POLITECNICO DI TORINO
Repository ISTITUZIONALE

A C1 stabilization-free Virtual Element Method for the biharmonic equation: Source and spectral problems

Original
A C1 stabilization-free Virtual Element Method for the biharmonic equation: Source and spectral problems / Fassino, D., Marcon, F., Mora, D., Silgado, A.. - In: APPLIED MATHEMATICS LETTERS. - ISSN 0893-9659. - 185:(2027), pp. 1-6. [10.1016/j.aml.2026.110166]

Availability:
This version is available at: 11583/3016548 since: 2026-10-12T08:08:32Z

Publisher:
Elsevier

Published
DOI:10.1016/j.aml.2026.110166

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

08 November 2026

Applied Mathematics Letters 185 (2027) 110166

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Regular article

A C^1 stabilization-free Virtual Element Method for the biharmonic equation: Source and spectral problems

Davide Fassino [a,*], Francesca Marcon [a], David Mora [b,c], Alberth Silgado [d]

[a] Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Politecnico di Torino, Torino, Italy
[b] GIMNAP, Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Concepción, Chile
[c] CI^{2}MA, Universidad de Concepción, Concepción, Chile
[d] Centro de Ciencias Exactas, GIMNAP, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Chillán, Chile

ARTICLE INFO

Keywords:
C^1 virtual element method
Stabilization-free
Biharmonic equation
Eigenvalue problems

ABSTRACT

In this work, we introduce a novel C^1 stabilization-free virtual element discretization for elliptic fourth-order problems, defined on triangular meshes. We propose a discrete approximation of the energy bilinear form that does not require an arbitrary non-polynomial stabilization term, based on the definition of a new polynomial projection, and we address the biharmonic eigenproblem. A representative set of numerical experiments shows the local stability and robustness of the method. Notably, the proposed approach does not produce spurious eigenvalues, which highlights its reliability in spectral approximations.

1. Introduction

The construction of H^2-conforming FEMs is difficult in general; some methods have been proposed in the past, such as the composite and reduced Hsieh–Clough–Tocher (HCT) element, the Zienkiewicz and Argyris elements. One of their limitations is that they usually involve a large number of degrees of freedom or a complicated structure. In particular, the implementation of HCT elements (both the complete and reduced versions) requires the subdivision of each triangle into three sub-triangles by connecting the barycenter to its vertices. This internal decomposition, along with the need to construct piecewise cubic basis functions that ensure global C^1-continuity, adds considerable complexity to the method. Similarly, the Argyris element relies on quintic polynomials to achieve C^1-conformity, further increasing the analytical and computational burden. Although some specialized codes and implementations are available in the literature, the main limitations of C^1-conforming methods lie in their intricate construction and the lack of low-order elements, which restricts the availability of general-purpose or ready-to-use libraries. We refer to [1] for further details. A recent and promising alternative to solve fourth-order problems is the H^2-conforming Virtual Element Method (VEM), proposed in [2–4], which reduces the number of degrees of freedom with respect to the classical FEMs, thus decreasing the computational cost of the method, and gives the advantage of implementing highly regular discrete spaces. The main characteristic of VEMs relies on the definition of discrete spaces that contain polynomials and other functions (whose expression is unknown), in order to satisfy the required Sobolev regularity of the scheme and to define exploitable sets of degrees of freedom. The discretization of the bilinear forms is structured as the sum of a bilinear form providing consistency on polynomials and another bilinear form, ensuring stability, whose choice remains somewhat arbitrary over the years. In recent years, a large literature has been produced on the role of the stability term, in particular, it has been shown that its isotropic nature, arbitrariness, and parameter dependence can cause some issues in several applications [5].

* Corresponding author.
E-mail addresses: davide.fassino@polito.it (D. Fassino), francesca.marcon@polito.it (F. Marcon), dmora@ubiobio.cl (D. Mora), asilgado@ubiobio.cl (A. Silgado).

https://doi.org/10.1016/j.aml.2026.110166
Received 23 June 2026; Received in revised form 30 September 2026; Accepted 5 October 2026
Available online 8 October 2026
0893-9659/© 2026 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

Recently, new VEMs, named Stabilization-free VEM (SFVEM), have been proposed and discussed to allow the definition of coercive discrete bilinear forms based only on polynomial projections of the basis functions, see for instance [6–12] and references therein.

The main objectives and contributions of the present work are the following: (i) the development of a novel lowest-order C^1 stabilization-free VEM discretization of the biharmonic operator for simplicial 2D: the method is based on the definition of a new computable H^2-orthogonal projection operator, as in the philosophy of SFVEM. (ii) by using the new projector, we propose a self-stabilized discretization for the right-hand side of the eigenvalue problem. Then, with these ingredients at hand, the discrete eigenvalue problem is formulated. (iii) the assessment of the numerical performance of the proposed approach: we report a representative set of numerical experiments that highlight the strong performance of the new numerical schemes for both source and eigenvalue problems on unstructured and structured triangles, while using a reduced number of degrees of freedom. In particular, we observe numerically that the method does not introduce spurious modes in the spectral approximation, in contrast to the behavior observed with stabilized VEMs, that need the tuning of a non physical parameter scaling the stabilization (see, for instance, [13]). These results suggest that our approach offers an efficient and competitive alternative for the numerical solution of fourth-order systems.

2. The source problem

Let Ω be an open bounded domain of $\mathbb{R}^2$, with polygonal boundary $\partial\Omega$. We are interested in solving the following fourth-order elliptic problem: given $f \in L^2(\Omega)$, find $u : \Omega \to \mathbb{R}$ such that

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \partial_{\mathbf{n}} u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\partial_{\mathbf{n}} v$ denotes the normal derivative of v.

We consider the Hilbert space $H^2_0(\Omega) = \left\{ v \in H^2(\Omega) : v = \partial_{\mathbf{n}} v = 0 \quad \text{on} \quad \partial\Omega \right\}$, endowing it with the standard norm $\|v\|_{H^2_0(\Omega)} := |v|_{H^2(\Omega)} = \|\mathbf{D}^2 v\|_{L^2(\Omega)}$, $\forall v \in H^2_0(\Omega)$, with $\mathbf{D}^2 v$ being the Hessian matrix of v. Then a variational formulation of (1) read as:

Problem 1. Find $u \in H^2_0(\Omega)$ such that $A(u, v) = \left(\mathbf{D}^2 u, \mathbf{D}^2 v\right)_\Omega = (f, v)_\Omega \qquad \forall v \in H^2_0(\Omega)$.

Since the bilinear form A is elliptic, Problem 1 is well-posed as a consequence of the Lax–Milgram Theorem. We state the following additional regularity result for the solution to Problem 1 (see [14]).

Theorem 1. *There exist $s \in (1/2, 1]$ and a constant $C > 0$, such that the solution u to Problem 1 satisfies $u \in H^{2+s}(\Omega) \cap H^2_0(\Omega)$ and $\|u\|_{H^{2+s}(\Omega)} \le C \|f\|_{L^2(\Omega)}$.*

2.1. The discrete formulation

Let Ω_h be a regular family of triangulations of the polygonal domain Ω, and let E denote a generic triangle in Ω_h. Moreover, let h_E denote the diameter of E, e and h_e denote respectively an edge of E and its length. Let $h = \max_{E \in \Omega_h} h_E$ be the mesh size. In the following, let $\mathbf{n}$ denote the unit outward normal, whose reference will be clear from the context. Furthermore, for any open bounded subdomain $\omega \subset \mathbb{R}^d$, with $d = 1$ or 2, given $t \in \mathbb{N}$ we define the space $\mathbb{P}_t(\omega)$ as the space of polynomials of degree up to t defined on ω. For each triangle $E \in \Omega_h$, we consider the lowest order VE space introduced in [2]:

$$\mathcal{V}_2^h(E) := \left\{ v_h \in H^2(E) : \Delta^2 v_h = 0,\ v_h|_e \in \mathbb{P}_3(e),\ \partial_{\mathbf{n}} v_h \in \mathbb{P}_1(e)\ \forall e \in \partial E \right\}.$$

Given $v_h \in \mathcal{V}_2^h(E)$ the chosen degrees of freedom (DoFs) of this space are:

D.1 the values of $v_h(\mathbf{v}_i)$ for all vertex $\mathbf{v}_i$ of E;
D.2 the values of $h_{\mathbf{v}_i} \nabla v_h(\mathbf{v}_i)$ for all vertex $\mathbf{v}_i$ of E,

where for each vertex $\mathbf{v}_i$ we set $h_{\mathbf{v}_i}$ as the average of the diameters of the triangles having $\mathbf{v}_i$ as a vertex.

In order to construct the stabilization-free scheme, we introduce the operator $\mathtt{P}_0 : H^1(E) \longrightarrow \mathbb{P}_0(E)$, defined for any $w \in H^1(E)$ by $\mathtt{P}_0 w := (w, 1)_{\partial E}$. Then, for each $E \in \Omega_h$, we define the projector $\Pi_E^{\mathbf{D},3} : H^2(E) \longrightarrow \mathbb{P}_3(E)$, such that, for any $v \in H^2(E)$, $\Pi_E^{\mathbf{D},3} v$ is the unique solution of

$$A_E(v - \Pi_E^{\mathbf{D},3} v, q) = 0 \qquad \forall q \in \mathbb{P}_3(E), \tag{2}$$

$$\mathtt{P}_0(v - \Pi_E^{\mathbf{D},3} v) = 0, \qquad \mathtt{P}_0(\nabla(v - \Pi_E^{\mathbf{D},3} v)) = 0. \tag{3}$$

Here, $A_E(\cdot,\cdot)$ denotes the local restriction of the global bilinear form $A(\cdot,\cdot)$ to the element E.

The following result shows that the projector $\Pi_E^{\mathbf{D},3}$ is computable using the degrees of freedom **D.1**–**D.2**.

Lemma 1. *The operator $\Pi_E^{\mathbf{D},3} : \mathcal{V}_2^h(E) \to \mathbb{P}_3(E)$ is computable for every $v_h \in \mathcal{V}_2^h(E)$, using only the DoFs **D.1**–**D.2**.*

Proof. Following [2,3], from (2) and the integration by parts rule, we have that, for all $q \in \mathbb{P}_3(E)$,

$$\left(\mathbf{D}^2 \Pi_E^{\mathbf{D},3} v_h, \mathbf{D}^2 q\right)_E = \left(\Delta^2 q, v_h\right)_E - \left(\operatorname{div}\left(\mathbf{D}^2 q\right) \cdot \mathbf{n}, v_h\right)_{\partial E} + \left(\mathbf{D}^2 q, \partial_{\mathbf{n}} v_h\right)_{\partial E} = -\left(\operatorname{div}\left(\mathbf{D}^2 q\right) \cdot \mathbf{n}, v_h\right)_{\partial E} + \left(\mathbf{D}^2 q, \partial_{\mathbf{n}} v_h\right)_{\partial E},$$

since $q \in \mathbb{P}_3(E)$, $\Delta^2 q = 0$ in E. Furthermore, for each $e \subset \partial E$ $v_h|_e \in \mathbb{P}_3(e)$ and $\partial_{\mathbf{n}} v_h \in \mathbb{P}_1(e)$, hence the remaining terms can be fully determined by the degrees of freedom on each edge e.

Now, we construct the discrete version of the bilinear form $A(\cdot,\cdot)$. Indeed, for all $v_h, w_h \in \mathcal{V}_2^h$, where $\mathcal{V}_2^h := \left\{v_h \in \mathrm{H}_0^2(\Omega) : \ v_h|_E \in \mathcal{V}_2^h(E), \quad \forall E \in \Omega_h\right\}$, we define the global bilinear form $A^h : \mathcal{V}_2^h \times \mathcal{V}_2^h \to \mathbb{R}$ as

$$A^h(v_h, w_h) := \sum_{E \in \Omega_h} A_E^h(v_h, w_h) = \sum_{E \in \Omega_h} A_E\left(\Pi_E^{\mathbf{D},3} v_h, \Pi_E^{\mathbf{D},3} w_h\right) \qquad \forall v_h, w_h \in \mathcal{V}_2^h. \tag{4}$$

Finally, the proposed approximation of Problem 1 reads as follows.

Problem 2. Find $u_h \in \mathcal{V}_2^h$, such that

$$A^h(u_h, v_h) = \sum_{E \in \Omega_h} \left(f, \Pi_E^{\mathbf{D},3} v_h\right)_E \quad \forall v_h \in \mathcal{V}_2^h.$$

Remark 1 (*Standard VEM Formulation*)**.** The lowest-order standard C^1 VEM consists in the use of the projector $\Pi_E^{\mathbf{D},2} : \mathrm{H}^2(E) \longrightarrow \mathbb{P}_2(E)$, defined as

$$A_E(v - \Pi_E^{\mathbf{D},2} v, q) = 0 \qquad \forall q \in \mathbb{P}_2(E), \tag{5}$$

$$\mathtt{P}_0(v - \Pi_E^{\mathbf{D},2} v) = 0, \qquad \mathtt{P}_0(\nabla(v - \Pi_E^{\mathbf{D},2} v)) = 0, \tag{6}$$

which results computable using the degrees of freedom **D.1–D.2**. In that case (even with Ω_h made only by triangles) the resulting discrete bilinear form is not locally stable and, as described in [2], an extra bilinear form needs to be added in order to obtain coercivity. Hence, let $\tilde{A}^h : \mathcal{V}_2^h \times \mathcal{V}_2^h \to \mathbb{R}$ be defined, for all $v_h, w_h \in \mathcal{V}_2^h(E)$, by

$$\tilde{A}^h(v_h, w_h) := \sum_{E \in \Omega_h} A_E\left(\Pi_E^{\mathbf{D},2} v_h, \Pi_E^{\mathbf{D},2} w_h\right) + h_{dof}^{-2} S_E\left(v_h - \Pi_E^{\mathbf{D},2} v_h, w_h - \Pi_E^{\mathbf{D},2} w_h\right), \tag{7}$$

where S_E is a proper defined stabilization term. Usually, the *dofi-dofi* stabilization is used, with h_{dof}, depending on $h_{\mathbf{v}_i}$, as described in [2]. Then, the standard VEM counterpart of Problem 2 results to be: find $u_h \in \mathcal{V}_2^h$, such that

$$\tilde{A}^h(u_h, v_h) := \sum_{E \in \Omega_h} \left(f, \Pi_E^{\mathbf{D},2} v_h\right)_E \quad \forall v_h \in \mathcal{V}_2^h. \tag{8}$$

Remark 2. We observe that extending the proposed formulation to general polygonal meshes and higher-order approximations is nontrivial. In particular, it would require the design of a computable projection operator onto polynomial spaces beyond $\mathbb{P}_3$, which is therefore left for future investigation.

3. The spectral problem

Let us assume that $\Omega \subseteq \mathbb{R}^2$ be a polygonal bounded domain occupied by the mean surface of a thin plate, clamped on its whole boundary $\partial\Omega$. We denote by u the transverse displacement of the mean surface of the plate. Thus, the vibration eigenvalue problem reads as follows: find $(\lambda, u) \in \mathbb{R} \times \mathrm{H}_0^2(\Omega)$, $u \neq 0$, such that

$$\Delta^2 u = \lambda u \quad \text{in } \Omega, \qquad u = \partial_{\mathbf{n}} u = 0 \quad \text{on } \partial\Omega. \tag{9}$$

By testing the first equation above with $v \in \mathrm{H}_0^2(\Omega)$ and integrating by parts, we arrive at the following weak formulation:

Problem 3. Find $(\lambda, u) \in \mathbb{R} \times \mathrm{H}_0^2(\Omega)$, $u \neq 0$, such that

$$A(u, v) = \lambda B(u, v) \quad \forall v \in \mathrm{H}_0^2(\Omega),$$

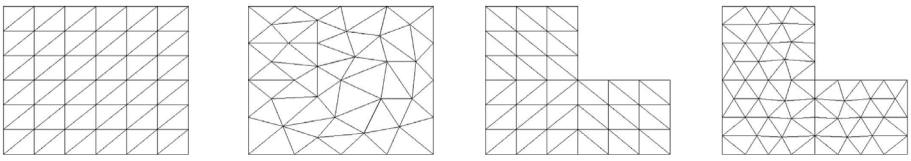
where the bilinear form $B : \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega) \to \mathbb{R}$ is defined by $B(u, v) := (u, v)_\Omega, \forall u, v \in \mathrm{L}^2(\Omega)$.

It is easy to see that the previous form is bounded and to prove that the eigenvalues of Problem 3 are real and positive. Next, we define the solution operator associated with Problem 3: $T : \mathrm{H}_0^2(\Omega) \longrightarrow \mathrm{H}_0^2(\Omega), \quad f \longmapsto Tf := w$, where $w \in \mathrm{H}_0^2(\Omega)$ is the unique solution (as a consequence of the ellipticity of A and the Lax–Milgram Theorem) of the source Problem 1, i.e., $A(w, v) = (f, v)_\Omega$, $\forall v \in \mathrm{H}_0^2(\Omega)$. Thus, the linear operator T is well defined and bounded. In addition, we notice that $(\lambda, u) \in \mathbb{R} \times \mathrm{H}_0^2(\Omega)$ solves Problem 3 (and hence (9)) if and only if $Tu = \mu u$ with $\mu \neq 0$ and $u \neq 0$, in which case $\mu := 1/\lambda$.

Theorem 1 states an additional regularity result for the solution of Problem 1 and consequently for the eigenfunctions of T. Moreover, as a consequence of the compact inclusion $\mathrm{H}^{2+s}(\Omega) \hookrightarrow \mathrm{H}_0^2(\Omega)$, T is a compact operator.

We have the following spectral characterization for the operator T.

Lemma 2. *The spectrum of T satisfies $sp(T) = \{0\} \cup \{\mu_k\}_{k \in \mathbb{N}}$ is a sequence of real positive eigenvalues which converges to 0. The multiplicity of each eigenvalue is finite and their corresponding eigenspaces lie in $\mathrm{H}^{2+s}(\Omega)$. Moreover, there exists $C > 0$ (depending on λ) such that $\|u\|_{\mathrm{H}^{2+s}(\Omega)} \leq C \|u\|_{\mathrm{H}^2(\Omega)}$.*



(a) Tricart (b) Tri (c) L-Tricart (d) L-Tri

Fig. 1. Meshes used in the numerical tests on the source and spectral problems.

Table 1
Test 1. Computed minimum $(h_E\sigma_E)_{\min}$ and maximum $(h_E\sigma_E)_{\max}$ not-vanishing singular value on different refinement of the Tri meshes made up of NP triangles.

	$NP = 60$	$NP = 319$	$NP = 1539$	$NP = 6244$	$NP = 77574$
$(h_E\sigma_E)_{\min}$	0.594	0.464	0.402	0.495	0.343
$(h_E\sigma_E)_{\max}$	0.838	0.864	0.867	0.898	0.932

3.1. The discrete formulation

In this section, we propose the SFVEM discretization of Problem 3. The setting considered is the one presented in Section 2.1.

Now, we want to construct a self-stabilized approximation of the bilinear form $B(\cdot,\cdot)$. In fact, for each $v_h, w_h \in \mathcal{V}_2^h$, we define the global bilinear form $B^h : \mathcal{V}_2^h \times \mathcal{V}_2^h \to \mathbb{R}$ as

$$B^h(v_h, w_h) := \sum_{E\in\Omega_h} B_E^h(v_h, w_h) \equiv \sum_{E\in\Omega_h} B_E\big(\Pi_E^{\mathbf{D},3} v_h, \Pi_E^{\mathbf{D},3} w_h\big), \tag{10}$$

where B_E is the restriction to each element of the continuous form B.

Remark 3. We observe that the bilinear forms $A^h(\cdot,\cdot)$ defined in (4) and $B^h(\cdot,\cdot)$ defined in (10) can be extended to $\mathrm{H}_0^2(\Omega)$, only taking the projections of the continuous functions $v, w \in \mathrm{H}_0^2(\Omega)$.

Now, we are in a position to write the virtual element discretization of Problem 3.

Problem 4. Find $(\lambda_h, u_h) \in \mathbb{R} \times \mathcal{V}_2^h$, $u_h \neq 0$, such that $A^h(u_h, v_h) = \lambda_h B^h(u_h, v_h) \quad \forall v_h \in \mathcal{V}_2^h$.

Remark 4. The analysis of Problem 4 is closely related to that of Problem 2. In particular, due to the definition of the bilinear forms $A^h(\cdot,\cdot)$ and $B^h(\cdot,\cdot)$, standard techniques allow one to derive the local coercivity of $B^h(\cdot,\cdot)$ from the corresponding stability properties of $A^h(\cdot,\cdot)$. For this reason, the study of Problem 4 essentially relies on the analysis carried out for Problem 2.

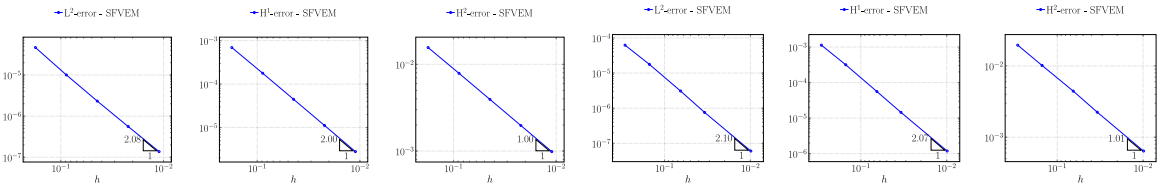
4. Numerical results

4.1. The source problem

In this section, we first carry out a numerical investigation on the local stability of the proposed method, then we present a numerical experiment to validate the performance of the proposed method for the solution to Problem 1. For these tests, we consider two families of triangular meshes: uniform triangles labeled Tricart and general triangles labeled Tri, as shown in Figs. 1(a) and 1(b), respectively.

Test 1. In this test, we take into consideration each element of the families of meshes, we build the corresponding local stiffness matrix related to the local discrete bilinear form A_E^h. For each element, we compute the square root of the smallest non-vanishing singular value σ_E to estimate the square root of the local coercivity constant. Since the matrix scales as h_E^{-2}, we check if $h_E\sigma_E$ is well detached from zero. For each considered element, we obtain that three singular values are numerically zero. Then, for the elements in Tricart we obtain that $h_E\sigma_E = 0.68$, and for each refinement of Tri mesh, we report in Table 1 the minimum and the maximum value of $h_E\sigma_E$ obtained. We observe that the reported values of $h_E\sigma_E$ are well-detached from zero, hence the proposed discretization results to be locally stable.

Test 2. We solve the bilaplacian Eq. (1) on a square domain $\Omega = (0,1)^2$, imposing the loading term and the Dirichlet boundary conditions such that the exact solution to the problem is $u(x,y) = x^2\,y^2\,(x-1)^2\,(y-1)^2$. In Fig. 2, we report the trend of the L^2, H^1 and the H^2 errors for the proposed SFVEM for the Tricart and Tri meshes. The measures of the errors are computed as $\mathrm{H}^i\text{-error}^2 = \sum_{E\in\Omega_h} |u - \Pi_E^{\mathbf{D},2} u_h|^2_{\mathrm{H}^i(E)} / |u|^2_{\mathrm{H}^i(\Omega)}$, $i = 0,1,2$, where H^0-error denotes the L^2-error with a slight abuse of notation. The results show optimal convergence rates.



(a) Tricart (b) Tricart (c) Tricart (d) Tri (e) Tri (f) Tri

Fig. 2. Test 2. Convergence plots of the L^2, H^1 and H^2 errors on the Tricart mesh and the Tri mesh. The reported convergence rate is obtained using the four finest refinements.

Table 2
Test 3. Computed lowest eigenvalues λ_h^i, with $1 \le i \le 5$ on the L-Tricart mesh, with methods SFVEM (Problem 4) and VEM (11). λ^{ref} is a set of tabulated values [15, Table 2]. The λ_h^i in the light blue boxes are spurious. Order is obtained using the four finest refinements.

	λ_h^i	$N^{\text{dof}} = 48$	$N^{\text{dof}} = 99$	$N^{\text{dof}} = 483$	$N^{\text{dof}} = 1155$	$N^{\text{dof}} = 2115$	$N^{\text{dof}} = 4899$	$N^{\text{dof}} = 36099$	λ^{ref}	Order
SFVEM	λ_h^1	527.0071	479.3039	437.2519	428.973	425.6866	422.9234	420.1809	418.9753	1.2564
	λ_h^2	829.5035	763.1221	706.7746	697.6771	694.6455	692.5435	691.1363	690.9065	2.0185
	λ_h^3	1054.3902	1004.9802	949.5507	939.3107	935.8365	933.4265	931.8335	931.5793	2.0379
	λ_h^4	1840.2607	1763.176	1667.1649	1648.5674	1642.2433	1637.8636	1634.9835	1634.5338	2.0538
	λ_h^5	2437.9076	2333.7547	2167.3862	2128.63	2114.1192	2102.9856	2093.7964	2090.8394	1.5126
VEM	λ_h^1	22.3812	67.9983	408.697	420.2917	420.8416	420.4853	419.5833	418.9753	0.56609
	λ_h^2	23.7633	69.9776	643.593	681.6301	686.4386	688.9747	690.6116	690.9065	2.0304
	λ_h^3	23.9292	70.2726	840.1895	917.3345	925.3338	929.0482	931.2287	931.5793	2.1726
	λ_h^4	23.9864	70.6796	1040.1190	1592.6648	1617.9205	1628.0802	1633.6294	1634.5338	2.2339
	λ_h^5	24.6566	71.6484	1046.7727	2040.7539	2079.7369	2089.4046	2091.4893	2090.8394	2.4494

Table 3
Test 3. Computed lowest eigenvalues λ_h^i, with $1 \le i \le 5$ on the L-Tri mesh, with methods SFVEM (Problem 4) and VEM (11). λ^{ref} is a set of tabulated values [15, Table 2]. The λ_h^i in the light blue boxes are spurious. Order is obtained using the four finest refinements.

	λ_h^i	$N^{\text{dof}} = 90$	$N^{\text{dof}} = 225$	$N^{\text{dof}} = 609$	$N^{\text{dof}} = 1338$	$N^{\text{dof}} = 4500$	$N^{\text{dof}} = 13746$	$N^{\text{dof}} = 69297$	λ^{ref}	Order
SFVEM	λ_h^1	469.7896	448.7857	437.2989	429.9121	423.4966	421.1363	419.876	418.9753	1.6249
	λ_h^2	729.9468	708.0875	699.325	696.1068	692.3836	691.4257	691.0186	690.9065	2.4918
	λ_h^3	989.1818	952.4072	946.1464	939.8027	933.533	932.2972	931.7225	931.5793	2.6201
	λ_h^4	1729.682	1677.6128	1661.5223	1649.9914	1638.56	1635.9706	1634.8217	1634.5338	2.5815
	λ_h^5	2282.1532	2180.6547	2161.4418	2130.3332	2104.0959	2096.6311	2092.9229	2090.8394	1.9106
VEM	λ_h^1	81.8000	254.4784	411.4242	421.3324	420.5042	420.0064	419.4052	418.9753	1.1012
	λ_h^2	87.0675	256.4910	652.7408	683.6979	688.7422	690.2374	690.7599	690.9065	2.5397
	λ_h^3	88.4496	257.5626	862.1414	919.8118	928.3649	930.5692	931.3814	931.5793	2.657
	λ_h^4	91.6861	264.2973	1267.1503	1605.8805	1628.0414	1632.5456	1634.124	1634.5338	2.7588
	λ_h^5	95.0884	265.1287	1317.4778	2049.1891	2086.4214	2090.5536	2091.2551	2090.8394	3.0995

4.2. The eigenproblem

In this section, we investigate the numerical results of the biharmonic eigenvalue problem, focusing in particular on the comparison between the proposed method and stabilized VEM [2,3]. In this latter case, the discretization of (9) reads as: find $(\tilde{\lambda}_h, \tilde{u}_h) \in \mathbb{R} \times \mathcal{V}_2^h$, $\tilde{u}_h \neq 0$, such that

$$\tilde{A}^h(\tilde{u}_h, v_h) = \tilde{\lambda}_h \tilde{B}^h(\tilde{u}_h, v_h) \quad \forall v_h \in \mathcal{V}_2^h, \tag{11}$$

where the bilinear forms $\tilde{A}^h : \mathcal{V}_2^h \times \mathcal{V}_2^h \to \mathbb{R}$ and $\tilde{B}^h : \mathcal{V}_2^h \times \mathcal{V}_2^h \to \mathbb{R}$ are defined, for all $v_h, w_h \in \mathcal{V}_2^h(E)$, by (7) and

$$\tilde{B}^h(v_h, w_h) := \sum_{E \in \Omega_h} B_E\left(\Pi_E^{\mathbf{D},2} v_h, \Pi_E^{\mathbf{D},2} w_h\right) + h_{dof}^2 S_E\left(v_h - \Pi_E^{\mathbf{D},2} v_h, w_h - \Pi_E^{\mathbf{D},2} w_h\right). \tag{12}$$

Test 3. Now, we consider the bilaplacian eigenvalue problem (9) on the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$. The reference values λ^{ref} are taken from [15, Table 2], computed with an Adaptive Argyris Finite Element Method. In Table 2, the first five eigenvalues for seven refinements of the L-Tricart, as depicted in Fig. 1(c), for the SFVEM and the VEM are tabulated. Using the stabilized VEM, we see the presence of spurious eigenvalues (in the blue boxes) on the coarsest meshes with $N^{\text{dof}} = 48$, $N^{\text{dof}} = 99$, and $N^{\text{dof}} = 483$, that does not appear in the case of the SFVEM. Analogously, the results shown in Table 3 display in the blue boxes the spurious eigenvalues in the case of general triangulations of the domain (depicted in Fig. 1(d)) with $N^{\text{dof}} = 90$, $N^{\text{dof}} = 225$ and $N^{\text{dof}} = 609$ applying the VEM, while good approximations of the eigenvalues are always obtained with the SFVEM. The presence of spurious eigenvalues is detected again by plotting the eigenfunctions. Fig. 3 shows the eigenfunctions u_h^i related to the eigenvalues λ_h^i, $i = 1, \dots, 5$ in the SFVEM and the stabilized VEM on the L-Tricart mesh with $N^{\text{dof}} = 483$. For the VEM case, the eigenfunctions u_h^i, $i = 4, 5$ are spurious.

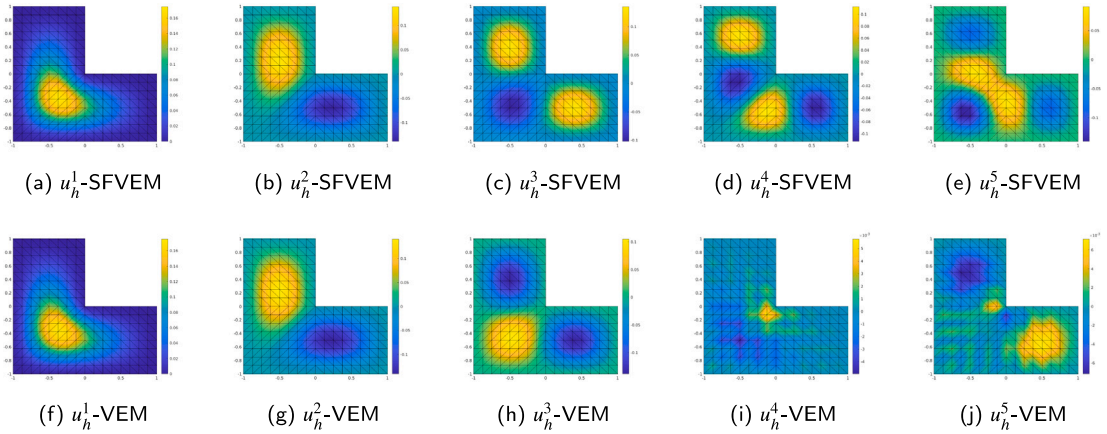


Fig. 3. Test 2. Eigenfunctions u_h^i, corresponding to the eigenvalues λ_h^i, $i = 1, \dots, 5$, on the `L-Tricart` mesh with $N^{\text{dof}} = 483$, solving Problem 4 (SFVEM) and solving (11) (VEM).

Acknowledgments

DF and FM are members of the Gruppo Nazionale Calcolo Scientifico (GNCS) at Istituto Nazionale di Alta Matematica (INdAM). They kindly acknowledge partial financial support by INdAM-GNCS Project 2026 (CUP: E53C25002010001). FM kindly acknowledges financial support by the Italian Ministry of University and Research (MUR) through the project MUR-M4C2-1.1-PRIN 2022 (CUP: 53D23005820006). DM was partially supported by ANID-Chile through FONDECYT project 1261217 and by Centro de Modelamiento Matemático (CMM), FB210005, BASAL funds for centers of excellence.

Data availability

Data will be made available on request.

References

[1] P.G. Ciarlet, The finite element method for elliptic problems, vol. 40, SIAM, Philadelphia, PA, 2002, p. xxviii+530, http://dx.doi.org/10.1137/1.9780898719208.
[2] F. Brezzi, L.D. Marini, Virtual element methods for plate bending problems, Comput. Methods Appl. Mech. Engrg. 253 (2013) 455–462, http://dx.doi.org/10.1016/j.cma.2012.09.012.
[3] C. Chinosi, L.D. Marini, Virtual element method for fourth order problems: L^2-estimates, Comput. Math. Appl. 72 (8) (2016) 1959–1967, http://dx.doi.org/10.1016/j.camwa.2016.02.001.
[4] P.F. Antonietti, L. Beirão da Veiga, S. Scacchi, M. Verani, A C^1 virtual element method for the cahn-hilliard equation with polygonal meshes, SIAM J. Numer. Anal. 54 (1) (2016) 34–56, http://dx.doi.org/10.1137/15M1008117.
[5] L. Mascotto, The role of stabilization in the virtual element method: a survey, Comput. Math. Appl. 151 (2023) 244–251, http://dx.doi.org/10.1016/j.camwa.2023.09.045.
[6] S. Berrone, A. Borio, F. Marcon, Lowest order stabilization free virtual element method for the 2D Poisson equation, Comput. Math. Appl. 177 (2025) 78–99, http://dx.doi.org/10.1016/j.camwa.2024.11.017.
[7] A. Lamperti, M. Cremonesi, U. Perego, A. Russo, C. Lovadina, A hu-washizu variational approach to self-stabilized virtual elements: 2D linear elastostatics, Comput. Mech. 71 (5) (2023) 935–955, http://dx.doi.org/10.1007/s00466-023-02282-2.
[8] B. Xu, Y. Wang, P. Wriggers, Stabilization-free virtual element method for 2D elastoplastic problems, Internat. J. Numer. Methods Engrg. 125 (15) (2024) http://dx.doi.org/10.1002/nme.7490, Paper No. e7490.
[9] S. Berrone, A. Borio, F. Marcon, A stabilization-free virtual element method based on divergence-free projections, Comput. Methods Appl. Mech. Engrg. 424 (2024) 116885, http://dx.doi.org/10.1016/j.cma.2024.116885.
[10] S. Berrone, A. Borio, D. Fassino, F. Marcon, Stabilization-free Virtual Element Method for 2D second order elliptic equations, Comput. Methods Appl. Mech. Engrg. 438 (2025) 117839, http://dx.doi.org/10.1016/j.cma.2025.117839.
[11] F. Marcon, D. Mora, A stabilization-free virtual element method for the convection-diffusion eigenproblem, J. Sci. Comput. 102 (2) (2025) http://dx.doi.org/10.1007/s10915-024-02765-1, Paper No. 46, 33.
[12] P.P. Foligno, D. Boffi, F. Credali, R. Vescovini, Benchmarking stabilized and self-stabilized p-virtual element methods with variable coefficients, Comput. Methods Appl. Mech. Engrg. 455 (2026) 118863, http://dx.doi.org/10.1016/j.cma.2026.118863.
[13] D. Mora, G. Rivera, I. Velásquez, A virtual element method for the vibration problem of Kirchhoff plates, ESAIM Math. Model. Numer. Anal. 52 (4) (2018) 1437–1456, http://dx.doi.org/10.1051/m2an/2017041.
[14] P. Grisvard, Elliptic Problems in Nonsmooth Domains, vol. 69, SIAM, Philadelphia, PA, 2011, p. xx+410, http://dx.doi.org/10.1137/1.9781611972030.ch1.
[15] C. Carstensen, B. Gräßle, Rate-optimal higher-order adaptive conforming FEM for biharmonic eigenvalue problems on polygonal domains, Comput. Methods Appl. Mech. Engrg. 425 (2024) http://dx.doi.org/10.1016/j.cma.2024.116931, Paper No. 116931, 18.