Abstract

This thesis is devoted to the study of *twisted* and *intermediate* eigenvalue problems in the context of shape optimization. These problems generalize the classical Dirichlet eigenvalue problems on bounded (quasi-)open sets with additional orthogonality constraints. On the one hand, several well-known properties of the Dirichlet setting, such as the L^∞-bounds for the eigenfunctions à la Davies, extend to this more general framework. On the other hand, not all classical results persist: for instance a Courant-type nodal domain theorem does not hold in general, contrary to prevailing expectations in the mathematical community. Indeed, it is shown a bounded domain (connected open set) whose first twisted (or intermediate) eigenvalue is simple and the corresponding eigenfunction possesses an arbitrary number of nodal domains.

The core of the thesis focuses on the analysis of shape optimization problems for twisted and intermediate eigenvalues. The main results concern both the existence and the non-existence of optimal shapes for the minimization of the k-th twisted and intermediate eigenvalues under volume constraint. Some existence results are obtained in a *local* setting, namely among sets contained in a fixed *box*, while others hold *globally*, over the entire space. These existence results are derived from the introduction of a generalized notion of γ-convergence: while in the twisted case the resulting γ-convergence theory closely parallels the classical Dirichlet setting, in the intermediate case only some of its features persist. Moreover, non-existence results arise from a loss of compactness in the class of admissible sets and can be established via suitable comparison arguments. In addition, a methodology for the numerical computation of optimal shapes within the twisted framework will be presented.

Then, the minimization of the first twisted eigenvalue is studied, not only with respect to the set but also to a single orthogonality function. The minimum value, among sets of fixed volume and functions of given L^∞-bounds, is uniquely reached by a suitable pair of balls and a *bang-bang* function. As a consequence, a one-parameter family of isoperimetric inequalities which, in a sense, interpolates between the classical Faber-Krahn and the Hong-Krahn-Szego inequalities is derived.

Finally, a proof of the Alt–Caffarelli–Friedman monotonicity formula is provided. While this formula is primarily applied in free-boundary problems, it is also relevant for the Lipschitz regularity of (sign-changing) Dirichlet eigenfunctions, thereby suggesting additional applications to the regularity problems considered in this thesis.