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ABSTRACT

We study the ground states for the Schrödinger equation with a focusing nonlinearity and a point interaction in dimension three. We establish that ground states exist for every value of the mass; moreover, they are positive, radially symmetric, and decreasing along the radial direction and show a Coulombian singularity at the location of the point interaction. Remarkably, the existence of the ground states is independent of the attractive or repulsive character of the point interaction.

I. INTRODUCTION

The standard Nonlinear Schrödinger Equation (NLSE) perturbed by a point interaction,

$$i\frac{\partial\psi}{\partial t} = (-\Delta + \alpha\delta_0)\psi \pm |\psi|^{p-2}\psi, \qquad \alpha \neq 0, \quad p > 2, \tag{1}$$

has recently been proposed as an effective model for a Bose–Einstein Condensate (BEC) in the presence of defects or impurities (see, e.g., Refs. 1–3).

While in dimension one, the delta interaction is a bounded perturbation of the Laplacian in the sense of the quadratic forms and the corresponding solutions are widely studied (see, e.g., Refs. 4–10), the analogous problem in higher dimension has been addressed only recently. In particular, well-posedness has been studied in dimensions two and three,[11] whereas properties of the standing waves have been investigated in dimension two[12,13] only.

Here, we extend the results obtained in Ref. 12 to the three-dimensional setting. In particular, we establish the existence and some qualitative properties of the ground states and show that such features are insensitive to the sign of the parameter α. This is in contrast with the case of a particle subject to a point interaction in the absence of a nonlinearity, for which ground states exist only for negative α. This contrast persists even if the point interaction itself bears a nonlinearity, namely, it is of the form $\alpha|\psi|^{p-2}\delta_0$ with $2 < p < 4$ (see Refs. 14–25).

A. Setting and main results

Here, we treat Eq. (1) in $\mathbb{R}^3$ in the *focusing* case. Like in dimension two,[12] in three dimensions, Eq. (1) is formal since the delta interaction is not a bounded perturbation of the Laplacian. The operator $-\Delta + \alpha\delta_0$ is, then, constructed through the theory of self-adjoint extensions of Hermitian operators, which guarantees (see, e.g., Ref. 26) the existence of a family $(H_\alpha)_{\alpha\in\mathbb{R}}$ of self-adjoint operators that realize all point perturbations of $-\Delta$. As a result, the domain of H_α is

$$D(H_\alpha) = \left\{v \in L^2(\mathbb{R}^3) : \exists q \in \mathbb{C} \text{ s.t. } v - \frac{q}{4\pi|x|} = \phi \in \dot{H}^2(\mathbb{R}^3),\ \nabla\phi \in H^1(\mathbb{R}^3) \text{ and } \phi(0) = \alpha q\right\},$$

and its action reads

$$H_\alpha v = -\Delta\phi.$$

The complex number q is called the *charge* of the function v and represents the size of the Coulombian singularity at the origin.

It proves convenient to represent the domain of H_α in an alternative way. One chooses an arbitrary positive number λ, and denoted by $\mathcal{G}_\lambda$ Green's function of $-\Delta + \lambda$, namely,

$$\mathcal{G}_\lambda(\mathbf{x}) := \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}, \tag{2}$$

one gets

$$D(H_\alpha) := \left\{v \in L^2(\mathbb{R}^3) : \exists q \in \mathbb{C} : \text{ s.t. } v - q\mathcal{G}_\lambda =: \phi_\lambda \in H^2(\mathbb{R}^3) \text{ and } \phi_\lambda(0) = \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)q\right\}$$

and

$$H_\alpha v := -\Delta\phi_\lambda - q\lambda\mathcal{G}_\lambda.$$

Note that $\mathcal{G}_\lambda$ is not in $H^1(\mathbb{R}^3)$ and belongs to $L^p(\mathbb{R}^3)$ if and only if $1 \leqslant p < 3$. This entails $D(H_\alpha) \subset L^p(\mathbb{R}^3)$ only if $2 \leqslant p < 3$, which is one of the major differences with the two-dimensional case where the embedding holds for $p > 2$.

In addition, one can see that functions in $D(H_\alpha)$ consist of a *regular part* ϕ_λ, on which the operator H_α acts as the standard Laplacian, and a *singular part* $q\mathcal{G}_\lambda$, on which the operator acts as the multiplication by $-\lambda$. The two components are connected by the boundary condition $\phi_\lambda(0) = \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)q$. We stress that λ is a dumb parameter that does not affect the definition of H_α since for every $\lambda > 0$, any function in $D(H_\alpha)$ can be equivalently decomposed into regular and singular parts.

The quadratic form associated with H_α has domain

$$D := \left\{v \in L^2(\mathbb{R}^3) : \exists q \in \mathbb{C},\ \lambda > 0 \text{ s.t. } v - q\mathcal{G}_\lambda =: \phi_\lambda \in H^1(\mathbb{R}^3)\right\} \tag{3}$$

and action

$$Q(v) := \langle H_\alpha v, v\rangle = \|\nabla\phi_\lambda\|_2^2 + \lambda\left(\|\phi_\lambda\|_2^2 - \|v\|_2^2\right) + \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)|q|^2, \qquad \forall v \in D, \tag{4}$$

where we denote by $\langle\cdot,\cdot\rangle$ the Hermitian product in $L^2(\mathbb{R}^3)$ and by $\|\cdot\|_p$ the usual norm in $L^p(\mathbb{R}^3)$. The value of $Q(v)$ is independent of the choice of λ. Note that in the form domain, no boundary condition is prescribed.

Finally, we denote by $-\omega_\alpha$ the bottom of the spectrum of H_α, so

$$-\omega_\alpha := \inf_{v\in D\setminus\{0\}} \frac{Q(v)}{\|v\|_2^2} = \begin{cases} -(4\pi\alpha)^2 & \text{if } \alpha < 0, \\ 0 & \text{if } \alpha \geqslant 0. \end{cases} \tag{5}$$

Therefore, the continuous spectrum of H_α is $[0, +\infty)$, while the point spectrum of H_α is empty if $\alpha \geqslant 0$ and has the sole negative eigenvalue $-(4\pi\alpha)^2$ if $\alpha < 0$.

We are ready to introduce the rigorous version of Eq. (1), namely,

$$i\frac{\partial\psi}{\partial t} = H_\alpha\psi - |\psi|^{p-2}\psi, \qquad \alpha \in \mathbb{R}, \quad 2 < p < 3. \tag{6}$$

Through this paper, we shall refer to such an equation as δ-NLSE. It is well-known[11] that its flow shows two conservation laws: mass, i.e., L^2-norm, and the *energy*

$$E(v) := \frac{1}{2}Q(v) - \frac{1}{p}\|v\|_p^p, \tag{7}$$

defined for any v in the form domain D.

Hereafter, we focus on the ground states of Eq. (6), according to the following definition.

Definition I.1. Given $\alpha \in \mathbb{R}$, $2 < p < 3$, and $\mu > 0$, we call *ground state at mass* μ every global minimizer of the δ-NLS energy functional defined in (7), among the functions belonging to

$$D_\mu := \{v \in D : \|v\|_2^2 = \mu\}.$$

The main result of this paper is as follows.

Theorem I.2 (δ-NLS ground states). *Let $p \in (2, 3)$ and $\alpha \in \mathbb{R}$. Therefore, for every $\mu > 0$, we have the following.*

(i) *There exists a ground state for the δ-NLS at mass μ.*
(ii) *If $u = \phi_\lambda + q\mathcal{G}_\lambda$ is a ground state, then we have the following.*

(a) *There does not exist $\lambda > 0$ such that ϕ_λ or $q\mathcal{G}_\lambda$ are identically zero.*
(b) *u is positive, radially symmetric, and decreasing along the radial direction, up to multiplication by a constant phase; moreover, ϕ_λ is non-negative if $\lambda = \omega := \mu^{-1}(\|u\|_p^p - Q(u))$ and positive if $\lambda > \omega$.*

Ground states are particular cases of *bound states*, i.e., functions that satisfy

$$u \in D(H_\alpha), \tag{8}$$

$$H_\alpha u + \omega u - |u|^{p-2}u = 0. \tag{9}$$

Bound states are the spatial profiles of *standing waves* in the sense that u is a bound state if and only if $\psi(t, \mathbf{x}) = e^{i\omega t}u(\mathbf{x})$ is a solution to Eq. (6).

In order to prove the qualitative features (iia) and (iib) in Theorem I.2, we will study a class of bound states larger than that of ground states, namely, the set of the minimizers of the action functional S_ω defined as

$$S_\omega : D \to \mathbb{R} \qquad \text{such that} \qquad S_\omega(v) := E(v) + \frac{\omega}{2}\|v\|_2^2, \tag{10}$$

among the functions belonging to Nehari's manifold

$$N_\omega := \{v \in D\backslash\{0\} : I_\omega(v) = 0\}, \tag{11}$$

where $I_\omega : D \to \mathbb{R}$ is given by

$$I_\omega(v) := \langle S'_\omega(v), v\rangle = \|\nabla\phi_\lambda\|_2^2 + \lambda\|\phi_\lambda\|_2^2 + (\omega - \lambda)\|v\|_2^2 + \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)|q|^2 - \|v\|_p^p.$$

The result on the minimizers of the action functional reads as follows.

Theorem I.3 (δ-NLS action minimizers). *Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$. Then, we have the following.*

(i) *A minimizer of the action of the δ-NLS at frequency ω does exist if and only if $\omega > \omega_\alpha$ [defined in (5)].*
(ii) *If $u = \phi_\lambda + q\mathcal{G}_\lambda$ is a minimizer of the action of the δ-NLS at frequency $\omega > \omega_\alpha$, then we have the following.*

(a) *There does not exist $\lambda > 0$ such that ϕ_λ or $q\mathcal{G}_\lambda$ are identically zero.*
(b) *u is positive, radially symmetric, and decreasing along the radial direction, up to multiplication by a constant phase factor; in particular, ϕ_λ is non-negative when $\lambda = \omega$ and positive when $\lambda > \omega$.*

Through the following lemma [whose proof can be found in Ref. 12 (Appendix B) and is an adaptation of what has been established in Ref. 27 for the standard NLSE], we can connect minimizers of the action with ground states.

Lemma I.4. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\mu > 0$. If u is a ground state of mass μ, then it is also a minimizer of the action at the frequency $\omega = \mu^{-1}(\|u\|_p^p - Q(u))$.

In view of this and of Lemma I.4, one can see that point (ii) of Theorem I.2 is a straightforward consequence of Theorem I.3. We also mention that establishing (ii)(b) of Theorem I.3 requires an equivalent formulation of the problem of the minimization of the action. We shall, in fact, minimize

$$Q_\omega(v) := Q(v) + \omega\|v\|^2$$

among the functions in D with the fixed L^p norm. This technique is purely variational and does not retraces the classical ones used for the standard NLSE.

Finally, from Theorems I.2 and I.3, it appears that the sign of α does not affect the behavior of the ground states and of the minimizers of the action. As mentioned at the beginning, this robustness is at first sight, surprising. More precisely, while in dimension two, this is natural as the sign of α does not even affect the existence of ground states for the linear problem, in dimension three, ground states of the linear problem exist only for negative values of α. In other words, the intuitive idea that ground states are deformations of the linear ground state due to the ignition of a nonlinearity is misleading. Such a description is inspired by the fact that minimizers of the action exist if and only if the frequency exceeds the bottom of the spectrum of H_α, which in dimension two coincides with its only eigenvalue. Yet in dimension three, the analogy fails since when $\alpha \geqslant 0$, the eigenvalue disappears, but the minimizers of the action still exist if and only if the frequency exceeds the bottom of the spectrum of H_α.

An intuitive explanation of this phenomenon can be drawn by describing the problem from another point of view. If one interprets the model as a delta perturbation of the NLSE, then one immediately sees that $D \supset H^1(\mathbb{R}^3)$, and so, even perturbing the standard NLSE with a repulsive delta interaction, the infimum of the action gets lower with respect to the infimum of unperturbed action. Thus, the perturbed problem is in any case energetically convenient with respect to the standard one.

Notation. In the following, we use the expressions of *δ-NLS ground states* and *NLS ground states* to refer to the global minimizers of the δ-NLS energy and the standard NLS energy, respectively. We use *δ-NLS action minimizers* and *NLS action minimizers* in an analogous way.

The organization of this paper is as follows.

- Section II introduces some preliminary results that are useful throughout this paper; more precisely, we have the following:
    - in Sec. II A, we recall some well-known features of Green's function of $-\Delta + \lambda$;
    - in Sec. II B, we establish two extensions of the Gagliardo–Nirenberg inequality (Proposition II.1).
- Section III addresses the existence of ground states [Theorem I.2 (i)].
- Section IV addresses the existence of action minimizers [Theorem I.3 (i)].
- Section V establishes the main features of both the ground states and the action minimizers [Theorem I.2 (ii)/Theorem I.3 (ii)].

II. PRELIMINARY RESULTS

In this section, we collect some preliminary results, which will be exploited in the proofs of Theorems I.2 and I.3.

A. Properties of Green's function

First, by (2), one can easily check that $\mathcal{G}_\lambda \in L^r(\mathbb{R}^3)$ for every $r \in [1,3)$, with

$$\|\mathcal{G}_\lambda\|_2^2 = \frac{1}{8\pi\sqrt{\lambda}} \qquad \text{and} \qquad \|\mathcal{G}_\lambda\|_r^r = \frac{\|\mathcal{G}_1\|_r^r}{\lambda^{\frac{3-r}{2}}} \quad \text{when} \quad r \in [1,3), \tag{12}$$

and that

$$\|\mathcal{G}_\lambda - \mathcal{G}_\nu\|_2^2 = \frac{1}{8\pi}\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\nu}} - \frac{4}{\sqrt{\lambda}+\sqrt{\nu}}\right). \tag{13}$$

We recall that $\mathcal{G}_\lambda$ is positive, radially symmetric, and decreasing along the radial direction, exponentially decaying at infinity and smoothening up to the origin. Moreover, $\mathcal{G}_\lambda \notin H^1(\mathbb{R}^3)$ and

$$\|\nabla(\mathcal{G}_\lambda - \mathcal{G}_\nu)\|_2^2 = \frac{1}{8\pi}\left(\frac{3\lambda\sqrt{\nu} - 3\nu\sqrt{\lambda} + \nu\sqrt{\nu} - \lambda\sqrt{\lambda}}{\nu - \lambda}\right). \tag{14}$$

Finally, whenever $\nu < \lambda$,

$$\mathcal{G}_\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|} < \frac{e^{-\sqrt{\nu}|x|}}{4\pi|x|} = \mathcal{G}_\nu(x), \qquad \forall x \in \mathbb{R}^3\backslash\{0\}. \tag{15}$$

B. Gagliardo–Nirenberg inequalities

Here, we aim at finding a version of Gagliardo–Nirenberg inequality for the energy space D, defined by (3).

First, recall the standard three-dimensional Gagliardo–Nirenberg inequality [see, e.g., Ref. 28 (Theorem 1.3.7)]: for every $p \in (2,6)$, there exists $C_p > 0$ such that

$$\|v\|_p^p \leqslant C_p \|\nabla v\|_2^{\frac{3(p-2)}{2}} \|v\|_2^{\frac{6-p}{2}}, \qquad \forall\, v \in H^1(\mathbb{R}^3). \tag{16}$$

Second, every function with $q \neq 0$ can be decomposed into a regular and a singular part according to the choice $\lambda = \varepsilon\frac{|q|^4}{\|u\|_2^4}$, with $\varepsilon > 0$ arbitrarily chosen, i.e.,

$$u = \phi + q\mathcal{G}_{\varepsilon\frac{|q|^4}{\|u\|_2^4}}, \quad \phi \in H^1(\mathbb{R}^3). \tag{17}$$

Using such decomposition, one can prove the following result.

Proposition II.1 (Gagliardo–Nirenberg inequalities). For every $p \in (2,3)$, *there exists* $K_p > 0$ *such that*

$$\|v\|_p^p \leqslant K_p\left(\|\nabla\phi_\lambda\|_2^{\frac{3(p-2)}{2}}\|\phi_\lambda\|_2^{\frac{6-p}{2}} + \frac{|q|^p}{\lambda^{\frac{3-p}{2}}}\right), \qquad \forall v = \phi_\lambda + q\mathcal{G}_\lambda \in D, \quad \forall \lambda > 0. \tag{18}$$

Moreover, there exists $M_{p,\varepsilon} > 0$ *such that*

$$\|v\|_p^p \leqslant M_{p,\varepsilon}\left(\|\nabla\phi\|_2^{\frac{3(p-2)}{2}}\|v\|_2^{\frac{6-p}{2}} + |q|^{3(p-2)}\|v\|_2^{2(3-p)}\right), \qquad \forall v = \phi + q\mathcal{G}_{\varepsilon\frac{|q|^4}{\|v\|_2^4}} \in D\backslash H^1(\mathbb{R}^3). \tag{19}$$

Proof. If we fix $v = \phi_\lambda + q\mathcal{G}_\lambda \in D$, for some $\lambda > 0$, then (16) and (12) yield

$$\|v\|_p^p = \|\phi_\lambda + q\mathcal{G}_\lambda\|_p^p \leqslant 2^{p-1}\left(\|\phi_\lambda\|_p^p + |q|^p\|\mathcal{G}_\lambda\|_p^p\right) \leqslant K_p\left(\|\nabla\phi_\lambda\|_2^{\frac{3(p-2)}{2}}\|\phi_\lambda\|_2^{\frac{6-p}{2}} + \frac{|q|^p}{\lambda^{\frac{3-p}{2}}}\right),$$

that is, (18). On the other hand, if we also assume that $q \neq 0$ and set $\lambda = \lambda_{q,\varepsilon} := \varepsilon\frac{|q|^4}{\|v\|_2^4}$, for some $\varepsilon > 0$, then by (12), (18), and the triangle inequality, we have

$$\begin{aligned}\|v\|_p^p &\leqslant 2^{\frac{p-2}{2}}K_p\left(\|\nabla\phi\|_2^{\frac{3(p-2)}{2}}\|v\|_2^{\frac{6-p}{2}} + \|\nabla\phi\|_2^{\frac{3(p-2)}{2}}\frac{|q|^{\frac{6-p}{2}}}{\lambda_{q,\varepsilon}^{\frac{6-p}{8}}} + \frac{|q|^p}{\lambda_{q,\varepsilon}^{\frac{3-p}{2}}}\right)\\ &\leqslant M_{p,\varepsilon}\left(\|\nabla\phi\|_2^{\frac{3(p-2)}{2}}\|v\|_2^{\frac{6-p}{2}} + |q|^{3(p-2)}\|v\|_2^{2(3-p)}\right),\end{aligned}$$

which concludes the proof. □

Remark II.2. Note that, whenever $q = 0$, i.e., $v \in H^1(\mathbb{R}^3)$, (18) reduces to (16). Note also that, in contrast to the standard Gagliardo–Nirenberg inequality, we must limit ourselves to the powers $p < 3$ since $\mathcal{G}_\lambda \notin L^p(\mathbb{R}^3)$ when $p \geqslant 3$. Finally, we highlight that $M_{p,\varepsilon} := 2^{\frac{p-2}{2}}K_p\max\left\{1, \varepsilon^{\frac{p-p}{8}}, \varepsilon^{\frac{p-3}{2}}\right\}$ and, thus, $M_{p,\varepsilon} \to +\infty$ as $\varepsilon \downarrow 0$.

III. EXISTENCE OF GROUND STATES

Here, we prove point (i) of Theorem I.2, which is the existence of ground states of mass μ for every $\mu > 0$. To this aim, some further notation is required: we denote by $\mathcal{E}(\mu)$ the δ-NLS energy infimum at mass μ, i.e.,

$$\mathcal{E}(\mu) := \inf_{v\in D_\mu} E(v),$$

with E defined in (7), and we denote by $\mathcal{E}^0(\mu)$ the NLS energy infimum at mass μ, i.e.,

$$\mathcal{E}^0(\mu) := \inf_{v\in H^1_\mu(\mathbb{R}^3)} E^0(v),$$

where

$$E^0(v) := \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{p}\|v\|_p^p \qquad \text{and} \qquad H^1_\mu(\mathbb{R}^3) := \{v \in H^1(\mathbb{R}^3) : \|v\|_2^2 = \mu\}.$$

As a preliminary step, we establish boundedness from below of E restricted to D_μ whenever $p \in (2,3)$. By the decomposition introduced in (17), the functional E reads

$$E(u) = \begin{cases} \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{\varepsilon|q|^4\|\phi\|_2^2}{2\|u\|_2^4} + \frac{\alpha|q|^2}{2} + \frac{|q|^4}{2\|u\|_2^2}\left(\frac{\sqrt{\varepsilon}}{4\pi} - \varepsilon\right) - \frac{\|u\|_p^p}{p} & \text{if } u \in D\backslash H^1(\mathbb{R}^3),\\ \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p & \text{if } u \in H^1(\mathbb{R}^3).\end{cases} \tag{20}$$

In addition, if one fixes $\varepsilon < \frac{1}{16\pi^2}$, then the coefficient in front of $\frac{|q|^4}{2\|u\|_2^2}$ is positive. For instance, we choose $\varepsilon = \frac{1}{64\pi^2}$ so that

$$E(u) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{|q|^4\|\phi\|_2^2}{128\pi^2\|u\|_2^4} + \frac{\alpha|q|^2}{2} + \frac{|q|^4}{128\pi^2\|u\|_2^2} - \frac{\|u\|_p^p}{p}, \qquad \forall u \in D\backslash H^1(\mathbb{R}^3). \tag{21}$$

Proposition III.1. For any fixed $p \in (2,3)$ and $\alpha \in \mathbb{R}$,

$$\mathcal{E}(\mu) > -\infty, \qquad \forall \mu > 0.$$

Proof. Let $u \in D_\mu$. We manage separately the cases $u \in H_\mu^1(\mathbb{R}^3)$ and $u \in D_\mu\backslash H_\mu^1(\mathbb{R}^3)$. In the former case, combining (20) and (16), we have

$$E(u) \geqslant \frac{1}{2}\|\nabla u\|_2^2 - \frac{C_p}{p}\|\nabla u\|_2^{\frac{3(p-2)}{2}}\mu^{\frac{6-p}{4}},$$

and thus, E is bounded from below on $H_\mu^1(\mathbb{R}^3)$ as $p \in (2,3)$ by assumption. In the latter case, combining (19) and (21) and denoting by M_p the constant $M_{p,\varepsilon}$ for $\varepsilon = \frac{1}{64\pi^2}$, we get that

$$E(u) \geqslant \frac{\|\nabla\phi\|_2^2}{2} - \frac{M_p\|\nabla\phi\|_2^{\frac{3(p-2)}{2}}\mu^{\frac{6-p}{4}}}{p} + \frac{|q|^4\|\phi\|_2^2}{128\pi^2\mu^2} + \frac{\alpha|q|^2}{2} + \frac{|q|^4}{128\pi^2\mu} - \frac{M_p|q|^{3(p-2)}\mu^{3-p}}{p},$$

and thus, E is also bounded from below on $D_\mu\backslash H_\mu^1(\mathbb{R}^3)$ as again $p \in (2,3)$ by assumption. □

In the following proposition, we compare the infima of the energy of the δ-NLS and of the NLS.

Proposition III.2. For any fixed $p \in (2,3)$ and $\alpha \in \mathbb{R}$,

$$\mathcal{E}(\mu) < \mathcal{E}^0(\mu) < 0, \qquad \forall \mu > 0. \tag{22}$$

The proof of such a proposition requires the following well-known result about the NLS ground states (see, e.g., Refs. 29 and 30).

Proposition III.3. Let $p \in \left(2, \frac{10}{3}\right)$. There exists an NLS ground state of mass μ for every $\mu > 0$. In addition, such a minimizer is unique, positive, radially symmetric, and decreasing along the radial direction, up to multiplication by a constant phase and translations.

Throughout this paper, we denote the positive symmetric NLS ground state of mass μ, usually called *soliton*, by S_μ.

Proof of Proposition III.2. For any $\mu > 0$, S_μ cannot be a δ-NLS ground state of mass μ. Indeed, a δ-NLS ground state has to satisfy the boundary condition in (8), namely, $\phi_\lambda(0) = (\alpha + \frac{\sqrt{\lambda}}{4\pi})q$, but since $S_\mu \in H^1(\mathbb{R}^3)$, $q = 0$ and $S_\mu \equiv \phi_\lambda$ so that $S_\mu(0) = 0$, which contradicts the positivity of S_μ. As a consequence, there must exists $v \in D_\mu$ such that $E(v) < E(S_\mu) = \mathcal{E}^0(\mu)$, which concludes the proof of the former inequality in (22).

Concerning the latter inequality, fix $\mu > 0$ and consider $v \in H_\mu^1(\mathbb{R}^3)$. Using the mass-preserving transformation

$$v_\sigma(x) = \sigma^{\frac{3}{2}}v(\sigma x),$$

one obtains

$$E^0(v_\sigma) = \frac{\sigma^2}{2}\|\nabla v\|_2^2 - \frac{\sigma^{\frac{3(p-2)}{2}}}{p}\|v\|_p^p,$$

and thus, since $p \in (2,3)$, choosing a small enough σ, one gets $\mathcal{E}^0(\mu) \leqslant E^0(v_\sigma) < 0$, and the proof is complete. □

The last two preliminary tools necessary for the proof of point (i) of Theorem I.2 are provided by the next two lemmas and concern the minimizing sequences at mass μ of the δ-NLS energy.

Lemma III.4. Let $p \in (2,3)$ and $\alpha \in \mathbb{R}$. For every minimizing sequence $u_n = \phi_{\lambda,n} + q_n\mathcal{G}_\lambda$ of the δ-NLS energy at mass μ, there exist $\bar{n} \in \mathbb{N}$ and $C > 0$ such that

$$|q_n| > C, \qquad \forall n \geqslant \bar{n}.$$

Proof. Assume by contradiction that there exists a minimizing sequence for the δ-NLS energy at mass μ such that $q_n \to 0$. Then, $\|\phi_{\lambda,n}\|_2^2$ is bounded since it converges to μ. Moreover, combining (7) and (18), one obtains

$$\begin{aligned} E(u_n) &\geqslant \frac{1}{2}\|\nabla\phi_{\lambda,n}\|_2^2 + \frac{\lambda}{2}(\|\phi_{\lambda,n}\|_2^2 - \mu) + \frac{(\alpha + \frac{\sqrt{\lambda}}{4\pi})}{2}|q_n|^2 \\ &\quad - \frac{K_p}{p}\left(\|\nabla\phi_{\lambda,n}\|_2^{\frac{3(p-2)}{2}}\|\phi_{\lambda,n}\|_2^{\frac{6-p}{2}} + \frac{|q_n|^p}{\lambda^{\frac{3-p}{2}}}\right) \\ &= \frac{1}{2}\|\nabla\phi_{\lambda,n}\|_2^2 - \frac{K_p}{p}\|\nabla\phi_{\lambda,n}\|_2^{\frac{3(p-2)}{2}}\|\phi_{\lambda,n}\|_2^{\frac{6-p}{2}} + o(1) \qquad \text{as} \quad n \to +\infty. \end{aligned}$$

Thus, as $E(u_n)$ is bounded from above and $p < 3$, $\|\nabla\phi_{\lambda,n}\|_2$ is bounded.

Now, define the sequence $\xi_n := \frac{\sqrt{\mu}}{\|\phi_{\lambda,n}\|_2}\phi_{\lambda,n}$ so that $\|\xi_n\|_2^2 = \mu$ for every $n \in N$, and $\|\nabla\xi_n\|_2^2 = \frac{\mu}{\|\phi_{\lambda,n}\|_2^2}\|\nabla\phi_{\lambda,n}\|_2^2$ is bounded. Then, as $\phi_{\lambda,n} - u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in [2,3)$, using the properties of ξ_n, we find that

$$\begin{aligned} E(u_n) = E^0(\phi_{\lambda,n}) + o(1) &= E^0(\xi_n) + o(1) \\ &\geqslant E^0(S_\mu) + o(1) \qquad \text{as} \quad n \to +\infty, \end{aligned}$$

where S_μ again denotes the soliton of mass μ. Hence, passing to the limit,

$$\mathcal{E}(\mu) \geqslant \mathcal{E}^0(\mu),$$

which contradicts (22), thus implying that $q_n \nrightarrow 0$. Since this is true for every subsequence of any minimizing sequence of the δ-NLS energy, this concludes the proof. □

Lemma III.5. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, $\mu > 0$, and $(u_n)_n$ be a minimizing sequence of the δ-NLS energy at mass μ. Then, $(u_n)_n$ is bounded in $L^r(\mathbb{R}^3)$ for every $r \in [2,3)$, and there exists $u \in D\backslash H^1(\mathbb{R}^3)$ such that, up to subsequences, the following holds.

- *$u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and*
- *$u_n \to u$ a.e. in $\mathbb{R}^3$.*

Moreover, if one fixes $\lambda > 0$ and considers the decomposition $u_n = \phi_{n,\lambda} + q_n\mathcal{G}_\lambda$, then $(\phi_{n,\lambda})_n$ and $(q_n)_n$ are bounded in $H^1(\mathbb{R}^3)$ and $\mathbb{C}$, respectively, and there exist $\phi_\lambda \in H^1(\mathbb{R}^3)$ and $q \in \mathbb{C}\backslash\{0\}$ such that $u = \phi_\lambda + q\mathcal{G}_\lambda$ such that, up to subsequences, the following holds.

- *$\phi_{n,\lambda} \rightharpoonup \phi_\lambda$ in $L^2(\mathbb{R}^3)$,*
- *$\nabla\phi_{n,\lambda} \rightharpoonup \nabla\phi_\lambda$ in $L^2(\mathbb{R}^3)$, and*
- *$q_n \to q$ in $\mathbb{C}$*

as $n \to +\infty$.

Proof. The proof follows from classical arguments and retraces that of Ref. 12 (Lemma 3.5). We sketch it here for the sake of completeness only.

By the Banach–Alaoglu theorem, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ up to subsequences. Moreover, owing to Lemma III.4, it is not restrictive to assume that $|q_n| > C > 0$ for every $n \in \mathbb{N}$. As a consequence, we can use the decomposition introduced by (17), namely, $u_n = \phi_n + q_n\mathcal{G}_{\nu_n}$ with $\lambda = \nu_n := \frac{1}{64\pi^2}\frac{|q_n|^4}{\|u_n\|_2^4}$, and (19).

Now, arguing as in Ref. 12 (Lemma 3.5) and using (19) and (21), one gets that ϕ_n is bounded in $H^1(\mathbb{R}^3)$ and q_n is bounded in $\mathbb{C}$. Finally, in order to prove the thesis for every $\lambda > 0$ fixed, we argue again as in Ref. 12 (Lemma 3.5), recalling that $\phi_{\lambda,n} := \phi_n + q_n(\mathcal{G}_{\nu_n} - \mathcal{G}_\lambda)$, using (13), (14), and the fact that q_n is bounded from above and away from zero and distinguishing the cases $\lambda \geqslant 1 + (8\pi\mu)^{-2}\sup_n|q_n|^4$ and $\lambda < 1 + (8\pi\mu)^{-2}\sup_n|q_n|^4$. □

Eventually, we can prove the existence of the δ-NLS ground states.

Proof of Theorem I.2 (i). Let $(u_n)_n$ be a minimizing sequence of the δ-NLS energy at mass μ. As we saw before, it is not restrictive to assume that $u_n = \phi_{n,\lambda} + q_n\mathcal{G}_\lambda$ with $q_n \neq 0$ and $\lambda > 0$. Hence, Lemma III.5 applies.

In order to conclude the proof, we argue as follows (note that all the limits below have to be meant up to subsequences). Set $m := \|u\|_2^2$. Weak lower semicontinuity of the $L^2(\mathbb{R}^3)$-norm implies $m \leqslant \mu$, while $q \neq 0$ implies $m \neq 0$. Then, suppose, by contradiction, that $m \in (0, \mu)$. Since $p > 2$ and $\frac{\mu}{\|u_n - u\|_2^2} > 1$ for n sufficiently large, we have that

$$\liminf_n E(u_n - u) \geqslant \frac{\mu - m}{\mu}\mathcal{E}(\mu). \tag{23}$$

On the other hand, it is possible to show in an analogous way that

$$E(u) > \frac{m}{\mu}\mathcal{E}(\mu). \tag{24}$$

Moreover, recalling that $u_n \rightharpoonup u, \phi_{n,\lambda} \rightharpoonup \phi_\lambda, \nabla\phi_{n,\lambda} \rightharpoonup \nabla\phi_\lambda$ in $L^2(\mathbb{R}^3)$ and $q_n \to q$ and using the Brezis–Lieb lemma,[31] we have that

$$E(u_n) = E(u_n - u) + E(u) + o(1) \qquad \text{as} \quad n \to +\infty. \tag{25}$$

Combining (23)–(25), one can see that

$$\mathcal{E}(\mu) = \liminf_n E(u_n) = \liminf_n E(u_n - u) + E(u) > \frac{\mu - m}{\mu}\mathcal{E}(\mu) + \frac{m}{\mu}\mathcal{E}(\mu) = \mathcal{E}(\mu),$$

which is a contradiction. Therefore, $m = \mu$, which means that $u \in D_\mu$ and that $u_n \to u, \phi_{n,\lambda} \to \phi_\lambda$ in $L^2(\mathbb{R}^3)$. Finally, by (18), $u_n \to u$ in $L^p(\mathbb{R}^3)$, and thus,

$$E(u) \leqslant \liminf_n E(u_n) = \mathcal{E}(\mu),$$

which completes the proof. □

IV. EXISTENCE OF THE MINIMIZERS OF THE ACTION

In this section, we prove (i) of Theorem I.3, which is the existence/nonexistence of the minimizers of the δ-NLS action at frequency ω. It is convenient to introduce some notation. First, we denote by $d(\omega)$ the infimum of the δ-NLS action at frequency ω, i.e.,

$$d(\omega) := \inf_{v \in N_\omega} S_\omega(v),$$

where S_ω, N_ω are given by (10) and (11), and we define

$$Q_\omega(v) := Q(v) + \omega\|v\|_2^2,$$

where Q is given by (4), so that

$$S_\omega(v) = \frac{1}{2}Q_\omega(v) - \frac{1}{p}\|v\|_p^p \qquad \text{and} \qquad I_\omega(v) = Q_\omega(v) - \|v\|_p^p.$$

On the other hand, we denote by $d^0(\omega)$ the infimum of the NLS action at frequency ω, i.e.,

$$d^0(\omega) := \inf_{v \in N_\omega^0} S_\omega^0(v),$$

where

$$S_\omega^0(v) := E^0(v) + \frac{\omega}{2}\|v\|_2^2,$$

$$N_\omega^0 := \{v \in H^1(\mathbb{R}^3)\backslash\{0\} : I_\omega^0(v) = 0\}, \qquad I_\omega^0(v) := \langle S_\omega^{0\,\prime}(v), v\rangle.$$

Preliminarily, we note that

$$S_\omega(v) = \widetilde{S}(v) > 0, \qquad \forall v \in N_\omega, \qquad \text{with} \qquad \widetilde{S}(v) := \frac{p-2}{2p}\|v\|_p^p. \tag{26}$$

As a consequence, since $S_\omega|_{H^1(\mathbb{R}^3)} = S_\omega^0$ and $N_\omega \cap H^1(\mathbb{R}^3) = N_\omega^0$, we get that

$$0 \leqslant d(\omega) \leqslant d^0(\omega), \qquad \forall \omega \in \mathbb{R}. \tag{27}$$

Furthermore, since $d^0(\omega) = 0$ for every $\omega \leqslant 0$ [Ref. 27 (Lemma 2.4 and Remark 2.5)], it is straightforward that $d(\omega) = 0$, for every $\omega \leqslant 0$, so that there are no minimizers of the δ-NLS action at $\omega \leqslant 0$. As a consequence, we only address the case $\omega > 0$.

The former step of our proof is to investigate when inequalities (27) are strict. We introduce the set

$$\widehat{N}_\omega := \{q\mathcal{G}_\lambda : \lambda > 0,\ q \in \mathbb{C}\backslash\{0\},\ I_\omega(q\mathcal{G}_\lambda) = 0\},$$

which is the subset of N_ω of the functions that admit a representation with the sole singular part for a value of $\lambda > 0$. We can characterize $\widehat{N}_\omega$ as follows.

Lemma IV.1. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > 0$. Then, $q\mathcal{G}_\lambda \in \widehat{N}_\omega$ if and only if

$$\frac{\omega}{\sqrt{\lambda}} + 8\pi\alpha + \sqrt{\lambda} > 0 \tag{28}$$

and

$$|q| = \frac{\lambda^{\frac{3-p}{2(p-2)}}}{\kappa_p}\left[\frac{\omega}{\sqrt{\lambda}} + 8\pi\alpha + \sqrt{\lambda}\right]^{\frac{1}{p-2}} \tag{29}$$

with $\kappa_p = (8\pi\|\mathcal{G}_1\|_p^p)^{\frac{1}{p-2}}$.

Proof. Let $q \neq 0$ and $\lambda > 0$. By (12), $I_\omega(q\mathcal{G}_\lambda) = 0$ if and only if

$$\frac{\omega-\lambda}{8\pi\sqrt{\lambda}}|q|^2 + \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)|q|^2 - \frac{\kappa}{\lambda^{\frac{3-p}{2}}}|q|^p = 0,$$

with $\kappa := \|\mathcal{G}_1\|_p^p$, so that

$$|q|^{p-2} = \frac{\lambda^{\frac{3-p}{2}}}{\kappa}\left(\frac{\omega-\lambda}{8\pi\sqrt{\lambda}} + \alpha + \frac{\sqrt{\lambda}}{4\pi}\right) = \frac{\lambda^{\frac{3-p}{2}}}{8\pi\kappa}\left(\frac{\omega}{\sqrt{\lambda}} + 8\pi\alpha + \sqrt{\lambda}\right).$$

Since $|q|^{p-2} > 0$, (28) and (29) follow. □

Lemma IV.2. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > 0$. Then, we have the following.

(i) *If $\alpha < 0$ and $\omega \in (0, \omega_\alpha)$, then there exists $\lambda_1(\omega) \in (0, \omega_\alpha)$ and $\lambda_2(\omega) > \omega_\alpha$ such that*

$$\widehat{N}_\omega = \{q\mathcal{G}_\lambda : \lambda \in (0, \lambda_1(\omega)) \cup (\lambda_2(\omega), +\infty) \textit{ and } q \in \mathbb{C}\backslash\{0\} \textit{ and satisfies } (29)\}.$$

In particular, $\lambda_1(\omega)$ and $\lambda_2(\omega)$ are the sole solutions of the following equation:

$$\frac{\omega}{\sqrt{\lambda}} + 8\pi\alpha + \sqrt{\lambda} = 0.$$

(ii) *If $\alpha < 0$ and $\omega = \omega_\alpha$, then*

$$\widehat{N}_\omega = \{q\mathcal{G}_\lambda : \lambda > 0,\ \lambda \neq \omega_\alpha,\ \textit{and } q \in \mathbb{C}\backslash\{0\} \textit{ and satisfies } (29)\}.$$

(iii) *If $\omega > \omega_\alpha$, then*

$$\widehat{N}_\omega = \{q\mathcal{G}_\lambda : \lambda > 0 \textit{ and } q \in \mathbb{C}\backslash\{0\} \textit{ and satisfies } (29)\}$$

[We recall that ω_α was defined in (5)].

Remark IV.3. We highlight that, in the previous result, the case $\alpha \geqslant 0$ is always taken into account by (iii) since in this case, $\omega_\alpha = 0$.

Proof of Lemma IV.2. Let $\omega > 0$ and

$$g(\lambda) := \frac{1}{\lambda^{\frac{p-2}{2}}}\left(\lambda + 8\pi\alpha\sqrt{\lambda} + \omega\right).$$

Recall that in view of Lemma IV.1, $q\mathcal{G}_\lambda \in \widehat{N}_\omega$ if and only if $g(\lambda) > 0$ and q satisfies (29), namely, $|q| = \kappa_p^{-1} g^{\frac{1}{p-2}}(\lambda)$. First, we observe that when $\alpha \geqslant 0$, $\omega_\alpha = 0$ so that g is strictly positive on $\mathbb{R}^+$ for any $\omega > \omega_\alpha$. Thus, (iii) is straightforward in this case.

We focus, then, on $\alpha < 0$. One can easily check that as $p \in (2, 3)$,

$$\lim_{\lambda\to 0^+} g(\lambda) = +\infty, \qquad \lim_{\lambda\to+\infty} g(\lambda) = +\infty.$$

On the other hand, since

$$g'(\lambda) = \frac{(4-p)\lambda + 2(3-p)\pi\alpha\sqrt{\lambda} - (p-2)\omega}{2\lambda^{\frac{p}{2}}},$$

g has a unique critical point on $\mathbb{R}^+$. Hence, there exists $\lambda^* = \lambda^*(\omega)$ such that g is strictly decreasing for $\lambda \leqslant \lambda^*$ and strictly increasing for $\lambda \geqslant \lambda^*$, being $g(\lambda^*)$ the global minimum of g on $\mathbb{R}^+$. In particular, if $\omega > \omega_\alpha$, then g is strictly positive and (28) admits a solution for every $\lambda > 0$ so that (iii) is proved. On the contrary, if $\omega = \omega_\alpha$, then g vanishes for $\lambda = \omega_\alpha$ only and (28) admits a solution whenever $\lambda > 0$ and $\lambda \neq \omega_\alpha$ so that (ii) is proved. Finally, if $\omega < \omega_\alpha$, then g vanishes at two points $\lambda_{1,2}(\omega) = (\sqrt{\omega_\alpha} \mp \sqrt{\omega_\alpha - \omega})^2$ with $\lambda_1(\omega) < \omega_\alpha < \lambda_2(\omega)$ and is negative for $\lambda \in [\lambda_1(\omega), \lambda_2(\omega)]$. Hence, (28) does not admits any solution if and only if $\lambda \in [\lambda_1(\omega), \lambda_2(\omega)]$ so that (i) is proved. □

After this characterization of the set $\widetilde{N}_\omega$, we can estimate the value of $d(\omega)$ for $\alpha < 0$ and $\omega \in (0, \omega_\alpha]$ [note that, in view of Remark IV.3 and the comments after (27), the analogous for $\alpha \geqslant 0$ is trivial].

Proposition IV.4. Let $p \in (2, 3)$ and $\alpha < 0$. Then, $d(\omega) = 0$ for every $\omega \in (0, \omega_\alpha]$.

Proof. We give the proof in the cases $\omega \in (0, \omega_\alpha)$ and $\omega = \omega_\alpha$ separately. If $\omega \in (0, \omega_\alpha)$, then by Lemma IV.2,

$$\lim_{\substack{\lambda \to \lambda_1(\omega)^-,\\ q\mathcal{G}_\lambda \in N_\omega}} |q| = \lim_{\lambda \to \lambda_1(\omega)^-} \frac{\lambda^{\frac{3-p}{2(p-2)}}}{\kappa_p} \left[\frac{\omega}{\sqrt{\lambda}} + 8\pi\alpha + \sqrt{\lambda} \right]^{\frac{1}{p-2}} = 0.$$

Hence, combining with (26) and (12),

$$0 \leqslant d(\omega) \leqslant \inf_{q\mathcal{G}_\lambda \in N_\omega} S_\omega(q\mathcal{G}_\lambda) \leqslant \lim_{\substack{\lambda \to \lambda_1(\omega)^-,\\ q\mathcal{G}_\lambda \in N_\omega}} S_\omega(q\mathcal{G}_\lambda) = \lim_{\substack{\lambda \to \lambda_1(\omega)^-,\\ q\mathcal{G}_\lambda \in N_\omega}} \widetilde{S}(q\mathcal{G}_\lambda) = \lim_{\substack{\lambda \to \lambda_1(\omega)^-,\\ q\mathcal{G}_\lambda \in N_\omega}} \frac{p-2}{2p} \|\mathcal{G}_1\|_p^p \frac{|q|^p}{\lambda^{\frac{3-p}{2}}} = 0.$$

If, on the contrary, $\omega = \omega_\alpha$, then one finds the same chain of inequalities and concludes by replacing the limits for $\lambda \to \lambda_1(\omega)^-$ with the limits for $\lambda \to \omega_\alpha$. □

The first consequence of this result if the following (again, the analogous for $\alpha \geqslant 0$ is omitted since it is trivial).

Corollary IV.5. Let $p \in (2, 3)$ and $\alpha < 0$. If $\omega \in (0, \omega_\alpha]$, then there does not exist any minimizer of the δ-NLS action at frequency ω.

Proof. The claim follows by Proposition IV.4 and (26). □

Before showing the proof of point (i) of Theorem I.3, the last preliminary step is to discuss the behavior of $d(\omega)$ when $\omega > \omega_\alpha$. We start by recalling the following relation between S_ω and $\widetilde{S}$.

Lemma IV.6. For every $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > \omega_\alpha$,

$$d(\omega) = \inf_{v \in \widetilde{N}_\omega} \widetilde{S}(v)$$

with

$$\widetilde{N}_\omega := \{v \in D\backslash\{0\} : I_\omega(v) \leqslant 0\}.$$

In addition,

$$\begin{cases} \widetilde{S}(u) = d(\omega), \\ I_\omega(u) \leqslant 0, \end{cases} \iff \begin{cases} S_\omega(u) = d(\omega), \\ I_\omega(u) = 0, \end{cases} \qquad \forall u \in D\backslash\{0\}.$$

Proof. The proof follows from classical arguments and is analogous to that of Ref. 12 (Lemma 4.5). □

Remark IV.7. Note that this result implies that searching for minimizers of the δ-NLS action means searching for

$$u \in \widetilde{N}_\omega \qquad \text{such that} \qquad \widetilde{S}(u) = \inf_{v \in \widetilde{N}_\omega} \widetilde{S}(v) = d(\omega).$$

Now, the former point is to prove that the left inequality of (27) is strict.

Proposition IV.8. For every $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > \omega_\alpha$, we have that $d(\omega) > 0$.

Proof. We can start by assuming $u \in \widetilde{N}_\omega \cap H^1(\mathbb{R}^3)$. By the Sobolev inequality,

$$0 \geqslant I_\omega(u) = \|\nabla u\|_2^2 + \omega\|u\|_2^2 - \|u\|_p^p \geqslant C_p\|u\|_p^2 + \omega\|u\|_2^2 - \|u\|_p^p \geqslant C_p\|u\|_p^2 - \|u\|_p^p,$$

for some suitable $C_p > 0$ only depending on p. Therefore, $\|u\|_p^{p-2} \geqslant C_p$ so that

$$\widetilde{S}(u) \geqslant \frac{p-2}{2p} C_p^{\frac{p}{p-2}},$$

whence

$$\inf_{v \in \widetilde{N}_\omega \cap H^1(\mathbb{R}^3)} \widetilde{S}(v) \geqslant \frac{p-2}{2p} C_p^{\frac{p}{p-2}} > 0. \tag{30}$$

It is, then, left to study the case $u = \phi_\lambda + q\mathcal{G}_\lambda \in \widetilde{N}_\omega \backslash H^1(\mathbb{R}^3)$. Assume, without loss of generality, that $\lambda \in (\omega_\alpha, \omega)$. Clearly, $\alpha + \frac{\sqrt{\lambda}}{4\pi} > 0$, and thus, there exists a constant $C > 0$ such that

$$\|\nabla\phi_\lambda\|_2^2 + \lambda\|\phi_\lambda\|_2^2 + (\omega - \lambda)\|u\|_2^2 + |q|^2\left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right) \geqslant C\big(\|\phi_\lambda\|_{H^1}^2 + |q|^2\big). \tag{31}$$

In addition, by Sobolev inequality, again,

$$\|u\|_p^p \leqslant C_p\big(\|\phi_\lambda\|_p^p + |q|^p\big) \leqslant C_p\big(\|\phi_\lambda\|_{H^1}^p + |q|^p\big) \leqslant C_p\big(\|\phi_\lambda\|_{H^1}^2 + |q|^2\big)^{\frac{p}{2}},$$

which implies

$$\|\phi_\lambda\|_{H^1}^2 + |q|^2 \geqslant \frac{1}{C_p}\|u\|_p^2. \tag{32}$$

Then, combining (31) and (32),

$$0 \geqslant I_\omega(u) \geqslant C\big(\|\phi_\lambda\|_{H^1}^2 + |q|^2\big) - \|u\|_p^p \geqslant \frac{C}{C_p}\|u\|_p^2 - \|u\|_p^p,$$

and thus, as mentioned before, there exists $K_p > 0$, depending only on p, such that

$$\widetilde{S}(u) \geqslant K_p,$$

whence

$$\inf_{v \in \widetilde{N}_\omega \backslash H^1(\mathbb{R}^3)} \widetilde{S}(v) \geqslant K_p > 0. \tag{33}$$

Finally, the claim follows by combining (30) and (33). □

The latter point is to prove that the right inequality in (27) is strict. To this aim, we mention the following well-known result for the NLS action minimizers at frequency ω (see, e.g., Ref. 28).

Proposition IV.9. Let $p \in (2, 6)$. For every $\omega > 0$, there exists a minimizer of the NLS action at frequency ω. Such a minimizer is unique, positive, radially symmetric, and decreasing along the radial direction, up to the multiplication by a constant phase and translations.

Proposition IV.10. For every $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > \omega_\alpha$, we get that $d(\omega) < d^0(\omega)$.

Proof. The proof is analogous to that of the first part of Proposition III.2. □

Concluding this section, we prove the existence part of Theorem I.3.

Proof of Theorem I.3 (i). The case $\omega \leqslant \omega_\alpha$ is a straightforward consequence of the remarks at the beginning of this section and of Corollary IV.5. On the other hand, in order to treat the case $\omega > \omega_\alpha$, it is sufficient to use Propositions IV.8 and IV.10 and follow the steps of the proof of Ref. 12 (Theorem 1.11). We mention here a brief sketch for the sake of completeness.

Step 1: weak convergence of the minimizing sequences. Let $(u_n)_n$ be a minimizing sequence of the δ-NLS action at frequency $\omega > \omega_\alpha$. By Remark IV.7, it is not restrictive to assume that $(u_n)_n \subset \widetilde{N}_\omega$ and $\widetilde{S}(u_n) \to d(\omega)$. Now, since $\|u_n\|_p^p \to \frac{2p}{p-2}d(\omega)$, u_n is bounded in $L^p(\mathbb{R}^3)$. Hence, as $I_\omega(u_n) \leqslant 0$, setting, for instance, $\lambda = \frac{\omega+\omega_\alpha}{2}$ and using the decomposition $u_n = \phi_{n,\lambda} + q_n\mathcal{G}_\lambda$, one gets that $\phi_{n,\lambda}$ and u_n are bounded in $L^2(\mathbb{R}^3)$ and that q_n is bounded in $\mathbb{C}$. Thus, there exists $\phi_\lambda \in H^1(\mathbb{R}^3)$, $q \in \mathbb{C}$, and $u \in D$ such that $u = \phi_\lambda + q\mathcal{G}_\lambda$ and

$$\nabla\phi_{n,\lambda} \rightharpoonup \nabla\phi_\lambda, \quad \phi_{n,\lambda} \rightharpoonup \phi_\lambda \quad u_n \rightharpoonup u \qquad \text{in} \quad L^2(\mathbb{R}^3) \qquad \text{and} \qquad q_n \to q \qquad \text{in} \quad \mathbb{C}.$$

Step 2: $u \in D\backslash H^1(\mathbb{R}^3)$. Suppose, by contradiction, that $u \in H^1(\mathbb{R}^3)$ so that $q = 0$, and define the sequence $w_n := \sigma_n\phi_{n,\lambda} \in H^1(\mathbb{R}^3)$ in such a way that $I_\omega^0(\sigma_n\phi_{n,\lambda}) = 0$. It is possible to prove that $(\sigma_n^p)_n$ is bounded from above by a sequence $(a_n)_n$ converging to 1. Thus, as $I_\omega^0(w_n) = 0$ and $\widetilde{S}(u_n) \to d(\omega)$,

$$d^0(\omega) + o(1) = \widetilde{S}(w_n) = \sigma_n^p\widetilde{S}(\phi_{n,\lambda}) \leqslant a_n\big(\widetilde{S}(u_n) + o(1)\big) = \widetilde{S}(u_n) + o(1) = d(\omega) + o(1),$$

which implies that $d(\omega) \geqslant d^0(\omega)$, which contradicts Proposition IV.10.

Step 3: $u \in \widetilde{N}_\omega$. In view of step 2, it is left to prove that $I_\omega(u) \leqslant 0$. Assume by contradiction that $I_\omega(u) > 0$. Since u_n is bounded in $L^p(\mathbb{R}^3)$, using the Brezis–Lieb lemma,

$$\widetilde{S}(u_n) - \widetilde{S}(u_n - u) - \widetilde{S}(u) \to 0. \tag{34}$$

Moreover, since $q_n \to q$, $\nabla\phi_{n,\lambda} \rightharpoonup \nabla\phi_\lambda$, $\phi_{n,\lambda} \rightharpoonup \phi_\lambda$, and $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and Q_ω is quadratic, we also have that

$$I_\omega(u_n) - I_\omega(u_n - u) - I_\omega(u) \to 0. \tag{35}$$

Let us prove now that $I_\omega(u_n) \to 0$. Assume by contradiction that $I_\omega(u_n) \nrightarrow 0$. Without loss of generality, we can suppose that $I_\omega(u_n) \to -\beta$ with $\beta > 0$. Consider, then, the sequence $v_n := \theta_n u_n$ such that $I_\omega(v_n) = 0$. An easy computation shows that

$$\theta_n \to \ell < 1.$$

As a consequence,

$$\widetilde{S}(v_n) = \widetilde{S}(\theta_n u_n) = \theta_n^p\widetilde{S}(u_n) \to \ell^p d(\omega) < d(\omega),$$

which is a contradiction. Hence, $I_\omega(u_n) \to 0$. Finally, looking back at (35), since $I_\omega(u) > 0$ and $I_\omega(u_n) \to 0$,

$$I_\omega(u_n - u) = I_\omega(u_n) - I_\omega(u) + o(1) = -I_\omega(u) + o(1),$$

entailing that $I_\omega(u_n - u) \to -I_\omega(u) < 0$. Choose, then, $\bar{n}$ such that $I_\omega(u_n - u) < 0$ for every $n \geqslant \bar{n}$. Since $d(\omega) \leqslant \widetilde{S}(u_n - u)$ and $\widetilde{S}(u) > 0$, (34) yields

$$d(\omega) \leqslant \lim_n \widetilde{S}(u_n - u) = d(\omega) - \widetilde{S}(u) < d(\omega),$$

which is again a contradiction. Thus, $I_\omega(u) \leqslant 0$.

Step 4: conclusion. By the boundedness in $L^p(\mathbb{R}^3)$, $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^3)$, and so, by weak lower semicontinuity,

$$\widetilde{S}(u) \leqslant \liminf_{n\to+\infty} \widetilde{S}(u_n) = d(\omega),$$

which concludes the proof. □

V. Further properties of ground states and action minimizers

This section discusses point (ii) of Theorems I.2 and I.3. We start by proving (ii)(a) of Theorem I.3.

Proposition V.1. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, $\omega > \omega_\alpha$, and u be a minimizer of the δ-NLS action at frequency ω. Then, $q \neq 0$ and $\phi_\lambda := u - q\mathcal{G}_\lambda \neq 0$ for every $\lambda > 0$.

Proof. Consider the decomposition $u = \phi_\lambda + q\mathcal{G}_\lambda$ for a fixed $\lambda > 0$, and suppose by contradiction that $\phi_\lambda = 0$. As $u \neq 0$, it must be $q \neq 0$. In addition, u must also fulfil (8) so that $\alpha + \frac{\sqrt{\lambda}}{4\pi} = 0$. Now, whenever $\alpha \geqslant 0$, this is a contradiction. On the other hand, when $\alpha < 0$, the previous equality entails that $\lambda = \omega_\alpha$. However, since u must also satisfy (9), easy computations yield

$$\omega - \omega_\alpha + |q|^{p-2}|\mathcal{G}_{\omega_\alpha}(x)|^{p-2} = 0, \qquad \forall x \in \mathbb{R}^3\backslash\{0\},$$

which is a contradiction.

It is, then, left to show that $q \neq 0$. Suppose by contradiction that $q = 0$. This would imply that $d(\omega) = d^0(\omega)$, which denies Proposition IV.10. □

In order to prove point (ii)(b) of Theorem I.3, we recall preliminarily that, up to the multiplication by a phase factor, a δ-NLS action minimizer $u = \phi_\lambda + q\mathcal{G}_\lambda$ can be assumed to display a charge $q > 0$ [for details, see Ref. 12 (Sec. 5)]. In addition, we have to turn to the following equivalent minimum problem [for details, see Ref. 12 (Proposition 5.3)].

Proposition V.2. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > \omega_\alpha$. Then,

$$\inf_{v \in D^p_{C(\omega)}} Q_\omega(v) = C(\omega),$$

where $C(\omega) := \frac{2p}{p-2}d(\omega)$ *and* $D^p_{C(\omega)} := \{v \in D : \|v\|_p^p = C(\omega)\}$*, and there exists* $u \in D^p_{C(\omega)}$ *such that* $Q_\omega(u) = C(\omega)$*. In particular,*

$$\begin{cases} Q_\omega(w) = C(\omega), \\ w \in D^p_{C(\omega)}, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} S_\omega(w) = d(\omega), \\ w \in N_\omega. \end{cases}$$

Remark V.3. The main consequence of this result is that we can study the properties of δ-NLS action minimizers at frequency ω by studying the properties of the minimizers of Q_ω on $D^p_{C(\omega)}$ with $C(\omega) = \frac{2p}{p-2}d(\omega)$.

Now, we can prove the first part of (ii)(b), which states positivity up to multiplication by a constant phase.

Proposition V.4. Let $p \in (2,3)$*,* $\alpha \in \mathbb{R}$*, and* $\omega > \omega_\alpha$*. Then, every minimizer of the* δ*-NLS action at frequency* ω *is positive, up to multiplication by a constant phase.*

Proof. Let u be a minimizer of the δ-NLS action at frequency ω. As mentioned before, up to multiplication by a constant phase, we can assume $q > 0$. On the other hand, by Proposition V.2, it is also a minimizer of Q_ω on $D^p_{C(\omega)}$ with $C(\omega) = \frac{2p}{p-2}d(\omega)$. Now, set $\lambda = \omega$ and $\Omega := \{x \in \mathbb{R}^3 : \phi_\omega(x) \neq 0\}$. By Proposition V.1, $|\Omega| > 0$. As a consequence,

$$u(x) = \phi_\omega(x) + q\mathcal{G}_\omega(x) = e^{i\eta(x)}|\phi_\omega(x)| + q\mathcal{G}_\omega(x), \qquad \forall x \in \Omega\backslash\{0\},$$

for some $\eta : \Omega \to [0, 2\pi)$. Hence, showing that $\eta(x) = 0$ for a.e. $x \in \Omega \setminus \{0\}$ implies that $\phi_\omega(x) = |\phi_\omega(x)| \geqslant 0$ for every $x \in \mathbb{R}^3$, whence $u(x) > 0$ for every $x \in \mathbb{R}^3\backslash\{0\}$.

Suppose by contradiction that $\eta \neq 0$ on $\Omega_1 \subset (\Omega \setminus \{0\})$ with $|\Omega_1| > 0$, and define $\widetilde{u} := |\phi_\omega| + q\mathcal{G}_\omega$ (which coincides with u in $\mathbb{R}^3\backslash\Omega_1$). Easy computations yield

$$\begin{aligned} |u(x)|^2 &= |\phi_\omega(x)|^2 + q^2\mathcal{G}_\omega^2(x) + 2\cos(\eta(x))|\phi_\omega(x)|\mathcal{G}_\omega(x) \\ &< |\phi_\omega(x)|^2 + q^2\mathcal{G}_\omega^2(x) + 2|\phi_\omega(x)|\mathcal{G}_\omega(x) = |\widetilde{u}(x)|^2, \qquad \forall x \in \Omega_1, \end{aligned}$$

so that since $|\Omega_1| > 0$,

$$\|u\|_p^p = \int_{\mathbb{R}^3} \left(|u|^2\right)^{\frac{p}{2}} dx < \int_{\mathbb{R}^3} \left(|\widetilde{u}|^2\right)^{\frac{p}{2}} dx = \|\widetilde{u}\|_p^p. \tag{36}$$

On the other hand, one can simply verify that $Q_\omega(\widetilde{u}) \leqslant Q_\omega(u)$. Thus, from (36) and the positivity of Q_ω, there exists $\beta \in (0,1)$ such that $\|\beta\,\widetilde{u}\|_p^p = \|u\|_p^p = C(\omega)$ and

$$Q_\omega(\beta\,\widetilde{u}) = \beta^2 Q_\omega(\widetilde{u}) < Q_\omega(u),$$

which contradicts the fact that u minimizes Q_ω on $D^p_{C(\omega)}$. As a consequence, $\eta = 0$ a.e. on $\Omega \setminus \{0\}$, which concludes the proof. □

Note that the arguments before also imply that the regular part of a minimizer of the δ-NLS action at frequency ω is non-negative when $\lambda = \omega$. In addition, we can prove that whenever $\lambda > \omega$, it is, in fact, positive.

Corollary V.5. Let $p \in (2, 3)$*,* $\alpha \in \mathbb{R}$*,* $\omega > \omega_\alpha$*, and* u *be a minimizer of the* δ*-NLS action at frequency* ω*. Then, the regular part* $\phi_\lambda := u - q\mathcal{G}_\lambda$ *is positive for every* $\lambda > \omega$*, up to multiplication by a constant phase.*

Proof. Let u be a positive minimizer of the δ-NLS action at frequency ω, and consider the decomposition $u = \phi_\lambda + q\mathcal{G}_\lambda$ for a fixed $\lambda > \omega$. By (15), since $q > 0$, one obtains

$$\phi_\lambda(x) = \phi_\omega(x) + q(\mathcal{G}_\omega(x) - \mathcal{G}_\lambda(x)) > 0, \qquad \forall x \in \mathbb{R}^3\backslash\{0\}.$$

On the other hand, as

$$\lim_{x\to 0}(\mathcal{G}_\omega - \mathcal{G}_\lambda)(x) = \frac{\sqrt{\lambda} - \sqrt{\omega}}{4\pi},$$

the claim is proved. □

The last part of this section is devoted to the radially symmetric monotonicity of minimizers of the δ-NLS action. To this aim, we recall the definition and some important properties of the radially decreasing rearrangement of a function.

Given a measurable $A \subset \mathbb{R}^3$ with a finite Lebesgue measure, we denote by A^* the open ball centered at zero with a Lebesgue measure equal to $|A|$, that is,

$$A^* := \left\{x \in \mathbb{R}^3 : \frac{4\pi}{3}|x|^3 < |A|\right\}.$$

In addition, given $f : \mathbb{R}^3 \to \mathbb{R}$, a non-negative measurable function such that $|\{f > t\}| := |\{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) > t\}| < +\infty$ for every $t > 0$, its radially symmetric rearrangement $f^* : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$f^*(\mathbf{x}) = \int_0^\infty \mathbb{1}_{\{f>t\}^*}(\mathbf{x})dt,$$

with $\mathbb{1}_{\{f>t\}^*}$ being the characteristic function of $\{f > t\}^*$. Concerning such a radially symmetric rearrangement, we need in the sequel the following three properties: First,

$$\|f^*\|_p = \|f\|_p, \qquad \forall f \in L^p(\mathbb{R}^3),\ f \geqslant 0, \quad \forall p \geqslant 1. \tag{37}$$

Second, given two non-negative functions $f, g \in L^p(\mathbb{R}^3)$, with $p > 1$, we get that

$$\int_{\mathbb{R}^3} |f + g|^p\, dx \leqslant \int_{\mathbb{R}^3} |f^* + g^*|^p\, dx, \tag{38}$$

and, in particular, if f is radially symmetric and strictly decreasing along the radial direction, then the equality in (38) implies that $g = g^*$ a.e. on $\mathbb{R}^3$ [see Ref. 12 (Proposition 2.3) for the proof]. Finally, if $f \in H^1(\mathbb{R}^3)$, then $f^* \in H^1(\mathbb{R}^3)$ and

$$\|\nabla f^*\|_2 \leqslant \|\nabla f\|_2 \tag{39}$$

(which is usually called *Pólya–Szegö inequality).*

Using these three properties, we can establish the following proposition.

Proposition V.6. Let $p \in (2, 3)$, $\alpha \in \mathbb{R}$, and $\omega > \omega_\alpha$. Then, every minimizer of the δ-NLS action at frequency ω is radially symmetric and decreasing along the radial directions, up to multiplication by a constant phase.

Proof. Assume, without loss of generality, that u is a positive minimizer of the δ-NLS action at frequency ω and fix the decomposition $u = \phi_\omega + q\mathcal{G}_\omega$, with $\lambda = \omega$. We have to show that $\phi_\omega = \phi_\omega^*$. Suppose by contradiction that $\phi_\omega \neq \phi_\omega^*$, and define the function $\widetilde{u} = \phi_\omega^* + q\mathcal{G}_\omega$. By (37) and (39), we have that $\|\nabla\phi_\omega^*\|_2 \leqslant \|\nabla\phi_\omega\|_2$ and $\|\phi_\omega^*\|_2 = \|\phi_\omega\|_2$ so that

$$Q_\omega(\widetilde{u}) \leqslant Q_\omega(u).$$

Now, applying (38) with $f = q\mathcal{G}_\omega$ and $g = \phi_\omega$ in the strict case, we get that $\|\widetilde{u}\|_p^p > \|u\|_p^p$. Therefore, as Q_ω is positive, there exists $\beta < 1$ such that $\|\beta\,\widetilde{u}\|_p^p = \|u\|_p^p$ and

$$Q_\omega(\beta\,\widetilde{u}) = \beta^2 Q_\omega(\widetilde{u}) < Q_\omega(\widetilde{u}) \leqslant Q_\omega(u),$$

but, via Proposition V.2 (arguing as in the Proof of Proposition V.4), this contradicts that u is a δ-NLS action minimizer, thus concluding the proof. □

Finally, we put together all the information we have obtained so far to prove point (ii) of Theorems I.2 and I.3.

Proof of Theorems I.2 and I.3 (ii). Let u be a minimizer of the δ-NLS action at frequency $\omega > \omega_\alpha$. Then, by Proposition V.1, Proposition V.4, Corollary V.5, and Proposition V.6, u satisfies all the properties stated in item (ii).

Let, then, $p \in (2, 3)$ and u be a δ-NLS ground state of mass μ. Combining Lemma I.4 and point (i) of Theorem I.3, one sees that u is also a minimizer of the δ-NLS action at some frequency $\omega > \omega_\alpha$ [in particular, $\omega = \mu^{-1}(\|u\|_p^p - Q(u))$]. Then, one concludes by point (ii) of Theorem I.3. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] W. Caspers and P. Clément, "A different approach to singular solutions," Differ. Integr. Equations **7**(5–6), 1227–1240 (1994); available at https://link.springer.com/content/pdf/10.1007/s00208-022-02382-z.pdf.
[2] H. Sakaguchi and B. A. Malomed, "Singular solitons," Phys. Rev. E **101**(1), 012211 (2020).
[3] E. Shamriz, Z. Chen, B. A. Malomed, and H. Sakaguchi, "Singular mean-field states: A brief review of recent results," Condens. Matter **5**(1), 20 (2020).
[4] R. Adami, F. Boni, and S. Dovetta, "Competing nonlinearities in NLS equations as source of threshold phenomena on star graphs," J. Funct. Anal. **283**(1), 109483 (2022).
[5] R. Adami, T. Nakamura, and A. Ruighi, "Discontinuous ground states for NLSE on $\mathbb{R}$ with a Fülöp-Tsutsui δ interaction," arXiv:2010.00895 [math.AP] (2020).
[6] R. Adami and D. Noja, "Existence of dynamics for a 1D NLS equation perturbed with a generalized point defect," J. Phys. A: Math. Theor. **42**(49), 495302 (2009).
[7] R. Adami and D. Noja, "Stability and symmetry-breaking bifurcation for the ground states of a NLS with a δ' interaction," Commun. Math. Phys. **318**(1), 247–289 (2013).
[8] R. Adami, D. Noja, and N. Visciglia, "Constrained energy minimization and ground states for NLS with point defects," Discrete Contin. Dyn. Syst. Ser. B **18**(5), 1155–1188 (2013).
[9] F. Boni and S. Dovetta, "Prescribed mass ground states for a doubly nonlinear Schrödinger equation in dimension one," J. Math. Anal. Appl. **496**(1), 124797 (2021).
[10] S. Le Coz, R. Fukuizumi, G. Fibich, B. Ksherim, and Y. Sivan, "Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential," Physica D **237**(8), 1103–1128 (2008).
[11] C. Cacciapuoti, D. Finco, and D. Noja, "Well posedness of the nonlinear Schrödinger equation with isolated singularities," J. Differ. Equations **305**, 288–318 (2021).
[12] R. Adami, F. Boni, R. Carlone, and L. Tentarelli, "Ground states for the planar NLSE with a point defect as minimizers of the constrained energy," arXiv:2109.09482 [math.AP] (2021).
[13] N. Fukaya, V. Georgiev, and M. Ikeda, "On stability and instability of standing waves for 2d-nonlinear Schrödinger equations with point interaction," J. Differ. Equations **321**, 258–295 (2022).
[14] R. Adami, R. Carlone, M. Correggi, and L. Tentarelli, "Blow-up for the pointwise NLS in dimension two: Absence of critical power," J. Differ. Equations **269**(1), 1–37 (2020).
[15] R. Adami, R. Carlone, M. Correggi, and L. Tentarelli, "Stability of the standing waves of the concentrated NLSE in dimension two," Math. Eng. **3**(2), 1–15 (2021).
[16] R. Adami, G. Dell'Antonio, R. Figari, and A. Teta, "The Cauchy problem for the Schrödinger equation in dimension three with concentrated nonlinearity," Ann. Inst. Henri Poincare **20**(3), 477–500 (2003).
[17] R. Adami, G. Dell'Antonio, R. Figari, and A. Teta, "Blow-up solutions for the Schrödinger equation in dimension three with a concentrated nonlinearity," Ann. Inst. Henri Poincare **21**(1), 121–137 (2004).
[18] R. Adami, R. Fukuizumi, and J. Holmer, "Scattering for the L^2 supercritical point NLS," Trans. Am. Math. Soc. **374**(1), 35–60 (2021).
[19] R. Adami, D. Noja, and C. Ortoleva, "Orbital and asymptotic stability for standing waves of a nonlinear Schrödinger equation with concentrated nonlinearity in dimension three," J. Math. Phys. **54**(1), 013501 (2013).
[20] R. Adami, D. Noja, and C. Ortoleva, "Asymptotic stability for standing waves of a NLS equation with subcritical concentrated nonlinearity in dimension three: Neutral modes," Discrete Contin. Dyn. Syst. Ser. B **36**(11), 5837–5879 (2016).
[21] R. Adami and A. Teta, "A class of nonlinear Schrödinger equations with concentrated nonlinearity," J. Funct. Anal. **180**(1), 148–175 (2001).
[22] R. Carlone, M. Correggi, and L. Tentarelli, "Well-posedness of the two-dimensional nonlinear Schrödinger equation with concentrated nonlinearity," Ann. Inst. Henri Poincare **36**(1), 257–294 (2019).
[23] R. Carlone, D. Finco, and L. Tentarelli, "Nonlinear singular perturbations of the fractional Schrödinger equation in dimension one," Nonlinearity **32**(8), 3112–3143 (2019).
[24] J. Holmer and C. Liu, "Blow-up for the 1D nonlinear Schrödinger equation with point nonlinearity I: Basic theory," J. Math. Anal. Appl. **483**(1), 123522 (2020).
[25] J. Holmer and C. Liu, "Blow-up for the 1D nonlinear Schrödinger equation with point nonlinearity II: Supercritical blow-up profiles," Commun. Pure Appl. Anal. **20**(1), 215–242 (2021).
[26] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Texts and Monographs in Physics (Springer-Verlag, New York, 1988).
[27] S. Dovetta, E. Serra, and P. Tilli, "Action versus energy ground states in nonlinear Schrödinger equations," Math. Ann. (2022); available at https://link.springer.com/content/pdf/10.1007/s00208-022-02382-z.pdf..

[28]T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes Vol. 10 (American Mathematical Society, Providence, RI, 2003).
[29]P. L. Lions, "The concentration-compactness principle in the calculus of variations. The locally compact case. II," Ann. Inst. Henri Poincare **1**(4), 223–283 (1984).
[30]B. Gidas, W.-M. Ni, and L. Nirenberg, "Symmetry and related properties via the maximum principle," Commun. Math. Phys. **68**(3), 209–243 (1979).
[31]H. Brezis and E.H. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," Proc. Am. Math. Soc. **88**(3), 486–490 (1983).