - u(y')|^p \,dx'dy'. \end{aligned} \tag{24}$$

In view of (22), an application of Jensen's inequality combined with (23) and (24) leads to

$$\begin{aligned}\int_{S_3} |\Phi(u)(x)-\Phi(u)(y)|^p\, dxdy &\le 2^{p-1}\left(\int_{S_3} |F_1(x,y)|^p\, dxdy + \int_{S_3} |F_2(x,y)|^p\, dxdy\right)\\ &\le c\int_{(B\cap\omega)^2} |u(x)-u(y)|^p\, dxdy, \end{aligned}\tag{25}$$

where c denotes a positive constant depending only on p, B, ω, ω' and $\mathcal{R}$.
Take now $(x,y)\in S_4$. Applying Lemma 2.4 and using the change of variables $x'=\mathcal{R}^{-1}(x)$, from the definition (17) of Φ, we deduce that

$$\begin{aligned}\int_{S_4} |\Phi(u)(x)-\Phi(u)(y)|^p\, dxdy &= |\omega'\setminus A_t|\int_{A_t\cap\omega'} \left|u(\mathcal{R}^{-1}(x))-u_{B\cap\omega}\right|^p\, dx\\ &\le c_{\mathcal{R}}|\omega'\setminus A_t|\int_{B\cap\omega} |u(x')-u_{B\cap\omega}|^p\, dx'\\ &\le c_{\mathcal{R}}\frac{|\omega'\setminus A_t|}{|B\cap\omega|}\int_{(B\cap\omega)^2} |u(x)-u(y)|^p\, dxdy\end{aligned}$$

Similarly, we also get

$$\int_{S_4'} |\Phi(u)(x)-\Phi(u)(y)|^p\, dxdy \le c_{\mathcal{R}}\frac{|\omega'\cap A_t|}{|B\cap\omega|}\int_{(B\cap\omega)^2} |u(x)-u(y)|^p\, dxdy.$$

Now, take $(x,y)\in S_5$. Hence, we have that $\Phi(x)-\Phi(y)=0$ for a.e. $x,y\in\omega'\setminus A_t$. Finally, gathering all the previous estimates, we conclude that

$$\begin{aligned}\int_{(\omega'\times\omega')\cap D_R} |\Phi(u)(x)-\Phi(u)(y)|^p dxdy &= \sum_{i=1}^{5}\int_{S_i} |\Phi(u)(x)-\Phi(u)(y)|^p dxdy\\ &\quad+\int_{S_2'\cup S_4'} |\Phi(u)(x)-\Phi(u)(y)|^p dxdy\\ &\le c_2\int_{(B\cap\omega)^2} |u(x)-u(y)|^p dxdy,\end{aligned}$$

where c_2 is a costant depending on p,ω',ω and B. This shows (14) and concludes the proof. $\square$

The reflection argument that we used to construct the operator Φ cannot be used to prove the existence of a map $\Phi : L^p(B)\to L^p(\omega)$ since estimate (14) may not hold with $\omega'=\omega$, as showed in the following example.

Example 2.6. *Let B be the ball in $\mathbb{R}^2$ centered at 0 and of radius 1 and let ω be the set of $\mathbb{R}^2$ defined by*

$$\omega := \{(x,y)\in\mathbb{R}^2 : x\in(-1,2), -x+1\le y\le -x+2\}.$$

We define $u \in L^p(B)$ as

$$u(x) := \begin{cases} 1, & x \in B \setminus \omega, \\ 0, & x \in B \cap \omega. \end{cases}$$

If $\Phi(u)$ is the extension of u out of B by reflection, then we have

$$\int_{\omega^2 \cap D_R} |\Phi u(x) - \Phi u(y)|^p dxdy > 0,$$

since u is not identically constant in the neighbourhood of the points $(1,0)$ and $(0,1)$, while

$$\int_{(B\cap\omega)^2 \cap D_R} |u(x) - u(y)|^p dxdy = 0,$$

so that the condition (14) is not satisfied.

Lemma 2.7. *Let E be a periodic, connected open subset of $\mathbb{R}^d$ with Lipschitz boundary. Let Ω, Ω' be open subsets of $\mathbb{R}^d$ such that $\Omega' \subset\subset \Omega$ and $\operatorname{dist}(\Omega', \partial\Omega) > \tilde{C}$. Then there exist $R = R(E) > 0$ and a linear and continuous operator*

$$L : L^p(\Omega \cap E) \to L^p(\Omega')$$

such that for all $r > 0$ and for all $u \in L^p(\Omega \cap E)$,

$$Lu = u, \qquad \text{a.e. in } \ \Omega' \cap E, \tag{26}$$

$$\int_{\Omega'} |Lu|^p dx \leq c_1 \int_{\Omega \cap E} |u|^p dx, \tag{27}$$

$$\int_{(\Omega' \times \Omega') \cap D_R} |Lu(x) - Lu(y)|^p dxdy \leq c_2(r) \int_{(\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy, \tag{28}$$

where c_1 and c_2 are positive constants depending on E and d and, in addition, c_2 depends also on r. The constant R depends only on the set E.

Proof. In view of Lemma 2.3, there exists $k \in \mathbb{N}$, $k \geq 4$, such that $3Q \cap E$ is contained in a single connected component C of $kQ \cap E$. Since C has Lipschitz boundary at each point of $C \cap 3\overline{Q}$, we can apply Lemma 2.5 with $B = C$, $\omega' = 2Q$ and $\omega = 3Q$. Hence, there exist $R > 0$ and a linear and continuous operator $\Phi : L^p(C) \to L^p(2Q)$ defined by (17) such that, for any $u \in L^p(C)$,

$$\Phi(u) = u \qquad \text{a.e. in } \ C \cap 2Q, \tag{29}$$

$$\int_{2Q} |\Phi(u)|^p dx \leq c_1 \int_{C \cap 3Q} |u|^p dx, \tag{30}$$

$$\int_{(2Q \times 2Q) \cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p dxdy \leq c_2 \int_{(C\cap 3Q)\times(C \cap 3Q)} |u(x) - u(y)|^p dxdy, \tag{31}$$

where the positive constants c_1 and c_2 depend on C and $2Q$.
Let $(Q_2^\alpha)_{\alpha\in\mathbb{Z}^d}$ be the open cover of $\mathbb{R}^d$ obtained by translating the cube $2Q$ by the vector $\alpha\in\mathbb{Z}^d$. For every set $\Omega\subset\mathbb{R}^d$, for every $\alpha\in\mathbb{Z}^d$ and for every real number $h>0$, we use the notation

$$\Omega_h^\alpha := \alpha + h\Omega. \tag{32}$$

For $h=1$ we simply write $\Omega^\alpha = \Omega_1^\alpha$, while, for $\alpha=0$, $\Omega_h=\Omega_h^0$. For every set $A\subseteq\mathbb{R}^d$, we define the set

$$I(A) := \{\alpha\in\mathbb{Z}^d : Q_2^\alpha\cap A\neq\emptyset\}.$$

Since $\operatorname{dist}(\Omega',\partial\Omega) > \tilde{C} = 2\sqrt{d}k$, for every $\alpha\in I(\Omega')$, we have that $Q_{2k}^\alpha\subset\Omega$.
For any $\alpha\in I(\Omega')$, we define the extension operator $\Phi^\alpha : L^p(C^\alpha)\to L^p(Q_2^\alpha)$ by translating the operator Φ by the integer vector α. In other words, for any $u\in L^p(C^\alpha)$,

$$\Phi^\alpha(u) := (\Phi(u\circ\pi^\alpha))\circ\pi^{-\alpha}, \tag{33}$$

where, for every $\alpha\in\mathbb{Z}^d$ and for every real number $h>0$, we use the notation

$$\pi_h^\alpha(x) := \alpha + hx \qquad \text{for } x\in\mathbb{R}^d. \tag{34}$$

If $h=1$, we write $\pi^\alpha=\pi_1^\alpha$ and if $\alpha=0$, we set $\pi_h=\pi_h^0$. For simplicity, for $u\in L^p(\Omega\cap E)$ we denote by u^α the function

$$u^\alpha := \Phi^\alpha(u_{|C^\alpha})\in L^p(Q_2^\alpha). \tag{35}$$

From (17) and (33), the explicit expression of u^α is given by

$$u^\alpha(x) := \begin{cases} u_{|C^\alpha}(x), & x\in(2Q\cap C)^\alpha,\\ \varphi(x-\alpha)u(\mathcal{R}^{-1}(x-\alpha)+\alpha) + (1-\varphi(x-\alpha))u_{(3Q\cap C)^\alpha}, & x\in(2Q\cap A_t)^\alpha,\\ u_{(3Q\cap C)^\alpha}, & x\in(2Q\setminus(C\cup A_t))^\alpha,\end{cases}$$

where A_t is given by (15) with $B=C^\alpha$, $\omega=3Q^\alpha$, and $u_{(3Q\cap C)^\alpha}$ is the mean value of $u_{|C^\alpha}$ over $(3Q\cap C)^\alpha$; *i.e.*,

$$u_{(3Q\cap C)^\alpha} := \fint_{(3Q\cap C)^\alpha} u_{|C^\alpha}(x)dx.$$

We now define the global extension operator $L : L^p(\Omega\cap E)\to L^p(\Omega')$. To this end, let $(\psi^\alpha)_{\alpha\in\mathbb{Z}^d}$ be a partition of unity associated to $(Q_2^\alpha)_{\alpha\in\mathbb{Z}^d}$ such that $\psi^\beta=\psi^\alpha\circ\pi^{\alpha-\beta}$, for any $\alpha,\beta\in\mathbb{Z}^d$. Then, the map $L : L^p(\Omega\cap E)\to L^p(\Omega')$ is defined by

$$Lu := \sum_{\alpha\in I(\Omega')} u^\alpha\psi^\alpha,$$

where u^α is given by (35). Note that L is a linear and continuous operator from $L^p(\Omega\cap E)$ to $L^p(\Omega')$ and that condition (26) is satisfied. Indeed, in view of (35) and due to (29), we have

$$Lu(x) = \sum_{\alpha\in I(\Omega')} u^\alpha(x)\psi^\alpha(x) = \sum_{\alpha\in I(\Omega')} u(x)\psi^\alpha(x) = u(x)$$

for a.e. $x \in \Omega' \cap E$.
Now, we show condition (27). To this end, fix $\beta \in I(\Omega')$ and note that, for any $\alpha \in I(Q_2^\beta)$, we have $Q_k^\alpha \subset Q_{2k}^\beta$. Combined with estimate (30) and Jensen's inequality, this implies that, for any $u \in L^p(\Omega \cap E)$,

$$\begin{aligned}\int_{Q_2^\beta} |Lu|^p dx &\leq N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_2^\beta \cap Q_2^\alpha} |u^\alpha|^p dx \leq c_1 N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{(C\cap 3Q)^\alpha} |u|^p dx \\ &\leq c_1 N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_k^\alpha \cap E} |u|^p dx \leq c_1 N^p \int_{Q_{2k}^\beta \cap E} |u|^p dx,\end{aligned}$$

where N denotes, henceforth, the cardinality of the set $I(Q_2^\beta)$. Taking the sum over $\beta \in I(\Omega')$ in the previous inequality, we deduce that

$$\begin{aligned}\int_{\Omega'} |Lu|^p dx &\leq \sum_{\beta \in I(\Omega')} \int_{Q_2^\beta} |Lu|^p dx \\ &\leq c_1 N^p \sum_{\beta \in I(\Omega')} \int_{Q_{2k}^\beta \cap E} |u|^p dx \leq N^p (2k)^d c_1 \int_{\Omega \cap E} |u|^p dx.\end{aligned}$$

The factor $(2k)^d$ is due to the fact that each point $x \in \mathbb{R}^d$ is contained in at most $(2k)^d$ cubes of the form $(Q_{2k}^\beta)_{\beta \in \mathbb{Z}^d}$.
To conclude the proof, it remains to show condition (28). To this end, we state the following estimate whose proof is given in Lemma 2.8 below: for all $r > 0$ there exists a positive constant $c = c(r)$ such that

$$\int_{((C\cap Q_3)^\alpha)^2} |u(x) - u(y)|^p dxdy \leq c(r) \int_{(Q_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy. \tag{36}$$

Fix $\beta \in \mathbb{Z}^d$. Since

$$Lu(x) - Lu(y) = \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha(x) - u^\alpha(y))\psi^\alpha(x) - \sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y)(\psi^\alpha(y) - \psi^\alpha(x))$$

for a.e. $x, y \in Q_2^\beta$, an application of Jensen's inequality leads to

$$\begin{aligned}&\int_{(Q_2^\beta)^2 \cap D_R} |Lu(x) - Lu(y)|^p dxdy \\ &\qquad \leq 2^{p-1} \int_{(Q_2^\beta)^2 \cap D_R} |\sum_{\alpha \in I(Q_2^\beta)} (u^\alpha(x) - u^\alpha(y))\psi^\alpha(x)|^p dxdy \\ &\qquad\quad + 2^{p-1} \int_{(Q_2^\beta)^2 \cap D_R} |\sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y)(\psi^\alpha(y) - \psi^\alpha(x))|^p dxdy.\end{aligned} \tag{37}$$

Due to Jensen's inequality and in view of (31) and (36), the first integral is estimated as follows

$$
\begin{aligned}
&\int_{(Q_2^\beta)^2\cap D_R} |\sum_{\alpha\in I(Q_2^\beta)} (u^\alpha(x)-u^\alpha(y))\psi^\alpha(x)|^p dxdy \\
&\quad\le\ N^{p-1}\sum_{\alpha\in I(Q_2^\beta)}\int_{(Q_2^\beta\cap Q_2^\alpha)^2\cap D_R}|u^\alpha(x)-u^\alpha(y)|^p dxdy \\
&\quad\le\ N^{p-1}\sum_{\alpha\in I(Q_2^\beta)}\int_{(Q_2^\alpha\times Q_2^\alpha)\cap D_R}|u^\alpha(x)-u^\alpha(y)|^p dxdy \\
&\quad\le\ c_2N^{p-1}\sum_{\alpha\in I(Q_2^\beta)}\int_{((Q_3\cap C)^\alpha)^2}|u(x)-u(y)|^p dxdy \\
&\quad\le\ c_2c(r)N^{p-1}\sum_{\alpha\in I(Q_2^\beta)}\int_{(Q_k^\alpha\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy \\
&\quad\le\ c_2c(r)N^{p}\int_{(Q_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy. \qquad (38)
\end{aligned}
$$

We evaluate the second integral. Since $\mathrm{supp}(\psi^\alpha)\subset Q_2^\alpha$ for any $\alpha\in\mathbb{Z}^d$, we have that, for any $x,y\in Q_2^\beta$,

$$
\sum_{\alpha\in I(Q_2^\beta)}(\psi^\alpha(x)-\psi^\alpha(y))=0,
$$

which implies that

$$
\begin{aligned}
\sum_{\alpha\in I(Q_2^\beta)} u^\alpha(y)(\psi^\alpha(y)-\psi^\alpha(x)) &= \sum_{\alpha\in I(Q_2^\beta)} u^\alpha(y)(\psi^\alpha(y)-\psi^\alpha(x)) - u^\beta(x)\sum_{\alpha\in I(Q_2^\beta)}(\psi^\alpha(y)-\psi^\alpha(x)) \\
&= \sum_{\alpha\in I(Q_2^\beta)}(u^\alpha(y)-u^\beta(x))(\psi^\alpha(y)-\psi^\alpha(x)),
\end{aligned}
$$

for a.e. $x,y\in Q_2^\beta$. Thanks to the Jensen inequality, we obtain that

$$
\begin{aligned}
&\int_{(Q_2^\beta)^2\cap D_R} |\sum_{\alpha\in I(Q_2^\beta)} u^\alpha(x)(\psi^\alpha(y)-\psi^\alpha(x))|^p dxdy \\
&\quad\le N^{p-1}\sum_{\alpha\in I(Q_2^\beta)}\int_{(Q_2^\beta\cap Q_2^\alpha)^2\cap D_R}|u^\alpha(y)-u^\beta(x)|^p|\psi^\alpha(y)-\psi^\alpha(x)|^p dxdy \\
&\quad\le cN^{p-1}\sum_{\alpha\in I(Q_2^\beta)}\int_{(Q_2^\beta\cap Q_2^\alpha)^2\cap D_R}|u^\alpha(y)-u^\beta(x)|^p dxdy. \qquad (39)
\end{aligned}
$$

In order to estimate the integral on the right-hand side of (39), we perform computations analogous to that of Lemma 2.5. The difference is that u^α and u^β are extensions of u which belong to two different translated cubes Q_2^α and Q_2^β. Hence, we separately evaluate the integral on the right-hand side of (39) on the following sets, which take into account the fact that u^α and u^β are the extension of $u \in L^p(\Omega' \cap E)$ on different translated cubes,

$$\begin{aligned}
S_1^{\alpha,\beta} &= (Q_2^\alpha \cap Q_2^\beta \cap C)^2 \cap D_R;\\
S_2^{\alpha,\beta} &= (((2Q \cap C)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap A_t)^\beta)) \cap D_R;\\
S_3^{\alpha,\beta} &= (((2Q \cap A_t)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap C)^\beta)) \cap D_R;\\
S_4^{\alpha,\beta} &= (((2Q \cap A_t)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap A_t)^\beta)) \cap D_R;\\
S_5^{\alpha,\beta} &= (((2Q \cap A_t)^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \setminus (C \cup A_t))^\beta)) \cap D_R;\\
S_6^{\alpha,\beta} &= (((2Q \setminus (C \cup A_t))^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \cap A_t)^\beta)) \cap D_R;\\
S_7^{\alpha,\beta} &= (((2Q \setminus (C \cup A_t))^\alpha \cap Q_2^\beta) \times (Q_2^\alpha \cap (2Q \setminus (C \cup A_t))^\beta)) \cap D_R.
\end{aligned}$$

Note that, as in Lemma 2.5, the other combinations do not occur since R is chosen such that $R < t$.
Consider the case $(x, y) \in S_1^{\alpha,\beta}$. Since $u^\alpha = u^\beta$ a.e. in $Q_2^\alpha \cap Q_2^\beta \cap C$ and due to estimate (36), we have

$$\begin{aligned}
\int_{S_1^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dxdy &= \int_{S_1^{\alpha,\beta}} |u(x) - u(y)|^p dxdy\\
&\leq \int_{(2Q\cap C)^\beta \times (2Q\cap C)^\beta} |u(x) - u(y)|^p dxdy\\
&\leq \int_{((Q_3\cap C)^\beta)^2} |u(x) - u(y)|^p dxdy\\
&\leq c(r) \int_{(Q_k^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy\\
&\leq c(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy.
\end{aligned}$$

Here, we have used the fact that $S_1^{\alpha,\beta} \subset (2Q \cap C)^\beta \times (2Q \cap C)^\beta$.
Now, take $(x, y) \in S_2^{\alpha,\beta}$. Hence,

$$\begin{aligned}
u^\alpha(x) - u^\beta(y) &= u(x) - \varphi(y-\beta)u(\mathcal{R}^{-1}(y-\beta)+\beta) - (1-\varphi(y-\beta))u_{(3Q\cap C)^\beta}\\
&= [u(x) - u_{(3Q\cap C)^\alpha}] + [u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}]\\
&\quad \varphi(y-\beta)[u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Q\cap C)^\beta}],
\end{aligned}$$

which implies that

$$
\begin{aligned}
\int_{S_2^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dxdy \le{}& 3^{p-1}|2Q \cap A_t| \int_{(2Q\cap C)^\alpha} |u(x) - u_{(3Q\cap C)^\alpha}|^p dx \\
&+ 3^{p-1}|2Q \cap C||2Q \cap A_t||u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}|^p \\
&+ 3^{p-1}|2Q \cap C| \int_{(2Q\cap A_t)^\beta} |\varphi(y-\beta)|^p |u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Q\cap C)^\beta}|^p dy.
\end{aligned} \tag{40}
$$

Taking Lemma 11 and estimate (36) into account, we immediately deduce that

$$
\begin{aligned}
\int_{(2Q\cap C)^\alpha} |u(x) - u_{(3Q\cap C)^\alpha}|^p dx &\le \int_{(3Q\cap C)^\alpha} |u(x) - u_{(3Q\cap c)^\alpha}|^p dx \\
&\le \frac{1}{|3Q \cap C|} \int_{((Q_3\cap C)^\alpha)^2} |u(x) - u(y)|^p dxdy \\
&\le \frac{c(r)}{|3Q \cap C|} \int_{(Q_k^\alpha\cap E)^2\cap D_r} |u(x) - u(y)|^p dxdy \\
&\le \frac{c(r)}{|3Q \cap C|} \int_{(Q_{2k}^\beta\cap E)^2\cap D_r} |u(x) - u(y)|^p dxdy.
\end{aligned} \tag{41}
$$

By (19), we already know that $\mathcal{R}$ has bounded Jacobian and $R^{-1}(2Q \cap A_t) \subset (3Q \cap C)$. Then, in view of (36) and Lemma 11, it follows, after the changes of variables $y' = y - \beta$ and then $y'' = \mathcal{R}^{-1}(y') + \beta$, that

$$
\begin{aligned}
&\int_{(2Q\cap A_t)^\beta} |\varphi(y-\beta)|^p |u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Q\cap C)^\beta}]|^p dy \\
&\qquad = \int_{2Q\cap A_t} |\varphi(y')|^p |u(\mathcal{R}^{-1}(y')+\beta) - u_{(3Q\cap C)^\beta}]|^p dy' \\
&\qquad \le c_{\mathcal{R}} \int_{(3Q\cap C)^\beta} |u(y'') - u_{(3Q\cap C)^\beta}|^p dy'' \\
&\qquad \le \frac{c_{\mathcal{R}}}{|3Q \cap C|} \int_{((Q_3\cap C)^\beta)^2} |u(x) - u(y)|^p dxdy \\
&\qquad \le \frac{c_{\mathcal{R}}}{|3Q \cap C|} c(r) \int_{(Q_k^\beta\cap E)^2\cap D_r} |u(x) - u(y)|^p dxdy \\
&\qquad \le \frac{c_1}{|3Q \cap C|} c(r) \int_{(Q_{2k}^\beta\cap E)^2\cap D_r} |u(x) - u(y)|^p dxdy.
\end{aligned} \tag{42}
$$

In order to estimate the term $|u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}|^p$, note that

$$\begin{aligned}|u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}|^p &= \frac{1}{|3Q\cap C|^p}\left|\int_{(3Q\cap C)^\alpha\times(3Q\cap C)^\beta} u_{|_{C^\alpha}}(x) - u_{|_{C^\beta}}(y)dxdy\right|^p \\ &\leq \frac{1}{|3Q\cap C|^p}\int_{(3Q\cap C)^\alpha\times(3Q\cap C)^\beta} |u_{|_{C^\alpha}}(x) - u_{|_{C^\beta}}(y)|^p dxdy. \qquad (43)\end{aligned}$$

Since $u_{|_{C^\alpha}} = u_{|_{C^\beta}}$ a.e. on $Q_3^\alpha \cap Q_3^\beta \cap C$, the last integral can be estimated as follows

$$\begin{aligned}&\int_{(3Q\cap C)^\alpha\times(3Q\cap C)^\beta} |u_{|_{C^\alpha}}(x) - u_{|_{C^\beta}}(y)|^p dxdy \\ &\quad = \frac{1}{|Q_3^\alpha\cap Q_3^\beta\cap C|}\int_{Q_3^\alpha\cap Q_3^\beta\cap C}\int_{(3Q\cap C)^\alpha\times(3Q\cap C)^\beta} |u_{|_{C^\alpha}}(x) - u(z) + u(z) - u_{|_{C^\beta}}(y)|^p dxdydz \\ &\quad \leq \frac{2^{p-1}|3Q\cap C|}{|Q_3^\alpha\cap Q_3^\beta\cap C|}\int_{(Q_3^\alpha\cap Q_3^\beta\cap C)\times(3Q\cap C)^\alpha} |u_{|_{C^\alpha}}(x) - u(z)|^p dxdz \\ &\qquad + \frac{2^{p-1}|3Q\cap C|}{|Q_3^\alpha\cap Q_3^\beta\cap C|}\int_{(Q_3^\alpha\cap Q_3^\beta\cap C)\times(3Q\cap C)^\beta} |u_{|_{C^\beta}}(y) - u(z)|^p dydz.\end{aligned}$$

Since $Q_3^\alpha \cap Q_3^\beta \cap C$ is contained in $(3Q\cap C)^\alpha$, an application of estimate (36) leads to

$$\begin{aligned}\int_{(Q_3^\alpha\cap Q_3^\beta\cap C)\times(3Q\cap C)^\alpha} |u_{|_{C^\alpha}}(x) - u(z)|^p dxdz &\leq \int_{((Q_3\cap C)^\alpha)^2} |u(x) - u(z)|^p dxdz \\ &\leq c(r)\int_{(Q_k^\alpha\cap E)^2\cap D_r} |u(x) - u(z)|^p dxdz \\ &\leq c(r)\int_{(Q_{2k}^\beta\cap E)^2\cap D_r} |u(x) - u(z)|^p dxdz.\end{aligned}$$

Similarly, we also deduce that

$$\int_{(Q_3^\alpha\cap Q_3^\beta\cap C)\times(3Q\cap C)^\beta} |u_{|_{C^\beta}}(y) - u(z)|^p dydz \leq c(r)\int_{(Q_{2k}^\beta\cap E)^2\cap D_r} |u(y) - u(z)|^p dydz.$$

Finally, from (43) we get

$$|u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}|^p \leq \frac{2^p c(r)}{|3Q\cap C|^{p-1}|Q_3^\alpha\cap Q_3^\beta\cap C|}\int_{(Q_{2k}^\beta\cap E)^2\cap D_r} |u(x) - u(y)|^p dxdy. \quad (44)$$

Gathering estimates (41), (42) and (44), from (40) we conclude that

$$\int_{S_2^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dxdy \leq c_1(r)\int_{(Q_{2k}^\beta\cap E)^2\cap D_r} |u(x) - u(y)|^p dxdy,$$

where $c_1(r)$ is a positive constant depending on p, E and r. The same arguments also show that

$$\int_{S_3^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dxdy \leq c_1(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy.$$

Now consider $(x, y) \in S_4^{\alpha,\beta}$. We have that

$$\begin{aligned} u^\alpha(x) - u^\beta(y) = \varphi(x-\alpha)[u(\mathcal{R}^{-1}(x-\alpha)+\alpha) - u_{(3Q\cap C)^\alpha}] + (u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}) \\ \varphi(y-\beta)[u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Q\cap C)^\beta}]. \end{aligned}$$

In view of inequalities (42) and (44), we obtain that

$$\begin{aligned} \int_{S_4^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)| dxdx \leq\ & 3^{p-1}|2Q \cap A_t| \int_{(2Q\cap A_t)^\alpha} |\varphi(x-\alpha)|^p |u(\mathcal{R}^{-1}(x-\alpha)+\alpha) - u_{(3Q\cap C)^\alpha}|^p dx \\ & 3^{p-1}|2Q \cap A_t|^2 |u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}|^p \\ & 3^{p-1}|2Q \cap A_t| \int_{(2Q\cap A_t)^\beta} |\varphi(y-\beta)|^p |u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Q\cap C)^\beta}|^p dy \\ \leq\ & c_1(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy, \end{aligned}$$

where c_1 is a positive constant depending on p, E and r.
Now, consider $(x, y) \in S_5^{\alpha,\beta}$. Hence,

$$u^\alpha(x) - u^\beta(y) = \varphi(x-\alpha)[u(\mathcal{R}^{-1}(x-\alpha)+\alpha) - u_{(3Q\cap C)^\alpha}] + (u_{(3Q\cap C)^\alpha} - u_{(3Q\cap C)^\beta}),$$

which, thanks to (42) and (44), implies that

$$\int_{S_5^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)| dxdx \leq c(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy.$$

Similarly, if $(x, y) \in S_6^{\alpha,\beta}$, we have

$$\int_{S_6^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)| dxdx \leq c(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy.$$

If $(x, y) \in S_7^{\alpha,\beta}$, then (44) shows the desired inequality on $S_7^{\alpha,\beta}$. Finally, gathering all the previous estimate on $S_i^{\alpha,\beta}$, $i = 1, \ldots, 7$, from (39) it follows that

$$\int_{(Q_2^\beta)^2 \cap D_R} |\sum_{\alpha \in I(Q_2^\beta)} u^\alpha(y)(\psi^\alpha(x) - \psi^\alpha(y))|^p dxdy \leq c_2(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy,$$

where c_2 denotes a positive constant depending on E, p and r. In view of (37), the previous estimate combined with (38) leads us to

$$\int_{(Q_2^\beta \times Q_2^\beta)\cap D_R} |Lu(x)-Lu(y)|^p dxdy \le c_2(r) \int_{(Q_{2k}^\beta \cap E)^2 \cap D_r} |u(x)-u(y)|^p dxdy,$$

with $c_2(r)$ being a positive constant depending on p, E and r. Finally, summing up over $\beta \in I(\Omega')$ in the last inequality, we conclude the

$$\begin{aligned}\int_{(\Omega'\times\Omega')\cap D_R} |Lu(x)-Lu(y)|^p dxdy &\le \sum_{\beta\in I(\Omega')} \int_{(Q_2^\beta \times Q_2^\beta)\cap D_R} |Lu(x)-Lu(y)|^p dxdy \\ &\le c_2(r) \sum_{\beta\in I(\Omega')} \int_{(Q_{2k}^\beta \cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy \\ &\le (2k)^{2d} c_2(r) \int_{(\Omega\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy,\end{aligned}$$

where $c_2(r)$ denotes the positive constant depending on p, E and r and the factor $(2k)^{2d}$ is due to the fact that each point $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ is contained in at most $(2k)^{2d}$ cubes of the form $(Q_{2k}^\beta \times Q_{2k}^\beta)_{\beta\in\mathbb{Z}^d}$. This concludes the proof. □

The next result proves estimate (36).

Lemma 2.8. *Let C be the connected component of $kQ\cap E$, $k\ge 4$, such that $3Q\cap E\subset C$ and C has Lipschitz boundary at each point of $\partial C\cap 3\overline{Q}$. For any $r>0$ there exists a constant $c(r)>0$ such that the following inequality holds*

$$\int_{(3Q\cap C)^2} |u(x)-u(y)|^p dxdy \le c(r) \int_{(kQ\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy. \tag{45}$$

Proof. We adapt the proof of [8, Lemma 3.3].
Note that for any function u the integral on the right-hand side of (45) is an increasing function of r. Hence, it is sufficient to prove (45) for $r>0$ small enough. For fixed $r>0$, there exists $r_1\in\left(0,\frac{1}{3}r\right)$ and $\nu\in(0,1]$ which depends on the Lipschitz constant of $\partial C\cap 3\overline{Q}$ such that for any two points $\eta',\eta''\in 3Q\cap C$ there exists a discrete path from η' to η''; *i.e.*, a set of points

$$\eta_0=\eta',\eta_1,\dots,\eta_N,\eta_{N+1}=\eta''$$

such that

i) $|\eta_{j+1}-\eta_j|\le r_1$, for $j=0,1,\dots,N$;

ii) for any $j=1,\dots,N$ the ball $B_{\nu r_1}(\eta_j)=\{\eta\in\mathbb{R}^d : |\eta-\eta_j|\le \nu r_1\}$ is contained in $kQ\cap C$;

iii) there exists $\overline{N}=\overline{N}(r_1)$ such that $N\le\overline{N}$ for all $\eta',\eta''\in 3Q\cap C$.

Let $\xi_j \in B_{\nu r_1}(\eta_j)$, for $j = 1, \dots, N$. Hence, thanks to the Jensen inequality and the condition $ii)$ above, we deduce, for $\eta', \eta'' \in 3Q \cap C$,

$$\begin{aligned}
&\int_{(3Q\cap C)\cap B_{\nu r_1}(\eta')\times(3Q\cap C)\cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} \\
&= c_d(\nu r_1)^{-dN} \int_{B_{\nu r_1}(\eta_1)} \cdots \int_{B_{\nu r_1}(\eta_N)} \int_{(3Q\cap C)\cap B_{\nu r_1}(\eta')\times(3Q\cap C)\cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_1) + u(\xi_1) - \dots \\
&\qquad\qquad - u(\xi_N) + u(\xi_N) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} d\xi_N \dots d\xi_1 \\
&\le (N+1)^{p-1} c_d(\nu r_1)^{-dN} \int_{(kQ\cap E)\cap B_{\nu r_1}(\eta_0)} \cdots \int_{(kQ\cap E)\cap B_{\nu r_1}(\eta_{N+1})} \sum_{j=1}^{N+1} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_0 d\xi_{N+1} \dots d\xi_1 \\
&= c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kQ\cap E)\cap B_{\nu r_1}(\eta_j)\times(kQ\cap E)\cap B_{\nu r_1}(\eta_{j-1})} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_j d\xi_{j-1}. \qquad (46)
\end{aligned}$$

In view of assumption (i), for $\xi_{j-1} \in (kQ \cap E) \cap B_{\nu r_1}(\eta_{j-1})$ and $\xi_j \in (kQ \cap E) \cap B_{\nu r_1}(\eta_j)$, we have

$$|\xi_j - \xi_{j-1}| \le |\xi_j - \eta_j| + |\eta_j - \eta_{j-1}| + |\eta_{j-1} - \xi_{j-1}| \le 2\nu r_1 + r_1 \le r,$$

which implies that $(kQ\cap E)\cap B_{\nu r_1}(\eta_j)\times(kQ\cap E)\cap B_{\nu r_1}(\eta_{j-1})$ is contained in $(kQ\cap E)^2\cap D_r$. In view of (46) and due to item (iii), we get

$$\begin{aligned}
c(N+1)^{p-1} &\sum_{j=1}^{N+1} \int_{(kQ\cap E)\cap B_{\nu r_1}(\eta_j)\times(kQ\cap E)\cap B_{\nu r_1}(\eta_{j-1})} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_j d\xi_{j-1} \\
&\le c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kQ\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta \\
&\le c(N+1)^p \int_{(kQ\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta \\
&\le c(\overline{N}+1)^p \int_{(kQ\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta.
\end{aligned}$$

This implies that

$$\begin{aligned}
\int_{(3Q\cap C)\cap B_{\nu r_1}(\eta')\times(3Q\cap C)\cap B_{\nu r_1}(\eta'')} &|u(\xi_0) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} \\
&\le c(\overline{N}+1)^p \int_{(kQ\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta.
\end{aligned}$$

Covering $3Q \cap C$ with a finite number of balls of radius νr_1 and summing up the last inequality over all pairs of these balls gives the desired estimate (28). □

Now, we may prove Theorem 2.2.

Proof of Theorem 2.2. The proof follows the lines of that of Theorem 2.1 in [1].
Fix $\varepsilon > 0$ and set $k_0 = 2\tilde{C}$. First, let us show that there exist $R = R(E) > 0$, independent of ε, and a linear and continuous extension operator $L_\varepsilon : L^p(\Omega \cap \varepsilon E) \to L^p(\Omega(\varepsilon k_0/2))$ such that, for all $r > 0$ and for any $u \in L^p(\Omega \cap \varepsilon E)$,

$$L_\varepsilon(u) = u \quad \text{a.e. in } \Omega(\varepsilon k_0/2) \cap \varepsilon E, \tag{47}$$

$$\int_{\Omega(\varepsilon k_0/2)} |L_\varepsilon(u)|^p dx \leq c_1 \int_{\Omega \cap \varepsilon E} |u|^p dx, \tag{48}$$

$$\int_{(\Omega(\varepsilon k_0/2))^2 \cap D_{\varepsilon R}} |L_\varepsilon(u)(x) - L_\varepsilon(u)(y)|^p dxdy \leq c_2(r) \int_{(\Omega \cap \varepsilon E)^2 \cap D_{\varepsilon r}} |u(x) - u(y)|^p dxdy. \tag{49}$$

To this end, note that for every $u \in L^p(\Omega \cap \varepsilon E)$, we have $u \circ \pi_\varepsilon \in L^p(\varepsilon^{-1}\Omega \cap E)$, where we use the notation (34) for the map π_ε. Moreover, $\mathrm{dist}(\varepsilon^{-1}\Omega(\varepsilon k_0/2), \partial(\varepsilon^{-1}\Omega)) > k_0 = 2\tilde{C}$. Hence, we can apply Lemma 2.7, so that there exist $R = R(E) > 0$, independent of ε, and a linear and continuous operator $L : L^p(\varepsilon^{-1}\Omega \cap E) \to L^p(\varepsilon^{-1}\Omega(\varepsilon k_0/2))$ such that, for all $r > 0$ and for all $u \in L^p(\varepsilon^{-1}\Omega \cap E)$,

$$L(u) = u, \quad \text{a.e. in } \varepsilon^{-1}\Omega(\varepsilon k_0/2) \cap E,$$

$$\int_{\varepsilon^{-1}\Omega(\varepsilon k_0/2)} |L(u)|^p dx \leq c_1 \int_{\varepsilon^{-1}\Omega \cap E} |u|^p dx,$$

$$\int_{(\varepsilon^{-1}\Omega(\varepsilon k_0/2))^2 \cap D_R} |L(u)(x) - L(u)(y)|^p dxdy \leq c_2(r) \int_{(\varepsilon^{-1}\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy,$$

where the constants c_1 and c_2 are given by Lemma (2.7) and they are, in particular, independent of ε. Hence, we set $L_\varepsilon u = (L(u \circ \pi_\varepsilon)) \circ \pi_{1/\varepsilon}$. Note that $L_\varepsilon u \in L^p(\Omega(\varepsilon k_0/2))$ and (47), (48), (49) are satisfied.
Now, we define the extension operator $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \to L^p(\Omega)$ by $T_\varepsilon(u) := L_\varepsilon(u)$ a.e. in $\Omega(\varepsilon k_0)$ and extended by zero out of $\Omega(\varepsilon k_0)$. Hence, we have that $T_\varepsilon(u) \in L^p(\Omega)$ and (8), (9) and (10) follow directly from (47), (48) and (49) and this concludes the proof. □

2.2 Compactness

In this section we prove a compactness result which in particular implies the equi-coerciveness of families of non-local functionals as those in the homogenization result in the next section. The proof is based on the extension Theorem 2.2 and on the following compactness result proved in [9] for the case $p = 2$ and in [4] for general $p > 1$.

Theorem 2.9. *Let Ω be an open set with Lipschitz boundary, and assume that for a family $\{w_\varepsilon\}_{\varepsilon>0}$, $w_\varepsilon \in L^p(\Omega)$, the estimate*

$$\int_{\Omega(\varepsilon k)} \int_{D_R} \left| \frac{w_\varepsilon(x+\xi) - w_\varepsilon(x)}{\varepsilon} \right|^p d\xi\, dx \leq c \tag{50}$$

is satisfied with some $k > 0$ and $R > 0$. Assume moreover that the family $\{w_\varepsilon\}$ is bounded in $L^p(\Omega)$. Then for any sequence $\varepsilon_j \to 0$ as $j \to +\infty$, and for any open subset $\Omega' \subset\subset \Omega$ the set $\{w_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is relatively compact in $L^p(\Omega')$ and every its limit point is in $W^{1,p}(\Omega)$.

Corollary 2.10. *Let u_ε be a family of functions in $L^p(\Omega \cap \varepsilon E)$ such that there exists $c > 0$ and $r > 0$ such that $||u_\varepsilon||_{L^p(\Omega\cap\varepsilon E)} \leq c$ and*

$$\int_{\{|\xi|\leq r\}} \int_{(\Omega\cap\varepsilon E)_\varepsilon(\xi)} \left| \frac{u_\varepsilon(x+\varepsilon\xi) - u_\varepsilon(x)}{\varepsilon} \right|^p dx\, d\xi \leq c, \tag{51}$$

for all $\varepsilon > 0$, with $(\Omega\cap\varepsilon E)_\varepsilon(\xi) = \{x \in \Omega\cap\varepsilon E : x+\varepsilon\xi \in \Omega\cap\varepsilon E\}$. Then, for any sequence $\varepsilon_j \to 0$ as $j \to +\infty$, and for any open subset $\Omega' \subset\subset \Omega$ the set $\{T_{\varepsilon_j} u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is relatively compact in $L^p(\Omega')$ and every its limit point is in $W^{1,p}(\Omega)$.

Proof. Let u_ε be such that $||u_\varepsilon||_{L^p(\Omega\cap\varepsilon E)} \leq c$ and (51) hold for every $\varepsilon > 0$. From Theorem 2.2, the extended functions $T_\varepsilon u_\varepsilon$ satisfy the estimates

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon u_\varepsilon|^p\, dx \leq c \tag{52}$$

and

$$\begin{aligned} &\frac{1}{\varepsilon^{d+p}} \int_{(\Omega(\varepsilon k_0))^2\cap D_{\varepsilon R}} |T_\varepsilon u_\varepsilon(y) - T_\varepsilon u_\varepsilon(x)|^p\, dy\, dx \\ &\qquad \leq c(r) \int_{|\xi|\leq r} \int_{(\Omega\cap E)_\varepsilon(\xi)} \left| \frac{u_\varepsilon(x+\varepsilon\xi) - u_\varepsilon(x)}{\varepsilon} \right|^p dx\, d\xi \leq c\,, \end{aligned}$$

for some $R > 0$ independent of ε. The latter, after the change of variables $y = x + \varepsilon\xi$, is equivalent to

$$\int_{\Omega(\varepsilon k_0)} \int_{|\xi|\leq R} \left| \frac{T_\varepsilon u_\varepsilon(x+\varepsilon\xi) - T_\varepsilon u_\varepsilon(x)}{\varepsilon} \right|^p d\xi\, dx \leq c, \tag{53}$$

which corresponds to (50), for $w_\varepsilon = T_\varepsilon u_\varepsilon$. Using Theorem 2.9 for $w_\varepsilon = T_\varepsilon u_\varepsilon$ and (52), (53), we can conclude that for any sequence $\varepsilon_j \to 0$ as $j \to +\infty$, and for any open subset $\Omega' \subset\subset \Omega$, $T_{\varepsilon_j} u_{\varepsilon_j}$ is relatively compact in $L^p(\Omega')$ and every its limit point is in $W^{1,p}(\Omega)$. □

Remark 2.11. The limit u in the previous corollary does not depend on the choice of the extension. In fact, if $\tilde{v}_\varepsilon$ is another extension of u_ε and v is its limit, then for any $\Omega'' \subset\subset \Omega' \subset\subset \Omega$

$$\int_{\Omega''\cap\varepsilon E} |u - v|^p\, dx \leq c \int_{\Omega'} |u - \tilde{u}_\varepsilon|^p\, dx + c \int_{\Omega'} |\tilde{v}_\varepsilon - v|^p\, dx$$

Passing to the limit as $\varepsilon \to 0$, one gets

$$|(0,1)^d \cap E| \int_{\Omega''} |u - v|^p \, dx \leq 0$$

and concludes that $u = v$, by the arbitrariness of Ω''.

3 An application to homogenization

In this section we present an application of the Extension Theorem 2.2 to the homogenization of non-local functional. Specifically, we consider a periodic integrand $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty)$; *i.e.*, a Borel function such that $h(\cdot, \xi, z)$ is $[0,1]^d$-periodic for all $\xi \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$ and satisfies the following growth conditions: there exist positive constants c_0, c_1, r_0 and non-negative function $\psi : \mathbb{R}^d \to [0, +\infty)$ such that

$$h(x, \xi, z) \leq \psi(\xi)(|z|^p + 1) \tag{54}$$

$$h(x, \xi, z) \geq c_0(|z|^p - 1) \quad \forall |\xi| \leq r_0 \tag{55}$$

with

$$\int_{\mathbb{R}^d} \psi(\xi)(|\xi|^p + 1) \, d\xi \leq c_1. \tag{56}$$

Let $\Omega \subset \mathbb{R}^d$ be an open set with Lipschitz boundary. For any $\varepsilon > 0$, we introduce the non-local functional $H_\varepsilon : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined as

$$H_\varepsilon(u) = \int_{\mathbb{R}^d} \int_{(\Omega \cap \varepsilon E)_\varepsilon(\xi)} h\left(\frac{x}{\varepsilon}, \xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\right) dx \, d\xi, \tag{57}$$

where for each set B, $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, we use the notation

$$B_\varepsilon(\xi) = \{x \in B : x + \varepsilon\xi \in B\} \tag{58}$$

Note that the integration in (57) is performed for x, ξ such that both x and $x + \varepsilon\xi$ belong to the perforated domain $\Omega \cap \varepsilon E$. Conditions (54)–(56) guarantee that functionals H_ε are estimated from above and below by functionals of the type (4).

Thanks to Corollary 2.10, our functionals H_ε are equi-coercive with respect to the $L^p_{\mathrm{loc}}(\Omega)$-convergence upon identifying functions with their extensions from the perforated domain. More precisely, from each sequence $\{u_\varepsilon\}$ with equi-bounded energy $H_\varepsilon(u_\varepsilon)$ we can extract a subsequence such that the corresponding extensions converge in L^p_{loc} to some limit $u \in W^{1,p}(\Omega)$. This is implied by Corollary 2.10 applied with $r = r_0$ to each component of the vector-valued functions u_ε, upon noting that (55) implies (51).
We now may state the homogenization result for the functional H_ε with respect to the $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ convergence.

Theorem 3.1. *The functionals H_ε defined by* (57) Γ*-converge with respect to $L^p_{\rm loc}(\Omega;\mathbb{R}^m)$-convergence to the functional*

$$H_{\rm hom}(u) = \begin{cases} \displaystyle\int_\Omega h_{\rm hom}(Du(x))\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases} \tag{59}$$

with $h_{\rm hom}$ satisfying the asymptotic formula

$$\begin{aligned} h_{\rm hom}(\Xi) &= \lim_{T\to+\infty} \frac{1}{T^d} \inf\Big\{ \int_{(0,T)^d\cap E} \int_{(0,T)^d\cap E} h(x, y-x, v(y)-v(x))\,dx\,dy : \\ &\qquad v(x) = \Xi x \text{ if } \operatorname{dist}(x, \partial(0,T)^d) < k_0 \Big\} \end{aligned} \tag{60}$$

for all $\Xi \in \mathbf{M}^{m\times d}$. Furthermore, if h is convex in the third variable, the cell-problem formula

$$h_{\rm hom}(\Xi) = \inf\Big\{ \int_{(0,1)^d\cap E} \int_E h(x, y-x, v(y)-v(x))\,dx\,dy : v(x) - \Xi x \ \textit{is 1-periodic} \Big\} \tag{61}$$

holds.

Proof. In [4] this theorem is proved when $E = \mathbb{R}^d$. We will prove Theorem 3.1 reducing to that case by a perturbation argument. For every $\delta \geq 0$ we set

$$h^\delta(x, \xi, z) = \chi_E(x)\chi_E(x+\xi)\, h(x, \xi, z) + \delta \chi_{B_{R_0}}(\xi)|z|^p,$$

where $R_0 > 0$ is fixed but arbitrary, and

$$H^\delta_\varepsilon(u) = \int_{\mathbb{R}^d} \int_{\Omega_\varepsilon(\xi)} h^\delta\left(\frac{x}{\varepsilon}, \xi, \frac{u(x+\varepsilon\xi) - u(x)}{\varepsilon}\right) dx\,d\xi$$

is defined for $u \in L^p(\Omega;\mathbb{R}^m)$, where we use the notation in (58) for the set $\Omega_\varepsilon(\xi)$. Note that $H^\delta_\varepsilon \geq H_\varepsilon$, and for $\delta = 0$ we have $H^0_\varepsilon = H_\varepsilon$. In the following, for any open set A and $\delta \geq 0$, we also consider the 'localized' functionals

$$H^\delta_\varepsilon(v, A) = \int_{\mathbb{R}^d} \int_{A_\varepsilon(\xi)} h\left(\frac{x}{\varepsilon}, \xi, \frac{u(x+\varepsilon\xi) - u(x)}{\varepsilon}\right) dx\,d\xi,$$

where we use the notation in (58) for the set $A_\varepsilon(\xi)$. If $\delta = 0$ we write $H_\varepsilon(v, A)$ in the place of $H^0_\varepsilon(v, A)$.

The homogenization theorem in [4] ensures that for all $\delta > 0$ there exists the Γ-limit

$$H^\delta_{\rm hom}(u) = \Gamma\text{-}\lim_{\varepsilon\to 0} H^\delta_\varepsilon(u)$$

with domain $W^{1,p}(\Omega;\mathbb{R}^m)$, on which it is represented as

$$H^\delta_{\rm hom}(u)=\int_\Omega h^\delta_{\rm hom}(Du)\,dx.$$

The energy density $h^\delta_{\rm hom}$ satisfies

$$\begin{aligned} h^\delta_{\rm hom}(\Xi) &= \lim_{T\to+\infty}\frac{1}{T^d}\inf\Big\{\int_{(0,T)^d}\int_{(0,T)^d} h^\delta(x,y-x,v(y)-v(x))\,dx\,dy : \\ &\qquad v(x)=\Xi x \text{ if } \mathrm{dist}(x,\partial(0,T)^d)<r\Big\}, \end{aligned}$$

for any fixed $r>0$, and

$$c_1(|\Xi|^p-1)\le h^\delta_{\rm hom}(\Xi)\le c_2(1+|\Xi|^p)$$

with c_1, c_2 independent of δ, for $\delta\in[0,1]$. Note that the independence of c_1 from δ is an immediate consequence of the Extension Theorem. Indeed, let $u^\delta_\varepsilon\to\Xi x$ be such that

$$h^\delta_{\rm hom}(\Xi)=\lim_{\varepsilon\to0}H^\delta_\varepsilon(u^\delta_\varepsilon,(0,1)^d).$$

Applying Corollary 2.10 with $\Omega=(0,1)^d$, we deduce that $T_\varepsilon u^\delta_\varepsilon$ converge to Ξx locally in $(0,1)^d$ (in particular the convergence is strong *e.g.* in $(\frac14,\frac34)^d$). Hence, using (55), the Extension Theorem, and the liminf inequality of the Γ-limit (see *e.g.* [7]) we have

$$\begin{aligned} \lim_{\varepsilon\to0}H^\delta_\varepsilon(u^\delta_\varepsilon,(0,1)^d) &\ge \liminf_{\varepsilon\to0}H_\varepsilon(u^\delta_\varepsilon,(0,1)^d)\\ &\ge c_0\liminf_{\varepsilon\to0}\Big(\frac{1}{\varepsilon^{p+d}}\int_{((0,1)^d\cap\varepsilon E)^2\cap D_{r_0}}|u^\delta_\varepsilon(x)-u^\delta_\varepsilon(y)|^p dxdy-1\Big)\\ &\ge \frac{c_0}{c_2(r_0)}\liminf_{\varepsilon\to0}\Big(\frac{1}{\varepsilon^{p+d}}\int_{((\frac14,\frac34)^d)^2\cap D_R}|T_\varepsilon u^\delta_\varepsilon(x)-T_\varepsilon u^\delta_\varepsilon(y)|^p dxdy-1\Big)\\ &\ge \frac{c_0}{c_2(r_0)}\min\Big\{\frac{1}{2^d}c_R,1\Big\}(|\Xi|^p-1), \end{aligned}$$

where in the last inequality we have used that

$$\Gamma\text{-}\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{p+d}}\int_{((\frac14,\frac34)^d)^2\cap D_R}|v(x)-v(y)|^p dxdy=c_R\int_{(\frac14,\frac34)^d}|\nabla v|^p dx,$$

where $c_R=\int_{\{|\xi|\le R\}}|\xi_1|^p d\xi$ (see [4]).

Since $h^\delta_{\rm hom}$ is increasing with δ, we may define

$$h_0(\Xi)=\inf_{\delta>0}h^\delta_{\rm hom}(\Xi)=\lim_{\delta\to0^+}h^\delta_{\rm hom}(\Xi),$$

and deduce (here we use the usual notation for the upper Γ-limit) that

$$\int_\Omega h_0(Du)\,dx \geq \Gamma\text{-}\limsup_{\varepsilon\to 0} H_\varepsilon(u)\,. \tag{62}$$

If $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $u_\varepsilon \to u$ with $\sup_\varepsilon H_\varepsilon(u_\varepsilon) < +\infty$ then for all fixed Ω' compactly contained in Ω, if $R_0 < R$, upon identifying u_ε with its extension given by the Extension Theorem, we obtain that,

$$\int_{\{|\xi|\leq R_0\}}\int_{(\Omega')_\varepsilon(\xi)}\left|\frac{u_\varepsilon(x+\varepsilon\xi)-u_\varepsilon(x)}{\varepsilon}\right|^p dx\,d\xi \leq c,$$

so that

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon,\Omega') \geq \liminf_{\varepsilon\to 0} H^\delta_\varepsilon(u_\varepsilon,\Omega') - \delta c.$$

From this inequality we obtain (in terms of the lower Γ-limit)

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} H_\varepsilon(u) \geq \int_\Omega h_0(Du)dx$$

by the arbitrariness of δ and $\Omega' \subset\subset \Omega$. Hence, recalling (62), we have proved that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} H_\varepsilon(u) = \int_\Omega h_0(Du)dx,$$

and in particular that the Γ-limit exists as $\varepsilon \to 0$ (no subsequence is involved) and it can be represented as an integral functional with a homogeneous integrand. Note moreover that the lower-semicontinuity of the Γ-limit implies that h_0 is quasiconvex (see [7]).

We now prove that h_0 coincides with h_{hom} given by the asymptotic formula. First, note that

$$\begin{aligned} h_0(\Xi) \quad\geq\quad & \limsup_{T\to+\infty}\frac{1}{T^d}\inf\Big\{\int_{(0,T)^d\cap E}\int_{(0,T)^d\cap E} h(x,y-x,v(y)-v(x))\,dx\,dy : \\ & \qquad v(x) = \Xi x \text{ if } \mathrm{dist}(x,\partial(0,T)^d) < r\Big\}\,. \end{aligned} \tag{63}$$

If we take $r = k_0$, we obtain a lower bound for h_0.

To prove the opposite inequality, for any diverging sequence $\{T_j\}$ we can consider (almost-)minimizers v_j of the problems in (63) with $r = k_0$ and $T = T_j$. By Lemma 2.7 (applied componentwise) with $\Omega = (0,T)^d$ and $\Omega' = (\frac{k_0}{2}, T_j - \frac{k_0}{2})^d$, recalling that $k_0 = 2\widetilde{C}$, we can consider $\widetilde{v}_j = L(v_j) \in L^p((\frac{k_0}{2}, T_j - \frac{k_0}{2})^d;\mathbb{R}^m)$ with $\widetilde{v}_j = v_j$ on $\Omega = (0,T)^d \cap E$ and

$$\begin{aligned} &\int_{(\frac{k_0}{2},T_j-\frac{k_0}{2})^d\cap D_R} |\widetilde{v}_j(\xi)-\widetilde{v}_j(\eta)|^p d\xi d\eta \\ &\qquad\leq c_2(r_0)\int_{(0,T_j)^d\cap E)^2\cap D_{r_0}} |v_j(\xi)-v_j(\eta)|^p d\xi d\eta \leq c\,T_j^d(1+|\Xi|^p) \end{aligned}$$

for some $c > 0$ independent of j. Upon choosing a larger $k_0 > 2$ we may suppose that $\lfloor \frac{k_0}{2} \rfloor + 1 < k_0$ so that we may consider $w_j \in L^p((0,T_j - n)^d; \mathbb{R}^m)$, where $n = 2\lfloor \frac{k_0}{2} \rfloor + 2$, defined by

$$w_j(x) = L(v_j)\Big(x + \Big(\lfloor \frac{k_0}{2} \rfloor + 1\Big)(1,\dots,1)\Big) - \Big(\lfloor \frac{k_0}{2} \rfloor + 1\Big)\Xi(1,\dots,1).$$

Having set $\varepsilon_j = T_j - n$ we can consider the scaled functions

$$u_j(x) = \varepsilon_j w_j\Big(\frac{x}{\varepsilon_j}\Big).$$

By the boundedness of the energies above and noting that there exists $c > 0$ such that $w_j(x) = \Xi x$ if $x \in E$ and $\operatorname{dist}(x, \partial(0,T_j - n)^d) < c$, upon extracting a subsequence, we may suppose that $u_j \to u$ and $u \in \Xi x + W_0^{1,p}((0,1)^d; \mathbb{R}^m)$. We may then use the quasiconvexity inequality for h_0 to obtain

$$\begin{aligned}
h_0(\Xi) &\le \int_{(0,1)^d} h_0(Du)dx \\
&\le \liminf_j H^\delta_{\varepsilon_j}(u_j,(0,1)^d) \\
&\le \liminf_j H_{\varepsilon_j}(u_j,(0,1)^d) + c\delta \\
&\le \liminf_j \frac{1}{(T_j - n)^d} H_1(w_j,(0,T_j - n)^d) + c\delta \\
&\le \liminf_j \frac{1}{(T_j - n)^d} H_1(v_j,(0,T_j)^d) + c\delta \\
&= \liminf_j \frac{1}{(T_j - n)^d} \inf\Big\{\int_{(0,T_j)^d\cap E}\int_{(0,T_j)^d\cap E} h(x, y-x, v(y)-v(x))\,dx\,dy : \\
&\qquad v(x) = \Xi x \text{ if } \operatorname{dist}(x,\partial(0,T_j)^d) < k_0\Big\} + c\delta \\
&= \liminf_j \frac{1}{T_j^d} \inf\Big\{\int_{(0,T_j)^d\cap E}\int_{(0,T_j)^d\cap E} h(x, y-x, v(y)-v(x))\,dx\,dy : \\
&\qquad v(x) = \Xi x \text{ if } \operatorname{dist}(x,\partial(0,T_j)^d) < k_0\Big\} + c\delta.
\end{aligned}$$

By the arbitrariness of δ and of the sequence T_j we obtain the desired upper bound for h_0, which, together with (63), proves the asymptotic formula.

In the convex case, again by the homogenization results in [4], we may repeat the arguments used to get (63) to obtain the lower bound for h_0

$$h_0(\Xi) \ge \inf\Big\{\int_{(0,1)^d\cap E}\int_E h(x, y-x, v(y)-v(x))\,dx\,dy : v(x) - \Xi x \text{ is 1-periodic}\Big\}. \quad (64)$$

Note that this implies that the right-hand side is bounded from above by $c_2(1 + |\Xi|^p)$.

Now, let v be an (almost) minimizing function for (64), and set $v_\varepsilon(x) = \varepsilon v(\frac{x}{\varepsilon})$. After applying Theorem 2.2 to any set Ω compactly containing $(0,1)^d$ to possibly redefine v_ε outside εE, we can suppose that v_ε converge in $L^p((0,1)^d;\mathbb{R}^m)$ to Ξx and that

$$\frac{1}{\varepsilon^{p+d}} \int_{((0,1)^d\times(0,1)^d)\cap D_{\varepsilon R_0}} |v_\varepsilon(x) - v_\varepsilon(y)|^p \; dx\, dy \le c(1+|\Xi|^p).$$

We then estimate

$$\begin{aligned} h^\delta_{\rm hom}(\Xi) &\le \liminf_{\varepsilon\to 0} H^\delta_\varepsilon(v_\varepsilon) \\ &\le \int_{(0,1)^d\cap E}\int_E h(x, y-x, v(y)-v(x))\, dx\, dy + c\delta(1+|\Xi|^p). \end{aligned}$$

Taking the limit as $\delta \to 0$, we obtain the converse inequality of (64), and conclude the proof. $\square$

Remark 3.2. The function $h_{\rm hom}$ obtained in the asymptotic formula (60) also satisfies

$$h_{\rm hom}(\Xi) = \lim_{T\to+\infty} \frac{1}{T^d} \inf\Big\{ \int_{(0,T)^d\cap E}\int_{(0,T)^d\cap E} h(x, y-x, v(y)-v(x))\, dx\, dy : \\ v(x) - \Xi x \,\text{is}\, (0,T)^d - \text{periodic}\Big\}.$$

Remark 3.3. An example is given by the convolution functional

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+p}} \int_{(\Omega\cap E_\varepsilon)\times(\Omega\cap E_\varepsilon)} a\left(\frac{y-x}{\varepsilon}\right) |u(x)-u(y)|^p\, dy\, dx.$$

Since the integrand function $h(x,\xi,z) = a(\xi)|z|^p$ is convex in z, then Theorem 3.1 and (61) ensure that the integrand of the Γ-limit (59) of F_ε is given by

$$\inf\left\{ \int_{(0,1)^d\cap E}\int_{E-\{x\}} a(\xi)|v(x+\xi)-v(x)|^p\, d\xi\, dx : v(x) - \Xi x \text{ is } 1\text{−periodic} \right\}.$$

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