# Lyapunov Central Limit Theorem: Theoretical Properties and Applications in Big-Data-Populated Smart City Settings

# Lyapunov Central Limit Theorem: Theoretical Properties and Applications in Big-Data-Populated Smart City Settings

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## ABSTRACT

Central Limit Theorems have a fundamental role in statistics and in a wide range of practical applications. The most famous formulation was proposed by Lindeberg–Lévy and it requires the variables to be independent and identically distributed. In the real setting these conditions are rarely matched, though. The Lyapunov Central Limit Theorem overcomes this limitation, since it does not require the same distribution of the random variables. However, the cost of this generalization is an increased complexity, moderately limiting its effective applicability. In this paper, we resume the main results on the Lyapunov Central Limit Theorem, providing an easy-to-prove condition to put in practice, and demonstrating its uniform convergence. These theoretical results are supported by some relevant applications in the field of big data in smart city settings.

## CCS CONCEPTS

• **Theory of computation** → Design and analysis of algorithms; • **Information systems** → Information systems applications; Mobile information processing systems.

## KEYWORDS

Lyapunov Central Limit Theorem, Uniform Convergence Condition, Smart City

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## 1 INTRODUCTION

Central limit theorems (CLTs) cover a central role in statistics and still attract interest (see [1, 2]). The most used version is the Lindeberg–Lévy theorem, which asserts that a sequence of independent and identically distributed random variables $\{X_1, X_2, \ldots, X_n\}$, under the conditions of $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}ar[X_i] = \sigma^2 < \infty$ then, $\sqrt{n}\sum_{i=1}^{n}(\overline{X_i} - \mu)$ converges to a normal distribution $N(0, \sigma^2)$, where $\overline{X_i} = \sum_{i=1}^{n} X_i/n$ .

For this theorem, the convergence of the empirical cumulative distribution function (cdf) to the standard normal cdf (reported as $\Phi(x)$) is uniform. Thus, the following statement holds:

$$\lim_{n\to\infty} sup_{z\in\mathbb{R}} \left| \Pr\left[\sqrt{n}\left(\sum_{i=1}^{n}\overline{X_i} - \mu\right) \le z\right] - \phi\left(\frac{z}{\sigma}\right)\right| = 0 \qquad (1)$$

It is worth noting that the Lindeberg–Lévy CLT is valid for both discrete, continuous, and mixed random variables.

In the literature, there are several central limit theorems that generalized the Lindeberg–Lévy theorem for non-identically distributed variables and for non-independent ones. Nevertheless, they are not widely known and thus less applied. In particular, the Lyapunov Central Limit Theorem does not require the identical distribution of the random variables but it requires to check the difficult Lyapunov's condition. Therefore, it is mainly used in theory and not in real applications. In this paper, we fill the gap by summarizing the main results about the Lyapunov counterpart, providing an easy-to-prove condition for its applicability and by presenting some real world applications.

The paper is organized as follows. In Section 2, we report the theoretical development by providing an easy-to-prove sufficient assertion for the Lyapunov's condition. Furthermore, we prove that if such a sufficient condition holds, then the convergence of the Lyapunov's theorem is uniform. In Section 3, we apply the analysis in a real setting, by also including a preliminary experimental evaluation that focuses on this setting. Finally, in Section 4 we present the conclusion of the current work and possible future extensions.

## 2 THE LYAPUNOV CENTRAL LIMIT THEOREM: EXTENSIONS TO UNIFORM CONVERGENCE CONDITION

In this Section, we present the main results about the Lyapunov CLT [3]. In order to clarify all the definitions, we recall the following statement.

Definition 1. *$F_n \to F$ means that for each $x$, $F_n(x) \to F(x)$. In other words*

$$\forall x,\ \epsilon > 0\ \exists N \text{ such that } |F_n(x) - F(x)| \langle \epsilon\ \forall n \rangle N \qquad (2)$$

For the sake of clarity, for a given $\epsilon$, a value of $N$, making statement in Eq. (2) true for some $x$, might not work for some other $x$. However, the idea of uniform convergence implies that we can choose $N$ without any regard to the value of $x$. Thus, the concept of uniform convergence is needed.

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Definition 2. *$F_n(x)$ converges uniformly to $F(x)$ if for every $\epsilon > 0$ , there exists $N$ such that $|F_n(x) - F(x)| < \epsilon$ for all $n > N$ and for all $x$.*

It is clear that, in general, point-wise convergence does not imply uniform convergence. However, the following theorem gives a special case in which it does.

Theorem 1. *If $F_n(x)$ and $F(x)$ are cdf's and $F(x)$ is continuous, then the pointwise convergence of $F_n$ to $F$ implies uniform convergence of $F_n$ to $F$.*

In order to introduce the Lyapunov CLT result we recall the following definition.

Definition 3. ***Lyapunov's Condition.*** *If a sequence of independent random variables $X_k$ $k = 1, \ldots, n$ is such that $E[X_k] = \mu_k < \infty$, $E[(X_k - \mu_k)^2] = \sigma_k^2 < \infty$, then for some $\delta > 0$:*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{n} \mathbb{E}\left[|X_k - \mu_k|^{2+\delta}\right] = 0 \tag{3}$$

*where $s_n^2 = \sum\limits_{k=1}^{n} \sigma_k^2 > 0$ the Lyapunov's condition holds for $\{X_k,\ k = 1,\ \ldots,\ n\ \}$.*

By using the Lyapunov condition it is possible to state the following known results.

Theorem 2. ***Lyapunov's Central Limit Theorem.*** *If the Lyapunov's condition holds, then for $n \to \infty$,*

$$\frac{1}{s_n} \sum_{k=1}^{n} X_k - \mu_k \xrightarrow{d} \mathcal{N}(0,1) \tag{4}$$

*where $s_n = \sqrt{\sum\limits_{k=1}^{n} \sigma_k^2}$ and $\xrightarrow{d}$ indicates the convergence in distribution.*

The proof of Theorem 2 is here omitted since it is out of the scope of the paper, however the interested reader is referred to [3].

Condition in Eq. (3) is, in general, difficult to prove, hence we recall an easier sufficient condition that holds for several distributions.

Theorem 3. *Given a sequence of independent random variables $\{X_k,\ k = 1,\ \ldots,\ n\ \}$ such that $\mathbb{E}[(X_k - \mu_k)^2] = \sigma_k^2 \geq \sigma_k^2 > 0\ \forall\ k$ holds and the centered 3-rd moments $\mathbb{E}[|X_k - \mu_k|^3] = \eta_k \leq \psi < \infty$, then the Lyapunov condition holds*

***Proof.*** *By considering the limit in (3) for $\delta = 1$ we have*

$$0 \leq \lim_{n\to\infty} \frac{1}{s_n^3} \sum_{k=1}^{n} \mathbb{E}\left[|X_k - \mu_k|^3\ \right] = \lim_{n\to\infty} \frac{\sum_k^n \eta_k^3}{\left[\sum_k^n \sigma_k^2\right]^2} \leq \lim_{n\to\infty} \frac{n\eta^*}{n^2\sigma_*^4} \to 0 \tag{5}$$

*where $\eta^* = \max\limits_k \eta_k^3$ and $\sigma_*^2 = \min\limits_k \sigma_k^2$ . Then, the Lyapunov condition holds.*

In order to study the convergence of Theorem (2) we recall another useful result, namely the Esseen inequality (see [4]).

Theorem 4. ***Esseen inequality.*** *Let $\{X_k,\ k = 1,\ \ldots,\ n\ \}$ be independent random variables with $\mathbb{E}[X_k] = 0$, $\mathbb{V}ar[X_k] = \sigma_k^2 > 0$ and $\mathbb{E}[|X_k|^3] = \eta_k < \infty$ . Also, let*

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}} \tag{6}$$

*Denote $F_n$ the cdf of $S_n$, and $\Phi$ the cdf of the standard normal distribution, then for all $n$ there exists an absolute constant $C_0$ such that*

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \leq \frac{C_0 \sum_{k=1}^{n} v_k}{\sqrt{\left(\sum_{k=1}^{n} \sigma_k^2\right)^3}} \tag{7}$$

***Proof.*** *see [5] and [6].*

By using Theorem (5), it is possible to prove the following result.

Theorem 5. *If a set of random variables $\{X_k\}_{k=1}^n$ with means $\mu_k$, variances $\sigma_k^2$ and 3-rd moments $\eta_k$ is such that $\sigma_k^2 \geq \sigma^2 > 0,\ \ \forall k$ and $\eta_k \leq \eta < +\infty,\ \forall k$ then the cdf of*

$$\frac{Z_1 + Z_2 + \cdots Z_n}{\sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \ \sigma_2^n}} \tag{8}$$

*where $Z_k = X_k - \mu_k$, converges uniformly to the standard normal cdf.*

***Proof.*** *The convergence to the standard normal distribution is ensured by verifying the hypothesis of Theorem (3); furthermore, the uniform convergence can be checked by using the Esseen inequality to the variables $Z_k$. In fact $\mathbb{E}[Z_k] = \mathbb{E}[X_k - \mu_k] = 0$, $Var[Z_k] = \mathbb{V}ar[X_k - \mu_k] = \sigma_k^2$ and finally*

$$\begin{aligned}\mathbb{E}\left[|Z_k|^3\right] \quad &= \quad \mathbb{E}\left[|X_k - \mu_k|^3\right] \leq \mathbb{E}\left[|X_k|^3\right] \\ &\quad + 3\mathbb{E}\left[|X_k|^2\right]\mu_k + 3\mathbb{E}\left[|X_k|\right]\mu_k^2 + \mu_k^3 < \infty\end{aligned}$$

Remark 1. *It is worth noting that the rate of uniform convergence is $\frac{1}{\sqrt{n}}$. In fact,*

$$\sup_{x\in\mathbb{R}} \left|F_{n(x)} - \Phi(x)\right| \leq \frac{C_0 \sum_{k=1}^{n} v_k}{\sqrt{\left(\sum_{k=1}^{n} \sigma_k^2\right)^3}} \leq \frac{C_0 \max_k v_k}{\sqrt{\min_k \sigma_k^3}} \frac{1}{\sqrt{N}} \tag{9}$$

*This rate does not depend by the nature of distributions. Thus, it also holds for discrete and mixed distributions.*

The concept of uniform convergence is important for several applications. In particular, from 1, we know that the convergence to the limiting distribution is $\frac{1}{\sqrt{n}}$. Hence, for a $N$ sufficiently high the simulation of a real case scenario can be done through the asymptotic case, leading to an estimation of the error.

This is particular important in contexts such as stochastic programming (see [7]), where small errors in the distribution can cause huge economical losses (see [8]).

# 3 APPLYING THEORETICAL RESUTLS TO BIG-DATA-POPULATED SMART CITY SETTINGS

In this Section, we test the result of Section 2 by considering the problem of approximating the number of people in a mobile phone cell during a certain time interval. This need comes from the Coiote project by TIM (Telecom Italia Mobile) and the ICT (Information and Communication Technology) for City Logistics and Enterprises Lab of Politecnico di Torino (see [9]). The goal of this project is to develop a mobile phone application, enabling TIM to ask users to share their Internet connection with smart sensors installed in the dumpsters, according to the users' positions in the mobile phone cell. In this way, the sensors can transmit to a central server data regarding the collect amount of waste, and the company in charge of the waste collection can plan the operations in an optimal way (see [10]). In exchange for the Internet connection which users share with the dumpster, TIM offers a reward. This is an example of application of the more general social engagement paradigm, reversing the usual direction of economic transaction in change of rewards. More and more applications of these new business models have been assuming central roles: some examples are the so-called crowd-shipping and the opportunistic Internet of Things (oIoT), respectively (e.g., [11-13]).

In particular, in crowd shipping, the company asks people to carry packages from one point to another in the city. By including enough people, it is possible to make the package reach the final destination. Using these methods, the company can save part of the cost of the standard workforce. Furthermore, it is likely that the participants have already planned to do a similar trip, thus reducing the environmental impact of travels. Among the several applications of this business model, it assumes relevance the Walmart (a grocery retailer) case, that asks in-store customers to carry packages to on-line customers in exchange for discounts.

IoT aims to establish interactions between sensors-equipped objects, through Internet networks. Despite its novelty, this paradigm is reshaping the world by offering a wide array of new applications and services. Information provided by data gathered by these sensors is of central importance for the development of smart cities, because they enable both the public administration and private companies to better provide their services and to more effectively manage their offers and activities. However, the drawback of this technology is about the requirements in terms of infrastructures (i.e., 5G, 6G, etc.), in order to ensure the connection of a considerable number of such items. An alternative approach to building such network architectures is to ask people to share their internet connection with the sensors, so that they could send data. This model is called opportunistic IoT (oIoT).

Those applications have been considered in the stochastic optimization framework in [14, 15]. The mathematical model of the problem deals with a minimization of the total amount of rewards, while still performing all the needed tasks, statistically modelling the locations of people in mobile network cells. Thus, a set of random variables $X_{ip}$ is defined, for each single person $p$ and each time instant $t$ such that:

$$X_{ip} = \begin{cases} 1, & \text{if person } p \text{ is in cell } i, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

We assume that $X_{ip}$ $\forall i, p$ are independent. This is due to two main reasons: firstly, TIM can easily identify people with similar behavior and exclude them for future calculation; secondly, the set of people $p$ is not the whole city population but just the people willing to be involved in the TIM system. From the independence assumption, $\theta_i^{\cdot\cdot}(w) = \sum_p X_{ip}$ is distributed according to a generalized binomial distribution (also known as Poisson binomial distribution). The probability mass function of the generalized binomial distribution is:

$$\sum_{\mathcal{A} \in \mathcal{F}_k} \prod_{i \in \mathcal{A}} p_i \prod_{j \in \mathcal{A}^c} \left(1 - p_j\right) \tag{11}$$

where $\mathcal{F}_k$ is the set of all the subsets of $k$ integers that can be selected from 1, ..., $n$ and $\mathcal{A}^c$is the complementary set of the set $\mathcal{A}$ (the distribution of a sum of Bernoulli random variables with different probabilities).

It is important to notice that we do not require that $\sum_{i=1}^{I} X_{ip} = 1$ because person $p$ can also be outside the city.

In an urban context, such as the one considered in this study, the number of people can be huge, hence we can apply the Lyapunov CLT.

Corollary 1. *Given a set of Bernoulli's random variables $X_k \sim \mathcal{B}(\pi_k)$, $k = 1, \ldots, n$ such that $0 < \pi_k < 1$ for all $k = 1, \ldots, n$, then*

$$\frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}(X_k - \pi_k)}{\sqrt{\sum_{k=1}^{n}\frac{\pi_k(1-\pi_k)}{n}}} \tag{12}$$

*converges in distribution to the standard normal distribution.*

***Proof.*** *If each $X_k \sim \mathcal{B}(\pi_k)$, $k = 1, \ldots, n$ has finite 4-th order moment and a strictly positive variance, then we can apply* **??**

By using Theorem 4, it is possible to prove the following corollary:

Corollary 2. *Given a set of Bernoulli's random variables $X_k \sim \mathcal{B}(\pi_k)$, $k = 1, \ldots, n$ such that $0 < \pi_k < 1$ for all $k = 1, \ldots, n$, then*

$$\left| \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}(X_k - \pi_k)}{\sqrt{\sum_{k=1}^{n}\frac{\pi_k(1-\pi_k)}{n}}} - \Phi(x) \right| \leq \frac{C_1}{\sqrt{n}} \tag{13}$$

*where $\Phi(x)$ is the standard normal distribution.*

***Proof.*** *Let us consider variables $X_l - \pi_k$, $k = 1, \ldots, n$; they are such that $\mathbb{E}[X_k - \pi_k] = 0$, $\sigma^2 = \mathbb{V}ar[X_k - \pi_k] = \mathbb{V}ar[X_k] = \pi_k(1-\pi_k) < \infty$ and $\eta_k = \mathbb{E}[|X_k - \pi_k|^3] = \pi_l - 3\pi_k^2 + 4\pi_k^3 - 2\pi_k^4 < \infty$. Furthermore,*

$$\frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}(X_k - \pi_k)}{\sqrt{\sum_{k=1}^{n}\frac{\pi_k(1-\pi_k)}{n}}} = \frac{\sum_{k=1}^{n}(X_k - \pi_k)}{\sqrt{\sum_{k=1}^{n}\sigma_k}} \tag{14}$$

*Hence, by applying Theorem (5) we have that (15) holds.*

$$\sup_{x \in \mathbb{R}} \left| F_{n(x)} - \Phi(x) \right| \leq \frac{C_0 \sum_{k=1}^{n} v_k}{\sqrt{\left(\sum_{k=1}^{n}\sigma_k^2\right)^3}} \tag{15}$$

*The right hand side of the inequality can be reduced to (16).*

$$\frac{C_0 \sum_{k=1}^{n} v_k}{\sqrt{\left(\sum_{k=1}^{n}\sigma_k^2\right)^3}} \leq \frac{C_0 n\, \rho_*}{\sqrt{n^3 \sigma_*^6}} \tag{16}$$

*where $\eta_* = \max_{k=1,\ldots,n}\eta_i$ and $\sigma_*^2 = \min_{k=1,\ldots,n}\sigma_k^2$. Since we assume that $\pi_k \neq 0, 1$ $\forall k = 1, \ldots, n$, then $\sigma_*^2 \neq 0$. . Finally, we obtain Eq (17).*

$$\sup_{x \in R} \left| F_{n(x)} - \Phi(x) \right| \leq \frac{C_0 \sum_{k=1}^{n} v_k}{\sqrt{\left(\sum_{k=1}^{n}\sigma_k^2\right)^3}} \leq \frac{C_0 n\, \rho_*}{\sqrt{n^3 \sigma_*^6}} = \frac{C_1}{\sqrt{n}} \tag{17}$$

Remark 2. *It is worth noting that Corollary (2) proves that the convergence of the statistics in (12) to the standard normal cdf is uniform.*

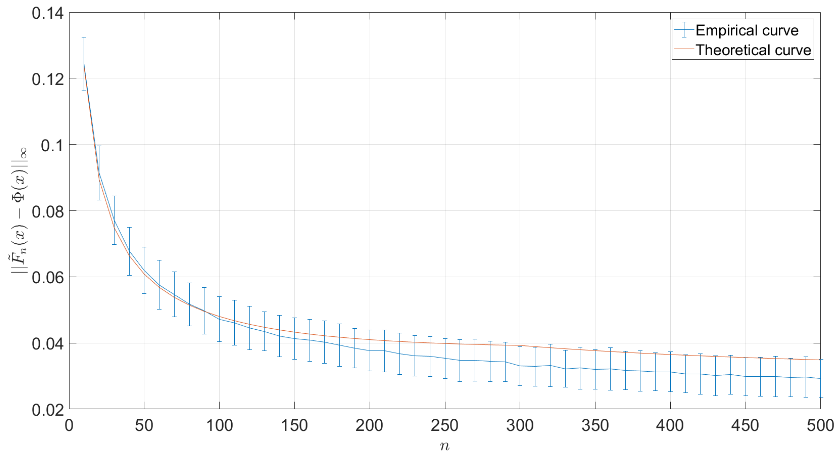


**Figure 1: The figure shows the decrease of e norm $||\tilde{F}_n - \Phi(x)||_\infty$ with respect to the number of observations. In blue the empirical curve and in red the theoretical one $C/\sqrt{n}$**

In Corollary 1, the assumption that $\pi_k \neq 0, 1\ \forall k$ is not strict because in the application that we are considering, means that person $p$ is certainly in cell $k$, while if $\pi_k = 0$ person $p$ is certainly not in cell $k$. Both cases are not good model choices because of Cromwell's rule (see [16]).

Owing Corollary 1 and since we are considering a crowded environment, we can simulate the number of people in a node by using a normal distribution. This result gives us a distribution to use for the simulation of the number of people in a network node. Furthermore, given data about the number of people in a cell in a certain hour, we can fit these values by using a normal distribution.

In order to further prove our results, we perform some examples in order to verify the theorems stated in Section 2. We consider the speed of convergence of the sum of Bernoulli random variables with different probabilities (i.e., $X_i \sim \mathcal{B}(p_i)$). We simulate the probability for each random variable from a uniform distribution between 0 and 1, i.e., $p_i \sim \mathcal{U}[0, 1]$. Then, for several numbers of observations ($n$), we compute the maximum error between the empirical cdf of random variable in Eq. (10) (we call it $\tilde{F}_n(x)$) and the standard normal cdf ($\Phi(x)$). In Figure 1 we report the results.

As the reader can notice, the theoretical curve (in red) is really close to the empirical one and, in particular, for big values of $n$, the empirical error is bounded above by the theoretical one ($\frac{1}{\sqrt{n}}$).

## 4 CONCLUSIONS AND FUTURE WORK

In this paper we resume a set of useful properties for the application of the Lyapunov CLT in the practical field. Furthermore, we prove that the convergence to a normal distribution is uniform for every type of distributions and we provide some easy-to-prove conditions to ensure the applicability of the Lyapunov CLT in the real setting. We hope that the results in this paper will lead more researchers to use this CLT, thus exploiting the possible range of applications that this variant encompasses. A final contribution of the paper is to show that the Lyapunov CLT has several applications relevant to the economic and social sciences and, in particular, in the Smart City and Gig Economy branches. Future work is mainly oriented on the enrichment of our framework with special features of big data management and analytics (e.g., [17-25]).

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