Doctoral Dissertation
Doctoral Program in Pure and Applied Mathematics (38.th cycle)

Nonlinear Schrödinger equation on graphs with physical potentials

Ground state existence and nonexistence

Ivan Gallo
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Supervisors
Prof. Riccardo Adami, Supervisor
Prof. Lamberto Rondoni and Marco Pizzi, Co-supervisors

Politecnico di Torino
January 31, 2026

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This work is funded by the Ministerial Decrees 351 and 352/2022 originated in the context of the PNRR with the aim of supporting the promotion and strengthening of higher education and postgraduate specialization at doctoral level, responding to the innovation needs of enterprises and promoting the recruitment of researchers from enterprises. Therefore, it complies with the principle of not doing significant harm to environmental objectives, in accordance with Article 17 of Regulation (EU) 2020/852

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Ivan Gallo

Turin, January 31, 2026

Summary

This thesis investigates the existence and nonexistence of ground states for the focusing nonlinear Schrödinger equation (NLS) on noncompact metric graphs in the presence of external physical potentials. Metric graphs provide a natural framework for modeling wave propagation in branched structures such as quantum wires, optical networks, and Bose–Einstein condensates confined in ramified geometries. Here, the interaction between nonlinearity, graph geometry, and external potentials produces genuinely new effects, making the study of stationary states an interesting variational problem with clear physical motivation.

After an introduction to the topic in Chapter 1, Chapter 2 revisits the classical variational problem associated with the NLS on graphs. In the absence of potentials, the existence of ground states, the minimizers of the NLS energy under a fixed mass constraint, depends delicately on the topology of the graph. A fundamental tool is the comparison with the soliton on the real line: if a function on the graph attains an energy strictly below the soliton energy, then a ground state exists; otherwise, minimizing sequences escape to infinity and no minimizer is achieved. This mechanism is governed by concentration–compactness and by geometric properties such as the presence of multiple unbounded directions.

Chapter 3 focuses on attractive potentials supported on compact subsets of the graph. The physical motivation is inspired by the case of a potential induced by the curvature of a graph. These potentials lower the energy of functions concentrating on their support and therefore favor the formation of bound states. We extend the classical existence criterion to this setting and establish two complementary existence results: ground states exist for sufficiently large mass and also for sufficiently small mass. In contrast, we show that for intermediate masses, nonexistence may occur when the compact core of the graph is sufficiently intricate and the potential is too weak to counterbalance the dispersive effect of the topology.

Chapter 4 examines magnetic potentials, introduced through the gauge-covariant derivative. While magnetic fields have no effect on trees, where they can be removed by a gauge transformation, they play a crucial role on graphs containing loops, in accordance with the Aharonov–Bohm effect. We show that the magnetic energy can be reformulated as an NLS with an effective repulsive potential supported on the cycles of the graph, with strength determined by the magnetic flux. This reduction

allows us to extend the existence criterion to the magnetic setting and to analyze how magnetic fluxes influence the formation of ground states. As an application, we study the tadpole graph and prove that, depending on the magnitude of the effective magnetic potential, ground states may exist or fail to exist.

Overall, the thesis clarifies how external interactions, attractive or repulsive, influence the variational landscape of the NLS on graphs.

Acknowledgements

I am a person of few words, yet it would take many to properly thank all those who helped me reach this point. I try to write a few of them.

Firstly, I want to thank my supervisor, Riccardo Adami. You taught me how to think like a mathematician, even when it went against all my instincts. This thesis would be very different without your vision.

Thanks to my co-supervisors, Lamberto Rondoni and Marco Pizzi. Although our work did not find its place in this thesis, I truly appreciated your guidance and support.

I want to thank the Analysis and Quantum Theory research group and Prof. Pavel Exner for valuable suggestions and for showing me the passion behind this topic.

Thank you to Nicolò and David for their help throughout this work and, most of all, for being wrong with me so many times. I am sure we will see each other again very soon.

A big thank you goes to the Dismini for being my companions in this adventure. I could not have wished for better colleagues and I am truly grateful. A special thanks to Marco, Emanuele, and Andrea for your friendship in the most demanding moments, I know you will have a wonderful career.

Lastly, I would like to thank everyone who loves me outside academia: those who were with me before this PhD began (and long before that, and even before) and those others I met along the way, my sister for her constant support and Jo, who also designed the cover of this thesis. This has been one of the most challenging experiences of my life, and it would have been impossible without the light you brought into this journey.

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Albert Einstein

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Chapter 1

Introduction

1.1 Physical background and motivation

Quantum mechanics is the theory that reveals the hidden rules that govern matter and energy at their most fundamental scale. It would not exist without the rigorous framework provided by mathematics and analysis. Analysis, in particular, provides the means to understand the existence of the results, the stability of states, the dynamics of solutions, and the transitions between different regimes, transforming physical insights into rigorous results.

A fundamental tool of the theory is the *Schrödinger equation* [62]

$$i\frac{\partial \psi}{\partial t}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t), \tag{1.1}$$

which describes the evolution of the wave function ψ of a particle under the effect of a potential $V(x)$. This equation is linear in ψ, reflecting the principle of superposition and allowing different states to combine and interfere as experiments show. However, in some physical contexts, a nonlinear dynamics provides an effective model more powerful than the linear one. A notable example is a gas of bosons cooled below a critical low temperature that forms a *Bose-Einstein condensate*. Predicted a century ago by Bose and Einstein [24, 39], the phenomenon was experimentally confirmed in 1995 by Cornell, Wieman and Ketterle [21] that won the Nobel Prize in Physics in 2001. In principle, such a system should be described by a large number of coupled Schrödinger equations. Yet, it is well-established that the condensate is accurately approximated by a single macroscopic wavefunction, at the cost of introducing a nonlinear term in the equation. The rigorous mean field derivation is known as the Gross-Pitaevskii model [60, 48] that leads to the Gross-Pitaevskii equation:

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi + g|\psi|^2\psi, \tag{1.2}$$

where V denotes the external trapping potential and the sign of g determines whether the gas is *focusing* (attractive interactions) or defocusing (repulsive interactions).

Therefore, an effective way to study the wavefunction of the system is through a variational formulation, i.e. looking for the solution of the variational problem

$$\min_{u \in H^1(\Omega),\ \int u^2 = N} E_{\mathrm{GP}}, \tag{1.3}$$

where the functional E_{GP} is defined (in dimensionless units) as

$$E_{GP} = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + V(x)|u|^2 + \frac{g}{2}|u|^4 \right) dx, \tag{1.4}$$

Ω is the magnetic trap in which the particles are confined, and N is the number of the particles of the system. Similar forms of nonlinear Schrödinger equations also emerge in different fields, such as optics, plasma physics, and hydrodynamics [44].

Particular attention is given to the existence of *ground states*, which represent the solution to the variational problem (1.3). These results strongly depend on the shape of the domain of the system. In this work the domains are *metric graphs*, i.e. one-dimensional objects made of edges, with finite or infinite length, connected by vertices where suitable boundary conditions are imposed. Such domains are chosen as they provide the natural framework for studying the dynamics of systems confined in ramified structure, for example to model a Bose-Einstein condensate in quantum wires [20, 54] or light propagation in waveguides [27, 47].

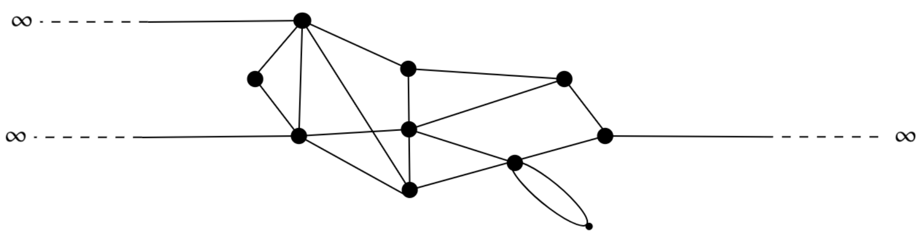

Figure 1.1: A noncompact metric graph

Early attempts to model quantum dynamics on graphs treated molecular bonds as graph edges and the Schrödinger operator on those edges [58, 61], laying the groundwork for the modern theory of *quantum graphs*. A rigorous classification of admissible vertex boundary conditions via self-adjoint extensions of the Laplacian was subsequently developed [51], followed by a systematic treatment of point interactions and self-adjoint Schrödinger operators [18]. Quantum graphs have also been used as tractable models in the study of quantum chaos [46]. To explore the subject of quantum graphs the monograph by Berkolaiko and Kuchment [23] has become the standard reference in the field.

For what concerns nonlinearity, the first nonlinear theory of wave propagation on networks was developed by Ali Mehmeti [19], but the field gained momentum after a series of work on the nonlinear Schrödinger equation on metric graphs started with the analysis on simple graphs [11, 63], and then widely developed. The existence and qualitative properties of ground states have been extensively investigated in a variety of settings [14, 15, 3, 8, 5], together with their stability and dynamical behavior [55]. This line of research has subsequently been extended to other classes of graphs, including periodic graphs [9, 34] as well as to hybrid and more complex geometries [12, 10, 13]. More recently, attention has turned on models with concentrated or nonstandard nonlinearities [66, 36, 59] and on the distinction between action and energy ground states [35, 33].

1.2 The variational problem

A central theme in the analysis of nonlinear equations is the variational structure of the focusing NLSE

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi - |\psi|^{p-2}\psi, \tag{1.5}$$

on noncompact metric graphs, where nonlinearity satisfies $p \in (2, +\infty)$. The analysis focuses on stationary solutions of the form $\psi(x,t) = e^{i\lambda t}u(x)$ with $\lambda \in \mathbb{R}$ which satisfy the associated stationary equation:

$$u'' + |u|^{p-2}u = \lambda u\,. \tag{1.6}$$

The equation must hold on every edge $e \in \mathbb{E}(\mathcal{G})$ of the graph $\mathcal{G}$ and is complemented at the vertices $v \in \mathbb{V}(\mathcal{G})$ by coupling conditions. In this work we adopt the standard Kirchhoff conditions, although several other types of vertex conditions are available in the literature [3, 4, 1].

The goal then is to investigate the existence of the ground state in different graphs $\mathcal{G}$, for the energy functional associated with the focusing NLSE

$$E_{\mathrm{NLS}}(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,\mathrm{d}x\,. \tag{1.7}$$

Since E_{NLS} is not bounded from below on $H^1(\mathcal{G})$, we impose a fixed mass condition

$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} |u|^2 dx = \mu > 0. \tag{1.8}$$

In other words: we look for solutions of the problem

$$\mathcal{E}_{\mathcal{G}}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathrm{NLS}}(u,\mathcal{G}), \tag{1.9}$$

in the mass constrained space

$$H^1_\mu(\mathcal{G}) := \{u \in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}. \tag{1.10}$$

To show that critical points of the constrained functional E_{NLS}, subject to Kirchhoff conditions at the vertices, indeed satisfy the stationary Schrödinger equation (1.6), we compute the associated Euler–Lagrange equation.

$$\nabla\left(E_{\text{NLS}}(u,\mathcal{G}) - \frac{\lambda}{2}\left(\mu - \|u\|^2_{L^2(\mathcal{G})}\right)\right) = 0,$$

for some $\lambda \in \mathbb{R}$. This means that, for every $\eta \in H^1(\mathcal{G})$,

$$\begin{aligned} 0 &= \int_{\mathcal{G}} (u'\eta' - |u|^{p-2}u\eta + \lambda u\eta)\mathrm{d}x \\ &= \sum_{e\in\mathbb{E}(\mathcal{G})} u'_e\eta_e\big|_0^{l_e} + \sum_{e\in\mathbb{E}(\mathcal{G})} \int_0^{l_e} (-u''_e - |u_e|^{p-2}u_e + \lambda u_e)\eta_e \mathrm{d}x_e\,. \end{aligned}$$

By the arbitrariness of η in the interior of each edge, the second term implies that the nonlinear stationary equation (1.6) holds in every edge e. To extract the vertex conditions we now chose η that vanish in all vertices except at a fixed vertex v. The first term becomes

$$\sum_{e\in\mathbb{E}(\mathcal{G})} u'_e\eta_e\big|_0^{l_e} = \eta(\mathrm{v})\sum_{e\prec \mathrm{v}} u'_e(\mathrm{v}) = 0.$$

Since η and v are arbitrary we obtain the *Kirchhoff's condition* for u at every vertex

$$\sum_{e\prec \mathrm{v}} u'_e(\mathrm{v}) = 0. \tag{1.11}$$

The geometry of the graph plays a crucial role in the minimization problem. For example, if $\mathcal{G}$ is compact, then every minimizing sequence is compact in $H^1(\mathcal{G})$ and therefore a minimizer always exists [7]. For this reason, we restrict our attention to noncompact graphs, i.e. graphs containing at least one unbounded edge.

Let us consider metric graphs that are invariant under stretching, fix $u \in H^1(\mathcal{G})$ with prescribed mass $\|u\|^2_{L^2(\mathcal{G})} = \mu$, and for $\tau > 0$ define the rescaled function $u_\tau(x) = \sqrt{\tau}\, u(\tau x)$. which has the same mass. The energy of u_τ is then given by

$$E_{\text{NLS}}(u_\tau;\mathcal{G}) = \frac{\tau^2}{2}\,\|u'\|^2_{L^2(\mathcal{G})} - \frac{\tau^{\frac{p}{2}-1}}{p}\,\|u\|^p_{L^p(\mathcal{G})}.$$

From this expression, the role of the exponent p becomes clear: the kinetic term dominates as $\tau \to +\infty$, and the energy is bounded from below if and only if $p \in (2{,}6]$. The case $p = 6$ is critical and in this regime the lower boundedness of the energy depends delicately on the value of the mass μ. Throughout this work, we consider exclusively the subcritical case.

1.3 Ground State existence

The first and most fundamental case in the study of the NLS on metric graphs is the real line, the simplest noncompact graph. Here the functional $E_{\mathrm{NLS}}(\cdot, \mathbb{R})$, possesses a ground state for every mass, given by the *soliton*

$$\phi_\mu(x) = \mu^\alpha C_p \operatorname{sech}^{\frac{\alpha}{\beta}}(c_p \mu^\beta x), \tag{1.12}$$

where c_p, C_p, α, β are positive constants depending only on p. It is known [67, 22] that up to translations and multiplication by a constant phase, ϕ_μ is the unique ground state on $\mathbb{R}$.

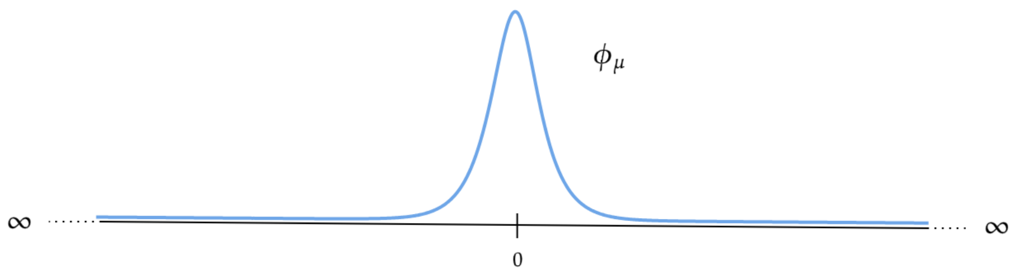

Figure 1.2: Soliton ϕ_μ

This result plays a crucial role for the analysis of the ground state in noncompact graphs. Indeed, a fundamental tool for proving the existence of ground states in graphs with infinite edges can be summarized in the following criterion

Existence Criterion. *Fix $2 < p < 6$ and let $\mathcal{G}$ be a noncompact graph with at least one unbounded edge. If there exists a function $v \in H^1_\mu(\mathcal{G})$ such that*

$$E_{\mathrm{NLS}}(v, \mathcal{G}) \leq E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R}), \tag{1.13}$$

then a ground state of mass μ exists on $\mathcal{G}$.

Thus, the existence problem on a noncompact graph reduces to a constructive task: it is enough to find a function on $\mathcal{G}$ whose energy lies strictly below the soliton energy on the line. Whenever such a competitor exists, the graph admits a ground state. The proof of the criterion is an application of the concentration-compactness principle [53], which provides the description of how a minimizing sequence may lose compactness. In the present setting, the only way a minimizing sequence can lose compactness is by escaping to infinity along an unbounded edge. In that case, the graph becomes indistinguishable from $\mathbb{R}$ and the sequence asymptotically behaves like a translated soliton. Therefore, finding a function with energy that lies below the soliton energy rules out the loss of compactness and it proves the existence of a minimizer.

However, such a competitor does not exist on every graph. For instance, if the graph is such that from any point one can move along a continuous trail that eventually reaches infinity in two distinct directions, then a nonexistence criterion holds. This geometric requirement is known as Assumption (H) [6]. If $\mathcal{G}$ satisfies Assumption (H), then for every $\mu > 0$

$$\mathcal{E}_{\mathcal{G}}(\mu) = \mathcal{E}_{\mathbb{R}}(\mu)$$

and the infimum is never attained.

1.4 Towards the results: NLS with potentials

Beyond the standard nonlinear Schrödinger dynamics, many physically relevant models require the inclusion of external interactions acting on the system. In this thesis, we focus on two different types of potentials: attractive potentials and magnetic potentials.

In *Chapter* 3 we investigate the existence of ground states for the energy functional

$$I(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p dx - \frac{1}{2}\int_{\mathcal{K}} w(x)|u|^2 dx, \tag{1.14}$$

defined on a metric graph $\mathcal{G}$ and involving a compact subset $\mathcal{K} \subset \mathcal{G}$. The potential $w(x)$ is assumed to be nonnegative, continuous, and supported on $\mathcal{K}$. We refer to it as *attractive* since it lowers the energy of functions that concentrate on its support, thereby favoring the formation of bound states.

Several mechanisms may produce such an attractive effect. The motivation for this chapter comes from the case of a potential induced by the curvature of a graph [42]. In fact, for a waveguide modeled as a curved tube, the Laplacian with Dirichlet boundary conditions can be rewritten as a standard Laplacian plus an effective potential. In the thin-waveguide limit, this potential is purely attractive, providing a natural geometric source of trapping.

A first step in our analysis is to extend the Existence Criterion to the setting with the potential w. This generalization allows us to establish two results of existence and one of nonexistence of ground states. Our main result of the chapter is the following.

Theorem 1.4.1. *Let $\mathcal{G}$ be a noncompact graph and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact set $\mathcal{K} \subset \mathcal{G}$. Then*

- *if the mass μ is sufficiently large, then there exists a ground state for $I(\cdot, \mathcal{G})$ at mass μ.*
- *if the mass μ is sufficiently small, then there exists a ground state for $I(\cdot, \mathcal{G})$ at mass μ.*

Finally, we show that in the intermediate regime of masses, there exists a class of graphs for which no ground state exists. The nonexistence is driven by a combination of weak potential and a highly intricate compact core.

The analysis of the NLS in the presence of external potentials is not a new topic. In particular, the small-mass existence theorem was already established in [29] for general attractive potentials. In that setting, ground states were shown to emerge through a nonlinear bifurcation from the linear ground state. In this thesis we adopt a different variational approach, which allows us to obtain existence results in other regimes.

In *Chapter* 4 we analyze the effect of a magnetic potential in the NLS on graphs setting.

In the presence of an external magnetic field $B(x)$, the dynamics of the system are conveniently described through the introduction of a magnetic vector potential $A(x)$. In this framework the standard momentum operator $-\nabla$ is replaced by the gauge-covariant derivative $-\nabla + iA$ leading to the stationary equation

$$-\left(\frac{d}{dx} - iA\right)^2 u - |u|^{p-2}u = \lambda u\,. \tag{1.15}$$

This modification encodes the influence of the magnetic field entirely through the phase of the wavefunction, in accordance with the usual principles of quantum mechanics.

The study of the magnetic effect on quantum graphs has been developed extensively over the years particularly in connection with models of quantum wires and mesoscopic systems [43, 51]. In the nonlinear setting, existence and stability results for stationary states which relied on concentration–compactness techniques were obtained [31, 40] and more recently normalized solutions have been investigated [32].

A standard result is that, in the absence of cycles (closed paths), the magnetic potential does not affect the energy or the dynamics of the problem and it can be removed by a gauge transformation. This mechanism is actually a manifestation of the Aharonov-Bohm effect [16], a quantum phenomenon showing that charged particles may be influenced by electromagnetic potentials even in regions where the magnetic field vanishes.

In this chapter we show that, when loops are present, the research for ground state of the energy in the magnetic setting is equivalent to one with an effective potential supported on loops. The corresponding energy functional takes the form:

$$I_A(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \mathrm{d}x + \int_{\mathcal{C}(\mathcal{G})} \Phi_\gamma(A)|u|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p dx\,. \tag{1.16}$$

where $\mathcal{C}(\mathcal{G})$ denotes the set of cycles of $\mathcal{G}$ and $\Phi_\gamma(A)$ is a positive quantity depending on the average of A along each loop. In this sense, the magnetic flux acts as a repulsive potential concentrated on the cycles.

This reformulation allows us to extend once again the Existence Criterion for repulsive potentials. As an application, we analyze the special case of the tadpole, a graph made by one vertex and two edges where one edge is a loop that connects the vertex to itself, and the other is a halfline, i.e. an edge with infinite length. For this simple case we establish both the existence and nonexistence of ground states depending on the strength of the effective magnetic potential.

Parts of this thesis are based on results previously obtained in [2, 30] . The two problems are motivated by the same physical question: how external interactions shape the behavior of nonlinear waves on networks. Taken together, these results highlight how on metric graphs, the competition between nonlinearity, geometry, and external fields produces mechanisms that have no analogue in simpler domains. Attractive potentials can enhance localization, magnetic fluxes can inhibit it, and the topology of the graph determines which of these effects prevails. The variational methods developed here provide a flexible approach for analyzing these interactions and can be adapted to a wide range of related problems. The analysis is not exhaustive, and many physical situations involve additional effects beyond the scope of this work. Nevertheless, the results presented here capture some of the main mechanisms that govern the formation of nonlinear bound states on networks.

The thesis is organized as follows. *Chapter* 2 introduces the mathematical setting and tools used in the standard variational problem. *Chapter* 3 is devoted to attractive potentials supported on compact subsets of the graph, where we establish existence and nonexistence results depending on the mass and on the geometry of the compact core. *Chapter* 4 focuses on magnetic potentials, showing how magnetic fluxes on loops can be reformulated as effective repulsive potentials and applying this perspective to the tadpole graph.

Chapter 2

Preliminaries

2.1 Metric graphs

A metric graph $\mathcal{G}$ is a metric space defined by the pair $(\mathbb{E}, \mathbb{V})$ where $\mathbb{V}(\mathcal{G}) \subset \mathbb{R}^d$ is a set of points called *vertices* and $\mathbb{E}(\mathcal{G}) \subset \mathbb{V} \times \mathbb{V}$ is a set of line segments called *edges*. The metric structure is defined by establishing that every edge $e \in \mathbb{E}$ is homeomorphic to a real interval $I_e = [0, \ell_e]$. If $\ell_e = \infty$, the edge is called a *halfline* and in this case $\mathcal{G}$ is *noncompact*. Throughout this work, we assume that $\mathcal{G}$ is connected, and that both $\mathbb{V}$ and $\mathbb{E}$ are finite. Thus, we always deal with connected graphs with a finite number of edges and vertices.

A function $u : \mathcal{G} \to \mathbb{C}$ is defined through its restriction to the edges, that is

$$u \equiv \{u_e\}_{e \in \mathbb{E}}, \quad u_e : I_e \to \mathbb{C}.$$

We say that a function u is continuous on $\mathcal{G}$ if each u_e is continuous on I_e, for every $e \in \mathbb{E}(\mathcal{G})$, and if it is continuous at the vertices, i.e., the value attained by u at any vertex v does not depend on the edge chosen to reach v.

Differential operators are defined edge by edge, namely $u' = \{u'_e\}_{e \in \mathbb{E}}$, and integrals on $\mathcal{G}$ are defined by

$$\int_{\mathcal{G}} u(x)\, dx \;=\; \sum_{e \in \mathbb{E}} \int_0^{\ell_e} u(x_e)\, dx_e.$$

The standard functional spaces are defined in the natural way. For $1 \leq p \leq \infty$

$$L^p(\mathcal{G}) = \{u : \mathcal{G} \to \mathbb{C},\ \|u_e\|_{L^p(I_e)} < \infty \ \forall e \in \mathbb{E}\ \},$$

endowed with the norm

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{e \in \mathbb{E}} \|u_e\|^p_{L^p(I_e)}.$$

Analogously for the Sobolev spaces, in particular

$$H^1(\mathcal{G}) = \{u : \mathcal{G} \to \mathbb{C},\ \|u_e\|_{H^1(I_e)} < \infty \ \forall e \in \mathbb{E}\},$$

with the norm
$$\|u\|^2_{H^1(\mathcal{G})} = \sum_{e\in\mathbb{E}} \|u_e\|^2_{H^1(I_e)}.$$
Introducing a prescribed *mass* μ, given by
$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} |u|^2 dx = \mu > 0,$$
we define
$$H^1_\mu(\mathcal{G}) = \{u,\ u \in H^1_\mu(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}.$$
the space of functions in $H^1(\mathcal{G})$ that fulfills the mass constraint.

There are different ways to fix the vertex condition. Here we choose the so-called *Kirchhoff conditions*
$$\begin{cases} u \text{ continuous}; \\ \displaystyle\sum_{e\succ v} \frac{du}{dx_e}(v) = 0 \quad \forall v \in \mathbb{V}, \end{cases} \tag{2.1}$$
where we denote $e \succ$ v an edge e incident at a vertex v.

2.2 Variational framework

The main objective in this framework is to study the mass-constrained minimization problem of the energy functional associated with the focusing NLSE
$$E_{\mathrm{NLS}}(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p \,\mathrm{d}x\,. \tag{2.2}$$
in the mass constrained space
$$H^1_\mu(\mathcal{G}) := \{u \in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}. \tag{2.3}$$
We denote this minimization problem as
$$\mathcal{E}_{\mathcal{G}}(\mu) := \inf_{u\in H^1_\mu(\mathcal{G})} E_{\mathrm{NLS}}(u,\mathcal{G}). \tag{2.4}$$
We also note that a function $u \in H^1_\mu(\mathcal{G})$ is called a *ground state* of E_{NLS} with mass μ if
$$E_{\mathrm{NLS}}(u,\mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu).$$

On the line, ground states for $p \in (2{,}6)$ exist and are all translations of the *soliton*
$$\phi_\mu(x) = \mu^\alpha C_p \operatorname{sech}^{\frac{\alpha}{\beta}}(c_p \mu^\beta x), \tag{2.5}$$

where c_p and C_p are positive constants and the powers α and β are

$$\alpha = \frac{2}{6-p}, \qquad \beta = \frac{p-2}{6-p}. \tag{2.6}$$

A classical way to prove that the solitons and their translations are the only stationary solutions on $\mathbb{R}$ is to rewrite the stationary equation (1.6) on the line in the form

$$u'' = -\frac{dV}{du}(u), \qquad V(u) = \frac{1}{p}u^p - \frac{\lambda}{2}u^2,$$

which corresponds to the equation of motion in the potential $V(u)$.

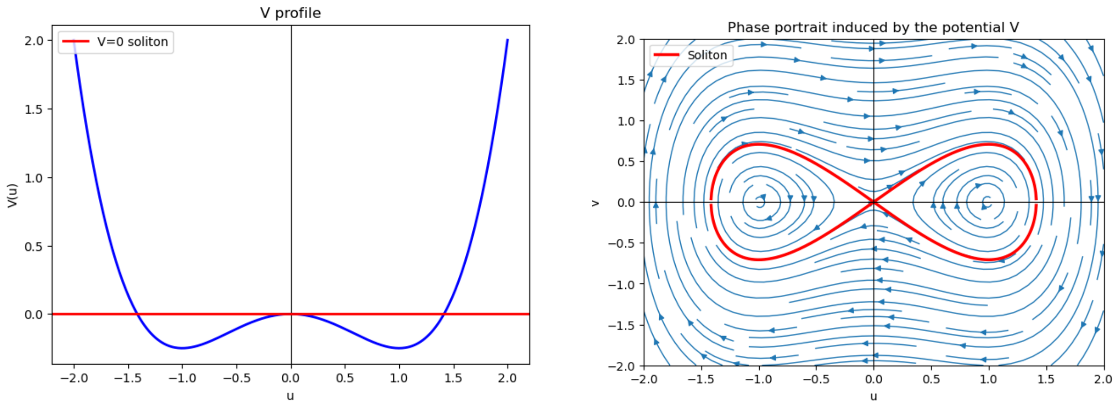

Figure 2.1: $V(u)$ profile and phase portrait

From the phase portrait (Fig. 2.1) of the mechanical analogy, it is clear that all solutions are periodic (and therefore do not belong to $H^1(\mathbb{R})$) except for those lying on the separatrix, corresponding to solutions with vanishing mechanical energy. These solutions are precisely the soliton and its translations. Therefore, we have

$$\mathcal{E}_{\mathbb{R}}(\mu) = E_{\text{NLS}}(\phi_\mu, \mathbb{R}) = -\theta_p \mu^{2\beta+1} < 0, \tag{2.7}$$

Another case of special interest is the halfline $\mathbb{R}^+$. When $\mathcal{G} = \mathbb{R}^+$ the ground state is the *half-soliton*, i.e. $\phi_{2\mu}$ restricted to $\mathbb{R}^+$

$$\mathcal{E}_{\mathbb{R}^+}(\mu) = E(\phi_{2\mu}, \mathbb{R}^+) = \frac{1}{2}E(\phi_{2\mu}, \mathbb{R}) = -2^{2\beta}\theta_p\mu^{2\beta+1}.$$

In [6], it was proved that any ground states satisfy

$$\mathcal{E}_{\mathbb{R}^+}(\mu) \leq \mathcal{E}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathbb{R}}(\mu). \tag{2.8}$$

The dependence of the lower boundedness in $H^1(\mathcal{G})$ on the nonlinearity p is well known. The key point is the validity of the *Gagliardo–Nirenberg inequalities* [26].

First, we introduce the preliminary notion of a monotone and symmetric rearrangement of a function on a metric graph. The concept was introduced in [45], where the monotone rearrangement is defined for functions on metric graphs by closely following the classical construction for functions on an interval (see, for instance, [50]). Moreover, in the same work they also established the Pólya-Szegő inequality, showing that the monotone rearrangement does not increase the kinetic energy.

Proposition 2.2.1 (Monotone rearrangement)**.** *Let $\mathcal{G}$ be a connected metric graph and let $u \in H^1(\mathcal{G})$ be nonnegative. Denote by u^* the monotone rearrangement of u on the interval $I^* = [0, |\mathcal{G}|)$, where $|\mathcal{G}|$ denotes the total length of $\mathcal{G}$, i.e. the sum, possibly infinite, of the lengths of all its edges. Then $u^* \in H^1(I^*)$ and the following properties hold:*

$$\|u^*\|_{L^r(I^*)} = \|u\|_{L^r(\mathcal{G})}, \qquad r \in [1, +\infty], \tag{2.9}$$

$$\|(u^*)'\|_{L^2(I^*)} \leq \|u'\|_{L^2(\mathcal{G})}. \tag{2.10}$$

For a general complex-valued function $u \in H^1(\mathcal{G})$ one has $\||u|\|_{L^r(\mathcal{G})} = \|u\|_{L^r(\mathcal{G})}$ and $\||u|'\|_{L^2(\mathcal{G})} \leq \|u'\|_{L^2(\mathcal{G})}$. Therefore, taking $u^* = |u|^*$, both the equimeasurability identity (2.9) and the Pólya–Szegő inequality (2.10) remain valid.

In [6], the theory was extended to the symmetric rearrangement.

Proposition 2.2.2 (Symmetric rearrangement)**.** *Let $u \in H^1(\mathcal{G})$ be a nonnegative function on a connected non-compact metric graph $\mathcal{G}$, and let u^* be its monotone rearrangement on $[0, |\mathcal{G}|)$. The* symmetric rearrangement *of u is the function $\tilde{u} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\tilde{u}(x) := u^*(2|x|), \qquad x \in \mathbb{R}.$$

Then $\tilde{u}$ is even, nonnegative, and equimeasurable with u, i.e.

$$\rho_{\tilde{u}}(t) = \rho_u(t) \qquad \text{for all } t \geq 0.$$

Moreover, for every $p \in [1, +\infty)$,

$$\int_{\mathbb{R}} |\tilde{u}(x)|^p \, dx = \int_{\mathcal{G}} |u(x)|^p \, dx.$$

The following property holds.

Proposition 2.2.3. *Let $u \in H^1(\mathcal{G})$ be a nonnegative function on a connected non-compact metric graph $\mathcal{G}$ and define*

$$N(t) := \#\{x \in \mathcal{G} : u(x) = t\}, \qquad t \in (0, \max u]$$

Then, the monotone rearrangement u^* satisfies

$$\int_0^{+\infty} |(u^*)'(x)|^2\,dx \;\le\; \int_{\mathcal{G}} |u'(x)|^2\,dx,$$

with strict inequality unless $N(t) = 1$ for a.e. t.
If $N(t) \ge 2$ for almost every $t \in (0, \max u]$, the symmetric rearrangement $\tilde{u}$ satisfies

$$\int_{\mathbb{R}} |\tilde{u}'(x)|^2\,dx \;\le\; \int_{\mathcal{G}} |u'(x)|^2\,dx,$$

and equality implies $N(t) = 2$ for a.e. t. Since rearrangements preserve L^p norms, we have

$$E_{\mathrm{NLS}}(u, \mathcal{G}) \ge E_{\mathrm{NLS}}(\tilde{u}, \mathbb{R}) \ge \mathcal{E}_{\mathbb{R}}(\mu) \;=\; E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R}),$$

where ϕ_μ is the soliton of mass μ on $\mathbb{R}$.

We are now ready to introduce the Gagliardo-Nirenberg inequality.

Proposition 2.2.4 (Gagliardo-Nirenberg inequalities)**.** *For every $p \in [2, \infty)$, there exists $C_p > 0$ such that*

$$\|u\|^p_{L^p(\mathcal{G})} \le C_p \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}. \tag{2.11}$$

Moreover

$$\|u\|^2_{L^\infty(\mathcal{G})} \le 2\|u\|_{L^2(\mathcal{G})} \|u'\|_{L^2(\mathcal{G})}, \tag{2.12}$$

for every $u \in H^1(\mathcal{G})$ and every metric graph $\mathcal{G}$ with $n \ge 1$ infinite edges.

Proof. We refer for proof to [66]. Consider a nonnegative function $u \in H^1(G)$. We define its decreasing rearrangements the function $u^* : \mathbb{R}^+ \to \mathbb{R}$ such that

$$u^*(x) := \inf\{t \ge 0 : \rho(t) \le x\},$$

where the distribution function of u is

$$\rho(t) := \sum_{e \in E} \mathrm{meas}\Big(\{x_e \in I_e : u_e(x_e) > t\}\Big), \qquad t \ge 0.$$

It is well known that

$$\int_{\mathcal{G}} |u(x)|^r\,dx = \int_0^\infty |u^*(x)|^r\,dx, \qquad \|u\|_{L^\infty(\mathcal{G})} = \|u^*\|_{L^\infty(\mathbb{R}^+)}, \tag{2.13}$$

for every $r \ge 1$. Moreover, $u^* \in H^1(\mathbb{R}^+)$ and

$$\int_0^\infty |(u^*)'(x)|^2\,dx \le \int_{\mathcal{G}} |u'(x)|^2\,dx. \tag{2.14}$$

Applying the classical Gagliardo-Nirenberg inequality on $\mathbb{R}^+$, we obtain

$$\|u\|^p_{L^p(\mathcal{G})} = \|u^*\|^p_{L^p(\mathbb{R}^+)} \leq C_p\, \|u^*\|^{\frac{p}{2}+1}_{L^2(\mathbb{R}^+)} \|(u^*)'\|^{\frac{p}{2}-1}_{L^2(\mathbb{R}^+)}.$$

Using (2.13) and (2.14), we conclude

$$\|u\|^p_{L^p(\mathcal{G})} \leq C_p\, \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})},$$

which proves (2.11). The proof of (2.12) is analogous. □

Using (2.11), it is therefore immediate that

$$E_{\mathrm{NLS}}(u,\mathcal{G}) \geq \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{C_p}{p}\mu^{\frac{p+2}{4}}\|u'\|^{\frac{p-2}{2}}_{L^2(\mathcal{G})} \tag{2.15}$$

and E_{NLS} is coercive in $H^1(\mathcal{G})$ only if $p \in (2{,}6)$.

In the next result we use the Gagliardo–Nirenberg inequalities to show that all the relevant quantities of the energy E_{NLS} can be controlled in terms of the mass μ, independently of $\mathcal{G}$ on a suitable subset of the energy space. In particular, the estimates hold for every ground state. For notational convenience we set:

$$T(u) = \int_{\mathcal{G}} |u'|^2\, dx, \qquad V(u) = \int_{\mathcal{G}} |u|^p\, dx. \tag{2.16}$$

Proposition 2.2.5. *Let $\mathcal{G}$ be a noncompact metric graph. For all $u \in H^1_\mu(\mathcal{G})$ such that*

$$E_{\mathrm{NLS}}(u,\mathcal{G}) \;\leq\; \frac{1}{2}\mathcal{E}_{\mathcal{G}}(\mu) \;<\; 0, \tag{2.17}$$

the following estimates hold

$$C_1\mu^{2\beta+1} \leq T(u) \leq C_2\mu^{2\beta+1} \tag{2.18}$$

$$C_1\mu^{2\beta+1} \leq V(u) \leq C_2\mu^{2\beta+1} \tag{2.19}$$

$$C_1\mu^{\beta+1} \leq \|u\|^2_{L^\infty(\mathcal{G})} \leq C_2\mu^{\beta+1}, \tag{2.20}$$

for some constants $C_1, C_2 > 0$ depending only on p.

Proof. From the Gagliardo-Nirenberg (2.11) we have that

$$V \leq C_p \mu^{\frac{p+2}{4}} T^{\frac{p-2}{4}},$$

for some consant C_p depending only on p. On the other hand, by hypothesis (2.17) and the soliton energy (2.7) we have

$$\frac{1}{2}T - \frac{1}{p}V \leq \frac{1}{2}\theta_p\mu^{2\beta+1} < 0. \tag{2.21}$$

For some constant C, combining the two, we obtain $V \leq C\mu^{2\beta+1}$ and therefore $T \leq C\mu^{2\beta+1}$. From (2.12) and the later we have $\|u\|^2_{L^\infty(\mathcal{G})} \leq C\mu^{\beta+1}$. For the estimate from below, since $T \geq 0$ we can use (2.7) to obtain $V \geq C\mu^{2\beta+1}$. Then, from (2.2) we have $T \geq C\mu^{2\beta+1}$. Finally, since $V \leq \mu\|u\|^{p-2}_{L^\infty(\mathcal{G})}$ we have $C\mu^{\beta+1} \leq \|u\|^2_{L^\infty(\mathcal{G})}$. Since every constant is only depending on p, the estimates follow. □

For some estimates in Chapter 3 we also need the following.

Proposition 2.2.6. *Let $\mathcal{K}$ be a compact metric graph with total length $|\mathcal{K}|$, $1 \leq r \leq p \leq \infty$ and $s = 1/r - 1/p$. Then for every $u \in H^1(\mathcal{K})$*

$$\|u\|_{L^r(\mathcal{K})} \leq \|u\|_{L^p(\mathcal{K})}|\mathcal{K}|^s.$$

Proof. This is a direct consequence of the Hölder inequality

$$\int_{\mathcal{K}} |fg|dx \leq \left(\int_{\mathcal{K}} |f|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathcal{K}} |g|^q dx\right)^{\frac{1}{q}},$$

where $p, q > 0$ and $1/p + 1/q = 1$, if we consider $f = |u|^r$ and $g \equiv 1$ on $\mathcal{K}$. □

2.3 Existence Criterion

In this section we present the proof of the Existence Criterion for ground states of the energy functional E_{NLS}. The argument relies on the concentration–compactness principle, which provides a tool to overcome the lack of compactness inherent to noncompact metric graphs. The strategy follows the approach developed in [6], although similar proofs can also be found (for example in [9]).

First, we show the concavity property of the energy functional.

Proposition 2.3.1. *The function $\mathcal{E}_{\mathcal{G}}(\mu) = \inf\{E(u,\mathcal{G}) : u \in H^1_\mu(\mathcal{G})\}$ is strictly concave and subadditive.*

Proof. For $u \in H^1(\mathcal{G})$, we define

$$U := \left\{u \in H^1(\mathcal{G}) : \int_{\mathcal{G}} |u|^2 = 1, \quad \mu^{\frac{p}{2}} V(u) \geq C_1\mu^{2\beta+1}\right\} \tag{2.22}$$

Then we consider the family of concave functions given by

$$f_u(\mu) := E(\sqrt{\mu}u, \mathcal{G}) = \frac{\mu}{2}\int_{\mathcal{G}} |u'|^2 dx - \frac{\mu^{\frac{p}{2}}}{p} V(u), \quad u \in U, \tag{2.23}$$

with $\mu > 0$. By Proposition 2.2.5 the value of $\mathcal{E}_{\mathcal{G}}(\mu)$ is unaltered, if the infimum is further restricted to functions satisfying the lower bound in (2.19) or, which is the same, to functions of the form $\sqrt{\mu}u$ with $u \in U$. Therefore we have

$$\mathcal{E}_{\mathcal{G}}(\mu) := \inf_{u\in U} f_u(\mu), \quad \mu \geq 0.$$

so that $\mathcal{E}_{\mathcal{G}}(\mu)$ inherits concavity from the functions f_u. In fact

$$f_u''(\mu) = -\frac{p-2}{4}\mu^{\frac{p}{2}-2}V(u) < 0,$$

and we see from (2.22) that $f_u''(\mu) < 0$, so that the strict concavity of f_u on every interval $[a,b] \subset (0,\infty)$ is uniform with respect to u. Hence $\mathcal{E}_{\mathcal{G}}$ is strictly concave and also strictly subadditive since $\mathcal{E}_{\mathcal{G}}(0) = 0$. □

Then we explicitly show the behaviour of the minimizing sequence.

Proposition 2.3.2. *Any minimizing sequence $\{u_n\} \subset \mathcal{H}^1_\mu(\mathcal{G})$ is weakly compact in $H^1(\mathcal{G})$.*

Proof. Using the Banach-Alaoglu theorem we need to prove that the sequence $\{u_n\}$ is bounded. Thanks to the Gagliardo-Nirenberg inequality and coercive estimate (2.15) that is the case if $2 < p < 6$. □

Proposition 2.3.3. *Let $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$. Then, either*
(i) $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$ and $u \equiv 0$ or
(ii) $u \in H^1_\mu(\mathcal{G})$ u is a minimizer and $u_n \to u$ strongly in $H^1(\mathcal{G}) \cap L^p(\mathcal{G})$

Proof. The boundedness of $\{u_n\}$ follows from Proposition 2.3.2. Assuming weak convergence in $L^2(\mathcal{G})$, let

$$m = \mu - \|u\|^2_{L^2(\mathcal{G})} \in [0,\mu] \tag{2.24}$$

be the loss of mass in the limit.
In general, thanks to the concentration-compactness principle, either $m = \mu$, $0 < m < \mu$, or $m = 0$. If $m = \mu$ then $u_n \to 0$ strongly in $L^\infty_{loc}(\mathcal{G})$, hence $u \equiv 0$ which is case (i).

Assuming $0 < m < \mu$, since $u_n \to u$ pointwise and $||u||^p_{L^p(\mathcal{G})}$ is uniformly bounded, the Brezis-Lieb Lemma [25] gives

$$\frac{1}{p}\int_{\mathcal{G}} |u_n|^p dx - \frac{1}{p}\int_{\mathcal{G}} |u_n - u|^p dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p dx = o(1),$$

as $n \to \infty$. Weak convergence of u_n' yields

$$\frac{1}{2}\int_{\mathcal{G}} |u_n'|^2 dx - \frac{1}{2}\int_{\mathcal{G}} |u_n' - u'|^2 dx - \frac{1}{2}\int_{\mathcal{G}} |u'|^2 dx = o(1).$$

Therefore

$$E_{\text{NLS}}(u_n,\mathcal{G}) = E_{\text{NLS}}(u_n - u,\mathcal{G}) + E_{\text{NLS}}(u,\mathcal{G}) + o(1) \quad as \quad n \to \infty.$$

Now from $u_n \rightharpoonup u$ in $L^2(\mathcal{G})$ we have

$$\|u_n - u\|^2_{L^2(\mathcal{G})} \to \mu + \|u\|_2^2 - 2\langle u,u\rangle_2 = \mu - \|u\|^2_{L^2(\mathcal{G})} = m \tag{2.25}$$

Since $\mathcal{E}_\mathcal{G}$ is continuous (from concavity in Proposition 2.3.1) we obtain

$$\mathcal{E}_\mathcal{G}(\mu) \geq \mathcal{E}_\mathcal{G}(m) + E_{\text{NLS}}(u,\mathcal{G}) \geq \mathcal{E}_\mathcal{G}(m) + \mathcal{E}_\mathcal{G}(\|u\|^2_{L^2(\mathcal{G})}) \tag{2.26}$$

but since $\mathcal{E}_\mathcal{G}$ is also strictly subadditive (from Proposition 2.3.1), if $0 < m < \mu$ we would have

$$\mathcal{E}_\mathcal{G}(\mu) < \mathcal{E}_\mathcal{G}(\mu - \|u\|^2_{L^2(\mathcal{G})}) + \mathcal{E}_\mathcal{G}(\|u\|^2_{L^2(\mathcal{G})}),$$

which is a contradiction.
If $m = 0$ then $u_n \to u$ strongly in $L^2(\mathcal{G})$ for (2.25). Since $p > 2$ and $\|u_n\|_{L^\infty} \leq C$ convergence in $L^2(\mathcal{G})$ implies strong convergence in $L^p(\mathcal{G})$. In fact

$$\int |u_n - u|^p dx = \int |u_n - u|^2 |u_n - u|^{p-2} dx \leq \|u_n - u\|_\infty^{p-2} \int |u_n - u|^2 dx$$

and

$$\|u_n - u\|^{p-2}_{L^\infty(\mathcal{G})} \leq \left(\|u_n\|_{L^\infty(\mathcal{G})} + \|u\|_\infty\right)^{p-2} \leq C\left(\|u_n\|_{H^1(\mathcal{G})} + \|u\|_{H^1(\mathcal{G})}\right)^{p-2}.$$

Finally, strong convergence in $L^2(\mathcal{G})$ and $L^p(\mathcal{G})$ implies convergence of the L^2-norms of u'. In fact, by definition $E_{\text{NLS}}(u,\mathcal{G}) \geq \mathcal{E}_\mathcal{G}(\mu)$, but since

$$\frac{1}{2}||u'||_2^2 \leq \frac{1}{2}\lim_{n\to\infty} ||u_n'||_2^2,$$

and

$$E_{\text{NLS}}(u,\mathcal{G}) = \frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p \leq \lim_{n\to\infty} E_{\text{NLS}}(u_n,\mathcal{G}) = \mathcal{E}_\mathcal{G}(\mu),$$

we must have $\|u_n' - u\| \to 0$. Hence $u_n \to u'$ strongly in $H^1(\mathcal{G})$ and u is a minimizer. □

The previous proposition shows that every minimizing sequence either completely vanishes or converges strongly to a minimizer. In order to obtain the existence of a ground state, it is therefore crucial to exclude the vanishing scenario. The next theorem provides a sufficient condition ensuring that this loss of mass cannot occur, and consequently that the infimum of the energy is attained.

Theorem 2.3.4. *Let $\mathcal{G}$ be a noncompact graph. If*

$$\inf_{v\in H^1_\mu(\mathcal{G})} E_{\text{NLS}}(v,\mathcal{G}) < \min_{\phi\in H^1_\mu(\mathbb{R})} E_{\text{NLS}}(\phi,\mathbb{R}), \tag{2.27}$$

then the infimum is attained.

Proof. Let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence. To apply Proposition 2.3.3, it suffices to exclude case (i). Assume by contradiction that $u_n \to 0$ in $L^\infty_{\text{loc}}(\mathcal{G})$. If ϵ_n denotes the maximum of u_n on the compact core of $\mathcal{G}$, clearly $\epsilon_n \to 0$. Therefore, letting $v_n := \max\{0, u_n - \epsilon_n\}$, since $\mathcal{G}$ contains at least one half-line along which $u_n \to 0$, we see that the number of preimages $v_n^{-1}(t)$ contains at least two points for every $t \in (0, \max v_n)$. Let $\widehat{v}_n$ be the symmetric rearrangement of v_n. Proposition 2.2.3 shows that after rearranging a function the kinetic energy diminishes and the other terms remain the same. Hence

$$E_{\mathrm{NLS}}(v_n, G) \geq E_{\mathrm{NLS}}(\widehat{v}_n, \mathbb{R}) \geq \min_{\phi \in H^1_\mu} E_{\mathrm{NLS}}(\phi, \mathbb{R}).$$

Finally, since $\|u_n - v_n\|_{H^1(\mathcal{G})} = o(1)$, the sequence $\{v_n\}$ is still a minimizing sequence. Letting $n \to \infty$ contradicts the strict inequality (2.27). Thus case (i) cannot occur, and the infimum is attained. □

The previous theorem provides a general mechanism for the existence of ground states. It can be reformulated as a practical existence criterion, which reduces the problem to finding a single competitor with sufficiently low energy. This yields the following.

Lemma 2.3.5 (Existence Criterion)**.** *Fix $2 < p < 6$ and let $\mathcal{G}$ be a noncompact metric graph. If there exists $v \in H^1_\mu(\mathcal{G})$ such that*

$$E_{\mathrm{NLS}}(v, \mathcal{G}) \leq E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R}), \tag{2.28}$$

then $E(\cdot, \mathcal{G})$ admits a ground state at mass μ.

Proof. If v is not a ground state, then

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathrm{NLS}}(u, \mathcal{G}) < E_{\mathrm{NLS}}(v, \mathcal{G}) < \min_{\phi \in H^1_\mu(\mathbb{R})} E_{\mathrm{NLS}}(\phi, \mathbb{R}),$$

and by Theorem 2.3.4 a ground state still exists. □

A simple example where the criterion can be applied is the *pendant graph*. The key geometric feature that allows the ground state to exist is the presence of a compact that allows the rearrangements.

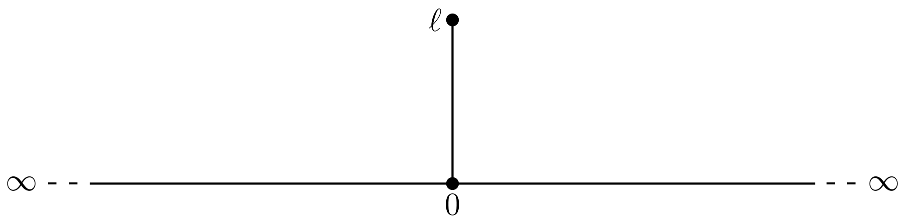

Figure 2.2: Pendant graph.

Remark 2.3.6 (Ground states on the pendant graph)**.** Let us consider the pendant graph, namely a metric graph consisting of a compact edge of length ℓ attached to two half-lines (see Figure 2.2). Let $\mathcal{G}$ be such a graph, fix $\mu > 0$ and let $\phi_\mu(x)$ be the soliton of mass μ on $\mathbb{R}$. We now construct a competitor $v \in H^1_\mu(\mathcal{G})$ as follows:

- we take the "head" $\phi_\mu|_{[-\frac{\ell}{2},\frac{\ell}{2}]}$ and place it on the compact edge of $\mathcal{G}$;
- we monotonically rearrange this head to a function on the interval $[0,\ell]$;
- we attach the two tails $\phi_\mu|_{(-\infty,-\frac{\ell}{2})}$ and $\phi_\mu|_{(\frac{\ell}{2},+\infty)}$ consecutively along the halflines.

By construction, v has the same L^p−norms as ϕ_μ, and its kinetic energy differs from that of ϕ_μ only by the geometric rearrangement of the head on the compact edge. Therefore,

$$E_{\text{NLS}}(v,\mathcal{G}) < E_{\text{NLS}}(\phi_\mu,\mathbb{R}).$$

Thus, condition (2.28) of the Existence Criterion is satisfied, and the pendant graph admits a ground state (not necessarily equal to v) for every $2 < p < 6$ and every mass $\mu > 0$.

Finally, we show a nonexistence results. A standard tool for proving nonexistence of ground states is a topological condition introduced in [6] called Assumption (H). Recall that a *trail* is a path consisting of adjacent edges in which every edge is run through exactly once. Assumption (H) can be stated as follows.

Assumption (H): Every $x \in \mathcal{G}$ lies on a trail that contains two half-lines.

Under this assumption, one obtains the following result.

Theorem 2.3.7 (Nonexistence)**.** *Assume that $\mathcal{G}$ satisfies Assumption (H). Then, for every $\mu > 0$*

$$\mathcal{E}_{\mathcal{G}}(\mu) = \mathcal{E}_{\mathbb{R}}(\mu),$$

and the infimum is never attained. In particular, no ground state exists, except in the special case where $\mathcal{G}$ is a "bubble tower".

Proof. We do not reproduce the proof here and refer to [6]. The key ingredients are the use of symmetric rearrangements and the construction of runaway soliton sequences, which show that minimizing sequences escape to infinity along the unbounded trails and therefore cannot converge to a minimizer on $\mathcal{G}$. □

It is immediate to verify that Assumption (H) is violated by the pendant graph (Figure 2.2), where ground states do exist, as well as by any graph containing fewer than two halflines.

Chapter 3

Attractive potentials

3.1 Introduction

In this chapter we investigate the existence of ground states for the energy functional

$$I(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'(x)|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |u(x)|^p dx - \frac{1}{2}\int_{\mathcal{K}} w(x)|u(x)|^2 dx, \tag{3.1}$$

on a noncompact metric graph $\mathcal{G}$, in the mass constrained space

$$H^1_\mu(\mathcal{G}) := \{u \in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}.$$

The energy functional (3.1) is obtained from the standard NLS energy

$$E_{\mathrm{NLS}}(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\, dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\, dx,$$

where, in addition to the usual kinetic and nonlinear terms, we introduce an external potential supported on the compact part of the graph $\mathcal{K}$. This potential is assumed to be nonnegative and continuous, so that it contributes negatively to the energy and energetically favors localization on the compact core. Such *attractive* potentials arise naturally in several contexts, including models of trapping defects and curvature induced interactions. We limit our analysis to the subcritical case of $2 < p < 6$, the critical case $p = 6$ is more delicate and needs to be treated separately (for instance [28]).

Our first goal is to extend the classical existence criterion for ground states of the free NLS energy in Lemma 2.3.5 to the case where an attractive potential is present. This generalized criterion becomes the main tool for the three results proved in this chapter: two existence theorems and one nonexistence theorem. We then prove Theorem 3.4.1, which establishes that if one concentrates a soliton in a region with an attractive potential then the existence criterion is fulfilled and a ground state exists. At the other end of the range of the mass, Theorem 3.4.2 establishes the existence of ground states for small masses. This fact is not new, as it was already proven in [29] for every attractive potential. Such ground states were proven to arise through a nonlinear bifurcation from the linear ground states. Here we give a different proof based on the extension of the existence criterion.

The last achievement, proven in Theorem 3.5.1, is a nonexistence result that holds for a class of graphs and some interval of masses, under the hypothesis of a weak potential. This generalizes a nonexistence result in [8] that holds in the absence of potentials. Combining Theorems 3.4.1 and 3.4.2, the overall picture is the following: in the presence of an attractive potential there exist two thresholds such that ground states exist for masses below or up to them, while in between there may be an interval where no ground states exist. Constructing explicit examples where this phenomenon occurs is delicate, and we provide one such example.

Besides the case of an external potential, our analysis applies to the case of a potential induced by the curvature of the graph [38]. In fact, a fundamental result in Dirichlet waveguides of quantum particles is that geometry induced bound states [42]. More specifically, we recall that for a waveguide modeled as a curved tube Ω_Γ around a curve Γ, one can express the Laplacian with Dirichlet boundary conditions by using a transformation from the ordinary cartesian coordinates to a system centered in a straightened tube Ω_0. The procedure can be summorize as follows. We suppose that the region Ω has a sufficiently regular boundary so that we can represent the Dirichlet Laplacian operator with partial derivatives

$$-\Delta_D^\Omega \psi = \sum_{j=1}^{d} \frac{\partial^2 \psi}{\partial x_j^2},$$

and take its domain as all ψ from the local Sobolev space $H_0^1(\Omega)$, such that $-\Delta\psi$ belongs to L^2. Furthermore, we define the subset of such $\psi \in C^\infty(\Omega)$ in order to satisfy the Dirichlet condition

$$\psi(x) = 0 \text{ for } x \in \partial\Omega.$$

Let Γ be a curve with natural curvilinear coordinates along which we can build the tube Ω_Γ. We define its signature curvature γ (the inverse radius of the osculation circle) with some regularity assumptions: its map is injective, γ is C^k-smooth and regular at infinity. Then we can go from the curved tube Ω_Γ to a straight tube Ω_0 using the straightening transformation $(\tilde{U}\psi)(s,u) := \psi(x,y)$ where we use the unitary operator $\tilde{U} : L^2(\Omega_\Gamma) \to L^2(\Omega_0, h^{\frac{1}{2}} ds du)$, thus

$$\tilde{H} := \tilde{U}(-\Delta_D^{\Omega_\Gamma})\tilde{U}^{-1} = -h^{-\frac{1}{2}}\partial_s h^{-\frac{1}{2}}\partial_s - h^{-\frac{1}{2}}\partial_u h^{-\frac{1}{2}}\partial_u,$$

where $h = h_{ss} = (1 + u\gamma)$ is the square of the Jacobian of the coordinate transformation $(x, y) \to (s, u)$ with the metric $dx^2 + dy^2 = h_{ss}ds^2 + h_{uu}du^2$ and $h_{uu} = 1$.

Additionally, we can eliminate the weight in the Hilbert space using the unitary operator $U : L^2(\Omega_\Gamma) \to L^2(\Omega_0)$ to transform as $U\psi := h^{\frac{1}{4}}\tilde{U}\psi$. Finally we get

$$H := U(-\Delta_D^{\Omega_\Gamma})U^{-1} = -\partial_s(1+u\gamma)^{-2}\partial_s - \partial_u^2 + V(s,u)$$

where

$$V(s,u) := -\frac{\gamma^2}{4(1+u\gamma)^2} + \frac{u\ddot{\gamma}}{2(1+u\gamma)^3} + \frac{5}{4}\frac{u^2\dot{\gamma}}{(1+u\gamma)^4}$$

is the curvature induced effective potential. Now we can point out that we are interested only in the thin waveguide limit, where the tube can be modeled by a graph. In this case the coordinate u does not have a role and the only term that survives in $V(s,u)$ is

$$V(s,u) \approx -\frac{\gamma^2}{4}. \tag{3.2}$$

The main source of issues when studying the limits of these branched networks comes from the vertices and the behaviour of eigenfunctions in and locally around them. We will avoid this problem by not adding curvature in the vertices and simply get *inspired* by this limit to study the effects of adding the potential to the edges of a generic graph $\mathcal{G}$.

3.2 General properties

Typically one defines the *compact core* of a metric graph, denoted by $\mathcal{K}$, as the subgraph obtained by removing all halflines from $\mathcal{G}$. In principle, this notion does not coincide with that of the support of a given compactly supported potential. However, one can consider a larger compact set by taking the union $\mathcal{K}'$ of the compact core of the graph with the support of the potential. The complement of $\mathcal{K}'$ contains then n halflines $\mathcal{H}_i'$, which in general are subsets of the original n halflines $\mathcal{H}_i$. The complement of the union of the halflines $\mathcal{H}_i'$ contains both the compact core of the graph $\mathcal{G}$ and the support of w, and can be used in the place of $\mathcal{K}$. For the sake of simplicity, in the following we refer to $\mathcal{K}$, the compact core of the graph, as the support of the potential. Of course, all results remain valid if the compact support of w does not coincide with $\mathcal{K}$.

The hypotheses on w are the following:

$$w \text{ is continuous, nonnegative, and supported on } \mathcal{K}.$$

For notational purposes we introduce the symbol

$$W(u) = \int_{\mathcal{K}} w(x)|u|^2\,dx,$$

so that

$$I(u,\mathcal{G}) = \frac{1}{2}T(u) - \frac{1}{p}V(u) - \frac{1}{2}W(u)$$

with $T(u)$ and $V(u)$, defined in equation (2.16).

Furthermore, we denote

$$\mathcal{I}_{\mathcal{G}}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} I(u, \mathcal{G}),$$
$$\mathcal{E}_{\mathcal{G}}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathrm{NLS}}(u, \mathcal{G}).$$

Since we are considering a nonnegative w, we get

$$\mathcal{I}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathcal{G}}(\mu) \leq -\theta_p \mu^{2\beta+1}. \tag{3.3}$$

Occasionally we use the shorthand notation

$$\|w\|_\infty := \max_{x \in \mathcal{K}} |w(x)|.$$

In analogy to Proposition 2.2.5 we want to show how the the relevant quantities of the energy I can be controlled in terms of the mass μ, independently of $\mathcal{G}$. We prove the following result.

Proposition 3.2.1. *Let $\mathcal{G}$ be a noncompact metric graph. For all $u \in H^1_\mu(\mathcal{G})$ such that*

$$I(u, \mathcal{G}) \;\leq\; \frac{1}{2}\mathcal{I}_{\mathcal{G}}(\mu) \;<\; 0, \tag{3.4}$$

the following estimates hold:

$$\min\{C_1 \mu^{2\beta+1}, C_2 \Lambda^{-\frac{2p}{p-2}} \mu^{3(2\beta+1)}\} \leq T(u) \leq C_3 \mu^{2\beta+1} + C_4 \|w\|_\infty \mu; \tag{3.5}$$

$$\min\{C_1 \mu^{2\beta+1}, C_2 \Lambda^{-\frac{p}{2}} \mu^{(2\beta+1)\frac{p}{2}}\} \leq V(u) \leq C_3 \mu^{2\beta+1} + C_4 \|w\|_\infty^{\frac{p-2}{4}} \mu^{\frac{p}{2}}; \tag{3.6}$$

$$\min\{C_1 \mu^{\beta+1}, C_2 \Lambda^{-\frac{p}{p-2}} \mu^{3\beta+2}\} \leq \|u\|_\infty^2 \leq C_3 \mu^{\beta+1} + C_4 \|w\|_\infty^{\frac{1}{2}} \mu, \tag{3.7}$$

for some constants $C_1, C_2, C_3, C_4 > 0$, where $\Lambda := \|w\|_\infty |\mathcal{K}|^{\frac{p-2}{p}}$.

Proof. Let $u \in H^1_\mu(\mathcal{G})$ satisfy hypothesis (3.4). If

$$W(u) < \frac{1}{2}\theta_p \mu^{2\beta+1}, \tag{3.8}$$

then by (3.4), (3.8), and (3.3)

$$\begin{aligned} \frac{1}{2}T(u) - \frac{1}{p}V(u) \;&=\; I(u, \mathcal{G}) + \frac{1}{2}W(u) \;<\; \frac{1}{2}\mathcal{I}_{\mathcal{G}}(\mu) + \frac{1}{4}\theta_p \mu^{2\beta+1} \\ &\leq\; -\frac{1}{2}\theta_p \mu^{2\beta+1} + \frac{1}{4}\theta_p \mu^{2\beta+1} \;=\; -\frac{1}{4}\theta_p \mu^{2\beta+1}. \end{aligned} \tag{3.9}$$

So the estimates of Proposition 2.2.5 hold, namely

$$\begin{aligned} C_1\mu^{2\beta+1} \leq & T(u) \leq C_3\mu^{2\beta+1}; \\ C_1\mu^{2\beta+1} \leq & V(u) \leq C_3\mu^{2\beta+1}; \\ C_1\mu^{\beta+1} \leq & \|u\|_\infty^2 \leq C_3\mu^{\beta+1}. \end{aligned} \tag{3.10}$$

On the other hand, if

$$W(u) \geq \frac{1}{2}\theta_p\mu^{2\beta+1}, \tag{3.11}$$

then we use the inequality (2.11) and obtain

$$V(u) \leq C_p\mu^{\frac{p+2}{4}}T(u)^{\frac{p-2}{4}}, \tag{3.12}$$

that, combined with (3.4), yields

$$T(u) - \frac{2C_p}{p}\mu^{\frac{p+2}{4}}T(u)^{\frac{p-2}{4}} < W(u) \leq \|w\|_\infty\mu.$$

Since $\frac{p-2}{4} < 1$, using Young's inequality there exists $C_4 > 0$ such that

$$T(u) \leq C_4\|w\|_\infty\mu + C_4\mu^{2\beta+1},$$

thus by (2.11)

$$V(u) \leq C_4\|w\|_\infty^{\frac{p-2}{4}}\mu^{\frac{p}{2}} + C_4\mu^{2\beta+1},$$

and by (2.12)

$$\|u\|_\infty^2 \leq C_4\|w\|_\infty^{\frac{1}{2}}\mu + C_4\mu^{\beta+1}.$$

To obtain the lower bound of $V(u)$ we apply Proposition 2.2.6 with $r = 2$. Denoting $\Lambda = \|w\|_\infty|\mathcal{K}|^{\frac{p-2}{p}}$ it holds that

$$W(u) \leq \Lambda V(u)^{\frac{2}{p}},$$

and from (3.11) one gets

$$C_2\Lambda^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}} \leq V(u).$$

Using (3.12) yields

$$C_2\Lambda^{-\frac{2p}{p-2}}\mu^{3(2\beta+1)} \leq T(u).$$

Finally, using the fact that $V(u) \leq \mu\|u\|_\infty^{p-2}$ we obtain

$$C_2\Lambda^{-\frac{p}{p-2}}\mu^{3\beta+2} \leq \|u\|_\infty^2.$$

Summing up, in the case (3.11) we obtain the inequalities

$$\begin{aligned} C_2\Lambda^{-\frac{2p}{p-2}}\mu^{3(2\beta+1)} &\leq T(u) \leq C_4\mu^{2\beta+1} + C_4\|w\|_\infty\mu; \\ C_2\Lambda^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}} &\leq V(u) \leq C_4\mu^{2\beta+1} + C_4\|w\|_\infty^{\frac{p-2}{4}}\mu^{\frac{p}{2}}; \\ C_2\Lambda^{-\frac{p}{p-2}}\mu^{3\beta+2} &\leq \|u\|_\infty^2 \leq C_4\mu^{\beta+1} + C_4\|w\|_\infty^{\frac{1}{2}}\mu. \end{aligned} \tag{3.13}$$

By (3.10) and (3.13) the proof is complete.

□

Remark 3.2.2. If u is a ground state at mass μ, then it satisfies (3.4), so the estimates in Proposition 2.2.5 hold. Moreover, from (3.8) one sees that the effect of the potential does not appear in the estimates if $\mu^{2\beta} > 2\frac{\|w\|_\infty}{\theta_p}$. Then, the presence of the potential is effective in the small mass regime or, equivalently, weak potential.

The relevance of the next remark lies in the fact that curvature generates exactly the kind of potential that is compatible with the intrinsic scaling of the NLS on graphs, making it the physically and mathematically "natural" external field to include in the model.

Remark 3.2.3. (Potential induced by curvature) From Remark 2.3 in [8] one has that if $u \in H^1(\mathcal{G})$, then the quantities

$$\mu^{-2\beta-1}\|u'\|^2_{L^2(\mathcal{G})}, \quad \mu^{-2\beta-1}\|u\|^p_{L^p(\mathcal{G})}, \quad \mu^{-\beta-1}\|u\|^2_\infty,$$

are invariant under the following rescaling of $\mathcal{G}$ and u:

$$\mathcal{G} \mapsto t^{-\beta}\mathcal{G}, \quad u(\cdot) \mapsto t^\alpha u(t^\beta \cdot).$$

Notice that u is rescaled with the mass as solitons do. The potential term shows the same invariance if one imposes the scaling

$$w(\cdot) \to t^{2\beta}w(\cdot), \tag{3.14}$$

i.e., if it scales as the inverse of the square of a length. In fact

$$\begin{aligned} \int_{\mathcal{K}} w(x)|u(x)|^2dx &\mapsto \int_{t^{-\beta}\mathcal{K}} t^{2\beta}w(x)|t^\alpha u(t^\beta x)|^2 t^{-\beta} d(t^\beta x) \\ &= t^{\beta+2\alpha}\int_{\mathcal{K}} w(x)|u(x)|^2dx, \end{aligned}$$

and since $\beta + 2\alpha = 2\beta + 1$ from (2.6), one recovers the same scaling law and thus the quantity $\mu^{-2\beta-1}E(u, \mathcal{G})$ is invariant while the mass is mapped from μ to $t\mu$.

In the already mentioned case of the potential V in (3.2) induced by the presence of a curvature in the edges this is exactly the case. Indeed the curvature γ scales as the inverse of a length, namely $\gamma \mapsto t^\beta\gamma$, which is consistent with the fact that, by definition, the curvature is the inverse of the radius of the osculating circle. This gives the insight that the curvature potential is not an arbitrary addition but it is dictated by the geometry itself, and it transforms exactly as required for the full energy functional to remain invariant under the soliton-type rescaling.

3.3 Existence criterion for $I(\cdot,\mathcal{G})$

Here we extend the existence criterion given in Chapter 2 to the functional (3.1) by proving the following

Lemma 3.3.1 (Existence criterion for $I(\cdot,\mathcal{G})$)**.** *Fix $2<p<6$ and let $\mathcal{G}$ be a noncompact metric graph. If there exists $v\in H^1_\mu(\mathcal{G})$ such that*

$$I(v,\mathcal{G})\leq E_{\mathrm{NLS}}(\phi_\mu,\mathbb{R}), \tag{3.15}$$

then $I(\cdot,\mathcal{G})$ admits a ground state at mass μ.

In order to prove Lemma 3.3.1 we follow the line of the argument of the previous Chapter. Thus, as a first step we show that the strict concavity of $\mathcal{I}_{\mathcal{G}}$ as a function of μ is preserved in the presence of the potential term.

Proposition 3.3.2. *$\mathcal{I}_{\mathcal{G}}$ is strictly concave and strictly subadditive as a function of the mass μ.*

Proof. Consider a sequence $\{v_n\}\subset H^1_\mu(\mathcal{G})$ that minimizes $I(\cdot,\mathcal{G})$ at mass μ. Since $\mathcal{I}_{\mathcal{G}}(\mu)<0$, one has $I(v_n,\mathcal{G})\leq\frac{1}{2}\mathcal{E}_{\mathcal{G}}(\mu)$. We restrict our attention to the elements of the sequence that satisfy such inequality.

By Proposition 3.2.1, one has $V(v_n)\geq\min(C_1\mu^{2\beta+1},C_2\Lambda^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}})$. Let us now introduce the sequence $\{u_n\}\subset H^1_1(\mathcal{G})$, defined as $u_n=\frac{v_n}{\sqrt{\mu}}$. Since $V(u_n)=\mu^{-\frac{p}{2}}V(v_n)$, one has

$$\mu^{\frac{p}{2}}V(u_n)=V(v_n)\geq\min\{C_1\mu^{2\beta+1},C_2\Lambda^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}}\},$$

that implies that every function u_n belongs to the set

$$U:=\left\{u\in H^1(\mathcal{G}),\int_{\mathcal{G}}|u|^2dx=1,\ \mu^{\frac{p}{2}}V(u)\geq\min\{C_1\mu^{2\beta+1},\ C_2\Lambda^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}}\}\right\},$$

with C_1,C_2 and Λ as in Proposition 3.2.1. Then we consider the family of functions f_u defined by

$$f_u(\mu):=I\left(\sqrt{\mu}u,\mathcal{G}\right)=\frac{\mu}{2}\int_{\mathcal{G}}|u'|^2dx-\frac{\mu^{\frac{p}{2}}}{p}V(u)-\frac{\mu}{2}\int_{\mathcal{K}}w(x)|u|^2dx,\ \ u\in U,$$

so that the following identities hold:

$$\mathcal{I}_{\mathcal{G}}(\mu)\ =\ \lim_{n\to\infty}I(v_n,\mathcal{G})=\lim_{n\to\infty}f_{u_n}(\mu), \tag{3.16}$$

thus

$$\mathcal{I}_{\mathcal{G}}(\mu)=\inf_{u\in U}f_u(\mu).$$

Now, since $u \in U$ implies $V(u) > 0$,

$$f_u''(\mu) = -\frac{p-2}{4}\mu^{\frac{p}{2}-2}V(u) < 0,$$

which implies that $\mathcal{I}_{\mathcal{G}}$ is concave. Furthermore, since f_u is uniformly strictly concave, $\mathcal{I}_{\mathcal{G}}$ is strictly concave too on every interval $[a,b] \subset (0,\infty)$ and also strictly subadditive since $\mathcal{I}_{\mathcal{G}}(0) = 0$. □

Now we analyze the behaviour of the minimizing sequences.

Proposition 3.3.3. *Any minimizing sequence $\{u_n\} \subset H^1_\mu(\mathcal{G})$ for the functional $I(\cdot,\mathcal{G})$, defined in (3.1), with $2 < p < 6$, is weakly compact in $H^1(\mathcal{G})$.*

Proof. Using (2.11), the fact that $2 < p < 6$ and a straightforward estimate for the potential term, we get the lower bound

$$E(u,\mathcal{G}) \;\geq\; \frac{1}{2}\|u_n'\|^2_{L^2(\mathcal{G})} - \frac{C_p}{p}\mu^{\frac{p}{4}+\frac{1}{2}}\|u_n'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})} - \frac{\mu}{2}\|w\|_\infty,$$

so that $\|u_n'\|_{L^2(\mathcal{G})}$ is bounded, otherwise the sequence $I(u_n,\mathcal{G})$ would diverge as $p < 6$ and the sequence u_n could not be minimizing. Since $\|u_n\|_{L^2(\mathcal{G})} = \sqrt{\mu}$, the sequence is bounded in $H^1(\mathcal{G})$ and then weakly compact by the Banach-Alaoglu Theorem. □

In the next result we characterise the behaviour of weakly convergent minimizing sequences.

Proposition 3.3.4. *If $\{u_n\}$ is a minimizing sequence for the functional $I(\cdot,\mathcal{G})$ at mass μ, and $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, then one of the two following cases occurs:*
(i) $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$ and $u \equiv 0$;
(ii) $u \in H^1_\mu(\mathcal{G})$ is a minimizer and $u_n \to u$ strongly in $H^1(\mathcal{G}) \cap L^p(\mathcal{G})$.

Proof. The sequence u_n converges weakly to u in $L^2(\mathcal{G})$ too, so let

$$m = \mu - \|u\|^2_{L^2(\mathcal{G})} \in [0,\mu]$$

be the loss of mass in the limit. If $m = \mu$, then case (i) occurs because $u_n \to 0$ strongly in $L^\infty_{loc}(\mathcal{G})$ and so $u \equiv 0$.

Furthermore, following Proposition 2.3.3, we can state that the case $0 < m < \mu$ never occurs. Indeed, first we note that from the fact that $u_n \rightharpoonup u$ in $L^2(\mathcal{G})$ one has $\|u_n - u\|^2_{L^2(\mathcal{G})} \to m$

Moreover,

$$\frac{1}{2}\int_{\mathcal{K}} w(x)|u_n|^2dx - \frac{1}{2}\int_{\mathcal{K}} w(x)|u_n - u|^2dx - \frac{1}{2}\int_{\mathcal{K}} w(x)|u|^2dx = o(1).$$

Therefore
$$I(u_n,\mathcal{G}) = I(u_n - u,\mathcal{G}) + I(u,\mathcal{G}) + o(1)$$
as $n \to \infty$. Now, since by concavity $\mathcal{I}_{\mathcal{G}}$ is continuous, one obtains

$$\mathcal{I}_{\mathcal{G}}(\mu) \;\geq\; \mathcal{I}_{\mathcal{G}}(m) + I(u,\mathcal{G}) \;\geq\; \mathcal{I}_{\mathcal{G}}(m) + \mathcal{I}_{\mathcal{G}}\left(\|u\|^2_{L^2(\mathcal{G})}\right).$$

From Proposition 3.3.2 the function $\mathcal{I}_{\mathcal{G}}$ is strictly subadditive, therefore if $0 < m < \mu$ then
$$\mathcal{I}_{\mathcal{G}}(\mu) < \mathcal{I}_{\mathcal{G}}\left(\mu - \|u\|^2_{L^2(\mathcal{G})}\right) + \mathcal{I}_{\mathcal{G}}\left(\|u\|^2_{L^2(\mathcal{G})}\right)$$
which is a contradiction.

If $m = 0$ then $u_n \to u$ strongly in $L^2(\mathcal{G})$. Now, since $p > 2$ and $\|u_n\|_{L^\infty} \leq C$, from

$$\|u_n - u\|^p_{L^p(\mathcal{G})} \leq \|u_n - u\|^{p-2}_{L^\infty(\mathcal{G})}\|u_n - u\|^2_{L^2(\mathcal{G})} \leq C\|u_n - u\|^2_{L^2(\mathcal{G})} \to 0, \tag{3.17}$$

one gets $u_n \to u$ strongly in $L^p(\mathcal{G})$ too.

Furthermore
$$\begin{aligned}\int_{\mathcal{K}} w(x)\left||u_n(x)|^2 - |u(x)|^2\right| dx \;&\leq C\int_{\mathcal{K}} w(x)|u_n(x) - u(x)|^2 dx \\ &\leq C\|w\|_\infty\|u_n - u\|^2_{L^2(\mathcal{G})} \to 0\end{aligned} \tag{3.18}$$

so $W(u_n)$ converges to $W(u)$.

Then, from Eq (3.17), Eq (3.18), and the fact that u_n is a minimizing sequence,

$$\begin{aligned}\|u_n'\|^2_{L^2(\mathcal{G})} \;&= I(u_n,\mathcal{G}) + \frac{1}{p}\|u_n\|^p_{L^p(\mathcal{G})} + \frac{1}{2}\int_{\mathcal{K}} w(x)|u_n(x)|^2 dx \\ &\to \mathcal{I}(\mu) + +\frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} + \frac{1}{2}\int_{\mathcal{K}} w(x)|u(x)|^2 dx \\ &\leq I(u,\mathcal{G}) + \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} + \frac{1}{2}\int_{\mathcal{K}} w(x)|u(x)|^2 dx \;=\; \|u'\|^2_{L^2(\mathcal{G})}.\end{aligned} \tag{3.19}$$

where the last passage exploits $u \in H^1_\mu(\mathcal{G})$. It follows

$$\lim \|u_n'\|^2_{L^2(\mathcal{G})} \;\leq\; \|u'\|^2_{L^2(\mathcal{G})}. \tag{3.20}$$

On the other hand, since u' is the weak limit in $L^2(\mathcal{G})$ of u_n', it must be

$$\liminf \|u_n'\|^2_{L^2(\mathcal{G})} \;\geq\; \|u'\|^2_{L^2(\mathcal{G})}. \tag{3.21}$$

From Eqs (3.20) and (3.21) one concludes $\lim \|u_n'\|^2_{L^2(\mathcal{G})} = \|u'\|^2_{L^2(\mathcal{G})}$, so u_n converges to u strongly in $H^1(\mathcal{G})$, and therefore $I(u,\mathcal{G}) = \lim I(u_n,\mathcal{G})$ and u is a ground state for $I(\cdot,\mathcal{G})$ at mass μ. □

We are now ready to prove Lemma 3.3.1.

Proof of Lemma 3.3.1. Given a minimizing sequence $\{u_n\} \subset H^1_\mu(\mathcal{G})$ we need to exclude case (i) of Proposition 4.4.5. Assume (i). Then $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$, that implies $u_n \to 0$ in $L^\infty(\mathcal{K})$. Since the potential is supported on $\mathcal{K}$, then $I(u_n, \mathcal{G}) = E_{\mathrm{NLS}}(u_n, \mathcal{G}) + o(1)$ and one can repeat the argument in Theorem 2.27. Such argument shows that such a minimizing sequence leads to an energy level not lower than the threshold $-\theta_p \mu^{2\beta+1}$. Therefore, if there exists $v \in H^1_\mu(\mathcal{G})$ such that the condition (3.15) is satisfied, then case (i) of Proposition 4.4.5 is ruled out, so case (ii) is verified. Thus we can assume the existence of a minimizing sequence $\{u_n\}$ that strongly converges to u in $H^1 \cap L^p(\mathcal{G})$ and u is a ground state. □

Owing to this theorem we can state that if there is a competitor function in $H^1_\mu(\mathcal{G})$ such that its energy is lower than the energy of the soliton, then the graph $\mathcal{G}$ admits a ground state with mass μ. Therefore, finding such competitor provides a tool for proving the existence of these states.

3.4 Existence results

It is known that a graph $\mathcal{G}$ does not support ground states for $E_{\mathrm{NLS}}(\cdot, \mathcal{G})$ at any mass, provided that it satisfies the so-called Assumption (H). This is the case, for instance, of the graph in Figure 3.1, called the 2-bridge.

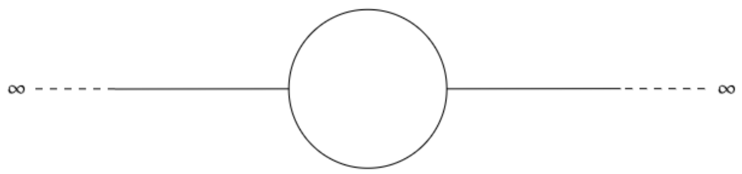

Figure 3.1: The 2-bridge graph satisfies the Assumption (H) that prevents the existence of ground states for E_{NLS} at every mass.

On the other hand, the presence of an attractive potential can make it energetically convenient to concentrate the mass in the support of the potential and therefore, at least for some values of the mass, ground states can exist. In the present section we show that this is indeed the case if the mass is large or small enough. We achieve the results by directly applying Lemma 3.3.1, i.e., exhibiting a function of mass μ whose energy is below the threshold $-\theta_p \mu^{2\beta+1}$.

3.4.1 Existence for large mass

Here we give the first result on existence of ground states.

Theorem 3.4.1. *Let $\mathcal{G}$ be a graph with $n \geq 1$ halflines and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact core $\mathcal{K}$ of $\mathcal{G}$.*

If μ is large enough, then there exists a Ground State for $I(\cdot, \mathcal{G})$ at mass μ.

Proof. Consider a point $\bar{x}$ in $\mathcal{K}$ such that $\bar{x}$ is not a vertex of $\mathcal{G}$ and $w(\bar{x}) > 0$. Such a point exists, since otherwise by continuity w would be zero in all vertices too, and then identically zero. With a slight abuse of notation, we denote by $\bar{x}$ the coordinate of the point $\bar{x}$ as an element of the interval $I_{\bar{e}}$ that represents the edge $\bar{e}$ in which $\bar{x}$ lies. By continuity, there is $\ell > 0$ such that the interval $[\bar{x} - \ell/2, \bar{x} + \ell/2]$ belongs to $\bar{e}$ and $w > 0$ in $[\bar{x} - \ell/2, \bar{x} + \ell/2]$. We denote $\kappa := \min_{[\bar{x}-\ell/2, \bar{x}+\ell/2]} w(x)$, so that $\kappa > 0$.

Consider now the family of functions $v_\mu \in H^1(\mathbb{R})$ defined in the following way:

$$v_\mu(x) = (\phi_\mu(x - \bar{x}) - \phi_\mu(\ell/2))\chi_{[\bar{x}-\ell/2, \bar{x}+\ell/2]}(x),$$

where χ_A denotes the characteristic function of the real subset A.

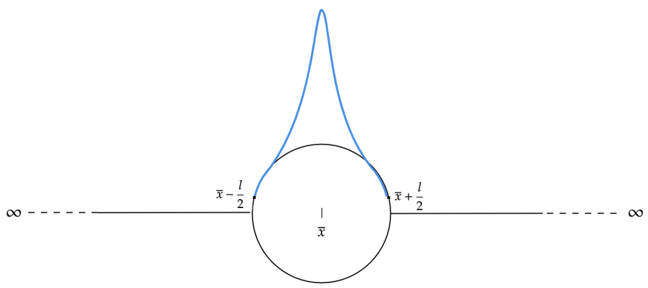

Figure 3.2: A representation of the function $\widetilde{f}_\mu$ on the 2-bridge graph.

One has

$$\begin{aligned}
\int_{\mathbb{R}} |v_\mu|^2 dx &= \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |\phi_\mu - \phi_\mu(\frac{\ell}{2})|^2 dx = 2\int_0^{\frac{\ell}{2}} |\phi_\mu - \phi_\mu(\frac{\ell}{2})|^2 dx \\
&= 2\int_0^{\frac{\ell}{2}} \phi_\mu(x)^2 dx + 2\int_0^{\frac{\ell}{2}} \phi_\mu(\frac{\ell}{2})^2 dx - 4\int_0^{\frac{\ell}{2}} \phi_\mu(x)\phi_\mu(\frac{\ell}{2}) dx \qquad (3.22) \\
&= \mu - 2\int_{\frac{\ell}{2}}^{\infty} \phi_\mu(x)^2 dx + \ell\phi_\mu(\frac{\ell}{2})^2 - 4\phi_\mu(\frac{\ell}{2})\int_0^{\frac{\ell}{2}} \phi_\mu(x) dx.
\end{aligned}$$

From the explicit form of the soliton (2.5), since $\ell > 0$ one immediately has $\phi_\mu(\ell/2) \leq C\mu^\alpha e^{-c\mu^\beta}$. Moreover, since $2\alpha - \beta = 1$,

$$\int_{\frac{\ell}{2}}^{\infty} \phi_\mu(x)^2 dx \leq \phi_\mu(\frac{\ell}{2}) \int_0^{\infty} \phi_\mu(x) dx \leq C\mu e^{-c\mu^\beta}, \qquad (3.23)$$

where C and c are positive constants independent of μ. Then, recalling that v_μ is a cut and lowered version of ϕ_μ, and that in (3.22) both $\phi_\mu(\ell/2)$ and $\phi_\mu^2(\ell/2)$ appear, we conclude

$$0 \leq \ \mu - \|v_\mu\|^2_{L^2(\mathbb{R})} \ \leq \ C(\mu^{2\alpha} + \mu)e^{-c\mu^\beta}. \tag{3.24}$$

Furthermore,

$$\|v_\mu - \phi_\mu(\cdot - \bar{x})\|^2_{L^2(\mathbb{R})} = 2\int_0^{\ell/2} \phi_\mu^2(\ell/2)dx + 2\int_{\ell/2}^\infty \phi_\mu^2(x)\,dx \ \leq \ C(\mu^{2\alpha}+\mu)e^{-c\mu^\beta}, \tag{3.25}$$

and

$$\|v'_\mu - \phi'_\mu(\cdot - \bar{x})\|^2_{L^2(\mathbb{R})} = 2\int_{\ell/2}^\infty (\phi'_\mu)^2(x)\,dx \leq C\mu^{2\beta}\int_{\ell/2}^\infty \phi_\mu^2(x)\,dx \leq C\mu^{2\beta+1}e^{-c\mu^\beta} \tag{3.26}$$

where the last estimate is obtained from inequality

$$|\phi'_\mu(x)| \ = \ C\mu^{\alpha+\beta}\frac{|\sinh(\mu^\beta c_p x)|}{\cosh^{\frac{\alpha}{\beta}+1}(\mu^\beta c_p x)} \leq C\mu^\beta\phi_\mu(x)$$

and by (3.23). Now we define the function $f_\mu = \frac{\sqrt{\mu}}{\|v_\mu\|_{L^2(\mathbb{R})}}v_\mu$ and compute

$$\begin{aligned}\|f_\mu - v_\mu\|_{L^2(\mathbb{R})} = \sqrt{\mu} - \|v_\mu\|_{L^2(\mathbb{R})} \ &= \ \frac{\mu - \|v_\mu\|^2_{L^2(\mathbb{R})}}{\sqrt{\mu} + \|v_\mu\|_{L^2(\mathbb{R})}} \ \leq \ C(\sqrt{\mu} + \mu^{2\alpha-\frac{1}{2}})e^{-c\mu^\beta} \\ \|f'_\mu - v'_\mu\|_{L^2(\mathbb{R})} \leq (\sqrt{\mu} - \|v_\mu\|_{L^2(\mathbb{R})})\frac{\|v'_\mu\|_{L^2(\mathbb{R})}}{\|v_\mu\|_{L^2(\mathbb{R})}} \ &\leq \ C(\sqrt{\mu} + \mu^{2\alpha-\frac{1}{2}})e^{-c\mu^\beta}\frac{\|v'_\mu\|_{L^2(\mathbb{R})}}{\|v_\mu\|_{L^2(\mathbb{R})}}.\end{aligned} \tag{3.27}$$

From (3.23)

$$\|v_\mu\|_{L^2(\mathbb{R})} \geq \sqrt{\mu - C(\mu + \mu^{2\alpha})e^{-c\mu^\beta}}$$

while, since $\|\phi'_\mu\|_{L^2(\mathbb{R})} \leq C\mu^{\beta+\frac{1}{2}}$, from (3.26) one gets

$$\|v'_\mu\|_{L^2(\mathbb{R})} \ \leq \ \|\phi'_\mu\|_{L^2(\mathbb{R})} + C\mu^{\beta+\frac{1}{2}}e^{-c\mu^\beta} \ \leq \ C\mu^{\beta+\frac{1}{2}}(1 + e^{-c\mu^\beta}).$$

Thus, for the second inequality in (3.27) one gets

$$\|f'_\mu - v'_\mu\|_{L^2(\mathbb{R})} \leq \ C(\sqrt{\mu} + \mu^{2\alpha-\frac{1}{2}})e^{-c\mu^\beta}\frac{\mu^{\beta+\frac{1}{2}}(1 + e^{-c\mu^\beta})}{\sqrt{\mu - C(\mu + \mu^{2\alpha})e^{-c\mu^\beta}}}, \tag{3.28}$$

where we used (3.23)–(3.26). From (3.27) and (3.28) one then concludes that both $\|f_\mu - v_\mu\|_{L^2(\mathbb{R})}$ and $\|f'_\mu - v'_\mu\|$ vanish as μ goes to infinity. Furthermore, from (2.11)

$$\|f_\mu - v_\mu\|^p_{L^p(\mathbb{R})} \leq C\|f'_\mu - v'_\mu\|^{\frac{p}{2}-1}_{L^2(\mathbb{R})}\|f_\mu - v_\mu\|^{\frac{p}{2}+1}_{L^2(\mathbb{R})} \to 0, \quad \mu \to \infty. \tag{3.29}$$

Let us introduce the function $\widetilde{f}_\mu$, defined as f_μ on the edge $\bar{e}$ and zero on all other edges. Obviously, $\widetilde{f}_\mu$ belongs to $H^1(\mathcal{G})$ and its $L^2(\mathcal{G})$, $L^p(\mathcal{G})$, and $H^1(\mathcal{G})$ norms are the same as the corresponding ones of f_μ as a function on $\mathbb{R}$. Now, from (3.27) and (3.26) one has

$$\begin{aligned} \left| \|\widetilde{f}'_\mu\|_{L^2(\mathcal{G})} - \|\phi'_\mu\|_{L^2(\mathbb{R})} \right| = \left| \|f'_\mu\|_{L^2(\mathbb{R})} - \|\phi'_\mu\|_{L^2(\mathbb{R})} \right| \leq \|f'_\mu - \phi'_\mu\|_{L^2(\mathbb{R})} \\ \leq \|f'_\mu - v'_\mu\|_{L^2(\mathbb{R})} + \|v'_\mu - \phi'_\mu\|_{L^2(\mathbb{R})} \to 0, \quad \mu \to \infty, \end{aligned} \tag{3.30}$$

and analogously from (3.29)

$$\|\widetilde{f}_\mu\|^p_{L^p(\mathcal{G})} - \|\phi_\mu\|^p_{L^p(\mathbb{R})} \to 0, \quad \mu \to \infty. \tag{3.31}$$

By (3.30) and (3.31)

$$\begin{aligned} I(\widetilde{f}_\mu, \mathcal{G}) - E_{\text{NLS}}(\phi_\mu, \mathbb{R}) &= E_{\text{NLS}}(\widetilde{f}_\mu, \mathcal{G}) - E_{\text{NLS}}(\phi_\mu, \mathbb{R}) - \frac{1}{2}\int_{\mathcal{G}} w(x)|\widetilde{f}_\mu(x)|^2 dx \\ &\leq -\frac{\kappa}{2}\int_{-\ell/2}^{-\ell/2} \phi^2_\mu(x)\, dx + o(1) \\ &\leq -\frac{\kappa}{2}\mu + o(1), \quad \mu \to \infty, \end{aligned}$$

so $I(\widetilde{f}_\mu, \mathcal{G}) < E_{\text{NLS}}(\phi_\mu, \mathbb{R})$ for μ large enough and by Lemma 3.3.1 the proof is complete. □

3.4.2 Existence for small mass

Here we give the second result on existence of ground states.

Theorem 3.4.2. *Let $\mathcal{G}$ be a graph with $n \geq 1$ infinite edges and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact core $\mathcal{K}$ of $\mathcal{G}$.*

If μ is small enough, then there exists a ground state for $I(\cdot, \mathcal{G})$ at mass μ.

Proof. Let $\mu > 0$. We define the function u_μ as follows:

$$u_\mu(x) = \begin{cases} \phi_m & \text{if } \ x \in \mathcal{H}_i, \ i = 1, \dots, n \\ \phi_m(0) & \text{if } \ x \in \mathcal{K}, \end{cases}$$

where $\mathcal{H}_i$ represents the halfline associated with the index i. The parameter m is uniquely determined by imposing $\|u_\mu\|^2_{L^2(\mathbb{R})} = \mu$, namely by the identity

$$\mu = \int_{\mathcal{G}} |u_\mu|^2 dx = \frac{n}{2}m + |\phi_m(0)|^2|\mathcal{K}| = \frac{n}{2}m + C_p^2|\mathcal{K}|m^{2\alpha},$$

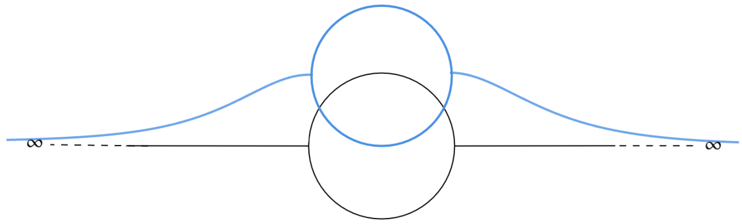

Figure 3.3: A representation of the function u_μ on the two-bridge graph.

where $|\mathcal{K}|$ is the total length of $\mathcal{K}$, while the energy of u_μ reads

$$I(u_\mu, \mathcal{G}) = -\frac{n}{2}\theta_p m^{2\beta+1} - \frac{1}{p}C_p^p m^{p\alpha}|\mathcal{K}| - C_p^2 m^{2\alpha}\frac{|\mathcal{K}|}{2}\int_{\mathcal{K}} w(x)dx.$$

In order to be less energetic than the soliton on the line with the same mass, u_μ must satisfy the condition

$$-\frac{n}{2}\theta_p m^{2\beta+1} - \frac{1}{p}C_p^p|\mathcal{K}|m^{p\alpha} - C_p^2\frac{|\mathcal{K}|}{2}m^{2\alpha}\int_{\mathcal{K}} w(x)dx < -\theta_p\left(\frac{n}{2}m + C_p^2|\mathcal{K}|m^{2\alpha}\right)^{2\beta+1}.$$

Since 2α is the smallest exponent in the last inequality, it turns out that the inequality is satisfied if μ is small enough, thus by Lemma 3.3.1 a ground state exists and the theorem is proved.

□

3.5 Nonexistence results

Here we extend the nonexistence result given in Theorem 5.1 of [8] to the case of the presence of a weak, attractive, compactly supported potential and give sufficient conditions for the nonexistence of ground states in some intervals of the values of the mass.

We preliminary recall a basic estimate for functions defined on a compact set $\mathcal{K} \subset \mathcal{G}$ ([49]):

$$\|u\|_{L^\infty(\mathcal{K})} \leq |\mathcal{K}|^{-\frac{1}{2}}\|u\|_{L^2(\mathcal{K})} + \text{diam}(\mathcal{K})^{\frac{1}{2}}\|u'\|_{L^2(\mathcal{K})}, \qquad \forall u \in H^1(\mathcal{K}) \tag{3.32}$$

where $|\mathcal{K}|$ is the total length of $\mathcal{K}$ and $\text{diam}(\mathcal{K})$ is its diameter.

Theorem 3.5.1. *Let $\mathcal{G}$ be a graph with $n \geq 1$ infinite edges and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact core $\mathcal{K}$ of*

$\mathcal{G}$. Furthermore denote by $|\mathcal{K}|$ the total length of $\mathcal{K}$ and by $\operatorname{diam}(\mathcal{K})$ *its diameter, i.e. the maximal distance between any pair of points of $\mathcal{K}$.*

Then there exists a number $\epsilon > 0$, that depends on p only, such that, if

$$\max\left(\mu^{\beta}\operatorname{diam}(\mathcal{K}), \frac{1}{\mu^{\beta}|\mathcal{K}|}, \frac{\|w\|_{\infty}}{\mu^{2\beta}}\right) < \epsilon \tag{3.33}$$

are satisfied, then the functional $I(\cdot, \mathcal{G})$ defined in (3.1) has no ground state at mass μ.

Proof. We proceed by contradiction, thus we consider $\mu > 0$ that satisfies the condition Eq (3.33) and suppose that there exists a ground state u at mass μ for $I(\cdot, \mathcal{G})$. Due to the invariance of $I(\cdot, \mathcal{G})$ under multiplication by a phase, we can assume without loss of generality that u is real and nonnegative.

First we assume that $\mathcal{G}$ contains only one halfline, i.e. $n = 1$. Taking $\epsilon < \theta_p/2$, condition Eq (3.33) guarantees that inequality Eq (3.8) holds, so that estimates Eq (3.10) are valid. It is therefore possible to follow the proof of Theorem 5.1 in [8] replacing $I(\phi_\mu, \mathbb{R})$ by $E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R})$ up to the last inequality.

We choose a coordinate $x \in [0, \infty)$ on the half-line, it is known [15] that the restriction of u to any half-line of $\mathcal{G}$ takes the form

$$u(x) = \phi_m(x + y), \qquad x \geq 0 \tag{3.34}$$

for some $y \in \mathbb{R}$ and $m > 0$.

(i) Proof that $y < 0$ and L^∞ estimate. Combining (3.32) and (3.33) yields

$$\|u\|^2_{L^\infty(\mathcal{K})} \leq 2\epsilon\left(\mu^{\beta}\int_{\mathcal{K}} u^2 dx + \mu^{-\beta}\int_{\mathcal{K}} |u'|^2 dx\right) \tag{3.35}$$

and, since $\|u\|^2_{L^2(\mathcal{K})} \leq \mu$ and $\|u'\|^2_{L^\infty(\mathcal{K})} \leq \|u'\|^2_{L^\infty(\mathcal{G})}$, using (3.10) we obtain

$$\|u\|^2_{L^\infty(\mathcal{K})} \leq \epsilon C \mu^{\beta+1} \tag{3.36}$$

We can notice that if $\|u\|^2_{L^\infty(\mathcal{G})} = \|u\|^2_{L^\infty(\mathcal{K})}$ the last relation (3.36) would violate the inequality in (3.7) for ϵ small enough. Therefore the maximum of u cannot be achieve in the compact core and we can conclude that $y < 0$.

(ii) We can estimate the difference between the mass of the function μ and the mass of the restriction of u in the halfline $\|\phi_m\|^2_{L^2(\mathbb{R}^+)} = m$:

$$\mu - m \geq \int_K |u|^2 dx - C\mu^{-\beta}\phi_m(y)^2 \tag{3.37}$$

(iii) We can estimate the energy in the halfline:

$$I(u, \mathcal{G}\backslash\mathcal{K}) - I(\phi_\mu, \mathbb{R}) \geq C^{-1}\mu^{2\beta}\int_{\mathcal{K}} |u|^2 dx - C\mu^{\beta}\phi_m(y)^2 \tag{3.38}$$

(iv) Energy estimate on $\mathcal{K}$. Using (3.36)

$$\int_{\mathcal{K}} |u|^p dx \le \|u\|^{p-2}_{L^\infty(\mathcal{K})} \int_{\mathcal{K}} |u|^2 dx \le C\epsilon^{\frac{p-2}{2}} \mu^{2\beta} \int_{\mathcal{K}} |u|^2 dx \tag{3.39}$$

Therefore

$$I(u,\mathcal{K}) \ge \frac{1}{2}\int_{\mathcal{K}} |u'|^2 dx - C\epsilon^{\frac{p-2}{2}} \mu^{2\beta} \int_{\mathcal{K}} |u|^2 dx - \frac{1}{2}\int_{\mathcal{K}} w(x)|u|^2 \tag{3.40}$$

Including the contribution of the potential we have

$$\begin{aligned}
&I(u,\mathcal{G}) - E_{\text{NLS}}(\phi_\mu,\mathbb{R}) \\
&\ge \left(C_1 - C_2\epsilon^{\frac{p-2}{2}}\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \frac{1}{2}\int_{\mathcal{K}}|u'|^2dx - C_3\mu^\beta\|u\|^2_{L^\infty(\mathcal{K})} - \frac{1}{2}\int_{\mathcal{K}} w(x)|u|^2dx \\
&\ge \left(C_1 - C_2\epsilon^{\frac{p-2}{2}}\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \frac{1}{2}\int_{\mathcal{K}}|u'|^2dx - C_3\mu^\beta\|u\|^2_{L^\infty(\mathcal{K})} - \frac{1}{2}\|w\|_\infty\int_{\mathcal{K}} |u|^2dx \\
&\ge \left(C_1 - C_2\epsilon^{\frac{p-2}{2}} - \frac{\epsilon}{2}\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \frac{1}{2}\int_{\mathcal{K}}|u'|^2dx - C_3\mu^\beta\|u\|^2_{L^\infty(\mathcal{K})}.
\end{aligned}$$

Using the inequality (3.32) to estimate $\|u\|_{L^\infty(\mathcal{K})}$ one finds

$$\begin{aligned}
&E(u,\mathcal{G}) - E_{\text{NLS}}(\phi_\mu,\mathbb{R}) \\
&\ge \left(C_1 - C_2\epsilon^{\frac{p-2}{2}} - \frac{\epsilon}{2} - 2C_3\epsilon\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \left(\frac{1}{2} - 2C_3\epsilon\right)\int_{\mathcal{K}}|u'|^2dx \\
&> 0,
\end{aligned}$$

uniformly in μ and provided that ϵ is small enough. By equation (3.3) this contradicts the fact that u is a Ground State of $I(\cdot,\mathcal{G})$, and the proof is complete for the case $n = 1$.

Suppose now $n > 1$. Let $\mathcal{H}_{\widetilde{ß}}$ be the halfline in which u attains $\max_{\mathcal{G}\setminus\mathcal{K}} u$, and let us call $\widetilde{x}$ the corresponding maximum point on $\mathcal{H}_{\widetilde{ß}}$. Moreover, let us define the halfline $\widetilde{\mathcal{H}}$ as the subset of $\mathcal{H}_{\widetilde{ß}}$ corresponding to the coordinate interval $[\widetilde{x}, +\infty)$. It is convenient to use on $\widetilde{\mathcal{H}}$ the coordinate system inherited by $\mathcal{H}_{\widetilde{ß}}$ ranging from $\widetilde{x}$ to $+\infty$.

For any $i \ne \widetilde{ß}$ let us set

$$\widetilde{y}_i = \min\{x \in [\widetilde{x}, +\infty),\ u_{\widetilde{ß}}(x) = u_i(0)\}, \tag{3.41}$$

which is well-defined since by definition of $\widetilde{x}$, for every $i \ne \widetilde{ß}$ one has $u_i(0) \le u_{\widetilde{ß}}(\widetilde{x})$, thus by continuity of $u_{\widetilde{ß}}$ there exists at least a point $\widetilde{z} \in \widetilde{\mathcal{H}}$ such that $u_{\widetilde{ß}}(\widetilde{z}) = u_i(0)$. The symbol $\widetilde{y}_i$ denotes then the minimum of such $\widetilde{z}$'s.

Now for every $i \ne \widetilde{ß}$ we attach the origin of $\mathcal{H}_i$ to the point of coordinate $\widetilde{y}_i$ in the halfine $\widetilde{\mathcal{H}}$. We obtain in this way a graph $\widehat{\mathcal{G}}$, made of one halfline $(\widetilde{\mathcal{H}})$, to which are attached by their origins $n-1$ other halflines $(\mathcal{H}_i,\ i \ne \widetilde{ß})$.

Let us consider the function $\widehat{u} : \widehat{\mathcal{G}} \to \mathbb{R}$, which is made of the restrictions of u to the halflines that constitute the graph $\widehat{\mathcal{G}}$. In symbols, $\widehat{u} = (\widehat{u}_{\widetilde{ß}}, \widehat{u}_i)_{i\neq\widetilde{ß}}$, with $\widehat{u}_{\widetilde{ß}}(x) = u_{\widetilde{ß}}(x)$ for every $x \in [\widetilde{x}, +\infty)$ and $\widehat{u}_i(x) = u_i(x)$ for every $x \in [0, +\infty)$. Since the restriction of $\widehat{u}$ to every halfline is in H^1, and since by definition of the points $\widetilde{y}_i$ the function $\widehat{u}$ is continuous at the vertices of $\widehat{\mathcal{G}}$, it follows that $\widehat{u} \in H^1(\widehat{\mathcal{G}})$. Moreover, $\widehat{\mathcal{G}}$ is connected and therefore one can apply Proposition 2.2.1 and then define the monotone rearrangement u^* of $\widehat{u}$, that is defined on $[0, +\infty)$.

Now we define the graph $\mathcal{G}'$ as the original graph $\mathcal{G}$, but with $\mathcal{H}_{\widetilde{ß}}$ as the only halfline, i.e.,

$$\mathcal{G}' = \mathcal{G}\backslash(\cup_{i\neq\widetilde{ß}}\mathcal{H}_i) = \mathcal{K} \cup \mathcal{H}_{\widetilde{ß}},$$

and construct on it the function $v : \mathcal{G}' \to \mathbb{R}$ as

$$v(x) := \begin{cases} u(x), & x \in \mathcal{K} \\ u_{\widetilde{ß}}(x), & x \in [0, \widetilde{x}] \subset \mathcal{H}_{\widetilde{ß}} \\ u^*(x - \widetilde{x}), & x \in (\widetilde{x}, +\infty) \subset \mathcal{H}_{\widetilde{ß}}. \end{cases}$$

From Proposition 2.2.1 the monotone rearrangement preserves the L^p-norms (see identity (2.9)), then

$$\begin{aligned} \|v\|^r_{L^r(\mathcal{G}')} =& \|v\|^r_{L^r(\mathcal{K})} + \|v\|^r_{L^r(\mathcal{H}_{\widetilde{ß}})} = \|v\|^r_{L^r(\mathcal{K})} + \int_0^{\widetilde{x}} |v_{\widetilde{ß}}|^r dx + \int_{\widetilde{x}}^{+\infty} |v_{\widetilde{ß}}|^r dx \\ =& \|u\|^r_{L^r(\mathcal{K})} + \int_0^{\widetilde{x}} |u_{\widetilde{ß}}|^r dx + \int_{\widetilde{x}}^{+\infty} |u^*(x - \widetilde{x})|^r dx \\ =& \|u\|^r_{L^r(\mathcal{K})} + \int_0^{\widetilde{x}} |u_{\widetilde{ß}}|^r dx + \sum_{i\neq\widetilde{ß}} \|u\|^r_{L^r(\mathcal{H}_i)} + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{ß}}|^r dx \\ =& \|u\|^r_{L^r(\mathcal{K})}, \end{aligned} \tag{3.42}$$

for every $r \in [1, +\infty]$. In particular, for $r = 2$ one has $\|v\|^2_{L^2(\mathcal{G}')} = \mu$.

Now, due to (3.42) and to $u \equiv v$ on $\mathcal{K}$, the difference $I(u, \mathcal{G}) - I(v, \mathcal{G}')$ reduces to $T(u) - T(v)$ outside $\mathcal{K}$, i.e., on the halflines only. Thus

$$\begin{aligned} & I(u, \mathcal{G}) - I(v, \mathcal{G}') \\ =& \sum_{i\neq\widetilde{ß}} \|u_i'\|^2_{L^2(\mathcal{H}_i)} + \int_0^{\widetilde{x}} |u_{\widetilde{ß}}'|^2 dx + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{ß}}'|^2 dx - \int_0^{\widetilde{x}} |v_{\widetilde{ß}}'|^2 dx - \int_{\widetilde{x}}^{+\infty} |v_{\widetilde{ß}}'|^2 dx \\ =& \sum_{i\neq\widetilde{ß}} \|u_i'\|^2_{L^2(\mathcal{H}_i)} + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{ß}}'|^2 dx - \int_{\widetilde{x}}^{+\infty} |v_{\widetilde{ß}}'|^2 dx \\ =& \sum_{i\neq\widetilde{ß}} \|u_i'\|^2_{L^2(\mathcal{H}_i)} + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{ß}}'|^2 dx - \int_{\widetilde{x}}^{+\infty} |(u^*)'(x - \widetilde{x})|^2 dx \\ \geq& \ 0, \end{aligned} \tag{3.43}$$

where in the last passage we used inequality (2.10).

Then, since u is supposed to be a ground state for $I(\cdot,\mathcal{G})$, by (3.43) it must be

$$I(v,\mathcal{G}') \leq I(u,\mathcal{G}) \leq -\theta_p \mu^{2\beta+1},$$

therefore by the existence criterion there exists a Ground State for $I(\cdot,\mathcal{G}')$ at mass μ, that contradicts the present proof in the case $n=1$. This concludes the proof. □

In the following we show an example where Theorem 3.5.1 can be applied. In particular, for the *n-fork* graph the condition Eq (3.33) singles out a significant interval of masses and therefore the Theorem is not empty.

Remark 3.5.2. As an application of Theorem 3.5.1 we consider the graph $\mathcal{G}$ made of one halfline and a compact core $\mathcal{K}$ consisting of n edges e_i, $i=1,\dots n$, each of length l, all attached at the origin of the halfline (see Fig. (3.4)). Thus $\mathrm{diam}(\mathcal{K}) = 2l$ and $|\mathcal{K}| = nl$.

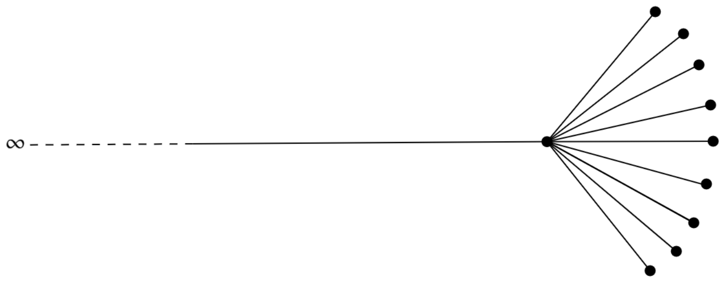

Figure 3.4: A n-fork graph consisting of one halfline and n edges of length l.

Moreover, we take in consideration a potential w supported on the edges e_i and defined as

$$w_i(x) := \frac{\epsilon^3}{4l^{2k+2}} x^{2k}, \quad i=1,\dots,n, \; x\in[0,l], \; k\in\mathbb{N},$$

where w_i denotes the restriction of w on the edge e_i. Obviously $w \geq 0$ and $\|w\|_\infty = \frac{\epsilon^3}{4l^2}$. The condition (3.33) rewrites then as

$$\epsilon > \max\left(2\mu^\beta l, \frac{1}{\mu^\beta nl}, \frac{\epsilon^3}{4l^2\mu^{2\beta}}\right),$$

that, by a straightforward computation, amounts to

$$\frac{1}{nl\epsilon} < \mu^\beta < \frac{\epsilon}{2l}. \tag{3.44}$$

The inequalities Eq (3.44) can be simultaneously satisfied for n large enough. In other words, if n is large enough, then there exists an interval of masses to which Theorem 3.5.1 applies.

Chapter 4

Magnetic potentials

4.1 Introduction

In this chapter, we extend the theory to the *Magnetic* Nonlinear Schrödinger Equation:

$$-\left(\frac{\mathrm{d}}{\mathrm{d}x} - iA\right)^2 u - |u|^{p-2}u = \lambda u\,, \tag{4.1}$$

where $A(x)$ is a magnetic vector potential representing an external field, $2 < p < 6$ and Kirchhoff magnetic boundary conditions are imposed at vertices. The introduction of a magnetic field generates rich physical phenomena and is motivated by the need to model realistic quantum transport in networks subject to external fields. As well known from the study of linear quantum graphs [42, 43, 52], magnetic potentials on networks manifest through the *Aharonov-Bohm effect* [17], a purely quantum phenomenon in which electromagnetic potentials influence the quantum phase of a charged particle even in regions where the magnetic field vanishes. On a metric graph, the magnetic field can always be removed by a gauge transformation, so it is topologically trivial, i.e. locally, it has no physical effect. However, in the presence of a cycle, the magnetic potential produces a non-removable phase shift in the wavefunction. This phase shift acts as an effective repulsive mechanism that competes with the focusing nonlinearity.

The primary goal of this work is to characterize the existence of ground states (i.e. global minimizers of the energy) for the magnetic NLS equation. This study builds on the established theoretical framework for Magnetic Nonlinear Schrödinger equations [31, 40, 65, 64], which encompasses both the foundational analysis of magnetic effects and the concentration-compactness approach to existence and stability of stationary states. Our contributions are twofold. We prove that the magnetic NLS equation on a metric graph is variationally equivalent to the standard (non-magnetic) NLS equation with an additional repulsive potential supported solely on the cycles of the graph. We derive an explicit formula for this effective potential,

$\Phi_\gamma(A)$, which depends on the distance between the magnetic flux and the nearest integer winding number. This reduction allows us to extend classical existence criteria in Lemma 2.3.5 to the magnetic setting using concentration-compactness arguments.

As a concrete application, we analyze the *tadpole* graph (a ring attached to a half-line). The graph naturally emerged as a simplified model for systems such as metallic rings coupled to external electron baths via a single conducting [41]. With Kirchhoff conditions, it satisfies the existence criterion of a ground state for the standard nonlinear setting [8]. Previous results on tadpole include the full classification of cubic solitons ($p = 4$) on the tadpole graph [56], the analysis of edge bifurcations for general nonlinearities [57], and the study of ground states in the presence of repulsive delta-type vertex conditions [37].

In the last section we identify an existence regime for ground states, providing a geometric condition in the case $p = 4$ on the underlying graph that guarantees their occurrence. In addition, we establish a nonexistence theorem showing that ground states cannot form when the magnetic flux is noninteger and sufficiently repulsive. The chapter is organized as follows. In Section 4.2, we introduce the variational framework and the magnetic Sobolev spaces. In Section 4.3, we prove the equivalence between the magnetic problem and the effective potential formulation. Section 4.4 establishes the general existence theory for ground states. Finally, in Section 4.5, we perform the detailed analysis of the tadpole graph, providing explicit existence and nonexistence results.

4.2 The Magnetic NLS formulation

In this section, we formulate the NLS equation on a metric graph $\mathcal{G}$ in the presence of a magnetic field. On a one-dimensional structure, the magnetic potential is described by a collection of real-valued functions $A = (A_e)_{e\in\mathbb{E}_\mathcal{G}}$ defined on each edge.

The stationary Magnetic Nonlinear Schrödinger Equation on the graph is given by:

$$-\left(\frac{\mathrm{d}}{\mathrm{d}x} - iA(x)\right)^2 u - |u|^{p-2}u = \lambda u\,, \tag{4.2}$$

where the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ is replaced by the covariant derivative

$$D = \frac{\mathrm{d}}{\mathrm{d}x} - iA\,. \tag{4.3}$$

The associated energy functional is

$$E_A(u,\mathcal{G}) = \frac{1}{2}\int_\mathcal{G} |Du|^2\,\mathrm{d}x - \frac{1}{p}\int_\mathcal{G} |u|^p\,\mathrm{d}x\,, \tag{4.4}$$

where $\mathcal{G}$ is assumed to have at least one unbounded edge. We seek ground states of this energy under the mass constraint

$$\|u\|^2_{L^2(\mathcal{G})} = \mu\,. \tag{4.5}$$

The natural domain for this problem is the magnetic Sobolev space

$$H^1_A(\mathcal{G}) := \{u \in L^2(\mathcal{G}) : Du \in L^2(\mathcal{G})\},$$

and the minimization is performed over the space:

$$H^1_{A,\mu}(\mathcal{G}) := \left\{u \in H^1_A(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu\right\}. \tag{4.6}$$

Consequently, the standard Kirchhoff vertex conditions are modified to preserve self-adjointness. A function u in the domain of the operator must satisfy:

$$\begin{cases} u(x) \text{ is continuous at every vertex } v \in \mathbb{V}_{\mathcal{G}}\,; \\ \sum_{e \succ v} \frac{\mathrm{d}u_e}{\mathrm{d}x_e}(v) - iA_e(v)u(v) = 0 \quad \forall v \in \mathbb{V}_{\mathcal{G}}\,, \end{cases} \tag{4.7}$$

where the sum runs over all edges e incident to the vertex v, and derivatives are taken in the outgoing direction (see, for instance, [23]).

The magnetic Sobolev space is endowed with the norm

$$\|u\|^2_{H^1_A(\mathcal{G})} := \int_{\mathcal{G}} \left(|Du|^2 + |u|^2\right) \mathrm{d}x. \tag{4.8}$$

In general, a function $u \in H^1_A(\mathcal{G})$ need not belong to $H^1(\mathcal{G})$, since continuity at the vertices is not implied. However, the modulus $|u|$ always belongs to $H^1(\mathcal{G})$. This follows from the diamagnetic inequality, which holds edge-wise on metric graphs.

Theorem 4.2.1 (Diamagnetic inequality on metric graphs)**.** *Let $A \in L^2_{\mathrm{loc}}(\mathcal{G})$ and let $u \in H^1_A(\mathcal{G})$. Then $|u| \in H^1(\mathcal{G})$ and the pointwise inequality*

$$\Big|(|u|)'(x)\Big| \le |Du(x)| \tag{4.9}$$

holds for almost every x on each edge of $\mathcal{G}$. Moreover, equality holds almost everywhere on an edge if and only if

$$Du = \frac{u}{|u|}\,(|u|)' \qquad \text{a.e. on that edge,} \tag{4.10}$$

where $\frac{u}{|u|}$ is defined wherever $u \neq 0$.

Proof. The result follows by applying the classical diamagnetic inequality on each edge of $\mathcal{G}$, viewed as an interval of $\mathbb{R}$, and summing over all edges. □

We conclude this section with the topological definitions required to describe the magnetic flux. We recall that a *cycle* is a finite sequence of connected edges where the initial and final vertices coincide, no edge is repeated, and which cannot be contracted to a point.

For a graph $\mathcal{G} = (\mathbb{V}_\mathcal{G}, \mathbb{E}_\mathcal{G})$, the *cyclomatic number* (or first Betti number) $\beta_\mathcal{G}$ is the minimum number of edges that must be removed from $\mathbb{E}_\mathcal{G}$ to transform $\mathcal{G}$ into a connected tree. For a connected graph, this is given by:

$$\beta_\mathcal{G} = |\mathbb{E}_\mathcal{G}| - |\mathbb{V}_\mathcal{G}| + 1\,.$$

An *independent cycle* is identified with a subset of edges $\gamma \subset \mathbb{E}_\mathcal{G}$ such that removing an edge from γ decreases the cyclomatic number $\beta_\mathcal{G}$ exactly by one. We denote by $\mathcal{C}(\mathcal{G})$ the set of the $\beta_\mathcal{G}$ independent cycles of the graph. This is the standard viewpoint when describing the effect of magnetic fluxes on metric graphs [23]. The length $|\gamma|$ of a cycle is defined as the sum of the metric lengths of its constituent edges:

$$|\gamma| = \sum_{e\in\gamma} |e|\,.$$

4.3 Effective Repulsion and Variational Reduction

In this section, we establish that the magnetic NLS equation is variationally equivalent to a non-magnetic problem augmented with a repulsive potential supported on the cycles of the graph. This reduction captures the Aharonov-Bohm effect purely through topological flux parameters.

Proposition 4.3.1. *Let $A \in C^1(\mathcal{G})$. Then the minimization problem*

$$\mathcal{E}_{A,\mathcal{G}}(\mu) := \inf_{u\in H^1_{A,\mu}(\mathcal{G})} E_A(u,\mathcal{G}), \tag{4.11}$$

is equivalent to the problem

$$\mathcal{I}_\mathcal{G}(\mu) := \inf_{v\in H^1_{\mu}(\mathcal{G})} I_A(v,\mathcal{G})\,, \tag{4.12}$$

where the effective functional I_A is defined as

$$I_A(v,\mathcal{G}) = \frac{1}{2}\int_\mathcal{G} |v'|^2 \mathrm{d}x + \int_{\mathcal{C}(\mathcal{G})} \Phi_\gamma(A)|v|^2 \mathrm{d}x - \frac{1}{p}\int_\mathcal{G} |v|^p \mathrm{d}x\,, \tag{4.13}$$

with $\mathcal{C}(\mathcal{G})$ denoting the set of independent cycles and $\Phi_\gamma(A)$ being a scalar repulsive potential determined by the magnetic flux (see (4.16)).

Proof. Consider a function $u \in H^1_A(\mathcal{G})$. It can be decomposed as $u(x) = v(x)e^{i\theta(x)}$ with $v = |u|$. A direct computation yields

$$\begin{aligned}\int_{\mathcal{G}} |Du|^2 &= \int_{\mathcal{G}} \left| v'e^{i\theta} + i\theta' v e^{i\theta} - iAve^{i\theta} \right|^2 \mathrm{d}x \\ &= \int_{\mathcal{G}} |v' - i(A-\theta')v|^2 \,\mathrm{d}x \\ &= \int_{\mathcal{G}} |v'|^2 \mathrm{d}x + \int_{\mathcal{G}} (A-\theta')^2 |v|^2 \mathrm{d}x \,.\end{aligned}$$

Since our goal is to characterize the solutions of $\mathcal{E}_{\mathcal{G}}(\mu)$, we reformulate the problem as

$$\inf_{v \in H^1_\mu(\mathcal{G}),\, \theta \in C^1(\mathcal{G})} E_A(u, \mathcal{G}) \,, \tag{4.14}$$

thereby decoupling the variables and focusing on the minimization with respect to θ. In particular, we seek the function that minimizes the quantity

$$\int_{\mathcal{G}} (A-\theta')^2 |v|^2 \mathrm{d}x \,,$$

while preserving the boundary conditions (4.7).

Let γ be a cycle parameterized by $[0, L]$. While the continuity conditions at the vertices require $u(0) = u(L)$, the phase function $\theta(x)$ must only satisfy the single-valued condition:

$$\theta(L) - \theta(0) = 2\pi m, \qquad m \in \mathbb{Z} \,.$$

Using the *Lagrange Multiplier Theorem* for fixed v, one can find the optimal θ' subject to the periodicity constraint

$$\int_0^L \theta'(x) \mathrm{d}x = 2\pi m.$$

For some $\lambda \in \mathbb{R}$, we have the variation:

$$\nabla (A-\theta')^2 - \lambda \nabla \left(\int_0^L \theta'(x) \mathrm{d}x - 2\pi m \right) = 0 \,.$$

Therefore we set $\theta'(x) = A(x) + \frac{1}{2}\lambda L$ for some constant λ to be determined. The single-valued constraint becomes:

$$\int_0^L \theta'(x)\,\mathrm{d}x = \int_0^L \left[A(x) + \frac{1}{2}\lambda L \right] \mathrm{d}x = 2\pi m \,.$$

This gives us:

$$\int_0^L A(x) \mathrm{d}x + \frac{1}{2}\lambda L^2 = 2\pi m \,,$$

which yields:

$$\lambda = \frac{4\pi m}{L^2} - \frac{2}{L^2}\int_0^L A(x)\mathrm{d}x\,.$$

Consequently, substituting λ back into the expression for $(A - \theta')$, we obtain:

$$A(x) - \theta'(x) = A(x) - \Big(A(x) + \frac{1}{2}\lambda L\Big) = \frac{1}{L}\int_0^L A(x)\mathrm{d}x - \frac{2\pi m}{L}\,,$$

and the magnetic energy contribution becomes:

$$\int_0^L |A-\theta'|^2|v|^2\mathrm{d}x = \int_0^L \left(\frac{1}{L}\int_0^L A(x)\mathrm{d}x - \frac{2\pi m}{L}\right)^2 |v|^2\mathrm{d}x\,.$$

Finally, we define m as the integer satisfying the minimization condition:

$$(\alpha_\gamma - m)^2 = \mathrm{dist}\,(\alpha_\gamma, \mathbb{Z})^2 := \min_{k\in\mathbb{Z}}(\alpha_\gamma - k)^2,$$

where we have set $\alpha_\gamma := \frac{1}{2\pi}\int_0^L A(x)\mathrm{d}x$. And thus we have:

$$\int_0^L |A-\theta'|^2\,|v|^2\mathrm{d}x = \int_0^L \frac{4\pi^2}{L^2}\mathrm{dist}\left(\frac{\int_0^L A\mathrm{d}x}{2\pi}, \mathbb{Z}\right)^2 |v|^2\mathrm{d}x\,. \tag{4.15}$$

Then, $(A(x) - \theta'(x))$ is constant on the loop, and depends only on the size of the loop and the average magnetic flux through the cycle.

We can now easily generalize this analysis for a general graph with more cycles. Let $\mathcal{C}(\mathcal{G})$ be the set of independent cycles of $\mathcal{G}$. Then for every $\gamma \in \mathcal{C}(\mathcal{G})$:

$$\Phi_\gamma(A) = \frac{4\pi^2}{|\gamma|^2}\mathrm{dist}\left(\frac{\int_\gamma A(x)\mathrm{d}x}{2\pi}, \mathbb{Z}\right)^2 \tag{4.16}$$

and the energy functional becomes:

$$I_A(v,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}}|v'|^2\mathrm{d}x + \int_{\mathcal{C}(\mathcal{G})}\Phi_\gamma(A)|v|^2\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}}|v|^p\mathrm{d}x\,. \tag{4.17}$$

Since in the definition (4.13) of I_A only $v = |u|$ appears, for Proposition 4.3.1 we have $v \in H^1_\mu(\mathcal{G})$ and the result follows. □

Remark 4.3.2. We highlight that if

$$\int_\gamma A(x)\mathrm{d}x = n\cdot 2\pi\,, \qquad n\in\mathbb{Z},$$

then there is no magnetic influence and the minimizing problem is the standard one.

The functional I_A can be viewed as an extension of the standard NLS energy, augmented with an effective magnetic potential term:

$$W(x) := \sum_{\gamma \in C(\mathcal{G})} \Phi_\gamma(A)\chi_\gamma(x)\,, \tag{4.18}$$

where χ_γ is the indicator function of the cycle γ. Consequently, any critical point of the functional must satisfy the corresponding Euler-Lagrange equation. By standard variational arguments (see Section 1.2), the minimizer u satisfies the differential equation:

$$-u'' - |u|^{p-2}u + W(x)u = \omega u\,, \tag{4.19}$$

on every edge, together with standard Kirchhoff conditions at the vertices. Specifically, on the non-magnetic edges $e \notin \mathcal{C}(\mathcal{G})$, the equation is the standard stationary NLS:

$$u'' + |u|^{p-2}u = -\omega u\,, \tag{4.20}$$

while on edges $e \in \gamma$ belonging to a cycle, the equation involves the effective potential:

$$u'' + |u|^{p-2}u = (\Phi_\gamma(A) - \omega)u\,. \tag{4.21}$$

4.4 Existence of ground states

Having reformulated the magnetic problem in terms of effective repulsive potentials, we address the question of ground state existence using concentration-compactness arguments. We seek solutions to the minimization problem:

$$\mathcal{I}_{\mathcal{G}}(\mu) := \inf_{u \in H^1_{A,\mu}(\mathcal{G})} I_A(u, \mathcal{G})\,. \tag{4.22}$$

We begin by stating the fundamental existence criterion.

Theorem 4.4.1. *Fix $2 < p < 6$ and let $\mathcal{G}$ be a metric graph with at least one infinite edge. If there exists a function $v \in H^1_\mu(\mathcal{G})$ such that*

$$I_A(v, \mathcal{G}) < E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R})\,, \tag{4.23}$$

then the functional $I_A(\cdot, \mathcal{G})$ admits a ground state at mass μ.

The proof of Theorem 4.4.1 relies on establishing properties of the energy functional and minimizing sequences. The strategy mirrors the arguments developed in Chapter 2 and Chapter 3.

First, we establish basic estimates.

Remark 4.4.2. Since the effective potential $\Phi_\gamma(A)$ is non-negative and locally constant (as shown in Section 4.3), it is always true that $I_A(v,\mathcal{G}) \geq E_{\mathrm{NLS}}(v,\mathcal{G})$. Moreover, by the definition of Φ_γ, for every γ

$$\Phi_\gamma \leq \min\left\{\frac{4\pi^2}{|\gamma|^2}, \|A\|^2_{L^\infty(\gamma)}\right\}. \tag{4.24}$$

Proposition 4.4.3. *Let $\mathcal{G}$ be a noncompact metric graph. For all $u \in H^1_\mu(\mathcal{G})$ satisfying the condition*

$$I_A(u,\mathcal{G}) \ \leq \ \frac{1}{2}\mathcal{I}_\mathcal{G}(\mu) \ < \ 0, \tag{4.25}$$

the following estimates hold for some constants $C_1, C_2 > 0$:

$$C_1\mu^{2\beta+1} \leq \|u'\|_2^2 \leq C_2\mu^{2\beta+1}\,; \tag{4.26}$$

$$C_1\mu^{2\beta+1} \leq \|u\|_p^p \leq C_2\mu^{2\beta+1}\,; \tag{4.27}$$

$$C_1\mu^{\beta+1} \leq \|u\|^2_{L^\infty(\mathcal{G})} \leq C_2\mu^{\beta+1}. \tag{4.28}$$

Proof. Let $u \in H^1_\mu(\mathcal{G})$ satisfy hypothesis (4.25). Thanks to Remark 4.4.2, it holds that

$$\frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p \leq \mathcal{I}_\mathcal{G}(\mu) - \int_{\mathcal{C}(\mathcal{G})}\Phi_\gamma|u|^2\mathrm{d}x \leq \frac{1}{2}\mathcal{I}_\mathcal{G}(\mu). \tag{4.29}$$

Using the Gagliardo-Nirenberg inequalities, the proof follows exactly as in Proposition 2.2.5 □

Next, we establish the concavity of the ground state energy function.

Proposition 4.4.4. *The function $\mathcal{I}_\mathcal{G}(\mu)$ defined in (4.22) is strictly concave and strictly subadditive for $\mu > 0$.*

Proof. Let U be the set of normalized profiles defined by:

$$U := \left\{u \in H^1(\mathcal{G}) : \|u\|_2^2 = 1,\ \mu^{p/2}\|u\|_p^p \geq C_1\mu^{2\beta+1}\right\}, \tag{4.30}$$

where C_1 is the constant from Proposition 4.4.3. Consider the family of functions $f_u(\mu)$ obtained by scaling $u \in U$:

$$f_u(\mu) := I_A(\sqrt{\mu}u,\mathcal{G}) = \frac{\mu}{2}\int_\mathcal{G}|u'|^2\mathrm{d}x - \frac{\mu^{\frac{p}{2}}}{p}\|u\|_p^p + \frac{\mu}{2}\int_{\mathcal{C}(\mathcal{G})}\Phi_\gamma|u|^2\mathrm{d}x. \tag{4.31}$$

We observe that $\mathcal{I}_\mathcal{G}(\mu) = \inf_{u\in U} f_u(\mu)$. Computing the second derivative with respect to μ:

$$f_u''(\mu) = -\frac{p-2}{4}\cdot\mu^{\frac{p}{2}-2}\cdot\|u\|_p^p < 0, \tag{4.32}$$

since $p > 2$ and $\|u\|_p > 0$. Thus, $\mathcal{I}_\mathcal{G}$ is the infimum of a family of strictly concave functions, which implies it is strictly concave. Strict subadditivity follows from strict concavity and the fact that $\mathcal{I}_\mathcal{G}(0) = 0$. □

Proposition 4.4.5. *Any minimizing sequence* $\{u_n\} \subset H^1_\mu(\mathcal{G})$ *is bounded. If* $\{u_n\}$ *is a minimizing sequence for* $I_A(\cdot, \mathcal{G})$ *and* $u_n \rightharpoonup u$ *weakly in* $H^1(\mathcal{G})$*, then one of the following alternatives holds:*

(i) $u_n \to 0$ *in* $L^\infty_{loc}(\mathcal{G})$*, which implies* $u \equiv 0$*;*

(ii) $u_n \to u$ *strongly in* $H^1(\mathcal{G}) \cap L^p(\mathcal{G})$*, and* u *is a minimizer.*

Proof. The boundedness of $\{u_n\}$ follows from Proposition 4.4.3. From the concentration compactness principle on graphs, three behaviors are possible: vanishing, dichotomy, and compactness. Dichotomy is ruled out by the strict subadditivity of the energy level $\mathcal{I}_{\mathcal{G}}(\mu)$. Thus, only vanishing (i) or compactness (ii) can occur. (see Proposition for more details). □

Proof of Theorem 4.4.1. Let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence. By Proposition 4.4.5, we must rule out the vanishing case (i). If vanishing occurs, then $u_n \to 0$ uniformly on compact sets. Since the effective magnetic potential Φ_γ is supported on a compact set, the term $\int \Phi_\gamma |u_n|^2 dx$ tends to 0. Consequently, the energy level of a vanishing sequence cannot be strictly below the threshold of the problem on the infinite line, $E_{\text{NLS}}(\phi_\mu, \mathbb{R})$. However, the hypothesis given by the inequality (4.23) assumes the existence of a state v with energy strictly below this threshold. This contradicts the vanishing scenario. Therefore, case (ii) must hold. □

This result extends the existence criterion to the case of bounded repulsive potentials defined on compact sets. In the following section, we apply this theorem to the tadpole graph.

4.5 The tadpole graph

The tadpole graph consists of a graph made by one vertex and two edges. One edge is a loop that connects the vertex to itself, the other is a halfline, i.e. an edge with infinite length (see Figure 4.1). It is known that with Kirchhoff conditions, it satisfies the existence criterion of a ground state proved in [8]. We recall that, in [56, 57], the authors classify all soliton solutions of the standard nonlinear Schrödinger equation on the tadpole graph, described via elliptic functions. In particular, if $p = 4$ the ground state is given by the *dnoidal* function on the ring and a piece of a soliton on the halfline.

Using the fact that the tadpole admits a ground state for the functional E_{NLS}, we obtain the following existence results.

Theorem 4.5.1 (Existence)**.** *Let* $\mathcal{T}$ *be a tadpole graph and fix* $\mu > 0$*. If* Φ_γ *is sufficiently small, then there exists a ground state for* $I_A(\cdot, \mathcal{T})$*.*

Figure 4.1: The tadpole graph.

Proof. Fix $\mu > 0$ and let u be the ground state for $E_{\text{NLS}}(u, \mathcal{T})$ with energy $\mathcal{E}_{\mathcal{T}}(\mu)$. Then, the energy of u can be written as

$$I_A(u, \mathcal{T}) = \mathcal{E}_{\mathcal{T}}(\mu) + \Phi_\gamma \int_\gamma |u|^2 .$$

Moreover, by the definition of ground state we have $\mathcal{E}_{\mathbb{R}}(\mu) - \mathcal{E}_{\mathcal{G}}(\mu) := R(\mu, \mathcal{G}) > 0$. Therefore, if Φ_γ is sufficiently small, then

$$\Phi_\gamma \int_\gamma |u|^2 \leq R(\mu, \mathcal{T}), \tag{4.33}$$

so the condition of Theorem (4.4.1) is satisfied and a ground state exists. □

Remark 4.5.2. In the previous proof we introduced the quantity

$$R(\mu, \mathcal{G}) := \mathcal{E}_{\mathbb{R}}(\mu) - \mathcal{E}_{\mathcal{G}}(\mu).$$

An explicit expression of $R(\mu, \mathcal{T})$ is available only if $p = 4$. However, since the lowest ground state on noncompacts is the one reached in the halfline $\mathcal{G} = \mathbb{R}^+$ [6], one always have the general bounds

$$0 \leq R(\mu, \mathcal{G}) \leq (2^{2\beta} - 1)\theta_p \mu^{2\beta+1}. \tag{4.34}$$

The small mass behaviour is immediate

$$R(\mu, \mathcal{G}) \to 0 \quad \text{if} \quad \mu \to 0.$$

We now show that $R(\mu, \mathcal{G}) \to 0$ also in the opposite regime $\mu \to \infty$. It is known [8] that every solution of the equation

$$u'' - |u|^{p-2} u = \lambda u \tag{4.35}$$

satisfy the scaling law

$$u_\tau(x) = \sqrt{\tau}\, u(\tau x). \tag{4.36}$$

Hence, as $\mu \to \infty$ the corresponding solutions become increasingly localized with characteristic length $\ell \sim \tau^{-1}$. Moreover, there exists $\mathcal{R} > \ell$ such that $u(x) \to 0$ for $|x| > \mathcal{R}$, In the tail region the nonlinearity is negligible and the solution behaves like solution of the linear equation $u'' = \lambda u$. In other words, the tail decays exponentially

$$u(x) \leq Ce^{-c|x|} \quad \text{for} \quad x > \mathcal{R}. \tag{4.37}$$

By scaling,

$$u_\tau(x) \leq C\sqrt{\tau}e^{-c\tau|x|} \quad \text{for} \quad x > \mathcal{R}. \tag{4.38}$$

The energy of the tail satisfies

$$\epsilon = \int_{|x|>\mathcal{R}} \frac{1}{2}|u'|^2 - \frac{1}{p}|u|^p \mathrm{d}x \leq C\tau^2 e^{-c\tau\mathcal{R}} \tag{4.39}$$

Therefore, in the large mass regime the energy of a ground state is arbitrary closed to the energy on the line. We conclude that

$$R(\mu, \mathcal{G}) \to 0 \quad \text{if} \quad \mu \to \infty.$$

Since $R(\mu, \mathcal{G}) \to 0$ both as $\mu \to 0$ and as $\mu \to \infty$, one should not expect ground states to appear in either extreme regime since the potential acts repulsively.

We finally establish a nonexistence result, showing that for sufficiently repulsive magnetic potentials no ground state can occur.

Theorem 4.5.3 (Nonexistence)**.** *Let $\mathcal{T}$ be a tadpole graph with cycle length L, and let $\mu > 0$ be the prescribed mass. If $\Phi_\gamma(A)$ exceeds a critical value depending on μ, then $I_A(\cdot, \mathcal{T})$ does not admit a ground state.*

Proof. Assume by contradiction that I_A admits a ground state u. Then

$$\mathcal{I}_\mathcal{T}(\mu) = \min_{\|v\|_2^2=\mu} \left\{ E_{\text{NLS}}(v, \mathcal{T}) + \int_\gamma \Phi_\gamma v^2 \,\mathrm{d}x \right\} \;\geq\; \mathcal{E}_\mathcal{T}(\mu) + \min_{\|v\|_2^2=\mu} \int_\gamma \Phi_\gamma v^2 \,\mathrm{d}x. \tag{4.40}$$

A ground state must place a strictly positive amount of mass $m > 0$ on the cycle. Assume by contradiction that there exists a ground state v of mass μ such that $m = 0$. Then $v = 0$ on the cycle and, by continuity at the vertex, $v(0) = 0$, so all the mass lies on the half-line. Extend v to the whole line by

$$w(x) := \begin{cases} v(x), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then $w \in H^1(\mathbb{R})$, $\|w\|_{L^2(\mathbb{R})}^2 = \mu$, and $E_{\text{NLS}}(w, \mathbb{R}) = E_{\text{NLS}}(v, \mathcal{T})$. Since v is a ground state on the tadpole $E_{\text{NLS}}(w, \mathbb{R}) \leq \mathcal{E}_\mathbb{R}(\mu)$. On the other hand, by definition of the ground-state energy on the line, $E_{\text{NLS}}(w, \mathbb{R}) \geq \mathcal{E}_\mathbb{R}(\mu)$. Therefore

$$E_{\text{NLS}}(w, \mathbb{R}) = \mathcal{E}_\mathbb{R}(\mu),$$

so w is a minimizer on $\mathbb{R}$ and must coincide with the soliton. However, the soliton on $\mathbb{R}$ is strictly positive and never vanishes, this is a contradiction, hence no ground state can have zero mass on the cycle.

Let $m > 0$ denote the minimal mass that any ground state must carry on the cycle. Assume that $\Phi_\gamma > 0$, then $\int_\gamma \Phi_\gamma |v|^2 \geq \Phi_\gamma m$. Combining this with (4.40) we obtain

$$\Phi_\gamma(A)\, m \;<\; R(\mu, \mathcal{T}), \tag{4.41}$$

If $\Phi_\gamma(A)$ is larger than the critical value for which (4.41) fails, then the inequality cannot hold. This contradicts the existence of a ground state, and therefore no ground state can exist in this regime. □

4.5.1 The hyperbolic secant competitor

Having established the general existence theory, we now turn to the concrete case of the tadpole graph with $p = 4$. The key step is to construct an admissible test function that beats the energy of the soliton on the real line. Drawing inspiration from the standard case, we build our competitor from hyperbolic secant profiles. The computation yields an explicit inequality involving the magnetic potential that assure the existence of ground states, in particular this is true for an intermediate regime mass (see Figure 4.3).

Lemma 4.5.4. *Let $\mathcal{T}$ be the tadpole graph with loop length $2L > 0$, and let $\mu > 0$ be the prescribed mass. Consider the nonlinear Schrödinger energy functional $I_A(\cdot, \mathcal{T})$ with $p = 4$. Let $m = m(\mu) > 0$ be the unique solution of the mass relation*

$$\mu = 2m\Big(1 + \tanh(mL)\Big). \tag{4.42}$$

If the magnetic potential satisfies

$$\Phi_\gamma(A) \;\leq\; m(\mu)^2\, \frac{1 + 3\tanh\big(m(\mu)L\big)}{\sinh\big(2m(\mu)L\big)}, \tag{4.43}$$

then the energy functional on $\mathcal{T}$ admits a ground state of mass μ.

Proof. We use the competitor u_μ constructed from hyperbolic secant profiles on the loop and on the half-line (see Figure 4.2):

$$u_\mu(x) = \begin{cases} \sqrt{2}\, m\, \operatorname{sech}\big(m(y+L)\big), & y \in [0, \infty), \\ \sqrt{2}\, m\, \operatorname{sech}(mx), & x \in [-L, L]. \end{cases}$$

A direct computation using $\int \operatorname{sech}^2(\xi)\, d\xi = \tanh(\xi) + C$ gives the mass identity

$$\mu = 2m^2 \int_{-L}^{L} \operatorname{sech}^2(mx)\, \mathrm{d}x + 2m^2 \int_0^\infty \operatorname{sech}^2\big(m(y+L)\big)\, dy = 2m\Big(1 + \tanh(mL)\Big),$$

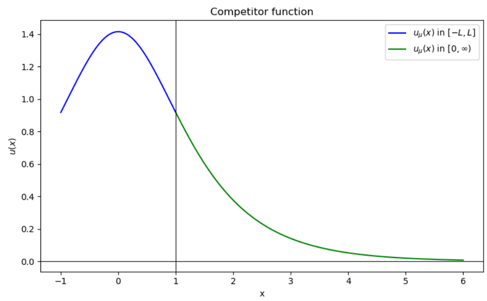

Figure 4.2: Competitor function $u_\mu(x)$ with $L = 1$ and $\mu = 1$

which determines $m = m(\mu)$ uniquely .
For the energy, we use the identity

$$\int \Big[\operatorname{sech}^2(\xi)\tanh^2(\xi) - \operatorname{sech}^4(\xi)\Big]\, d\xi = \frac{2\tanh^3(\xi) - 1}{3} - \tanh(\xi) + C.$$

Evaluating on the loop and the half-line yields the nonlinear contribution

$$E_{\mathrm{NLS}}(u_\mu, \mathcal{T}) = \frac{1}{3} m^3 \Big(2T^3 - 3T - 1\Big), \qquad T = \tanh(mL).$$

The magnetic term contributes $\Phi_\gamma 2mT$, so the total energy is

$$I_A(u_\mu) = \frac{1}{3} m^3 \Big(2T^3 - 3T - 1\Big) + 2\Phi_\gamma m T.$$

The soliton on $\mathbb{R}$ with the same mass μ has energy

$$\mathcal{E}_{\mathbb{R}}(\mu) = -\frac{1}{12} m^3 (1 + T)^3,$$

where the same mass relation links m and μ. Requiring $I_A(u_\mu, \mathcal{T}) \le \mathcal{E}_{\mathbb{R}}(\mu)$ gives

$$4\Phi_\gamma T \le m^2 \left(-3T^3 - T^2 + 3T + 1\right),$$

which simplifies to

$$\Phi_\gamma(A) \le m^2 \frac{1 + 3\tanh(mL)}{\sinh(2mL)}.$$

Under this condition, the competitor u_μ has strictly lower energy than any soliton on $\mathbb{R}$, and for Theorem 4.4.1 a ground state exists. □

The explicit relation (4.43) obtained in the Lemma is illustrated in Figure 4.3, where the intermediate mass regime for which the inequality is satisfied is highlighted.

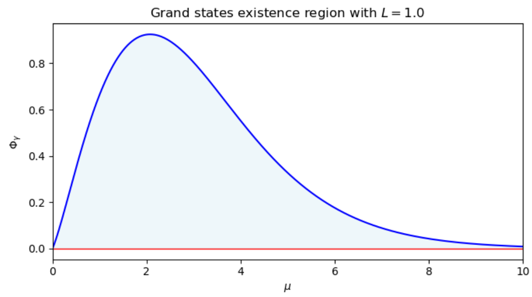

Figure 4.3: Parameter region in the (μ, Φ_γ)-plane where the condition of the inequality (4.43) holds with $L = 1$.

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