Bipartite consensus for a class of nonlinear multi-agent systems under switching topologies: A disturbance observer-based approach

Bipartite Consensus for a Class of Nonlinear Multi-agent Systems Under Switching Topologies: A Disturbance Observer-Based Approach

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Abstract

This paper considers the leader-following bipartite consensus for a class of nonlinear multi-agent systems (MASs) subject to exogenous disturbances under directed fixed and switching topologies, respectively. First, the scenario of a fixed topology is considered, and a new output feedback control protocol based on relative output measurements of neighboring agents is proposed. In order to estimate the disturbances produced by an exogenous system, a disturbance observer-based approach is developed and incorporated into the controller. Then, theoretical guarantees on the effectiveness of the proposed controller in steering the system to a bipartite leader-following consensus are derived. Second, the scenario of switching topologies is considered, and a disturbance observer-based controller is proposed, following a similar approach.

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Then, it is proved that the leader-following bipartite consensus can be realized with the designed output feedback control protocol if the dwell time is larger than a non-negative threshold. Finally, numerical simulations inspired by a real-world physical MAS are provided to illustrate the effectiveness of the proposed controllers.
Keywords: Nonlinear MASs, bipartite consensus, signed graphs, switching topologies, disturbance observer, observer-based control

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1. Introduction

In recent years, a tremendous amount of attention has been devoted to cooperative control of MASs [1, 2, 3]. This can be ascribed to its wide applications in varieties of fields such as sensor networks [4], unmanned air vehicles [5], opinion dynamics [6, 7, 8], formation control [9, 10], and so on. Among the dynamic behavior of MASs, consensus is a fundamental issue, which is committed to driving each participating agent to the same state by only collecting the local information from its neighbors [9, 11, 12, 13]. Leader-following consensus reduces the tracking consensus when there is a leader in MASs, where the followers' states will gradually become consistent with that of the leader over time [14, 15, 16, 17, 18]

While most of these works focus on agents that cooperatively work toward a common goal [9], in many practical situations competitive and antagonistic interaction are also present [19]. For instance, in [20], it was observed that, in the industrial market, companies not only collaborate, but also compete for market resources. The co-existence of cooperative and competitive interactions has been observed as a key feature in government formation process in parliamentary democracies [21]. In a leader-follower framework, the authors of [22] investigated the cooperation and competition between employer and employees in management control systems. Signed graphs, originally proposed in [19], became a universal tool, widely used to describe and study networks with both cooperative and competitive relationships.

The study on dynamic behavior of MASs over signed graphs can be traced back to the seminal work on linear systems in [23], where it is studied how the network structure determines whether the system converges to a collective agreement or a polarized scenario, termed bipartite consensus. In particular, it is proved that the states of the agents converge to two subgroups with different consensus states if the signed graph is structurally balanced, or they converge to a common state if the signed graph is structurally unbalanced. Inspired by this pioneering work, many efforts have been made toward addressing the dynamic behaviors of more complex dynamics of MASs on signed graphs [24, 25, 26, 27, 28, 29]. Concerning leader-following dynamics, we mention leader-following bipartite consensus under discrete-time [30, 31], fractional-order nonlinear dynamics [32], adaptive control [33], and finite-time consensus [34].

Most of works on competitive interactions proposed in the literature rely on the assumption that communication topologies are undirected and fixed [35]. However, for many practical situations, communication channels among agents may be directional and change in time, due to possible disruptions in the communication pattern, or changes in the agents' displacement. Therefore, a growing literature on consensus on time-varying topologies have been reported [36, 37, 38, 39, 40]. Within this research field, switching topologies has recently started becoming popular. Concerning its application to leader-following consensus, in [41], based on Lyapunov theory and the property of M-matrix, the leader-following consensus control problem was solved with nonlinear MASs under switching topologies. Directed topologies were considered in [42]. In [43], the authors addressed the leader-following bipartite consensus problem of multiple uncertain Euler-Lagrange systems over signed switching networks by means of a distributed observer, and leveraging the certainty equivalence principle. Other works dealt with bipartite consensus over fixed and switching topologies [44], utilizing pinning control strategies [45, 46] and distributed adaptive control algorithms [47]. However, the above mentioned literatures only study bipartite consensus problem under switching topologies without exogenous disturbances, and the limited information for unknown disturbances makes it difficult to consider the bipartite consensus, involving how to construct the output feedback control strategy without using any state information, how to combine the nonlinear control condition, and how to deal with the effects of competitive relationship between agents. These are challenging problems for leader-following bipartite consensus for nonlinear MASs subject to exogenous disturbances.

In this paper, we fill in this gap by considering a leader-following bipartite consensus for a class of nonlinear MASs under directed fixed and switching topologies. While many existing works on control of bipartite utilize the relative states of neighboring agents to construct control laws, this information cannot be obtained in many real-world applications. Therefore, we decided to implement a distributed controller [48, 49], which have been already adopted for nonlinear dynamics, and for which promising results have been found in [50, 51]. However, differently from these works, in our formulation, we assume that the MAS is subject to exogenous disturbances. The study of MASs with exogenous disturbances has received growing attention recently [52, 53]. For leader-follower bipartite consensus, results have been derived for fixed [54, 55] and switching topologies [56]. In particular, disturbance observer-based control strategies were proposed in [57, 58]. Motivated by these works, we propose two disturbance observer-based control laws to steer the system to a bipartite consensus when agents interacts over fixed and switching topologies, respectively.

After having formally defined the two controllers and illustrated the algorithms to set the gain matrices, we performed a theoretical analysis of the proposed approaches. Through a Lyapunov-based argument, we prove that the two controllers are able to guarantee convergence of the system to a leader-following bipartite consensus. Our algorithms and theoretical findings are then illustrated via numerical simulations on some case studies based on a real-world MAS inspired by [46, 52] and formed by single-link manipulators with revolute joints actuated by DC motors, which interact over different (static or switching) topologies. The numerical findings show the good performances of the proposed controllers in different scenarios, corroborating our theoretical results. The main novelties of our approach with respect to the literature discussed in the above is summarized as follows:

- As opposite to many existing works, in which full relative states of neighboring agents are used [16, 41], we propose two distributed bipartite consensus control strategies based on output measurements, in which only the relative output information of neighboring agents is utilized.

- Different from [31, 42], where the dynamic of the system state is linear, we consider a large class nonlinear dynamics, which allows to represent more complex and realistic real-world dynamical systems.

- In real-world systems, disturbances are inevitably preset, as an effect of the presence of exogenous dynamics that may impact the MAS under study. In order to deal with this problem, a disturbance observer-based control protocol is formulated to estimate the exogenous disturbances and system states, which provides an effective solution to the MASs subject to exogenous disturbances.

- The patterns of real-world interactions between agents are typically complex, as they might be time-varying, cooperative or antagonistic. In this setting, the proposed controllers are able to deal with the presence of all these features in the context of leader-following, guaranteeing convergence to a leader-following bipartite consensus.

The rest of the paper is organized as follows. In Section 2 we report some definitions and preliminaries used in this paper. In Section 3, we formulate the problem. In Section 4, we present our main results, with proofs reported in Appendices A and B. In Section 5, we discuss three numerical examples. Section 6 concludes the paper and outlines future research.

2. Notation and Preliminaries

2.1. Notation

We gather here some notational conventions. We denote by $\mathbb{N}_+$, $\mathbb{R}$, and $\mathbb{R}_+$ the sets of positive integers, real numbers, and real positive numbers. Given a positive integer $N \in \mathbb{N}^+$, the N-dimensional Euclidean space is denoted by $\mathbb{R}^N$, $\mathbf{1}_N$ denotes the N-dimensional column vectors of all 1 and I_N is the $N \times N$ identity matrix. Given a scalar $x \in \mathbb{R}$, we denote by $|x|$ its absolute value. Given a vector $x \in \mathbb{R}^N$, $||x||$ denotes its 2-norm. Given a matrix $A \in \mathbb{R}^{N\times N}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent its the minimum and maximum eigenvalues. The symbol $\otimes$ denotes the Kronecker product, $\text{sgn}(\cdot)$ is the sign function, and $\text{diag}(\cdot)$ is the diagonalization operator. We also define the set of $N\times N$ matrices $\mathcal{S}_N = \text{diag}\{s_1, \dots, s_N\}$, where $s_i \in \{+1, -1\}$.

2.2. Preliminaries

We consider a system of N follower agents (also referred to as *followers*), labeled by positive integer numbers as $\mathcal{V} = \{1, \dots, N\}$, connected through a (signed di-)graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, which represents the communication topology between the followers. When not differently denoted, the graph is assumed to be fixed (that is time-invariant). Specifically, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges, where $(j, i) \in \mathcal{E}$ means that agent i can receive information from agent j, and $W \in \mathbb{R}^{N\times N}$ is the (signed) weighted adjacency matrix, in which its generic entry w_{ij} measures the information that i receives from j; hence $w_{ij} \neq 0$ if and only if $(j, i) \in \mathcal{E}, i \neq j$, and $w_{ij} = 0$ otherwise. We define the (signed) Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ of the graph as

$$l_{ij} := \begin{cases} \sum_{k\in\mathcal{V}\setminus\{i\}} |w_{ik}| & \text{if } i = j, \\ -w_{ij} & \text{if } i \neq j. \end{cases} \tag{1}$$

Define $\mathcal{N}_i := \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$ as the set of (in-)*neighbors* of agent i. A path from agent v_i to agent v_j is a sequence of edges $e_1 = (v_1, w_1), \dots e_m = (v_m, w_m) \in \mathcal{E}$ such that i) $v_1 = v_i$, ii) $w_{k-1} = v_k$, $k = 2, \dots, m$ and iii) $w_m = v_j$. In the following, we list some definitions and some technical results that will be used in the following of the paper.

Definition 1 (Spanning tree). *A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is said to contain a spanning tree if there exists a agent i such that there exists a path from j to any other agent $j \in \mathcal{V} \setminus \{i\}$. The agent i is said to be the* root *of the spanning tree.*

Definition 2 (Structural balance). *A signed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is said to be structurally balanced if there exists a partition of the agent set $\mathcal{V}_1$, $\mathcal{V}_2$ satisfying i) $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, ii) $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, iii) $w_{ij} \geq 0, \forall\, v_i, v_j \in \mathcal{V}_k\, (k \in \{1,2\})$, and iv) $w_{ij} \leq 0, \forall\, v_i \in \mathcal{V}_k, v_j \notin \mathcal{V}_k\, (k \in \{1,2\})$. Otherwise, it is called structurally unbalanced.*

Note that the notion of structural balance derives from the balance theory in the social psychology literature [19], for which a social system is in psychological balance if friends of friends are friends, enemies of friends are enemies, and enemies of enemies are friends.

Lemma 1 (Lemma 1 from [23]). *For a structurally balanced graph $\mathcal{G}$, there exists a diagonal matrix $S \in \mathcal{S}_N$ such that all diagonal elements of SWS are* *nonnegative. Moreover, the diagonal entries of SLS are nonnegative and all off-diagonal entries of SLS are nonpositive. Besides, $S \in \mathcal{S}_N$ induces a partition $\mathcal{V}_1 = \{i : s_i > 0\}$ and $\mathcal{V}_2 = \{i : s_i < 0\}$ that satisfies properties i)–iv) in Definition 2.*

In this paper, we will consider an augmented graph $\mathcal{G}_R$ formed by the set of N followers and a leader, which will be labeled as v_0. Some of the followers may be access the leader's information. We define by $\mathcal{N}_0$ the set of neighbors of the leader. The augmented graph has thus agent set $\mathcal{V}_R = \mathcal{V} \cup \{v_0\}$ and edge set $\mathcal{E}_R = \mathcal{E} \cup \{(j, 0) : j \in \mathcal{N}_0\}$. We define a nonnegative $N \times N$-dimensional diagonal matrix $R = \text{diag}([a_{10}, \dots a_{N0}])$, whose entry $w_{i0} \geq 0$ measures how much follower i interacts with the leader, with the understanding that $w_{i0} > 0$ if and only if $i \in N_0$, whereas $w_{i0} = 0$ if $i \notin N_0$. Then, we can define the *augmented* Laplacian matrix of the leader-follower system $L_R \in \mathbb{R}^{N \times N}$ as $L_R = L + R$.

Lemma 2 (Lemma 6 from [41]). *Suppose that the graph $\mathcal{G}_R$ consisting of a leader and the N followers contains a directed spanning tree with the leader located at the root, then L_R is positive definite and there exists a matrix $\Theta = \text{diag}\{\varphi_1, \cdots, \varphi_{\mathrm{N}}\}$, with $\varphi_i > 0$ such that $\Theta L_R + L_R^\top \Theta > 0$, where $L_R^\top \varphi = \mathbf{1}_N$ and $\varphi = [\varphi_1, \cdots, \varphi_N]^\top \in \mathbb{R}^N$.*

In the second part of this paper, we will deal with switching topologies. A (directed, signed) switching topology $(\tilde{\mathcal{G}}, \varpi)$ is defined by a set of τ directed and signed graphs $\tilde{\mathcal{G}} = \{\mathcal{G}^1 = (\mathcal{V}, \mathcal{E}^1, W^1), \cdots, \mathcal{G}^\tau = (\mathcal{V}, \mathcal{E}^\tau, W^\tau)\}$ and a *switching signal* $\varpi : \mathbb{R}_+ \to \{1, \cdots, \tau\}$, which assigns to each nonnegative time $t \in \mathbb{R}_+$ a communication topologies $\mathcal{G}^{\varpi(t)}$. The switching signal is piecewise constant, over the time intervals $[t_k, t_{k+1}), k \in \mathbb{N}$, with $t_1 = 0, \hat{\tau}_0 \leq t_{k+1} - t_k \leq \hat{\tau}_1$, in which the positive constant $\hat{\tau}_0$ is termed *dwell time* and denotes the minimum time between two changes in the topology.

3. Problem Formulation

We consider a MAS consisting of N follower agents and one leader agent. Each agent $i \in \mathcal{V}$ is characterized by a n-dimensional *state vector* $x_i(t) \in \mathbb{R}^n$ and a r-dimensional *output measurement* $z_i(t) \in \mathbb{R}^r$, which evolve in continuous time $t \in \mathbb{R}$. We assume that the all the followers have the same internal dynamics, which may then be subject to external inputs and exogenous disturbances. Specifically, the dynamics of the ith follower agent and is described by the following system of nonlinear ordinary differential equations:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + f(x_i(t), t) + Bu_i(t) + Dw_i(t), \\ z_i(t) &= Cx_i(t), \end{aligned} \tag{2}$$

where $u_i(t) \in \mathbb{R}^m$ is the m-dimensional input vector; $f(\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear function, common to all the agents, which is continuous and differentiable in t; $w_i(t) \in \mathbb{R}^p$ denotes the exogenous disturbance, which is generated by

$$\dot{w}_i(t) = Mw_i(t), \tag{3}$$

in which $M \in \mathbb{R}^{p\times p}$ is an external matrix; and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $D \in \mathbb{R}^{n\times p}$ are constant matrices.

We assume that the leader's state $x_0(t) \in \mathbb{R}^n$ is not influenced by external inputs and disturbances. Hence, the dynamic of the leader, labeled by 0, is defined as

$$\dot{x}_0(t) = Ax_0(t) + f(x_0(t), t). \tag{4}$$

In the rest of this paper, we will make the following assumptions on the agents' dynamics.

Assumption 1. *The pair (A, B, C) is stabilizable and detectable.*

Assumption 2. *There exists a matrix $F \in \mathbb{R}^{m \times p}$ such that $D = BF$.*

Assumption 3. *The matrix M has k distinct eigenvalues and the real part of each eigenvalue is zero. Moreover, (M, D) is observable.*

Assumption 4. *There exists a positive constant $\rho > 0$ such that*

$$\|f(a_1, t) - s_i f(a_2, t)\| \leq \rho \|a_1 - s_i a_2\|, \ \ \forall\, a_1, a_2 \in \mathbb{R}^n. \tag{5}$$

In our analysis, we focus on the study of the convergence to a leader-following bipartite consensus for the MAS of N followers and one leader defined in (2) and (4), which is defined as follows.

Definition 3 (Leader-following bipartite consensus). *Consider a MAS with N followers $\mathcal{V}$ and one leader, denoted by index 0. Then, we say that the MAS converges to a leader-following bipartite consensus if there exists a partition $\mathcal{V}_1, \mathcal{V}_2$ with $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that*

$$\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0, \quad \forall\, i \in \mathcal{V}_1, \tag{6}$$

$$\lim_{t\to\infty} \|x_i(t) + x_0(t)\| = 0, \quad \forall\, i \in \mathcal{V}_2, \tag{7}$$

which can be written in a general form as

$$\lim_{t\to\infty} \|x_i(t) - s_i x_0(t)\| = 0, \quad \forall\, i \in \mathcal{V}, \tag{8}$$

for some $\mathrm{diag}([s_1, \ldots, s_n]) \in \mathcal{S}_n$.

Note that, differently from other notions of consensus [3], in Definition 3 we are not requiring that the leader converges to a fixed point, but we say that a leader-following bipartite consensus is achieved if the entire system synchronizes toward a trajectory in which a set of followers has the same state of the leader, and the remaining followers have the opposite state.

Remark 1. *As indicated in [42], Assumption 2 presents a matching condition under which the disturbance effects can be compensated through the control action. A sufficient criterion for the existence of the matrix F is that* $\mathrm{rank}(B, D) = \mathrm{rank}(B)$. *Since F may not be equal to I_m, the disturbances can be imposed on some channels other than the control input channels.*

Remark 2. *Equation* (3) *is mainly motivated by the disturbance observer-based approach for MASs in the work of [57]. It is to reflect the deterministic disturbances such as constants and sinusoidal disturbances, and it covers a wide range of periodic disturbances such as the sinusoidal functions upon which many other functions can be approximated with a bias. Moreover, Assumption 3 on the eigenvalues of M is commonly used for disturbance rejection and output regulation. If the eigenvalues of the matrix M are strictly located in the left-half plane, the disturbance is stable.*

4. Main Results

In this section, we solve the leader-following bipartite consensus problem for nonlinear MASs with deterministic exogenous disturbances over directed topologies by utilizing an observer-based approach. Specifically, we will start considering fixed topologies. Then, we will generalize our results to switching topologies.

4.1. Bipartite consensus under directed fixed topologies

In this subsection, we consider the bipartite consensus for the MASs in (2)–(4) over a directed fixed topology containing a spanning tree. Specifically, we make the following assumption.

Assumption 5. *Suppose that the augmented graph $\mathcal{G}_R$ consisting of the N followers and the leader is static and contains a directed spanning tree with the leader located at the root and that the signed network is structurally balanced.*

An observer-based controller based on the output measurements is developed by defining the following input functions for the followers:

$$\begin{aligned} u_i(t) =& \beta K_1 [\sum_{j\in\mathcal{V}} |w_{ij}| \left(\hat{x}_i(t) - \operatorname{sgn}\left(w_{ij}\right) \hat{x}_j(t)\right) \\ &+ w_{i0}\left(\hat{x}_i(t) - s_i x_0\left(t\right)\right)] - F\hat{w}_i(t), \quad i \in \mathcal{V}, \end{aligned} \tag{9}$$

where $\beta > 0$ is a coupling strength, K_1 is the feedback gain matrix, $\hat{x}_i(t)$ is the state observer, and $\hat{w}_i(t)$ is the disturbance estimation vector. The evolution of these two functions is defined as follows:

$$\begin{aligned}\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + f\left(\hat{x}_i\left(t\right), t\right) + Bu_i(t) + D\hat{w}_i\left(t\right) - \alpha\tilde{F}\Bigg[\sum_{j\in\mathcal{V}}|w_{ij}|\cdot \\ &\cdot\left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{ij}\right)\tilde{\xi}_j\left(t\right)\right) + w_{i0}\left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{i0}\right)\tilde{\xi}_0\left(t\right)\right)\Bigg],\end{aligned} \tag{10}$$

and

$$\begin{aligned}\dot{\hat{w}}_i(t) =& M\hat{w}_i(t) - G_1\Bigg[\sum_{j\in\mathcal{V}}|w_{ij}|\left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{ij}\right)\tilde{\xi}_j\left(t\right)\right) \\ &+ w_{i0}\left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{i0}\right)\tilde{\xi}_0\left(t\right)\right)\Bigg],\end{aligned} \tag{11}$$

for all $i \in \mathcal{V}$, respectively, where $\alpha > 0$ is a coupling strength, $\tilde{F}$ is the state observer gain matrix, G_1 is the disturbance observer gain matrix, and $\tilde{\xi}_i\left(t\right) =: z_i\left(t\right) - C\hat{x}_i\left(t\right)$ is the error between the measurement output, $z_i(t)$, and the corresponding quantity computed from the state observer, $C\hat{x}_i\left(t\right)$. Since the leader acts as a reference signal generator, it is supposed that $\hat{x}_0\left(t\right) = x_0\left(t\right)$, i.e., the leader does not need to observe its own state as it knows it, and, consequently, it holds $\tilde{\xi}_0\left(t\right) = z_0\left(t\right) - C\hat{x}_0\left(t\right) = z_0\left(t\right) - Cx_0\left(t\right) = 0$.

Being the signed graph structurally balanced (Assumption 5), let $S = \operatorname{diag}([s_1, \ldots, s_N]) \in \mathcal{S}_N$ be the diagonal matrix induced by the partition defined in Lemma 1. In order to analyze the MAS in (2)–(4) under the observer-based controller defined in 9–11, we define the following three errors:

$$\bar{\xi}_i(t) = x_i(t) - \hat{x}_i(t), \quad \hat{\xi}_i(t) = \hat{x}_i(t) - s_i x_0\left(t\right), \quad e_i(t) := w_i(t) - \hat{w}_i(t), \tag{12}$$

between the state of agent i and its observer (*observer error*), between the state observer of agent i and the state of the leader or its opposite depending on the set $i \in \mathcal{V}$ belongs to (*consensus tracking error*), and between the disturbance and its observer, respectively. Note that, if the three quantities in (12) converges to 0 as $t \to \infty$ for all $i \in \mathcal{V}$, then the observer-based controller is well-defined and a bipartite leader-following consensus is achieved according to Definition 3). Hence, we will utilize these three quantities to study the system.

Specifically, utilizing the definitions in (12), we can write the dynamics for the state and its observer as

$$\begin{aligned}\dot{x}_i(t) =& Ax_i(t) + f\left(x_i(t), t\right) + \beta BK_1\Bigg[\sum_{j\in\mathcal{V}} |w_{ij}| \left(\hat{x}_i(t) - \operatorname{sgn}\left(w_{ij}\right)\hat{x}_j(t)\right) \\ &+ w_{i0}\left(\hat{x}_i(t) - s_i x_0\left(t\right)\right)\Bigg] + De_i(t),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + \beta BK_1\Big[\sum_{j\in\mathcal{V}} |w_{ij}| \left(\hat{x}_i(t) - \operatorname{sgn}\left(w_{ij}\right)\hat{x}_j(t)\right) \\ &+ w_{i0}\left(\hat{x}_i(t) - s_i x_0\left(t\right)\right)\Big] + f\left(\hat{x}_i\left(t\right), t\right) \\ &- \alpha\tilde{F}\Big[\sum_{j=\in\mathcal{V}} |w_{ij}| \left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{ij}\right)\tilde{\xi}_j\left(t\right)\right) + w_{i0}\tilde{\xi}_i\left(t\right)\Big],\end{aligned} \tag{14}$$

respectively. From which we can derive the following dynamics for the three errors in (12):

$$\dot{\bar{\xi}}_i(t) = A\bar{\xi}_i(t) + f(x_i(t), t) - f(\hat{x}_i(t), t) + De_i(t) \tag{15}$$
$$+ \alpha\tilde{F}\left[\sum_{j\in\mathcal{V}} |w_{ij}| \left(\tilde{\xi}_i(t) - \operatorname{sgn}(w_{ij})\tilde{\xi}_j(t)\right) + w_{i0}\tilde{\xi}_i(t)\right],$$
$$\dot{\hat{\xi}}_i(t) = A\hat{\xi}_i(t) + f(\hat{x}_i(t), t) - s_i f(x_0(t), t) + Bu_i(t) \tag{16}$$
$$- \alpha\tilde{F}\left[\sum_{j\in\mathcal{V}} |w_{ij}| \left(\tilde{\xi}_i(t) - \operatorname{sgn}(w_{ij})\tilde{\xi}_j(t)\right) + w_{i0}\tilde{\xi}_i(t)\right],$$
$$\dot{e}_i(t) = Me_i(t) + G_1\left[\sum_{j\in\mathcal{V}} |w_{ij}| \left(\tilde{\xi}_i(t) - \operatorname{sgn}(w_{ij})\tilde{\xi}_j(t)\right) + w_{i0}\tilde{\xi}_i(t)\right], \tag{17}$$

where the latter is obtained using that $\operatorname{sgn}(w_{ij})s_j = s_i$, which holds true due to Lemma 1 and property iv) in Definition 3.

In the following, we will show that, if Assumptions 1–3 and 5 hold true, then one can establish an algorithmic procedure to design the gain matrices for the observer-based control law in (9)–(11). Specifically, we propose the following algorithm.

Algorithm 1. *Assume Assumptions 1–3 and 5 hold. Then, we define the following steps:*

1. *Set four positive scalars* $\mu_0 > 0$*,* $\mu_1 > 0$*,* $c_1 > 0$*, and* $c_2 > 0$*, and choose coupling strengths* $\alpha > \mu_0/\lambda_0$ *and* $\beta > \mu_1/\lambda_0$*, where* $\lambda_0 := \lambda_{\min}\left(L_R + \Theta^{-1}L_R^\top\Theta\right)$*, with* Θ *from Lemma 2.*
2. *Solve the following two matrix inequalities to get a matrix* P *that veri-*

fies:

$$\begin{bmatrix} A^TP + PA - \mu_0 C^TC + \gamma_1 I + c_1 P & P & D^TP \\ P & -I & 0 \\ * & 0 & -I \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} A^TP + PA - \mu_1 PB^TBP + \gamma_2 I + c_2 P & P \\ P & -I \end{bmatrix} < 0, \tag{19}$$

with $\gamma_1 > (\alpha+1)\,\lambda_{\max}\left(\Theta^{-1}L_R^\top \Theta L_R\right) + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2$*,* $\gamma_2 > \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 + \alpha\lambda_{\max}(C^\top CC^\top C)$*, and* $\rho > 0$ *such that verifies Assumption 4.*

3. *Solve the following linear matrix inequality to get a matrix* Q *that verifies:*

$$QM + M^\top Q + \gamma_3 I + c_3 Q < 0, \tag{20}$$

with $c_3 > 0$ *and* $\gamma_3 > \lambda_{\max}(C^\top CC^\top C) + 1$*.*

4. *Define* $K_1 = -B^TP$*,* $G_1 = Q^{-1}C^T$*, and* $\tilde{F} = -P^{-1}C^\top$*.*

At this stage, we can analytically prove that the observer-based control law in (9)–(11) with gain matrices defined via Algorithm 1 solves the leader-following bipartite consensus for nonlinear MAS in (2)–(4) on a structurally balanced communication network with a directed spanning tree with leader located at the root (Assumption 5). The following result formally guarantees our claim. The proof, which is based on a Lyapunov argument to show convergence to 0 for the three quantities in (12), is quite cumbersome and is thus reported in Appendix A, for the sake of readability.

Theorem 1. *Consider the nonlinear MAS in* (2) *and* (4) *on a (static) signed communication network with the deterministic disturbances from* (3)*. Let*

Assumptions 1–3 and 5 hold. Then, the observer-based control law in (9)–(11) *with gain matrices defined via Algorithm 1 solves the leader-following bipartite consensus, as defined in Definition 3.*

Remark 3. *For general nonlinear MASs, it may be challenging to design distributed protocols only based on relative states of neighboring agents over directed networks to eliminate the effects of the nonlinear term, and the state feedback control approach is no longer applicable. To amend the drawback of this fact, the disturbance observer approach and output feedback control approach have been proved to be significant in dealing with the bipartite consensus problem of nonlinear MASs with exogenous disturbances. Furthermore, Assumption 4 is the so-called Lipschitz condition and all linear and piecewise-linear time-invariant continuous functions satisfy this condition. However, it is still an open topic that how to ensure bipartite consensus for MASs under directed switching topologies and unknown continuous-time nonlinear dynamics which do not satisfy the Lipschitz condition. The design approaches presented in* [52, 48, 50] might be useful for investigating this topic.

4.2. Bipartite consensus under directed switching topologies

In this subsection, we consider the leader-following bipartite consensus for MAS in (2)–(4) over switching topologies $(\tilde{\mathcal{G}}, \varphi)$, where $\tilde{\mathcal{G}} = \{\mathcal{G}_1, \ldots, \mathcal{G}_\tau\}$ is the set of signed graphs and ϖ is the switching signal, with dwell time $\hat{\tau}_0 > 0$. We will consider the scenario in which the switching topology that characterizes the augmented graph has always a spanning tree and is structurally balanced, that is, the following assumption is verified.

Assumption 6. *Assume that the all the augmented graphs in $\tilde{\mathcal{G}}_R$ contain a directed spanning tree with the leader located at the root node and they are structurally balanced and admit the same partitioning $\mathcal{V}_1$ and $\mathcal{V}_2$.*

A direct consequence of applying Lemma 1 to all the graphs in $\tilde{\mathcal{G}}_R$ is that the following result can be directly claimed.

Lemma 3. *If Assumption 6 holds, then there exists a diagonal matrix $\Theta^\Gamma = \operatorname{diag}\{\varphi_1^\Gamma, \cdots, \varphi_N^\Gamma\}$, with $\varphi_i^\Gamma > 0$ such that $\Theta^\Gamma L_R^\Gamma + \left(L_R^\Gamma\right)^\top \Theta^\Gamma > 0$, where $\left(L_R^\Gamma\right)^\top \varphi^\Gamma = \mathbf{1}_N$ and $\varphi^\Gamma = \left[\varphi_1^\Gamma, \cdots, \varphi_N^\Gamma\right]^\top \in \mathbb{R}^N$, for any $\Gamma = 1, \ldots, \tau$.*

Similar to the scenario of static topologies, we propose an observer-based controller to guarantee leader-follower bipartite consensus for the MAS. In particular, for each agent $i \in \mathcal{V}$, we propose the following input function:

$$
\begin{aligned}
u_i(t) =& \tilde{\beta} K_2 \Bigg[\sum_{j\in\mathcal{V}} \left| w_{ij}^{\varpi(t)} \right| \left(\hat{x}_i(t) - \operatorname{sgn}\left(w_{ij}^{\varpi(t)} \right) \hat{x}_j(t) \right) \\
&+ w_{i0}^{\varpi(t)} \left(\hat{x}_i(t) - s_i x_0(t) \right) \Bigg] - F \hat{w}_i(t),
\end{aligned} \tag{21}
$$

where $\tilde{\beta} > 0$ is a coupling strength, K_2 is the feedback gain matrix, $\hat{x}_i(t)$ is the state observer, and $\hat{w}_i(t)$ is the disturbance estimation vector. The evolution of these two functions is defined as follows:

$$
\begin{aligned}
\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + f\left(\hat{x}_i(t), t\right) + Bu_i(t) + D\hat{w}_i(t) - \tilde{\alpha}\hat{F} \Bigg[\sum_{j\in\mathcal{V}} \left| w_{ij}^{\varpi(t)} \right| \cdot \\
&\cdot \left(\left(\tilde{\xi}_i(t) - \operatorname{sgn}\left(w_{ij}\right) \tilde{\xi}_j(t) \right) + w_{i0}^{\varpi(t)} \left(\tilde{\xi}_i(t) - \operatorname{sgn}\left(w_{i0}^{\varpi(t)} \right) \tilde{\xi}_0(t) \right) \right] ,
\end{aligned} \tag{22}
$$

and

$$\begin{aligned}\dot{\hat{w}}_i(t) =& M\hat{w}_i(t) - G_2\Bigg[\sum_{j\in\mathcal{V}}\left|w_{ij}^{\varpi(t)}\right|\left(\left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{ij}\right)\tilde{\xi}_j\left(t\right)\right)\right.\\ &+w_{i0}^{\varpi(t)}\left(\tilde{\xi}_i\left(t\right) - \operatorname{sgn}\left(w_{i0}^{\varpi(t)}\right)\tilde{\xi}_0\left(t\right)\right)\Bigg],\end{aligned} \tag{23}$$

respectively, where $\tilde{\alpha} > 0$ is a coupling strength, $\hat{F}$ is the state observer gain matrix, G_2 is the disturbance observer gain matrix, and $\tilde{\xi}_i\left(t\right) =: z_i\left(t\right) - C\hat{x}_i\left(t\right)$ is the error between the measurement output, $z_i(t)$, and the corresponding quantity computed from the state observer, $C\hat{x}_i\left(t\right)$. Similar to the scenario with fixed topology, since the leader acts as a reference signal generator, it is supposed that $\hat{x}_0\left(t\right)=x_0\left(t\right)$, i.e., the leader does not need to observe its own state as it knows it, and, consequently, $\tilde{\xi}_0\left(t\right) = 0$. Similar to the analysis of the scenario with static topologies, we can study the convergence of the system to a leader-following bipartite consensus by studying the convergence of the errors in (12) to 0.

In the following, we will show that, if Assumptions 1–3 and 6 hold true, then a procedure to design the gain matrices for the observer-based control law in (21)–(23) can be designed, according to the following algorithm.

Algorithm 2. *Assume Assumptions 1–3 and 6 hold. Then, we define the following steps:*

1. *Set four positive scalars $\tilde{\mu}_0 > 0$, $\tilde{\mu}_1 > 0$, $\tilde{c}_1 > 0$, and $\tilde{c}_2 > 0$, and choose coupling strengths $\tilde{\alpha} > \tilde{\mu}_0/\tilde{\lambda}_0$ and $\tilde{\beta} > \tilde{\mu}_1/\tilde{\lambda}_0$, where $\tilde{\lambda}_0 := \min_{\Gamma=1,\cdots,\tau}\left(\lambda_{\min}\left(L_R^\Gamma + \left(\Theta^\Gamma\right)^{-1}\left(L_R^\Gamma\right)^T\Theta^\Gamma\right)\right)$, with Θ^Γ from Lemma 3.*

2. *Solve the following two matrix inequalities:*

$$\begin{bmatrix} A^\top \bar{P} + \bar{P}A - \tilde{\mu}_0 C^\top C + \gamma_4 I + \tilde{c}_1 \bar{P} & \bar{P} & D^\top \bar{P} \\ \bar{P} & -I & 0 \\ * & 0 & -I \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} A^\top \bar{P} + \bar{P}A - \tilde{\mu}_1 \bar{P} B^\top B \bar{P} + \gamma_5 I + \tilde{c}_2 \bar{P} & \bar{P} \\ \bar{P} & -I \end{bmatrix} < 0, \tag{25}$$

to get a matrix $\bar{P}$*, with* $\gamma_5 > \max_{\Gamma=1,\ldots,\tau} \frac{\lambda_{\max}(\Theta^\Gamma)}{\lambda_{\min}(\Theta^\Gamma)} \rho^2 + \tilde{\alpha} \lambda_{\max}(C^\top C C^\top C)$*,* ρ *is defined in Assumption 4, and*

$$\gamma_4 > \max_{\Gamma=1,\ldots,\tau} \frac{\lambda_{\max}\left(\Theta^\Gamma\right)}{\lambda_{\min}\left(\Theta^\Gamma\right)} \rho^2 + (\tilde{\alpha}+1)\, \lambda_{\max}\left((\Theta^\Gamma)^{-1} \left(L_R^\Gamma\right)^\top \Theta^\Gamma L_R^\Gamma \right).$$

3. *Solve the LMI as follows:*

$$\bar{Q}M + M^\top \bar{Q} + \gamma_6 I + \tilde{c}_3 \bar{Q} < 0, \tag{26}$$

to get a matrix $\bar{Q}$*, with scalars* $\tilde{c}_3 > 0$*,* $\gamma_6 > \lambda_{\max}(C^\top C C^\top C) + 1$*.*

4. *Set* $K_2 = -B^T \bar{P}$*,* $\hat{F} = -\bar{P}^{-1} C^T$*, and* $G_2 = \bar{Q}^{-1} C^\top$*.*

Similar to the scenario with fixed topologies, we can analytically prove that the observer-based control law in (21)–(23) with gain matrices defined via Algorithm 2 solves the leader-following bipartite consensus for nonlinear MAS in (2)–(4) on switching topologies that verify Assumption 6. The following result, whose proof is reported in Appendix B, formally guarantees our claim, under some conditions on the dwell time $\hat{\tau}_0$.

Theorem 2. *Consider the nonlinear MAS in* (2) *and* (4) *on switching topologies with deterministic disturbances from* (3)*. Let Assumptions 1–3 and*

6 hold and let the dwell time satisfy the inequality $\hat{\tau}_0 > \ln(\ell_0/\hat{c}_0)$*, where* $\hat{c}_0 = \min_{i\in\{1,2,3\}}\{\hat{c}_i\}$*,* $\ell_0 = \varphi_{\max}/\varphi_{\min}$*,* $\varphi_{\min} = \min_{\Gamma\in\{1,\dots,\tau\},i\in\{1,\dots,N\}}\{\varphi_i^\Gamma\}$*, and* $\varphi_{\max} = \max_{\Gamma\in\{1,\dots,\tau\},i\in\{1,\dots,N\}}\{\varphi_i^\Gamma\}$*. Then, the observer-based control law in* (21)–(23) *with gain matrices defined via Algorithm 2 solves the leader-following bipartite consensus, as defined in Definition 3.*

We conclude this section with some general remarks on the algorithms and the theoretical findings presented in this section and a discussion of their relation with the existing literature.

Remark 4. *The control protocols* (9) *and* (21) *are partly motivated by the observer-type protocols for MASs proposed in [42]. However, the observer designed here is indeed different from that in [42]. In fact, our assumptions are less restrictive, as we only require that a follower can access the information on its state observer, and not the actual state, as in [42].*

Remark 5. *According to Algorithms 1 and 2, the existence of the gain matrices to be used in the controller depends on the possibility to solve a set of matrix inequalities. It is easy to observe that the matrix inequalities in Algorithm 2 are feasible if and only if the matrix inequalities in Algorithm 1 are feasible. However, observe that the positive scalars* γ_4*,*γ_5*,* $\tilde{\lambda}_0$*, and matrix* Θ^Γ *defined in Algorithm 2 depends on the characteristics of the switching topologies, so the choices of parameters and the solutions of the matrix inequalities are in general different for the two algorithms.*

5. Numerical examples

In this section, we propose and discuss three examples to illustrate our theoretical findings and demonstrate the performance of the algorithms we

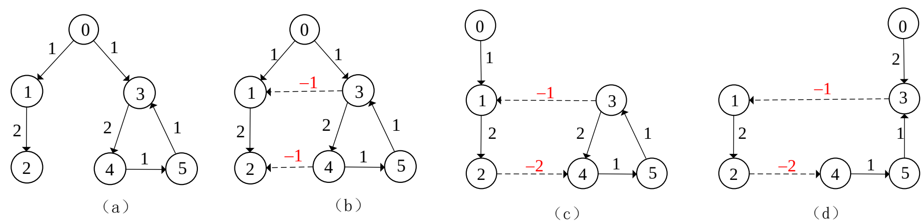


Figure 1: The four different topologies considered in the two examples: (a) $\mathcal{G}_1$, (b), $\mathcal{G}_2$, (c) $\mathcal{G}_3$, and (d) $\mathcal{G}_4$. Solid arrows with black text denote positive entries of the signed weighted adjacency matrix W, dashed arrows with red text denote negative entries of W.

developed. In the first example, we consider a scenario with a fixed topology and we show that the observer-based controller proposed in Algorithm 1 is able to steer the system to leader-following bipartite consensus. We consider the scenario of a linear dynamics and we discuss the characteristics of the consensus state reached, depending on the topology structure. In the second example, we consider a nonlinear dynamics on a static signed topology, and we show that our algorithm is also able to deal with nonlinear scenarios, where other methods proposed in the literature fail [31, 42]. In the third example, we consider a scenario of nonlinear dynamics on switching topologies, and we illustrate how Algorithm 2 can be used to design an observer-based controller for the system, whereas linear controllers and observers proposed in the literature cannot be used [52].

In the examples, we consider a network of six agents interacting according to four different topologies, labeled as $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$, and illustrated in Fig. 1. Observe that all the six topologies are structurally balanced, with the same partition equal to $\mathcal{V}_1 = \{1,2\}, \mathcal{V}_2 = \{3,4,5\}$.

Similar to [46, 52], we consider a MAS consisting of six single-link manipulators with revolute joints actuated by DC motors: $N = 5$ follower agents and one leader agent, labeled as agent 0. A schematic illustration of the physical system is reported in Fig. 2. The state of each agent is characterized by a 4-dimensional vector $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t), x_{i4}(t))^\top$, where $x_{i1}(t)$ is the angular rotation of the motor, $x_{i2}(t)$ is the angular velocity of the motor, $x_{i3}(t)$ is the angular rotation of the link of the ith manipulator, and $x_{i4}(t)$ is the angular velocity of the link of the ith manipulator. Similar to [46, 52], the dynamic of the ith manipulator can be written in the form of (2), with

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & 0 & 10 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}^\top, \quad D = B \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top,
\end{aligned}
\tag{27}
$$

and nonlinear function $f(x_i(t), t) = (0, 0, 0, 0.333 \sin(x_{i3}(t)))^\top$, $i \in \mathcal{V}$, when present. The disturbances are generated by (3) with $M = \begin{bmatrix} 0 & 1.5 \\ -1.5 & 0 \end{bmatrix}$. It is easy to obtain that $D = BF$. Also, it is easy to verify that Assumptions 1–3 hold, and that Assumptions 4 holds with $\rho = 0.333$. In all the simulations, the initial conditions are assigned randomly.

Example 1. *We consider the MAS made of six single-link manipulators with dynamics defined in* (2) *and* (27)*, interacting on the fixed topologies* $\mathcal{G}_1$ *and*

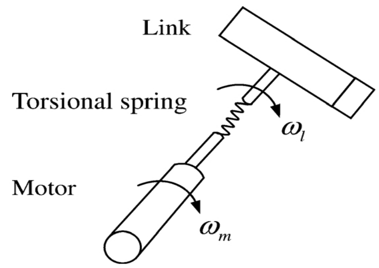
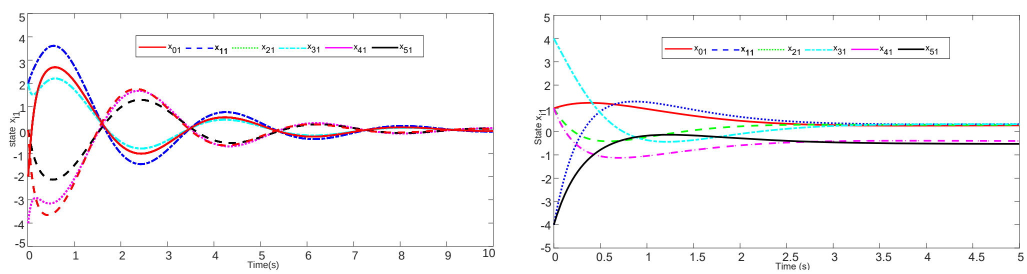


Figure 2: Schematic of the single-link manipulator with a flexible joint.



(a) Topology $\mathcal{G}^1$ (b) Topology $\mathcal{G}^2$

Figure 3: Temporal evolution of $x_{i1}(t)$ under the linear dynamics in Example 1, for two different topologies.

$\mathcal{G}_2$, *illustrated in Figs. 1(a) and (b), respectively, where the first topology does not contain antagonistic edges, whereas the second does. We construct the observer-based controller utilizing the proposed Algorithm 1, for which convergence is guaranteed by Theorem 1. Specifically, by simple calculations, we get* $\Theta = \mathrm{diag}\{2.0422, 1.0060, 0.9012, 2.3980\}$. *We take* $c_1 = 1$, $c_2 = 1$, $c_3 = 3$, $\gamma_1 = 5$, $\gamma_2 = 7$, $\gamma_3 = 6$, *and following step 1) of Algorithm 1, we set* $\alpha = 34$, $\beta = 21$, $\mu_0 = 10$, *and* $\mu_1 = 6$, *it yields that* $\lambda_0 = 0.3173$. *Note that step 2) in Algorithm 1 holds if*

$$\begin{bmatrix} A^\top P + PA + \gamma_2 I + c_2 P & P \\ P & -I \end{bmatrix} < 0. \tag{28}$$

Then, based on (28) *and following steps 2) and 3) of Algorithm 1, we compute the following matrices* P *and* Q

$$P = \begin{bmatrix} 0.7373 & 0.0147 & -0.6210 & -0.2588 \\ 0.0147 & 0.0162 & -0.0063 & 0.0175 \\ -0.6210 & -0.0063 & 1.0708 & -1.0625 \\ -0.2588 & 0.0175 & -1.0625 & 3.7443 \end{bmatrix},$$

$$Q = \begin{bmatrix} 3.9492 & 0.1854 & -1.2871 & -0.6210 \\ 0.1854 & 4.5754 & -1.2934 & -0.1762 \\ -1.2871 & -1.2934 & 3.6787 & -1.2641 \\ -0.6210 & -0.1762 & -1.2641 & 3.3694 \end{bmatrix},$$

respectively, which are used in step 4) of Algorithm 1 to compute the gain *matrices:*

$$G_1 = \begin{bmatrix} 0.3679 & 0.2960 & 0.3001 & 0.1959 \\ 0.2997 & 0.1963 & 0.6546 & 0.6079 \end{bmatrix}^\top,$$

$$\tilde{F} = \begin{bmatrix} -1.1711 & -60.8337 & -1.1634 & -0.1267 \\ -15.5325 & 14.2425 & -15.6654 & -5.8523 \end{bmatrix}^\top,$$

and $K_1 = [-0.3167\ -0.3508\ 0.1363\ -0.3781]$*. We consider the linear dynamics, that is, excluding the term* f*, and we implement the proposed observer-based controller. Figure 3 shows that the followers asymptotically approach the leader's state in the absence of antagonistic edges (*$\mathcal{G}_1$*), while in their presence (*$\mathcal{G}_2$*), the system converges to a leader-following bipartite consensus. Interestingly, when considering the trajectories with antagonistic edges reported in Fig. 3b, it can be observed that the states of agents 1 and 2 can asymptotically approach those of the leader, while the states of agents 3, 4 and 5 asymptotically track the opposite values of those of the leader, which is in accordance with the cluster partition* $\mathcal{V}_1$ *and* $\mathcal{V}_2$*.*

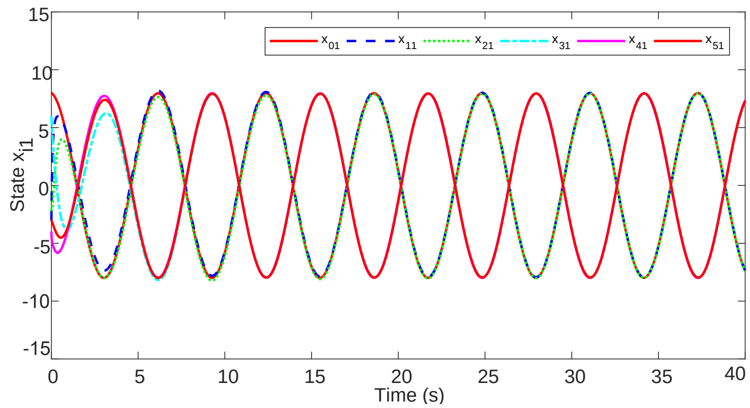
Example 2. *Then, we consider the nonlinear MAS made of the six single-link manipulators with dynamics defined in* (2) *and* (27)*, interacting on the fixed topology* $\mathcal{G}_2$*. The results are reported in Fig. 4. Figure 4a reports the temporal evolution of the state of the agents, when the observer-based controller is enacted, showing that the state of the followers converges to a leader-follower bipartite consensus. Note that many existing controllers proposed in the literature cannot handle this scenario [31, 42]. Figures 4b and 4c depicts the error between the (signed) leader's state and each follower agent under the proposed control protocol for different coupling gains* α *and* β*; these figures* *shows how the errors quickly vanish. Figures. 4d and 4e shows the temporal evolution of the disturbances and disturbance observers. These figures shows that the disturbance observer exhibits an excellent estimation performance.*

Example 3. *Finally, we consider the nonlinear MAS in the scenario of switching topologies, and we provide some simulation results to illustrate the effectiveness of the control protocol proposed in Algorithm 2, and whose effectiveness is discussed in Theorem 2. We consider the MAS of six single-link manipulators under switching topologies with antagonistic edges shown in panels (c) and (d) of Fig. 1. Note that it is not difficult to get* (A, B, C) *stabilizable and detectable. Assume that the network topologies switches periodically between* $\mathcal{G}_3$ *and* $\mathcal{G}_4$ *every* $0.4s$. *Assume Assumption 6 holds, then, following step 1) of Algorithm 2, let* $\tilde{\alpha} = 29$, $\tilde{\beta} = 32$, $\tilde{\mu}_0 = 20$, $\tilde{\mu}_1 = 8$, $\gamma_4 = 9$, $\gamma_5 = 6$, $\gamma_6 = 5$, $\tilde{c}_1 = 1$, $\tilde{c}_2 = 2$, *and* $\tilde{c}_3 = 4$, *it yields that* $\tilde{\lambda}_0 = 0.7218$. *Similar to Example 2, following steps 2) and 3) of Algorithm 2, we compute*

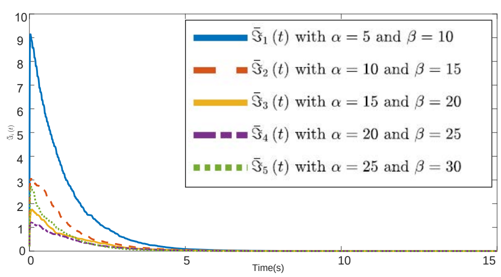
$$\bar{P} = \begin{bmatrix} 0.3109 & 0.0083 & -0.2887 & -0.0387 \\ 0.0083 & 0.0526 & -0.0283 & 0.0652 \\ -0.2887 & -0.0283 & 0.8722 & -1.0888 \\ -0.0387 & 0.0652 & -1.0888 & 2.7165 \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} 4.1127 & 0.1678 & -1.3310 & -0.6656 \\ 0.1678 & 4.7540 & -1.3341 & -0.2042 \\ -1.3310 & -1.3341 & 3.8466 & -1.3126 \\ -0.6656 & -0.2042 & -1.3126 & 3.5213 \end{bmatrix},$$

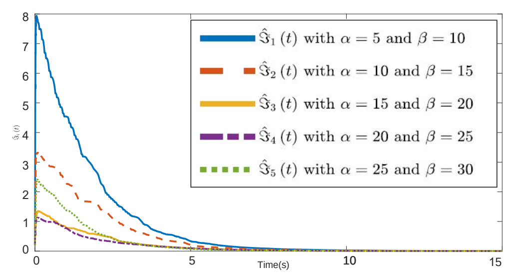
respectively. Then, following step 4) of Algorithm 2, we compute the three



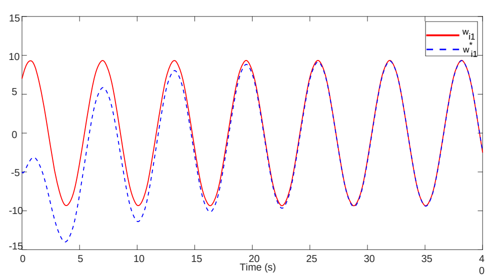
(a) State variable



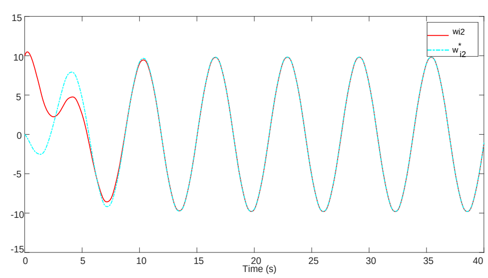
(b) Observer error



(c) Consensus tracking error



(d) Disturbance



(e) Estimated disturbance

Figure 4: Temporal evolution of (a) state $x_{i1}(t)$, (b) error $\bar{\xi}_i(t)$, (c) error $\hat{\xi}_i(t)$, (d) disturbance $\omega_i(t)$, and (e) disturbance estimation $\tilde{\omega}_i(t)$ under the nonlinear dynamics in Example 2, with topology $\mathcal{G}^2$.

gain matrices, obtaining

$$G_2 = \begin{bmatrix} 0.3554 & 0.2867 & 0.2876 & 0.1910 \\ 0.2886 & 0.1901 & 0.6244 & 0.5823 \end{bmatrix}^\top,$$

$$\hat{F} = \begin{bmatrix} -8.7655 & -18.2778 & -6.2112 & -2.1757 \\ -10.2680 & 1.8811 & -10.3730 & -4.7172 \end{bmatrix}^\top,$$

and $K_2 = [0.3303\ -0.7992\ 0.2282\ -0.1146]$.

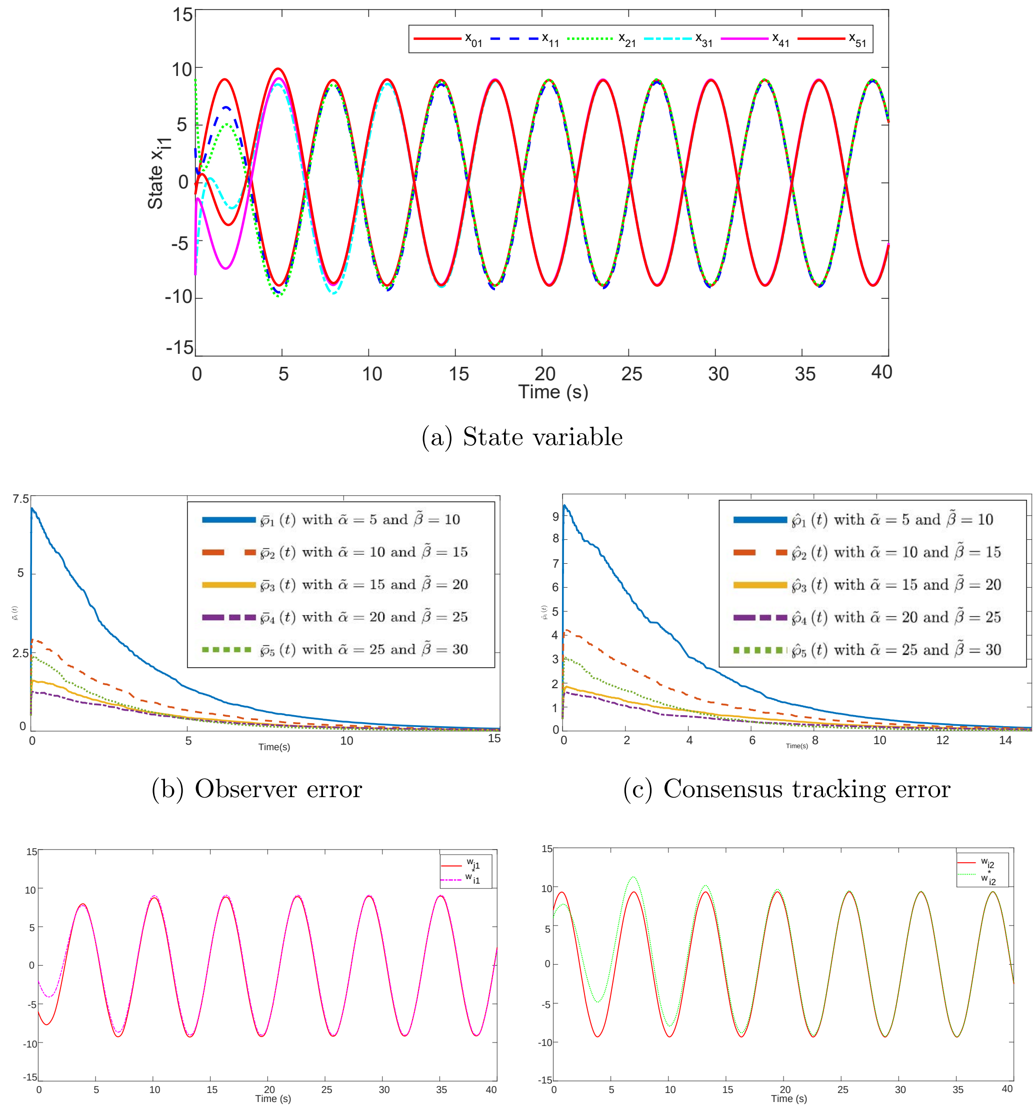
Based on Theorem 2, bipartite consensus control subject to the deterministic disturbances can be realized if the dwell time $\hat{\tau}_0 > \ln(\ell_0/\hat{c}_0) = 0.3715$*; in our scenario* $\hat{\tau}_0 = 0.4 > 0.3715$*, hence this condition is verified. Results are reported in Fig. 5. In particular, in Fig. 5a we show that we can solve the consensus problem for nonlinear MASs under directed switching topologies by means of the controller proposed in Algorithm 2, differently from most linear controller and observer designed in the literature [52], which cannot deal with nonlinear scenario. To illustrate the performance of our controller, in Fig. 5b and 5c we report the profiles of the observer error* $\bar{\xi}_i(t)$ *and of the bipartite consensus tracking error* $\hat{\xi}_i(t)$*, respectively for different coupling strengths. Our numerical simulations illustrate that both the convergence for leader-following bipartite consensus and state's observer are typically fast and that their rates can be improved by enlarging the coupling strength* $\tilde{\alpha}$ *and* $\tilde{\beta}$*. This indicates that, even though the theoretical guarantees in Theorem 2 ensures that the bipartite consensus can be solved by setting any coupling strengths* $\tilde{\alpha} > \tilde{\mu}_0/\tilde{\lambda}_0$ *and* $\tilde{\beta} > \tilde{\mu}_1/\tilde{\lambda}_0$*, the convergence rates may be quite small if the coupling strengths are only slightly larger than the requirements, suggesting the use of larger couplings to speed up the convergence process. The evolution of disturbances and disturbance observers are plotted in Fig. 5d and 5e, re-*

spectively, which shows the good performance of our algorithm in estimating the exogenous disturbances.

6. Conclusion

In this paper, we have investigated the leader-following bipartite consensus control of nonlinear MASs subject to exogenous disturbances, connected through directed signed topologies, which may be fixed or switching. In both scenarios, we utilized an observer-based approach. Specifically, we propose a control law based on relative output measurements of neighboring agents and on an observer-based estimator of the disturbances generated from the exogenous system. Then, by assuming that each fixed or switching topology contains a directed spanning tree, we proved that the leader-following bipartite consensus can be achieved with the designed output feedback control law. For switching topologies, we determined a further condition on the dwell time to be sufficiently large to guarantee convergence. Finally, the effectiveness of the developed algorithms in different scenarios is verified via three numerical examples, based on a real-world physical system.

Our work advances the literature along several directions. Compared to previous works [43, 57], our approach is more general, as it consider nonlinear MASs, antagonistic interactions, output feedback control, disturbance observer, and switching topologies simultaneously, allowing to deal with more general and realistic scenarios. In particular, different from [31, 33, 35], the disturbance observer is incorporated into the controller to actively compensate for the disturbance effects on leader-following bipartite consensus. This



(a) State variable

(b) Observer error

(c) Consensus tracking error

(d) Disturbances

(e) Estimated disturbance

Figure 5: Temporal evolution of (a) state $x_{i1}(t)$, (b) error $\bar{\xi}_i(t)$, (c) error $\hat{\xi}_i(t)$, (d) disturbance $\omega_i(t)$, and (e) disturbance estimation $\tilde{\omega}_i(t)$ under the nonlinear dynamics in Example 3, with switching topologies $\mathcal{G}_3$ and $\mathcal{G}_4$, with deterministic switches every $0.4s$.

enables our controller to be robust to disturbances produced by an exogenous system, making our algorithms suitable for direct application in many control areas, such as game control, formation control, containment control, and flocking control for MASs [4, 53, 49].

Our promising results, supported by the examples illustrated in Section 5, suggest the possible extension of our methodology to different scenarios. In particular, following [28, 44], it would be interesting to investigate event-based bipartite consensus of MASs under directed switching topologies, and further extending our algorithms to deal with disturbances that cannot be estimated or may be unbounded. Furthermore, following [35], a promising idea can be that of implementing a reduced-order dynamic gain observer into a distributed disturbance observer to utilize the distributed information of the agents under directed switching topologies. This idea will be investigated in our future study.

Acknowledgment

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Appendix A. Proof of Theorem 1

In the proof, we will show that $\bar{\xi}(t) \to 0$, $\hat{\xi}(t) \to 0$, and $e(t) \to 0$ as $t \to \infty$, guaranteeing convergence to the leader-following bipartite consensus.

First, utilizing the Kronecker product notation, we can write the dynamics in (16)–(17) in a compact matrix form as

$$\dot{\bar{\xi}}(t) = \left[I_N \otimes A + \alpha\left(L_R \otimes \tilde{F}C\right)\right]\bar{\xi}(t) + (I_N \otimes D)\, e(t) \tag{A.1}$$
$$+ I_N \otimes (f(x(t), t) - f(\hat{x}(t), t)),$$
$$\dot{\hat{\xi}}(t) = [I_N \otimes A + \beta(L_R \otimes BK_1)]\,\hat{\xi}(t) - \alpha\left(L_R \otimes \tilde{F}C\right)\bar{\xi}(t) \tag{A.2}$$
$$+ (f(\hat{x}(t), t) - (SI_N \otimes f(x_0(t), t))),$$
$$\dot{e}(t) = (I_N \otimes M)e(t) + (L_R \otimes G_1C)\bar{\xi}(t), \tag{A.3}$$

where we use the notation $f(x(t), t) := \left[f^\top(x_1(t), t), \cdots, f^\top(x_N(t), t)\right]^\top$, $f(\hat{x}(t), t) := \left[f^\top(\hat{x}_1(t), t), \cdots, f^\top(\hat{x}_N(t), t)\right]^\top$, and $e(t) := \left[e_1^\top(t), e_2^\top(t), \cdots, e_N^\top(t)\right]^\top$.

Then, we choose a Lyapunov function candidate $V_1(t)$ as follows

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t), \tag{A.4}$$

where $V_{11}(t) = \bar{\xi}^T(t)(\Theta \otimes P)\bar{\xi}(t), V_{12}(t) = \hat{\xi}^T(t)(\Theta \otimes P)\hat{\xi}(t)$, and $V_{13}(t) = e^T(t)(\Theta \otimes Q)e(t)$. Taking the derivatives of $V_{11}(t)$, $V_{12}(t)$ and $V_{13}(t)$, we obtain

$$\begin{aligned}
\dot{V}_{11}(t) =& \bar{\xi}^T(t)\left[\Theta \otimes \left(A^\top P + PA\right) + 2\alpha\left(\Theta L_R \otimes P\tilde{F}C\right)\right]\bar{\xi}(t) \\
&+ 2\bar{\xi}^T(t)(\Theta \otimes P) \times (f(x(t), t) - f(\hat{x}(t), t)), \\
&+ 2\bar{\xi}^T(t)(\Theta \otimes PD)\, e(t) \\
\dot{V}_{12}(t) =& \hat{\xi}^T(t)\left[\Theta \otimes \left(A^\top P + PA\right) + 2\beta(\Theta L_R \otimes (PBK_1))\right]\hat{\xi}(t) \\
&+ 2\hat{\xi}^T(t)(\Theta \otimes P) \times (f(\hat{x}(t), t) - (SI_N \otimes f(x_0(t), t))), \\
&- 2\alpha\hat{\xi}^T(t)\left(\Theta L_R \otimes P\tilde{F}C\right)\bar{\xi}(t) \\
\dot{V}_{13}(t) =& e^\top(t)\left[\Theta \otimes \left(QM + M^\top Q\right)\right]e(t) \\
&+ 2e^\top(t)(\Theta L_R \otimes QG_1C)\,\bar{\xi}(t).
\end{aligned} \tag{A.5}$$

According to Assumption 4 and Young's inequality, we have

$$
\begin{aligned}
&2\bar{\xi}^{\top}(t)(\Theta\otimes P)(f(x(t),t)-f(\hat{x}(t),t))\\
\leq&\bar{\xi}^{\top}(t)\left(\Theta\otimes PP^{\top}\right)\bar{\xi}(t)+\Big[(f(x(t),t)-f(\hat{x}(t),t))^{\top}\\
&(\Theta\otimes I)(f(x(t),t)-f(\hat{x}(t),t))\Big]\\
\leq&\bar{\xi}^{\top}(t)\left(\Theta\otimes PP^{\top}\right)\bar{\xi}(t)+[\lambda_{\max}(\Theta)\\
&(f(x(t),t)-f(\hat{x}(t),t))^{\top}(f(x(t),t)-f(\hat{x}(t),t))]\\
\leq&\bar{\xi}^{\top}(t)\left(\Theta\otimes PP^{\top}\right)\bar{\xi}(t)+\lambda_{\max}(\Theta)\rho^{2}\bar{\xi}^{\top}(t)\bar{\xi}(t)\\
\leq&\bar{\xi}^{\top}(t)\left[\Theta\otimes\left(PP^{\top}+\frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^{2}I\right)\right]\bar{\xi}(t),
\end{aligned}
\tag{A.6}
$$

where

$$
\lambda_{\max}(\Theta)\rho^{2}\bar{\xi}^{\top}(t)\bar{\xi}(t)\leq\frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^{2}\bar{\xi}^{\top}(t)(\Theta\otimes I)\bar{\xi}(t).
$$

Similar to (A.6), we obtain

$$
\begin{aligned}
&2\hat{\xi}^{\top}(t)(\Theta\otimes P)(f(\hat{x}(t),t)-SI_N\otimes f(x_0(t),t))\\
&\quad\leq\hat{\xi}^{\top}(t)(\Theta\otimes PP^{\top})\hat{\xi}(t)+[(f(\hat{x}(t),t)-SI_N\otimes f(x_0(t),t))^{\top}\cdot\\
&\qquad\cdot(\Theta\otimes I)(f(\hat{x}(t),t)-SI_N\otimes f(x_0(t),t))]\\
&\leq\hat{\xi}^{\top}(t)(\Theta\otimes PP^{\top})\hat{\xi}(t)\\
&\quad+[\lambda_{\max}(\Theta)(f(\hat{x}(t),t)-SI_N\otimes f(x_0(t),t))^{\top}\cdot\\
&\qquad\cdot(f(\hat{x}(t),t)-SI_N\otimes f(x_0(t),t))]\\
&\quad\leq\hat{\xi}^{\top}(t)(\Theta\otimes PP^{\top})\hat{\xi}(t)+\lambda_{\max}(\Theta)\rho^{2}\hat{\xi}^{\top}(t)\hat{\xi}(t)\\
&\quad\leq\hat{\xi}^{\top}(t)\Big[\Theta\otimes\Big(PP^{\top}+\frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^{2}I\Big)\Big]\hat{\xi}(t).
\end{aligned}
\tag{A.7}
$$

Substituting $K_1=-B^TP$, $G_1=Q^{-1}C^T$, and $\tilde{F}=-P^{-1}C^{\top}$ into (A.5), and using (A.6) and (A.7), we obtain the following inequalities:

$$\begin{aligned}
\dot{V}_{11}(t) \leq & \bar{\xi}^{\top}(t)\Big[\Theta \otimes \left(PA + A^{\top}P\right) - \alpha\left(\left(\Theta L_R + L_R^{\top}\Theta\right) \otimes C^{\top}C\right) \\
& + \Theta \otimes \left(PP^{\top} + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 I\right)\Big]\bar{\xi}(t) + 2\bar{\xi}^T(t)\left(\Theta \otimes PD\right)e(t), \\
\dot{V}_{12}(t) \leq & \hat{\xi}^T(t)\Big[\Theta \otimes \left(A^{\top}P + PA\right) - \beta\left(\left(\Theta L_R + L_R^{\top}\Theta\right) \otimes PBB^{\top}P\right) \\
& + \Theta \otimes \left(PP^{\top} + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 I\right)\Big]\hat{\xi}(t) + 2\alpha\hat{\xi}^T(t)\left(\Theta L_R \otimes C^{\top}C\right)\bar{\xi}(t), \\
\dot{V}_{13}(t) = & e^{\top}(t)\left[\Theta \otimes \left(QM + M^{\top}Q\right)\right]e(t) + 2e^{\top}(t)\left(\Theta L_R \otimes C^TC\right)\bar{\xi}(t).
\end{aligned} \tag{A.8}$$

Let $\hat{A} = L_R + \Theta^{-1}L_R^{\top}\Theta$ and $H = -\alpha\left(\left(\Theta L_R + L_R^{\top}\Theta\right) \otimes C^{\top}C\right)$, by utilizing Lemma 2, we obtain $H \leq -\alpha\Theta\hat{A} \otimes C^{\top}C \leq -\alpha\lambda_0\Theta \otimes C^{\top}C$, where $\lambda_0 \triangleq \lambda_{\min}\left(L_R + \Theta^{-1}L_R^{\top}\Theta\right)$ and $\Theta = \mathrm{diag}\{\varphi_1, \cdots, \varphi_N\}$. According to Lemma 4 in [23], we obtain

$$\begin{aligned}
\dot{V}_{11}(t) \leq & \bar{\xi}^T(t)\Big[\Theta \otimes \Big(PA + A^{\top}P - \alpha\lambda_0 C^{\top}C + PP^{\top} \\
& + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 I\Big)\Big]\bar{\xi}(t) + 2\bar{\xi}^T(t)(\Theta \otimes PD)e(t).
\end{aligned} \tag{A.9}$$

Similarly, calculating the derivative of $V_{12}(t)$, we obtain

$$\begin{aligned}
\dot{V}_{12}(t) \leq & \hat{\xi}^T(t)\Big[\Theta \otimes \Big(A^{\top}P + PA - \beta\lambda_0 PBB^{\top}P + PP^{\top} \\
& + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 I\Big)\Big]\hat{\xi}(t) + 2\alpha\hat{\xi}^T(t)(\Theta L_R \otimes C^{\top}C)\bar{\xi}(t).
\end{aligned} \tag{A.10}$$

Based on Lemma 2 and the facts $\alpha > \mu_0/\lambda_0$, $\beta > \mu_1/\lambda_0$, we obtain

$$\begin{aligned}
\dot{V}_{11}(t) \leq & \bar{\xi}^T(t)\Bigg[\Theta \otimes \Big(PA + A^\top P - \mu_0 C^\top C + PP^\top \\
& + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 I\Big)\Bigg]\bar{\xi}(t) + 2\bar{\xi}^T(t)(\Theta \otimes PD)\, e(t), \\
\dot{V}_{12}(t) \leq & \hat{\xi}^T(t)\Bigg[\Theta \otimes \Big(A^\top P + PA - \mu_1 PBB^\top P + PP^\top \\
& + \frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 I\Big)\Bigg]\hat{\xi}(t) + 2\alpha\hat{\xi}^T(t)\left(\Theta L_R \otimes C^\top C\right)\bar{\xi}(t).
\end{aligned}$$

Furthermore, under Assumptions 1 and 2, by utilizing Young's inequality and Lemma 1, we have

$$\begin{aligned}
& 2\bar{\xi}^T(t)(\Theta \otimes PD)\, e(t) \\
& \leq \bar{\xi}^T(t)\left(\Theta \otimes D^\top PPD\right)\bar{\xi}(t) + e^\top(t)(\Theta \otimes I)\, e(t), \\
& 2\alpha\hat{\xi}^T(t)\left(\Theta L_R \otimes C^\top C\right)\bar{\xi}(t) \leq \alpha\hat{\xi}^T(t)\left(\Theta \otimes C^\top C C^\top C\right)\hat{\xi}(t) \\
& + \alpha\bar{\xi}^\top(t)(\Theta \otimes I)\left(\Theta^{-1}L_R^\top \Theta L_R \otimes I\right)\bar{\xi}(t) \\
& \leq \alpha\lambda_{\max}(C^\top C C^\top C)\hat{\xi}^T(t)(\Theta \otimes I)\,\hat{\xi}(t) \\
& + \alpha\lambda_{\max}\left(\Theta^{-1}L_R^\top \Theta L_R\right)\bar{\xi}^\top(t)(\Theta \otimes I)\,\bar{\xi}(t), \\
& 2e^\top(t)\left(\Theta L_R \otimes C^\top C\right)\bar{\xi}(t) \leq e^\top(t)\left(\Theta \otimes C^\top C C^\top C\right) e(t) \\
& + \bar{\xi}^\top(t)(\Theta \otimes I)\left(\Theta^{-1}L_R^\top \Theta L_R \otimes I\right)\bar{\xi}(t) \\
& \leq \lambda_{\max}(C^\top C C^\top C) e^\top(t)(\Theta \otimes I)\, e(t) \\
& + \lambda_{\max}\left(\Theta^{-1}L_R^\top \Theta L_R\right)\bar{\xi}^\top(t)(\Theta \otimes I)\,\bar{\xi}(t).
\end{aligned} \tag{A.11}$$

Substituting (27) into (A.8), we have $\dot{V}_1(t) \leq \dot{\bar{V}}_{11}(t) + \dot{\bar{V}}_{12}(t) + \dot{\bar{V}}_{13}(t)$, where

$$\begin{aligned}
\dot{\bar{V}}_{11}(t) =& \bar{\xi}^T(t) \left[\Theta \otimes \left(PA + A^\top P - \mu_0 C^\top C + D^\top PPD + PP^\top \right.\right. \\
& \left.\left. + \left(\frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 + (\alpha+1)\lambda_{\max}\left(\Theta^{-1}L_R^\top \Theta L_R\right)\right) I\right)\right] \bar{\xi}(t), \\
\dot{\bar{V}}_{12}(t) =& \hat{\xi}^T(t) \left[\Theta \otimes \left(A^\top P + PA - \mu_1 PBB^\top P + PP^\top \right.\right. \\
& \left.\left. + \left(\frac{\lambda_{\max}(\Theta)}{\lambda_{\min}(\Theta)}\rho^2 + \alpha\lambda_{\max}(C^\top CC^\top C)\right) I\right)\right] \hat{\xi}(t), \\
\dot{\bar{V}}_{13}(t) =& e^\top(t) \left[\Theta \otimes \left(QM + M^\top Q + \left(\lambda_{\max}(C^\top CC^\top C) + 1\right) I\right]\right. e(t).
\end{aligned}$$

It can be yielded from (18), (11) and (20) that

$$\begin{aligned}
\dot{\bar{V}}_{11}(t) \leq& \bar{\xi}^T(t) \left[\Theta \otimes \left(PA + A^\top P - \mu_0 C^\top C + D^\top PPD + \gamma_1 I + PP^\top\right)\right] \bar{\xi}(t), \\
\dot{\bar{V}}_{12}(t) \leq& \hat{\xi}^T(t) \left[\Theta \otimes \left(A^\top P + PA - \mu_1 PBB^\top P + PP^\top + \gamma_2 I\right)\right] \hat{\xi}(t), \\
\dot{\bar{V}}_{13}(t) \leq& e^\top(t) \left[\Theta \otimes \left(QM + M^\top Q + \gamma_3 I\right)\right] e(t).
\end{aligned}$$

According to the Schur complement lemma, with $\rho > 0$, we can conclude there exist three nonnegative parameters c_1, c_2, and c_3 such that $\dot{\bar{V}}_{11}(t) < -c_1\bar{\xi}^\top(t)(\Theta \otimes P)\bar{\xi}(t)$, $\dot{\bar{V}}_{12}(t) < -c_2\hat{\xi}^\top(t)(\Theta \otimes P)\hat{\xi}(t)$, and $\dot{\bar{V}}_{13}(t) < -c_3 e^\top(t)$ $(\Theta \otimes Q)e(t)$. Then, we obtain $\dot{V}_1(t) < -c_0 V_1(t)$, where $c_0 = \min_{i\in\{1,2,3\}}\{c_i\}$. Thus, one has $V_1(t) < e^{-c_0 t}V_1(0)$. By summing the three terms in (A.4), we conclude that when $t \to 0$, $V_1(t) \to 0$, which implies $\bar{\xi}(t) \to 0$, $\hat{\xi}(t) \to 0$, and $e(t) \to 0$. This completes the proof. ■

Appendix B. Proof of Theorem 2

Under Assumption 1, and utilizing the definitions in (12), we can write the following dynamics for the state and its estimator:

$$\begin{aligned}\dot{x}_i(t) =& Ax_i(t) + \tilde{\beta}BK_2 \left[\sum_{j\in\mathcal{V}} \left|w_{ij}^{\varpi(t)}\right| \left(\hat{x}_i(t) - \operatorname{sgn}\left(w_{ij}^{\varpi(t)}\right)\hat{x}_j(t)\right)\right. \\ & \left. + w_{i0}^{\varpi(t)} \left(\hat{x}_i(t) - s_i x_0(t)\right)\right] + f\left(x_i(t), t\right) + De_i(t),\end{aligned} \tag{B.1}$$

and

$$\begin{aligned}\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + \tilde{\beta}BK_2 \left[\sum_{j\in\mathcal{V}} \left|w_{ij}^{\varpi(t)}\right| \left(\hat{x}_i(t) - \operatorname{sgn}\left(w_{ij}^{\varpi(t)}\right)\hat{x}_j(t)\right)\right. \\ & \left. + w_{i0}^{\varpi(t)} \left(\hat{x}_i(t) - s_i x_0(t)\right)\right] + f\left(\hat{x}_i(t), t\right) \\ & - \tilde{\alpha}\hat{F}\left[\sum_{j\in\mathcal{V}} \left|w_{ij}^{\varpi(t)}\right| \left(\tilde{\xi}_i(t) - \operatorname{sgn}\left(w_{ij}^{\varpi(t)}\right)\tilde{\xi}_j(t)\right) + w_{i0}^{\varpi(t)}\tilde{\xi}_i(t)\right],\end{aligned} \tag{B.2}$$

respectively, from which we derive the following dynamics for the three errors in (12):

$$\begin{aligned}\dot{\bar{\xi}}(t) =& \left[I_N \otimes A + \tilde{\alpha}\left(L_R^{\varpi(t)} \otimes \hat{F}C\right)\right]\bar{\xi}(t) + \left(I_N \otimes D\right)e(t) \\ & + I_N \otimes \left(f\left(x(t), t\right) - f\left(\hat{x}(t), t\right)\right), \\ \dot{\hat{\xi}}(t) =& \left[I_N \otimes A + \tilde{\beta}\left(L_R^{\varpi(t)} \otimes BK_2\right)\right]\hat{\xi}(t) - \tilde{\alpha}\left(L_R^{\varpi(t)} \otimes \hat{F}C\right)\bar{\xi}(t) \\ & + \left(\mathrm{f}\left(\mathrm{x}(\mathrm{t}), \mathrm{t}\right) - \left(\mathrm{SI_N} \otimes \mathrm{f}\left(\mathrm{x_0}(\mathrm{t}), \mathrm{t}\right)\right)\right), \\ \dot{e}(t) =& \left(I_N \otimes M\right)e(t) + \left(L_R^{\varpi(t)} \otimes G_2 C\right)\bar{\xi}(t).\end{aligned}$$

We will now show that these three quantities converge to 0 under the control proposed in Algorithm 2. For $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, the multiple

Lyapunov functions $V_2(t)$ can be constructed as follows:

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t), \tag{B.3}$$

where $V_{21}(t) = \bar{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}\right)\bar{\xi}(t), V_{22}(t) = \hat{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}\right)\hat{\xi}(t)$, and $V_{23}(t) = e^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{Q}\right)e(t)$. Taking the derivatives of $V_{21}(t)$, $V_{22}(t)$ and $V_{23}(t)$, which yield

$$\begin{aligned}
\dot{V}_{21}(t) =& \bar{\xi}^\top(t)\left[\Theta^{\varpi(t)} \otimes \left(A^\top\bar{P} + \bar{P}A\right) + 2\tilde{\alpha}\left(\Theta^{\varpi(t)}L_R^{\varpi(t)} \otimes \bar{P}\hat{F}C\right)\right]\bar{\xi}(t) \\
&+ 2\bar{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}\right)\left(f(x(t),t) - f(\hat{x}(t),t)\right) \\
&+ 2\bar{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}D\right)e(t), \\
\dot{V}_{22}(t) =& \hat{\xi}^\top(t)\left[\Theta^{\varpi(t)} \otimes \left(A^\top\bar{P} + \bar{P}A\right) + 2\tilde{\beta}\left(\Theta^{\varpi(t)}L_R^{\varpi(t)} \otimes \left(\bar{P}BK_2\right)\right)\right]\hat{\xi}(t) \\
&+ 2\hat{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}\right)\left(f(\hat{x}_i(t),t) - (SI_N \otimes f(x_0(t),t))\right), \\
&- 2\tilde{\alpha}\hat{\xi}^\top(t)\left[\Theta^{\varpi(t)}L_R^{\varpi(t)} \otimes \left(\bar{P}\hat{F}C\right)\right]\bar{\xi}(t) \\
\dot{V}_{23}(t) =& e^\top(t)\left[\Theta^{\varpi(t)} \otimes \left(\bar{Q}M + M^\top\bar{Q}\right)\right]e^\top(t) + 2e^\top(t)\left(\Theta^{\varpi(t)}L_R^{\varpi(t)} \otimes \bar{Q}G_2C\right)\bar{\xi}(t).
\end{aligned} \tag{B.4}$$

To simplify the analysis, based on Assumption 4 and Young's inequality, we obtain

$$\begin{aligned}
&2\bar{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}\right)\left(f(x(t),t) - f(\hat{x}(t),t)\right) \\
&\leq \bar{\xi}^\top(t)\left[\Theta^{\varpi(t)} \otimes \left(\bar{P}\bar{P}^\top + \frac{\lambda_{\max}\left(\Theta^{\varpi(t)}\right)}{\lambda_{\min}\left(\Theta^{\varpi(t)}\right)}\rho^2 I\right)\right]\bar{\xi}(t), \\
&2\hat{\xi}^\top(t)\left(\Theta^{\varpi(t)} \otimes \bar{P}\right)\left(f(\hat{x}(t),t) - SI_N \otimes f(x_0(t),t)\right) \\
&\leq \hat{\xi}^\top(t)\left[\Theta^{\varpi(t)} \otimes \left(\bar{P}\bar{P}^\top + \frac{\lambda_{\max}\left(\Theta^{\varpi(t)}\right)}{\lambda_{\min}\left(\Theta^{\varpi(t)}\right)}\rho^2 I\right)\right]\hat{\xi}(t).
\end{aligned} \tag{B.5}$$

Substituting $K_2 = -B^T\bar{P}$, $G_2 = \bar{Q}^{-1}C^T$, and $\hat{F} = -\bar{P}^{-1}C^\top$ into (B.4), and in view of (B.5), we obtain

$$\begin{aligned}\dot{V}_{21}(t) \leq & \bar{\xi}^{\top}(t)\Big[\Theta^{\varpi(t)}\otimes\left(\bar{P}A+A^{\top}\bar{P}\right)-\tilde{\alpha}\Big(\Big(\Theta^{\varpi(t)}L_R^{\varpi(t)} \\ & +\left(L_R^{\varpi(t)}\right)^{\top}\Theta^{\varpi(t)}\Big)\otimes C^{\top}C\Big)+\Theta^{\varpi(t)}\otimes(\bar{P}\bar{P}^{\top} \\ & +\frac{\lambda_{\max}\left(\Theta^{\varpi(t)}\right)}{\lambda_{\min}\left(\Theta^{\varpi(t)}\right)}\rho^2 I)\Big]\bar{\xi}(t)+2\bar{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes\bar{P}D\right)e(t), \\ \dot{V}_{22}(t) \leq & \hat{\xi}^{\top}(t)\Big[\Theta^{\varpi(t)}\otimes\left(A^{\top}\bar{P}+\bar{P}A\right)-\tilde{\beta}\Big(\Big(\Theta^{\varpi(t)}L_R^{\varpi(t)}+ \\ & \left(L_R^{\varpi(t)}\right)^{\top}\Theta^{\varpi(t)}\Big)\otimes\bar{P}BB^{\top}\bar{P}\Big)+\Theta^{\varpi(t)}\otimes(\bar{P}\bar{P}^{\top} \\ & +\frac{\lambda_{\max}\left(\Theta^{\varpi(t)}\right)}{\lambda_{\min}\left(\Theta^{\varpi(t)}\right)}\rho^2 I)\Big]\hat{\xi}(t)+2\tilde{\alpha}\hat{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}L_R^{\varpi(t)}\otimes C^{\top}C\right)\bar{\xi}(t), \\ \dot{V}_{23}(t) = & e^{\top}(t)\left[\Theta^{\varpi(t)}\otimes\left(\bar{Q}M+M^{\top}\bar{Q}\right)\right]e(t)+2e^{\top}(t)\left(\Theta^{\varpi(t)}L_R^{\varpi(t)}\otimes C^{\top}C\right)\bar{\xi}(t).\end{aligned}$$

Since $\tilde{\alpha} > \tilde{\mu}_0/\tilde{\lambda}_0$ and $\tilde{\beta} > \tilde{\mu}_1/\tilde{\lambda}_0$, and according to Lemma 4 in [45], we obtain

$$\begin{aligned}\dot{V}_{21}(t) \leq & \bar{\wp}^{\mathrm{T}}(t)\Big[\Theta^{\varpi(t)}\otimes(\bar{P}A+A^{\mathrm{T}}\bar{P}-\tilde{\mu}_0 C^{\mathrm{T}}C+\bar{P}\bar{P}^{\mathrm{T}} \\ & +\frac{\lambda_{\max}\left(\Theta^{\varpi(t)}\right)}{\lambda_{\min}\left(\Theta^{\varpi(t)}\right)}\rho^2 I)\Big]\bar{\wp}(t), \\ \dot{V}_{22}(t) \leq & \hat{\xi}^{\top}(t)\Big[\Theta^{\varpi(t)}\otimes(\bar{P}A+A^{\top}\bar{P}-\tilde{\mu}_1\bar{P}BB^{\top}\bar{P}+\bar{P}\bar{P}^{\top} \\ & +\frac{\lambda_{\max}\left(\Theta^{\varpi(t)}\right)}{\lambda_{\min}\left(\Theta^{\varpi(t)}\right)}\rho^2 I)\Big]\hat{\xi}(t)+2\tilde{\alpha}\hat{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}L_R^{\varpi(t)}\otimes C^{\top}C\right)\bar{\xi}(t).\end{aligned}$$

Furthermore, under Assumptions 2 and 4, by utilizing Young's inequality and Lemma 1, one has

$$
\begin{aligned}
&2\bar{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes\bar{P}D\right)e(t)\\
&\leq\bar{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes D^{\top}\bar{P}\bar{P}D\right)\bar{\xi}(t)+e^{\top}(t)\left(\Theta^{\varpi(t)}\otimes I\right)e(t),\\
&2\tilde{\alpha}\hat{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}L_R\otimes C^{\top}C\right)\bar{\xi}(t)\\
&\leq\tilde{\alpha}\lambda_{\max}(C^{\top}CC^{\top}C)\hat{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes I\right)\hat{\xi}(t)\\
&+\tilde{\alpha}\lambda_{\max}\left((\Theta^{\varpi(t)})^{-1}\left(L_R^{\varpi(t)}\right)^{\top}\Theta^{\varpi(t)}L_R^{\varpi(t)}\right)\bar{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes I\right)\bar{\xi}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&2e^{\top}(t)\left((\Theta^{\varpi(t)})^{-1}L_R^{\varpi(t)}\otimes C^{\top}C\right)\bar{\xi}(t)\\
&\leq\lambda_{\max}(C^{\top}CC^{\top}C)e^{\top}(t)\left(\Theta^{\varpi(t)}\otimes I\right)e(t)\\
&+\lambda_{\max}\left((\Theta^{\varpi(t)})^{-1}\left(L_R^{\varpi(t)}\right)^{\top}\Theta^{\varpi(t)}L_R^{\varpi(t)}\right)\ \bar{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes I\right)\bar{\xi}^{\top}(t).
\end{aligned}
$$

Similarly, one can conclude $\dot{V}_2(t)\leq\dot{\bar{V}}_{21}(t)+\dot{\bar{V}}_{22}(t)+\dot{\bar{V}}_{23}(t)$, where

$$
\begin{aligned}
\dot{\bar{V}}_{21}(t)\leq&\bar{\xi}^{\top}(t)\left[\Theta^{\varpi(t)}\otimes\left(A\bar{P}+\bar{P}A^{\top}-\tilde{\mu}_0C^{\top}C+D^{\top}\bar{P}\bar{P}D\right.\right.\\
&\left.\left.+\gamma_4I+\bar{P}^{\top}\bar{P}\right)\right]\bar{\xi}(t),\\
\dot{\bar{V}}_{22}(t)\leq&\hat{\xi}^{\top}(t)\left[\Theta^{\varpi(t)}\otimes\left(A^{\top}\bar{P}+\bar{P}A-\tilde{\mu}_1\bar{P}BB^{\top}\bar{P}\right.\right.\\
&\left.\left.+\gamma_5I+\bar{P}\bar{P}^{\top}\right)\right]\hat{\xi}(t),\\
\dot{\bar{V}}_{23}(t)\leq&e^{\top}(t)\left[\Theta^{\varpi(t)}\otimes\left(\bar{Q}M+M^{\top}\bar{Q}+\gamma_6I\right)\right]e(t).
\end{aligned}
$$

Based on the Schur's complement lemma, one can conclude there exist three non-negative parameters $\tilde{c}_1,\tilde{c}_2$ and $\tilde{c}_3$ such that $\dot{\bar{V}}_{21}(t)<-\tilde{c}_1\bar{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes\bar{P}\right)$ $\bar{\xi}(t),\dot{\bar{V}}_{22}(t)<-\tilde{c}_2\hat{\xi}^{\top}(t)\left(\Theta^{\varpi(t)}\otimes\bar{P}\right)\hat{\xi}(t)$, and $\dot{\bar{V}}_{23}(t)<-\tilde{c}_3e^{\top}(t)\left(\Theta^{\varpi(t)}\otimes\bar{Q}\right)e(t)$. Hence, one has $\dot{V}_2(t)<-\tilde{c}_0V_2(t),t\in[t_k,t_{k+1})$, for $k\in\mathbb{N}$, where $\tilde{c}_0=\min_{i\in\{1,2,3\}}\{\tilde{c}_i\}$. It is noted that the MASs with control law (21) switches when $t = t_k, k \in \mathbb{N}$. Then one can obtain $V_2\left(t_2^-\right) < V_2\left(t_1\right) e^{-\tilde{c}_0(t_2-t_1)} < e^{-\tilde{c}_0\hat{\tau}_0} V_2\left(t_1\right)$, $t \in [t_1, t_2)$, where $V_2\left(t_2^-\right) = \lim\limits_{t\to t_2} V_2\left(t\right)$. Let $\Gamma = \varpi\left(t_k\right)$, $\bar{\Gamma} = \varpi\left(t_{k^-}\right)$, $\hat{A}_{\min} = \lambda_{\min}\left(\Theta^{\Gamma}\right)\otimes\bar{P}$, $\hat{A}_{\max} = \lambda_{\max}\left(\Theta^{\Gamma}\right)\otimes\bar{P}$, $\tilde{A}_{\max} = \lambda_{\max}\left(\Theta^{\bar{\Gamma}}\right)\otimes\bar{P}$, $\tilde{A}_{\min} = \lambda_{\min}\left(\Theta^{\bar{\Gamma}}\right)\otimes\bar{P}$, $\hat{B}_{\min} = \lambda_{\min}\left(\Theta^{\Gamma}\right)\otimes\bar{Q}$, $\hat{B}_{\max} = \lambda_{\max}\left(\Theta^{\Gamma}\right)\otimes\bar{Q}$, $\tilde{B}_{\max} = \lambda_{\max}\left(\Theta^{\bar{\Gamma}}\right)\otimes\bar{Q}$, and $\tilde{B}_{\min} = \lambda_{\min}\left(\Theta^{\bar{\Gamma}}\right)\otimes\bar{Q}$, according to (B.3), one gets $\bar{\xi}^{\mathrm{T}}\left(t_k\right)\hat{A}_{\min}\bar{\xi}\left(t_k\right) \le V_{21}\left(t_k\right) \le \bar{\xi}^{\mathrm{T}}\left(t_k\right)\hat{A}_{\max}\bar{\xi}\left(t_k\right), \hat{\xi}^{\mathrm{T}}\left(t_k\right)\hat{A}_{\min}\hat{\xi}\left(t_k\right) \le V_{22}\left(t_k\right) \le \hat{\xi}^{\mathrm{T}}\left(t_k\right)\hat{A}_{\max}\hat{\xi}\left(t_k\right), e^{\mathrm{T}}\left(t_k\right)\hat{B}_{\max}e\left(t_k\right) \le V_{23}\left(t_k\right) \le e^{\mathrm{T}}\left(t_k\right)\hat{B}_{\min}e\left(t_k\right)$, $\bar{\xi}^{\mathrm{T}}\left(t_{k^-}\right)\tilde{A}_{\min}\bar{\xi}\left(t_{k^-}\right) \le V_{21}\left(t_{k^-}\right) \le \bar{\xi}^{\mathrm{T}}\left(t_{k^-}\right)\tilde{A}_{\max}\bar{\xi}\left(t_{k^-}\right), \hat{\xi}^{\mathrm{T}}\left(t_{k^-}\right)\tilde{A}_{\min}\hat{\xi}\left(t_{k^-}\right) \le V_{22}\left(t_{k^-}\right) \le \hat{\xi}^{\mathrm{T}}\left(t_{k^-}\right)\tilde{A}_{\max}\hat{\xi}\left(t_{k^-}\right)$, and $e^{\mathrm{T}}\left(t_{k^-}\right)\tilde{B}_{\max}e\left(t_{k^-}\right) \le V_{23}\left(t_{k^-}\right) \le e^{\mathrm{T}}\left(t_{k^-}\right)\tilde{B}_{\min}e\left(t_{k^-}\right)$, In summary, we obtain $V_{21}\left(t_k\right) \le \ell_0 V_{21}\left(t_{k^-}\right)$, $V_{22}\left(t_k\right) \le \ell_0 V_{22}\left(t_{k^-}\right)$, and $V_{23}\left(t_k\right) \le \ell_0 V_{23}\left(t_{k^-}\right)$, where

$$\ell_0 = \frac{\max_{\Gamma=1,\cdots,L}\left(\lambda_{\max}\left(\Theta^{\Gamma}\right)\right)}{\min_{\bar{\Gamma}=1,\cdots,L}\left(\lambda_{\min}\left(\Theta^{\bar{\Gamma}}\right)\right)} = \frac{\varphi_{\max}}{\varphi_{\min}}.$$

Thus, we obtain $V_2\left(t_2\right) < \ell_0 e^{-\tilde{c}_0\hat{\tau}_0} V_2\left(t_1\right)$, i.e., $V_2\left(t_2\right) < e^{(-\tilde{c}_0\hat{\tau}_0+\ln\ell_0)} V_2\left(0\right)$. Hence, if the dwell time satisfies $\hat{\tau}_0 > \ln(\ell_0/\tilde{c}_0)$, then the following holds $V_2\left(t_2\right) < e^{-\kappa\hat{\tau}_0} V_2\left(0\right)$, where $\kappa = \tilde{c}_0 - (\ln\ell_0)/\hat{\tau}_0 > 0$. For $t > t_2$, there exists a non-negative integer $s \ge 2$ such that $t_s < t \le t_{s+1}$. In addition, for an arbitrary nonnegative integer $w \in N$, one ohas $V_2\left(t_{w+1}\right) < e^{-\kappa\hat{\tau}_0} V_2\left(t_w\right) < e^{-\kappa w\hat{\tau}_0} V_2\left(0\right)$. Similarly, when $t \in (t_s, t_{s+1})$, one has

$$\begin{aligned} V_2\left(t\right) &< e^{-\tilde{c}_0(t-t_s)} V_2\left(t_s\right) < e^{-[\tilde{c}_0(t-t_s)+(s-1)\kappa\hat{\tau}_0]} V_2\left(0\right) \\ &< e^{-\frac{(s-1)\hat{\tau}_0}{s\hat{\tau}_1}\kappa t} V_2\left(0\right) < e^{-\frac{\hat{\tau}_0\kappa}{2\hat{\tau}_1}t} V_2\left(0\right), \end{aligned}$$

where $s \ge 2$ and $\hat{\tau}_0 \le t_{k+1} - t_k \le \hat{\tau}_1$. When $t = t_{s+1}$, we obtain $V_2\left(t\right) < e^{-\frac{\hat{\tau}_0\kappa}{\hat{\tau}_1}t} V_2\left(0\right)$, which implies $\bar{\xi}(t) \to 0$, $\hat{\xi}(t) \to 0$, and $e\left(t\right) \to 0$ as $t \to \infty$. ■

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