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Solutions of Singular Integral Equations from Gas Dynamics and Plasma Physics

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In this paper we give the explicit form of the solutions of the singular integral equations associated with some models of gas dynamics and plasma physics which are extensively investigated in the existing literature. In particular, we deal with equations on infinite and semi-infinite contours, where the data are assumed to be meromorphic functions. In this context we rederive some published results and present some new results which show how our method can be successfully used to obtain the explicit form of the solutions in much more general cases than those found in the literature.

KEY WORDS: Plasma oscillations; electron swarms; slip-flow problem.

1. INTRODUCTION

In two previous papers,[1,2] we have obtained considerable simplifications to the solution of singular integral equations on closed contours[1] and on intervals.[2] In particular, the principal value integrals which appear in the usual form of the solution[3,4] can be carried out explicitly under the assumption that the inhomogeneous term in the equation is a meromorphic function; we have treated the polynomial and rational function cases as examples. In ref. 2 an application was made to one-speed transport theory and, in particular, a simple expression was derived for the reflected distribution. In this paper, we present similar results for singular integral equations on infinite and semi-infinite contours such as those which arise in gas dynamics[5-7] problems and plasma physics.[8-12] Our result for the slip-flow problem has been obtained by Siewert and Thomas[13] for a matrix gas dynamics problem, but they do not explain how they managed

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to perform the principal value integral which appears in the usual presentation of the slip-flow result.[5,6] We speculate that they used an identity similar to that used by Cercignani in obtaining a similar result for a special case of the equation described in ref. 2. Our general method has allowed us to solve the "generalized slip-flow problem" (cf. Section 3).

2. BACKGROUND

For many problems of linear transport theory, the solution can be reduced to solving a singular integral equation. A full treatment has been given in ref. 11. Here we sketch some of the details to motivate the subsequent discussion.

In general, a linear transport equation is an equation of the form[14]

$$h(\mu)\frac{\partial f}{\partial x}+Af=0, \qquad x\in\mathbb{R} \tag{2.1a}$$

where $\mu\in I\subset\mathbb{R}$, I being a closed interval. I is equipped with a measure $d\mu$; in the Hilbert space $H=L^2(I, d\mu)$, A is a self-adjoint and quasipositive operator (i.e., positive except perhaps on a finite-dimensional subspace of H); $h(\mu)$ is some function. Equation (2.1) is to be solved subject to boundary conditions, for example,

$$f\to 0 \qquad \text{as} \quad x\to\infty \tag{2.2a}$$

or possibly

$$f=0(1) \qquad \text{as} \quad |x|\to\infty \tag{2.2b}$$

depending on the physics. At $x=0$ a typical condition might be the "full-range" condition

$$f(0, \mu)=f_0(\mu) \tag{2.3a}$$

or the "half-range condition"

$$f(0, \mu)=f_0(\mu), \qquad \mu\in I^+ \tag{2.3b}$$

$$I^+=\{\mu\in I; h(\mu)>0\} \tag{2.3c}$$

Strictly speaking, a true "transport problem" requires the conditions (2.3), which express the physical statement that the particle distribution is determined by the flux *incident* at $x=0$, but by abuse of nomenclature we often refer to problems involving (2.2) also as transport problems.

The forms of I, h, and A in the abstract equation (2.1) define the physical problem under consideration. For example, the "standard problem" (one-speed transport with isotropic scattering) has[11]

$$
\begin{aligned}
I &= [-1, 1] \quad \text{or} \quad [0, 1] \\
h(\mu) &= \mu \\
(Af)(x, \mu) &= f(x, \mu) - \frac{c}{2}\int_{-1}^{1} f(x, \mu')\, d\mu'
\end{aligned}
\tag{2.4}
$$

In this paper we consider two different transport problems. The problems describing plasma oscillations[11,15] are of the full-range type, with x representing the time. We have

$$
\begin{aligned}
I &= \mathbb{R} \\
h &= 1 \\
(Af)(x, \mu) &= ik\left(f - \eta\int_{\mathbb{R}} f(s)\, ds\right)
\end{aligned}
\tag{2.5}
$$

where η is a constant involving the plasma frequency and the constant k is the wave number of the oscillation.

One problem describing gas dynamics in the so-called "linearized BGK model"[7,5] is of the half-range type. It has

$$
\begin{aligned}
I &= \mathbb{R}^{+} \\
h &= \mu \\
(Af)(x, \mu) &= \nu_0\left[f - \eta(\mu)\int_{-\infty}^{\infty} f(x, s)\, ds\right]
\end{aligned}
\tag{2.6}
$$

where ν_0 is the (constant) collision frequency of gas molecules and $\eta(\mu)$ is the Maxwellian distribution shifted by the drift velocity of the gas.[7]

Case introduced in 1960[15] the idea of treating Eq. (2.1) by separation of variables:

$$
f(x, \mu) = \phi_\nu(\mu)\, e^{-x/\nu}, \qquad \nu \in \mathbb{R} \tag{2.7}
$$

with ν playing the role of an "eigenvalue" (the spectrum actually contains both point and continuous spectra). The ϕ_ν are called "Case eigenfunctions." Ultimately, the solution is expressed as a superposition of the fundamental solutions (2.7)

$$
f(x, \mu) = \int_{I_1} A(\nu)\, \phi_\nu(\mu)\, e^{-x/\nu}\, d\nu, \qquad I_1 \subset I \tag{2.8}
$$

The coefficient $A(\nu)$ is determined by the boundary conditions (2.2) or (2.3) as the case may be,

$$f_0 = \int_{I_1} A(\nu)\,\phi_\nu(\mu)\,d\nu \tag{2.9}$$

Since typically[11,15] $\phi_\nu(\mu)$ contains factors like $(\nu-\mu)^{-1}$, Eq. (2.9) is a singular integral equation of Cauchy type which must be solved for $A(\nu)$, after which the solution is given by (2.8). The solution of such equations is well known[3,4,16]; it is expressed in terms of principal value integrals which must in general be evaluated numerically. Our contribution in refs. 1 and 2 was to show that for a large (and dense) class of data f_0, these integrals could be evaluated analytically, thus drastically simplifying the numerical evaluation of the solutions to singular integral equations. In the present paper we apply these methods specifically to the two problems expressed in (2.5) and (2.6). No knowledge of transport theory is required to understand our analysis; it is all based on rather simple applications of Cauchy's theorem! A rather crucial function which enters all of transport theories is the so-called "dispersion function" $\Lambda(z)$, whose zeros given the discrete spectrum [see Eq. (2.7)]. It is always analytic on $\mathbb{C}\backslash I$ with boundary values on the branch cut I denoted by $\Lambda^{\pm}(\mu) = \lim_{\varepsilon\to 0} \Lambda(\mu \pm i\varepsilon)$. Another crucial function is the analytic Riemann–Hilbert function $X(z)$, which is related to $\Lambda^{\pm}$ in that its boundary values on I_1, $X^{\pm}$, obey

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)}, \qquad \mu \in I_1 \tag{2.10}$$

If $I_1 = I$, the solution to (2.10) is trivially $X = \Lambda$; otherwise a solution must be constructed; since for any solution $X_0(z)$ of Eq. (2.10) $P(z)\,X_0(z)$ is also a solution for any polynomial $P(z)$, the determination of the proper solution is a somewhat delicate matter which we do not discuss here; we assume in the subsequent discussion that the appropriate solution has been obtained; it depends on the so-called "index" of the singular integral equation (2.9).[3,4,16,17]

3. APPLICATIONS

3.1. Full Range

A well-known example of full-range problems is the Vlasov–Poisson system describing plasma oscillations, which can be solved in terms of Case eigenfunctions.[9–12] After Fourier transformation with respect to space, one seeks a solution $f_k(v, t)$ with initial value $f_0(v)$ which represents a perturbation from the equilibrium distribution (v is the dimensionless velocity variable and t the time). In the simplest case (no discrete spectrum) the solution for the electric field $E_k(t)$ is given by[10,11]

$$E_k(t) \propto \int_{-\infty}^{\infty} A_k(s)\, e^{-ikst}\, ds \tag{3.1}$$

where $A_k(s)$ is a solution of the singular integral equation[10,11]

$$f_0(s) = \lambda(s)\, A_k(s) + \eta_k(s)\, \mathscr{P} \int_{-\infty}^{\infty} A_k(s') \frac{ds'}{s'-s} \tag{3.2}$$

Here η is related to the equilibrium distribution F by

$$\eta_k(v) = -\frac{\omega_p^2}{k^2} F'(v) \tag{3.2a}$$

where ω_p is the plasma frequency. (Typically, but not necessarily, F is a Maxwellian, in which case η is an odd function.) Defining the plasma dispersion function as

$$\Lambda_k(z) = 1 + \int_{-\infty}^{\infty} \eta_k(s) \frac{ds}{s-z}, \qquad x+iy = z \in \mathbb{C} \tag{3.2b}$$

we find that its boundary values for $y \to 0^{\pm}$, $\Lambda_k^{\pm}(s)$, are

$$\Lambda_k^{\pm}(s) = \lambda_k(s) \pm \pi i \eta_k(s)$$

with

$$\lambda_k(s) = 1 + \mathscr{P} \int_{-\infty}^{\infty} \eta_k(s') \frac{ds'}{s'-s}$$

In terms of these quantities, the solution to Eq. (2) is found to be[11,12]

$$A_k(s) = \frac{\lambda_k(s)\, f_0(s)}{\Lambda_k^{+}(s)\, \Lambda_k^{-}(s)} - \frac{\eta_k(s)}{\Lambda_k^{+}(s)\, \Lambda_k^{-}(s)}\, \mathscr{P} \int_{-\infty}^{\infty} f_0(s') \frac{ds'}{s'-s} \tag{3.3}$$

where the symbol $\mathscr{P}$ denotes the Cauchy principal value.

If we use the following, somewhat bizarre identity

$$1 = \frac{\Lambda_k^{+}(s) - \Lambda_k^{-}(s)}{2\pi i \eta_k(s)} \tag{3.4a}$$

as well as the Plemelj formulas[3]

$$\mathscr{P}\frac{ds'}{s'-s}=\lim_{\varepsilon\downarrow 0}\frac{ds'}{s'\pm i\varepsilon-s}\pm\pi i\delta(s'-s)\,ds' \tag{3.4b}$$

in Eq. (3.3), we easily obtain, after evaluating the integrals containing the δ functions,

$$A_k(s)=-\frac{\eta_k(s)}{\Lambda_k^+(s)\,\Lambda_k^-(s)}\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left[\frac{\Lambda_k^+(s')}{s'+i\varepsilon-s}-\frac{\Lambda_k^-(s')}{s'-i\varepsilon-s}\right]\frac{f_0(s')}{\eta_k(s')}\,ds' \tag{3.5}$$

The integral in Eq. (3.5) can be carried out analytically for a suitable choice of $f_0(s)$. Here we choose a perturbation from the equilibrium distribution of the form

$$f_0(v)=\frac{\eta(v)}{1+v^2}$$

Then A_k can be reexpressed in terms of contour integrals:

$$A_k(s)=-\frac{\eta_k(s)}{\Lambda_k^+(s)\,\Lambda_k^-(s)}\frac{1}{2\pi i}\left[\oint_{C_1}\frac{\Lambda_k(z)}{z-s}\frac{dz}{1+z^2}-\oint_{C_2}\frac{\Lambda_k(z)}{z-s}\frac{dz}{1+z^2}\right]$$

Here, the closed contour C_1 runs on the positive direction of the real axis and thence in the positive sense around the upper half-plane, while C_2 runs along the negative side of the real axis and closes in a negative sense about the lower half-plane. These two integrals can be evaluated by residues to yield

$$A_k(s)=\frac{\eta_k(s)}{\Lambda_k^+(s)\,\Lambda_k^-(s)}\frac{1}{2i}\left(\frac{\Lambda_k(i)}{s-i}-\frac{\Lambda_k(-i)}{s+i}\right) \tag{3.6}$$

Of course, the expression for A_k is manifestly real; also, $\Lambda_k(i)$ is real for odd $F(v)$. Equation (3.6) can now be inserted into Eq. (3.1) and integrated by residues. Let us assume that $k<0$. Then, the contour in Eq. (3.1) can be closed in the upper-half plane. Substituting

$$\Lambda_k^-(s)=\Lambda_k^+(s)-2\pi i\eta_k(s)$$

in Eq. (3.6), we find

$$E_k(t)\propto\frac{\Lambda_k(i)}{2i}\oint_{C_1}\frac{\eta_k(z)}{\Lambda_k(z)[\Lambda_k(z)-2\pi i\eta_k(z)]}e^{-ikzt}\frac{dz}{z-i} \tag{3.7}$$

Note that the analytic function

$$\frac{\eta_k(z)}{\Lambda_k(z)[\Lambda_k(z)-2\pi i\eta_k(z)]}$$

is bounded on C_1. Then, evaluating (3.7) by residues and assuming Λ_k and $\Lambda_k-2\pi i\eta_k$ have no zeros, we find

$$E_k(t) \propto e^{kt}$$

that is, Landau damping[8–10] is explicitly exhibited. (In other words, the electric field introduced into the plasma by the initial perturbation decays exponentially.) Naturally, for $k>0$, we would have expressed $\Lambda_k^+(s)\,\Lambda_k^-(s)$ [Eq. (3.5)] in terms of $\Lambda_k^-(s)$, and closed the contour in the lower half-plane.

The assumption that Λ_k has no zeros is not essential. If there are zeros, the electric field is given by discrete terms[11,12] plus the same contour integral (3.6) as here. [Additional residues are introduced by the zeros of $\Lambda_k(z)-2\pi i\eta_k(z)$, but these also are damped out.] The discrete terms in general lead to linear instabilities of the electric field.[11,12]

More generally, suppose

$$f_0(v)=\eta(v)\,\frac{\prod_{i=1}^{N}(v-a_i)}{\prod_{j=1}^{M}(v-b_j)} \tag{3.8}$$

with the b_i distinct. Then, calculating as above, we get

$$\Lambda_k(v)=\frac{\eta_k(v)}{\Lambda_k^+(v)\,\Lambda_k^-(v)}\left[-\sum_{\substack{l=0\\ \operatorname{Im} b_l\neq 0}}^{M}\frac{p_l}{v-b_l}+v^{N-M}\sum_{m=1}^{N-M}v^{-m}\Lambda_k^{(m)}\right] \tag{3.9}$$

where

$$p_l=\frac{\prod_{i=1}^{N}(b_l-a_i)}{\prod_{j=1,j\neq l}^{M}(b_l-b_j)}\,\Lambda_k(b_l) \tag{3.10a}$$

and

$$\begin{aligned}\Lambda_k^{(0)}&=1\\ \Lambda_k^{(m)}&=-\int_{-\infty}^{\infty}\eta_k(s)\,s^{m-1}\,ds, \qquad m>0\end{aligned} \tag{3.10b}$$

Then, a simple application of the residue theorem, assuming $k<0$ as above, proves

$$E_k(t)\propto\sum_{\substack{l=0\\ \operatorname{Im} b_l>0}}^{M}\frac{p_l b_l^{N-M}\eta_k(b_l)\,e^{-ikb_lt}}{\Lambda_k(b_l)-2\pi i\eta_k(b_l)} \tag{3.10c}$$

If $\Lambda_k(z)$ has zeros, or if $\Lambda_k(z) - 2\pi i\eta_k(z)$ has zeros, there will be additional terms in the above sum coming from the zeros in the upper half-plane, as well as discrete modes. As in the simpler case discussed previously, the continuum modes exhibit Landau damping explicitly. The calculations which have been presented here are reminiscent of the argument used by Landau,[8] who used an entirely different (Laplace transform) method. In particular, he stressed the necessity for f_0 to be an analytic function.

The result, Eq. (3.10c), can be used to prove Landau damping for the initial value problem in $\mathbb{R}^3$. The constant of proportionality in this equation is actually ik[10,11]; noting Eqs. (3.2a) and (3.2b), we can see that Eq. (3.10c) is of the form

$$E_k(t) = \sum_{\operatorname{Im} b_l > 0} \frac{C_l k}{k^2 + D_l} e^{-i|k| b_l t}$$

for constants C_l and D_l. Then

$$E(x, t) = \int_{-\infty}^{\infty} e^{ikx} E_k(t)\, dk$$

tends to zero as $t \to \infty$. I am indebted to Robert Glassey for suggesting this computation; he has actually obtained this result another way.

We next consider the full-range gas dynamics problem described in ref. 7. A solution is sought of the following singular integral equation:

$$f(v) - a_0 \eta(v \mid w) = \lambda(v \mid w)\, B(v) + \eta(v \mid w)\, \mathscr{P} \int_{-\infty}^{\infty} B(s) \frac{s}{s-v}\, ds \quad (3.11a)$$

where v is the dimensionless particle velocity and $\eta(v \mid w)$ is the Maxwellian shifted by the peculiar velocity w:

$$\eta(v \mid w) = \frac{1}{\sqrt{\pi}} e^{(v-w)^2} \quad (3.11b)$$

The dispersion function $\Lambda(z \mid w)$ is given by

$$\Lambda(v \mid w) = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-(z-w)}\, dt, \qquad z \in \mathbb{C} \backslash \mathbb{R} \quad (3.11c)$$

and its boundary values on $\mathbb{R}^{\pm}$ are

$$\Lambda(t \mid w) = \lambda(t \mid w) \pm i\pi t \eta(t \mid w) \quad (3.11d)$$

with

$$\lambda(t \mid w) = 1 + \mathscr{P} \int_{-\infty}^{\infty} \frac{t}{s-t} \eta(s \mid w)\, ds \quad (3.11e)$$

It is further required[7] that

$$0 < \int_{-\infty}^{\infty} vf(v) < \infty \tag{3.11f}$$

The solution as given in ref. 7, Eqs. (3.9) and (3.10), is

$$B(s) = \frac{\lambda(s \mid w) f(s)}{\Lambda^+(s \mid w) \Lambda^-(s \mid w)} - \frac{\eta(s \mid w)}{\Lambda^+(s \mid w) \Lambda^-(s \mid w)} \mathscr{P} \int_{-\infty}^{\infty} \frac{vf(v)}{v - s} dv \tag{3.12a}$$

and

$$a_0 = \frac{1}{w} \int_{-\infty}^{\infty} vf(v)\, dv \tag{3.12b}$$

If we choose

$$f(v) = v^n \eta(v \mid w), \qquad n \text{ odd} \tag{3.13}$$

and apply the same technique as in the case of the Vlasov equation [except that here we have

$$v = \frac{\Lambda^+(v \mid w) - \Lambda^-(v \mid w)}{2\pi i \eta(v \mid w)} \tag{3.14}$$

instead of Eq. (3.4a)], Eq. (3.12a) reduces to

$$B(s) = \frac{\eta(s \mid w)}{2\pi i \Lambda^+(s \mid w) \Lambda^-(s \mid w)} \oint_{C_\infty} \frac{z^n \Lambda(z \mid w)}{z - s} dz \tag{3.15}$$

The contour is taken around an infinite circle, in the positive sense. The integrals are easily evaluated in terms of the moments of Λ as in the case of the Vlasov equation. Then

$$B(s) = \frac{\eta(s \mid w)}{\Lambda^+(s \mid w) \Lambda^-(s \mid w)} \sum_{l=1}^{n} \Lambda^{(l)}(w)\, s^{n-l} \tag{3.16a}$$

and

$$a_0 = \frac{1}{w} \eta_{n+1}(w) \tag{3.16b}$$

with

$$\begin{aligned} \Lambda^{(l)}(w) &= 0, \qquad & l = 0 \\ &= -\eta_l(w), \qquad & l > 0 \end{aligned} \tag{3.17a}$$

and

$$\eta_l(w) = \int_{-\infty}^{\infty} \eta(t \mid w)\, t^l\, dt \tag{3.17b}$$

Again, this procedure can be easily generalized to the case in which $f(v)$ is of the form of $f_0(v)$ in Eq. (3.8). In this case, the solution takes the form

$$B(s) = -\frac{\eta(s \mid w)}{2\pi i \Lambda^+(s \mid w)\, \Lambda^-(s \mid w)} \times \int_{-\infty}^{\infty} \frac{\prod_{i=1}^{N}(v-a_i)}{\prod_{j=1}^{M}(v-b_j)} [\Lambda^+(v \mid w) - \Lambda^-(v \mid w)] \frac{dv}{v-s} \tag{3.18}$$

The contribution from the circle at infinity is exactly what we have previously calculated, Eq. (3.16a), except for the change $n \to N-M$. Such a term vanishes if $M \geqslant N$. In addition, there is a contribution from the poles of f, which is

$$-\frac{\eta(s \mid w)}{\Lambda^+(s \mid w)\, \Lambda^-(s \mid w)} \sum_{\substack{l=1 \\ \operatorname{Im} b_l \neq 0}}^{M} \frac{p_l \Lambda(b_l \mid w)}{b_l - s} \tag{3.19}$$

Also, for the coefficient a_0 we get

$$a_0 = \frac{1}{w} \sum_{\substack{l=1 \\ \operatorname{Im} b_l \neq 0}}^{M} p_l \Lambda(b_l \mid w) + \frac{1}{w} \eta_{N-M+1}(w) \tag{3.20}$$

3.2. Half Range

The equation we now deal with is found in ref. 7, Section 5:

$$f(v) - a_0 \eta(v \mid w) = \lambda(v \mid w)\, B(v) + \eta(v \mid w)\, \mathscr{P} \int_0^{\infty} B(s) \frac{s}{s-v}\, ds \tag{3.21}$$

Note that, for $w = 0$, this reduces to the BGK equation considered in refs. 5 and 6. The solution is given by[7]

$$B(s) = \frac{\lambda(s \mid w)\, f(s)}{\Lambda^+(s \mid w)\, \Lambda^-(s \mid w)} + \frac{s\eta(s \mid w)}{X_1^+(s \mid w)\, \Lambda^-(s \mid w)} \mathscr{P} \int_0^{\infty} \frac{X_1^+(v \mid w)\, f(v)}{\Lambda^+(v \mid w)(s-v)}\, dv \tag{3.22a}$$

$$a_0 = -\int_0^{\infty} f(v)\, X_1^+(v \mid w)\, \Lambda^+(v \mid w)^{-1}\, dv \tag{3.22b}$$

Here, $X_1(z \mid w)$ is the solution of the homogeneous Riemann–Hilbert problem [cf. Eq. (2.1a)]

$$\frac{X_1^+(s \mid w)}{X_1^-(s \mid w)} = \frac{\Lambda^+(s \mid w)}{\Lambda^-(s \mid w)}, \qquad s \in \mathbb{R} \tag{3.23}$$

which is analytic in $\mathbb{C} \backslash \mathbb{R}$ and bounded at infinity.

Using methods as discussed above (cf. also ref. 2), we find

$$B(s) = \frac{s\eta(s \mid w)}{X_1^+(s \mid w)\, \Lambda^-(s \mid w)} \oint_{C_\infty} \frac{X_1(z \mid w)}{z - s} \frac{f(z)}{z\eta(z \mid w)}\, dz \tag{3.24}$$

where C_∞ is a positively oriented contour at infinity. In obtaining (3.24) from (3.22a) we have used the Plemelj formulas, (3.4b), as well as the identity

$$\frac{X_1^+(v \mid w)}{\Lambda^+(v \mid w)} = \frac{X_1^+(v \mid w) - X_1^-(v \mid w)}{2\pi i \eta(v \mid w)}$$

Note that there is a difference of notation between refs. 5 and 6 and ref. 7. In particular, the dependent variable in the dynamic equation (2.1) of ref. 7 differs by a Maxwellian from that of refs. 5 and 6; this is merely a matter of notation, and we have adopted the same notation as in ref. 7.

The "generalized slip-flow" problem assumes $f(z) = z^n \eta(z \mid w)$, where $n = 1$ and $w = 0$ correspond to the usual slip-flow problem. For the above f, the contour integral in Eq. (3.24) can be evaluated pretty much as in ref. 2, i.e., in terms of the moments of $X_1(z)$. We find

$$B(s) = \frac{s\eta(s \mid w)}{X_1^+(s \mid w)\, \Lambda^-(s \mid w)} \sum_{l=0}^{n-1} X_1^{(l)} s^{n-l} \tag{3.25a}$$

and

$$a_0 = X_1^{(n)}(w) \tag{3.25b}$$

The dependence of $B(s)$ and the moments $X_1^{(l)}(w)$ on $X_1^+(w)$ can be removed by utilizing the Wiener–Hopf factorization of $\Lambda(z \mid w)$.[17] Since this differs for $w > 0$ [$\Lambda(z \mid w)$ is nonsymmetric] and $w = 0$ [$\Lambda(z \mid 0)$ is symmetric], we derive both cases in the Appendix. The results are as follows.

(a) $w = 0$

$$B(s) = -\frac{\eta(s \mid 0)\, X_1(-s \mid 0)}{2s\Lambda^+(s \mid 0)\, \Lambda^-(s \mid 0)} \sum_{l=0}^{n-1} X_1^{(l)}(0)\, s^{n-l} \tag{3.26}$$

and

$$X_1^{(0)}(0) = 1$$
$$X_1^{(l)}(0) = 2\int_0^\infty \frac{s^{2+l}\eta(s\,|\,0)}{X_1(-s\,|\,0)}\,ds \tag{3.27}$$

a_0 is, of course, still given by (3.25b).

(b) $w \neq 0$

$$B(s) = -\frac{w\eta(s\,|\,w)\,Y_1(-s\,|\,w)}{\Lambda^+(s\,|\,w)\,\Lambda^-(s\,|\,w)} \sum_{l=0}^{n-1} X_1^{(l)}(w)\,s^{n-l} \tag{3.28}$$

$Y_1(s\,|\,w)$ is an auxiliary function needed in nonsymmetric problems.[17] See the Appendix. The moments of $X_1(z\,|\,w)$ are

$$X_1^{(0)}(w) = 1$$
$$X_1^{(l)}(w) = \frac{1}{w}\int_0^\infty s^{l+1}\eta(s\,|\,w)\,\frac{ds}{Y_1(-s\,|\,w)} \tag{3.29}$$

For $w = 0$, X_1 can be calculated from a nonsingular, nonlinear integral equation, which is a modified version of the Chandrasekhar H-equation for the transport problem considered in ref. 7. See the Appendix. For $w \neq 0$, there are two coupled equations for X_1 and Y_1, also given in the Appendix.

For the classical slip-flow problem ($w = 0$, $n = 1$) one gets

$$B(s) = -\frac{\eta(s\,|\,0)\,X_1(-s\,|\,0)}{2s\Lambda^+(s\,|\,0)\,\Lambda^-(s\,|\,0)}$$

and

$$a_0 = X_1^{(1)}(0)$$

This result also generalizes readily to the case in which $f(v) = v\eta(v)\,R(v)$, where R is a rational function of the form given in the previous section [see Eq. (3.8)]. We find for $w > 0$

$$B(s) = -\frac{w\eta(s\,|\,w)\,Y_1(-s\,|\,w)}{\Lambda^+(s\,|\,w)\,\Lambda^-(s\,|\,w)} \left[\sum_{l=0}^{N-M} X_1^{(l)}(w)\,s^{N-M-l+1} - \sum_{\substack{l=1 \\ b_l \in \mathbb{C}\setminus[0,\infty)}}^{M} \frac{X_1(b_l\,|\,w)}{b_l - s}\,p_l \right] \tag{3.30}$$

while for $w=0$ the term in brackets is unchanged and the factor multiplying it is replaced by

$$-\frac{\eta(s)\,X_1(-s)}{2s\Lambda^+(s)\,\Lambda^-(s)}$$

For the coefficient a_0, we get

$$a_0 = X_1^{(N-M+1)}(w) - \sum_{\substack{l=1 \\ b_l \in \mathbb{C}\setminus[0,\infty)}}^{M} X_1(b_l \mid w)\,p_l$$

APPENDIX

For the half-range gas dynamics problems discussed in Section 3, boundary values $X_1^{\pm}$ of the solution to the Riemann–Hilbert problem, Eq. (3.23) or (2.10), enter the solution. These are difficult to evaluate directly because they involve numerical principal value integrations. However, the fact that the dispersion function $\Lambda(z)$ has a Wiener–Hopf factorization can be utilized to simplify the calculation. It should be stressed that the Wiener–Hopf technique cannot be applied to all singular integral equations. If we refer to refs. 1, 2, 4, and 16, we observe that the combinations $\lambda \pm i\pi\eta := \Lambda^{\pm}$ enter the solution. In general, an analytic function $\Lambda(z)$ with arbitrary boundary values $\Lambda^{\pm}$ does not exist. That it does in transport theory is a serendipitous happenstance.

In many cases, the dispersion function $\Lambda(z)$ is an even function: $\Lambda(z) = \Lambda(-z)$. This is true in the case of the gas dynamics considered here only if $w=0$. So we consider the cases $w=0$ and $w \neq 0$ separately.

(a) $w=0$. Equation (6) from ref. 17 [with $X(z) = Y(z)$] expresses the Wiener–Hopf factorization of $\Lambda(z) = \Lambda(z \mid 0)$ as

$$\Lambda(z) = -\frac{2}{z^2}\,X_1(z)\,X_1(-z) \tag{A.1}$$

Then, since $X_1(-z)$ is continuous on $[0, \infty)$,

$$\frac{X_1^+(s)}{\Lambda^+(s)} = -\frac{s^2}{2X_1(-s)} \tag{A.2}$$

$X_1(s)$, and the moments $X_1^{(k)}$ which enter Eqs. (3.26) and (3.27), can be calculated in the following way. Since $X_1(z) \to 1$ at infinity, Cauchy's theorem implies [with $\eta(s) = \eta(s \mid 0)$]

$$X_1(z) = 1 + \frac{1}{2\pi i}\int_0^\infty [X_1^+(s) - X_1^-(s)]\frac{ds}{s-z} \tag{A.3a}$$

$$= 1 + \int_0^\infty s\eta(s)\frac{X_1^+(s)}{\Lambda_1^+(s)}\frac{ds}{s-z} \tag{A.3b}$$

or, using Eq. (A.2),

$$X_1(z) = 1 - \frac{1}{2}\int_0^\infty \frac{s^3\eta(s)}{X_1(-s)}\frac{ds}{s-z} \tag{A.4}$$

From this we see immediately that

$$\begin{aligned} X_1^{(0)} &= 1 \\ X_1^{(l)} &= \frac{1}{2}\int_0^\infty \frac{s^{2+l}\eta(s)}{X_1(-s)}\,ds \end{aligned} \tag{A.5}$$

Equation (A.5) has already been given, without proof, in Section 3. Equation (A.4) is itself a nonlinear, nonsingular integral equation for $X_1(-t)$ for $t \in [0, \infty)$:

$$X_1(-t) = 1 + \frac{1}{2}\int_0^1 \frac{s^3\eta(s)}{X_1(-s)}\frac{ds}{s+t} \tag{A.6}$$

It is a modified version of the so-called Chandrasekar H-equation which has been derived previously in a number of places.

(b) $w \neq 0$. The analysis of ref. 17 now implies that two functions X_1 and Y_1 are required for the Wiener–Hopf factorization of $\Lambda(z \mid w)$, which is no longer symmetric:

$$\Lambda(z \mid w) = -\frac{w}{z} X_1(z \mid w)\, Y_1(-z \mid w) \tag{A.7}$$

As is shown in ref. 17, Y_1 is a solution of the associated Riemann–Hilbert problem

$$\frac{Y_1^-(v \mid w)}{Y_1^+(v \mid w)} = \frac{\Lambda^+(-v \mid w)}{\Lambda^-(-v \mid w)}, \qquad v \in [0, \infty) \tag{A.8}$$

$X_1(z \mid w)$ still obeys Eq. (A.3b); if we now use Eq. (A.7) in that equation, we find, instead of (A.4),

$$X_1(z \mid w) = 1 - \frac{1}{w}\int_0^\infty \frac{s^2\eta(s \mid w)}{Y_1(-s \mid w)}\frac{ds}{s-z} \tag{A.9a}$$

Using Cauchy's theorem for $Y_1(z \mid w)$ gives

$$Y_1(z \mid w) = 1 + \frac{1}{2\pi i} \int_0^\infty \left[Y_1^+(v \mid w) - Y_1^-(v \mid w) \right] \frac{dv}{v - z} \tag{A.9b}$$

Now, it is easily checked, using (A.8), that

$$\frac{Y^+(v \mid w) - Y^-(v \mid w)}{2\pi i} = \frac{Y^+(v \mid w)}{\Lambda^-(-v \mid w)} v\eta(-v \mid w) \tag{A.10a}$$

while, from (A.7),

$$\frac{Y^+(v \mid w)}{\Lambda^-(v \mid w)} = \frac{v}{w} X_1(-v \mid w) \tag{A.10b}$$

Thus Eq. (A.9b) becomes

$$Y_1(z \mid w) = 1 + \frac{1}{w} \int_0^\infty \frac{v^2 \eta(-v \mid w)}{X_1(-v \mid w)} \frac{dv}{v - z} \tag{A.11}$$

Then $X_1(-v \mid w)$ and $Y_1(-v \mid w)$ can be obtained by numerically solving Eqs. (A.9a) and (A.11) for $z = -v$.

The expression for the moments also changes from (A.9a) to

$$X_1^{(0)} = 1$$

$$X_1^{(l)} = \int_0^\infty \frac{s^{1+l} \eta(s \mid w)}{Y_1(-s \mid w)} ds$$

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