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Trajectory generation for nonholonomic control systems using reconstruction techniques on $SE(2)$

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Abstract. In this note we investigate the trajectory generation problem for nonholonomic mechanical shape-control systems, focusing in the case in which the symmetry group is $SE(2)$, by using techniques from reconstruction theory.

Keywords: Controlled nonholonomic systems · trajectory generation · reconstruction theory.

1 Introduction

Nonholonomic mechanical systems, that are central to robotics, wheeled vehicle dynamics, and motion generation, continue to pose fundamental challenges, particularly due to their nontrivial geometric structure and nonlinear dynamics [16, 3, 2, 4, 7, 8]. While these systems have attracted significant interest, especially for their connection with control theory [13, 17, 14, 15, 18], many foundational aspects remain open, especially regarding the role of symmetry in understanding and controlling their behavior.

Symmetries in dynamical systems provide a foundational mechanism for reducing system complexity via reduction by symmetry, which allows one to obtain a lower-dimensional quotient system whose dynamics can be studied independently. The complete dynamics of the original system can subsequently be reconstructed, particularly from equilibria and relative periodic orbits, offering profound insights into the global behaviour of the system [1, 10]. Developments in this direction have been achieved for purely kinematic control systems, such as the kinematic car, bicycle models and micro-swimmers in [9, 11], whose dynamics are naturally modelled as driftless affine control systems.

In this work we propose an extension of that framework, aiming to generalise the results in [11] to a broader class of systems. We focus on the case of time–independent nonholonomic systems with linear constraints and natural

Lagrangian and symmetry group $SE(2)$. In these systems the control inputs often correspond directly to some of the system's coordinates, the so called *shape variables*, resulting in a nonlinear dynamics that complicate both controllability analysis and motion planning. Using reconstruction techniques from equilibria and relative periodic orbits, we are able to systematically derive effective control inputs capable of steering the system along, or modifying, a given trajectory.

2 Basic notions on nonholonomic systems, and reconstruction theory

Definition 1. *A nonholonomic mechanical system with symmetry is a quadruple $(Q, L, \mathcal{D}, G)$, where $(Q, L, \mathcal{D})$ is a nonholonomic mechanical system, in which Q is an n–dimensional configuration manifold, $L : TQ \longrightarrow \mathbb{R}$ $L = T - V$ is a mechanical Lagrangian, with T and V the kinetic and potential energy, respectively, and $\mathcal{D}$ a constant rank non–integrable distribution on Q. G is a Lie group that acts on Q leaving both the Lagrangian and the constraint invariant.*

In bundle coordinates $(q, \dot{q})$ the potential energy only depends on the configuration variables, $V = V(q)$, and the kinetic energy is a quadratic form on the velocities: $T(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot A(q)\dot{q}$,[3] with $A(q)$ a symmetric and positive definite matrix, called kinetic matrix. The kinetic energy induces a Riemannian metric on Q. The distribution $\mathcal{D}$, called the *constraint distribution*, represents the ideal nonholonomic constraint. We will regard $\mathcal{D}$ as a sub-bundle of TQ, but it may be as well regarded as a submanifold $D \subset TQ$ of dimension $n + r$, that will be called the *constraint submanifold*.
When symmetry is present it is often convenient to introduce coordinates adapted to the symmetry. More precisely it is convenient to split the configuration variables $q \in Q$ in *group variables* g, which describe the overall position (attitude) of the system, and *shape variables* s, which describe the positions of the system's components relative to each other. The equations of motions (see [3, 17]) are then

$$\begin{cases} \ddot{s} = f(s, \dot{s}, p) \\ \dot{g} = g(J(s)p + A(s)\dot{s}) \\ \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, \dot{s}\rangle + \langle C(s)\dot{s}, \dot{s}\rangle \end{cases} \tag{1}$$

where p is the nonholonomic momentum, $f(s, \dot{s}, p)$ represents the total force acting on the system, $J(s)$, $A(s)$, $M(S)$, $N(s)$, $C(s)$ are tensors that depend only on the shape variables, and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and co-vectors. (see e.g. [3, 2] for details on equations (1)). The first equation of (1) is called *shape equation*, the second, *reconstruction equation* and the third one, *momentum equation*. The G–invariance of equations (1) allows them to induce a dynamical system on the reduced space $\mathcal{D}/G$.

A dynamical system

$$\dot{q} = \mathcal{Z}(q, u)\,, \tag{2}$$

[3] The dot denotes the canonical scalar product of $\mathbb{R}^n$.

where q are local coordinates on a smooth finite dimensional manifold Q and $u : [0, T] \to U$ is a measurable function taking values in an open subset U of $\mathbb{R}^h$, with h a positive integer, is called a *control system*. We call the function u an admissible control and U the control manifold. A solution of the control system (2) is a pair (q, u), where $q : I \to Q$, with $I \subseteq \mathbb{R}$, is a locally absolutely continuous curve and $u : I \to U$ is an admissible control, that satisfies (2) (possibly in the in the weak sense [6]).

We consider the notion of nonholonomic mechanical shape control system, in which the control input are given in terms in term of an internal 'shape' variable

Definition 2. *A nonholonomic mechanical shape–control system is a sextuple $(Q, L, \mathcal{D}, \mathcal{S}, U, G)$, where $(Q, L, \mathcal{D}, G)$ is a nonholonomic mechanical system with symmetry. $\mathcal{S} := Q/G$, the so–called shape space, represents the controlled variables, and is assumed to be diffeomorphic to $\mathbb{R}^h$, for some positive integer h and U diffeomorphic to $\mathbb{R}^h$ as well, is the control space.*

Let $(g, p, s, \dot{s})$ be coordinates on $\mathcal{D}$ adapted to the constraint and to the symmetry. Following [17], the equations of motion of a nonnholonomic shape control system are then

$$\begin{cases} \ddot{s} = f(s, \dot{s}, p) + u \\ \dot{g} = g(J(s)p + A(s)\dot{s}) \\ \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, \dot{s}\rangle + \langle C(s)\dot{s}, \dot{s}\rangle \end{cases}, \tag{3}$$

where $u : t \longrightarrow U$ an admissible control, and define a dynamical system on $\mathcal{D}$.[4] The invariance properties of the constraint and of the Lagrangian, guarantee that equations of motion (3) define a dynamical system on the principal bundle $\mathcal{D} \longrightarrow \mathcal{D}/G$. If one assumes to directly control the shape variables, the equations of motion read:

$$\begin{cases} \dot{s} = u \\ \dot{g} = g(J(s)p + A(s)u) \\ \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, u\rangle + \langle C(s)u, u\rangle \end{cases}, \tag{4}$$

with $u : t \longmapsto u(t)$ is an admissible control. As above the invariance of both the Lagrangian and the constraint ensures that equations (4) define a dynamical system on the principal bundle $\mathcal{D} \to \mathcal{D}/G$

$$\begin{cases} \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, u\rangle + \langle C(s)u, u\rangle \\ \dot{s} = u \end{cases}, \tag{5}$$

called *reduced nonholonomic shape-control system* and that well be the main objective of study of this note.

[4] Observe that equations (3) are equations (1) in which an external (admissible) control is added at the right hand side as it is usually done for Lagrange equations.

2.1 Basic reconstruction techniques

The reconstruction of the dynamics from reduced equilibria and reduced periodic orbits has been well studied in [10, 12], when the symmetry group is compact and in [1] in the non–compact case. In this subsection we shortly review the basic results of reconstruction theory in the simplest framework, of free and proper group actions. We consider a Lie group G that acts on a manifold M. The freeness and properness of the action guarantee that the quotient space M/G has a manifold structure and $\pi : M \longrightarrow M/G$ is a principal bundle with structural group G. Let X be a G–equivariant vector field on M, then there exists a vector field $\hat{X}$ on M/G, π–related to X.

Let us now introduce the following definitions

Definition 3. – *A flow of a vector field is called quasi-periodic with k frequencies if there exist a differentiable map which conjugates it to a linear flow on a torus $\mathbb{T}^k$.*
- *A flow of a vector field is called spiral flow if it is an action of $\mathbb{R}$ on $\mathbb{T}^k \times \mathbb{R}^n$ as $(t, \alpha, z) \to (\alpha + tw \,(mod\ 1), z + tv)$ with $w \in \mathbb{R}^k$ and $v \in \mathbb{R}^n$, $v \neq 0$.*
- *Let $m_0 \in M$. A G–orbit $\mathcal{O}_{m_0} = G \cdot m_0$ is a relative equilibrium for X, if it is invariant with respect to the flow of X.*
- *A G–invariant subset $\mathcal{P}$ of M is called a relative periodic orbit for X, if its projection, $\hat{\mathcal{P}}$, by $\pi_{\mathcal{P}}$ on the quotient manifold M/G is a periodic orbit of $\hat{X}$. We call a periodic orbit on M/G a loop.*

Let $G \cdot m_0$ be a relative equilibrium, it is well characterised in terms of the Lie algebra $\mathfrak{g}$ of G and of the flow of X. Precisely a G–orbit $G \cdot m_0$ is a relative equilibrium for X if and only if there exists $\xi \in \mathfrak{g}$, called *infinitesimal generator of the relative equilibrium*, such that for all $\tilde{m} \in \mathcal{O}_m$, $X(\tilde{m}) = \xi_M(\tilde{m})$ or equivalently if and only if there exists $\xi \in \mathfrak{g}$, such that for all $\tilde{m} \in \mathcal{O}_m$, $\phi_t^X(\tilde{m}) = \exp(t\xi)\, \tilde{m}$, where ϕ_t^X denotes the flow of X.

Proposition 1. *[10, 12, 1] Let ξ be the infinitesimal generator of a relative equilibrium $\mathcal{O}_{m_0}$ and consider the closed connected abelian subgroup*

$$K(\xi) := clos\{\exp(t\xi) | t \in \mathbb{R}\}$$

of G. Then

(i) if the Lie group G is compact, $K(\xi)$ is a torus, the relative equilibrium is foliated by copies of $K(\xi) \cdot m_0$ and the flow of X on $K(\xi) \cdot m_0$ is quasi–periodic with at most $rank(G)$ frequencies;

(ii) if the Lie group G is non–compact, $K(\xi)$ is a subgroup of G isomorphic either to a torus or to $\mathbb{R}$, the relative equilibrium is foliated by minimal closed flow–invariant subsets conjugate to $K(\xi) \cdot m_0$ and consisting of tori $\mathbb{T}^\kappa$, where $\kappa \leq \dim K(\xi)$, or of copies of $\mathbb{R}$. Therefore the flow of X on the relative equilibrium is either quasi–periodic or a spiral flow.

Let $\mathcal{P}$ be a relative periodic orbit and γ a curve in $\mathcal{P}$. By the periodicity of the reduced dynamics, the integral curves of the complete system, that pass through $\gamma(0)$, returns periodically, with period $\tau > 0$, to the G–orbit through $\gamma(0)$. The freeness of the action of G on M guarantees that $\forall\gamma$ in $\mathcal{P}$ there exists a unique $\mathfrak{p}(\hat{\gamma})$ in G and $\xi \in \mathfrak{g}$ such that

$$\phi_\tau^X(\gamma) = \psi_{\mathfrak{p}(\hat{\gamma})}(\gamma) = \exp(\tau\xi)\gamma(0)\,,$$

where ϕ_τ^X is the flow of X at time τ, ψ_g is the action of G on M, $\hat{\gamma}$ is the projection of γ on M/G with respect to π, the map $\mathfrak{p} : \mathcal{P} \to G$, $\gamma \mapsto \mathfrak{p} = \mathfrak{p}(\hat{\gamma})$ is the so–called *phase* [9], and ξ is the so called *infinitesimal generator of the phase*. The phase p is a piecewise smooth map, constant along the orbits of X (i.e. $\mathfrak{p} \circ \phi_t^X = \mathfrak{p}$, $\forall t$) and it is equivariant with respect to conjugation, that is $\mathfrak{p}(h \cdot \gamma) = h\,\mathfrak{p}(\hat{\gamma})h^{-1}, \quad \forall h \in G, \forall\gamma \in \mathcal{P}$. Then the following Proposition holds.

Proposition 2. *[10, 12, 1] Let $\mathcal{P}$ be a relative periodic orbit of X and let $\hat{\mathcal{P}}$ the projection of $\mathcal{P}$ on M/G with respect to $\pi_{\mathcal{P}}$. Let ξ be the infinitesimal generator of a relative of the phase p of the relative periodic orbit, and consider the closed connected abelian subgroup*

$$K(\xi) := clos\{\exp(t\xi)|t \in \mathbb{R}\}$$

of G. Then

(i) if the group G is compact, $K(\xi)$ is a torus, and the flow of X over $\hat{\mathcal{P}}$ is quasi–periodic with at most $rankG + 1$ frequencies;
(ii) if G is non–compact, $K(\xi)$ is a subgroup of G isomorphic either to a torus or to $\mathbb{R}$, and the flow of X over $\hat{\mathcal{P}}$ is either quasi–periodic, or a spiral flow.

The non–compact case is the most frequent and also the most interesting. For example one can say more on which of the two behaviours (quasi–periodic or spiral) is "generic", by investigating the group G. A behaviour is called *generic* depending on the *codimension* of the two sets

$$\begin{aligned} \mathfrak{g}_{\mathfrak{T}} &:= \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a torus}\} \\ \mathfrak{g}_{\mathfrak{R}} &:= \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a subgroup isomorphic to } \mathbb{R}\}\,. \end{aligned}$$

where ξ is either the infinitesimal generator of a relative equilibrium or of a relative periodic orbit. More precisely, according to [1], if the codimension of $\mathfrak{g}_{\mathfrak{T}}$ (respectively $\mathfrak{g}_{\mathfrak{R}}$) is low, one relative to the other, the corresponding quasi–periodic flow (respectively spiral flow) is *generic*, on the other hand, if its codimension is high, we call the corresponding flow *special*. (For more details and examples see [1]).

3 Trajectory generation via reconstruction theory

Our aim is now to study the trajectory generation for nonholonomic mechanical shape-control systems using reconstruction techniques, in the case in which the

symmetry group is $SE(2)$, which is one of the most common non-compact groups occurring in nonholonomic mechanical systems. If one consider generic controls it is difficult to get information on the reconstructed dynamics. Nevertheless some hints on their qualitative behaviours can be obtained when the flow of the reduced dynamics (6) is an equilibrium or periodic.

3.1 Reconstruction from reduced equilibria and asymptotic controllability

We start by investigating the free dynamics of a nonholonomic mechanical shape control system, that is the dynamics of (4) with frozen shape, i.e. $u = 0$. The behavior of this system can be investigated by applying Proposition 1: if the symmetry group of the system is non-compact the reconstructed trajectory can be either quasiperiodic, if the infinitesimal generator ξ of the relative equilibrium belongs to $\mathfrak{g}_{\mathfrak{T}}$, or spiral if ξ belongs to $\mathfrak{g}_{\mathfrak{R}}$:

- If the initial conditions p_0 and s_0 are such that $(J(s_0) \cdot p_0)_3 = 0$ the system moves along a straight line in the (x, y) plane;
- Otherwise if $(J(s_0) \cdot p_0)_3 \neq 0$ the system moves along a circle in the (x, y) plane.

Starting from these uncontrolled trajectories it is possible to design an algorithm to change trajectory type by controlling the shape parameters.

Theorem 1. *Let us suppose that the group of symmetry is $SE(2)$ and to start from a certain group configuration (x_A, y_A, θ_A) in the plane and suppose that, there exist two reduced equilibria (s_{0_1}, p_{0_1}) and (s_{0_2}, p_{0_2}) such that the flow of the vector field defined by (5) over (s_{0_1}, p_{0_1}) is a circle and the flow over (s_{0_2}, p_{0_2}) is a spiral flow along a certain direction. Moreover suppose that the equilibrium (s_{0_2}, p_{0_2}) is asymptotically stable. Then it is possible to steer the system on a trajectory that in mean asymptotically approaches a straight line motion along a prescribed direction in the (x, y) plane.*

Proof. Without loss of generality assume that $\theta_A = 0$. Let α be the angle between the asymptotic direction of the spiral flow above the equilibrium (s_{0_2}, p_{0_2}) as $t \to \infty$ and the x–axis, and β be the angle between the desired direction and the positive direction of the x–axis. The strategy is the following: use a null control starting from the equilibrium (s_{0_1}, p_{0_1}) until a time $T_1 > 0$ such that $\theta(T_1) = \beta - \alpha$. Then consider the following function

$$f(t) = \begin{cases} s_{0_1} & \text{if } t < 0 \\ s_{0_1} + \frac{2s_{0_2}}{T^2} t^2 & \text{if } 0 \leq t \leq \frac{T}{2} \\ \frac{4s_{0_1}(T_1)}{T^2}(t-T)^2 - \frac{s_{0_2}(2t^2 - 5tT + T^2)}{2T^2} & \text{if } \frac{T}{2} \leq t \leq T \\ s_{0_2} & \text{if } t \geq T \end{cases}.$$

Setting $s(t) = f(t)$ at time $t = T$ the value of the shape $s(T)$ becomes s_{0_2}, thus, for $t > T$, according to our assumption, the asymptotic direction of the

trajectory forms the angle α with the x-axis. Thus using as shape parameter $s(t) = f(t - T_1)$ we will obtain a trajectory that asymptotically approaches a straight line in the desired direction. □

Now let us suppose that the tensor $M(s)$ in (5) is skew-symmetric, fact that holds for many physical examples. Under this assumption, the terms quadratic in p in the momentum equation vanish, and equations (5) re-write

$$\begin{cases} \dot{p} = \langle N(s)p, u\rangle + \langle C(s)u, u\rangle \\ \dot{s} = u \end{cases} . \tag{6}$$

We call the reduced system (6) a *skew reduced nonholonomic mechanical shape-control system.*

Theorem 2. *Consider a skew reduced nonholonomic mechanical shape-control system, let $SE(2)$ be the group of symmetry, and assume that the reduced system (6) is controllable. Then it is possible to steer the overall system (6) from a trajectory which is a circle, to a trajectory that is a straight line motion along a certain direction in the (x, y) plane.*

Proof. The strategy to prove the theorem is the following: exploit the controllability of the reduced system to move from an equilibria that generates a circular trajectory to one which generates a straight line motion along a certain direction.
First of all note that all points $(s, p, u) = (s, p, 0)$ are equilibria of the reduced system (6). Moreover note that for any s for sure there exist $p_1 \neq 0$ such that $(J(s) \cdot p_1)_3 = 0$, indeed p_1 is the orthogonal vector to the third row of the matrix $J(s)$ in (4). Thus fixing $s = s_1$, the equilibrium (s_1, p_1) of the uncontrolled system generates a reconstructed trajectory which is a straight line motion along a certain direction in the plane, which forms an angle α with the x-axis. Since we have seen that for $u = 0$ all points (s, p) are equilibria of the uncontrolled system, and since the codimension of the set $\mathfrak{g}_{\mathfrak{T}}$ is higher than the codimension of $\mathfrak{g}_{\mathfrak{R}}$ a generic equilibrium (s, p) with $p \neq p_1$ will generate a trajectory which is a circle. Let us consider one of these equilibria, (s_2, p_2).
Let us now suppose to start from a certain group configuration (x_A, y_A, θ_A) and without loss of generality we can suppose $\theta_A = 0$. Let β be the angle between the desired direction and the positive direction of the x–axis. The strategy is the following: use a null control starting from the equilibrium (s_2, p_2) until a time $T_1 > 0$ such that $\theta(T_1) = \beta - \alpha$.Then exploiting the controllability of the reduced system, it is possible to find a control $\bar{u}(t) \neq 0$ that make the reduced system moving between (s_2, p_2) to the equilibrium (s_1, p_1) in a time $T > 0$. Therefore summarizing the control to use to steer the system from the circular trajectory to the straight line one should be

$$u(t) = \begin{cases} 0 \quad \text{for } t \leq T_1 \\ \bar{u}(t - T_1) \quad \text{for } T_1 < t \leq T_1 + T \\ 0 \quad \text{for } t > T_1 + T \end{cases} .$$

□

3.2 Reconstruction from relative periodic orbits

Let us now investigate which kind of trajectories can be obtained using periodic controls.

Assumption 1. *For any periodic control, there exist a periodic solution of the reduced equations* (5).

Under this hypothesis, the existence of a reduced periodic orbit for a nonholonomic shape-control system is granted, and the kind of trajectory attainable for the complete system can be investigated using Proposition 2:

- if the infinitesimal generator $\xi := (v, \omega)$ of the phase $\mathfrak{p}$ of the relative periodic orbit is such that $\omega = 0$, the system moves in mean along a straight line in the (x, y) plane;
- otherwise if $\omega \neq 0$ the system moves along a circle in the (x, y) plane.

Theorem 3. *Consider a skew reduced nonholonomic mechanical shape-control system and let $SE(2)$ be the symmetry group, and assume that*

i. Assumption 1 holds;
ii. there exist two reduced periodic orbits $\hat{\mathcal{P}}^{\tau} = (s^{\tau}(t), p^{\tau}(t))$, $\hat{\mathcal{P}}^{\mathfrak{R}} = (s^{\mathfrak{R}}(t), p^{\mathfrak{R}}(t))$ such that the flow of the vector field defined by (5) *over $\hat{\mathcal{P}}^{\tau}$ is a circle and the flow over $\hat{\mathcal{P}}^{\mathfrak{R}}$ is a spiral flow along a certain direction.*

Then:

(a) if the reduced periodic orbit $\hat{\mathcal{P}}^{\mathfrak{R}}$ is an attractor starting from a certain group configuration (x_A, y_A, θ_A), it is possible to steer the system on a trajectory that in mean asymptotically approaches a straight line motion along a prescribed direction in the (x, y) plane;
(b) if the reduced system is controllable, starting from a certain group configuration (x_A, y_A, θ_A), it is possible to steer the system on a trajectory that is a straight line motion along a prescribed direction in the (x, y) plane.

Proof. a) The proof follows the ideas of the proof of Theorem 1, the strategy is the following at first use the periodic control s^{τ} until a time T_1 to change orientation and then use a control function

$$s(t) = \begin{cases} s^{\tau}(T_1) & \text{if } t < 0 \\ s^{\tau}(T_1) + \frac{2s^{\mathfrak{R}}(0)}{T^2} t^2 & \text{if } 0 \leq t \leq \frac{T}{2} \\ \frac{4s^{\tau}(T_1)}{T^2}(t-T)^2 - \frac{s^{\mathfrak{R}}(0)(2t^2 - 5tT + T^2)}{2T^2} & \text{if } \frac{T}{2} \leq t \leq T \\ s^{\mathfrak{R}}(t-T) & \text{if } t \geq T \end{cases},$$

which will bring the system on a reduced trajectory that as t tends to infinity converges to the reduced periodic orbit $\hat{\mathcal{P}}^{\mathfrak{R}}$, thus the global trajectory asymptotically approaches a straight line motion along the desired direction.

b) The proof follows the ideas of Theorem 2, where to change between the orbits we use a control that steers the system to the initial $p^{\mathfrak{R}}(0)$.

□

Remark 1. Note that having a reconstructed trajectory that is a circle over the relative periodic orbit $\hat{\mathcal{P}}^{\tau}$ requires that the frequency of the relative periodic orbit is resonant, so that the quasi periodic reconstructed trajectory, predicted using Proposition 2, is actually periodic with only one frequency.

We stress that Assumption 1 is satisfied under certain conditions for a generic symmetry group G.

Proposition 3. *Consider a skew reduced nonholonomic mechanical shape-control system, and assume that there exists a periodic control $u(t)$ such that the monodromy matrix of the system* (5) *has not eigenvalue one. Then there exists at least one periodic solution $p(t)$ of* (5).

Proof. If the tensor $M(s)$ is skew symmetric, the the term quadratic in p vanishes ending up with reduced equations of type (6). Choosing a periodic control function the equation for p becomes an affine system with periodic coefficients for which we know from [5] that under the assumption that monodromy matrix $\exp(N^T u)$ has not eigenvalue one, there exists at least a periodic solution. □

The presented theorems and propositions are powerful tools for trajectory generation and motion planning in nonholonomic systems. By exploiting the $SE(2)$ symmetry group, together with reconstruction theory from equilibria and relative periodic orbits, and integrating methods from control theory, we establish a framework for characterizing the admissible trajectories in nonholonomic mechanical shape control systems. Furthermore, this approach facilitates the systematic derivation of control inputs capable of steering the system along, or modifying, a given trajectory.

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