Results on a Generalized Fractional Cumulative Entropy

# Results on a Generalized Fractional Cumulative Entropy

Farid Foroghi[1], Saeid Tahmasebi[1],
Mahmoud Afshari[1], Francesco Buono *[2]

[1] Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science, Persian Gulf University, Bushehr, Iran
[2] Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Università degli Studi di Napoli Federico II, Naples, Italy,
Now at Institute of Statistics, RWTH Aachen University, Aachen, Germany

**Abstract**

Recently, a modification of fractional entropy based on the inverse Mittag-Leffler function (MLF) was proposed by Zhang and Shang (2021). In this paper, we present an extension of the fractional cumulative entropy (FCE) and obtain some further results about this measure. We study new equivalent expressions, bounds, stochastic ordering, and properties of dynamic generalized FCE. By using the empirical approach, we give an estimator of this measure and study large sample properties of it. In addition, the validity of this new measure is supported by numerical simulations on logistic map equations. Finally, an application of this measure is proposed in the evaluation of MRI scans for brain cancer.

**Keywords:** Cumulative entropy, Evaluation of MRI scans, Fractional entropy, Financial stock, Inverse Mittag-Leffler function, Logistic map equations, Studying chaos
**AMS Mathematical Subject Classification [2020]:** 60E15, 62B10, 94A17

## 1 Introduction and background

Let $X$ be a discrete random variable which takes values in $\{x_1, x_2, \dots, x_m\}$ with probability mass function vector $\{p_1, p_2, \dots, p_m\}$. Jumarie [13] introduced a new fractional entropy of order $\alpha$ as

$$\tilde{H}_\alpha(X) = -\sum_{i=1}^{m} p_i [Ln_\alpha p_i]^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1, \tag{1}$$

where $Ln_\alpha$ is the inverse Mittag-Leffler function, satisfying $Ln_\alpha 1 = 0$, $Ln_\alpha 0 = -\infty$, $0(Ln_\alpha 0)^{\frac{1}{\alpha}} = 1(Ln_\alpha 1)^{\frac{1}{\alpha}} = 0$, and $Ln_\alpha x < 0$ when $x < 1$. Note that $\tilde{H}_\alpha(X)$ can generate negative entropy. For instance, if $\alpha = 0.5$, then $\tilde{H}_\alpha(X) < 0$. The parameter $\alpha$ considered here for the fractional entropy is related to fractals, even though the fractal dimension is another and different measure of complexity.

For a non-negative and absolutely continuous random variable $X$, with pdf $f$, Ubriaco [20] proposed a new entropy measure known as the fractional entropy

$$H_q(X) = \int_0^{+\infty} f(x)[-\log f(x)]^q dx, \quad 0 < q \leq 1. \tag{2}$$

*francesco.buono3@unina.it

Ubriaco [20] established that the fractional entropy is concave, positive and non-additive. From a physical point of view, it satisfies Lesche and thermodynamic stability. Then, following his work, Machado and Lopes [15] proposed the fractional Rényi entropy and obtained some results by the properties of fractional calculus. In analogy with (2), we can define the fractional cumulative entropy of $X$ by

$$\mathcal{CE}_q(X) \;=\; \int_0^s F(x)[-\log F(x)]^q dx, \;\; 0 < q \leq 1, \tag{3}$$

where $(0, s)$ is the support of $X$. Note that $\mathcal{CE}_1(X)$ is the cumulative entropy of $X$ (for details on the cumulative entropy one may refer to Longobardi [14] and Balakrishnan et al. [2]). This version of the cumulative entropy is defined on the idea of fractional cumulative residual entropy studied by Xiong et al. [23]. Recently, Di Crescenzo et al. [3] proposed a new measure of entropy named fractional generalized cumulative entropy and analyzed several of its properties. In the literature, there are many other formulations of cumulative entropies with interesting properties and applications, see, for instance, Di Crescenzo and Longobardi [5], Irshad et al. [12] and Psarrakos and Toomaj [16].

Recently, Zhang and Shang [26] introduced and studied a modification of the fractional entropy as

$$H_\alpha(X) = \sum_{i=1}^{m} p_i[-Ln_\alpha p_i]^{\frac{1}{\alpha}}, \;\; 0 < \alpha < 1. \tag{4}$$

Note that $H_\alpha(X) \geq 0$. Moreover, the corresponding continuous version is defined by

$$H_\alpha(X) = \int f(x)[-Ln_\alpha f(x)]^{\frac{1}{\alpha}} dx. \tag{5}$$

Here, we recall the following important approximation $Ln_\alpha p \approx \log p^{\alpha!}$ $(0 < \alpha! < 1)$ . Hence, another fractional entropy is proposed to measure the information content

$$K_\alpha(X) = -\sum_{i=1}^{m} p_i(Ln_\alpha p_i).$$

The corresponding version of (5) for the cumulative residual entropy has been studied by Foroghi et al. [9]. Fractional versions of information measures with their applications in complex systems have been proposed and extensively studied in literature (see, for instance, Dong and Zhang [8], Wang and Shang [21] and Zhang and Shang [25]).

In this paper we try to investigate some applications of the generalized FCE. The rest of this paper is organized as follows. Section 2 contains some properties of a new generalized FCE and its dynamic version. Section 3 gives an estimator of generalized FCE by using empirical approach and provides numerical simulations on logistic map equations to show the validity of the generalized FCE. Moreover, the empirical measure of a generalized FCE is applied in financial stock data. Finally, an application of generalized FCE is presented in evaluation of MRI scans for brain cancer. Throughout this paper, it is assumed that the expectation exists when it appears and the terms ‘increasing’ and ‘decreasing’ are used in non-strict sense.

## 2 New generalized FCE and its properties

In this section, we provide some properties of a new generalized FCE. Also, a dynamic (past) version of this measure is considered and studied.

**Definition 1.** Let $X$ be a non-negative continuous random variable with support $(0, s)$. Then, in analogy with the measure given in (4), a generalized FCE (GFCE) is defined by

$$\mathcal{CE}_{\alpha}(X) = \int_0^s F(x)[-Ln_{\alpha}F(x)]^{\frac{1}{\alpha}}dx, \quad 0 < \alpha < 1. \tag{6}$$

**Remark 1.** Let $X$ be an absolutely continuous random variable with support $(0, s)$ and cdf $F$. Another extension of FCE can be proposed in the following way

$$\widetilde{\mathcal{CE}}_{\alpha}(X) = -\int_0^s F(x)\left(Ln_{\alpha}F(x)\right)dx \approx \alpha!\mathcal{CE}(X), \quad 0 < \alpha < 1. \tag{7}$$

In the following proposition, the behavior of GFCE is studied for linear transformations. Moreover, it is shown that GFCE equals 0 if, and only if, the distribution is degenerate. The proof is straightforward and hence it is omitted.

**Proposition 2.1.** *(i) Let $X$ be a random variable with cdf $F$ and let $Y$ be a linear transformation of $X$, $Y = aX + b$, with $a > 0$ and $b \geq 0$. Then $\mathcal{CE}_{\alpha}(Y) = a\mathcal{CE}_{\alpha}(X)$.*

*(ii) Vanishing GFCE characterizes degenerate distributions, i.e., $\mathcal{CE}_{\alpha}(X) = 0$, if and only if, $X$ is degenerate.*

In the following, we give a few examples of the GFCE for some well-known distributions.

**Example 1.** (i) Let $X$ be a random variable with uniform distribution in $(0, b)$, $F_X(x) = \frac{x}{b}$, $0 < x < b$ . Hence, $\mathcal{CE}_{\alpha}(X) \approx b\,(\alpha!)^{\frac{1}{\alpha}}\left(\frac{\Gamma(\frac{\alpha+1}{\alpha})}{2^{\frac{\alpha+1}{\alpha}}}\right)$.

(ii) Let $X$ be a random variable with distribution function $F_X(x) = \exp\left(-\frac{\theta}{x}\right)$, $0 < x, \theta$ . Here, we have $\mathcal{CE}_{\alpha}(X) \approx (\alpha!)^{\frac{1}{\alpha}}\,\theta\Gamma(\frac{1}{\alpha} - 1)$.

(iii) If $X$ has a inverse Weibull distribution with cdf $F_X(x) = \exp(-\lambda x^{-\gamma})$, $0 \leq x$, $\lambda, \gamma > 0$ , then $\mathcal{CE}_{\alpha}(X) \approx (\alpha!)^{\frac{1}{\alpha}}\left(\frac{\lambda^{\frac{1}{\gamma}}\Gamma(\frac{1}{\alpha} - \frac{1}{\gamma})}{\gamma}\right)$.

In the following proposition, a connection between GFCE and the extended fractional cumulative residual entropy is explained for symmetric distributions. The proof is straightforward and hence it is omitted.

**Proposition 2.2.** *Let $X$ be a bounded random variable on $(0, s)$ with a symmetric distribution, i.e., $F(x) = \bar{F}(s - x)$ for all $0 < x < s$. Then, we have*

$$\mathcal{CE}_{\alpha}(X) = \mathcal{E}_{\alpha}(X),$$

*where $\mathcal{E}_{\alpha}(X) = \int_0^s \bar{F}(x)[-Ln_{\alpha}\bar{F}(x)]^{\frac{1}{\alpha}}dx$ is the extended fractional cumulative residual entropy defined and studied in Foroghi et al. [9].*

**Proposition 2.3.** *Let $X$ be a non-negative random variable with cdf $F$ and reversed hazard rate function $r(z), z > 0$, with finite GFCE, i.e., $\mathcal{CE}_{\alpha}(X) < \infty$. Then, for any $0 < \alpha < 1$,*

$$\mathcal{CE}_{\alpha}(X) \approx (\alpha!)^{\frac{1}{\alpha}}\int_0^{+\infty} r(z)\left\{\int_0^z F(x)[-\log F(x)]^{\frac{1}{\alpha}-1}dx\right\}dz. \tag{8}$$

*Proof.* By (6) and the relation $-\log F(x) = \int_x^{\infty} r(z)dz$, we have

$$\mathcal{CE}_{\alpha}(X) \approx (\alpha!)^{\frac{1}{\alpha}} \int_0^{+\infty} \int_x^{\infty} r(z)F(x)[-\log F(x)]^{\frac{1}{\alpha}-1} dzdx.$$

By using Fubini's theorem, we get

$$\mathcal{CE}_{\alpha}(X) \approx (\alpha!)^{\frac{1}{\alpha}} \int_0^{+\infty} \int_0^{z} r(z)F(x)[-\log F(x)]^{\frac{1}{\alpha}-1} dxdz,$$

and the result follows. □

In the following propositions, we obtain some bounds and results of stochastic ordering based on the GFCE. The proof follows on the same lines of the one given by Psarrakos and Toomaj [16].

**Proposition 2.4.** *Let $X$ be an absolutely non-negative random variable with finite GFCE, $\mathcal{CE}_{\alpha}(X) < \infty$. Then, we have*

$$\mathcal{CE}_{\alpha}(X) \geq [\widetilde{\mathcal{CE}}_{\alpha}(X)]^{\frac{1}{\alpha}}, \ \ 0 < \alpha < 1,$$

*where* $\widetilde{\mathcal{CE}}_{\alpha}(X) = -\int_0^{+\infty} F(x)(Ln_{\alpha}F(x))dx.$

*Proof.* From (6) we have

$$\mathcal{CE}_{\alpha}(X) = \int_0^{+\infty} F(x)[-Ln_{\alpha}F(x)]^{\frac{1}{\alpha}} dx \geq \int_0^{+\infty} [-F(x)Ln_{\alpha}F(x)]^{\frac{1}{\alpha}} dx.$$

Since $g(x) = x^{\frac{1}{\alpha}}$, $0 < \alpha < 1$ is a convex function, the Jensen inequality gives

$$\int_0^{+\infty} [-F(x)Ln_{\alpha}F(x)]^{\frac{1}{\alpha}} dx \geq \left(-\int_0^{+\infty} F(x)Ln_{\alpha}F(x)dx\right)^{\frac{1}{\alpha}},$$

and the result follows. □

**Proposition 2.5.** *Let $X$ be a non-negative random variable with cdf $F$ and finite mean $\mu = \mathbb{E}(X) < \infty$. Then,*

$$\widetilde{\mathcal{CE}}_{\alpha}(X) \geq \alpha! gini(X)\mu, \ \ 0 < \alpha < 1,$$

*where $gini(\cdot)$ is the Gini index, a measure of income inequality which can be expressed by*

$$gini(X) = 1 - \frac{\int_0^{\infty} [\bar{F}(x)]^2 dx}{\mu},$$

*see Wang [22].*

*Proof.* The proof follows of Proposition 1 of Rao [18] and the result given in (7). □

**Proposition 2.6.** *Let $X$ be a non-negative random variable with absolutely continuous cdf $F(x)$ and finite GFCE, $\mathcal{CE}_{\alpha}(X) < \infty$. Then,*

$$\mathcal{CE}_{\alpha}(X) = \mathbb{E}\left(Q_{\alpha}(X)\right), \ \ 0 < \alpha < 1, \tag{9}$$

*where* $Q_{\alpha}(x) = \int_x^{\infty} [-Ln_{\alpha}F(w)]^{\frac{1}{\alpha}} dw.$

*Proof.* From (6) and Fubini's theorem, we get

$$\mathcal{CE}_\alpha(X) = \int_0^{+\infty} \left[\int_0^w f(x)dx\right] [-Ln_\alpha F(w)]^{\frac{1}{\alpha}} dw = \int_0^{+\infty} f(x) \left[\int_x^{\infty} [-Ln_\alpha F(w)]^{\frac{1}{\alpha}} dw\right] dx,$$

so that the relation (9) holds. □

**Proposition 2.7.** *Let $X$ be a non-negative random variable with finite mean $\mu = \mathbb{E}(X) < \infty$. Then, we have*

$$\mathcal{CE}_\alpha(X) \geq Q_\alpha(\mu), \quad 0 < \alpha < 1,$$

*where $Q_\alpha(\cdot)$ is defined in Proposition 2.6.*

*Proof.* Since $Q_\alpha(.)$ is a convex function, the proof follows by applying Jensen's inequality in (9). □

Before obtaining a lower bound for $\mathcal{CE}_\alpha$ based on $H_\alpha$, we recall the following important expression introduced by Jumarie [13] in the theory of the inverse MLF

$$[Ln_\alpha uv]^{\frac{1}{\alpha}} = [Ln_\alpha u]^{\frac{1}{\alpha}} + [Ln_\alpha v]^{\frac{1}{\alpha}}.$$

**Proposition 2.8.** *Let $X$ be a non-negative and absolutely continuous random variable with pdf $f$. Then, a lower bound for the GFCE is given by*

$$\mathcal{CE}_\alpha(X) \geq \exp\left(\frac{-1}{\alpha!}\left[-H_\alpha(X) + \int_0^1 \left(-Ln_\alpha u(-Ln_\alpha u)^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}} du\right]^\alpha\right), \tag{10}$$

*where $H_\alpha(X)$ is the fractional entropy defined in* (5).

*Proof.* Using Theorem 3 of Zhan and Shang (2020), we get

$$\begin{aligned}
&\int_0^{\infty} f(x)\left(-Ln_\alpha \frac{f(x)}{F(x)(-Ln_\alpha F(x))^{\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}} dx \\
&= H_\alpha(X) + \int_0^{\infty} f(x)\left(-Ln_\alpha \frac{1}{F(x)(-Ln_\alpha F(x))^{\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}} dx \\
&= H_\alpha(X) - \int_0^{\infty} f(x)\left[-Ln_\alpha F(x)(-Ln_\alpha F(x))^{\frac{1}{\alpha}}\right]^{\frac{1}{\alpha}} dx \\
&\leq \left(-Ln_\alpha \frac{1}{\int_0^\infty F(x)(-Ln_\alpha F(x))^{\frac{1}{\alpha}} dx}\right)^{\frac{1}{\alpha}} \\
&= -\left(-Ln_\alpha \mathcal{CE}_\alpha(X)\right)^{\frac{1}{\alpha}},
\end{aligned} \tag{11}$$

and the result follows. □

In the following definition, we introduce the two-dimensional version of $\mathcal{CE}_\alpha(X)$ .

**Definition 2.** Let $X$ and $Y$ be two random variables with support $(0, s_1)$ and $(0, s_2)$, respectively, and with joint cdf $F(x, y)$. Then, the bivariate GFCE is expressed by

$$\mathcal{CE}_\alpha(X, Y) \;=\; \int_0^{s_2}\int_0^{s_1} F(x,y)[-Ln_\alpha F(x,y)]^{\frac{1}{\alpha}} dxdy, \;\; 0 < x < s_1, \;\; 0 < y < s_2, \; 0 < \alpha < 1.$$

In the following proposition, the two-dimensional version of GFCE is analyzed for independent random variables. The proof follows by applying (10) and hence it is omitted.

**Proposition 2.9.** *Suppose that the non-negative random variables $X$ and $Y$ are independent with supports $(0, s_1)$, $(0, s_2)$, respectively. Then*

$$\mathcal{CE}_\alpha(X,Y) = \mathcal{CE}_\alpha(X)\int_0^{s_2} F(y)dy + \mathcal{CE}_\alpha(Y)\int_0^{s_1} F(x)dx, \quad 0<\alpha<1. \tag{12}$$

In analogy with the normalized CE studied in Di Crescenzo and Longobardi [4], we define the normalized GFCE's as follows:

$$\mathcal{NCE}_\alpha(X) = \frac{\mathcal{CE}_\alpha(X)}{\mathbb{E}(X)}, \qquad \widetilde{\mathcal{NCE}}_\alpha(X) = \frac{\widetilde{\mathcal{CE}}_\alpha(X)}{\mathbb{E}(X)}, \quad 0<\alpha<1.$$

Note that $0 \le \widetilde{\mathcal{NCE}}_\alpha(X) \le 1$. In the following definition, we recall some of the most important stochastic orders which will be useful in the sequel (for more details, see Shaked and Shanthikumar [19]).

**Definition 3.** Let $X$ and $Y$ be two random variables with cdf's $F(x)$ and $G(x)$, respectively. $X$ is said to be smaller than $Y$

1. in the hazard rate order (denoted by $X \overset{hr}{\le} Y$), if $\lambda_X(x) \ge \lambda_Y(x)$ for all $x$.

2. in the decreasing convex order (denoted by $X \overset{dcx}{\le} Y$), if $\mathbb{E}(\phi(X)) \le \mathbb{E}(\phi(Y))$ for any decreasing convex functions $\phi$.

3. in the dispersive order (denoted by $X \overset{disp}{\le} Y$), if $F^{-1}(v) - F^{-1}(v) \le G^{-1}(v) - G^{-1}(u)$ for all $0 < u \le v < 1$, where $F^{-1}$ and $G^{-1}$ are the right-continuous inverses functions of $F$ and $G$, respectively.

Furthermore, in the following definition, we introduce a new stochastic order based on the comparison among GFCE.

**Definition 4.** Let $X$ and $Y$ be two random variables. $X$ is said to be smaller than $Y$ in the GFCE order (denoted by $X \overset{gfce}{\le} Y$) if, for all $0<\alpha<1$, we have $\mathcal{CE}_\alpha(X) \le \mathcal{CE}_\alpha(Y)$.

**Proposition 2.10.** *Let $X$ and $Y$ be two non-negative random variables such that $X \overset{disp}{\le} Y$. Then $X \overset{gfce}{\le} Y$.*

*Proof.* The proof follows in analogy with Proposition 2.16 in Foroghi et al. [9], where it is studied about the fractional cumulative residual entropy, and hence it is omitted. □

**Corollary 2.1.** *Let $X$ and $Y$ be two non-negative random variables such that $X \overset{dcx}{\le} Y$. Then, $X \overset{gfce}{\le} Y$.*

**Corollary 2.2.** *Let $X$ and $Y$ be two non-negative random variables such that $X \overset{hr}{\le} Y$ and let $X$ or $Y$ be DFR (decreasing failure rate). Then, we have $\mathcal{CE}_\alpha(X) \le \mathcal{CE}_\alpha(Y)$.*

*Proof.* If $X \overset{hr}{\leq} Y$ and $X$ or $Y$ is DFR, then $X \overset{disp}{\leq} Y$ (see Bagai and Kochar [1]). Hence, the result follows by Proposition 2.10. □

**Proposition 2.11.** *Let $X$ and $Y$ be two independent non-negative random variables with log-concave densities. Then,*

$$\mathcal{CE}_\alpha(X+Y) \geq max\left\{\mathcal{CE}_\alpha(X), \mathcal{CE}_\alpha(Y)\right\}. \tag{13}$$

*Proof.* The proof is similar to that of Theorem 3.2 of Di Crescenzo and Toomaj [6] and hence it is omitted. □

**Proposition 2.12.** *Let $\phi : (0,\infty) \to (0,\infty)$ be a strictly increasing function such that $\phi'(u) \geq 1$ for all $u > 0$ and define $Y$ as $Y = \phi(X)$. Then, $\mathcal{CE}_\alpha(X) \leq \mathcal{CE}_\alpha(Y)$.*

Hereafter, consider two non-negative random variables $X$ and $X_\theta$ with cdf's $F(x)$ and $F^*(x)$, respectively. These variables satisfy the proportional reversed hazard rate model (PRHM), with proportionality constant $\theta > 0$, if

$$F^*_{X_\theta}(x) = [F(x)]^\theta, \quad x > 0, \tag{14}$$

see Gupta et al. [11]. Now, by using (6) and the formula $Ln_\alpha(x^b) = b^\alpha Ln_\alpha(x)$, we get

$$\mathcal{CE}_\alpha(X_\theta) = \theta \int_0^{+\infty} [F(x)]^\theta [-Ln_\alpha F(x)]^{\frac{1}{\alpha}} dx.$$

**Proposition 2.13.** *Let $X$ be a non-negative absolutely continuous random variable with cdf $F$. Then, the GFCE of $X$ and $X_\theta$ are related, for $\theta \geq 1$, by*

$$\mathcal{CE}_\alpha(X_\theta) \leq \theta \mathcal{CE}_\alpha(X).$$

Suppose that $X$ is the lifetime of a system with cdf $F$. Clearly, the past lifetime $\tilde{X}_t = [X|X < t]$ is a non-negative random variable representing the lifetime of the system given that it has been found down at time $t$. Hence, the cdf of $\tilde{X}_t$ is given by $F_{\tilde{X}_t}(x) = \frac{F(x)}{F(t)}, \ \ x < t$. Analogously, we can also consider the dynamic version of GFCE as

$$\begin{aligned}\mathcal{CE}_\alpha(\tilde{X}_t) := \mathcal{CE}_\alpha(X;t) &= \int_0^t \frac{F(x)}{F(t)} \left[-Ln_\alpha \frac{F(x)}{F(t)}\right]^{\frac{1}{\alpha}} dx \\ &= \frac{1}{F(t)} \int_0^t F(x)\left[-Ln_\alpha F(x)\right]^{\frac{1}{\alpha}} dx - \left[-Ln_\alpha F(t)\right]^{\frac{1}{\alpha}} \int_0^t \frac{F(x)}{F(t)} dx \\ &= \frac{\mathcal{CE}_\alpha(t-X)}{F(t)} - \tilde{M}_X(t)[-Ln_\alpha F(t)]^{\frac{1}{\alpha}}, \ \ t > 0, \ \ 0 < \alpha < 1, \end{aligned} \tag{15}$$

where $\tilde{M}_X(t)$ is the mean past lifetime function. Naturally, $\mathcal{CE}_\alpha(X;0) = \mathcal{CE}_\alpha(X)$. Another version of dynamic $FCE$ is proposed as

$$\widetilde{\mathcal{CE}}_\alpha(\tilde{X}_t) := \widetilde{\mathcal{CE}}_\alpha(X;t) = -\int_0^t \frac{F(x)}{F(t)} Ln_\alpha \frac{F(x)}{F(t)} dx \approx \alpha! \widetilde{\mathcal{CE}}(\tilde{X}_t), \ \ 0 < \alpha < 1, \tag{16}$$

where $\widetilde{\mathcal{CE}}(\tilde{X}_t) = -\int_0^t \frac{F(x)}{F(t)} \log\left(\frac{F(x)}{F(t)}\right) dx$.

**Remark 2.** Let $X$ be a non-negative absolutely continuous random variable with cdf $F$. Then, for any $t > 0$, we have

$$\mathcal{CE}_\alpha(t - X) \geq \tilde{M}_X(t)F(t)[-Ln_\alpha F(t)]^{\frac{1}{\alpha}}.$$

**Proposition 2.14.** *Let $X$ be a non-negative and absolutely continuous random variable with cdf $F$. Then, for any $t > 0$ and $0 < \alpha < 1$, we have*

$$\mathcal{CE}_\alpha(X; t) = \mathbb{E}[Q_\alpha(X; t)|X < t],$$

*where*

$$Q_\alpha(x; t) = \int_x^t \left[-Ln_\alpha \frac{F(z)}{F(t)}\right]^{\frac{1}{\alpha}} dz, \quad 0 < x < t.$$

**Remark 3.** If $X$ is a non-negative and absolutely continuous random variable with cdf $F$, then, for $0 < \alpha < 1$, we have

$$\mathcal{CE}_\alpha(X_t) \geq [\widetilde{\mathcal{CE}}_\alpha(X_t)]^{\frac{1}{\alpha}}.$$

**Remark 4.** Suppose that $X$ is a non-negative random variable with cdf $F$. Then, for any $\theta \geq 1$, we obtain

$$\mathcal{CE}_\alpha(X_\theta; t) \leq \theta \mathcal{CE}_\alpha(X; t).$$

# 3 Applications of empirical measure of $\mathcal{CE}_\alpha(X)$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a continuous cdf $F(x)$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ represent the order statistics of the sample $X_1, X_2, \ldots, X_n$. Then, the empirical measure of $F(x)$ is defined by

$$\hat{F}_n(x) = \begin{cases} 0, & x < X_{(1)}, \\ \frac{i}{n}, & X_{(i)} \leq x < X_{(i+1)}, \quad i = 1, 2, \ldots, n-1 \\ 1, & x \geq X_{(n+1)}. \end{cases}$$

Thus, the empirical measure of $\mathcal{CE}_\alpha(X)$ is obtained as

$$\begin{aligned} \mathcal{CE}_\alpha(\hat{F}_n) &\approx (\alpha!)^{\frac{1}{\alpha}} \int_0^{+\infty} \hat{F}_n(x) \left(-\log \hat{F}_n(x)\right)^{\frac{1}{\alpha}} dx \\ &\approx (\alpha!)^{\frac{1}{\alpha}} \sum_{i=1}^{n-1} U_{i+1} \left(\frac{i}{n}\right) \left[-\log(\frac{i}{n})\right]^{\frac{1}{\alpha}}, \quad 0 < \alpha < 1, \end{aligned} \tag{17}$$

where $U_{i+1} = X_{(i+1)} - X_{(i)}$. Similarly, the empirical measure of $\widetilde{\mathcal{CE}}_\alpha(X)$ is given by

$$\widetilde{\mathcal{CE}}_\alpha(\hat{F}_n) \approx -\alpha! \sum_{i=1}^{n-1} U_{i+1} \left(\frac{i}{n}\right) \log\left(\frac{i}{n}\right).$$

The following theorem asserts that $\mathcal{CE}_\alpha(\hat{F}_n)$ converges almost surely to the $\mathcal{CE}_\alpha(F) := \mathcal{CE}_\alpha(X)$.

**Theorem 3.1.** *Let $X$ be a non-negative and absolutely continuous random variable with cdf $F$ such that $X \in L^p$ for some $p > 2$. Then, we have*

$$\mathcal{CE}_\alpha(\hat{F}_n) \stackrel{a.s}{\to} \mathcal{CE}_\alpha(F) \quad as \quad n \to \infty.$$

*Proof.* The proof follows by using Glivenko-Cantelli theorem and in analogy with Theorem 4.1 in Foroghi et al. [9]. □

In the following examples, we obtain the $\mathcal{CE}_\alpha(\hat{F}_n)$ for uniform and exponential distributions.

**Example 2.** Let $X_1, X_2, \ldots, X_n$ be a random sample with a uniform distribution in $[0,1]$ as parent distribution. Then $U_{i+1}, \quad i = 1, 2, \ldots, n-1$ are independent and follow the beta distribution with parameters 1 and $n$ (see, for instance, Pyke [17]). Hence, by using (17), we obtain the mean and the variance of $\mathcal{CE}_\alpha(\hat{F}_n)$

$$\mathbb{E}\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] \approx \frac{(\alpha!)^{\frac{1}{\alpha}}}{n+1}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)\left[-\log\left(\frac{i}{n}\right)\right]^{\frac{1}{\alpha}}, \tag{18}$$

$$Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] \approx \frac{n(\alpha!)^{\frac{2}{\alpha}}}{(n+1)^2(n+2)}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^2\left[-\log\left(\frac{i}{n}\right)\right]^{\frac{2}{\alpha}}. \tag{19}$$

Note that

$$\lim_{n\to\infty}\mathbb{E}\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] \approx (\alpha!)^{\frac{1}{\alpha}}\left(\frac{\Gamma(\frac{\alpha+1}{\alpha})}{2^{\frac{\alpha+1}{\alpha}}}\right), \qquad \lim_{n\to\infty} Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] = 0.$$

Due to the symmetric properties of the parent distribution and based on Proposition 2.2, the results obtained here for the empirical GFCE are the same of the ones obtained in Example 4.2 in Foroghi et al. [9]. By using the numerical results given there, we can observe that $\mathbb{E}\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ is increasing in $n$ and decreasing in $\alpha$, while $Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ is decreasing in $n$ and $\alpha$. By plotting the estimation of GFCE for a random variable with uniform distribution in $[0,1]$ with the theoretical values, we can observe that $\mathcal{CE}_\alpha(\hat{F}_n)$ is close to $\mathcal{CE}_\alpha(F)$ of the uniform distribution for each value of $\alpha$ (see also Figure 2 in Foroghi et al. [9]).

**Example 3.** Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with parameter $\lambda$ (mean $\frac{1}{\lambda}$). Then, $U_{i+1}, \quad i = 1, 2, \ldots, n-1$ are independent and follow the exponential distribution with mean $\frac{1}{\lambda(n-i)}$ (see, for instance, Pyke [17]). By using (17), the mean and the variance of the $\mathcal{CE}_\alpha(\hat{F}_n)$ are respectively given by

$$\mathbb{E}\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] \approx \frac{(\alpha!)^{\frac{1}{\alpha}}}{n\lambda}\sum_{i=1}^{n-1}\frac{i}{n-i}\left[-\log\left(\frac{i}{n}\right)\right]^{\frac{1}{\alpha}}, \tag{20}$$

and

$$Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] \approx \frac{(\alpha!)^{\frac{2}{\alpha}}}{(n\lambda)^2}\sum_{i=1}^{n-1}\frac{i^2}{(n-i)^2}\left[-\log\left(\frac{i}{n}\right)\right]^{\frac{2}{\alpha}}. \tag{21}$$

In Table 1, the values of the mean (20) and the variance (21) are obtained for sample sizes $n = 15, 20, 30$ with $\alpha = 0.2, 0.3, 0.4, 0.5$ and $\lambda = 0.5, 1, 2$ . Note that $\mathbb{E}\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ is increasing (decreasing) in $n$ ($\alpha$), while $Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ is decreasing in $n$ and $\alpha$. In addition, we remark that

$$\lim_{n\to\infty} Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right] = 0.$$

Table 1: Computed values of $E\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ and $Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ for exponential distribution presented in Example 3.

| | $E\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | 0.5 | 1 | 2 | 0.5 | 1 | 2 | 0.5 | 1 | 2 |
| $n$ | $\alpha=0.5$ | | | $\alpha=0.6$ | | | $\alpha=0.8$ | | |
| 5 | 0.5648 | 0.2824 | 0.1412 | 0.6455 | 0.3228 | 0.1614 | 0.8462 | 0.4231 | 0.2116 |
| 10 | 0.6103 | 0.3052 | 0.1526 | 0.6880 | 0.3440 | 0.1720 | 0.9120 | 0.4560 | 0.2280 |
| 20 | 0.6266 | 0.3133 | 0.1566 | 0.7022 | 0.3511 | 0.1755 | 0.9375 | 0.4688 | 0.2344 |
| 30 | 0.6305 | 0.3153 | 0.1576 | 0.7055 | 0.3527 | 0.1764 | 0.9445 | 0.4722 | 0.2361 |
| 40 | 0.6321 | 0.3161 | 0.1580 | 0.7068 | 0.3534 | 0.1767 | 0.9475 | 0.4738 | 0.2369 |
| 50 | 0.6330 | 0.3165 | 0.1582 | 0.7074 | 0.3537 | 0.1769 | 0.9492 | 0.4746 | 0.2373 |
| 100 | 0.6342 | 0.3171 | 0.1586 | 0.7084 | 0.3542 | 0.1771 | 0.9522 | 0.4761 | 0.2380 |

| | $Var\left[\mathcal{CE}_\alpha(\hat{F}_n)\right]$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | 0.5 | 1 | 2 | 0.5 | 1 | 2 | 0.5 | 1 | 2 |
| $n$ | $\alpha=0.5$ | | | $\alpha=0.6$ | | | $\alpha=0.8$ | | |
| 5 | 0.0913 | 0.0228 | 0.0057 | 0.1083 | 0.0271 | 0.0068 | 0.0057 | 0.0014 | 0.0004 |
| 10 | 0.0490 | 0.0123 | 0.0031 | 0.0561 | 0.0140 | 0.0035 | 0.0956 | 0.0239 | 0.0060 |
| 20 | 0.0250 | 0.0062 | 0.0016 | 0.0282 | 0.0071 | 0.0018 | 0.0486 | 0.0121 | 0.0030 |
| 30 | 0.0167 | 0.0042 | 0.0010 | 0.0188 | 0.0047 | 0.0012 | 0.0325 | 0.0081 | 0.0020 |
| 40 | 0.0125 | 0.0031 | 0.0008 | 0.0141 | 0.0035 | 0.0009 | 0.0244 | 0.0061 | 0.0015 |
| 50 | 0.0100 | 0.0025 | 0.0006 | 0.0113 | 0.0028 | 0.0007 | 0.0196 | 0.0049 | 0.0012 |
| 100 | 0.0050 | 0.0013 | 0.0003 | 0.0057 | 0.0014 | 0.0004 | 0.0098 | 0.0024 | 0.0006 |

In the following, we show a central limit theorem for the empirical measure of $\mathcal{CE}_\alpha(X)$ from an exponential distribution with parameter $\lambda$ (mean $\frac{1}{\lambda}$).

**Theorem 3.2.** *Let $X_1, X_2, \ldots, X_n$ be a random sample from exponential distribution with parameter $\lambda$. Then,*

$$\frac{\mathcal{CE}_\alpha(\hat{F}_n) - \mathbb{E}[\mathcal{CE}_\alpha(\hat{F}_n)]}{\left(Var[\mathcal{CE}_\alpha(\hat{F}_n)]\right)^{1/2}} \quad \overset{d}{\to} \quad \mathcal{N}(0,1).$$

*Proof.* The empirical measure $\mathcal{CE}_\alpha(\hat{F}_n)$ can be approximated by the following sum of independent random variables

$$\mathcal{CE}_\alpha(\hat{F}_n) \approx \sum_{i=1}^{n-1} Y_i, \tag{22}$$

where $Y_i = (\alpha!)^{\frac{1}{\alpha}} U_{i+1}[\frac{i}{n}][-\log(\frac{i}{n})]^{\frac{1}{\alpha}}, \quad i = 1, 2, \ldots, n-1$ are independent with mean and variance given by

$$\mathbb{E}(Y_i) = \frac{(\alpha!)^{\frac{1}{\alpha}} i}{n\lambda(n-i)} \left[-\log\left(\frac{i}{n}\right)\right]^{\frac{1}{\alpha}}, \tag{23}$$

$$Var(Y_i) = \frac{(\alpha!)^{\frac{2}{\alpha}} i^2}{(n\lambda)^2(n-i)^2} \left[-\log\left(\frac{i}{n}\right)\right]^{\frac{2}{\alpha}}. \tag{24}$$

By defining $\beta_{i,r} = \mathbb{E}[|Y_i - E(Y_i)|^r]$, $r = 2, 3$, we get the following approximations for large $n$

$$\begin{aligned}\sum_{i=1}^{n-1} \beta_{i,2} = \sum_{i=1}^{n-1} \mathbb{E}[|Y_i - E(Y_i)|^2] &= \frac{(\alpha!)^{\frac{2}{\alpha}}}{(n\lambda)^2} \sum_{i=1}^{n-1} \frac{i^2}{(n-i)^2} \left[-\log\left(\frac{i}{n}\right)\right]^{\frac{2}{\alpha}} \\ &\approx \frac{(\alpha!)^{\frac{2}{\alpha}} \Gamma(\frac{2}{\alpha}+1)}{\lambda^2 n}.\end{aligned}$$

Hence, by recalling that for the exponential distribution we have $\mathbb{E}[|Y_i - \mathbb{E}(Y_i)|^3] = \frac{2(6-e)[\mathbb{E}(Y_i)]^3}{e}$, we get

$$\begin{aligned}\sum_{i=1}^{m-1} \beta_{i,3} &= \frac{(\alpha!)^{\frac{3}{\alpha}}}{(n\lambda)^3} \sum_{i=1}^{n-1} \left[-\log\left(\frac{i}{n}\right)\right]^{\frac{3}{\alpha}} \\ &\approx \frac{(\alpha!)^{\frac{3}{\alpha}} 2(6-e)\Gamma(\frac{3}{\alpha}+1)}{e\lambda^3 n^2},\end{aligned}$$

for $0 < \alpha < 1$ and large $n$. Finally, the proof is completed by observing that Lyapunovâs condition of the central limit theorem is satisfied since

$$\frac{\left(\sum_{i=1}^m \beta_{i,3}\right)^{1/3}}{\left(\sum_{i=1}^m \beta_{i,2}\right)^{1/2}} \approx \frac{[2(6-e)\Gamma(\frac{3}{\alpha}+1)]^{\frac{1}{3}}}{e^{\frac{1}{3}}[\Gamma(\frac{2}{\alpha}+1)]^{\frac{1}{2}}} (n)^{-1/6} \to 0 \qquad \text{as } n \to \infty.$$

□

Now, in analogy with Xiong et al. [23], we consider the stability of the empirical GFCE.

**Definition 5.** Let $\{\acute{X}_i\}_{i=1,2,\ldots,n}$ be a small deformation of the random sample $\{X_i\}_{i=1,2,\ldots,n}$ taken from a population with cdf $F$. Then, the empirical GFCE is stable if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |X_i - \acute{X}_i| < \delta \Rightarrow |\mathcal{CE}_\alpha(\hat{F}_n(X)) - \mathcal{CE}_\alpha(\hat{F}_n(\acute{X}))| < \epsilon.$$

for all $n \in \mathbb{N}$

In the following theorem we present sufficient condition for the stability of $\mathcal{CE}_\alpha(\hat{F}_n)$.

**Theorem 3.3.** *Let $X$ be a non-negative and absolutely continuous random variable distributed on a finite interval. Then, the empirical GFCE of $X$ is stable.*

*Proof.* Suppose $X$ is distributed on a non-negative finite interval. Then, the empirical GFCE is given by

$$\mathcal{CE}_\alpha(\hat{F}_n) \approx (\alpha!)^{\frac{1}{\alpha}} \sum_{i=1}^{n-1} [X_{(i+1)} - X_{(i)}](F_n(X_{(i)}))[-\log(F_n(X_{(i)}))]^{\frac{1}{\alpha}}, \tag{25}$$

Hence, the proof follows by Theorem 5 of Xiong et al. [23]. □

### 3.1 Numerical simulations of logistic map equations

In the following cases, we consider the validity of the GFCE through simulations on discrete logistic map equations.

**Case 1:** The logistic map has excellent properties in studying chaos. The conventional logistic map equation, which is a polynomial mapping with degree 2, is defined by

$$x_n = \beta x_{n-1}(1 - x_{n-1}), \tag{26}$$

where $x_0 \in [0, 1]$ and $\beta \in [0, 4]$. It is usually regarded as a typical nonlinear dynamic system with chaotic characteristics. Figure 1 (left) illustrates the bifurcation diagram of the logistic map, and Figure 1 (right) presents the GFCE measure for generated series with $\beta = 3.2, 3.4, 3.6, 3.8, 4$, $n = 2000$ and the initial value chosen as $x_0 = 0.1$. If the logistic map is fully chaotic, that is $\beta = 4$, then the degree of randomness is the highest for all possible values of $\alpha$. Observe that the GFCE increases with the increasing of the parameter $\beta$, and so it can properly characterize the difference of uncertainty between chaotic and periodic series. As $\alpha$ increases, the curve gradually tends to a stable level. For the logistic map exhibiting chaotic behavior (i.e., $\beta = 3.8$ and 4), we have higher entropy values than periodic ones. So, these results demonstrate that the GFCE is a valid measure of uncertainty in applications.

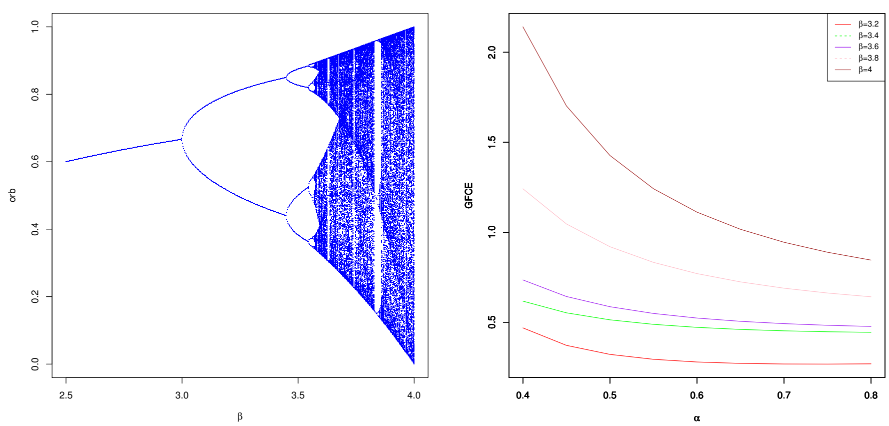


Figure 1: Bifurcation diagram for the logistic map (left). The GFCE of logistic map with varied parameters $\beta$ and sample size is $n = 2000$. The fractional order $\alpha$ is chosen from 0 to 1 with step-size 0.02 (right).

**Case 2:** The best example to study discrete chaotic systems is the fractional order logistic map. The equation of this logistic map is introduced as

$$x_n = x_{n-1} + \frac{r^a}{\Gamma(1+a)} \beta x_{n-1}(1 - x_{n-1}), \tag{27}$$

where $x_0 \in [0, 1]$, $\beta \in [4, 9]$, $r = 0.25$ and $a = 0.8$. With the change of $\beta$, the data have different characteristics like chaotic states and periodical series. The bifurcation diagram for the fractional order logistic map can be seen in Figure 2 (left). Now, we use the sample size $n = 2000$ with the initial value of $x_0 = 0.1$ and the step size 0.02. From Figure 2 (right), it can be clearly observed that the GFCE increases with the increasing of the parameter $\beta$, and then it can properly characterize the difference of uncertainty between chaotic and periodic series.

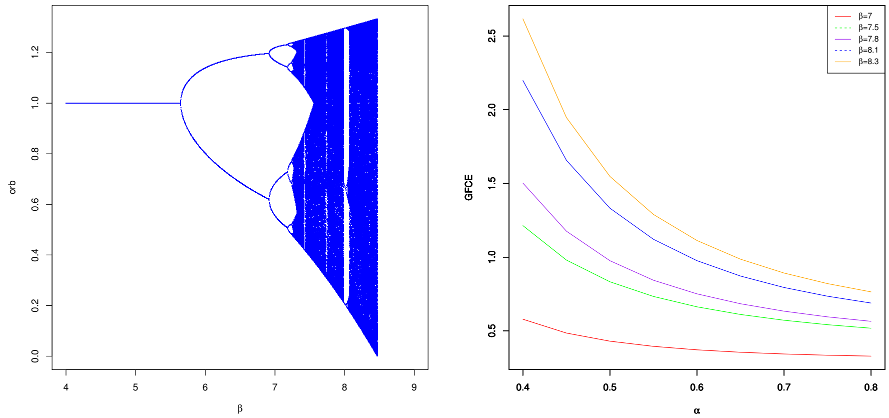


Figure 2: Bifurcation diagram for the order logistic map (left). The GFCE of order logistic map with varied parameters $\beta$ and sample size is $n = 2000$. The fractional order $\alpha$ is chosen from 0 to 1 with step-size 0.02 (right).

As $\alpha$ increases, the curve gradually tends to a stable level. Thus, it is efficient when applied to chaotic systems.

**Case 3:** Another example to study discrete chaotic systems is the fractional generalized logistic map. The equation of this logistic map is given by

$$x_n = \beta x_{n-1}^{a}(1 - x_{n-1}^{b}), \tag{28}$$

where $x_0 \in [2, 4]$, $\beta \in [0, 4]$ and $a = b = 0.6$. By varying the value of $\beta$, the data have different characteristics like chaotic states, periodical series and clusters points. The bifurcation diagram for the fractional order logistic map is given in Figure 3 (left). Here, we choose $n = 2000$ with the initial value $x_0 = 0.1$ and the step size 0.02. Figure 3 (right) illustrates the corresponding fractional GFCE for the generated series. Note that the GFCE can properly characterize the difference of uncertainty between chaotic and periodic series. The chaotic dynamics is produced with $\beta = 3.93$ and $\beta = 4$. They have higher entropy values than periodic ones. When the logistic map is fully chaotic ($\beta = 4$), the degree of randomness is the highest for all $\beta$. Hence, these results show that the GFCE is a credit measure of uncertainty in applications.

### 3.2 Application of $\widetilde{\mathcal{CE}}_\alpha(\hat{F}_n)$ in financial stock data

In this subsection, we apply the empirical measure $\widetilde{\mathcal{CE}}_\alpha(\hat{F}_n)$ to the price returns of Dow Jones Average (DJIA) from 1997 to 2014 (a total of $T = 4500$ data points). All the data are daily closing prices collected from Yahoo Finance [24]. We first calculate the price return as $r_t = \log(x_t) - \log(x_{t-1})$, where $x_t$ is the closing price of the current date and $x_{t-1}$ is the price of previous day. Before calculating the empirical measure $\widetilde{\mathcal{CE}}_\alpha(\hat{F}_n)$ of DJIA, we transform the returns in non-negative values by $y_t = r_t - \min\{r_t\}_{t=1}^{T}$. In Figure 4 it is shown a time series of $y_t$. For evaluating the GFCE of $y_t$, an overlapping sliding time window of $W = 200$ data points with shift size of 100 points is considered. Figure 5 represents the contour plot of $\widetilde{\mathcal{CE}}_\alpha(\hat{F}_n)$ for the transformed returns $y$. The fractional order $\alpha$ is chosen from 0 to 1 with step-size 0.02. On the vertical axis, the time index is the center of each sliding time window. Note that $\widetilde{\mathcal{CE}}_\alpha$, for $0 < \alpha < 1$, has a high sensitivity to the dynamics of the series. There was a crisis during

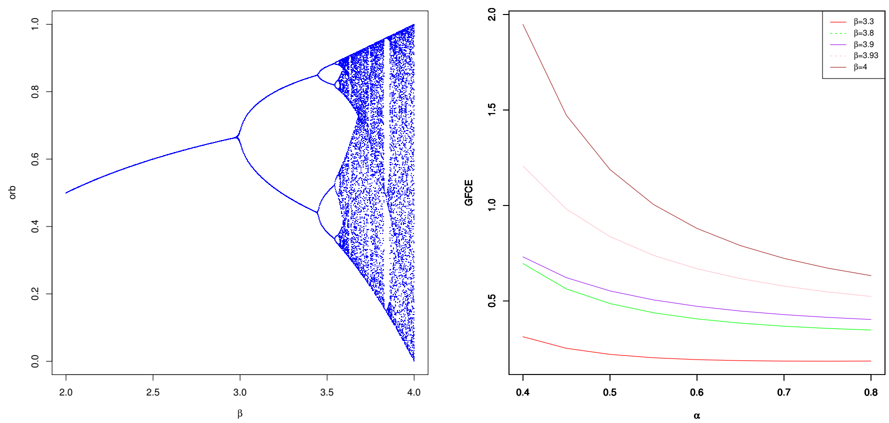


Figure 3: Bifurcation diagram for the generalized logistic map (left). The GFCE of generalized logistic map with arbitrary power with varied parameters $\beta$ and sample size is $n = 2000$. The fractional order $\alpha$ is chosen from 0 to 1 with step-size 0.02 (right).

the years 2008-2010, and the entropy values fluctuate a lot in the region $\alpha < 0.5$. Hence, when $\alpha$ is close to 1, the variations become much less significant, so the classic cumulative entropy is unable to reveal that much information on the financial system compared to $\widetilde{\mathcal{CE}}_\alpha$.

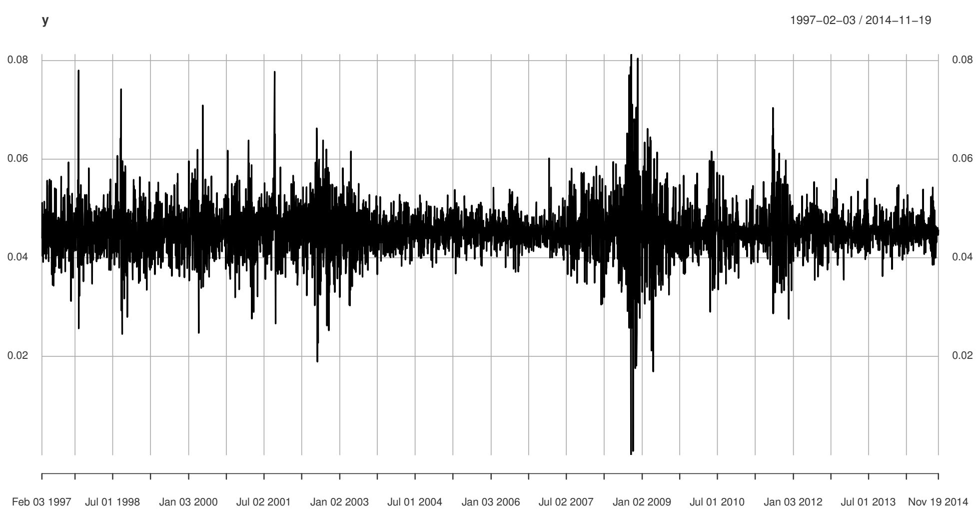


Figure 4: A time series of the transformed returns $y$.

### 3.3 Application in evaluation of MRI scans for brain cancer

In this subsection, we use of $\mathcal{CE}_\alpha(\hat{F}_n)$ to evaluate the MRI scans for a type of brain cancer. It is an aggressive type of brain cancer, glioblastoma, and has no cure. Patients survive an average of 15 months after the diagnosis. A recent study from Washington University in St. Louis has pointed out that timing of chemotherapy could improve treatment for deadly brain cancer. A minor adjustment to the current standard treatment giving chemotherapy in the

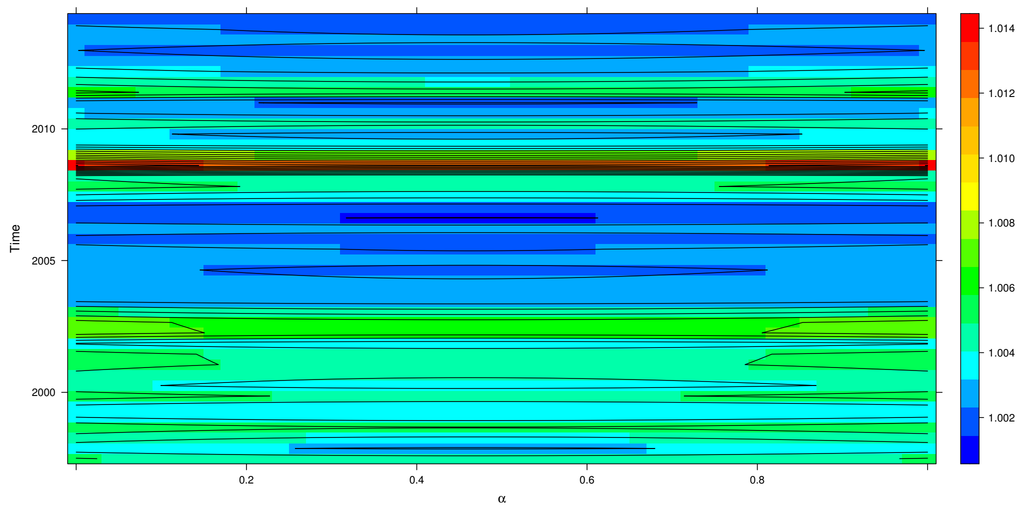


Figure 5: The contour plot of $\widetilde{\mathcal{CE}}_\alpha$ with fractional order $\alpha$ and time index $t$.

morning rather than the evening, could add a few months to patients' survival (see Damato et al. [7]). By resorting to a study made in Foroghi et al. [10] based on a weighted version of entropy, we want to use the GFCE measure to support this hypothesis. We use two MRI brain images of size $377 \times 507$ pixels given in Figure 1 in Foroghi et al. [10] representing MRI scans of the brain of a participant in evaluating chronotherapy based on circadian rhythms for glioblastoma, one for the morning and one for the evening. Figure 6 represents the plots of $\mathcal{CE}_\alpha(\hat{F}_n)$ for MRI scans. The fractional order $\alpha$ is chosen from 0.2 to 0.8 with step-size 0.01. It can be seen that there is significant difference between GFCE measures of two MRI scans. So, this result is interesting because is suggests that chemotherapy or the timed delivery of drugs, for glioblastoma may work better in morning than evening. Also, this result shows that the GFCE is a credit measure of uncertainty in evaluation of MRI scans for brain cancer.

## Conclusions

In this paper, we have proposed and studied a new version of FCE and its dynamic formulation. Some properties of this information measure have been analyzed. Moreover, we have obtained several results including the bivariate version of GFCE, some bounds and connections with stochastic orders. The validity of the new measure has been supported by numerical simulations on logistic map equations. Finally, we have presented an application of this measure in the evaluation of MRI scans for brain cancer, by showing a significant difference between GFCE measures of two MRI scans. So, this result has shown that chemotherapy or the timed delivery of drugs, for glioblastoma may work better in morning than evening, supporting the hypothesis of other recent studies.

## Declarations

**Conflict of interest** The authors declare no conflict of interest.

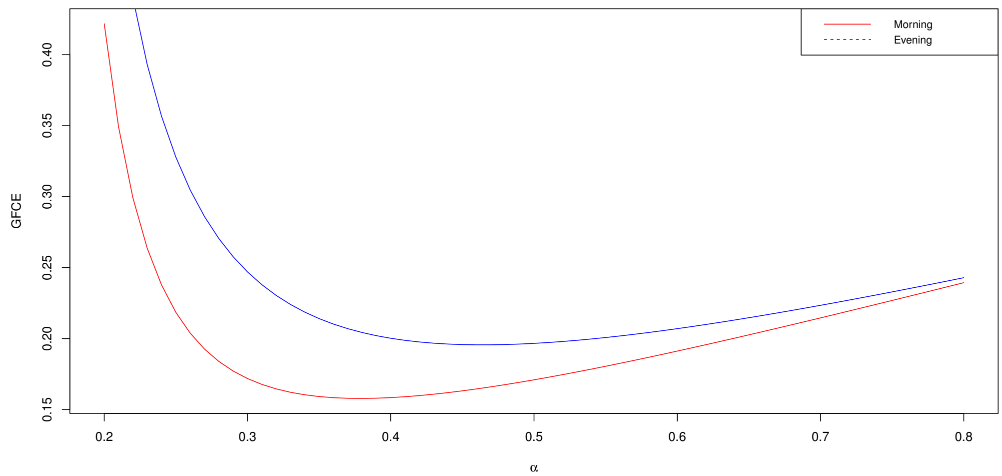


Figure 6: The empirical measures of GFCE for MRI scans of the brain of a participant in a clinical trial.

# Acknowledgements

Francesco Buono is member of the research group GNAMPA of INdAM (Istituto Nazionale di Alta Matematica)and is partially supported by MIUR-PRIN 2017, project "Stochastic Models for Complex Systems", no. 2017 JFFHSH.

# References

[1] Bagai, I., Kochar, S.C. (1986). On tail-ordering and comparison of failure rates. Communications in Statistics - Theory and Methods, 15, 1377-1388.

[2] Balakrishnan, N., Buono, F., Longobardi, M. (2022). On Cumulative Entropies in Terms of Moments of Order Statistics. Methodology and Computing in Applied Probability, 24, 345-359.

[3] Di Crescenzo, A., Kayal, S., Meoli, A. (2021). Fractional generalized cumulative entropy and its dynamic version. Communications in Nonlinear Science and Numerical Simulation, 102 (3), 105899.

[4] Di Crescenzo, A., Longobardi, M. (2009). On cumulative entropies. Journal of Statistical Planning and Inference, 139, 4072-4087.

[5] Di Crescenzo, A., Longobardi, M. (2013). Stochastic Comparisons of Cumulative Entropies. In: Li, H., Li, X. (eds) Stochastic Orders in Reliability and Risk. Lecture Notes in Statistics(), vol 208. Springer, New York, NY. https://doi.org/10.1007/978-1-4614-6892-9_8.

[6] Di Crescenzo, A., Toomaj, A. (2017). Furher results on the generalized cunulative entropy. KYBERNETIKA 53(5), 959-982.

[7] Damato, A. R., Luo, J., Katumba, R. G., Talcott, G. R., Rubin, J. B., Herzog, E. D., Campian, J. L. (2021). Temozolomide chronotherapy in patients with glioblastoma: a retrospective single-institute study. Neuro-Oncology Advances, 3(1), vdab041.

[8] Dong, K., Zhang, X. (2020). Multiscale fractional cumulative residual entropy of higher-order moments for estimating uncertainty. Fluctuation and Noise Letters, 2050038.

[9] Foroghi, F., Tahmasebi, S., Afshari, M., Lak, F. (2022). Extensions of fractional cumulative residual entropy with applications. Communications in Statistics - Theory and Methods, doi:10.1080/03610926.2022.2044493.

[10] Foroghi, F., Tahmasebi, S., Daneshi, S. (2022b). Some Results on a Weighted Fractional Entropy, Proceedings of the 16th Iranian Statistics Conference, University of Mazandaran.

[11] Gupta, R.C., Kannan, N., Raychaudhari, A. (1997). Analysis of lognormal survival data. Math. Biosci. 139, 103-115.

[12] Irshad, M.R., Maya, R., Buono, F., Longobardi, M. (2022). Kernel Estimation of Cumulative Residual Tsallis Entropy and Its Dynamic Version under $\rho$-Mixing Dependent Data. Entropy, 24, 9.

[13] Jumarie, G. (2012). Derivation of an amplitude of information in the setting of a new family of fractional entropies. Information Sciences, 216, 113-137.

[14] Longobardi, M. (2014). Cumulative measures of information and stochastic orders. Ricerche di Matematica, 63(1), 209-223.

[15] Machado, J.A.T., Lopes, A.M. (2019). Fractional Rényi entropy. The European Physical Journal Plus, 134, 217.

[16] Psarrakos, G., Toomaj, A. (2017). On the generalized cumulative residual entropy with applications in actuarial science. Journal of Computational and Applied Mathematics 309, 186–199.

[17] Pyke, R. (1965). Spacings. Journal of the Royal Statistical Society: Series B (Methodological), 27(3), 395-436.

[18] Rao, M. (2005). More on a new concept of entropy and information. Journal of Theoretical Probability, 18(4), 967-981.

[19] Shaked, M., Shanthikumar, J. G. (2007). Stochastic orders. NewYork: Springer.

[20] Ubriaco, M. R. (2009). Entropies based on fractional calculus. Physics Letters A 373(30), 2516–2519.

[21] Wang, Y., Shang, P. (2020). Complexity analysis of time series based on generalized fractional order cumulative residual distribution entropy. Physica A: Statistical Mechanics and its Applications, 537, 122582.

[22] Wang, S. (1998). An actuarial index of the right-tail risk. N. Am. Actuar. J. 2, 88-101.

[23] Xiong, H., Shang, P., Zhang, Y. (2019). Fractional cumulative residual entropy. Communications in Nonlinear Science and Numerical Simulation 78, 104879.

[24] Yahoo Finance, https://finance.yahoo.com.

[25] Zhang, B., Shang, P. (2019). Uncertainty of financial time series based on discrete fractional cumulative residual entropy. Chaos: An Interdisciplinary Journal of Nonlinear Science, 29(10), 103104.

[26] Zhang, B., Shang, P. (2021). Cumulative permuted fractional entropy and its applications. IEEE Transactions on Neural Networks and Learning Systems,32 (11), 4946–55.