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THE MOMENT MAP ON THE SPACE OF SYMPLECTIC 3D MONGE-AMPÈRE EQUATIONS

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Abstract. For any 2^{nd} order scalar PDE $\mathcal{E}$ in one unknown function, we construct, by means of the characteristics of $\mathcal{E}$, a contact sub–bundle of the underlying contact manifold J^1, consisting of conic varieties, called the *contact cone structure* associated with $\mathcal{E}$. We then focus on symplectic Monge–Ampère equations in 3 independent variables, that are naturally parametrized, over $\mathbb{C}$, by the projectivization of the 14–dimensional irreducible representation of the simple Lie group $\mathsf{Sp}(6,\mathbb{C})$. The associated moment map allows to define a rational map ϖ from the space of symplectic 3D Monge-Ampère equations to the projectivization of the space of quadratic forms on a 6–dimensional symplectic vector space. We study the relationship between the variety $\varpi(\mathcal{E})=0$, herewith called the *co-characteristic variety* of $\mathcal{E}$, and the contact cone structure of a 3D Monge-Ampère equation $\mathcal{E}$, by obtaining a complete list of mutually non–equivalent quadratic forms on a 6–dimensional symplectic space.

1. Introduction

1.1. **Starting point: 2^{nd} order PDEs and their symbol.** A (scalar) 2^{nd} order PDE in one unknown function $u=u(x^1,\ldots,x^n)$ and n independent variables $\boldsymbol{x}=(x^1,\ldots,x^n)$, henceforth called n–dimensional PDEs (nD

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PDEs), can be written as

$$\mathcal{E} := \left\{ F(\boldsymbol{x}, u, \nabla u, \nabla^2 u) = F(x^i, u, u_i, u_{ij}) = 0 \right\}, \tag{1.1}$$

where F is a real function on a domain of

$$\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \tag{1.2}$$

and

$$u_i := \frac{\partial u}{\partial x^i}, \quad u_{ij} := \frac{\partial^2 u}{\partial x^i x^j}. \tag{1.3}$$

Equation ((1.1)) is *elliptic* at a point

$$\theta_2 = \big(\boldsymbol{x}_0, u(\boldsymbol{x}_0), \nabla u(\boldsymbol{x}_0), \nabla^2 u(\boldsymbol{x}_0)\big) = (x^i_0, u_0, u^0_i, u^0_{ij})$$

of the space (1.2) lying in the subset $\mathcal{E}$ given by (1.1), if the matrix

$$\frac{1}{2-\delta_{ij}} \frac{\partial F}{\partial u_{ij}}\Big|_{\theta_2} \tag{1.4}$$

is definite (either positive or negative), i.e., if the quadratic form

$$\sum_{i \leq j} \frac{\partial F}{\partial u_{ij}}\Big|_{\theta_2} \eta_i \eta_j, \tag{1.5}$$

that we call also the *symbol* of equation (1.1) at the point $\theta_2 \in \mathcal{E}$, is either greater or less than zero for all vectors $(\eta_1, \ldots, \eta_n) \in \mathbb{R}^n \smallsetminus \{0\}$.

A point $\theta_2 \in \mathcal{E}$ is called *regular* if the matrix (1.4) is not zero. In Sections 1–5, points of $\mathcal{E}$ are always assumed to be regular, unless otherwise specified.

1.2. **Context.** It is well known that the notion of symbol of a 2$^{\text{nd}}$ order PDE $\mathcal{E}$, more precisely its rank, is closely related to the notion of characteristic of $\mathcal{E}$. There are PDEs that are completely characterized by the behavior of their characteristics: for instance, 2D hyperbolic (resp., parabolic) Monge–Ampère equations are those 2$^{\text{nd}}$ order PDEs whose characteristic lines are arranged in a couple of different (resp. coincident) 2-dimensional linear subspaces (see, for instance, [1, 3] and reference therein). One can ask if a similar phenomenon occurs also in the multidimensional situation. The oldest paper regarding a multidimensional generalization of Monge–Ampère equations, to the authors' best knowledge, dates back to a 1899 work by Goursat [20], where the Monge–Ampère equations with an arbitrary number of independent variables were introduced as those PDEs whose characteristic cones "degenerate" into linear subspaces. This phenomenon corresponds to the degeneration of the the symbol of the Monge–Ampère equations. The equations obtained by Goursat are indeed of type

$$\det \|u_{ij} - b_{ij}\| = 0\,, \quad b_{ij} = b_{ij}(x^1, \dots, x^n, u, u_1, \dots, u_n)\,,$$

and a straightforward computation shows that the rank of their symbols is less or equal to 2: as such, these PDEs are a proper subclass of a larger class of Monge–Ampère equations that were introduced later by Boillat in [6], as the only PDEs whose characteristic velocities behave in the "completely exceptional" way in the sense of P. Lax. [7, 8, 21, 28, 29, 32].

V. Lychagin, who proposed studying 2^{nd} order PDEs by means of the underlying contact geometry of the $(2n+1)$–dimensional first–order jet space J^1 (i.e., the space locally parametrized by (x^i, u, u_i), defined in [30] n–dimensional Monge–Ampère equations in terms of certain differential n–forms on the $2n$–dimensional contact distribution $\mathcal{C}$ of J^1, which he called *effective n–forms* and whose set we will be denoting by $\Lambda_0^n(\mathcal{C}^*)$; see also [27]. Lychagin's approach leads directly to the general expression of an n–dimensional Monge–Ampère equation:

$$M_n + M_{n-1} + \cdots + M_0 = 0\,,$$

where M_k is a linear combination of all $k \times k$ minors of the Hessian matrix $\|u_{ij}\|$, with coefficients in $C^\infty(J^1)$. Moreover, the equivalence problem for Monge–Ampère equations, thanks to the aforementioned correspondence, can be recast in terms of effective forms, which is especially advantageous in the case of *symplectic* Monge–Ampère equations, i.e., Monge–Ampère equations of type[1]

$$F(u_{ij}) = 0\,, \tag{1.6}$$

with F not depending neither on x^i, u, nor u_i. For instance, in the case $n = 3$, the authors of [27] associated with any effective 3–form $\Phi \in \Lambda_0^3(\mathcal{C}^*)$ the following simplistically invariant quadratic form on $\mathcal{C}$:

$$\operatorname{trace}(\omega^{-1} \circ \Phi)^2\,, \tag{1.7}$$

where ω is a prescribed representative of the conformal symplectic structure of $\mathcal{C}$, and employed it to obtain the normal forms of symplectic 3D Monge–Ampère equations with non–degenerate symbol. A peculiar feature of the 3–dimensional case is that the 20–dimensional vector space $\Lambda^3(\mathcal{C}^*_{\theta_1})$, where $\theta_1 \in J^1$, is equipped with a natural symplectic structure, which makes the natural $\mathsf{Sp}(\mathcal{C}_{\theta_1})$–action a Hamiltonian one; the corresponding moment map, first studied by N. Hitchin in [23], descends to a quadratic map between

[1] Such equation are known also as "Hirota type", see, e.g., [16, 36, 13].

the 14–dimensional vector space $\Lambda^3_0(\mathcal{C}^*_{\theta_1})$ and the Lie algebra $\mathfrak{sp}(\mathcal{C}_{\theta_1})$, which in turns identifies naturally with $S^2(\mathcal{C}^*_{\theta_1})$, the space of quadratic forms on $\mathcal{C}_{\theta_1}$. It has been proved in [4] that Hitchin's restricted moment map and the quadratic form (1.7) lead to the same quadric in $\mathcal{C}_{\theta_1}$, see also [27, Prop. 8.1.5]. Yet another possibility of seeing symplectic n–dimensional Monge–Ampère equations is as *hyperplane sections* of the Lagrangian Grassmannian $\mathrm{LGr}(n, 2n)$: such is the perspective adopted, among others, by E. Ferapontov and his collaborators, who were mainly interested in the hydrodynamic integrability property of PDEs of the form (1.6), see [18, 19]. One of the results of [18] is the existence of a "master equation" in the class of symplectic hydrodynamically integrable 2^{nd} order PDEs in 3 independent variables: in terms of group actions, this means that the 21–dimensional group $\mathsf{Sp}(6, \mathbb{R})$, that acts naturally on the Lagrangian Grassmannian $\mathrm{LGr}(3, 6)$, has an open orbit in the space that parametrizes such PDEs. Another result is that the intersection of the latter space with that parametrizing (symplectic 3D) Monge–Ampère equations turns out to be the class of $\mathsf{Sp}(6, \mathbb{R})$–linearizable Monge–Ampère equations.

1.3. **Structure of the paper and description of the main results.** In Section 2, after refreshing the basics concerning the characteristics and the symbol of a scalar 2^{nd} order PDE in n independent variables, we introduce the k–order jet spaces J^k of (smooth) functions in n independent variables and then we interpret 2^{nd} order PDEs as hyper surfaces of J^2; the fibers of the latter are open dense subsets of the Lagrangian Grassmannian $\mathrm{LGr}(n, 2n)$.

Next, we define a general (symplectic) Monge–Ampère equation as a hyperplane section of a Lagrangian Grassmannian and, in the particular case $n = 3$, we define the *cocharacteristic variety* of a (3D) Monge–Ampère equation as the zero locus of the aforementioned map (1.7). We also introduce the notion of a *contact cone structure on* J^1, that is an assignment $\theta_1 \in J^1 \to \mathcal{V}_{\theta_1} \subset \mathcal{C}_{\theta_1}$ of affine conic varieties, where $\mathcal{C}$ is the contact distribution on J^1; *cone structures* have been long known, both in the real–differentiable and in the complex–analytic contexts, and have recently risen to a certain attention as they cast a bridge between the two categories, e.g., in the works of J.–M. Hwang [25]: a *contact* cone structure can be then regarded as a cone structure that is compatible with a preexisting contact structure.

In Section 3, we show that the notion of symbol, of a characteristic line, and of a characteristic hyperplane, when properly framed in the theory of

2^{nd} order PDEs based on the contact manifold J^1, can be naturally combined to construct a contact cone structure on J^1, which we call the *contact cone structure associated with the considered PDE*. This supplies a common footing to both Goursat's idea of defining a Monge–Ampère equation via a linear sub–distribution of the contact distribution, and the KLR invariant (1.7).

In Section 4, we narrow our attention to the class of (symplectic) 3D Monge–Ampère equations and we compute both the contact cone structure and the cocharacteristic variety of four particular equations that represents almost all possible $\mathsf{Sp}(6,\mathbb{R})$–equivalence classes of such PDEs—or all of them, without "almost", once we will have passed to the field of complex numbers. These computations will be used to prove that, for a Monge–Ampère equation with non–degenerate symbol, the notions of contact cone structure and the notion of cocharacteristic variety are the same (in particular, they are quadric hyper-surfaces of the contact plane of J^1), whereas for Monge–Ampère equations with degenerate symbol, the cocharacteristic variety and the contact cone structure are quite different. Such results are contained in Theorem 4.1, which is reformulated in Section 9 over the field of complex numbers: Corollary 9.1 represents indeed a coarse proof of Theorem 4.1.

In Section 5, we give an answer to the natural question whether it is possible to revert the above procedure, i.e., to construct a 3D 2^{nd} order PDE starting from a contact cone structure in dimension seven and to what extent the correspondence between a PDE and its contact cone structure is one–to–one. This leads to the study of the integral manifolds of certain vector distributions defined on the fiber of the 2^{nd} order jet space J^2. As an example of computations, we consider the contact cone structures associated with the four particular Monge–Ampère equations studied in Section 4 above.

The results of Sections 4 (in particular Theorem 4.1) and Section 5 show that the contact cone structures of 3D Monge–Ampère equations form a narrow sub–class of the class of all quadrics in the contact distribution $\mathcal{C}$: even a dimensional inspection shows that the space of 3D symplectic Monge–Ampère equations is 13–dimensional, whereas the space of all quadrics in $\mathcal{C}_{\theta_1}$ is 20–dimensional. We also stress that, to obtain a PDE from a contact cone structure, as shown in Section 5, one generally has to pass through an integration procedure: such a procedure would be very difficult to carry out successfully without a complete list of normal forms of quadratic forms on $\mathcal{C}$ with respect to the action of the symplectic group $\mathsf{Sp}(\mathcal{C}_{\theta_1}) = \mathsf{Sp}(6,\mathbb{R})$. This motivates looking for the normal forms of the quadratic forms on a

6–dimensional symplectic space, up to symplectic equivalence. This classification problem is classical, since, in view of the identification of $\mathfrak{sp}(\mathcal{C}_{\theta_1})$ with $S^2(\mathcal{C}^*_{\theta_1})$, it coincides with the classification problem of adjoint orbits in a semi-simple Lie algebra, about which there is a lot of excellent literature (besides the classical book [12] we mention [10, 14, 24] and related works, such as [15, 33]), even though an explicit list of normal forms seems yet to be missing.

After fixing some notation and introducing some necessary tools in Section 6, in Section 7, we indeed give, over the field of complex numbers, a complete list of mutually non–equivalent (up to symplectic transformations) quadratic forms on a 6–dimensional symplectic space.

In Section 8 the Hitchin's moment map is reviewed and its equivalence with the simplistically invariant form (1.7) is proved. Next equivalence, that is, the one with the cocharacteristic variety, is proved in Section 9, together with a review of the four geometries of the hyperplane sections of the Lagrangian Grassmanian $\mathrm{LGr}(3,6)$, which is, in part, already contained in [26, Proposition 2.5.1].

Notation and conventions. The Einstein convention for repeated indices will be used throughout the text, unless otherwise specified. The linear span of vectors $v_1, \ldots, v_k$ is denoted by $\langle v_1, \ldots, v_k\rangle$. The cofactor matrix of a matrix A is denoted by $A^\sharp$. The linear dual of a linear space (real or complex) V is denoted by V^*, whereas $X^\vee$ stands for the *projective dual* of the projective variety X. Symbol X_{sm} stands for the subset of smooth points of a variety X. By a vector distribution $\mathcal{X}$ on a manifold M, we mean a smooth assignment $p \in M \to \mathcal{X}_p \subseteq T_pM$ (not necessarily of constant rank). We say that $\mathcal{X}$ is *integrable* if it is such in the Frobenius sense, i.e., if $[Y, Z] \in \mathcal{X}$ for any vector fields $Y, Z \in \mathcal{X}$, where $[Y, Z]$ is the Lie bracket of Y and Z.

2. Preliminaries

2.1. **Cauchy data and characteristics of $2^{\mathbf{nd}}$ order PDEs.** In the present section, as well as in the Sections 3, 4, and 5, we mainly deal with PDEs (1.1) of *non–elliptic type*, i.e., equations that are non–elliptic in the subset $\mathcal{E}$ of (1.2); indeed, non–elliptic PDEs (and, in particular, the hyperbolic ones) are abundant in *real* characteristics: this is a choice of pure convenience, just to visually explain, via tangible examples, how to use the characteristics to construct our main object, that is the quadric cone structure associated with a PDE. Starting from Section 6, we switch to the field of *complex* numbers and the very notion of ellipticity becomes meaningless.

Let us consider the space consisting of the first three factors appearing in (1.2), that is, the space with coordinates (x^i, u, u_i): a *Cauchy datum* is a particular $(n-1)$–dimensional sub-manifold of such a space; one way to give it explicitly is via a parametrization:

$$\Phi(\mathbf{t}) = \big(x^1(\mathbf{t}), \dots, x^n(\mathbf{t}), u(\mathbf{t}), u_1(\mathbf{t}), \dots, u_n(\mathbf{t})\big)\,, \tag{2.1}$$

$\mathbf{t} = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$, and this is the point of view we will adopt in this paper. By a "particular sub-manifold", we meant that the functions $u_i(\mathbf{t})$ cannot be arbitrarily chosen: one should take into account that each u_i must be the derivative of u with respect to x^i, cf. (1.3); this heuristic requirement will be made formal in Section 3: as we shall see, the general condition for (2.1) to be a Cauchy datum can be given geometrically in terms of sub-manifolds of the (natural) contact structure the $(2n+1)$–dimensional (x^i, u, u_i)–space is equipped with.

Given a Cauchy datum (2.1), we can formulate a *Cauchy problem*: this entails finding a solution $u = f(x^1, \dots, x^n)$ to (1.1) under the additional requirement that it satisfies the conditions

$$f\big(x^1(\mathbf{t}), \dots, x^n(\mathbf{t})\big) = u(\mathbf{t})\,, \quad \frac{\partial f}{\partial x^i}\big(x^1(\mathbf{t}), \dots, x^n(\mathbf{t})\big) = u_i(\mathbf{t})\,, \tag{2.2}$$

$\forall \mathbf{t} \in \mathbb{R}^{n-1}$. If the Cauchy datum (2.1) is *non–characteristic* for the PDE (1.1), then all the derivatives of arbitrary order of $u = f(x^1, \dots, x^n)$ are determined by conditions (2.2) and (1.1). This means that the full Taylor expansion of f is well determined, i.e., there exists a unique *formal* solution; otherwise, the Cauchy datum (2.1) is called *characteristic*. Of course the above definitions and reasonings can be localized in a neighborhood of a considered point. The classical literature about geometry of PDEs and their characteristics comprises, among others, [5, 35]; see also the recent reviews [17, 39].

Example 2.1. Consider the wave equation in one spacial dimension:

$$u_{12} = 0\,. \tag{2.3}$$

The Cauchy datum

$$u(x^1, 0) = x^1\,,\ u_1(x^1, 0) = 1\,,\ u_2(x^1, 0) = 0\,, \quad x^1 \in \mathbb{R}\,, \tag{2.4}$$

can then be seen as a (parametric) curve $\Phi(x^1)$ in the (x^1, x^2, u, u_1, u_2)–space:

$$\Phi(x^1) = \big(x^1, 0, x^1, 1, 0\big)\,. \tag{2.5}$$

It is characteristic for (2.3), since we cannot determine all the derivatives of arbitrary order of a solution along the curve (2.4); indeed, the function

$f(x^1,x^2)=x^1+k(x^2)^2$ is a solution to (2.3), whose first jet (see below) along the curve $x^1\mapsto(x^1,0)$ coincides with (2.5) for each k. On the contrary, the Cauchy datum

$$u(x^2,x^2)=x^2+(x^2)^2,\quad u_1(x^2,x^2)=1,\quad u_2(x^2,x^2)=2x^2,$$

whose corresponding parametric curve is

$$\Psi(x^2)=\left(x^2,x^2,(x^2)^2+x^2,1,2x^2\right), \tag{2.6}$$

is not characteristic for (2.3): indeed, the function $f(x^1,x^2)=x^1+(x^2)^2$ is the only solution to (2.3), whose first jet along the curve $x^2\mapsto(x^2,x^2)$ coincides with (2.6).

2.2. **Spaces of k–jets of functions in n variables.** Let $f:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ be a smooth function on an open domain $\Omega\subseteq\mathbb{R}^n$. The *$k$–jet* of $f=f(\boldsymbol{x})=f(x^1,\dots,x^n)$ at a point $\boldsymbol{x}_0=(x_0^1,\dots,x_0^n)\in\Omega$ is defined as the Taylor expansion of f at $\boldsymbol{x}_0$ up to order k:

$$j^k_{\boldsymbol{x}_0}f:=\sum_{h=0}^{k}\sum_{i_1\dots i_h}\frac{1}{h!}\frac{\partial^h f}{\partial x^{i_1}\cdots\partial x^{i_h}}(\boldsymbol{x}_0)\,(x^{i_1}-x_0^{i_1})\cdots(x^{i_h}-x_0^{i_h})\,.$$

The totality of such polynomials is denoted by

$$J^k:=\{j^k_{\boldsymbol{x}_0}f\mid f:\Omega\to\mathbb{R},\,\boldsymbol{x}_0\in\Omega\}$$

and it is called the *space of k–jets of functions* on $\mathbb{R}^n$. Note that $J^0=\Omega\times\mathbb{R}$, whereas (1.2) is nothing but J^2 with $\Omega=\mathbb{R}^n$. Since $j^k_{\boldsymbol{x}_0}f$ is unambiguously defined by the coefficients of the above polynomial, we can unambiguously write

$$\begin{aligned}j^k_{\boldsymbol{x}_0}f&=\Big(\boldsymbol{x}_0\,,\,f(\boldsymbol{x}_0)\,,\,\nabla f(\boldsymbol{x}_0)\,,\dots,\,\nabla^k f(\boldsymbol{x}_0)\Big)\\&=\Big(\boldsymbol{x}_0\,,\,f(\boldsymbol{x}_0)\,,\,\frac{\partial f}{\partial x^i}(\boldsymbol{x}_0)\,,\dots,\,\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}(\boldsymbol{x}_0)\Big)\,,\end{aligned}$$

where $1\le i_1\le i_2\le\cdots\le i_k\le n$, i.e., one can regard $j^k_{\boldsymbol{x}_0}f$ as the equivalence class $[f]^k_{\boldsymbol{x}_0}$ of functions having the same derivatives of f at $\boldsymbol{x}_0$, up to order k. The space J^k admits a coordinate system

$$(x^i,u,u_i,u_{ij},\dots,u_{i_1\cdots i_k})\,,\quad 1\le i_1\le i_2\le\cdots\le i_k\le n\,, \tag{2.7}$$

which can be thought of as an extension of a coordinate system $(x^1,\dots,x^n,u)$ on $\Omega\times\mathbb{R}$ in the following sense: each coordinate function[2] $u_{i_1\cdots i_h}$ on J^k, with

[2]The $u_{i_1\cdots i_k}$'s are symmetric in the lower indices.

$h \leq k$, is unambiguously defined by

$$u_{i_1\cdots i_h}(j^k_{\boldsymbol{x}_0} f) = \frac{\partial^h f}{\partial x^{i_1} \cdots \partial x^{i_h}}(\boldsymbol{x}_0)\,.$$

To keep the notation light, from now on, a particular point $j^k_{\boldsymbol{x}_0} f \in J^k$ will be denoted by the symbol

$$\theta_k = (x^i_0, u_0, u^0_i, u^0_{ij}, \ldots, u^0_{i_1\cdots i_k}), \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n.$$

The natural projections $\pi_{k,m} : J^k \to J^m$, $\theta_k \mapsto \theta_m$, $k > m$, define a tower of bundles

$$\ldots \longrightarrow\ J^k \longrightarrow J^{k-1} \longrightarrow \ldots \longrightarrow J^1 \longrightarrow J^0 = \Omega \times \mathbb{R}\,.$$

We denote the fiber of $\pi_{k,k-1}$ over the point $\theta_{k-1} \in J^{k-1}$ by

$$J^k_{\theta_{k-1}} := \pi^{-1}_{k,k-1}(\theta_{k-1})\,. \tag{2.8}$$

The (truncated to order k) total derivative operators are defined as follows:

$$D^{(k)}_i := \partial_{x^i} + u_i \partial_u + u_{ij_1}\partial_{u_{j_1}} + \cdots + \sum_{j_1 \leq \cdots \leq j_{k-1}} u_{ij_1\cdots j_{k-1}} \partial_{u_{j_1\cdots j_{k-1}}}\,. \tag{2.9}$$

2.3. 2^{nd} order PDEs via hyper-surfaces in a Lagrangian Grassmanian. From now on, we will be considering only 2^{nd} order PDEs, i.e., we set $k = 2$ in the framework given in Section 2.2: the general machinery of jet spaces briefly sketched above gives way to the more specific, and yet equivalent, formalism based on contact manifolds, see, e.g., [27]. In view of such a choice, there will appear some terminology and gadgets typical of Exterior Differential Systems, such as *integral elements* or *Pfaffian systems*, see [9, 31].

The *integral element* associated with $\theta_2 \in J^2$, denoted by L_{θ_2}, is, by definition, the n–dimensional vector subspace of $T_{\theta_1} J^1$ spanned by the operators (2.9) (with $k = 2$) evaluated at

$$\theta_2 = \left(x^i_0, u_0, u^0_i, u^0_{ij}\right) = \left(\theta_1, u^0_{ij}\right). \tag{2.10}$$

More precisely,

$$\begin{aligned} L_{\theta_2} :&= \left\langle D^{(2)}_i\big|_{\theta_2}\right\rangle_{i=1,2,\ldots,n} \\ &= \left\langle \partial_{x^i}\big|_{\theta_1} + u^0_i \partial_u\big|_{\theta_1} + \sum_{i\leq j} u^0_{ij}\partial_{u_j}\big|_{\theta_1}\right\rangle_{i=1,2,\ldots,n}. \end{aligned} \tag{2.11}$$

The key remark of this section is that the $(2n+1)$–dimensional jet space J^1 is endowed with a natural *contact structure*, i.e., a (completely non–integrable) vector distribution of rank $2n$, which we denote by $\mathcal{C}$ and can locally be described as the kernel of the *contact form* $\theta = du - u_i dx^i$. It turns out that

$$\mathcal{C} = \left\langle D_i^{(1)}, \partial_{u_i} \right\rangle_{i=1,2,\dots,n}. \tag{2.12}$$

Note that

$$(d\theta)_{\theta_1} = (dx^i)_{\theta_1} \wedge (du_i)_{\theta_1}$$

is a symplectic form on $\mathcal{C}_{\theta_1}$ for any $\theta_1 \in J^1$. A (local) diffeomorphism $g : J^1 \longrightarrow J^1$ preserving the contact distribution is called a *contactmorphism* and induces a *symplectomorphism* $g_* : \mathcal{C}_{\theta_1} \longrightarrow \mathcal{C}_{g(\theta_1)}$ between the corresponding contact spaces. The contactomorphisms that leave the point θ_1 invariant constitute a group isomorphic to the conformal symplectic group $\mathsf{CSp}(2n)$. Any contactomorphism can be prolonged to J^2 by taking the corresponding tangent map.

Example 2.2. Let $m \leq n$. The (partial) Legendre transformation

$$\begin{aligned} &(x^i, u, u_i) \longmapsto (\tilde{x}^i, \tilde{u}, \tilde{u}_i) \\ &= \left(u_1, \dots, u_m, x^{m+1}, \dots, x^n, u - \sum_{i=1}^m x^i u_i, -x^1, \dots, -x^m, u_{m+1}, \dots, u_n\right) \end{aligned} \tag{2.13}$$

is a contactomorphism. If $m = n$, then we have a *total* Legendre transformation.

A *Lagrangian* subspace is an n–dimensional and isotropic (with respect to the symplectic form $d\theta$) subspace of the symplectic space $\mathcal{C}_{\theta_1}$. The set

$$\mathrm{LGr}(n, \mathcal{C}_{\theta_1}) := \{\text{Lagrangian subspaces of } \mathcal{C}_{\theta_1}\} \tag{2.14}$$

of all such subspaces is the *Lagrangian Grassmannian variety* of $\mathcal{C}_{\theta_1}$; since all the Lagrangian Grassmannian varieties of $2n$–dimensional symplectic spaces are isomorphic, one can use the collective symbol $\mathrm{LGr}(n, 2n)$, when there is no need to stress the base point θ_1. It is worth stressing that in $\mathrm{LGr}(n, \mathcal{C}_{\theta_1})$ there are, in particular, the integral elements L_{θ_2} corresponding to the points θ_2 of the fiber $J^2_{\theta_1}$, cf. (2.8): these integral elements do not, however, fill out the whole of (2.14), but just an open subset of it.

A 2^{nd} order PDE $\mathcal{E}$ can be then seen as a hyper-surface of J^2 whose local expression, in a system of coordinates (2.7), is (1.1): put differently, a 2^{nd} order PDE $\mathcal{E}$ is a sub–bundle of J^2 over $\pi_{2,1}(\mathcal{E})$, where the latter subset of J^1 can be always assumed, save for a few exceptional cases, to be coinciding with J^1 itself. It is called *symplectic* (or *dispersion-less Hirota type*, see [19]) if the function F of (1.1) does not depend neither on coordinates x^i, nor on u and its derivatives u_i. In other words, a symplectic PDE $\mathcal{E}$ has the structure of a product of the (x^i, u, u_i)–space, by a fiber

$$\mathcal{E}_{\theta_1} := \mathcal{E} \cap J^2_{\theta_1} \,. \tag{2.15}$$

Accordingly, the equivalence problem for symplectic PDEs becomes a problem of (conformal) symplectic equivalence. In fact, the study of symplectic PDEs up to contactomorphisms coincides with the study of codimension–one subsets of the fiber $J^2_{\theta_1}$ up to $\mathsf{CSp}(2n)$, which in turn boils down to studying codimension–one subsets of $\mathrm{LGr}(n, 2n)$, see [22] for more details.

2.4. **3D symplectic Monge–Ampère equations as hyperplane sections of** $\mathrm{LGr}(3,6)$**.** The main concern of this paper are 3D Monge–Ampère equations, i.e., Monge–Ampère equations with 3 independent variables: we set then $n = 3$. In the framework we have just outlined, a general 3D Monge–Ampère equation, regarded as a hyper-surface of J^2, can be given in terms of integral elements L_{θ_2} as follows

$$\mathcal{E} := \{\theta_2 \in J^2 \mid \Phi|_{L_{\theta_2}} = 0\} \,, \tag{2.16}$$

where

$$\Phi = \Phi_{ijk} dy^i \wedge dy^j \wedge dy^k \,, \quad (y^1, \ldots, y^6) = (x^1, x^2, x^3, u_1, u_2, u_3) \,, \tag{2.17}$$

$\Phi_{ijk} \in C^\infty(J^1)$, is a 3–differential form. By using the system of coordinates (2.7), it suffices to substitute

$$u_i \to u_{ij} dx^j \tag{2.18}$$

in (2.17) in order to obtain a local coordinate description of the equation (2.16): indeed, substitution (2.18) gives us a multiple of the "volume form" $dx^1 \wedge dx^2 \wedge dx^3$, whose coefficient, equated to zero, locally represents $\mathcal{E}$.

Example 2.3. Replacement (2.18) above, performed on the differential 3–form

$$\Phi = du_1 \wedge du_2 \wedge du_3 - k dx^1 \wedge dx^2 \wedge dx^3 \,, \quad k \in \mathbb{R} \,,$$

yields

$$\big(\det \|u_{ij}\| - k\big) dx^1 \wedge dx^2 \wedge dx^3,$$

whose coefficient, equated to zero, is the Monge–Ampère equation

$$\det \|u_{ij}\| = k.$$

After a total Legendre transform (2.13) $(m = n = 3)$, Φ reads

$$\widetilde{\Phi} = k d\widetilde{u}_1 \wedge d\widetilde{u}_2 \wedge d\widetilde{u}_3 + d\widetilde{x}^1 \wedge d\widetilde{x}^2 \wedge d\widetilde{x}^3,$$

whose associated Monge–Ampère equation is $k \det \|\widetilde{u}_{ij}\| + 1 = 0$. Had we considered the transformation (2.13) with $m = 1$ and $n = 3$, then Φ would have become

$$\widetilde{\Phi} = k d\widetilde{u}_1 \wedge d\widetilde{x}^2 \wedge d\widetilde{x}^3 + d\widetilde{x}^1 \wedge d\widetilde{u}_2 \wedge d\widetilde{u}_3,$$

whose associated Monge–Ampère equation would be

$$k\widetilde{u}_{11} + \widetilde{u}^{\sharp}_{11} = k\widetilde{u}_{11} + \widetilde{u}_{22}\widetilde{u}_{33} - \widetilde{u}^2_{23} = 0.$$

It turns out that a Monge–Ampère equation is described by (1.1), where F is a linear combination of the minors of the Hessian matrix $\|u_{ij}\|$ with coefficients in $C^\infty(J^1)$.

Definition 2.1. *An equation* (1.1)*, where F is a linear combination of the minors of the Hessian matrix $\|u_{ij}\|$ with coefficients in $C^\infty(J^1)$, is called a* Monge–Ampère equation*; if the coefficients are constant, then it is called* symplectic*.*

A general Monge–Ampère equation $\mathcal{E}$ with 3 independent variables, in view of Definition 2.1, can be then written down as

$$\mathcal{E} := \Big\{ A \det \|u_{ij}\| + \sum_{i\leq j} B_{ij} u^{\sharp}_{ij} + \sum_{i \leq j} C^{ij} u_{ij} + D = 0 \Big\}, \tag{2.19}$$

where we recall that $\|u^{\sharp}_{ij}\|$ is the cofactor matrix of $\|u_{ij}\|$ and $A, B_{ij}, C^{ij}, D \in C^\infty(J^1)$. Thus, 3D symplectic Monge–Ampère equations are subsets (2.19) with $A, B_{ij}, C^{ij}, D \in \mathbb{R}$. Note that, as codimension–one subsets of the Lagrangian Grassmanian $\mathrm{LGr}(3,6)$, cf. (2.14), symplectic Monge–Ampère equations are hyper-surfaces of the simplest kind, that is, *hyperplane sections* of $\mathrm{LGr}(3,6)$; in the last Section 9, we show that, at least over the field of complex number, there are only four possible geometries for such hyperplane sections.

In order to obtain a faithful parametrization, it is convenient to consider a special subclass of differential 3–forms (2.17), namely the linear subspace consisting of those forms Φ, such that $\omega^{ij}\Phi_{ijk} = 0$, where $\omega = dx^i \wedge du_i$ locally represents (the conformal class of) the natural symplectic form on $\mathcal{C}$. Such forms, introduced in [27], are called *effective* and they are in one–to–one correspondence with 3D Monge–Ampère equations, up to a nowhere zero factor: this is the reason why, in Section 6, we will be considering the projectivization of the space of effective 3–forms (at a point θ_1); not only it leads to a strict one–to–one parametrization (of *symplectic* Monge–Ampère equations), but it also allows to work with the symplectic group $\mathsf{Sp}(\mathcal{C}_{\theta_1}) = \mathsf{Sp}(6, \mathbb{R})$, rather than its conformal counterpart. From Section

6 on, we will be working on $\mathbb{C}$ and then group $\mathsf{Sp}(\mathcal{C}_{\theta_1}) = \mathsf{Sp}(6,\mathbb{C})$ will be a simple one, thus making it possible to deploy the whole machinery of representation theory.

In the last Section 9 the reader will find another definition of a 3D symplectic Monge–Ampère equation, see Definition 9.1: it is formally analogous to Definition 2.1 above, only over the field of complex numbers.

2.5. **The Kushner–Lychagin–Rubtsov (KLR) invariant and the cocharacteristic variety.** In [27] the authors have showed that with any Monge-Ampère equation (2.19) (which has been defined by means of the 3–form (2.17)) one can associate a quadratic form via

$$\omega^{i_1 j_1}\omega^{i_2 j_2}\Phi_{a\, i_1 i_2}\Phi_{b\, j_1 j_2} dy^a dy^b\,, \tag{2.20}$$

where

$$\begin{aligned}
&\Phi_{456} = A\,, \quad \Phi_{156} = B_{11}\,, \quad \Phi_{146} = -\Phi_{256} = -B_{12}\,, \quad \Phi_{145} = \Phi_{356} = B_{13}\,,\\
&\Phi_{246} = -B_{22}\,, \quad \Phi_{346} = -\Phi_{245} = -B_{23}\,, \quad \Phi_{345} = B_{33}\,, \quad \Phi_{234} = C^{11}\,,\\
&\Phi_{235} = -\Phi_{134} = C^{12}\,, \quad \Phi_{124} = \Phi_{236} = C^{13}\,, \quad \Phi_{135} = -C^{22}\,,\\
&\Phi_{125} = -\Phi_{136} = C^{23}\,, \quad \Phi_{126} = C^{33}\,, \quad \Phi_{123} = D\,.
\end{aligned}$$

and $(y^1,\dots,y^6)$ are the same as in (2.17).

Remark 2.1. The quadratic form (2.20) above can be given without employing any coordinates, see (1.7); from now on, both of them will be referred to as the *KLR invariant* of the Monge-Ampère equation (2.19).

Theorem 8.1 below shows that the KLR invariant (2.20) is equivalent to the Hitchin moment map on the space parametrizing symplectic Monge–Ampère equations, and it will be thoroughly reviewed and deepened in Section 8.1; moreover, Corollary 9.1 shows that, if the invariant (2.20) is equated to zero, one obtains a quadratic hyper-surface in $\mathcal{C}$, which is naturally linked with the characteristics of the Monge–Ampère equation at hand via *projective duality*. Such a duality is the rationale behind the choice of the prefix "co" in the next definition.

Definition 2.2. *The zero locus of the homogeneous* 2^{nd} *order polynomial* (2.20) *is called the* cocharacteristic variety *of the 3D Monge–Ampère equation* (2.19).

We stress again that later on in Section 9, above Definition 2.2 will be reformulated in the complex setting for *symplectic* 3D Monge–Ampère equations, see Definition 9.4: to avoid uninteresting complications, the proofs of the main results of this paper will be given over the field of complex numbers.

2.6. **Monge–Ampère equations of Goursat–type and a non–linear generalization of his idea: contact cone structures.** At the very beginning, we mentioned Goursat pioneering paper [20]: there, he proposed the following way of obtaining Monge-Ampère equations: substituting

$$du_1 = u_{11}dx^1 + u_{12}dx^2 \quad \text{and} \quad du_2 = u_{12}dx^1 + u_{22}dx^2$$

into the Pfaffian system

$$\begin{cases} du_1 - b_{11}dx^1 - b_{12}dx^2 = 0\,, \\ du_2 - b_{21}dx^1 - b_{22}dx^2 = 0\,, \end{cases} \qquad b_{ij} = b_{ij}(x^1, x^2, u, u_1, u_2)\,,$$

and then requiring its (non trivial) compatibility. Such a procedure was then generalized by Goursat to any number n of independent variables, by considering the system

$$\alpha_i := du_i - \sum_{j=1}^{n} b_{ij}dx^j = 0\,, \quad i = 1, \dots, n\,, \tag{2.21}$$

$b_{ij} = b_{ij}(x^1, \dots, x^n, u, u_1, \dots, u_n)$, thus getting the equation

$$\det \|u_{ij} - b_{ij}\| = 0\,. \tag{2.22}$$

Definition 2.3. *An equation* (1.1), *where F is given by the left–hand side of* (2.22), *is called a Monge–Ampère equation of* Goursat type*.*

The same equation is obtained if a replacement $b_{ij} \to b_{ji}$ is performed in (2.21), i.e., if one starts from the Pfaffian system

$$\widetilde{\alpha}_i := du_i - \sum_{j=1}^{n} b_{ji}dx^j = 0\,, \;\; i = 1, \dots, n\,.$$

Equation (2.22) belongs to a special class of Monge-Ampère equations, i.e., those whose symbol has rank less or equal to 2; see [1] for more details.

An equivalent approach, leading to the same equation (2.22), goes as follows.

Let us consider the jet space J^1 with coordinates (x^i, u, u_i) and, within its canonical contact distribution $\mathcal{C}$, let us single out two n–dimensional sub–distributions:

$$\mathcal{D} := \langle \partial_{x^i} + u_i\partial_u + b_{ij}\partial_{u_j} \rangle_{i=1,\dots n} = \{\alpha_i = 0\,, \theta = 0\}\,, \tag{2.23}$$
$$\mathcal{D}^\perp = \langle \partial_{x^i} + u_i\partial_u + b_{ji}\partial_{u_j} \rangle_{i=1,\dots n} = \{\widetilde{\alpha}_i = 0\,, \theta = 0\}\,.$$

Then, the equation (2.22) is obtained by requiring that a general integral element $\langle \partial_{x^i} + u_i\partial_u + u_{ij}\partial_{u_j} \rangle_{i=1,\dots n}$ of $\mathcal{C}$ nontrivially intersects $\mathcal{D} \cup \mathcal{D}^\perp$. In the above cited paper [1] it is proved that any hyperplane containing a line of $\mathcal{D}\cup\mathcal{D}^\perp$ is a characteristic hyperplane for the equation (2.22), and vice versa.

Our idea for generalizing Goursat approach stems from a simple observation: the union $\mathcal{V}:=\mathcal{D}\cup\mathcal{D}^\perp$ is a distribution $\theta_1\in J^1\to\mathcal{V}_{\theta_1}\subset\mathcal{C}_{\theta_1}$ of (very degenerate) n–codimensional quadric affine conic varieties; therefore, if from $\mathcal{V}$ one gets a Monge–Ampère equation whose characteristic hyperplanes lie in $\mathcal{V}$, which PDEs would one obtain, starting from a more general quadric (for instance, non–degenerate and not necessarily of codimension n) in $\mathcal{C}$?

This question is the paper's backbone and it leads, in particular, to the problem of classifying all quadric hyper-surfaces in a 6–dimensional symplectic space, up to symplectomorphisms, which will be done in Section 7.

In Sections 3, 4, and 5 below, we will be rather focusing on some practical examples of how to pass from a 3D 2$^{\text{nd}}$ order PDE to a (quadratic) contact cone structure, accordingly to the following definition.

Definition 2.4. *A* contact cone structure $\mathcal{V}$ *on* J^1 *is a smooth assignment of an affine conic variety* $\mathcal{V}_{\theta_1}\subset\mathcal{C}_{\theta_1}$ *or, equivalently, of a projective variety* $\mathbb{P}\mathcal{V}_{\theta_1}\subset\mathbb{P}\mathcal{C}_{\theta_1}$*, in each point* $\theta_1\in J^1$*.*

The cocharacteristic variety associated with a Monge–Ampère equation (see Definition 2.2) is our first example of a *quadratic* generalization of a *linear* sub–distribution of a contact distribution, henceforth called a *quadric contact cone structure*. It turns out that there is more than one way to associate a quadric contact cone structure with a 2$^{\text{nd}}$ order PDE, see Section 3 below: nevertheless, at least for 3D Monge–Ampère equations with *non–degenerate symbol*, the cocharacteristic variety is the only natural one (this will follows from Corollary 9.1 later on).

3. THE CONTACT CONE STRUCTURE ASSOCIATED WITH A 2$^{\text{ND}}$ ORDER PDE

In order to associate a contact cone structure with a 2$^{\text{nd}}$ order PDE and, in particular, with a Monge–Ampère equation, we need to pass through the notion of a characteristic which is, in turn, related to that of a rank–one vector. To this purpose, we need the following definition.

Definition 3.1. *Any κ–dimensional subspace H_{θ_1} of the contact space $\mathcal{C}_{\theta_1}$, with $\kappa\le n$, can be prolonged to a sub-manifold $H^{(1)}_{\theta_1}$ of $J^2_{\theta_1}$ defined by*

$$H^{(1)}_{\theta_1}:=\{\theta_2\in J^2_{\theta_1}\ \mid\ L_{\theta_2}\supseteq H_{\theta_1}\}\,.$$

Remark 3.1. It is easy to see that if H_{θ_1} is a hyperplane of L_{θ_2}, then $H^{(1)}_{\theta_1}$ is a curve in $J^2_{\theta_1}$ passing through θ_2.

Remark 3.2. As we said in Section 1.1, we work with regular points of 2^{nd} order PDEs, so that any point $\theta_2 \in \mathcal{E}$ of a 2^{nd} order PDE $\mathcal{E}$ is assumed regular. For the sake of simplicity, and to not overload the notation, the set of regular points $\mathcal{E}_{\mathrm{reg}}$ of a 2^{nd} order PDE $\mathcal{E}$ will be denoted by the same symbol we use for the PDE, i.e., by $\mathcal{E}$. For the same reason, the projection $\check{J}^1 := \pi_{2,1}(\mathcal{E}_{\mathrm{reg}}) \subseteq J^1$ of $\mathcal{E}_{\mathrm{reg}}$ onto J^1 will be denoted by J^1.

From the point of view of contact geometry, the notions that have been introduced in Section 2 above can be recast, more geometrically, as follows.

Definition 3.2. *A* Cauchy datum *for a 2^{nd} order PDE $\mathcal{E} \subset J^2$ is an $(n-1)$–dimensional integral sub-manifold Σ of the contact distribution $\mathcal{C}$ on J^1. It is* characteristic *at a point $\theta_2 \in \mathcal{E}$ if the prolongation $(T_{\theta_1}\Sigma)^{(1)}$ is tangent to $\mathcal{E}_{\theta_1}$ at the point θ_2. In this case, $T_{\theta_1}\Sigma$ is also called a* characteristic *hyperplane (at $\theta_1 = \pi_{2,1}(\theta_2)$).*

3.1. **Rank of vertical vectors and characteristics.**

Definition 3.3. *The* rank *of a vector*

$$\nu_{\theta_2} = \sum_{i\leq j} \nu_{ij}\partial_{u_{ij}}\big|_{\theta_2}, \quad \nu_{ij} \in \mathbb{R}, \tag{3.1}$$

of $T_{\theta_2}J^2_{\theta_1}$ is the rank of the matrix

$$\begin{pmatrix} \nu_{11} & \dots & \nu_{1n} \\ \vdots & \ddots & \vdots \\ \nu_{1n} & \dots & \nu_{nn} \end{pmatrix}. \tag{3.2}$$

The rank of a line $\ell_{\theta_2} \subset T_{\theta_2}J^2_{\theta_1}$ is the rank of any vector v_{θ_2}, such that $\ell_{\theta_2} = \langle v_{\theta_2}\rangle$.

A direct computation shows that the rank of a line ℓ_{θ_2} in $T_{\theta_1}J^2_{\theta_1}$ is invariant with respect to contactomorphisms.

Below, we shall clarify the nature of such an invariant. Let us fix a point $\theta_2 \in J^2$ and a vector (3.1). Take a curve $\theta_2(t)$ in $J^2_{\theta_1}$ such that $\theta_2(0) = \theta_2$ and $\theta_2'(0) = \nu_{\theta_2}$: it will be given locally by

$$\theta_2(t) = \big(\theta_1, u_{ij}(t)\big), \tag{3.3}$$

with $u'_{ij}(0)=\nu_{ij}$ and it will correspond to a 1–parametric family of integral elements (cf. (2.11)):

$$\begin{aligned} L_{\theta_2(t)} &= \left\langle \partial_{x^i}\big|_{\theta_1} + u_i^0\partial_u\big|_{\theta_1} + u_{ij}(t)\partial_{u_j}\big|_{\theta_1}\right\rangle_{i=1,\dots,n} \\ &= \left\langle D_i^{(1)}\big|_{\theta_1} + u_{ij}(t)\partial_{u_j}\big|_{\theta_1}\right\rangle_{i=1,\dots,n}. \end{aligned}$$

Since $L_{\theta_2(t)}$ and L_{θ_2} are two n–dimensional subspaces of the $2n$–dimensional vector space $\mathcal{C}_{\theta_1}$, their intersection $L_{\theta_2(t)}\cap L_{\theta_2}$ is, generically, zero–dimensional. However, there are curves (3.3) for which $L_{\theta_2(t)}\cap L_{\theta_2}$ is κ–codimensional $\forall\, t$, i.e., $L_{\theta_2(t)}$ "rotates" around an $(n-\kappa)$–dimensional subspace of L_{θ_2}. From an infinitesimal viewpoint, this means that there are tangent directions (3.1) along which $L_{\theta_2(t)}$, with $\theta_2(t)$ given by (3.3), moves away from L_{θ_2} by retaining some "common piece". This motivates the following definition.

Definition 3.4. *We say that $L_{\theta_2(t)}$ has a deviation of order κ from L_{θ_2} if* $\dim\left(L_{\widetilde{\theta}_2(t)}\cap L_{\theta_2}\right)=n-\kappa$, *for small nonzero t and $\widetilde{\theta}_2(t)$ is the linear approximation of $\theta_2(t)$ at θ_2. In other words, if*

$$\dim\left(L_{\theta_2+\theta_2'(0)t}\cap L_{\theta_2}\right)=n-\kappa\,,\ \forall\, t\in(-\epsilon,\epsilon)\smallsetminus\{0\}\,.$$

Proposition 3.1. *The vector* (3.1) *has rank κ if and only if $L_{\theta_2(t)}$, with $\theta_2(t)$ given by* (3.3), *has a deviation of order κ from L_{θ_2}.*

Proof. Without any loss of generality, we can assume that θ_2 has all 2^{nd} order jet coordinates equal to 0, i.e., $u_{ij}(\theta_2)=0\ \forall i,j$. Then

$$L_{\theta_2+\theta_2'(0)t}=\left\langle D_i^{(1)}\big|_{\theta_1}+u'_{ij}(0)t\,\partial_{u_j}\big|_{\theta_1}\right\rangle_{i=1,\dots,n}$$

and the intersection $L_{\theta_2+\theta_2'(0)t}\cap L_{\theta_2}$ has dimension $n-\kappa$ if and only if the block matrix

$$\left(\begin{array}{c|c}\mathrm{Id} & u'_{ij}(0)t\\ \hline \mathrm{Id} & 0\end{array}\right),$$

has rank $n+\kappa$, which in turn is equivalent to the upper–right block having rank κ: but, in view of (3.3), this is the same as the rank of the matrix (3.2). □

Putting together Definition 3.4 and Proposition 3.1, we obtain that to each line $\ell_{\theta_2}\subset T_{\theta_2}J^2_{\theta_1}$ of rank κ we can associate an $(n-\kappa)$–dimensional subspace $H(\ell_{\theta_2})$ of the integral element L_{θ_2}:

$$\text{tangent lines of rank }\kappa\text{ in }T_{\theta_2}J^2_{\theta_1}\Longrightarrow\\ (n-\kappa)\text{–dimensional subspaces of }L_{\theta_2}\,. \quad (3.4)$$

The implication (3.4) is one–to–one only in the case of lines of rank 1. Indeed, the dimension of the prolongation $H^{(1)}$ of a subspace $H \subseteq L_{\theta_2}$ of codimension $\kappa \geq 2$ is greater than 1, whereas it is equal to 1 in the case $\kappa = 1$ (see Remark 3.1).

We now analyze in more detail the case $\kappa = 1$, i.e., rank–one lines, since they are closely related with characteristic Cauchy data. In this case (3.4) becomes

$$\text{tangent lines } \ell_{\theta_2} = \langle \nu_{\theta_2} \rangle \subset T_{\theta_2} J^2_{\theta_1} \text{ of rank 1} \Longleftrightarrow \text{hyperplanes } H(\ell_{\theta_2}) \subset L_{\theta_2} . \quad (3.5)$$

Taking into account Definition 3.2 and correspondence (3.5), it is easy to realize that with any rank–one line ℓ_{θ_2} is associated the hyperplane $H(\ell_{\theta_2})$ of L_{θ_2} and that $H(\ell_{\theta_2})$ is characteristic if the direction ℓ_{θ_2} is included in the tangent space of the considered PDE.

Definition 3.5. *A line $\ell_{\theta_2} \subset T_{\theta_2} J^2_{\theta_1}$ of rank 1 is* characteristic *for a PDE (1.1) in $\theta_2 \in \mathcal{E}$ if $\ell_{\theta_2} \subset T_{\theta_2}\mathcal{E}$.*

It is well known that matrices of rank 1 have the (i, j)–entry equal to $\eta_i \eta_j$, where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ is a vector in $\mathbb{R}^n$. Therefore, a rank–one vector (3.1) has the form

$$\nu_{\theta_2} = \sum_{i \leq j} \eta_i \eta_j \partial_{u_{ij}} \big|_{\theta_2} , \quad \eta_i \in \mathbb{R} . \quad (3.6)$$

Then the hyperplane $H(\ell_{\theta_2})$ corresponding to L_{θ_2} via (3.5) is locally described by

$$H(\ell_{\theta_2}) = \left\{ \xi^i D^{(2)}_i \big|_{\theta_2} \mid \xi^i \eta_i = 0 , \quad \xi^i \in \mathbb{R} \right\} \quad (3.7)$$

or equivalently, as

$$H(\ell_{\theta_2}) = \ker \eta , \quad \eta = \eta_i dx^i \in L^*_{\theta_2} . \quad (3.8)$$

The claim (3.8) is a direct consequence of an elementary property of symmetric rank–one $n \times n$ matrices:

$$\ker(\eta_i \eta_j) = \big\langle (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n \mid \xi^i \eta_i = 0 \big\rangle .$$

The converse reads as follows: given a hyperplane $H \subset L_{\theta_2}$, there is a covector $\eta \in T^*_{\boldsymbol{x}_0} \mathbb{R}^n$ (defined up to a nonzero factor) by an analogous formula to (3.8); the vector ν_{θ_2}, such that $H(\ell_{\theta_2}) = H$ is then constructed by means of formula 3.6, by using the components of η. If we look at Definition 3.5 in local coordinates (2.7), we see that a line ℓ_{θ_2} is of rank–one if it is spanned by a vector of type 3.6. Therefore, it is characteristic for PDE (1.1) at $\theta_2 \in \mathcal{E}$ if

$$\sum_{i\leq j} \frac{\partial F}{\partial u_{ij}}\Big|_{\theta_2} \eta_i\eta_j = 0\,, \tag{3.9}$$

which coincides with the equation of characteristics (see, for instance, [35]). In other words, the covector $\eta_i dx^i$ annihilates the symbol of the equation (cf. 1.5).

Example 3.1. In the case $n = 2$, i.e., with 2 independent variables, in view of correspondence (3.5), to the line ℓ_{θ_2} it corresponds a line $H(\ell_{\theta_2})$ in L_{θ_2} via (3.5). On account of (3.7), (3.9) reads as

$$\frac{\partial F}{\partial u_{11}}\Big|_{\theta_2}(\xi^1)^2 - \frac{\partial F}{\partial u_{12}}\Big|_{\theta_2}\xi^1\xi^2 + \frac{\partial F}{\partial u_{22}}\Big|_{\theta_2}(\xi^2)^2 = 0\,. \tag{3.10}$$

For instance, if we consider the equation $\mathcal{E} := \{u_{12} = 0\}$ from Example 2.1, the characteristic equation (3.9) reads $\eta_1\eta_2 = 0$ or, in the form (3.10), $\xi^1\xi^2 = 0$. So, characteristic lines ℓ_{θ_2}, where $\theta_2 = (x_0^1, x_0^2, u^0, u_1^0, u_2^0, u_{11}^0, 0, u_{22}^0) \in \mathcal{E}$, are spanned by 3.6 where either η_1 or η_2 is equal to zero, i.e., lines

$$\ell^+_{\theta_2} = \left\langle \partial_{u_{11}}\big|_{\theta_2}\right\rangle \quad \text{or} \quad \ell^-_{\theta_2} = \left\langle \partial_{u_{22}}\big|_{\theta_2}\right\rangle,$$

and the corresponding characteristic hyperplanes (that, in this case, are lines of the integral element L_{θ_2}) are

$$H(\ell^+_{\theta_2}) = \left\langle D_2^{(2)}\big|_{\theta_2}\right\rangle = \left\langle \partial_{x^2}\big|_{\theta_2} + u_2^0\partial_u\big|_{\theta_2} + u_{22}^0\partial_{u_2}\big|_{\theta_2}\right\rangle,$$
$$H(\ell^-_{\theta_2})\rangle = \left\langle D_1^{(2)}\big|_{\theta_2}\right\rangle = \left\langle \partial_{x^1}\big|_{\theta_2} + u_1^0\partial_u\big|_{\theta_2} + u_{11}^0\partial_{u_1}\big|_{\theta_2}\right\rangle.$$

In fact, in Example 2.1, according to the definitions given in the present section, the curve $\Sigma = \{\Phi(x^1) \mid x^1 \in \mathbb{R}\}$ is a Cauchy datum as it is a 1–dimensional integral sub-manifold of the contact distribution $\mathcal{C}$ (cf. Definition 3.2). Its prolongation (cf. Definition 3.1) is

$$\Sigma^{(1)}_{\theta_1} = \{(x_0^1, 0, x_0^1, 1, 0, 1, 0, t) \mid t \in \mathbb{R}\},$$

and $T_{\theta_2}\Sigma^{(1)}$ is a characteristic hyperplane, since $T_{\theta_2}\Sigma^{(1)} = \ell^-_{\theta_2} \subset T_{\theta_2}\mathcal{E}_{\theta_1}$.

3.2. **Contact cone structure associated with a 2^{nd} order PDE in 3 independent variables of non–elliptic type.** From now on, unless specified otherwise, we will be considering only PDEs in 3 independent variables, i.e.,

$$\mathcal{E} := \left\{ F(x^1, x^2, x^3, u, u_1, u_2, u_3, u_{11}, \ldots, u_{33}) = 0 \right\} \tag{3.11}$$

that are non–elliptic. Below, we will see how to construct, starting from a PDE (3.11), a contact cone structure $\mathcal{V}: \theta_1 \in J^1 \to \mathcal{V}_{\theta_1} \subset \mathcal{C}_{\theta_1}$, see Definition 2.4.

Such a construction breaks down into 4 steps.

(1) Fix a point $\theta_2 = \left(x_0^i, u_0, u_i^0, u_{ij}^0\right) \in \mathcal{E}$, i.e., such that

$$F\left(x_0^i, u_0, u_i^0, u_{ij}^0\right) = 0\,.$$

(2) Consider the set of rank–one lines at θ_2 that are tangent to $\mathcal{E}$. Such a set is given by vectors of type (3.6) such that $(\eta_1, \ldots, \eta_n)$ satisfies (3.9). Note that such vectors are organized either in two distinct families ν^+ and ν^-, if the PDE (3.11) is hyperbolic at θ_2, or in one single family $\nu^+ = \nu^- = \nu$ if it is parabolic at θ_2. In both cases, such families are 2-parametric. This implies that the corresponding families of rank–one *lines* are 1-parametric. We denote such lines by $\ell^+_{\theta_2}(t)$ and $\ell^-_{\theta_2}(t)$, where t is the aforementioned parameter.

(3) Let us consider the line $\ell^+_{\theta_2}(t)$ only, since the same reasoning works for $\ell^-_{\theta_2}(t)$ as well. To each line $\ell^+_{\theta_2}(t)$, we associate the hyperplane $H(\ell^+_{\theta_2}(t)) \subset L_{\theta_2}$, see (3.7). Then, by varying the parameter t, the hyperplanes $H(\ell^+_{\theta_2}(t))$ sweep a cone of L_{θ_2}, that we denote by $\mathcal{V}_{\theta_2}$, in the following sense: the generatrix $v(\ell^+_{\theta_2}(t))$ of $\mathcal{V}_{\theta_2}$, which is a line passing through $\theta_1 = \left(x_0^i, u_0, u_i^0\right)$, will be given as an infinitesimal intersection

$$v(\ell^+_{\theta_2}(t)) := \lim_{\epsilon \to 0} H(\ell^+_{\theta_2}(t)) \cap H(\ell^+_{\theta_2}(t+\epsilon))\,.$$

Summing up, to any point $\theta_2 \in \mathcal{E}$, we can associate two cones in L_{θ_2}:

$$\mathcal{V}^+_{\theta_2} := \bigcup_t v(\ell^+_{\theta_2}(t))\,, \quad \mathcal{V}^-_{\theta_2} := \bigcup_t v(\ell^-_{\theta_2}(t)).$$

(4) If now we let vary the point $\theta_2 \in \mathcal{E}$ over θ_1, we obtain a *conic variety* $\mathcal{V}_{\theta_1} \subseteq \mathcal{C}_{\theta_1}$:

$$\mathcal{V}_{\theta_1} := \mathcal{V}^+_{\theta_1} \cup \mathcal{V}^-_{\theta_1} = \bigcup_{\theta_2 \in \mathcal{E}_{\theta_1}} \mathcal{V}_{\theta_2}\,, \quad \mathcal{V}^\pm_{\theta_1} := \bigcup_{\theta_2 \in \mathcal{E}_{\theta_1}} \mathcal{V}^\pm_{\theta_2}\,, \quad \mathcal{V}_{\theta_2} := \mathcal{V}^+_{\theta_2} \cup \mathcal{V}^-_{\theta_2}\,. \quad (3.12)$$

Definition 3.6. *The conic sub–distribution* $\mathcal{V}: \theta_1 \in J^1 \to \mathcal{V}_{\theta_1} \subset \mathcal{C}_{\theta_1}$*, with* $\mathcal{V}_{\theta_1}$ *given by* (3.12)*, is called the* contact cone structure *of the PDE* (3.11).

4. QUADRIC CONTACT CONE STRUCTURES ASSOCIATED WITH 3D MONGE–AMPÈRE EQUATIONS

In this section, we work with 3D Monge–Ampère equations, i.e., Monge–Ampère equations in 3 independent variables.

As we will see in Section 8.2, up to symplectic equivalence and signature, there are only four types of symplectic 3D Monge–Ampère equations: below we pick a representative for each type and compute its contact cone structure, that will turn out to be a *quadric* contact cone structure. At the end of each subsection, we quickly comment on the relationship between the quadric contact cone structure and the cocharacteristic variety of each considered PDE.

Notation 4.1. From now on, the coordinates on $\mathcal{C}_{\theta_1}$ dual to the (truncated) total derivatives $D_i^{(1)}\big|_{\theta_1}$ and vectors $\partial_{u_i}|_{\theta_1}$ will be denoted by z^i and q_i, respectively: such a choice is dictated by a purely aesthetic concern.

4.1. **The quadric contact cone structure of the equation** $\det\|u_{ij}\| = 1$**.** Let us consider the equation

$$\mathcal{E} := \{\det\|u_{ij}\| = 1\} \tag{4.1}$$

and apply to it, step by step, the scheme given at the beginning of Section 3.2. Recall that $u^\sharp_{ij}$ denotes the (i,j)-entry of the cofactor matrix of $\|u_{ij}\|$.

(1) Let us fix a point $\theta_2 \in \mathcal{E}$, $\theta_2 = \left(x^i_0, u_0, u^0_i, u^0_{ij}\right)$ with

$$u^0_{11} = \frac{(u^0_{12})^2 u^0_{33} - 2u^0_{12}u^0_{13}u^0_{23} + (u^0_{13})^2 u^0_{22} + 1}{u^{0\sharp}_{11}},$$

assuming $u^{0\sharp}_{11} \neq 0$. We also assume that θ_2 is not an elliptic point for the equation (4.1).

(2) Equation (3.9), which reads now

$$\sum_{i,j} u^\sharp_{ij}\eta_i\eta_j = \sum_{i\leq j}(2-\delta_{ij})u^\sharp_{ij}\eta_i\eta_j = 0,$$

can be solved with respect to η_1, obtaining

$$\eta_1^\pm(\eta_2,\eta_3) = \frac{-u^{0\sharp}_{12}\,\eta_2 - u^{0\sharp}_{12}\,\eta_3 \pm \sqrt{B}}{u^{0\sharp}_{11}}, \tag{4.2}$$

with

$$B = B(\eta_2,\eta_3) := -\eta_2^2 u^0_{33} + 2\eta_2\eta_3 u^0_{23} - \eta_3^2 u^0_{22}\,.$$

All the rank–1 vectors ν_{θ_2} of the PDE (4.1) at the (non–elliptic) point θ_2, in view of formula (3.6), are described by the two 2–parametric families

$$\begin{aligned}\nu^\pm_{\theta_2}(\eta_2,\eta_3) = {}& (\eta_1^\pm)^2\partial_{u_{11}}\big|_{\theta_2} + \eta_1^\pm\eta_2\partial_{u_{12}}\big|_{\theta_2} + \eta_1^\pm\eta_3\partial_{u_{13}}\big|_{\theta_2} \\ & + \eta_2^2\partial_{u_{22}}\big|_{\theta_2} + \eta_2\eta_3\partial_{u_{23}}\big|_{\theta_2} + \eta_3^2\partial_{u_{33}}\big|_{\theta_2}\,,\end{aligned} \tag{4.3}$$

with $\eta_1^\pm$ given by (4.2). We focus now only on the family $\nu^+_{\theta_2}(\eta_2,\eta_3)$, since, for the other one, computations and reasoning are the same. Since we are interested in the line spanned by $\nu^+_{\theta_2}(\eta_2,\eta_3)$, we shall substitute $\eta_2=t$ and $\eta_3=1$ in (4.3), thus obtaining $\ell^+_{\theta_2}(t):=\langle\nu^+_{\theta_2}(t,1)\rangle$. Accordingly, we set $\eta_1^+(t):=\eta_1^+(t,1)$, $B(t)=B(t,1)$.

(3) With the line $\ell^+_{\theta_2}(t)$ is associated the hyperplane $H(\ell^+_{\theta_2}(t)\subset L_{\theta_2}$, which, in view of (3.7), is given by

$$H(\ell^+_{\theta_2}(t))=\left\langle -tD_1^{(2)}(\theta_2)+\eta_1^+(t)\,D_2^{(2)}(\theta_2)\ ,\ D_2^{(2)}(\theta_2)-tD_3^{(2)}(\theta_2)\right\rangle .$$

In order to study $\lim_{\epsilon\to 0} H(\ell^+_{\theta_2}(t))\cap H(\ell^+_{\theta_2}(t+\epsilon))$, we consider the system

$$\begin{cases}\eta_1^+(t)\,\xi^1+t\xi^2+\xi^3=0\,,\\ \eta_1^+(t+\epsilon)\,\xi^1+(t+\epsilon)\xi^2+\xi^3=0\,.\end{cases}\tag{4.4}$$

By solving system (4.4) with respect to $\xi^2=\xi^2(\xi^1,t,\epsilon)$, $\xi^3=\xi^3(\xi^1,t,\epsilon)$, and then by computing $\lim_{\epsilon\to0}\xi^2$ and $\lim_{\epsilon\to0}\xi^3$, we obtain

$$\xi^2=-\frac{\xi_1\left(-\sqrt{B(t)}u^{0\sharp}_{12}-u^0_{33}t+u^0_{23}\right)}{\sqrt{B(t)}u^{0\sharp}_{11}}\,,\quad \xi^3=\frac{\xi_1\left(\sqrt{B(t)}u^{0\sharp}_{13}-u^0_{23}t+u^0_{22}\right)}{\sqrt{B(t)}u^{0\sharp}_{11}}\,,$$

so that

$$\begin{aligned}v(\ell^+_{\theta_2}(t))=\Big\langle\ &\sqrt{B(t)}u^{0\sharp}_{11}\,D_1^{(2)}(\theta_2)+\left(\sqrt{B(t)}u^{0\sharp}_{12}+u^0_{33}t-u^0_{23}\right)D_2^{(2)}(\theta_2)\\ &+\xi_1\left(\sqrt{B(t)}u^{0\sharp}_{13}-u^0_{23}t+u^0_{22}\right)D_3^{(2)}(\theta_2)\Big\rangle\subset L_{\theta_2}\,.\end{aligned}\tag{4.5}$$

(4) Had we considered the family $\nu^-_{\theta_2}(\eta_2,\eta_3)$, we would have come to the line $v(\ell^-_{\theta_2}(t))$. If we let vary the parameter t and the point θ_2 on the fiber $\mathcal{E}_{\theta_1}$, the above–found lines $v(\ell^\pm_{\theta_2}(t))$ give a conic variety inside the contact hyperplane $\mathcal{C}_{\theta_1}$ of $T_{\theta_1}J^1$:

$$\mathcal{V}_{\theta_1}: z^1q_1+z^2q_2+z^3q_3=0\,.\tag{4.6}$$

By computing the cocharacteristic variety of the same equation (4.1), according to Definition 2.2, we obtain again (4.6).

Remark 4.1. The partial Legendre transformation

$$(x^i,u,u_i)\to(u_1,x^2,x^3,u-x^1u_1,-x^1,u_2,u_3)=\left(\tilde x^1,\tilde x^2,\tilde x^3,\tilde u_1,\tilde u_2,\tilde u_3\right)\,,$$

cf. (2.13), transforms equation (4.1) into (up to a renaming of coordinates)

$$u_{11}+u_{22}u_{33}-u_{23}^2=0\tag{4.7}$$

and the conic variety (4.6) into

$$z^1 q_1 - z^2 q_2 - z^3 q_3 = 0\,. \tag{4.8}$$

In fact, the conic variety (4.8) is the quadric contact cone structure of equation (4.7).

4.2. **The quadric contact cone structure of the equation** $u_{11} - u_{22} - u_{33} = 0$**.** Let us consider the wave equation

$$\mathcal{E} := \{u_{11} = u_{22} + u_{33}\}\,. \tag{4.9}$$

As we did in Section 4.1, we apply the same scheme to the equation (4.9).

(1) Let us fix a point $\theta_2 = \left(x_0^i, u_0, u_i^0, u_{22}^0 + u_{33}^0, u_{12}^0, \dots, u_{33}^0\right) \in \mathcal{E}$.

(2) Equation (3.9) reads $\eta_1^2 = \eta_2^2 + \eta_3^2$, so that the rank–1 vectors ν_{θ_2} of the PDE (4.9) at point θ_2, in view of formula (3.6), are described by the two 2–parametric families:

$$\begin{aligned}\nu^{+}_{\theta_2}(\eta_2,\eta_3) = (\eta_2^2+\eta_3^2)\partial_{u_{11}}\big|_{\theta_2} &\pm \sqrt{\eta_2^2+\eta_3^2}\,\eta_2\partial_{u_{12}}\big|_{\theta_2} \\ &\pm \sqrt{\eta_2^2+\eta_3^2}\,\eta_3\partial_{u_{13}}\big|_{\theta_2} + \eta_2^2\partial_{u_{22}}\big|_{\theta_2} + \eta_2\eta_3\partial_{u_{23}}\big|_{\theta_2} + \eta_3^2\partial_{u_{33}}\big|_{\theta_2}\,.\end{aligned}$$

Once again consider only the first family. Since we are interested in the line spanned by $\nu^{+}_{\theta_2}(\eta_2,\eta_3)$, we set $\eta_2 = t$, $\eta_3 = 1$, thus obtaining

$$\begin{aligned}\ell^{+}_{\theta_2}(t) = \langle \nu^{+}_{\theta_2}(t,1)\rangle = \Big\langle (1+t^2)\partial_{u_{11}}\big|_{\theta_2} &+ \sqrt{1+t^2}\,t\partial_{u_{12}}\big|_{\theta_2} \\ &+ \sqrt{1+t^2}\partial_{u_{13}}\big|_{\theta_2} + t^2\partial_{u_{22}}\big|_{\theta_2} + t\partial_{u_{23}}\big|_{\theta_2} + \partial_{u_{33}}\big|_{\theta_2}\Big\rangle\,.\end{aligned}$$

(3) With the line $\ell^{+}_{\theta_2}(t)$ is associated the hyperplane

$$H(\ell^{+}_{\theta_2}(t)) = \Big\langle -tD_1^{(2)}(\theta_2) + \sqrt{1+t^2}\,D_2^{(2)}(\theta_2)\ ,\ D_2^{(2)}(\theta_2) - tD_3^{(2)}(\theta_2)\Big\rangle \subset L_{\theta_2}\,.$$

In order to study the intersection of $H(\ell^{+}_{\theta_2}(t))$ with the plane $H(\ell^{+}_{\theta_2}(t+\epsilon))$ at the limit $\epsilon \to 0$, we consider the system

$$\begin{cases}\sqrt{1+t^2}\,\xi^1 + t\xi^2 + \xi^3 = 0\,, \\ \sqrt{1+(t+\epsilon)^2}\,\xi^1 + (t+\epsilon)\xi^2 + \xi^3 = 0\,.\end{cases} \tag{4.10}$$

By solving system (4.10) with respect to $\xi^2 = \xi^2(\xi^1, t, \epsilon)$, $\xi^3 = \xi^3(\xi^1, t, \epsilon)$, and then by computing $\lim_{\epsilon\to 0}\xi^2$ and $\lim_{\epsilon\to 0}\xi^3$, we obtain

$$\xi^2 = -\frac{t\xi^1}{\sqrt{1+t^2}}\,, \quad \xi^3 = -\frac{\xi^1}{\sqrt{1+t^2}}\,,$$

i.e., the limit solution of (4.10) is

$$v(\ell^+_{\theta_2}(t)) = \left\langle \sqrt{1+t^2}\, D^{(2)}_1(\theta_2) - tD^{(2)}_2(\theta_2) - D^{(2)}_3(\theta_2) \right\rangle \subset L_{\theta_2}\,.$$

(4) Had we considered the family $\nu^-_{\theta_2}(\eta_2, \eta_3)$, we would have come to the line $v(\ell^-_{\theta_2}(t))$. As in the step (4) of the previous Section 4.1, the lines $v(\ell^\pm_{\theta_2}(t))$ describe the conic variety

$$\mathcal{V}_{\theta_1} : (z^1)^2 - (z^2)^2 - (z^3)^2 = 0\,. \tag{4.11}$$

By computing the cocharacteristic variety of the same equation (4.9), according to Definition 2.2, we obtain again (4.11). This is the last case, when the two objects coincide.

4.3. **The quadric contact cone structure of the equation** $u_{12} = 0$**.** The equation

$$\mathcal{E} : \{u_{12} = 0\}\,, \tag{4.12}$$

considered in this section, is degenerate in the sense that such is its symbol. Indeed, according to formula (1.5), the symbol of (4.12) is equal to $\eta_1\eta_2$, that is a degenerate quadratic form. Let us compute the contact cone structure of such an equation by following the steps described in Section 3.2.

(1) Let us fix a point $\theta_2 = \left(x^i_0, u_0, u^0_i, u^0_{11}, 0, u^0_{13}, u^0_{22}, u^0_{23}, u^0_{33}\right) \in \mathcal{E}$.

(2) Equation (3.9) reads $\eta_1\eta_2 = 0$, which gives either $\eta_1 = 0$ or $\eta_2 = 0$. Below, we work out the case $\eta_2 = 0$ as the case $\eta_1 = 0$ can be treated in the same way. We are going to use "+" to indicate the case when $\eta_2 = 0$ and "−" to indicate the case when $\eta_1 = 0$. The rank–one directions at θ_2 that are tangent to $\mathcal{E}$ are $\eta_1^2\partial_{u_{11}}\big|_{\theta_2} + \eta_1\eta_3\partial_{u_{13}}\big|_{\theta_2} + \eta_3^2\partial_{u_{33}}\big|_{\theta_2}$, so that, by letting $t = \eta_3/\eta_1$, we have

$$\ell^+_{\theta_2}(t) = \partial_{u_{11}}\big|_{\theta_2} + t\partial_{u_{13}}\big|_{\theta_2} + t^2\partial_{u_{33}}\big|_{\theta_2}\,.$$

(3) With the line $\ell^+_{\theta_2}(t)$ is associated the hyperplane

$$H(\ell^+_{\theta_2}(t)) = \left\langle -tD^{(2)}_1(\theta_2) + D^{(2)}_3(\theta_2)\,,\, D^{(2)}_2(\theta_2)\right\rangle \subset L_{\theta_2}\,,$$

i.e., all planes $H(\ell^+_{\theta_2}(t))$ contain the line $\langle D^{(2)}_2(\theta_2)\rangle$. Thus, the line $v(\ell^+_{\theta_2}(t))$ is independent of t:

$$v(\ell^+_{\theta_2}(t)) = v(\ell^+_{\theta_2}) = \left\langle D^{(2)}_2(\theta_2)\right\rangle. \tag{4.13}$$

(4) If we let vary the point θ_2 on the fiber $J^2_{\theta_1}$, the line (4.13) describes a 3-dimensional subspace of the contact hyperplane $\mathcal{C}_{\theta_1}$:

$$\mathcal{W}_{\theta_1} := \mathcal{V}^+_{\theta_1} = \left\langle D^{(1)}_2(\theta_1)\,,\, \partial_{u_2}\big|_{\theta_1}\,,\, \partial_{u_3}\big|_{\theta_1}\right\rangle.$$

Had we considered, in the above step (2), the case $\eta_1 = 0$, we would have gotten the subspace $\mathcal{W}_{\theta_1}^{\perp}$ of $\mathcal{C}_{\theta_1}$, that is the symplectic orthogonal to $\mathcal{W}_{\theta_1}$:

$$\mathcal{W}_{\theta_1}^{\perp} := \mathcal{V}_{\theta_1}^{-} = \left\langle D_1^{(1)}(\theta_1)\,,\, \partial_{u_1}\big|_{\theta_1}\,,\, \partial_{u_3}\big|_{\theta_1} \right\rangle.$$

We come to the conclusion that the cone structure associated with PDE (4.12) is a pair of mutually symplectic–orthogonal distributions:

$$\mathcal{V} : \theta_1 \in J^1 \to \mathcal{V}_{\theta_1} = \mathcal{W}_{\theta_1} \cup \mathcal{W}_{\theta_1}^{\perp}. \tag{4.14}$$

Now, the notion of a cocharacteristic variety and that of a contact cone structure begin to diverge: indeed, the cocharacteristic variety of the equation (4.12), according to Definition 2.2, is the degenerate quadric hypersurface $\{(z^3)^2 = 0\} = \{z^3 = 0\}$, which turns out to be the linear span of $\mathcal{W}_{\theta_1} \cup \mathcal{W}_{\theta_1}^{\perp}$.

4.4. **The quadric contact cone structure of the equation $u_{11} = 0$.** The reasoning to get to the contact cone structure associated with the equation $u_{11} = 0$ is the same as the one we employed in Section 4.3 and then we omit it: the result is the Lagrangian distribution

$$\mathcal{V} : \theta_1 \in J^1 \to \mathcal{V}_{\theta_1} = \left\langle D_1^{(1)}(\theta_1)\,,\, \partial_{u_2}\big|_{\theta_1}\,,\, \partial_{u_3}\big|_{\theta_1} \right\rangle. \tag{4.15}$$

It is worth adding that, in this case the cocharacteristic variety becomes the most degenerate as possible: indeed, the cocharacteristic variety of the equation $u_{11} = 0$, according to Definition 2.2, is the whole contact space $\mathcal{C}_{\theta_1}$.

4.5. **Relation between the contact cone structure and the characteristic variety of a Monge–Ampère equation.** All the examples worked out in the previous four subsections point towards the existence of a natural relationship between the proposed construction of a quadric cone structure associated with a Monge–Ampère equation, cf. (2.19), and their cocharacteristic variety (see Definition 2.2): this is captured by the following theorem.

Theorem 4.1. *Let $\mathcal{E}$ be a Monge–Ampère equation* (2.19) *and let $\theta_2 \in \mathcal{E}_{\theta_1}$, cf.* (2.15)*, be its regular point, see also Remark* 3.2*. Then the following are true.*

(1) *If the symbol of $\mathcal{E}$ is not degenerate at θ_2, then the contact cone structure of $\mathcal{E}$ at θ_1 is the cocharacteristic variety of $\mathcal{E}$ at θ_1.*
(2) *If the symbol of $\mathcal{E}$ has rank* 2 *at θ_2 and it is hyperbolic in this point, then its contact cone structure at θ_1 is the union $D \cup D^{\perp}$ of two symplectic-orthogonal* 3*-dimensional subspaces of $\mathcal{C}_{\theta_1}$ whereas*

the cocharacteristic variety of $\mathcal{E}$ at θ_1 describes the smallest linear subspace of $\mathcal{C}_{\theta_1}$ containing $D \cup D^{\perp}$.

(3) *If the symbol of $\mathcal{E}$ has rank 1 at θ_2, then the contact cone structure of $\mathcal{E}$ at θ_1 is a Lagrangian subspace of $\mathcal{C}_{\theta_1}$ and the cocharacteristic variety of $\mathcal{E}$ at θ_1 is trivial.*

In Section 9, we reformulate Theorem 4.1 over the field of complex numbers: the so–obtained Corollary 9.1 represents then a coarse proof of Theorem 4.1; indeed, in the complex case, the four examples above exhaust all possible isomorphism types of Monge–Ampère equations. A finer and as such complete proof can be easily obtained by modifying the signature in the given examples.

5. Reconstructing a 2^{nd} order PDE from a contact cone structure

Now, we try to reverse the above recipe, i.e., starting from an arbitrary contact cone structure $\mathcal{V}$, we propose two different methods of associating a 2^{nd} order PDE with $\mathcal{V}$: the reader will immediately recognize in the second a "degenerate version" of the first. Since there is plenty of contactomorphism types of contact cone structures (even considering only the quadratic ones), in the face of only four contactomorphism types of (symplectic) Monge–Ampère equations, a general "inverse recipe" would necessarily exceed the class of PDE under consideration. This is why we propose below only two versions: they will be just enough to reconstruct all Monge–Ampère equations.

5.1. **The case of a 5-dimensional contact cone structure.** Let $\mathcal{V}$ be a contact cone structure on J^1 and let us assume that $\dim(\mathcal{V}_{\theta_1}) = 5$ $\forall\theta_1 \in J^1$. Starting from $\mathcal{V}_{\theta_1}$, we will be constructing a distribution (not necessarily of constant rank) on $J^2_{\theta_1}$ by working out the following steps, that represent a sort of inverse procedure to the one described in Section 3.2.

(1) Let us consider $\theta_2 \equiv L_{\theta_2} \in J^2_{\theta_1}$ and set $\mathcal{V}_{\theta_2} := L_{\theta_2} \cap \mathcal{V}_{\theta_1}$: then, generically, $\dim \mathcal{V}_{\theta_2} = 2$.

(2) If $\dim \mathcal{V}_{\theta_2} = 2$, then to point θ_2 we can associate the set

$$H_{\theta_2} := \{\text{Hyperplanes of } L_{\theta_2} \text{ tangent to } \mathcal{V}_{\theta_2} \text{ along its generatrices}\}\,.$$

The set H_{θ_2} depends on one parameter, i.e., we have a 1–parametric family $H_{\theta_2}(t)$ of hyperplanes.

(3) Let $\ell_{\theta_2}(t)$ be the line of rank 1 corresponding to $H_{\theta_2}(t)$ via (3.5).

(4) Let $\mathcal{D}_{\theta_2}$ be the smallest linear subspace containing $\ell_{\theta_2}(t)\ \forall\, t$. Then the correspondence $\theta_2 \to \mathcal{D}_{\theta_2}$ defines a distribution $\mathcal{D}$ on $J^2_{\theta_1}$: its integral sub-manifolds will be sub-manifolds of $J^2_{\theta_1}$, i.e., fibers of PDEs.

By starting from the contact cone structure associated with a symplectic Monge–Ampère equation, the above procedure leads to a foliation of J^2 and then, each leaf of it will be a PDE.

5.1.1. *Foliation of PDEs associated with the contact cone structure* (4.8)*.* At the end of Section 4.1, we have seen that equations $\det\|u_{ij}\| = 1$ and $u_{11}+u_{22}u_{33}-u_{23}^2 = 0$ are contactomorphic; therefore, we can consider them, as well as their contact cone structures, as equivalent. In particular, in this section, we will be working with the contact cone structure (4.8) because the computations are easier.

Let us apply the scheme explained above at the beginning of Section 5.1, in order to construct the 2$^{\text{nd}}$ order PDEs associated with the contact cone structure $\mathcal{V}_{\theta_1}$ given by (4.8). We employ the coordinates (z^i, q_i) on $\mathcal{C}_{\theta_1}$ introduced in Notation 4.1.

(1) Let us fix

$$\theta_2 = \left(x_0^i, u_0, u_i^0, u_{ij}^0\right) = \left(\theta_1, u_{ij}^0\right) \in J^2_{\theta_1},$$

cf. (2.10), or, equivalently,

$$L_{\theta_2} = \langle D_i^{(1)}(\theta_1) + u_{ij}^0 \partial_{u_j}\big|_{\theta_1}\rangle_{i=1,2,3} = \left\langle D_i^{(2)}(\theta_2)\right\rangle_{i=1,2,3}.$$

Since L_{θ_2}, as a vector subspace of $\mathcal{C}_{\theta_1}$, is locally described by

$$q_i - u_{ij}^0 z^j = 0,$$

$$\mathcal{V}_{\theta_2} = \{q_i - u_{ij}^0 z^j = 0\,,\; z^1 q_1 - z^2 q_2 - z^3 q_3 = 0\}\,.$$

(2) Then, we have

$$H_{\theta_2} = \left\{\xi^i D_i^{(2)}(\theta_2) = \xi^i D_i^{(1)}(\theta_1) + \xi^i u_{ij}^0 \partial_{u_j}\big|_{\theta_1}\right\} \tag{5.1}$$

with ξ^i satisfying

$$\begin{aligned} &\bar q_i - u_{ij}^0 \bar z^j = 0\,, \quad \bar z^1 \bar q_1 - \bar z^2 \bar q_2 - \bar z^3 \bar q_3 = 0\,, \\ &\bar q_1 \xi^1 - \bar q_2 \xi^2 - \bar q_3 \xi^3 + \bar z^1 \xi^i u_{i1}^0 - \bar z^2 \xi^i u_{i2}^0 - \bar z^3 \xi^i u_{i3}^0 = 0\,. \end{aligned} \tag{5.2}$$

By a direct computation, from the first four equations of the system (5.2), we obtain

$$\bar z^3 = \frac{-u_{23}^0 \bar z^2 \pm \sqrt{u_{11}^0 u_{33}^0 (\bar z^1)^2 - u_{22}^0 u_{33}^0 (\bar z^2)^2 + (u_{23}^0)^2 (\bar z^2)^2}}{u_{33}^0} \tag{5.3}$$

$$=: \frac{-u^0_{23}\bar z^2 \pm \sqrt{A(\bar z^1,\bar z^2)}}{u^0_{33}}$$

and, assuming $u^0_{33}\neq 0$, the last equation of the system (5.2) yields

$$u^0_{11}u^0_{33}\bar z^1\xi^1 - u^0_{22}u^0_{33}\bar z^2\xi^2 + (u^0_{23})^2\bar z^2\xi^2 \mp \sqrt{A(\bar z^1,\bar z^2)}u^0_{23}\xi^2 \mp \sqrt{A(\bar z^1,\bar z^2)}u^0_{33}\xi^3 = 0.$$

Let us consider the case with the plus sign in (5.3). By setting $\bar z^1 = t$ and $\bar z^2 = 1$, equation above becomes

$$u^0_{11}u^0_{33}t\,\xi^1 - u^0_{22}u^0_{33}\xi^2 + (u^0_{23})^2\xi^2 - \sqrt{A(t)}\,u^0_{23}\xi^2 - \sqrt{A(t)}\,u^0_{33}\xi^3 = 0\,,$$

$A(t) := A(t,1)$, whose solution $(\xi^1(t),\xi^2(t),\xi^3(t))$, substituted in (5.1), gives $H_{\theta_2}(t)$ that we were looking for.

(3) By looking at (3.6)–(3.7), the rank–one line $\ell_{\theta_2}(t)$ corresponding to $H_{\theta_2}(t)$ is

$$\ell_{\theta_2}(t) = \Big\langle \sum_{i\le j}\eta_i\eta_j\partial_{u_{ij}}\big|_{\theta_2}\Big\rangle\,,$$

where

$$\eta_1 = u^0_{11}u^0_{33}t\,,\quad \eta_2 = -u^0_{22}u^0_{33} + (u^0_{23})^2 - \sqrt{A(t)}\,u^0_{23}\,,\quad \eta_3 = -\sqrt{A(t)}\,u^0_{33}\,.$$

(4) The smallest linear subspace $\mathcal{D}_{\theta_2}$ containing $\ell_{\theta_2}(t)$ for any t is

$$\mathcal{D}_{\theta_2} = \big\langle X_1|_{\theta_2}, X_2|_{\theta_2}, X_3|_{\theta_2}, X_4|_{\theta_2}, X_5|_{\theta_2}\big\rangle,$$

where the vector fields X_i on $J^2_{\theta_1}$ are:

$$\begin{aligned}
X_1 &= u_{11}u_{23}u_{33}\frac{\partial}{\partial u_{12}} + u_{11}u_{33}^2\frac{\partial}{\partial u_{13}}\,,\\
X_2 &= -2u_{22}u_{23}u_{33}\frac{\partial}{\partial u_{22}} + 2u_{23}^3\frac{\partial}{\partial u_{22}} - u_{22}u_{33}^2\frac{\partial}{\partial u_{23}} + u_{23}^2u_{33}\frac{\partial}{\partial u_{23}}\,,\\
X_3 &= u_{11}^2u_{33}^2\frac{\partial}{\partial u_{11}} + u_{11}u_{23}^2u_{33}\frac{\partial}{\partial u_{22}} + u_{11}u_{23}u_{33}^2\frac{\partial}{\partial u_{23}} + u_{11}u_{33}^3\frac{\partial}{\partial u_{33}}\,,\\
X_4 &= -u_{11}u_{22}u_{33}^2\frac{\partial}{\partial u_{12}} + u_{11}u_{23}^2u_{33}\frac{\partial}{\partial u_{12}}\,,\\
X_5 &= u_{22}^2u_{33}^2\frac{\partial}{\partial u_{22}} - 3u_{22}u_{23}^2u_{33}\frac{\partial}{\partial u_{22}} + 2u_{23}^4\frac{\partial}{\partial u_{22}} - u_{22}u_{23}u_{33}^2\frac{\partial}{\partial u_{23}}\\
&\quad + u_{23}^3u_{33}\frac{\partial}{\partial u_{23}} - u_{22}u_{33}^3\frac{\partial}{\partial u_{33}} + u_{23}^2u_{33}^2\frac{\partial}{\partial u_{33}}\,.
\end{aligned}$$

A direct computation shows that the vector distribution

$$\theta_2 \in J^2_{\theta_1} \to \mathcal{D}_{\theta_2} \subseteq T_{\theta_2}J^2_{\theta_1}$$

is integrable, so it admits a 1-parametric family of integral sub-manifolds locally given by $f=0$, where f is the general solution to the system

$$\{X_1(f)=X_2(f)=X_3(f)=X_4(f)=X_5(f)=0\},$$

which is

$$f=K_1u_{11}+K_2(u_{22}u_{33}-u_{23}^2)\,,\quad K_i\in\mathbb{R}\,,$$

so that the PDEs we were looking for are given by

$$K_1u_{11}+K_2(u_{22}u_{33}-u_{23}^2)=0\,.$$

5.1.2. *Foliation of PDEs associated with the contact cone structure* (4.11). We consider now the cone structure (4.11) and we perform the same steps as in Section 5.1.1 above. We report only the final result of the computations, i.e., the distribution $\mathcal{D}$ on $J^2_{\theta_1}$ constructed starting from (4.11):

$$\mathcal{D}=\langle\partial_{u_{11}}+\partial_{u_{22}}\,,\,\partial_{u_{11}}+\partial_{u_{33}}\,,\,\partial_{u_{12}}\,,\,\partial_{u_{13}}\,,\,\partial_{u_{23}}\rangle\,.$$

The integral manifolds of $\mathcal{D}$ are described by

$$u_{11}-u_{22}-u_{33}=K\,,\quad K\in\mathbb{R}\,.$$

5.2. **The case of a degenerate 3-dimensional contact cone structure.** In step (3) of the recipe at the beginning of Section 5.1, we have seen that the hyperplanes $H_{\theta_2}(t)$ are tangent to $\mathcal{V}_{\theta_2}$ along its generatrices. If we do not assume particular properties of $\mathcal{V}$, one can have a unique generatrix of $\mathcal{V}_{\theta_2}$. This happens, for instance, in the case when $\mathcal{V}$ is a 3-dimensional vector distribution on J^1, i.e., a particular degenerate contact cone structure on J^1, that we study in details below. We give here a similar scheme to the one given in Section 5.1, that allows us to define a (non-constant rank) distribution $\mathcal{D}$ on $J^2_{\theta_1}$ starting from $\mathcal{V}_{\theta_1}$.

(1) Let us consider $\theta_2\equiv L_{\theta_2}\in J^2_{\theta_1}$ and let $\mathcal{V}_{\theta_2}:=L_{\theta_2}\cap\mathcal{V}_{\theta_1}$.
(2) To the point θ_2, we associate the set

$$H_{\theta_2}:=\{\text{Hyperplanes of } L_{\theta_2} \text{ containing } \mathcal{V}_{\theta_2}\}\,.$$

(3) To each element $h\in H_{\theta_2}$, we associate the rank–one line $\ell_{\theta_2}(h)$ via (3.5).
(4) Let $\mathcal{D}_{\theta_2}$ be the smallest linear subspace containing $\ell_{\theta_2}(h)\;\forall\, h\in H_{\theta_2}$. Then the correspondence $\theta_2\to\mathcal{D}_{\theta_2}$ defines a (non-constant rank) distribution $\mathcal{D}$ on $J^2_{\theta_1}$: its integral sub-manifolds will be sub-manifolds of $J^2_{\theta_1}$, i.e., fibers of PDEs.

5.2.1. *PDEs associated with the contact cone structure* (4.14). We work out the above steps in the case of the contact cone structure (4.14). A key remark is that L_{θ_2} intersects $\mathcal{W}_{\theta_1}$ if and only if it intersects also $\mathcal{W}_{\theta_1}^{\perp}$: in particular, it is enough to study the intersection $\mathcal{V}_{\theta_2}^{+} = L_{\theta_2} \cap \mathcal{W}_{\theta_1}$, since for $\mathcal{V}_{\theta_2}^{-} = L_{\theta_2} \cap \mathcal{W}_{\theta_1}^{\perp}$ the reasonings are the same.

(1) Let us fix $\theta_2 = \left(x_0^i, u_0, u_i^0, u_{ij}^0\right) = \left(\theta_1, u_{ij}^0\right) \in J_{\theta_1}^2$, cf. (2.10), or, equivalently, $L_{\theta_2} = \langle D_i^{(1)}(\theta_1) + u_{ij}^0 \partial_{u_j}\big|_{\theta_1} \rangle_{i=1,2,3} = \left\langle D_i^{(2)}(\theta_2) \right\rangle_{i=1,2,3}$. The dimension of $\mathcal{V}_{\theta_2}^{+} = L_{\theta_2} \cap \mathcal{W}_{\theta_1}$ is the corank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & u_{11}^0 & u_{12}^0 & u_{13}^0 \\ 0 & 1 & 0 & u_{12}^0 & u_{22}^0 & u_{23}^0 \\ 0 & 0 & 1 & u_{13}^0 & u_{23}^0 & u_{33}^0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5.4}$$

The rank of matrix (5.4) is 6 if and only if $u_{12}^0 \neq 0$ and it is 5 otherwise. No other cases can occur.

(2) If the rank of matrix (5.4) is 6, then $\dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}) = 0 = \dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}^{\perp})$ and H_{θ_2} consists of all the hyperplanes of L_{θ_2}. If the rank of matrix (5.4) is 5, then $\dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}) = 1$ and $H_{\theta_2}^{+}$ consists of the hyperplanes of L_{θ_2} containing the line $\mathcal{V}_{\theta_2}^{+} = L_{\theta_2} \cap \mathcal{W}_{\theta_1}$.

(3) In the case $\dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}) = 0$ $(= \dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}^{\perp}))$, the set $\{\ell_{\theta_2}(h)\}$ consists of all the rank–one lines of $J_{\theta_1}^2$.

In the case $\dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}) = 1$, we have that

$$\mathcal{V}_{\theta_2}^{+} = \left\langle D_2^{(2)}(\theta_2) \right\rangle = \left\langle D_2^{(1)}(\theta_2) + u_{22}^0 \partial_{u_2}\big|_{\theta_2} + u_{23}^0 \partial_{u_3}\big|_{\theta_2} \right\rangle.$$

In view of the previous point, we have that $H_{\theta_2}^{+} = \{H_{\theta_2}^{+}(t)\}$, where

$$H_{\theta_2}^{+}(t) = \left\langle -tD_1^{(2)}(\theta_2) + D_3^{(2)}(\theta_2)\,,\, D_2^{(2)}(\theta_2) \right\rangle,$$

so that the set $\{\ell_{\theta_2}(h)\}$ equals $\{\ell_{\theta_2}^{+}(t) \mid t \in \mathbb{R}\}$, where

$$\ell_{\theta_2}^{+}(t) = \left\langle \partial_{u_{11}}\big|_{\theta_2} + t\partial_{u_{13}}\big|_{\theta_2} + t^2 \partial_{u_{33}}\big|_{\theta_2} \right\rangle, \tag{5.5}$$

cf. (3.7). Had we considered the case $\dim(L_{\theta_2} \cap \mathcal{W}_{\theta_1}^{\perp}) = 1$, we would have gotten

$$\ell_{\theta_2}^{-}(t) = \left\langle \partial_{u_{22}}\big|_{\theta_2} + t\partial_{u_{23}}\big|_{\theta_2} + t^2 \partial_{u_{33}}\big|_{\theta_2} \right\rangle. \tag{5.6}$$

(4) Taking into account that the smallest linear space containing (5.5) and (5.6), for any $t \in \mathbb{R}$ is $\langle \partial_{u_{11}}|_{\theta_2}, \partial_{u_{13}}|_{\theta_2}, \partial_{u_{22}}|_{\theta_2}, \partial_{u_{23}}|_{\theta_2}, \partial_{u_{33}}|_{\theta_2} \rangle$, combining above points $(1)-(3)$, the distribution $\mathcal{D}$ on $J^2_{\theta_1}$ turns out to be

$$\mathcal{D} = \langle \partial_{u_{11}}\, ,\, u_{12}\partial_{u_{12}}\, ,\, \partial_{u_{13}}\, ,\, \partial_{u_{22}}\, ,\, \partial_{u_{23}}\, ,\, \partial_{u_{33}} \rangle \, .$$

The only 5–dimensional integral sub-manifold of $\mathcal{D}$ is described by $u_{12} = 0$.

5.2.2. *PDEs associated with the contact cone structure* (4.15). In the case of (4.15), the computations to get to the distribution $\mathcal{D}$ of step (4) of the scheme given at the beginning of Section 5.2 closely follow those of Section 5.2.1, so we omit them. The result is the distribution

$$\mathcal{D} = \langle u_{11}\partial_{u_{11}}\, ,\, u_{11}\partial_{u_{12}}\, ,\, u_{11}\partial_{u_{13}}\, ,\, \partial_{u_{22}}\, ,\, \partial_{u_{23}}\, ,\, \partial_{u_{33}} \rangle \, .$$

Note that $\dim D_{\theta_2} = 6$ if $u_{11} \neq 0$ and that the only 5–dimensional integral sub-manifold of $\mathcal{D}$ is described by $u_{11} = 0$.

6. THE SPACE $\mathbb{P}\Lambda^3_0(\mathcal{C})$ OF SYMPLECTIC 3D MONGE–AMPÈRE EQUATIONS

Warning. *From now on, we will be working over the field of complex numbers; we retain the symbol $\mathcal{C}$ for a 6–dimensional* (*complex*) *linear symplectic space but we no longer make a distinction between "total derivatives" and "vertical vectors", see* (2.12)*: we will have a generic bi–Lagrangian splitting of $\mathcal{C}$ instead.*

By regarding $\mathcal{C}$ as the contact plane $\mathcal{C}_{\theta_1}$ at a generic point $\theta_1 \in J^1$, and by replacing $\mathbb{C}$ with $\mathbb{R}$, the reader will immediately see how the constructions obtained below mirror analogous results in the real–differentiable case of symplectic Monge–Ampère equations and contact cone structures; with only one major caveat: structures that are equivalent over $\mathbb{C}$ need not to be equivalent over $\mathbb{R}$.

6.1. **The symplectic space $\mathcal{C}$.** We define $\mathcal{C}$ by fixing a subspace $V \subset \mathcal{C}$, such that $\mathcal{C} := V \oplus V^*$, and the symplectic form ω corresponds to $(0, \mathrm{id}_V, 0)$ in the splitting

$$\Lambda^2(V \oplus V^*) = \Lambda^2(V) \oplus \mathrm{End}(V) \oplus \Lambda^2(V^*)\, .$$

A choice of a basis of V, and its dual in V^*, for instance,

$$V = \langle e_1, e_2, e_3 \rangle\, , \quad V^* = \langle \epsilon^1, \epsilon^2, \epsilon^3 \rangle\, , \quad \epsilon^i(e_j) = \delta^i_j\, , \tag{6.1}$$

leads to the basis

$$e_1,\ e_2,\ e_3,\ e_4 := \epsilon^1,\ e_5 := \epsilon^2,\ e_6 := \epsilon^3 \tag{6.2}$$

of $\mathcal{C}$, such that the symplectic form ω looks like $\omega = \epsilon^i \wedge e_i \in \Lambda^2(\mathcal{C}^*)$. At risk of sounding redundant, we set

$$x^1 := \epsilon^1,\ x^2 := \epsilon^2,\ x^3 := \epsilon^3\ , x^4 := e_1\ , x^5 := e_2\ , x^6 := e_3 \tag{6.3}$$

and we regard these x^i's as linear functions on $\mathcal{C}$, that is, as basis elements of $\mathcal{C}^*$; the usefulness of such a choice will become clearer in the sequel. Observe that, by construction, $x^j(e_i) = \delta_i^j$ and

$$\omega = x^1 \wedge x^4 + x^2 \wedge x^5 + x^3 \wedge x^6\,. \tag{6.4}$$

The isomorphism $\mathcal{C} \simeq \mathcal{C}^*$ given by (6.4), acts on the basis elements $e_1, \ldots, e_6$ of $\mathcal{C}$ as follows:

$$e_1 \to x^4\,,\ e_2 \to x^5\,,\ e_3 \to x^6\,,\ e_4 \to -x^1\,,\ e_5 \to -x^2\,,\ e_6 \to -x^3\,. \tag{6.5}$$

Remark 6.1. If we regard $\mathcal{C}$ as $\mathcal{C}_{\theta_1}$, then $e_i \leftrightarrow D_i^{(1)}\big|_{\theta_1}$ and $\epsilon^i \leftrightarrow \partial_{u_i}|_{\theta_1}$. Therefore, each x^i of (6.3) corresponds precisely to $dy^i\big|_{\mathcal{C}_{\theta_1}}$ that appears in (2.17), $i = 1, \ldots, 6$: it follows that $\epsilon^i = z^i$ and $e_i = q_i$, $i = 1, 2, 3$, where z^i and q_i are as in Notation 4.1.

Remark 6.2. The assignment

$$(x^1, x^2, x^3, x^4, x^5, x^6) \to (x^4, x^5, x^6, -x^1, -x^2, -x^3) \tag{6.6}$$

defines a transformation of $\mathcal{C}$ that preserves the symplectic form (6.4). By borrowing the terminology from Example 2.2, we call (6.6) a *total Legendre transformation*: indeed, *partial* Legendre transformations can be defined as well, along the lines of (2.13).

6.2. **The moment map identifying $\mathfrak{sp}(\mathcal{C})$ with $S^2(\mathcal{C}^*)$.** The Lie group $\mathsf{Sp}(\mathcal{C})$ of symplectomorphisms of $\mathcal{C}$ is defined as usual:

$$\mathsf{Sp}(\mathcal{C}) := \{g \in \mathsf{GL}(\mathcal{C}) \mid g^*(\omega) = \omega\}\,.$$

For any $X \in \mathfrak{gl}(\mathcal{C}) = \mathrm{End}(\mathcal{C})$ the following contraction of X with ω, namely,

$$Q_X(a, b) := \omega(X(a), b)\,, \quad \forall a, b \in \mathcal{C}\,, \tag{6.7}$$

defines a quadratic form Q_X on $\mathcal{C}$; this allows for a transparent description of the Lie algebra

$$\mathfrak{sp}(\mathcal{C}) := \{X \in \mathfrak{gl}(\mathcal{C}) \mid Q_X \text{ is symmetric}\}$$

of the group $\mathsf{Sp}(\mathcal{C})$.

By regarding $\mathfrak{gl}(\mathcal{C}) = \mathcal{C}^* \otimes \mathcal{C}$ as the linear part of the (graded) algebra $\mathfrak{X}(\mathcal{C})$ of polynomial vector fields on the (linear) symplectic manifold $\mathcal{C}$, the natural embedding

$$j : \mathfrak{sp}(\mathcal{C}) \longrightarrow \mathfrak{X}(\mathcal{C}) \tag{6.8}$$

realizes an element $X \in \mathfrak{sp}(\mathcal{C})$ as a (linear) vector field $j(X)$ on $\mathcal{C}$; the fact that $X \in \mathfrak{sp}(\mathcal{C})$ translates into $j(X)$ being a symplectic, even Hamiltonian, vector field. It makes then sense to consider the associated *moment map*, that is the $\mathsf{Sp}(\mathcal{C})$–equivariant map $\mu : \mathcal{C} \longrightarrow \mathfrak{sp}(\mathcal{C})^*$ unambiguously defined by

$$d\langle \mu, X\rangle = j(X) \lrcorner\, \omega\,, \quad \forall X \in \mathfrak{sp}(\mathcal{C})\,. \tag{6.9}$$

It follows immediately that

$$\mu(a) : \mathfrak{sp}(\mathcal{C}) \quad \longrightarrow \mathbb{C}\,, \quad X \longmapsto \quad Q_X(a,a)\,,$$

for any $a \in \mathcal{C}$. Formula (6.9) tells precisely that the *linear* map j associating with any element of $\mathfrak{sp}(\mathcal{C})$ its Hamiltonian vector field $j(X)$ arises as the differential of the *quadratic* map μ; as such, the latter factors through the Veronese embedding v_2, i.e., diagram

$$\begin{tikzcd} S^2(\mathcal{C}) \arrow[r, "\phi^*"] & \mathfrak{sp}(\mathcal{C})^* \\ \mathcal{C} \arrow[u, "v_2"] \arrow[ur, "\mu"'] & \end{tikzcd} \tag{6.10}$$

commutes. We labeled the upper arrow by ϕ^*, because it is precisely the dual isomorphism to

$$\phi : \mathfrak{sp}(\mathcal{C}) \longrightarrow S^2(\mathcal{C}^*)\,, \quad X \longmapsto Q_X\,. \tag{6.11}$$

This is just another way to prove that $\mathfrak{sp}(\mathcal{C})$ is naturally identified with $S^2(\mathcal{C}^*)$: to obtain this well–known identification, we employed the moment map of the natural $\mathsf{Sp}(\mathcal{C})$–action on $\mathcal{C}$, whose quadratic character is responsible for the appearance of the second symmetric power of $\mathcal{C}^*$; we stressed this elementary phenomenon here, because it will reappear later in Section 8 when we will be performing an analogous construction on the space of Monge–Ampère equations.

6.2.1. *The identification in matrix form.* In view of the obvious decomposition

$$\mathfrak{gl}(\mathcal{C}) = \mathfrak{gl}(V) \oplus V^{\otimes 2} \oplus V^{*\otimes 2} \oplus \mathfrak{gl}(V^*)\,,$$

any element $X \in \mathfrak{gl}(\mathcal{C})$ can be presented, by employing bases (6.1), as

$$X = S_i^j \epsilon^i \otimes e_j + R^{ij} e_i \otimes e_j + T_{ij} \epsilon^i \otimes \epsilon^j + U_j^i e_i \otimes \epsilon^j\,,$$

that is

$$X = \begin{pmatrix} S & R \\ T & U \end{pmatrix}.$$

Easy computations shows that

$$Q_{\epsilon^i\otimes e_j} = e_j\otimes\epsilon^i\,,\quad Q_{e_i\otimes e_j} = -e_j\otimes e_i\,,\quad Q_{\epsilon^i\otimes\epsilon^j} = \epsilon^j\otimes\epsilon^i\,,\quad Q_{e_i\otimes\epsilon^j} = -\epsilon^j\otimes e_i\,,$$

cf. (6.7), whence

$$Q_X = S_i^j e_j\otimes\epsilon^i - R^{ij}e_j\otimes e_i + T_{ij}\epsilon^j\otimes\epsilon^i - U_j^i\epsilon^j\otimes e_i \tag{6.12}$$

is symmetric if and only if both R and T are symmetric and, moreover, $U = -S^t$; these are the conditions that single out the 21–dimensional Lie sub–algebra

$$\mathfrak{sp}(\mathcal{C}) = \left\{ \begin{pmatrix} S & R \\ T & -S^t \end{pmatrix} \mid R = R^t\,, T = T^t \right\}$$

of $\mathfrak{gl}(\mathcal{C})$. Therefore, if $X\in\mathfrak{sp}(\mathcal{C})$, then (6.12) reads

$$Q_X = T_{ij}\epsilon^i\epsilon^j + S_i^j\epsilon^i e_j - R^{ij}e_ie_j \in S^2(\mathcal{C}^*)$$

and (6.11) reads

$$\phi : \begin{pmatrix} S & R \\ T & -S^t \end{pmatrix} \longmapsto \begin{pmatrix} T & S \\ S^t & -R \end{pmatrix}.$$

It is easy to see that ϕ is a $\mathsf{Sp}(\mathcal{C})$–module isomorphism: indeed, for all $g\in\mathsf{Sp}(\mathcal{C})$, we have

$$\begin{aligned} g^*(Q_X)(a,b) &= Q_X(g\cdot a, g\cdot b)) = \omega(X(g\cdot a), g\cdot b)) = \omega(g\cdot X(a), g\cdot b)) \\ &= g^*(\omega)(X(a),b) = \omega(X(a),b) = Q_X(a,b)\,. \end{aligned}$$

6.3. **The space $\Lambda^3(\mathcal{C})$ and its subspace $\Lambda^3_0(\mathcal{C})$.** It is now convenient to introduce the notation

$$e_{i_1i_2\cdots i_k} := e_{i_1}\wedge\cdots\wedge e_{i_k}\,,\quad \forall i_1,\ldots,i_k = 1,2,\ldots,6\,,\quad k = 2,3,\ldots,6\,,$$

with the obvious identifications

$$e_{i_1i_2\cdots i_k} = \mathrm{sign}(\sigma)e_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)}\,.$$

By defining the dual symbols in an analogous way, i.e., $x^{i_1i_2\cdots i_k} := x^{i_1}\wedge\cdots\wedge x^{i_k}$, we see that, for example, the symplectic form (6.4) $\omega = x^{14}+x^{25}+x^{36}$. We warn the reader that the juxtaposition of symbols, e.g., $e_{123}e_{456}$ denotes the *symmetric* product and not the anti–symmetric one, i.e., e_{123456}.

Modulo these identifications, we have exactly 20 symbols, that correspond to as many generators of the 20–dimensional space $\Lambda^3(\mathcal{C})$; the latter is equipped with a naturally defined $\Lambda^6(\mathcal{C})$–valued symplectic form Ω:

$$\Omega : \Lambda^3(\mathcal{C})\times\Lambda^3(\mathcal{C}) \longrightarrow \Lambda^6(\mathcal{C})\,,\quad (\alpha,\beta)\longrightarrow\alpha\wedge\beta\,. \tag{6.13}$$

Skipping the twisting factor e_{123456}, the symplectic form Ω looks like

$$\Omega = x^{123}\wedge x^{456} - \begin{pmatrix} x^{423} & x^{143} & x^{124} \\ x^{523} & x^{153} & x^{125} \\ x^{623} & x^{163} & x^{126} \end{pmatrix} \wedge \begin{pmatrix} x^{156} & x^{256} & x^{356} \\ x^{416} & x^{426} & x^{436} \\ x^{451} & x^{452} & x^{453} \end{pmatrix} \in \Lambda^2(\Lambda^3(\mathcal{C}^*)),$$

having understood (by borrowing the notation from [26, Section 2.2]) the wedge product of the matrices above in the following way:

$$\|a^{ij}\| \wedge \|b^{ij}\| := \sum_{i,j} a^{ij}\wedge b^{ij}\,.$$

By linearity with respect to the wedge product, the natural $\mathsf{Sp}(\mathcal{C})$–action extends to the whole exterior algebra $\Lambda^\bullet(\mathcal{C})$: the resulting $\mathsf{Sp}(\mathcal{C})$–module $\Lambda^3(\mathcal{C})$ is not, however, irreducible since it contains the space of 3–forms that are multiple of ω, which is a copy of the 6–dimensional fundamental representation. The remaining 14–dimensional constituent, henceforth denoted by $\Lambda^3_0(\mathcal{C})$, is irreducible and can be described as follows.

Let $i_\omega : \Lambda^3(\mathcal{C}) \longrightarrow \mathcal{C}$ denote the insertion of ω and $m_{\omega^{-1}}$ the right multiplication by $\omega^{-1}\in\Lambda^2(\mathcal{C})$, i.e., the embedding

$$m_{\omega^{-1}} : \mathcal{C} \quad\longrightarrow \Lambda^3(\mathcal{C})\,, \quad e \longmapsto e\wedge\omega^{-1}\,.$$

Since $i_\omega(\omega^{-1}) = 1$, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "m_{\omega^{-1}}"] \arrow[dr, equal] & \Lambda^3(\mathcal{C}) \arrow[d, "i_\omega"] \\ & \mathcal{C} \end{tikzcd}$$

From now on, our main concern will be the 14–dimensional space of 3–forms

$$\Lambda^3_0(\mathcal{C}) := \ker i_\omega\,,$$

which, in the real–differentiable setting and up to the natural identification

$$\Lambda^3(\mathcal{C}) \simeq \Lambda^3(\mathcal{C}^*) \quad \text{via } \omega,$$

is the space of "effective 3–forms" mentioned in Section 2.4 above.

Using the above–defined coordinates (x^{123}, X, Y, x^{456}) on $\Lambda^3(\mathcal{C})$, where

$$X = \begin{pmatrix} x^{423} & x^{143} & x^{124} \\ x^{523} & x^{153} & x^{125} \\ x^{623} & x^{163} & x^{126} \end{pmatrix}, \; Y = \begin{pmatrix} x^{156} & x^{416} & x^{451} \\ x^{256} & x^{426} & x^{452} \\ x^{356} & x^{436} & x^{453} \end{pmatrix}, \tag{6.14}$$

we see that an element of $\Lambda^3(\mathcal{C})$ belongs to $\Lambda^3_0(\mathcal{C})$ if and only if equalities $X = X^t$ and $Y = Y^t$ hold on that element; in a similar way, the symplectic form Ω descends to $\Lambda^3_0(\mathcal{C})$ (therefore, we keep using the same symbol).

Definition 6.1. *The projectivization* $\mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$ *of the 14–dimensional irreducible representation* $\Lambda^3_0(\mathcal{C}^*)$ *of* $\mathsf{Sp}(\mathcal{C})$ *is the* space parametrizing 3D symplectic Monge–Ampère equations.

Remark 6.3. This sudden switch from $\mathcal{C}$ to $\mathcal{C}^*$ in Definition 6.1 will simplify matching it with the previously given one, as we shall see below; from a representation–theoretical standpoint, however, there is no difference, since the $\mathsf{Sp}(\mathcal{C})$–modules $\Lambda^3_0(\mathcal{C})$ and $\Lambda^3_0(\mathcal{C}^*)$ are isomorphic: a distinguished isomorphism, which in coordinates is given by (6.5), descends from the symplectic form (6.13). In other words, both the spaces $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$ and $\mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$ can be taken as the parametrizing space of 3D symplectic Monge–Ampère equations.

To see how above Definition 6.1 matches with the definition given above of a Monge–Ampère equation (see Definition 2.1) it is enough to take an element $[\eta]\in\mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$ and associate with it the *hyperplane section*

$$\mathcal{E}_\eta := \mathbb{P}(\ker\eta)\cap \mathrm{LGr}(3,\mathcal{C})\,. \tag{6.15}$$

Then it is enough to recall that $\mathrm{LGr}(3,\mathcal{C})$ is identified with $J^2_{\theta_1}$ and that a symplectic 2^{nd} order PDE is a trivial bundle over J^1, see Section 2.3; to see how $\mathcal{E}_\eta$ looks like in coordinates, let us write down η as a linear combination

$$\eta = \eta_{123}x^{123} + \mathrm{tr}(B_\eta X) + \mathrm{tr}(C_\eta Y) + \eta_{456}x^{456}\,, \tag{6.16}$$

where B_η and C_η are 3×3 matrices, cf. also (2.19): then, $\mathcal{E}_\eta=\{F_\eta=0\}$, with

$$F_\eta(u_{ij}) = \eta_{123} + \mathrm{tr}(B_\eta U) + \mathrm{tr}(C_\eta U^\sharp) + \eta_{456}\det U\,, \tag{6.17}$$

and $U=\|u_{ij}\|$. Observe that F_η depends upon η, whereas $\mathcal{E}_\eta$ only upon $[\eta]$.

Below, we give an example that will be useful later on.

Example 6.1. The element $e_{423}=e_4\wedge e_2\wedge e_3\in\Lambda^3_0(\mathcal{C})$ can be regarded, via (6.5), as the 3–form $\eta=-x^{156}=-x^1\wedge x^5\wedge x^6\in\Lambda^3_0(\mathcal{C}^*)$ on $\mathcal{C}$, see also Remark 6.3: in the coordinate representation (6.16) all the coefficients of η are equal to zero, save for the $(1,1)$–entry of the matrix C_η, which is equal to -1: therefore, dropping the negligible sign, formula (6.17) leads to the Monge-Ampère equation

$$F_\eta(u_{ij}) = u^\sharp_{11} = u_{22}u_{33}-u^2_{23} = 0\,. \tag{6.18}$$

It is worth stressing that transformation (6.6) sends $-x^{156}$ into x^{423} and that the Monge-Ampère equation associated with $\eta=x^{423}$ is

$$F_\eta(u_{ij}) = u_{11} = 0\,,$$

which turns out to be equivalent to the Monge-Ampère equation (6.18).

Having interpreted symplectic Monge–Ampère equations as hyperplane sections of the Lagrangian Grassmannian $\mathrm{LGr}(3,\mathcal{C})$, it is natural to expect that the cocharacteristic variety (see above Definition 2.2) be a geometric feature of the hyperplane section itself that can be computed by means of algebraic manipulations on η: this will be shown in the last Section 9, after a two–section hiatus. In the next Section 7, we find a list of normal forms of quadratic forms on $\mathcal{C}$ with respect to $\mathsf{Sp}(\mathcal{C})$ and then, in Section 8, we show which of these forms come, via the Hitchin moment map, from the four isomorphism classes of Monge–Ampère equations. At the very end of Section 8 it is shown that the KLR contraction map, see (2.20), is equivalent to the Hitchin moment map (Theorem 8.1).

representative	coordinate expression	type	dim
$q_{[6]}$	$\epsilon^1 e_2+\epsilon^2 e_3+e_3^2$	nil	18
$q^{(111)}$	$\lambda\epsilon^1 e_1+\mu\epsilon^2 e_2+\nu\epsilon^3 e_3$	ss	18
$q^{(21)}+\phi(X_{h_1-h_2})$	$\mu(\epsilon^1 e_1+\epsilon^2 e_2)+\nu\epsilon^3 e_3+\epsilon^1 e_2$	mix	18
$q^{(11)}+\phi(X_{-2h_1})$	$\mu\epsilon^2 e_2+\nu\epsilon^3 e_3+(\epsilon^1)^2$	mix	18
$q^{(2)}+\phi(X_{h_2-h_3}+X_{-2h_1})$	$\nu(\epsilon^2 e_2+\epsilon^3 e_3)+\epsilon^2 e_3+(\epsilon^1)^2$	mix	18
$q^{(3)}+\phi(X_{h_1-h_2}+X_{h_2-h_3})$	$\nu(\epsilon^1 e_1+\epsilon^2 e_2+\epsilon^3 e_3)+\epsilon^1 e_2+\epsilon^2 e_3$	mix	18
$q^{(1)}+\phi(X_{h_1-h_2}-X_{2h_2})$	$\nu\epsilon^3 e_3+\epsilon^1 e_2+e_2^2$	mix	18
$q_{[4,2]}$	$\epsilon^1 e_3+e_2^2+e_3^2$	nil	16
$q^{(21)}$	$\mu(\epsilon^1 e_1+\epsilon^2 e_2)+\nu\epsilon^3 e_3$	ss	16
$q^{(11)}$	$\mu\epsilon^2 e_2+\nu\epsilon^3 e_3$	ss	16
$q^{(2)}+\phi(X_{h_2-h_3})$	$\nu(\epsilon^2 e_2+\epsilon^3 e_3)+\epsilon^2 e_3$	mix	16
$q^{(2)}+\phi(X_{-2h_1})$	$\nu(\epsilon^2 e_2+\epsilon^3 e_3)+(\epsilon^1)^2$	mix	16
$q^{(3)}+\phi(X_{h_1-h_2})$	$\nu(\epsilon^1 e_1+\epsilon^2 e_2+\epsilon^3 e_3)+\epsilon^1 e_2$	mix	16
$q^{(1)}+\phi(-\frac{1}{2}X_{h_1+h_2})$	$\nu\epsilon^3 e_3+e_1 e_2$	mix	16
$q_{[4,1^2]}$	$\epsilon^1 e_2+e_2^2$	nil	14
$q_{[3^2]}$	$\epsilon^1 e_3+e_2 e_3$	nil	14
$q^{(2)}$	$\nu(\epsilon^2 e_2+\epsilon^3 e_3)$	ss	14
$q^{(1)}+\phi(-X_{2h_1})$	$\nu\epsilon^3 e_3+e_1^2$	mix	14
$q_{[2^3]}$	$e_1^2+e_2^2+e_3^2$	nil	12
$q^{(3)}$	$\nu(\epsilon^1 e_1+\epsilon^2 e_2+\epsilon^3 e_3)$	ss	12
$q_{[2^2,1^2]}$	$e_1^2+e_2^2$	nil	10
$q^{(1)}$	$\nu\epsilon^3 e_3$	ss	10
$q_{[2,1^4]}$	e_1^2	nil	6

7. NORMAL FORMS OF QUADRATIC FORMS ON $\mathcal{C}$ WITH RESPECT TO $\mathsf{Sp}(\mathcal{C})$

In this section, we work out the classification of all $\mathsf{Sp}(\mathcal{C})$–orbits in $S^2(\mathcal{C}^*)$. The main result, i.e., a list of normal forms, given in the basis (6.2) of $\mathcal{C}$, can immediately be seen in Section 7.1 below: as the table shows, there are three qualitative different types of quadratic forms, which we called *nilpotent* (discussed in Section 7.3), *semisimple* (see Section 7.4) and *mixed* (see Section 7.5); basic facts about the root structure of $\mathsf{Sp}(6)$ are collected in Section 7.2

7.1. **The complete classification.** The representatives of all nonzero $\mathsf{Sp}(\mathcal{C})$ –orbits in $S^2(\mathcal{C}^*)$ are listed in the table: they are all non–equivalent, up to a sign change of the coefficients $(\lambda, \mu, \nu) \in \mathbb{C}^3$ and, if possible, a their permutation. We stress that the basis elements appearing in the column labeled "coordinate expression" are to be regarded as elements of $\mathcal{C}^*$: a more homogeneous, though less explanatory coordinate representation, involving only the x^i's, may be obtained by means of the substitution (6.5).

7.2. **Representation–theoretic preliminaries.** With the standard choice of a Cartan subalgebra,

$$\mathfrak{h} := \{\mathrm{diag}(\lambda, \mu, \nu, -\lambda, -\mu, -\nu) \mid \lambda, \mu, \nu \in \mathbb{C}\}, \tag{7.1}$$

the 3–dimensional, eighteen–element root system Φ of $\mathfrak{sp}(\mathcal{C})$ is generated by the simple roots $\Delta := \{h_1 - h_2, h_2 - h_3, 2h_3\}$, where $2h_3$ is the long one: by $h_i \in \mathfrak{h}^*$, we mean the linear operator reading off the i^{th} entry of a diagonal matrix $H \in \mathfrak{h}$, i.e.,

$$h_1(H) := \lambda, \quad h_2(H) := \mu, \quad h_3(H) := \nu.$$

The corresponding set of positive roots will be then

$$\Phi^+ = \Delta \cup \{h_1 - h_3, 2h_1, 2h_2, h_1 + h_2, h_2 + h_3, h_1 + h_3\},$$

where $\Phi = \Phi^+ \cup \Phi^-$, with $\Phi^- = -\Phi^+$, see Figure (1); hence, the root space decomposition of $\mathfrak{sp}(\mathcal{C})$ is

$$\mathfrak{sp}(\mathcal{C}) = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha .$$

Remark 7.1. We let $E(i,j) \in \mathfrak{gl}(\mathcal{C})$ be the (i,j)–elementary matrix and we pick a generator $X_\alpha \in \mathfrak{sp}(\mathcal{C})$ of the (one–dimensional) root space $\mathfrak{g}_\alpha$, for all $\alpha \in \Phi$; it is then easy to see that:

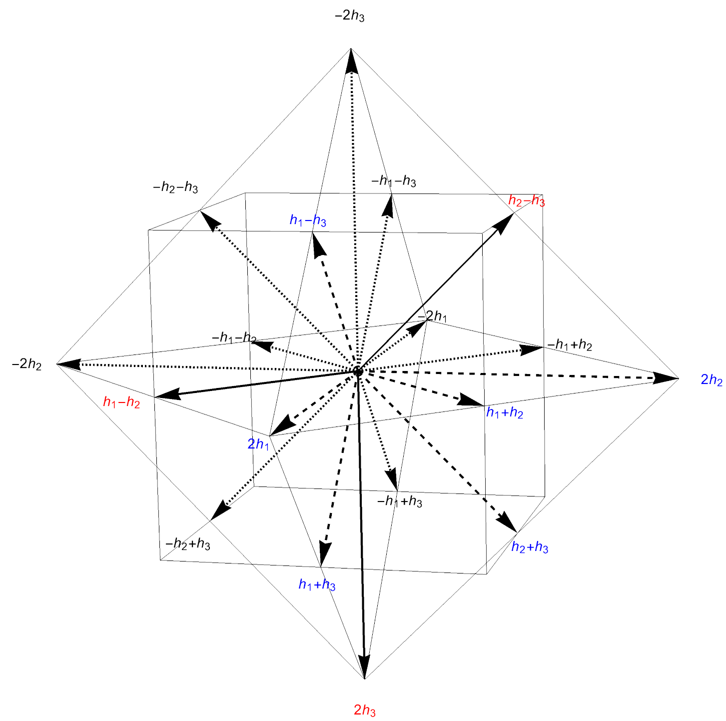

FIGURE 1. The root system of type C_3.

- matrices $E(1,4), E(2,5)$ and $E(3,6)$ will be the root vectors X_{2h_1}, X_{2h_2} and X_{2h_3} corresponding to the long positive roots $2h_1$, $2h_2$ and $2h_3$, respectively;
- matrices $E(1,2)-E(5,4), E(2,3)-E(6,5)$ and $E(1,3)-E(6,4)$ will be the root vectors $X_{h_1-h_2}, X_{h_2-h_3}$ and $X_{h_1-h_3}$ corresponding to the short positive roots h_1-h_2, h_2-h_3 and h_1-h_3, respectively;
- matrices $E(1,5)+E(2,4), E(2,6)+E(3,5)$ and $E(1,6)+E(3,4)$ will be the root vectors $X_{h_1+h_2}, X_{h_2+h_3}$ and $X_{h_1+h_3}$ corresponding to the short positive roots h_1+h_2, h_2+h_3 and h_1+h_3, respectively;
- matrix $X_\alpha := X^t_{-\alpha}$ will be the root vector corresponding to the negative root $\alpha \in \Phi^-$.

Remark 7.2. Recalling that the fundamental weights of $\mathfrak{sp}(\mathcal{C})$ are h_1, h_1+h_2, $h_1+h_2+h_3$ (see, e.g., [11, 2.2.13]), we denote by $W_{(a,b,c)}$ the irreducible $\mathfrak{sp}(\mathcal{C})$–representation whose highest weight is $(a+b+c)h_1+(b+c)h_2+ch_3$, for any non–negative integers a,b,c. In particular, $W_{(1,0,0)} = \mathcal{C}$

(resp., $W_{(2,0,0)} = \mathfrak{sp}(\mathcal{C})$) and the highest weight vector is e_1 (resp., X_{2h_1}); accordingly, $W_{(0,0,1)} = \Lambda^3_0(\mathcal{C})$ with highest weight $h_1 + h_2 + h_3$ and highest weight vector e_{123}: the irreducible representation $W_{(0,0,2)}$, whose highest weight vector e^2_{123} is the square of e_{123}, turns out to be 84–dimensional and will play some role in the sequel.

7.3. **Nilpotent orbits in $\mathfrak{sp}(\mathcal{C})$.** The Hasse diagram of nonzero nilpotent orbits in $\mathfrak{sp}(\mathcal{C})$ is well known (see, e.g., [12, Example 6.2.6]):

$$\begin{tikzcd}
 & \mathcal{O}_{[6]} \arrow[d, dash] & & \dim = 18 \\
 & \mathcal{O}_{[4,2]} \arrow[dl, dash] \arrow[dr, dash] & & \dim = 16 \\
\mathcal{O}_{[4,1^2]} \arrow[dr, dash] & & \mathcal{O}_{[3^2]} \arrow[dl, dash] & \dim = 14 \\
 & \mathcal{O}_{[2^3]} \arrow[d, dash] & & \dim = 12 \\
 & \mathcal{O}_{[2^2,1^2]} \arrow[d, dash] & & \dim = 10 \\
 & \mathcal{O}_{[2,1^4]} & & \dim = 6
\end{tikzcd}$$

Diagram above is a particular example of the Dynkin–Kostant classification [12, Chapter 3], which ultimately associates with any (nonzero) *nilpotent* orbit $\mathcal{O} := \mathsf{Sp}(\mathcal{C}) \cdot X$, the orbit of a distinguished *semisimple* element $H \in \mathfrak{h}$: these distinguished semisimple elements can be then labeled by *weighted Dynkin diagrams*, the admissible weights being $0, 1$ and 2.

Indeed, thanks to Jacobson–Morozov theorem, it is always possible to find, beside H, an appropriate $Y \in \mathfrak{sp}(\mathcal{C})$, such that the three–element subset $\{X, H, Y\} \subset \mathfrak{sp}(\mathcal{C})$ constitutes a so–called *standard* $\mathfrak{sl}_2$*–triple*, X, H and Y being its *nilpositive*, *neutral* and *nilnegative* element, respectively. Then a theorem by Kostant guarantees that any two standard $\mathfrak{sl}_2$–triple sharing the same nilpositive element are conjugate: this means that the orbit of the neutral element $\mathcal{O}_H := \mathsf{Sp}(\mathcal{C}) \cdot X$, is well defined and canonically associated with the nilpotent orbit $\mathcal{O}$. The Dynkin–Kostant classification gives us the (finite) list of all conjugacy classes of the so–obtained neutral elements: a representative for each class can be easily read off from the corresponding weighted Dynkin diagram.

Below, we start from a weighted Dynkin diagram, we construct the corresponding neutral element H, and then we compute all the possible standard $\mathfrak{sl}_2$–triples containing H: we finally pick a particular nilpositive element X, such that the corresponding quadratic form on $\mathcal{C}$ takes a particularly easy expression.

7.3.1. *The principal orbit* $\mathcal{O}_{prin} = \mathcal{O}_{[6]}$. The weighted Dynkin diagram is

$$\overset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!=\!\!<\!\!=\!\overset{2}{\bullet}\ ,$$

to which it corresponds the neutral element $H = \mathrm{diag}(5, 3, 1, -5, -3, -1)$: easy computations show that, if $\{X, H, Y\}$ is a standard $\mathfrak{sl}_2$–triple containing the aforementioned neutral element H, then its nilpositive and nilnegative elements must necessarily be

$$X = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 \end{pmatrix}, \tag{7.2}$$

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{\alpha_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{\alpha_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{8}{\alpha_2} \\ 0 & 0 & \frac{9}{\alpha_3} & 0 & 0 & 0 \end{pmatrix},$$

respectively. In particular, none of the three parameters α_1, α_2 and α_3 can be zero; therefore, we can set them to be 1, 1, and -1, respectively: this choice leads to a particularly simple expression for the associated quadratic form on $\mathcal{C}$, viz.

$$q_{[6]} := \phi(X) = \epsilon^1 e_2 + \epsilon^2 e_3 + e_3^2\,.$$

In the next cases, we will refrain from showing the explicit form of the nilpositive and nilnegative elements forming a standard $\mathfrak{sl}_2$–triple together with H; indeed, on a deeper level, what we have done above in (7.2), can be reformulated as the definition of a $\mathfrak{sl}_2$–gradation of $\mathfrak{sp}(\mathcal{C})$, that is

$$\mathfrak{sp}(\mathcal{C}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i\,, \tag{7.3}$$

where $g_i := \{Z \in \mathfrak{g} \mid [H, Z] = i\}$: the chosen nilpositive element X is then an arbitrary nonzero element of the 3–dimensional constituent $\mathfrak{g}_2$. In the case under consideration of the principal orbit $\mathcal{O}_{[6]}$, the nontrivial pieces of the $\mathfrak{sl}_2$–gradation are:

$$\begin{aligned}\mathfrak{g}_{-10} \oplus \mathfrak{g}_{-8} \oplus \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{h} \oplus \underbrace{\left\langle X_{h_1-h_2}, X_{h_2-h_3}, X_{2h_3}\right\rangle}_{\mathfrak{g}_2} \\ \oplus \underbrace{\left\langle X_{h_1-h_3}, X_{h_2+h_3}\right\rangle}_{\mathfrak{g}_4} \oplus \underbrace{\left\langle X_{h_1+h_3}, X_{2h_2}\right\rangle}_{\mathfrak{g}_6} \oplus \underbrace{\left\langle X_{h_1+h_2}\right\rangle}_{\mathfrak{g}_8} \oplus \underbrace{\left\langle X_{2h_1}\right\rangle}_{\mathfrak{g}_{10}}.\end{aligned} \tag{7.4}$$

In general, such a gradation is very useful in computing the Lie algebra $\mathfrak{g}^X := \mathfrak{stab}_{\mathfrak{sp}(\mathcal{C})}(X)$ of the subgroup $\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}(X)$ of $\mathsf{Sp}(\mathcal{C})$ stabilizing X: indeed, $\mathfrak{g}^X$ is compatible with the $\mathfrak{sl}_2$–gradation:

$$\mathfrak{g}^X = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i^X\,,\quad \mathfrak{g}_i^X := \mathfrak{g}_i \cap \mathfrak{g}^X\,,$$

see [12, (3.4.2)]. Then one realizes that solving the equation $[Z, X] = 0$ in $\mathfrak{g}^i$ is computationally easier than solving it in the whole of $\mathfrak{g}$: in the case of $\mathcal{O}_{[6]}$, we find out that only $\mathfrak{g}_2^X$, $\mathfrak{g}_6^X$ and $\mathfrak{g}_{10}^X$ do not vanish and, in fact, are one–dimensional. It follows that

$$\dim \mathfrak{g}^X = 1 + 1 + 1 = 3$$

and then

$$\dim \mathcal{O}_{[6]} = \dim \mathfrak{sp}(\mathcal{C}) - \dim \mathfrak{g}^X = 21 - 3 = 18;$$

on this concern it is worth observing that the dimension of the principal nilpotent orbit of a semisimple Lie group G is *always* equal to $\dim(G) - \mathrm{rank}(G)$, see [12, Lemma 4.1.3].

We will not insist here on the topology of nilpotent orbits of G, which is highly nontrivial [12, Chapter 6]; we only recall that there exists a unique orbit of codimension $\mathrm{rank}(G) + 2$, denoted by $\mathcal{O}_{\mathrm{subreg}}$ and called *subregular*, which is open and dense in $\overline{\mathcal{O}_{\mathrm{prin}}} \smallsetminus \mathcal{O}_{\mathrm{prin}}$, according to Steinber theorem [12, Theorem 4.2.1].

7.3.2. *The subregular orbit* $\mathcal{O}_{\mathrm{subreg}} = \mathcal{O}_{[4,2]}$. In the case $G = \mathsf{Sp}(\mathcal{C})$, we find a 16–dimensional orbit, corresponding to the weighted Dynkin diagram

$$\overset{2}{\bullet} \!-\!\!-\!\!-\!\! \overset{0}{\bullet} \Leftarrow \overset{2}{\bullet}\ .$$

The $\mathfrak{sl}_2$–gradation (7.3) above now reads

$$\mathfrak{g}_{-6} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-2} \oplus \underbrace{\mathfrak{h} \oplus \left\langle X_{h_2-h_3}, X_{-h_2+h_3}\right\rangle}_{\mathfrak{g}_0} \oplus \underbrace{\left\langle X_{h_1-h_2}, X_{h_1-h_3}, X_{h_2+h_3}, X_{2h_2}, X_{2h_3}\right\rangle}_{\mathfrak{g}_2} \oplus \underbrace{\left\langle X_{h_1+h_2}, X_{h_1+h_3}\right\rangle}_{\mathfrak{g}_4} \oplus \underbrace{\left\langle X_{2h_1}\right\rangle}_{\mathfrak{g}_6}$$

and $\mathcal{O}_{\mathrm{subreg}}$ will be generated by any nonzero element of $\mathfrak{g}_2$: a particular choice of the nilpositive element $X \in \mathfrak{g}_2$ gives us

$$q_{[4,2]} := \epsilon^1 e_3 + e_2^2 + e_3^2\,.$$

It is then easy to check that the only nontrivial constituents of $\mathfrak{g}^X$ are $\mathfrak{g}_2^X$, $\mathfrak{g}_4^X$ and $\mathfrak{g}_6^X$, and have dimensions 3, 1 and 1, respectively: indeed,

$$\dim \mathcal{O}_{\mathrm{subreg}} = 21 - (3+1+1) = 16 = \dim(\mathsf{Sp}(\mathcal{C})) - \mathrm{rank}(\mathsf{Sp}(\mathcal{C})) - 2.$$

7.3.3. *The* 14*–dimensional strata:* $\mathcal{O}_{[4,1^2]}$ *and* $\mathcal{O}_{[3,3]}$. The singular locus $\overline{\mathcal{O}_{\mathrm{subreg}}} \smallsetminus \mathcal{O}_{\mathrm{subreg}}$ consists of the closure of two 14–dimensional orbits:

$$\overline{\mathcal{O}_{\mathrm{subreg}}} \smallsetminus \mathcal{O}_{\mathrm{subreg}} = \overline{\mathcal{O}_{[4,1^2]}} \cup \overline{\mathcal{O}_{[3,3]}}\,,$$

one of which, namely, $\mathcal{O}_{[4,1^2]}$, will be of a paramount importance for us, as it contains the cocharacteristic variety of a "generic" Monge–Ampère equation.

The weighted Dynkin diagrams of $\mathcal{O}_{[4,1^2]}$ and $\mathcal{O}_{[3,3]}$ are

$$\overset{2}{\bullet} \!-\!\!-\!\!-\!\! \overset{1}{\bullet} \Leftarrow \overset{0}{\bullet}\ , \qquad \overset{0}{\bullet} \!-\!\!-\!\!-\!\! \overset{2}{\bullet} \Leftarrow \overset{0}{\bullet}\ ;$$

the $\mathfrak{sl}_2$–gradations are:

$$\mathfrak{g}_{-6} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{h} \oplus \left\langle X_{2h_3}, X_{-2h_3}\right\rangle}_{\mathfrak{g}_0} \oplus \underbrace{\left\langle X_{h_2-h_3}, X_{h_2+h_3}\right\rangle}_{\mathfrak{g}_1} \oplus \underbrace{\left\langle X_{h_1-h_2}, X_{2h_2}\right\rangle}_{\mathfrak{g}_2} \oplus \underbrace{\left\langle X_{h_1-h_3}, X_{h_1+h_3}\right\rangle}_{\mathfrak{g}_3} \oplus \underbrace{\left\langle X_{h_1+h_2}\right\rangle}_{\mathfrak{g}_4} \oplus \underbrace{\left\langle X_{2h_1}\right\rangle}_{\mathfrak{g}_6}$$

and

$$\mathfrak{g}_{-4} \oplus \mathfrak{g}_{-2} \oplus \underbrace{\mathfrak{h} \oplus \left\langle X_{h_1-h_2}, X_{2h_3}, X_{-h_1+h_2}, X_{-2h_3}\right\rangle}_{\mathfrak{g}_0}$$

$$\oplus \underbrace{\left\langle X_{h_1-h_3}, X_{h_2-h_3}, X_{h_1+h_3}, X_{h_2+h_3}\right\rangle}_{\mathfrak{g}_2} \oplus \underbrace{\left\langle X_{h_1+h_2}, X_{2h_1}, X_{2h_2}\right\rangle}_{\mathfrak{g}_4},$$

respectively. We finally obtain

$$q_{[4,1^2]} := \epsilon_1 e_2 + e_2^2 , \quad q_{[3^2]} := \epsilon^1 e_3 + e_2 e_3 .$$

7.3.4. *The 12–dimensional orbit $\mathcal{O}_{[2^3]}$.* The common singular locus

$$(\overline{\mathcal{O}_{[4,1^2]}} \smallsetminus \mathcal{O}_{[4,1^2]}) \cap (\overline{\mathcal{O}_{[3,3]}} \smallsetminus \mathcal{O}_{[3,3]})$$

is the closure of the 12–dimensional orbit $\mathcal{O}_{[2^3]}$; its weighted Dynkin diagram is

$$\overset{0}{\bullet} \text{———} \overset{0}{\bullet} \Leftarrow \overset{2}{\bullet} \ ,$$

and the corresponding $\mathfrak{sl}_2$–gradation:

$$\mathfrak{g}_{-2} \oplus \underbrace{\mathfrak{h} \oplus \left\langle X_{h_1-h_2}, X_{h_1-h_3}, X_{h_2-h_3}, X_{-h_1+h_2}, X_{-h_1+h_3}, X_{-h_2+h_3}\right\rangle}_{\mathfrak{g}_0}$$
$$\oplus \underbrace{\left\langle X_{h_1+h_2}, X_{h_1+h_3}, X_{h_2+h_3}, X_{2h_1}, X_{2h_2}, X_{2h_3}\right\rangle}_{\mathfrak{g}_2}.$$

The quadratic form is

$$q_{[2^3]} := e_1^2 + e_2^2 + e_3^2 .$$

7.3.5. *The 10–dimensional orbit $\mathcal{O}_{[2^2,1^2]}$.* The 10–dimensional stratum

$$\overline{\mathcal{O}_{[2^2,1^2]}} = \overline{\mathcal{O}_{[3^2]}} \smallsetminus \mathcal{O}_{[3^2]}$$

is the last non–smooth one and, in our further analysis, it will be parametrizing the Goursat–type symplectic Monge–Ampère equations, see Section 2.6. The weighted Dynkin diagram of $\mathcal{O}_{[2^2,1^2]}$ is

$$\overset{0}{\bullet} \text{———} \overset{1}{\bullet} \Leftarrow \overset{0}{\bullet} \ ,$$

and the corresponding $\mathfrak{sl}_2$–gradation:

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{h} \oplus \left\langle X_{h_1-h_2}, X_{2h_3}, X_{-h_1+h_2}, X_{-2h_3}\right\rangle}_{\mathfrak{g}_0}$$
$$\oplus \underbrace{\left\langle X_{h_1-h_3}, X_{h_2-h_3}, X_{h_1+h_3}, X_{h_2+h_3}\right\rangle}_{\mathfrak{g}_1} \oplus \underbrace{\left\langle X_{h_1+h_2}, X_{2h_1}, X_{2h_2}\right\rangle}_{\mathfrak{g}_2}.$$

The quadratic form is

$$q_{[2^2,1^2]} := e_1^2 + e_2^2.$$

7.3.6. *The cone over the adjoint variety $\mathcal{O}_{[2,1^4]}$.* The smallest nonzero nilpotent orbit is

$$\mathcal{O}_{[2,1^4]} = \overline{\mathcal{O}_{[2,1^4]}} = \overline{\mathcal{O}_{[2^2,1^2]}} \smallsetminus \mathcal{O}_{[2^2,1^2]}$$

and has dimension 6: unique, amongst all adjoint orbits, for being closed, its projectivization $\mathbb{P}(\mathcal{O}_{[2,1^4]})$ is a remarkable 5–dimensional homogeneous contact manifold, known as the *adjoint variety* of $\mathsf{Sp}(\mathcal{C})$; its weighted Dynkin diagram is

$$\overset{1}{\bullet} \text{———} \overset{0}{\bullet} = \!\!< \!\!= \overset{0}{\bullet}\ .$$

The $\mathfrak{sl}_2$–gradation of the subadjoint variety, namely,

$$\begin{aligned} &\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \\ &\quad \oplus \underbrace{\mathfrak{h} \oplus \left\langle X_{h_2-h_3}, X_{h_2+h_3}, X_{2h_2}, X_{2h_3}, X_{-h_2+h_3}, X_{-h_2-h_3}, X_{-2h_2}, X_{-2h_3}\right\rangle}_{\mathfrak{g}_0} \\ &\quad \oplus \underbrace{\left\langle X_{h_1-h_2}, X_{h_1-h_3}, X_{h_1+h_2}, X_{h_1+h_3}\right\rangle}_{\mathfrak{g}_1} \oplus \underbrace{\left\langle X_{2h_1}\right\rangle}_{\mathfrak{g}_2}, \end{aligned}$$

is the only one for which $\dim \mathfrak{g}_2 = 1$; moreover, the $\mathfrak{g}_2$–valued two–form $\Lambda^2(\mathfrak{g}_1^*) \longrightarrow g_2$, $(Z,W) \longmapsto [Z,W]$ is non–degenerate and this motivates calling such a gradation a *contact grading*, see [2] on this concern.

The following quadratic form ends our list of normal forms of quadratic forms on $\mathcal{C}$ corresponding to (nonzero) nilpotent elements of $\mathfrak{sp}(\mathcal{C})$:

$$q_{[2,1^4]} := e_1^2\,.$$

7.4. **Semisimple orbits in $\mathfrak{sp}(\mathcal{C})$.** The theory of semisimple orbits in semisimple Lie algebras is considerably simpler than that of nilpotent ones: their classification boils down to taking the quotient of a Cartan subalgebra $\mathfrak{h}$ with respect to the natural action of the Weyl group.

In the present case the elements of $\mathfrak{h}$ are unambiguously labeled by a vector $(\lambda,\mu,\nu) \in \mathbb{C}^3$, cf. (7.1), and the Weyl group is $W = S_3 \ltimes \mathbb{Z}_2^3$, acting naturally on $\mathbb{C}^3$: this means that we identify

$$(\lambda,\mu,\nu) \sim (\epsilon_1\sigma(\lambda), \epsilon_2\sigma(\mu), \epsilon_3\sigma(\nu))\,,$$

where $\epsilon_i = \pm 1$, for $i = 1,2,3$, and σ is a permutation of the set $\{\lambda,\mu,\nu\}$, and the corresponding equivalence class unambiguously identifies a semisimple orbit.

In what follows, symbol H denotes an element of $\mathfrak{h}$ labeled by the equivalence class of (λ,μ,ν), while $\mathfrak{g}^H$ denotes its stabilizing subalgebra, i.e., the

Lie algebra of the subgroup

$$G^H := \{x \in \mathsf{Sp}(\mathcal{C}) \mid xHx^{-1} = H\} = \{x \in \mathsf{Sp}(\mathcal{C}) \mid xH = Hx\}$$

of elements of $\mathsf{Sp}(\mathcal{C})$ commuting with H. Letting

$$\mathcal{C} = (\ker H) \oplus \bigoplus_{\substack{\alpha=\lambda,\mu,\nu \\ \alpha\neq 0}} (V_\alpha \oplus V_\alpha^*)\,,$$

where V_α denotes the eigenspace relative to the eigenvector $\alpha \in \mathbb{C} \setminus \{0\}$, we immediately see that

$$G^H = \mathsf{Sp}(\ker H) \times \prod_{\substack{\alpha=\lambda,\mu,\nu \\ \alpha\neq 0}} \mathsf{GL}(V_\alpha)\,. \tag{7.5}$$

Therefore, there are only six possible isomorphism types of stabilizers, collected below:

type of $\mathcal{O}$	dim V_λ	dim V_μ	dim V_ν	dim $\ker H$	stabilizer	dim $\mathcal{O}$
(111)	1	1	1	0	$\mathsf{GL}(V_\lambda) \times \mathsf{GL}(V_\mu) \times \mathsf{GL}(V_\nu) = \mathsf{GL}_1^3$	18
(21)	0	2	1	0	$\mathsf{GL}(V_\mu) \times \mathsf{GL}(V_\nu) = \mathsf{GL}_2 \times \mathsf{GL}_1$	16
(11)	0	1	1	2	$\mathsf{Sp}(\ker H) \times \mathsf{GL}(V_\mu) \times \mathsf{GL}(V_\nu)$ $= \mathsf{SL}_2 \times \mathsf{GL}_1^2$	16
(2)	0	0	2	2	$\mathsf{Sp}(\ker H) \times \mathsf{GL}(V_\nu) = \mathsf{SL}_2 \times \mathsf{GL}_2$	14
(3)	0	0	3	0	$\mathsf{GL}(V_\nu) = \mathsf{GL}_3$	12
(1)	0	0	1	4	$\mathsf{Sp}(\ker H) \times \mathsf{GL}(V_\nu) = \mathsf{Sp}_4 \times \mathsf{GL}_1$	10

Observe that, by the *type* of the orbit $\mathcal{O}$ passing through H, we meant the collection of the nonzero dimensions of V_λ, V_μ and V_ν.

7.4.1. *18–dimensional semisimple orbits.* The type of H is (111) if λ, μ, ν are different from zero and different form each other; this is the "non–degenerate" case, that is, when H commutes only with other elements of $\mathfrak{h}$: this means that $G^H = \mathsf{GL}_1^3$, $\mathfrak{g}^H = \mathfrak{h}$, and then $\dim(\mathsf{Sp}(\mathcal{C}) \cdot H) = 21 - 3 = 18$.

The quadratic form on $\mathcal{C}$ corresponding to H will be denoted by

$$q^{(111)} := \lambda\epsilon^1 e_1 + \mu\epsilon^2 e_2 + \nu\epsilon^3 e_3\,.$$

7.4.2. *16–dimensional semisimple orbits.* The type of H is (21) if $\lambda = \pm\mu$ and ν are different from zero and different form $\pm$ each other: in this case

$$G^H = \mathsf{GL}_2 \times \mathsf{GL}_1\,, \quad \mathfrak{g}^H = \mathfrak{h} \oplus \langle X_{h_1-h_2}, X_{-h_1+h_2}\rangle\,, \tag{7.6}$$

whence $\dim \mathfrak{g}^H = 5$ and then the orbit is 16–dimensional.

The type of H is (11) if $\lambda = 0$ and μ and ν are different from zero and different from $\pm$ each other: we obtain

$$G^H = \mathsf{Sp}_2 \times \mathsf{GL}_1^2 = \mathsf{SL}_2 \times \mathsf{GL}_1^2\,, \quad \mathfrak{g}^H = \mathfrak{h} \oplus \left\langle X_{2h_1}, X_{-2h_1} \right\rangle, \tag{7.7}$$

that is another 16–dimensional orbit. The corresponding quadratic forms are

$$q^{(21)} := \mu(\epsilon^1 e_1 + \epsilon^2 e_2) + \nu\epsilon^3 e_3, \quad q^{(11)} := \mu\epsilon^2 e_2 + \nu\epsilon^3 e_3,$$

respectively.

7.4.3. *14–dimensional semisimple orbits.* The type of H is (2) if $\lambda = 0$ and $\mu = \pm\nu$ is different from zero: we obtain 7–dimensional stabilizers

$$\begin{aligned} G^H &= \mathsf{Sp}_2 \times \mathsf{GL}_2 = \mathsf{SL}_2 \times \mathsf{GL}_2\,, \\ &\qquad \mathfrak{g}^H = \mathfrak{h} \oplus \left\langle X_{h_2-h_3}, X_{2h_1}, X_{-h_2+h_3}, X_{-2h_1} \right\rangle, \end{aligned} \tag{7.8}$$

whence a 14–dimensional orbit and the corresponding quadratic form is $q^{(2)} := \nu(\epsilon^2 e_2 + \epsilon^3 e_3)$.

7.4.4. *12–dimensional semisimple orbits.* The type of H is (3) if

$$\lambda = \pm\mu = \pm\nu$$

is different from zero; in this case the stabilizers are 9–dimensional:

$$\begin{aligned} G^H &= \mathsf{GL}_3\,, \\ \mathfrak{g}^H &= \mathfrak{h} \oplus \left\langle X_{h_1-h_2}, X_{h_1-h_3}, X_{h_2-h_3}, X_{-h_1+h_2}, X_{-h_1+h_3}, X_{-h_2+h_3} \right\rangle, \end{aligned} \tag{7.9}$$

whence a 12–dimensional orbit and the corresponding quadratic form is $q^{(3)} := \nu(\epsilon^1 e_1 + \epsilon^2 e_2 + \epsilon^3 e_3)\,.$

7.4.5. *10–dimensional semisimple orbits.* The smallest nontrivial semisimple orbits have dimension 10 and correspond to an H of type (1), i.e., with $\lambda = \mu = 0$ and $\nu \neq 0$: the stabilizers

$$\begin{aligned} G^H &= \mathsf{Sp}_4 \times \mathsf{GL}_1\,, \\ \mathfrak{g}^H &= \mathfrak{h} \oplus \left\langle X_{h_1-h_2}, X_{h_1+h_2}, X_{2h_1}, X_{2h_2}, X_{-h_1+h_2}, X_{-h_1-h_2}, X_{-2h_1}, X_{-2h_2} \right\rangle, \end{aligned} \tag{7.10}$$

are 11–dimensional, whence a 10–dimensional orbit and the corresponding quadratic form is $q^{(1)} := \nu\epsilon^3 e_3\,.$

7.5. **Mixed orbits in $\mathfrak{sp}(\mathcal{C})$.** We pass now to the generic case and we study orbits passing through (nonzero) elements of the form $Z = H + X$, where $H \in \mathfrak{g}$ is semisimple and $X \in \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}$ denotes the *nilpotent cone*, that is, the subset of nilpotent elements of the Lie algebra $\mathfrak{g}$, and, moreover, $[H, X] = 0$.

The strategy consists in bringing H to one of the normal forms above (corresponding to the quadratic forms labeled $q^{(111)}, q^{(21)}, q^{(11)}, q^{(2)}, q^{(3)}, q^{(1)}$): then, thanks to the Jordan's decomposition theorem, the nilpotent part X of Z must be coming from the set

$$\mathfrak{g}^H \cap \mathcal{N} = \mathfrak{stab}_{\mathfrak{sp}(\mathcal{C})}(H) \cap \mathcal{N} = \{X \in \mathfrak{sp}(\mathcal{C}) \mid [H, X] = 0\} \cap \mathcal{N}\,,$$

whence the problem is reduced to studying the G^H–orbits in $\mathfrak{g}^H \cap \mathcal{N}$, where G^H has the structure shown in (7.5) and acts naturally and componentwise on

$$g^H = \mathfrak{sp}(\ker H) \oplus \bigoplus_{\substack{\alpha=\lambda,\mu,\nu \\ \alpha\neq 0}} \mathfrak{gl}(V_\alpha)\,.$$

Such a particular structure of g^H is mirrored by an analogous decomposition of the nilpotent cone:

$$\mathfrak{g}^H \cap \mathcal{N} = \mathcal{N}(\mathfrak{sp}(\ker H)) + \sum_{\substack{\alpha=\lambda,\mu,\nu \\ \alpha\neq 0}} \mathcal{N}(\mathfrak{gl}(V_\alpha))\,,$$

whence it suffices to classify the orbit of

$$G^H_{\mathrm{ss}} := \mathsf{Sp}(\ker H) \times \prod_{\substack{\alpha=\lambda,\mu,\nu \\ \alpha\neq 0}} \mathsf{SL}(V_\alpha)$$

acting naturally and componentwise on

$$\mathfrak{g}^H \cap \mathcal{N} = \mathcal{N}(\mathfrak{sp}(\ker H)) + \sum_{\substack{\alpha=\lambda,\mu,\nu \\ \alpha\neq 0}} \mathcal{N}(\mathfrak{sl}(V_\alpha))\,.$$

7.5.1. *Mixed orbits with semisimple part of type* (111)*.* In this case, $\mathfrak{g}^H = \mathfrak{h}$, whence $\mathfrak{g}^H \cap \mathcal{N} = 0$: this means that the only mixed orbit with semisimple part of type $q^{(111)}$ is the orbit of $q^{(111)}$ itself.

7.5.2. *Mixed orbits with semisimple part of type* (21)*.* In this case, H corresponds to (μ, μ, ν): the space V_μ has dimension two and

$$G^H_{\mathrm{ss}} = \mathsf{SL}(V_\mu) \simeq \mathsf{SL}_2$$

acts naturally on

$$\mathfrak{g}^H \cap \mathcal{N} = \mathcal{N}(\mathfrak{sl}(V_\mu)) \subseteq \mathsf{Sp}(\mathcal{C}) \cdot \langle X_{h_1-h_2}, X_{-h_1+h_2}\rangle,$$

cf. (7.6). Since in $\mathfrak{sl}(V_\mu) \simeq \mathfrak{sl}_2$ there is only one nontrivial nilpotent orbit, namely, the one passing through $X_{h_1-h_2}$, we see that, besides $q^{(21)}$ itself, the only other orbit with semisimple part $q^{(21)}$ is

$$q^{(21)} + \phi(X_{h_1-h_2}) = \mu(\epsilon^1 e_1 + \epsilon^2 e_2) + \nu\epsilon^3 e_3 + \epsilon^1 e_2\,.$$

The aforementioned nilpotent orbit in $\mathfrak{sl}_2$ has dimension 2, whereas the orbit in $\mathfrak{sp}(\mathcal{C})$ of the semisimple element $q^{(21)}$ has dimension 16: it follows that the mixed orbit has dimension 18.

7.5.3. *Mixed orbits with semisimple part of type* (11). In this case, H corresponds to $(0,\mu,\nu)$: the spaces V_μ and V_ν have both dimension one and

$$G^H_{\mathrm{ss}} = \mathsf{Sp}(\ker H) = \mathsf{SL}(\ker H) \simeq \mathsf{SL}_2$$

acts naturally on

$$\mathfrak{g}^H \cap \mathcal{N} = \mathcal{N}(\mathfrak{sl}(\ker H)) \subseteq \mathsf{Sp}(\mathcal{C}) \cdot \langle X_{2h_1}, X_{-2h_1}\rangle\,,$$

cf. (7.7). Again, in $\mathfrak{sl}(\ker H) \simeq \mathfrak{sl}_2$ there is only one nontrivial nilpotent orbit: the one passing, e.g., through X_{-2h_1}, whence, besides $q^{(11)}$ itself, the only other orbit with semisimple part $q^{(11)}$ is

$$q^{(11)} + \phi(X_{-2h_1}) = \mu\epsilon^2 e_2 + \nu\epsilon^3 e_3 + (\epsilon^1)^2\,.$$

Recalling that also the semisimple element $q^{(11)}$ of $\mathfrak{sp}(\mathcal{C})$ has a 16–dimensional orbit, we get another mixed orbit of dimension 18.

7.5.4. *Mixed orbits with semisimple part of type* (2). In this case, H corresponds to $(0,\nu,\nu)$: the space V_ν has dimension two and

$$G^H_{\mathrm{ss}} = \mathsf{Sp}(\ker H) \times \mathsf{SL}(V_\nu) = \mathsf{SL}(\ker H) \times \mathsf{SL}(V_\nu) \simeq \mathsf{SL}_2 \times \mathsf{SL}_2$$

acts naturally and componentwise on

$$\begin{aligned}\mathfrak{g}^H \cap \mathcal{N} = \mathcal{N}(\mathfrak{sl}(\ker H)) + \mathcal{N}(\mathfrak{sl}(V_\nu)) \\ \subseteq \mathsf{Sp}(\mathcal{C}) \cdot (\langle X_{h_2-h_3}, X_{-h_2+h_3}\rangle + \langle X_{2h_1}, X_{-2h_1}\rangle)\,,\end{aligned}$$

cf. (7.8). Now, it is convenient to introduce two coefficients δ_1 and δ_2, that can be either 0 or 1: all mixed orbits with semisimple part $q^{(2)}$ will then pass trough one and only one of the four following elements:

$$q^{(2)} + \delta_1\phi(X_{h_2-h_3}) + \delta_2\phi(X_{-2h_1}) = \nu(\epsilon^2 e_2 + \epsilon^3 e_3) + \delta_1\epsilon^2 e_3 + \delta_2(\epsilon^1)^2\,.$$

The semisimple orbit in $\mathfrak{sp}(\mathcal{C})$ passing through $q^{(2)}$ has dimension 14; since both $\ker H$ and $\mathfrak{sl}(V_\nu)$ are isomorphic to $\mathfrak{sl}_2$, we easily get the following formula for the dimension of the mixed orbits: $\dim = 14 + 2(\delta_1+\delta_2)$.

7.5.5. *Mixed orbits with semisimple part of type* (3). In this case, H corresponds to (ν,ν,ν): the space V_ν has dimension three and $G^H_{\mathrm{ss}} = \mathsf{SL}(V_\nu) \simeq \mathsf{SL}_3$ acts naturally on

$$\begin{aligned}\mathfrak{g}^H \cap \mathcal{N} &= \mathcal{N}(\mathfrak{sl}(V_\nu))\\ &\subseteq \mathsf{Sp}(\mathcal{C}) \cdot \left\langle X_{h_1-h_2}, X_{h_1-h_3}, X_{h_2-h_3}, X_{-h_1+h_2}, X_{-h_1+h_3}, X_{-h_2+h_3}\right\rangle ,\end{aligned}$$

cf. (7.9). Now, in $\mathfrak{sl}(V_\nu) \simeq \mathfrak{sl}_3$ there are two nontrivial nilpotent orbits: one passing through $X_{h_1-h_2}$ and the other passing through $X_{h_1-h_2} + X_{h_2-h_3}$, whence, besides $q^{(3)}$ itself, the only other orbits with semisimple part $q^{(3)}$ are

$$q^{(3)} + \phi(X_{h_1-h_2}) = \nu(\epsilon^1 e_1 + \epsilon^2 e_2 + \epsilon^3 e_3) + \epsilon^1 e_2\,, \tag{7.11}$$

$$q^{(3)} + \phi(X_{h_1-h_2} + X_{h_2-h_3}) = \nu(\epsilon^1 e_1 + \epsilon^2 e_2 + \epsilon^3 e_3) + \epsilon^1 e_2 + \epsilon^2 e_3\,. \tag{7.12}$$

The aforementioned nilpotent orbits of $\mathfrak{sl}_3$ correspond, respectively, to the submaximal (which is the same as the minimal) and to the maximal nilpotent orbit and, therefore, have dimension 4 and 6, respectively; recalling that in $\mathfrak{sp}(\mathcal{C})$ the semisimple element $q^{(3)}$ generates a 12–dimensional orbit, the corresponding mixed orbits (7.11) and (7.12) have dimension 16 and 18, respectively.

7.5.6. *Mixed orbits with semisimple part of type* (1). In this case, H corresponds to $(0,0,\nu)$: the space V_ν has dimension one, whereas $\ker H$ is four–dimensional; therefore,

$$G^H_{\mathrm{ss}} = \mathsf{Sp}(\ker H) \simeq \mathsf{Sp}_4$$

acts naturally on

$$\begin{aligned}&\mathfrak{g}^H \cap \mathcal{N} = \mathcal{N}(\mathfrak{sp}(\ker H))\\ &\subseteq \mathsf{Sp}(\mathcal{C}) \cdot \left\langle X_{h_1-h_2}, X_{h_1+h_2}, X_{2h_1}, X_{2h_2}, X_{-h_1+h_2}, X_{-h_1-h_2}, X_{-2h_1}, X_{-2h_2}\right\rangle ,\end{aligned}$$

cf. (7.10). Now, in $\mathfrak{sp}(\ker H) \simeq \mathfrak{sp}_4$ there are three nontrivial nilpotent orbits, passing through, e.g., $X_{h_1-h_2} - X_{2h_2}$, $-\frac{1}{2}X_{h_1+h_2}$ and $-X_{2h_1}$, whence, besides $q^{(1)}$ itself, the only other orbits with semisimple part $q^{(1)}$ are

$$q^{(1)} + \phi(X_{h_1-h_2} - X_{2h_2}) = \nu\epsilon^3 e_3 + \epsilon^1 e_2 + e_2^2\,, \tag{7.13}$$

$$q^{(1)} + \phi(-\tfrac{1}{2}X_{h_1+h_2}) = \nu\epsilon^3 e_3 + e_1 e_2\,, \tag{7.14}$$

$$q^{(1)} + \phi(-X_{2h_1}) \quad = \quad \nu\epsilon^3 e_3 + e_1^2\,, \tag{7.15}$$

respectively.

The aforementioned nilpotent orbits of $\mathfrak{sp}_4$ correspond, respectively, to the maximal, the submaximal, and to the minimal nilpotent orbit and, therefore, have dimension 8, 6 and 4, respectively; since the semisimple element $q^{(1)}$ generates a 10–dimensional orbit in $\mathfrak{sp}(\mathcal{C})$, the corresponding mixed orbits (7.13), (7.14) and (7.15) have dimension 18, 16 and 14, respectively.

Remark 7.3. For any semisimple element $H \in \mathfrak{sp}(\mathcal{C})$ there is a nilpotent element X, commuting with H, such that the mixed orbit passing through $Z = H + X$ has dimension 18; therefore, such a dimension is the highest attainable by all the adjoint orbits in $\mathfrak{sp}(\mathcal{C}) \simeq \mathfrak{sp}_6$.

8. THE MOMENT MAP ϖ ON THE SPACE OF MONGE–AMPÈRE EQUATIONS

We resume here the study of the space $\mathbb{P}(\Lambda_0^3(\mathcal{C}))$, whose dual $\mathbb{P}(\Lambda_0^3(\mathcal{C}^*))$ parametrizes Monge–Ampère equations, that we have introduced earlier in Section 6.3. In Section 8.1, we sketch the construction of a quadratic $\mathsf{Sp}(\mathcal{C})$–equivariant moment map ϖ on the symplectic space $\Lambda_0^3(\mathcal{C})$, that has been observed in the first place by N. Hitchin in [23], in a similar fashion as we did for the canonical identification discussed in Section 6.2 above. In the next Section 8.2, we compute the image via ϖ of each of the four orbits of $\mathbb{P}(\Lambda_0^3(\mathcal{C}))$, as well as all the isomorphism types of the fibers of ϖ. In the last Section 8.3, we show that the Hitchin moment map is equivalent to the KLR contraction, cf. (2.20), up to the natural identification $\mathcal{C} \simeq \mathcal{C}^*$ via the symplectic form.

8.1. **The Hitchin moment map on $\Lambda_0^3(\mathcal{C})$.** By Leibniz's rule, the embedding j given in (6.8) extends to the algebra of polynomial vector fields on $\Lambda_0^3(\mathcal{C})$ and it is not hard to see that these are Hamiltonian with respect to Ω, cf. (6.13): therefore, it must exist, similarly as before, a *quadratic* moment map $\widetilde{\mu}$ closing the diagram

$$\begin{tikzcd} S^2(\Lambda_0^3(\mathcal{C})) \arrow[r, "\mathrm{pr}"] & \mathfrak{sp}(\mathcal{C})^* \\ \Lambda_0^3(\mathcal{C}) \arrow[u, "v_2"] \arrow[ur, "\widetilde{\mu}"'] & \end{tikzcd} \tag{8.1}$$

of $\mathsf{Sp}(\mathcal{C})$–equivariant polynomial maps. It is worth stressing that only the upper arrow is linear and it is in fact the projection of the 105–dimensional representation $S^2(\Lambda^3_0(\mathcal{C}))$ onto its unique 21–dimensional irreducible constituent:

$$S^2(\Lambda^3_0(\mathcal{C})) = W_{(2,0,0)} \oplus W_{(0,0,2)} = \mathfrak{sp}(\mathcal{C})^* \oplus W_{(0,0,2)}\,.$$

By identifying $\mathfrak{sp}(\mathcal{C})^*$ with $S^2(\mathcal{C})$, cf. (6.10), we finally obtain the $\mathsf{Sp}(\mathcal{C})$–equivariant quadratic map

$$\varphi := \phi^{*\,-1} \circ \widetilde{\mu} : \Lambda^3_0(\mathcal{C}) \longrightarrow S^2(\mathcal{C}) \tag{8.2}$$

the whole paper hinges around.

Observe that the map φ, which is clearly non–surjective, fails also to be injective: in particular, the smallest nonzero $\mathsf{Sp}(\mathcal{C})$–orbit in $\Lambda^3_0(\mathcal{C})$, that is the cone over the Lagrangian Grassmanian $\mathrm{LGr}(3,\mathcal{C})$, is mapped to zero; therefore, by projectivizing the above diagram (8.1) and taking into account (8.2), we obtain a commuting diagram of rational maps

$$\begin{tikzcd} \mathbb{P}(\Lambda^3_0(\mathcal{C})) \arrow[r, "v_2"] \arrow[dr, dotted, "\varpi"'] & \mathbb{P}(S^2(\Lambda^3_0(\mathcal{C}))) \arrow[d, dotted, "{[\mathrm{pr}]}"] \\ & \mathbb{P}(S^2(\mathcal{C}))\,. \end{tikzcd} \tag{8.3}$$

Definition 8.1. *The rational map ϖ, defined as the projectivization of the composition of the Veronese embedding and the projection onto the first factor or, equivalently, as the projectivization of the moment map on the space of symplectic 3D Monge–Ampère equations, will be called simply the* moment map*.*

8.2. **Images and fibers of the moment map across the four orbits of $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$.** An obvious and yet crucial remark is that the above defined map ϖ is $\mathsf{Sp}(\mathcal{C})$–equivariant, as it is a composition of two maps having such property. It is also well known that the natural action of $\mathsf{Sp}(\mathcal{C})$ on $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$ has precisely four orbits, that are linearly arranged with respect to the closure order.

Therefore, in order to fully describe ϖ, it will suffice to compute the image of chosen representatives of the four orbits of $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$, and $\varpi(\mathbb{P}(\Lambda^3_0(\mathcal{C})))$ is going to be a sum of orbits of $\mathbb{P}(S^2(\mathcal{C}))$: our main tool to spell out the structure of such sums will be the study of weights. Concerning the study of fibers, it is worth noticing that, if $[q] = \varpi([\eta])$, then $\varpi^{-1}([q])$ is naturally acted upon by the stabilizer $\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}([q])$ in a transitive way; in other words, $\varpi^{-1}([q])$ has the structure of a $\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}([q])$–homogeneous space.

8.2.1. *The orbit structure of* $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$. The details of the four orbits of $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$, that we are going to need, are summarized below (recall that symbol $X^\vee$ denotes the projective dual of the variety X):

	dim	structure	representative
O	13	$\mathbb{P}(\Lambda^3_0(\mathcal{C})) \smallsetminus \mathrm{LGr}(3,\mathcal{C})^\vee$	$[e_{123}+e_{456}]$
L	12	$\mathrm{LGr}(3,\mathcal{C})^\vee \smallsetminus \mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)$	$[e_{423}+e_{126}+e_{153}+e_{123}]$
G	9	$\mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee) \smallsetminus \mathrm{LGr}(3,\mathcal{C})$ $\simeq \mathrm{Gr}(3,\mathcal{C})/\mathbb{Z}_2$	$[e_{163}+e_{125}]$
P	6	$\mathrm{LGr}(3,\mathcal{C})$	$[e_{123}]$

The acronyms O, L, G, and P stand, respectively, for "open", "linearizable", "Goursat", and "parabolic".

The smallest one (P) is 6–dimensional, smooth and closed: it is the Lagrangian Grassmanian $\mathrm{LGr}(3,\mathcal{C})$; it is natural to pick as a representative (the projective class of) the highest weight vector $[e_{123}]$ of the irreducible representation $W_{(0,0,1)} = \Lambda^3_0(\mathcal{C})$, see Remark 7.2.

The projective dual of $\mathrm{LGr}(3,\mathcal{C})$ is the quartic hypersurface $\mathrm{LGr}(3,\mathcal{C})^\vee = \{f=0\}$, with

$$f = (x^{123}x^{456} - \mathrm{tr}(XY))^2 + 4x^{123}\det(Y) + 4x^{456}\det(X) \\ - 4\sum_{1\le i,j\le 3} \det\|X_{i,j}\|\det\|Y_{i,j}\|\,, \quad (8.4)$$

where X and Y are defined by (6.14) and the symbol $A_{i,j}$ denotes the 2×2 block that is complementary to the i^{th} row and the j^{th} column of the 3×3 matrix A.

Its smooth locus $\mathrm{LGr}(3,\mathcal{C})^\vee \smallsetminus \mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)$ is the second biggest orbit (L) and as its representative we take the projective class $[e_{423}+e_{126}+e_{153}+e_{123}]$. The biggest orbit is the open one (O), that is $\mathbb{P}(\Lambda^3_0(\mathcal{C})) \smallsetminus \mathrm{LGr}(3,\mathcal{C})^\vee$, which is the orbit passing through, e.g., $[e_{123}+e_{456}]$. Finally, the singular locus $\mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)$, that is the variety cut out by the Jacobian ideal of (8.4), is 9–dimensional and $\mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee) \smallsetminus \mathrm{LGr}(3,\mathcal{C})$ is the orbit (G) represented, e.g., by $[e_{163}+e_{125}]$.

8.2.2. *(P) The singular locus.*

Proposition 8.1. *The singular locus of* ϖ *is precisely* $\mathrm{LGr}(3,\mathcal{C})$.

Proof. Recall that we have chosen, as a representative for the smallest orbit, the element $[e_{123}]$: the vector e_{123} has weight $h_1+h_2+h_3$ and the image of $[e_{123}]$ under the Veronese embedding corresponds to a vector of weight

$2h_1+2h_2+2h_3$; the latter is nothing but the highest weight vector of the 84–dimensional constituent $W_{(0,0,2)}$ that is annihilated by the affine projection pr in (8.1). This shows that the map ϖ is not defined on $\mathrm{LGr}(3,6)$ and since in the next propositions we will find the images of remaining orbits, it suffices to prove the claim. □

8.2.3. *(G) Image of the 9–dimensional orbit.* It will be useful to recall that the minimal projective embedding $\mathrm{LGr}(2,V') \subset \mathbb{P}(\Lambda_0^2(V'))$, where V' is a 4–dimensional symplectic space, has only two orbits: the 3–dimensional (smooth, closed) Lagrangian Grassmanian $\mathrm{LGr}(2,V')=\mathrm{LGr}(2,4)$ itself and its (open) complement in $\mathbb{P}(\Lambda_0^2(V'))=\mathbb{P}^4$; one way to see this is the Dynkin diagram identification $\mathsf{C}_2\equiv\mathsf{B}_2$: it follows that the $\mathsf{Sp}(4)$–action on $\mathbb{P}^4$ coincides with the $\mathsf{SO}(5)$–action and then there is only one invariant quadric.

Proposition 8.2. *The image of* $\mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)\backslash\mathrm{LGr}(3,\mathcal{C})$ *via* ϖ *is the* (*5–dimensional, contact*) *adjoint variety* $\mathbb{P}(\mathcal{O}_{[2,1^4]})$ *of* $\mathsf{Sp}(\mathcal{C})$*, see Section* 7.3.6. *The fiber of* ϖ *restricted to* $\mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)\setminus\mathrm{LGr}(3,\mathcal{C})$ *is isomorphic to* $\mathbb{P}^4\setminus\mathrm{LGr}(2,4)$*. The "boundary" of the compactification of the fiber, i.e.,* $\mathrm{LGr}(2,4)$*, is a subset of* $\mathrm{LGr}(3,\mathcal{C})$*.*

Proof. Since the vector corresponding to the chosen representative $[e_{163}+e_{125}]$ has weight h_1, its square has weight $2h_1$. There are two other elements in $S^2\big(\Lambda_0^3(\mathcal{C})\big)$ with this weight, namely, $e_{123}\cdot e_{156}$ and $e_{153}\cdot e_{126}$ and some combination of these three vectors is the highest weight vector $w_{(2,0,0)}$ of $W_{(2,0,0)}=S^2(\mathcal{C})\subset S^2\big(\Lambda_0^3(\mathcal{C})\big)$. To determine $w_{(2,0,0)}$ recall that it is annihilated by all primitive positive root vectors. On the other hand, the images of elements with weights $2h_1+2h_3$ and $2h_1+h_2-h_3$ via negative root vectors belong to $W_{(0,0,2)}\subset S^2\big(\Lambda_0^3(\mathcal{C})\big)$ (i.e., to the complement). Direct computations show that $w_{(2,0,0)}=(e_{163}+e_{125})^2-4e_{123}e_{156}+4e_{153}e_{126}$ and then the 3–dimensional weight space $\big(S^2\big(\Lambda_0^3(\mathcal{C})\big)\big)_{2h_1}$ splits as

$$\begin{aligned}\big(S^2\big(\Lambda_0^3(\mathcal{C})\big)\big)_{2h_1}&=\big(W_{(2,0,0)}\big)_{2h_1}\oplus\big(W_{(0,0,2)}\big)_{2h_1}\\&=\langle w_{(2,0,0)}\rangle\oplus\langle e_{123}e_{156}+e_{153}e_{126},(e_{163}+e_{125})^2+2e_{126}e_{153}\rangle.\end{aligned}$$

Since $(e_{163}+e_{125})^2$ has nonzero component along $w_{(2,0,0)}$, it projects nontrivially onto $(W_{(2,0,0)})_{2h_1}$ and this subspace is spanned by the matrix $E(1,4)$, that is X_{2h_1}, which clearly belongs to $\mathcal{O}_{[2,1^4]}$, see Remark 7.1.

As for the second statement, recall that the fiber of ϖ that passes through $[e_{163}+e_{125}]$ is a homogeneous space of the subgroup $\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}([E(1,4)])$ acting on $\mathbb{P}(\Lambda_0^3(\mathcal{C}))$.

It is now convenient to introduce the symplectic space $V' = \langle e_2, e_3, e_5, e_6\rangle$: indeed, every element of $\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}([E(1,4)])$ leaves e_1 fixed and only e_4 gets sent to e_4, so that the stabilizer is isomorphic to $\mathsf{Sp}(V') \simeq \mathsf{Sp}(4)$. More precisely, the wedge–multiplication by e_1 realizes an embedding of $\mathbb{P}(\Lambda^2_0(V'))$ into $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$ that commutes with the action of $\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}([E(1,4)])$, understood as a subgroup of $\mathsf{Sp}(\mathcal{C})$: this reduces our problem to that of finding the $\mathsf{Sp}(V')$–orbit of the class of the two–form $e_{63}+e_{25}$ living in $\mathbb{P}(\Lambda^2_0(V'))$. The second claim follows then from the fact that $e_{63}+e_{25}$ is it not a decomposable two–form, i.e., $[e_{63}+e_{25}] \notin \mathrm{LGr}(2, V')$. Finally, the wedge multiplication by e_1 maps the Lagrangian subspaces of V' to Lagrangian subspaces of $\mathcal{C}$, and they belong to $\mathrm{LGr}(3, \mathcal{C})$. □

8.2.4. *(L) Image of the twelve–dimensional orbit.* We introduce now a special projective line in $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$:

$$\mathbb{P}^1 := \overline{\{[e_{423}+e_{126}+e_{153}+k\cdot e_{123}] \mid k\in\mathbb{C}\}}\ ,$$

understood as the compactification of its (one–dimensional) affine neighborhood

$$\mathbb{C}^1 := \{[e_{423}+e_{126}+e_{153}+k\cdot e_{123}] \mid k\in\mathbb{C}\}\,,$$

by means of the "point at infinity" $[e_{123}]$.

Proposition 8.3. *The image of* $\mathrm{LGr}(3,\mathcal{C})^\vee \setminus \mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)$ *via* ϖ *is* $\mathbb{P}(\mathcal{O}_{[2^3]})$*, which has dimension* 11*. The fiber of* ϖ *over*

$$\mathrm{LGr}(3,\mathcal{C})^\vee \setminus \mathrm{Sing}(\mathrm{LGr}(3,\mathcal{C})^\vee)$$

is isomorphic to $\mathbb{C}^1$*, and can be compactified to* $\mathbb{P}^1$ *by a point lying in* $\mathrm{LGr}(3,\mathcal{C})$*.*

Proof. Consider the square of $e_{423}+e_{126}+e_{153}+k\cdot e_{123}$. It can be decomposed into the sum of elements with weights $\pm 2h_1 \pm 2h_2 \pm 2h_3$ (pure squares), $2h_i+2h_j$ (elements with $k\cdot e_{123}$) and $2e_{126}e_{153}+2e_{423}e_{126}+2e_{423}e_{153}$. The projection onto $W_{(2,0,0)}$ kills the first two sets of elements, and as in Proposition 8.2, we can check that the three last elements project nontrivially onto the corresponding weight subspaces in $\mathfrak{sp}(\mathcal{C})$: they are spanned by $(e_{623}+e_{124})^2 - 4e_{123}e_{426}+4e_{126}e_{423}$ for $2h_2$ and $(e_{523}+e_{143})^2-4e_{123}e_{453}+4e_{423}e_{153}$ for $2h_3$. Therefore, the image of the chosen representative does not depend on k and is equal to the class of $E(1,4)+E(2,5)+E(3,6)$ and such matrices live in $\mathbb{P}(\mathcal{O}_{[2^3]})$.

In other words, we have just shown that the fiber contains $\mathbb{C}^1$; since the "point at infinity" $[e_{123}]$ lies in $\mathrm{LGr}(3,\mathcal{C})$, the closure $\mathbb{P}^1$ of $\mathbb{C}^1$ is no longer a subset of the fiber.

Now, to show that there are no other components than $\mathbb{C}^1$, we consider the stabilizer subgroup

$$\mathrm{Stab}_{\mathsf{Sp}(\mathcal{C})}([E(1,4)+E(2,5)+E(3,6)]) \subset \mathsf{Sp}(\mathcal{C}),$$

and recall that it acts transitively on the fiber: we will prove the connectedness of the fiber by showing that the stabilizer subgroup is connected. To this end, observe that

$$\begin{aligned}&\mathrm{Stab}([E(1,4)+E(2,5)+E(3,6)])\\ &\qquad = \left\{ \begin{pmatrix} A & B \\ 0 & a\cdot A \end{pmatrix} \mid a \in \mathbb{C}^*, a\cdot A\cdot A^t = I_3, A^t\cdot B = B^t \cdot A \right\}.\end{aligned}$$

Since we want to stabilize the projective class of an element we gain an additional parameter: the scalar a. Thanks to it, the determinant of A can be any nonzero complex number, therefore the stabilizer subgroup is connected, and that finishes the proof. □

8.2.5. *(O) Image of the open orbit.* As before, we introduce a special projective line in $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$: $\mathbb{P}^1 := \overline{\{[e_{123}+k\cdot e_{456}] \mid k\in\mathbb{C}\}}$, understood this time as the compactification of

$$\mathbb{C}^\times := \{[e_{123}+k\cdot e_{456}] \mid k \in \mathbb{C}\smallsetminus\{0\}\}\,, \tag{8.5}$$

by means of the "point at infinity" $[e_{456}]$, as well as well as the "zero point" $[e_{123}]$.

Proposition 8.4. *The image of $\mathbb{P}(\Lambda^3_0(\mathcal{C}))\smallsetminus \mathrm{LGr}(3,\mathcal{C})^\vee$ via ϖ is the projectivization*

$$\mathbb{P}\Big(\bigcup_{\nu\in\mathbb{C}\smallsetminus\{0\}} \mathsf{Sp}(\mathcal{C})\cdot\nu\cdot(\epsilon^1 e_1+\epsilon^2 e_2+\epsilon^3 e_3)\Big)$$

of the sum of a 1–parameter family of 12–dimensional orbits of type $q^{(3)}$. The fiber of ϖ, restricted to $\mathbb{P}(\Lambda^3_0(\mathcal{C}))\smallsetminus\mathrm{LGr}(3,\mathcal{C})^\vee$, is equal to $\mathbb{C}^\times$ and can be compactified by two points that belong to $\mathrm{LGr}(3,\mathcal{C})$.

Proof. Consider the element $e_{123}+k\cdot e_{456}$, whose projective class belongs to the open orbit for all $k\neq 0$. Its square is equal to $e^2_{123}+k^2e^2_{456}+2ke_{123}e_{456}$ and the weights of its two first summands clearly indicate that they belong to $W_{(0,0,2)}$. The mixed term has weight 0, so if it projects nontrivially onto $\mathfrak{sp}(\mathcal{C})$, then it lives in the Cartan subalgebra. We claim that in fact the image of $[e_{123}+ke_{456}]$ is precisely

$$[h_1+h_2+h_3] = [\mathrm{diag}(1,1,1,-1,-1,-1)].$$

To this end, we can compute how the weight vectors X_{2h_i} act upon $2ke_{123}e_{456}$, namely,

$$X_{2h_1} \cdot 2ke_{123}e_{456} = 2ke_{123}e_{156},$$
$$X_{2h_2} \cdot 2ke_{123}e_{456} = 2ke_{123}e_{426},$$
$$X_{2h_3} \cdot 2ke_{123}e_{456} = 2ke_{123}e_{453}.$$

Since all three are nonzero, it follows that

$$[2ke_{123}e_{456}] \mapsto [ah_1 + bh_2 + ch_3]$$

for $a, b, c \neq 0$, and moreover $a = b$ because

$$X_{h_2 - h_1} \cdot X_{2h_1} \cdot 2ke_{123}e_{456} = X_{h_1 - h_2} \cdot X_{2h_2} \cdot 2ke_{123}e_{456}.$$

In the same manner we can show that $b = c$. As before, what we have shown so far ensures that the fiber contains $\mathbb{C}^\times$ and we see that the points 0 and ∞ lie in $\mathrm{LGr}(3,6)$. Again, let us consider the stabilizer acting transitively on the fiber over $[h_1 + h_2 + h_3]$:

$$\begin{aligned} &\mathrm{Stab}([h_1 + h_2 + h_3]) \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{Sp}(6) \mid \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = \begin{pmatrix} aA & aB \\ -aC & -aD \end{pmatrix}, a \in \mathbb{C}^* \right\}. \end{aligned}$$

Since now a can only take values ± 1, the stabilizer has two connected components: $B = C = 0$ and $A = D = 0$, so we cannot conclude as before. However, it is straightforward that for any g belonging to the stabilizer,

$$g \cdot [e_{123} + e_{456}] = [(\det(A) + \det(B))e_{123} + (\det(C) + \det(D))e_{456}],$$

so that there are no other components. □

8.2.6. *Summary of the results.*

	dim	representative	image of the representative	fiber	PDE
O	13	$[e_{123} + e_{456}]$	$[\epsilon^1 e_1 + \epsilon^2 e_2 + \epsilon^3 e_3]$	$\mathbb{C}^\times$	$\det \Vert u_{ij} \Vert = 1$
L	12	$[e_{423} + e_{126} + e_{153} + e_{123}]$	$[e_1^2 + e_2^2 + e_3^2]$	$\mathbb{C}^1$	$u_{11} + u_{22} + u_{33} = 0$
G	9	$[e_{163} + e_{125}]$	$[e_1^2]$	$\mathbb{P}^4 \setminus \mathrm{LGr}(2,4)$	$u_{23} = 0$
P	6	$[e_{123}]$	$\emptyset$	$--$	$u_{11} = 0$

It is worth stressing that, by regarding each element in the "representative" column as a 3–form (2.17) on $\mathcal{C}$, see Remark 6.1, and then by applying

(2.16) to the so–obtained 3–from, one *does not* obtain exactly the corresponding element in the "PDE" column, but rather an $\mathsf{Sp}(\mathcal{C})$–equivalent to it. The following example clarifies this aspect.

Example 8.1. By employing the same procedure, we used in Example 6.1, we will associate a Monge-Ampère equation to all of the four 3–forms that appear in the "representative" column of the above table.

- The form $e_{123}+e_{456}$ gives the Monge-Ampère equation $\det\|u_{ij}\|=1$.
- The form $e_{423}+e_{126}+e_{153}+e_{123}$ gives the Monge-Ampère equation

$$\det\|u_{ij}\| = u^{\sharp}_{11}+u^{\sharp}_{22}+u^{\sharp}_{22}\,.$$

  The last equation, however, thanks to a transformation (6.6), is $\mathsf{Sp}(\mathcal{C})$–equivalent to $u_{11}+u_{22}+u_{33}+1=0$, which, in turn, is $\mathsf{Sp}(\mathcal{C})$–equivalent to $u_{11}+u_{22}+u_{33}=0$, by using, for instance, the additional transformation $x^4\to x^4-x^1$.
- The $e_{163}+e_{125}$ gives the Monge-Ampère equation $u^{\sharp}_{23}=0$, which is $\mathsf{Sp}(\mathcal{C})$–equivalent to $u_{23}=0$, again, by means of (6.6).
- The form e_{123} gives the Monge-Ampère equation $\det\|u_{ij}\|=0$, which is $\mathsf{Sp}(\mathcal{C})$–equivalent to $u_{11}=0$: to see this it suffices to use a partial Legendre transformation

$$(x^1,x^2,x^3,x^4,x^5,x^6)\to(x^1,x^5,x^6,x^4,-x^2,-x^3),$$

  see Remark 6.2.

8.3. **Equivalence of the KLR invariant with the Hitchin moment map.** We recast now, in the complex setting, the definition of the KLR invariant, introduced earlier in Section 2.5; the underlying idea is the same: a 3–form η on a 6–dimensional symplectic space $(\mathcal{C},\omega)$ can be contracted with (the inverse ω^{-1} of) the symplectic form ω, thus obtaining a quadratic form on $\mathcal{C}$. In the present coordinates, we have

$$\eta=\eta_{ijk}x^{ijk}\in\Lambda^3(\mathcal{C}^*)\,,$$

so that the output $q(\eta)$ of the contraction, i.e., formula (2.20), will now look like

$$q(\eta)_{ab}=\eta_{aij}\eta_{bhk}\omega^{ih}\omega^{jk}\in S^2(\mathcal{C}^*)\,.$$

Direct computations show that $g^*(q(\eta)) = q(g^*(\eta))$, for all $g \in \mathsf{Sp}(\mathcal{C})$; moreover, if $\eta = \alpha \wedge \omega$, then $q(\eta) = 3\alpha^2$: in other words,

$$\begin{tikzcd} \mathcal{C}^* \arrow[r, "m_\omega"] \arrow[dr, "3v_2"'] & \Lambda^3(\mathcal{C}^*) \arrow[d, "q"] \\ & S^2(\mathcal{C}^*) \end{tikzcd}$$

is a commutative diagram of $\mathsf{Sp}(\mathcal{C})$–invariant polynomial maps. We will denote by the same symbol q the restriction of q to the subspace $\Lambda^3_0(\mathcal{C}^*)$. Let us stress that $\varpi : \mathbb{P}(\Lambda^3_0(\mathcal{C})) \dashrightarrow \mathbb{P}(S^2(\mathcal{C}))$, cf. (8.3), whereas $[q] : \mathbb{P}(\Lambda^3_0(\mathcal{C}^*)) \dashrightarrow \mathbb{P}(S^2(\mathcal{C}^*))$, so that the next statement makes sense, provided that one identifies $\mathcal{C}$ with $\mathcal{C}^*$ via ω, see Section 6.1.

Theorem 8.1. *The rational quadratic map ϖ coincides with the projectivization of q.*

Proof. In view of the invariance of both ϖ and q it suffices to check the claim on the orbit representatives. We also observe that, in our coordinates, the inverse of ω (which is nothing but $-\omega$) reads $x^{41} + x^{52} + x^{63}$.

Let us start from the open orbit (O): if we take $\eta = e_{123} + e_{456}$, we have that

$$\begin{aligned} q(\eta) &= (e_{123} + e_{456})(e_{123} + e_{456})(x^{41} + x^{52} + x^{63})(x^{41} + x^{52} + x^{63}) \\ &= (e_{123} + e_{456})(e_{23}x^4 + e_{13}x^5 + e_{12}x^6 + e_{56}x^1 + e_{46}x^2 + e_{45}x^3) \\ &\cdot (x^{41} + x^{52} + x^{63}) \\ &= (e_{123} + e_{456}) \cdot 2 \\ &\cdot (e_1x^5x^6 + e_2x^4x^6 + e_3x^4x^5 + e_4x^2x^3 + e_5x^1x^3 + e_6x^1x^2) \\ &= 4 \cdot (e_1e_4 + e_2e_5 + e_3e_6) = 4 \cdot q^{(3)}\,. \end{aligned}$$

This means that $\varpi([\eta]) = [g(\eta)]$ and the claim is valid on the orbit O, see Proposition 8.4.

In order to study the orbit L, we take $\eta = e_{423} + e_{126} + e_{153} + e_{123}$ and we compute

$$\begin{aligned} q(\eta) &= (e_{423} + e_{126} + e_{153} + e_{123})(e_{423} + e_{126} + e_{153} + e_{123}) \\ &\cdot (x^{41} + x^{52} + x^{63})(x^{41} + x^{52} + x^{63}) \\ &= (e_{423} + e_{126} + e_{153} + e_{123}) \cdot \\ &\cdot \big(e_{23}x^1 + (e_{26} + e_{53} + e_{23})x^4 + (e_{43} + e_{16} + e_{13})x^5 \end{aligned}$$

$$
\begin{aligned}
&+ e_{13}x^2 + (e_{42} + e_{15} + e_{12})x^6 + e_{12}x^3\big) \cdot (x^{41} + x^{52} + x^{63})\\
&(e_{423} + e_{126} + e_{153} + e_{123}) \cdot\\
&\cdot 2 \cdot \big(e_1(x^2x^6 + x^5x^6 + x^3x^5) + e_2(x^1x^6 + x^4x^6 + x^3x^4)\\
&+ e_3(x^1x^5 + x^2x^4 + x^4x^5) + e_4x^5x^6 + e_5x^4x^6 + e_6x^4x^5\big)\\
&= 2 \cdot (e_1(e_1 + e_1) + e_2(e_2 + e_2) + e_3(e_3 + e_3))\\
&= 4 \cdot (e_1^2 + e_2^2 + e_3^2) = 4 \cdot q_{[2^3]}\,,
\end{aligned}
$$

and the claim follows now from Proposition 8.3.

We pass to the last orbit on which ϖ acts nontrivially, that is the orbit G: we take $\eta = e_{163} + e_{125}$ and we compute

$$
\begin{aligned}
q(\eta) &= (e_{163} + e_{125})(e_{163} + e_{125})(x^{41} + x^{52} + x^{63})(x^{41} + x^{52} + x^{63})\\
&= (e_{163} + e_{125})(e_{63}x^4 + e_{25}x^4 + e_{12}x^2 + e_{15}x^5 + e_{13}x^2 + e_{16}x^3)\\
&\cdot (x^{41} + x^{52} + x^{63})\\
&= (e_{163} + e_{125})(2(e_1x^2x^5 + e_2x^2x^4) + e_3(x^4x^6 + x^3x^4) + e_5(x^4x^5 + x^2x^4)\\
&+ e_6(x^3x^4 + x^4x^6) + e_1((x^6)^2 + (x^3)^2))\\
&= 2 \cdot e_1^2 = 2 \cdot q_{[2,1^4]}\,,
\end{aligned}
$$

see then Proposition 8.2.

To complete the proof, we recall Proposition 8.1 and observe that, on the representative of the orbit G, we have

$$
\begin{aligned}
q(e_{123}) &= e_{123}e_{123}(x^{41} + x^{52} + x^{63})(x^{41} + x^{52} + x^{63})\\
&= e_{123}(e_{23}x^4 + e_{13}x^5 + e_{12}x^6)(x^{41} + x^{52} + x^{63})\\
&= e_{123} \cdot 2 \cdot (e_1x^5x^6 + e_2x^4x^6 + e_3x^4x^5) = 0\,. \qquad \square
\end{aligned}
$$

8.4. **Fiber compactification via singular limits (over $\mathbb{R}$).** In this subsection, we momentarily go back to the real–differentiable setting.

Indeed, the two–points compactification of the fiber through a generic Monge–Ampère equation, discussed in Section 8.2.5 above, becomes particularly evident if we go back to the real case and employ a total Legendre transform (for reasons of clarity, below we shall adopt the notation of Example 2.2). Then, instead of (8.5), we shall have a real one–parametric family of generic, that is, non–linearizable, Monge–Ampère equations, corresponding to the following family of 3–forms:

$$
\{[e_{123} + k \cdot e_{456}] \mid k \in \mathbb{R} \setminus \{0\}\}\,.
$$

By computing the Monge–Ampère equation associated with each member of the family (for the procedure, see also Example 8.1), we obtain a one–parameter family of PDEs:

$$\det \|u_{ij}\| = k. \tag{8.6}$$

While the limit for $k \to 0$ is the parabolic Monge–Ampère equation

$$\det \|u_{ij}\| = 0\,, \tag{8.7}$$

in order to take the limit of (8.6) for $k \to \infty$, we can, for instance, perform a total Legendre transform (i.e., (2.13) with $m = n = 3$), thus obtaining (see also Example 2.3)

$$\det \|\widetilde{u}_{ij}\| = -\frac{1}{k}$$

in the new (tilded) coordinates. Now, by taking $k \to \infty$, we obtain

$$\det \|\widetilde{u}_{ij}\| = 0\,. \tag{8.8}$$

The two equations (8.7) and (8.8), though equivalent, are not the same, as they correspond to two mutually transversal 3D sub-distributions, namely,

$$\mathcal{D} = \langle D_1^{(1)}, D_2^{(1)}, D_3^{(1)} \rangle \quad \text{and} \quad \widetilde{\mathcal{D}} = \langle \widetilde{D}_1^{(1)}, \widetilde{D}_2^{(1)}, \widetilde{D}_3^{(1)} \rangle = \langle \partial_{u_1}, \partial_{u_2}, \partial_{u_3} \rangle,$$

see Section 2.6, in particular equation (2.23) with $b_{ij} = 0$. Recall also that $D_i^{(1)} = \partial_{x^i} + u_i \partial_u$, cf. (2.9).

The proposed framework has thus allowed visualizing the "singular limit" procedure mentioned by E. Ferapontov in the context of integrable PDEs as a two–points compactification of an affine line whose points parametrize a family of non–linearizable Monge–Ampère equations. Also, note that the two parabolic Monge–Ampère equations (8.7) and (8.8), related by a total Legendre transformation, or, in a matter of speaking, by a "flip" of the corresponding 3D sub-distribution, are actually the limit points of a family of non–linearizable equations.

We would have obtained similar results, had we considered the partial Legendre transformation discussed in Remark 4.1 above. It transforms equation (8.6) into (cf. also Example 2.3) $k\tilde{u}_{11} + \tilde{u}_{22}\tilde{u}_{33} - \tilde{u}_{23}^2 = 0$. In this case, the limits $k \to \infty$ and $k \to 0$ give, respectively, the equations $\tilde{u}_{11} = 0$ and $\tilde{u}_{22}\tilde{u}_{33} - \tilde{u}_{23}^2 = 0$.

9. Hyperplane sections of the Lagrangian Grassmannian $\mathrm{LGr}(3, \mathcal{C})$

We can finally pick up the thread we left at the end of Section 6 and provide a proof of Theorem 4.1 over the field of the complex numbers, see

Corollary 9.1 below. The reader must be warned that the definition of a Monge–Ampère equation, as well as that of its cocharacteristic variety, were already given above, see Definition 2.1 and Definition 2.2, respectively: we repeat them below (see Definition 9.1 and Definition 9.4, respectively) to stress that, despite their formal similarity, the two versions of the same definition pertain to different categories (real differentiable versus complex analytic). Moreover, the study carried out in the second part of the paper is strictly point–wise, i.e., it pertains a particular fiber of the bundle $J^2 \to J^1$, whereas in the first part the formalism is global, even though we have always considered *symplectic* (i.e., not depending on the point of J^1) Monge–Ampère equations: compare, for instance, the earlier Definition 2.3 of a Goursat type Monge–Ampère equation with the new Definition 9.5 given below.

9.1. **The Lagrangian Grassmannian and its tangent geometry.** In what follows, X will be denoting the 6–dimensional regular variety $\mathrm{LGr}(3, \mathcal{C})$ of 3–dimensional isotropic linear subspaces of $\mathcal{C}$, embedded into $\mathbb{P}(\Lambda^3_0(\mathcal{C}))$ via the Plücker embedding

$$X \longrightarrow \mathbb{P}(\Lambda^3_0(\mathcal{C}))\,, \quad L = \langle l_1, l_2, l_3 \rangle \longmapsto \mathrm{vol}(L) := [l_1 \wedge l_2 \wedge l_3]\,.$$

Let us fix $L \in X$: a nice description of the tangent geometry of X at L goes as follows [37]: we "perturb" the Lagrangian subspace L by means of a linear map $h \in \mathrm{Hom}(L, \mathcal{C})$, that is, we employ h to define a curve

$$\begin{aligned} \gamma_h(t) &:= [(l_1 + th(l_1)) \wedge (l_2 + th(l_2)) \wedge (l_3 + th(l_3))] \\ &= [l_1 \wedge l_2 \wedge l_3 + t(h(l_1) \wedge l_2 \wedge l_3 + l_1 \wedge h(l_2) \wedge l_3 + l_1 \wedge l_2 \wedge h(l_3)) + o(t^2)] \end{aligned} \tag{9.1}$$

passing through L at $t = 0$. Above formula (9.1) shows that, if h takes its values in L, then part that is linear in t is absorbed by the free term and, hence, the velocity $\dot{\gamma}_h(0)$ of γ_h at 0 vanishes: in order to obtain all tangent vectors to X at L it is then safe to assume that h is an element of

$$\mathrm{Hom}(L, \tfrac{\mathcal{C}}{L}) = L^* \otimes \tfrac{\mathcal{C}}{L} = L^* \otimes L^*\,,$$

where the identification $\frac{\mathcal{C}}{L} = L^*$ is given by the isomorphism $\widetilde{\omega}_L$ closing the commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\omega} & \mathcal{C}^* \\ \downarrow & & \downarrow \\ \frac{\mathcal{C}}{L} & \xrightarrow{\widetilde{\omega}_L} & L^*\,. \end{array}$$

Less evident is that $\gamma_h(t)$ is a curve in X if and only if $h \in S^2(L^*)$ (it can be proved by a direct coordinate approach); since the contraction

$$\begin{aligned} S^2(L^*) &\longrightarrow \Lambda^3(\mathcal{C}) \\ h &\longmapsto h\lrcorner(l_1\wedge l_2\wedge l_3) := h(l_1)\wedge l_2\wedge l_3 + l_1\wedge h(l_2)\wedge l_3 + l_1\wedge l_2\wedge h(l_3) \end{aligned}$$

is injective, it can be concluded that $h \longmapsto \dot\gamma_h(0)$ is a monomorphism of $S^2(L^*)$ into T_LX and then, for obvious dimensional reasons, an isomorphism.

9.2. **Hyperplane sections and their tangent geometry.** The moduli space of hyperplane sections of X is the 13–dimensional projective space $\mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$: in view of the applications to the theory of 2^{nd} order PDEs, let us recall, cf. (6.15), that $\mathcal{E}_\eta$ denotes the hyperplane section determined by $[\eta] \in \mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$: in other words, from now on, the name "(symplectic) Monge–Ampère equation" will be a synonym of "hyperplane section".

Definition 9.1. *The five–fold $\mathcal{E}_\eta \subset X$ is the* Monge–Ampère equation *associated with $\eta \in \Lambda^3_0(\mathcal{C}^*)$ (in the sense of Lychagin).*

Let us now fix a smooth point $L \in (\mathcal{E}_\eta)_{\mathrm{sm}}$ and consider the curve (9.1) determined by $h \in S^2(L^*)$: it will be tangent to the MAE $\mathcal{E}_\eta$ if and only if

$$\eta(h\lrcorner(l_1\wedge l_2\wedge l_3)) = 0\,.$$

Put differently, having defined the linear map

$$T_LX = S^2(L^*) \xrightarrow{\widetilde{\eta}_L} \Lambda^3 L^*\,, \quad h \longmapsto \eta(h\lrcorner\,\cdot\,)\,,$$

we obtain

$$T_L(\mathcal{E}_\eta) = \ker \widetilde{\eta}_L\,. \tag{9.2}$$

Definition 9.2. *The quadric $\sigma_L \subset \mathbb{P}(L^*)$ cut out by the equation $\widetilde{\eta}_L = 0$ is the* characteristic variety *of the MAE $\mathcal{E}_\eta$ at the point L.*

We can globalize the above reasoning by introducing the tautological bundle $\mathcal{L}$ on X (whose fiber at $L \in X$ is L itself): we have then a global identification $TX = S^2(\mathcal{L}^*)$ and $\widetilde{\eta}_L$ turns out to be the value at L of a section $\widetilde{\eta} \in \Gamma((\mathcal{E}_\eta)_{\mathrm{sm}}, S^2(\mathcal{L})\otimes\Lambda^3(\mathcal{L}^*))$, such that

$$\ker \widetilde{\eta}|_{(\mathcal{E}_\eta)_{\mathrm{sm}}} = T(\mathcal{E}_\eta)_{\mathrm{sm}}\,.$$

Definition 9.3. *The sub-variety $\sigma \subset \mathbb{P}(\mathcal{L}^*|_{(\mathcal{E}_\eta)_{sm}})$ cut out by the equation $\widetilde{\eta} = 0$ is the* characteristic variety *of the MAE $\mathcal{E}_\eta$.*

Observe that $\sigma = \bigcup_{L\in(\mathcal{E}_\eta)_{\mathrm{sm}}} \sigma_L$.

9.3. **Projective duality and the cocharacteristic variety.** We would like now to pass to the projective dual $\sigma_L^\vee \subset \mathbb{P}(L)$ of the characteristic varieties σ_L, because each $\mathbb{P}(L)$ embeds naturally into $\mathbb{P}(\mathcal{C})$, whereas $\mathbb{P}(L^*)$ does not posses any special embedding into $\mathbb{P}(\mathcal{C}^*)$: the passage from $\mathbb{P}(L^*)$ to $\mathbb{P}(L)$ will allow us to regard the sum of all the $\sigma_L^\vee$'s as a subset of $\mathbb{P}(\mathcal{C})$, viz.

$$\sigma^\vee := \overline{\bigcup_{L\in(\mathcal{E}_\eta)_{\mathrm{sm}}} \sigma_L^\vee} \subset \mathbb{P}(\mathcal{C})\,. \tag{9.3}$$

Definition 9.4. *The subset $\sigma^\vee$ of $\mathbb{P}(\mathcal{C})$ is called the* cocharacteristic variety *of the MAE $\mathcal{E}_\eta$.*

The equation cutting out $\sigma_L^\vee$ is easily obtained from $\widetilde{\eta}$, regarded as a map $\widetilde{\eta}_L : L^* \longrightarrow L \otimes \Lambda^3(L^*)$ and then passing to its second exterior power:

$$\Lambda^2(\widetilde{\eta}_L) : \Lambda^2(L^*) \longrightarrow \Lambda^2(L) \otimes (\Lambda^3(L^*))^2\,.$$

Then, in virtue of the identifications

$$\Lambda^2(L) = L^* \otimes \Lambda^3(L) \quad \text{and} \quad \Lambda^2(L^*) = L \otimes \Lambda^3(L^*),$$

we can regard $\Lambda^2(\widetilde{\eta}_L)$ as

$$\Lambda^2(\widetilde{\eta}_L) : L \otimes \Lambda^3(L^*) \longrightarrow L^* \otimes \Lambda^3(L) \otimes (\Lambda^3(L^*))^2 = L^* \otimes \Lambda^3(L^*)\,,$$

and finally

$$\Lambda^2(\widetilde{\eta}_L) \in L^* \otimes L^* \otimes \Lambda^3(L^*) \otimes \Lambda^3(L) = L^* \otimes L^*.$$

In view of the usual symmetry consideration, we have in fact that $\Lambda^2(\widetilde{\eta}_L) \in S^2(L^*)$. Since $\sigma_L^\vee$ is cut out by the 2×2 minors of the (3×3) matrix of the quadratic form $\widetilde{\eta}_L$, and the matrix of such minors is precisely $\Lambda^2(\widetilde{\eta}_L)$, we have that

$$\sigma_L^\vee = \{\Lambda^2(\widetilde{\eta}_L) = 0\} \subset \mathbb{P}(L)\,. \tag{9.4}$$

It is worth observing that

$$\Lambda^2(\widetilde{\eta}) \in \Gamma((\mathcal{E}_\eta)_{\mathrm{sm}}, S^2(\mathcal{L}^*))\,, \tag{9.5}$$

that is, $\Lambda^2(\widetilde{\eta})$ is a section of honest (untwisted) quadratic forms on the tautological bundle $\mathcal{L}$. In the last Section 9.1, we will prove that the cocharacteristic variety is, in the appropriate sense, cut out by section 9.5. Before passing to that, we study the rank of the characteristic variety across the four isomorphism types of Monge–Ampère equations.

9.4. **Singular loci of Monge–Ampère equations.** The Lagrangian Grassmanian $X \subset \mathbb{P}(\Lambda^3_0(\mathcal{C}))$ allows us to recast the stratification of $\mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$ in terms of dual varieties: in this perspective, the open (13–dimensional) orbit is the complement $\mathbb{P}(\Lambda^3_0(\mathcal{C}^*)) \smallsetminus X^\vee$, whereas the closed (6–dimensional) one is isomorphic to X itself via the isomorphism $\mathbb{P}(\Lambda^3_0(\mathcal{C})) \longrightarrow \mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$ induced by the symplectic form Ω, see Remark 6.3. In order to stress that X is embedded into $\mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$, we will use the symbol X^* for it (not to be confused with the projective dual $X^\vee$).

The remaining strata, of dimension 12 and 9, are $X^\vee \smallsetminus \mathrm{Sing}(X^\vee)$ and $\mathrm{Sing}(X^\vee) \smallsetminus X^*$, respectively.

As observed in [26, Proposition 2.5.1 (iii)] there is a finite map

$$\mathrm{Gr}(3, \mathcal{C}) \longrightarrow \mathrm{Sing}(X^\vee)\,, \quad D \longmapsto [\eta(D)]\,, \tag{9.6}$$

where $[\eta(D)]$ is the only element, such that the hyperplane section $\mathcal{E}_{\eta(D)}$ is the Schubert cycle

$$\mathcal{E}_{\eta(D)} = \mathcal{E}_D := \{L \in X \mid \dim(L \cap D) \geq 1\} \tag{9.7}$$

determined by D in X; see also [1, Section 3.3] for the real–differentiable setting.

Definition 9.5. *The Monge–Ampère equation $\mathcal{E}_D$ is called the* Goursat type Monge–Ampère equation *determined by the 3–dimensional subspace $D \subset \mathcal{C}$. If D is Lagrangian, then $\mathcal{E}_D$ is called* parabolic.

The map (9.6), restricted to the Lagrangian Grassmannian $X \subset \mathrm{Gr}(3, \mathcal{C})$, realizes an isomorphism between X and X^*, whereas the fiber over a point $[\eta(D)] \in \mathrm{Sing}(X^\vee) \smallsetminus X^*$ is the pair $\{D, D^\perp\}$: all of this reflects what happens in the real–differentiable setting, see Section 2.6 above.

The results of Theorem 9.1 below are partially contained in the above–quoted [26, Proposition 2.5.1].

Theorem 9.1. *Let $[\eta] \in \mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$. Up to $\mathsf{Sp}(\mathcal{C})$-action, we have four possibilities:*

(O) *if $[\eta] \notin X^\vee$, then $\mathcal{E}_\eta$ is nonsingular and* $\operatorname{rank}\widetilde{\eta} = 3$*;*
(L) *if $[\eta] \in X^\vee \smallsetminus \mathrm{Sing}(X^\vee)$, then $\mathcal{E}_\eta$ has a unique quadratic singularity and* $\operatorname{rank}\widetilde{\eta} = 3$ *on $(\mathcal{E}_\eta)_{sm}$;*
(G) *if $[\eta] \in \mathrm{Sing}(X^\vee) \smallsetminus X^*$, then $\mathrm{Sing}(\mathcal{E}_\eta)$ is isomorphic to the Schubert cycle determined by a non–Lagrangian two–dimensional subspace in* $\mathrm{LGr}(2,4)$ *and* $\operatorname{rank}\widetilde{\eta} = 2$ *on $(\mathcal{E}_\eta)_{sm}$;*
(P) *if $[\eta] \in X^*$, then $\mathrm{Sing}(\mathcal{E}_\eta)$ is the projective cone over the Veronese surface, and* $\operatorname{rank}\widetilde{\eta} = 1$ *on $(\mathcal{E}_\eta)_{sm}$.*

Before giving a proof, we recall the $\mathsf{Sp}(\mathcal{C})$–equivariant double fibration

$$\begin{tikzcd} & \mathrm{LFl}(1,3;\mathcal{C}) \arrow[dl, "\overline{p}"'] \arrow[dr, "\overline{q}"] & \\ \mathbb{P}\mathcal{C} & & X\,, \end{tikzcd} \tag{9.8}$$

where $\mathrm{LFl}(1,3;\mathcal{C})$ is the smooth, eight–dimensional $\mathsf{Sp}(\mathcal{C})$–homogeneous variety of *Lagrangian* (or, according to some authors, *ω–isotropic*) *flags* of $\mathcal{C}$. The fibers are easily described: $\overline{p}^{-1}(\ell) = \mathrm{LGr}(2, \frac{\ell^\perp}{\ell})$, that is, a copy of the 3–dimensional quadric $\mathrm{LGr}(2,4) \subset \mathbb{P}^4$, for all $\ell \in \mathbb{P}\mathcal{C}$, whereas $\overline{q}$ is nothing but the projectivized tautological bundle $\mathbb{P}\mathcal{L}$, i.e., $\overline{q}^{-1}(L) = \mathbb{P}L$, that is, a copy of $\mathbb{P}^2$, for all $L \in X$.

The projectivization $\mathbb{P}D$ of a 3–dimensional linear subspace $D \in \mathrm{Gr}(3,\mathcal{C})$ of $\mathcal{C}$ is an irreducible sub-variety of $\mathbb{P}\mathcal{C}$ of pure codimension three and degree one: by regarding the Schubert cycle (9.7) as the *Lagrangian Chow transform*

$$\mathcal{E}_D = \overline{q}(\widetilde{\mathcal{E}}_D)\,, \quad \widetilde{\mathcal{E}}_D := \overline{p}^{-1}(\mathbb{P}D) = \{(\ell, L) \mid \ell \subseteq L \cap D\} \tag{9.9}$$

of $\mathbb{P}D$, we come to the conclusion that $\mathcal{E}_D$ is a hypersurface of X of degree one, that is, a hyperplane section, see [2, Lemma 23]. This hypersurface, however, is not smooth.

Diagram (9.8) provides a singularity resolution of $\mathcal{E}_D$: the subset $\widetilde{\mathcal{E}}_D \subset \mathrm{LFl}(1,3;\mathcal{C})$, being the restriction of the bundle $\overline{p}$ to a smooth sub-variety of its base $\mathbb{P}\mathcal{C}$, is smooth as well and has dimension $5 = 2 + 3 = \dim(\mathbb{P}D) + \dim(\mathrm{LGr}(2,4))$, that is the same dimension of $\mathcal{E}_D \subset X$: by restricting (9.8) to $\mathbb{P}D$ and $\mathcal{E}_D$, we obtain then a double fibration

$$\begin{tikzcd} & \widetilde{\mathcal{E}}_D \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}D & & \mathcal{E}_D\,, \end{tikzcd} \tag{9.10}$$

where p has the same fibers as $\overline{p}$, but the restriction q of $\overline{q}$ is a surjective morphism between varieties of the same dimension five.

Notation

$$\mathcal{E}_{D,i} := \{L \in X \mid \dim(L \cap D) = i\}\,, \quad i = 1,2,3\,, \tag{9.11}$$

will come in handy in the proof below.

Proof of Theorem 9.1. In the case (O), the hyperplane $\mathbb{P}(\ker\eta)$ is nowhere tangent to X, therefore $\mathcal{E}_\eta = X \cap \mathbb{P}(\ker\eta)$ must be nonsingular.

In the case (L), there exists a unique line through $[\eta]$ that is tangent to X^*. The point that corresponds to the tangency point in X, via the isomorphism induced by the symplectic form, is precisely the unique singular point of $\mathcal{E}_\eta$. If we choose some local coordinates around this singular point, then the Hessian will be nonzero in it, so the singularity will be quadratic: see [38, Theorem 6.1]

In the case (G), there exists an element $D \in \mathrm{Gr}(3,\mathcal{C}) \smallsetminus X$, such that $\mathcal{E}_\eta = \mathcal{E}_D$: the dimension of $L \cap D$ must then be less than 3 and the union $\mathcal{E}_D = \mathcal{E}_{D,1} \cup \mathcal{E}_{D,2}$ is disjoint, cf. (9.11). We also observe that, since $\omega|_D$ must be degenerate and $\ker \omega|_D$ cannot be 3, the line $\ell_D := \ker \omega|_D$ will fulfill $\omega(\ell, D) = 0$, that is $\ell_D \subset D^\perp$: summing up,

$$\ell_D = D \cap D^\perp\,, \quad \dim \ell_D = 1\,, \quad D + D^\perp = \ell_D^\perp\,, \quad \dim \ell_D^\perp = 5\,. \tag{9.12}$$

If $L \in \mathcal{E}_{D,1}$, then $\ell(D) := L \cap D$ is a line and then $q^{-1}(L) = \{\ell(D)\} \subset \mathbb{P}L$ is a point, cf. (9.9): in other words,

$$q' : \mathcal{E}_{D,1} \longrightarrow \mathbb{P}\mathcal{L}|_{\mathcal{E}_{D,1}}\,, \quad L \longmapsto \ell(D)$$

defines a section of the projectivized tautological bundle over $\mathcal{E}_{D,1}$, i.e., a section of $\overline{q}$ over $\mathcal{E}_{D,1}$, that takes its values in $q^{-1}(\mathcal{E}_{D,1})$ and it is clearly an inverse of q. Summing up, q restricts to an isomorphism between the subsets $q^{-1}(\mathcal{E}_{D,1}) \subset \widetilde{\mathcal{E}}_D$ and $\mathcal{E}_{D,1} \subset \mathcal{E}_D$, which in turn implies that $\mathrm{Sing}(\mathcal{E}_D) \subseteq \mathcal{E}_D \smallsetminus \mathcal{E}_{D,1} = \mathcal{E}_{D,2}$.

If $L \in \mathcal{E}_{D,2}$, then $\pi := L \cap D$ is a plane and, by passing to the corresponding symplectic orthogonal subspaces, $\pi^\perp = L + D^\perp$, with $\dim \pi^\perp = 4$: it follows that $\dim L \cap D^\perp = 2$, i.e., L has a plane in common both with D and with $D^\perp$; but L has dimension three, so these two planes cannot be disjoint: they must necessarily have a line in common, and such a line must necessarily be ℓ_D. In other words, L contains ℓ_D and it is contained into $\ell_D^\perp$; it follows that the image $\widetilde{L}$ of L via the projection $\ell_D^\perp \longrightarrow \frac{\ell_D^\perp}{\ell_D}$ on the 4–dimensional symplectic space $\frac{\ell_D^\perp}{\ell_D}$ will be a 2–dimensional isotropic subspace. This means that $\widetilde{L} \in \mathrm{LGr}(2, \frac{\ell_D^\perp}{\ell_D})$; denoting by $\widetilde{D}$ the projection of D, it is then obvious that the projection itself induces an isomorphism between $\mathcal{E}_{D,2}$ and the (two–dimensional) Schubert cycle

$$\mathcal{E}_{\widetilde{D}} = \left\{ \widetilde{L} \in \mathrm{LGr}(2, \tfrac{\ell_D^\perp}{\ell_D}) \mid \dim(\widetilde{L} \cap \widetilde{D}) \geq 1 \right\}\,,$$

which is smooth, since the dimension of $\widetilde{L} \cap \widetilde{D}$ is ≥ 1 if and only if it is equal to one. We also observe that $q^{-1}(L) = \mathbb{P}(L \cap D) \subset \mathbb{P}L$ is a copy of $\mathbb{P}^1$, for all $L \in \mathcal{E}_{D,2}$.

To prove the first part of claim (G) it then suffices to show that $\mathrm{Sing}(\mathcal{E}_D) \supseteq \mathcal{E}_{D,2}$; let us take $L \in \mathcal{E}_{D,2}$ and assume that there is a smooth neighborhood $\mathcal{U} \subset \mathcal{E}_D$ of L: then q will be a surjective morphism between the five–folds $q^{-1}(\mathcal{E}_{D,2})$ and $\mathcal{E}_{D,2}$ and, therefore, there are two possibilities: either it is an isomorphism, or the locus

$$Z := \{z \in q^{-1}(\mathcal{U}) \mid \det(d_z q) = 0\} \subseteq q^{-1}(\mathcal{U}) \subseteq \widetilde{\mathcal{E}}_D \, , \tag{9.13}$$

where the rank of dq drops, is a hypersurface, that is, 4–dimensional. The first scenario is quickly discarded, since the fiber of q over L is a whole $\mathbb{P}^1$; in general,

$$q^{-1}(\mathcal{E}_{D,2}) \subseteq Z \tag{9.14}$$

is a proper subset, because $\dim \mathcal{E}_{D,2} = 2$ and q has one–dimensional fibers over $\mathcal{E}_{D,2}$, hence $q^{-1}(\mathcal{E}_{D,2})$ has dimension three: at the same time, since Z cannot contain the pre–images of points outside $\mathcal{E}_{D,2}$, i.e., $Z \cap q^{-1}(\mathcal{E}_D \smallsetminus \mathcal{E}_{D,2}) = \emptyset$, inclusion $Z \subseteq q^{-1}(\mathcal{E}_{D,2})$ should hold as well—a contradiction.

It was then forbidden to assume that $L \in \mathcal{E}_{D,2}$ admit a smooth neighborhood $\mathcal{U} \subset \mathcal{E}_D$: all points of $\mathcal{E}_{D,2}$ are singular, whence the sought–for inclusion $\mathrm{Sing}(\mathcal{E}_D) \supseteq \mathcal{E}_{D,2}$.

In the last case (P) the element D, such that $\mathcal{E}_\eta = \mathcal{E}_D$, is Lagrangian, i.e., $D \in X$: the dimension of $L \cap D$ can then attain the value 3 in the case when L coincides with D, so that in the disjoint union $\mathcal{E}_D = \mathcal{E}_{D,1} \cup \mathcal{E}_{D,2} \cup \mathcal{E}_{D,3}$, cf. (9.11), we have $\mathcal{E}_{D,3} = \{D\}$. As before, the restriction of q defines an isomorphism between $q^{-1}(\mathcal{E}_{D,1})$ and $\mathcal{E}_{D,1}$, so that $\mathrm{Sing}(\mathcal{E}_D) \subseteq \mathcal{E}_{D,2} \cup \mathcal{E}_{D,3}$.

Since D is Lagrangian, there always exists a $D' \in X$, such that $\mathcal{C} = D \oplus D'$ is a bi–Lagrangian decomposition; the latter determines the so–called *big cell* $\mathcal{V} := \{L \in X \mid L \cap D' = 0\}$, that is an open and dense subset of X isomorphic to S^2D^*: a quadratic form $h \in S^2D^*$ is identified with the Lagrangian subspace $\mathrm{graph}(h) := \langle x + h(x) \mid x \in D\rangle \subset \mathcal{C}$, having understood $h \in \mathrm{Hom}(D, D^*)$ as an element of $\mathrm{Hom}(D, D')$ by means of the natural identifications $D^* \simeq \frac{\mathcal{C}}{D} \simeq D'$, see [22, Section 1.2]. Obviously, $\mathcal{E}_{D,3} \subset \mathcal{V}$, because $\mathcal{E}_{D,3} = \{D\}$ and $D = \mathrm{graph}(0) \in \mathcal{V}$.

A key remark is that

$$L \cap D = \ker(h)$$

in the case when $L = \mathrm{graph}(h)$: it follows that

$$\mathcal{E}_D \cap \mathcal{V} = \{h \in S^2D^* \mid \det(h) = 0\};$$

therefore, the well–known formula $d_h(\det) = h^\sharp \cdot dh$ for the differential of the determinant $\det : S^2D^* \to \mathbb{C}$ at the point $h \in S^2D^*$ implies that

$$\mathrm{Sing}(\mathcal{E}_D \cap \mathcal{V}) = \mathrm{Sing}(\mathcal{E}_D) \cap \mathcal{V} = \{h \in S^2D^* \mid \mathrm{rank}(h) = 0, 1\} \, .$$

On the other hand, if $L \in \mathcal{E}_{D,2} \cap \mathcal{V}$, then $L = \mathrm{graph}(h)$ and $L \cap D = \ker(h)$ is two–dimensional, i.e., $\mathrm{rank}(h) = 1$, and vice–versa: in other words,

$$\mathcal{E}_{D,2} \cap \mathcal{V} = \{h \in S^2D^* \mid \mathrm{rank}(h) = 1\},$$

which leads to

$$\mathrm{Sing}(\mathcal{E}_D) \cap \mathcal{V} = (\mathcal{E}_{D,2} \cap \mathcal{V}) \cup \mathcal{E}_{D,3}\,. \tag{9.15}$$

To prove the first part of claim (P), we observe that the composition of the inclusion $\mathcal{V} \subset X$ with the Plücker embedding $X \longrightarrow \mathbb{P}(\Lambda^3_0\mathcal{C})$ is the map

$$\begin{aligned} S^2D^* \simeq \mathcal{V} &\stackrel{\iota}{\longrightarrow} \mathbb{P}(\Lambda^3_0\mathcal{C}) = \mathbb{P}(\mathbb{C}^* \oplus S^2D^* \oplus S^2D \oplus \mathbb{C})\,,\\ h &\longrightarrow [1 : h : h^\sharp : \det h]\,, \end{aligned} \tag{9.16}$$

where we decomposed $\Lambda^3_0\mathcal{C}$ into irreducible $\mathsf{SL}(D)$–modules, see [22, Section 1.5] (the reader will recognize in (9.16) the same expression that appeared in (2.19), after replacing h with u).

Therefore, we can regard the image of $\mathcal{V} \cap (\mathcal{E}_{D,2} \cup \mathcal{E}_{D,3})$ via (9.16) as a subset of the projective subspace $\mathbb{P}(\mathbb{C}^* \oplus S^2D^*)$:

$$\iota(\mathcal{V} \cap (\mathcal{E}_{D,2} \cup \mathcal{E}_{D,3})) = \{[1 : h] \mid \mathrm{rank}(h) = 0, 1\} \subset \mathbb{P}(\mathbb{C}^* \oplus S^2D^*)\,. \tag{9.17}$$

In particular,

$$\iota(D) = [1 : 0] = \mathbb{P}(\mathbb{C}^*)$$

is the unique point in the image of $\mathcal{E}_{D,3}$: the *projective* cone with vertex $[1 : 0]$ over the Veronese surface $v_2 : \mathbb{P}D^* \longrightarrow \mathbb{P}(S^2D^*)$ in $\mathbb{P}(S^2D^*)$, that is

$$\bigcup_{x \in v_2(\mathbb{P}D^*)} \mathbb{P}^1([1 : 0], x) = \underbrace{\{[1 : h] \mid h \in \widehat{v_2(\mathbb{P}D^*)}\}}_{\text{three–dimensional}} \cup \underbrace{v_2(\mathbb{P}D^*)}_{\text{two–dimensional}}\,, \tag{9.18}$$

contains then (9.17) as the (open and dense) three–dimensional subset, where by $\widehat{v_2(\mathbb{P}D^*)} \subset S^2D^*$, we meant the *affine* cone.

Since $\overline{\mathcal{V}} = X$ and $\mathcal{E}_{D,2} \cup \mathcal{E}_{D,3}$ is closed, by passing to the closures, we see that $\iota(\mathcal{E}_{D,2} \cup \mathcal{E}_{D,3})$ coincides with the whole of (9.18); similarly, from (9.15) we obtain $\mathrm{Sing}(\mathcal{E}_D) \supseteq \mathcal{E}_{D,2} \cup \mathcal{E}_{D,3}$ and then the equality. But $\iota(\mathcal{E}_{D,2} \cup \mathcal{E}_{D,3})$ is the projective cone over the Veronese surface and ι is an embedding, therefore, we are done.

In order to finish the proof, it suffices to compute the rank of $\widetilde{\eta}$ for each of the four 3–forms η listed in Section 8.2.6; to this end, we observe that the linear subspace (9.2) can be recast as $\ker dF$, where F is any function, such that, locally, $\mathcal{E}_\eta = \{F = 0\}$: a choice of such a function can be seen in the last column of the aforementioned table. Therefore, $\widetilde{\eta}$ and dF are proportional and, as such, regarded as quadratic forms, they have the same rank: it then remains to use formula (1.5) to compute the symbol of $\mathcal{E}_\eta$ at a smooth point of the open and dense subset where equality $\mathcal{E}_\eta = \{F = 0\}$ holds. In the cases (L), (G) and (P) one immediately obtains, retaining the same notation as in (1.5), $\eta_1^2+\eta_2^2+\eta_3^2$, $\eta_1\eta_2$ and η_1^2, respectively, whose rank is manifestly 3, 2 and 1, respectively. The last case (O) can be worked out in the same fashion. □

9.5. **The cocharacteristic variety via the momentum map.** We go back to the idea of the cocharacteristic variety of a Monge–Ampère equation that we have introduced earlier in Section 9.3, and we prove the next original result of the paper.

The construction that has led us to (9.5) is manifestly $\mathsf{Sp}(\mathcal{C})$–invariant, so that, if we compute $\Lambda^2(\widetilde{\eta})$ for any of the four representatives η listed in Section 8.2.6 above, we will have obtained all four possible isomorphism classes of such sections. The very same approach has been used in Theorem 8.1, when we established that the KLR invariant $q(\eta)$ is, up to a projective factor, the same as the Hitchin moment map $\varpi(\eta)$, both computed on the 3–form η: we finish up this paper by establishing another (projective) equality between $q(\eta)$ (or $\varpi(\eta)$), and $\Lambda^2(\widetilde{\eta})$, showing at the same time that all the three of them cut out the cocharacteristic variety of $\mathcal{E}_\eta$.

To make such last claim rigorous, it must be underlined that there is a $\mathsf{Sp}(\mathcal{C})$–equivariant linear embedding

$$S^2(\mathcal{C}^*) \xrightarrow{s} \Gamma((\mathcal{E}_\eta)_{\mathrm{sm}}, S^2(\mathcal{L}^*)) \tag{9.19}$$

sending a quadratic form q to the section $s(q)$ defined in the natural way:

$$s(q)(L) := i_L^*(q)\,, \quad \forall L \in (\mathcal{E}_\eta)_{\mathrm{sm}}\,,$$

where $i_L : L \subset \mathcal{C}$ denotes the embedding.

We have then the following diagram of $\mathsf{Sp}(\mathcal{C})$–equivariant maps:

$$\begin{tikzcd} & \mathbb{P}(\Lambda^3_0(\mathcal{C})) \arrow[dl, dashed, "\varpi={[q]}"'] \arrow[d, dashed, "{[\Lambda^2(\widetilde{\cdot})]}"] \\ \mathbb{P}(S^2(\mathcal{C}^*)) \arrow[r, "s"] & \mathbb{P}\big(\Gamma((\mathcal{E}_\eta)_{\mathrm{sm}}, S^2(\mathcal{L}^*))\big)\,. \end{tikzcd} \tag{9.20}$$

The commutativity of (9.20) can be checked manually by a case–by–case technique. The departing point is always an unknown quadratic form on $\mathcal{C}$:

$$q = r^{ij}e_{i+3}e_{j+3} + s^{ij}e_{i+3}e_j + t^{ij}e_ie_j\,,$$

that is, a 6×6 symmetric matrix presented in a 3×3 block form, cf. (6.1).

Since the other cases are formally analogous, we will be focusing only on the orbit "O", i.e., on the equation $\det\|u_{ij}\| = 1$ that corresponds to

$\eta = e_{123}+e_{456}$, cf. (8.6). As a first step, we choose a local parametrization of $\mathcal{E}_\eta$, for example, by calculating u_{11} as a function of the remaining parameters:

$$u_{11} = f(u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = \frac{u_{12}^2 u_{33} - 2u_{12}u_{13}u_{23} + u_{13}^2 u_{22} + 1}{u_{22}u_{33} - u_{23}^2}\,.$$

Next, for any point $L \in \mathcal{E}_\eta$ lying in this local parametrization, i.e., for any

$$L = \Big\langle e_1 + f(u_{12}, u_{13}, u_{22}, u_{23}, u_{33})e_4 + u_{12}e_5 + u_{13}e_6, e_2 + \sum_{i=1}^{3} u_{2i}e_{i+3}, e_3 + \sum_{i=1}^{3} u_{3i}e_{i+3}\Big\rangle, \quad (9.21)$$

we compute the restriction $i_L^*(q)$, which is then a 3×3 symmetric matrix depending upon the five parameters $u_{12}, u_{13}, u_{22}, u_{23}, u_{33}$.

Such a matrix can be computed by hand (better if aided by a computer algebra software) though, due to its lengthy expression, we leave it aside and we pass to the computation of the symbol of the equation $\mathcal{E}_\eta$ at the same point L given by (9.21): the result is again a 3×3 (symmetric) matrix, cf. (1.5), depending on the same five parameters $u_{12}, u_{13}, u_{22}, u_{23}, u_{33}$; it is not hard to see that its adjoint matrix is given by

$$\begin{pmatrix} \frac{u_{12}^2 u_{33} - 2u_{12}u_{13}u_{23} + u_{13}^2 u_{22} + 1}{u_{22}u_{33} - u_{23}^2} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{pmatrix} \quad (9.22)$$

The desired commutativity of (9.20) (at least on the orbit "O") means precisely that the requirement that $i_L^*(q)$ and the matrix (9.22) above cut out the same quadric in L, *for all the points* $L \in (\mathcal{E}_\eta)_{\mathrm{sm}}$, singles out a unique q (up to a projective class).

By imposing that $i_L^*(q)$ coincide with (9.22) for all the values of the five parameters $u_{12}, u_{13}, u_{22}, u_{23}, u_{33}$, one obtains a system of 83 linear equations in the 21 entries of q, whose left–hand sides are listed below:

$$\begin{aligned} &- t^{11}s^{11} - 1, -r^{11}, 1 - s^{11}, 2r^{11}, t^{13}, -s^{13}, -t^{13}, 2s^{13}, -t^{33}, -2t^{11}, 2t^{33}, t^{12},\\ &- s^{12}, -t^{12}, 2s^{12}, 4t^{11}, 2 - 2s^{11}, -t^{23}, 2s^{11} - 2, 2t^{23}, -2t^{13}, 2t^{13}, -t^{22}, 2t^{22},\\ &- 2t^{12}, 2t^{12}, -4t^{11}, \frac{s^{21}}{2}, \frac{t^{13}}{2}, -\frac{r^{12}}{2}, -\frac{s^{13}}{2}, -\frac{s^{21}}{2}, -\frac{t^{13}}{2}, \frac{t^{12}}{2}, r^{12}, s^{13}, -\frac{s^{12}}{2},\\ &- \frac{t^{12}}{2}, s^{12}, -\frac{s^{23}}{2}, s^{23}, -\frac{t^{23}}{2}, t^{23}, t^{11}, -\frac{s^{11}}{2} - \frac{s^{22}}{2} + 1, s^{11} + s^{22} - 2, -s^{21}, \end{aligned}$$

$$-\frac{3t^{13}}{2}, s^{21}, \frac{3t^{12}}{2}, 2t^{11}, \frac{s^{31}}{2}, -\frac{r^{13}}{2}, -\frac{s^{31}}{2}, r^{13}, -\frac{s^{11}}{2} - \frac{s^{33}}{2} + 1, s^{11} + s^{33} - 2,$$
$$\frac{3t^{13}}{2}, -\frac{s^{32}}{2}, s^{32}, -s^{31}, -\frac{3t^{12}}{2}, s^{31}, -r^{22}, -s^{23}, 2r^{22}, 2s^{23}, 1 - s^{22}, 2s^{22} - 2,$$
$$2s^{21}, -\frac{r^{23}}{2}, \frac{1}{2}(-s^{22} - s^{33} + 2), r^{23}, s^{22} + s^{33} - 2, \frac{t^{23}}{2}, -r^{33}, 1 - s^{33}, -s^{32},$$
$$2r^{33}, 2s^{33} - 2, 2s^{32}, 2s^{31}\,.$$

Miraculously, the system is compatible and its solution is $s^{11} = s^{22} = s^{33} = 1$, the other entries being zero, that is, q is proportional to a quadratic form of type $q^{(3)}$, which is precisely the image via the KLR–Hitchin map of the form η, see Section 8.2.6 above.

The computations for the representatives of the orbits "L" and "G" are analogous and we omit them; the orbit "P" is the singular locus of the map ϖ (Proposition 8.1) and, since $\widetilde{\eta}$ and rank one (Theorem 9.1), its cofactor matrix $\Lambda^2(\widetilde{\eta})$ must be zero: hence, the orbit "P" is the singular locus of the map $[\Lambda^2(\widetilde{\cdot})]$ as well, and the commutativity of (9.20) has been proved. On a deeper level, this surprising compatibility is a cohomological feature of homogeneous bundles over X: in the Appendix we prove that the map s of diagram (9.20) is actually an isomorphism, see Section 11 below.

Corollary 9.1. *For any $[\eta] \in \mathbb{P}(\Lambda^3_0(\mathcal{C}^*))$ the cocharacteristic variety $\sigma^\vee \subset \mathbb{P}(\mathcal{C})$ of the Monge–Ampère equation $\mathcal{E}_\eta$ is cut out by* (*the projective class of*) *the quadratic form $\varpi([\eta]) \in \mathbb{P}(S^2(\mathcal{C}^*))$. In particular:*

- (O) $\sigma^\vee$ *is an irreducible, nonsingular and non–degenerate* (*rank–six*) *quadric;*
- (L) $\sigma^\vee$ *is an irreducible, nonsingular and degenerate rank–three quadric;*
- (G) $\sigma^\vee$ *is a reducible, nonsingular and degenerate rank–one quadric: it coincides with the hyperplane $\mathbb{P}(\ell_D^\perp)$, where $D \in \mathrm{Gr}(3, \mathcal{C})$ is such that $\mathcal{E}_D = \mathcal{E}_\eta$;*
- (P) $\sigma^\vee$ *is trivial, i.e., it has rank zero and $\sigma^\vee = \mathbb{P}(\mathcal{C})$.*

Proof. Commutativity of diagram (9.20) tell us that, up to a projective factor, $\Lambda^2(\widetilde{\eta})$ is the section $s(q(\eta))$, for any of the four representatives η and, in virtue of its $\mathsf{Sp}(\mathcal{C})$–equivalency, for all the elements of $\Lambda^3_0(\mathcal{C}^*)$.

The cocharacteristic variety $\sigma^\vee$, that has been defined above by (9.3), can be equivalently given as the (closure of the) image of the natural projection

$$\begin{array}{ccc} (\mathcal{E}_\eta)_{\mathrm{sm}} \times \mathcal{C} \supseteq \mathcal{L}|_{(\mathcal{E}_\eta)_{\mathrm{sm}}} & \longrightarrow & \mathcal{C} \\ \downarrow & & \\ (\mathcal{E}_\eta)_{\mathrm{sm}} & & \end{array}$$

over $\mathcal{C}$ of the (conic) sub–bundle of $\mathcal{L}|_{(\mathcal{E}_\eta)_{\mathrm{sm}}}$ cut out by the section $s(q(\eta))$, that is the same as the section $\Lambda^2(\widetilde{\eta})$: indeed, at each nonsingular point $L \in \mathcal{E}_\eta$, the conic subset of L cut out by the equation $\Lambda^2(\widetilde{\eta}) = 0$ is the cone over the projective dual of the characteristic variety σ_L of $\mathcal{E}_\eta$ at L, see (9.4).

Since the section $s(q(\eta))$ comes from the quadratic form $q(\eta)$, the latter is precisely the quadratic form cutting out $\sigma^\vee$ in $\mathcal{C}$: this proves the first part of the claim.

The second part can be carried out by working out all the four cases listed in Section 8.2.6 above: for the orbit "P" the claim is evident; in the cases "O" the quadric has manifestly rank six: as such, it is nonsingular and non–degenerate and, therefore, it is irreducible; in the case "L" we see that $\sigma^\vee$ is the pre–image of a rank–three (and, as such, non–degenerate, nonsingular and irreducible) quadric on the first summand V of $\mathcal{C} = V \oplus V^*$.

In the last case "G" reducibility it obvious, since $\{e_1^2 = 0\}$ is a linear hyperplane; it remains to prove that such a hyperplane is the symplectic orthogonal to the line ℓ_D, where D is one of the only two elements of $\mathrm{Gr}(3, \mathcal{C})$ that determine the same equation $\mathcal{E}_\eta$, see (9.6). Direct computations, analogous to those carried out over $\mathbb{R}$ in Section 4.3 above, show that these elements are precisely $D = \langle e_1, e_3, \epsilon^3 \rangle$, $D^\perp = \langle e_1, e_2, \epsilon^2 \rangle$. One readily sees that $\mathcal{E}_\eta = \mathcal{E}_D = \mathcal{E}_{D^\perp} = \{u_{23} = 0\}$. Recalling the definition of ℓ_D, cf. (9.12), we see that $\ell_D = \langle e_1 \rangle$, whence

$$\ell_D^\perp = \langle e_1, e_2, e_3, \epsilon^2, \epsilon^3 \rangle = \ker e_1 = \{e_1^2 = 0\} = \sigma^\vee \,,$$

and the proof is complete. □

Observe that the Theorem 4.1that we have formulated before is the real–differentiable counterpart of the Corollary 9.1 we have just proved in the complex–analytic setting: claims (1), (2), and (3) of the former correspond to claims (O+L), (G) and (P) of the latter. As the (omitted) proof of Theorem 4.1 is formally analogous to that of Corollary 9.1, we can draw the following conclusion: for a Monge–Ampère equation $\mathcal{E}_\eta$ the KLR invariant and the Hitchin moment map of the 3–form η, equated to zero, gives the same quadric in $\mathcal{C}$, that we have called *cocharacteristic variety*, since it consists of the projective duals of the fibers of the well–known characteristic variety; in the case when $\mathcal{E}_\eta$ has *non–degenerate symbol*, the cocharacteristic variety coincides also with the *contact cone structure* associated with the PDE itself, which can be thought of as a non–linear generalization of Goursat's idea of describing a Monge–Ampère equation via a sub–distribution of the contact distribution on J^1; in case of a degenerated symbol, the cocharacteristic variety is the smallest liner subspace containing both D and $D^\perp$, when $\mathcal{E}_\eta = \mathcal{E}_D$, or it even becomes trivial, when $D = D^\perp$.

10. Conclusions and perspectives

We have seen that to any Monge–Ampère equation in three independent variables, that we understood as a hypersurface of the 2^{nd} order jet space J^2, is associated, at a fixed point of J^1, a quadric cone inside the contact space of J^1; the latter turns out to be a 6–dimensional symplectic vector space. We showed that, in the case when the symbol of the Monge–Ampère equation is non–degenerate, the aforementioned cone coincides with the cocharacteristic variety of the considered equation. This motivated the study of quadratic forms on a 6–dimensional symplectic vector space, up to symplectic equivalence.

It turned out that quadric cones that are contact cone structures of Monge–Ampère equations fill up a narrow subclass. Thus, it is natural to ask where the remaining cones come from, that is, whether there exists another class of PDEs, necessarily more general than those of Monge–Ampère type, that accounts for the remaining normal forms of the table of Section 7.1; in practise, this means solving the following problem: given *any* quadratic form from the aforementioned table, one can consider it zero locus and wonder whether the latter can be realized as the contact cone structure of some PDE and, moreover, to what extent such a PDE is characterized by its contact cone structure. Among the candidates for a larger class of PDEs there are the equations of Monge–Ampère type of "higher algebraic degree", that is, PDEs given by an analogous formula to (2.19), where, instead of a linear polynomial of the minors of the Hessian matrix, we find a polynomial of higher degree (see also [21, Section 5] for more details about this class of PDEs). For example, in Section 5.1.1, we have shown how to recover the equation (4.7) starting from the quadric cone (4.8): performing similar computations, starting from the cone variety corresponding to the normal form $q^{(2)}$ of the aforementioned table, that is,

$$z^2 q_2 + z^3 q_3 = 0\,, \tag{10.1}$$

we obtain the equation

$$u_{12}(u_{13}u_{23} - u_{12}u_{33}) + u_{13}(u_{12}u_{23} - u_{13}u_{22}) = 0, \tag{10.2}$$

i.e., the variety (10.1) is the contact cone structure of the PDE (10.2). As it happens, though, PDE (10.2) is no longer a Monge–Ampère equation (i.e., at any point of J^1, it is not a hyperplane section of the Lagrangian Grassmanian $\mathrm{LGr}(3,\mathcal{C})$), but a *quadratic* Monge–Ampère equation instead, that is a PDE of Monge–Ampère type of algebraic degree equal to 2 (or, in other words, at a point of J^1, it is a *hyperquadric* section of the Lagrangian Grassmanian $\mathrm{LGr}(3,6)$), being the left hand side of (10.2) a polynomial of second degree in the minors of the Hessian matrix $\|u_{ij}\|$.

It is then natural to ask, whether there exists a minimal algebraic degree d, such that the class of PDEs of Monge–Ampère type of algebraic degree equal to d contains *all* PDEs whose conic structure is given by one of the quadratic forms listed in the table of Section 7.1.

Another important class of 2^{nd} order PDEs is the class of so–called *hydrodynamically integrable* (2^{nd} order) PDEs, that have been intensively studied, from the point of view of hyper-surfaces in a Lagrangian Grassmannian, by E. Ferapontov and his collaborators [18]: in particular, they have discovered that the 21–dimensional group $\mathsf{Sp}(6,\mathbb{R})$ has an open orbit in the space that parametrizes these PDEs, which in turns implies that such a space has dimension 21. In spite of the fact that the space of integrable PDEs and the space of quadrics on $\mathcal{C}$ have the same dimension, the former cannot be the larger class of PDEs we are looking for, since it does not contain the class of Monge–Ampère equations: actually they intersect along the orbit made of the linearizable Monge–Ampère equations, that is, the orbit that we have labeled by "L" in Section 9 above.

One may then also wonder, whether it is possible to characterize the class of integrable 2^{nd} order PDEs in terms of their contact cone structures.

11. APPENDIX: COHOMOLOGY OF HOMOGENEOUS BUNDLES OVER $\mathrm{LGr}(3,\mathcal{C})$

To prove the following result, we employ the Borel–Weil–Bott theorem, as formulated in [34], and we will follow the notation therein.

Proposition 11.1. *The natural sheaf morphism*

$$S^2(\mathcal{O}\otimes\mathcal{C}^*)|_{\mathcal{E}_\eta} \xrightarrow{p} S^2(\mathcal{L}^*)|_{\mathcal{E}_\eta} \tag{11.1}$$

induces an isomorphism between the corresponding spaces of sections, and they can be further identified with the irreducible $\mathsf{Sp}(\mathcal{C})$*–representation* $W_{(2,0,0)}$*, that is,* $S^2(\mathcal{C})$*.*

Let us observe that, as a consequence of such a result, the arrow s of diagram (9.20) above must be an isomorphism: indeed, Proposition (11.1) was intended to provide a solid theoretical background to the computation performed throughout Section 9.5.

Proof of the Proposition 11.1. We will need the hyperplane exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\mathcal{E}_\eta} \longrightarrow 0\,, \tag{11.2}$$

together with the exact sequence

$$0 \longrightarrow Q^* \cdot \mathcal{C}^* \longrightarrow S^2(\mathcal{O} \otimes \mathcal{C}^*) \longrightarrow S^2(\mathcal{L}^*) \longrightarrow 0\,, \tag{11.3}$$

coming from the dual of the tautological sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O} \otimes \mathcal{C} \longrightarrow Q \longrightarrow 0\,. \tag{11.4}$$

By combining sequence (11.2) and (11.3), we obtain a commutative diagram of coherent sheaves on X with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q^* \cdot \mathcal{C}^*(-1) \arrow[r] \arrow[d] & Q^* \cdot \mathcal{C}^* \arrow[r] \arrow[d] & Q^* \cdot \mathcal{C}^*|_{\mathcal{E}_\eta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^2(\mathcal{O} \otimes \mathcal{C}^*)(-1) \arrow[r] \arrow[d] & S^2(\mathcal{O} \otimes \mathcal{C}^*) \arrow[r] \arrow[d] & S^2(\mathcal{O} \otimes \mathcal{C}^*)|_{\mathcal{E}_\eta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^2(\mathcal{L}^*)(-1) \arrow[r] \arrow[d] & S^2(\mathcal{L}^*) \arrow[r] \arrow[d] & S^2(\mathcal{L}^*)|_{\mathcal{E}_\eta} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{11.5}$$

The symbol $Q^* \cdot \mathcal{C}^*$ stands for the sheaf that fits into the exact sequence

$$0 \longrightarrow \Lambda^2(Q^*) \longrightarrow Q^* \otimes \mathcal{C}^* \longrightarrow Q^* \cdot \mathcal{C}^* \longrightarrow 0\,. \tag{11.6}$$

To each short exact sequence that appears in diagram (11.5) above, we can associate the long exact sequence in cohomology: we will show that the only nonzero cohomology groups are the zeroth groups of right lower 2×2 square of the diagram.

Since $\mathcal{L}$, Q, $\mathcal{O}_X(-1)$ and their various products are $\mathsf{Sp}(\mathcal{C})$–homogeneous, Borel–Weil–Bott theorem allows us to compute their cohomologies by weight considerations. We recall that the Lagrangian Grassmanian X is a $\mathsf{Sp}(\mathcal{C})$–homogeneous space, whose parabolic subgroup P corresponds to the marked Dynkin diagram

$$\bullet \text{——} \bullet = \!\!< \!\!= \circ\ .$$

We recall that the fundamental Weyl chamber $D \subset \mathfrak{h}^*$ is the positive cone spanned by the fundamental weights, see Remark 7.2, and we define δ as their sum, i.e., $\delta = 3h_1 + 2h_2 + h_3$. By the *index* $\mathrm{ind}(\lambda)$ of the weight λ we mean the smallest number of simple reflections needed to move λ to the fundamental Weyl chamber D; a representative of the orbit (with respect to the Weyl group action) of λ belonging to D will be denoted by the symbol $w(\lambda)$.

Since irreducible $\mathsf{Sp}(\mathcal{C})$–homogeneous bundles on X correspond to irreducible representations of P, they are determined by their highest weights. To begin with, the tautological bundle $\mathcal{L}$ has weights $-h_1, -h_2, -h_3$, the minus sign coming from the very definition of associated vector bundle, where we act on the fiber by the inverse. In this case, the highest weight is $-h_3$.

Since, in the case of $\mathrm{LGr}(3, 6)$, the bundle $\mathcal{L}$ is identified with Q^*, the weights are the same. For $\mathcal{L}^*$ the weights will be h_1, h_2, h_3, with h_1 being the highest; the line bundle $\mathcal{O}_X(-1)$ has (highest) weight $-h_1 - h_2 - h_3$. Therefore, we can obtain the highest weight of various products: for $S^2(\mathcal{L}^*)$ we obtain $2h_1$, for $S^2\mathcal{L}^*(-1)$, we obtain $h_1 - h_2 - h_3$, for $\Lambda^2(Q^*)$, we obtain $-h_2 - h_3$ and for $\Lambda^2 Q^*(-1)$, we obtain $-h_1 - 2h_2 - 2h_3$.

By Borel–Weil–Bott theorem, if $w(\lambda + \delta)$ belongs to the boundary δD of D, then the bundle associated to λ is cohomologically trivial (or, as some authors say, *immaculate*), i.e., $H^i = 0$ for all i. In our case, easy computations show that $\mathcal{L} = Q^*$, $\Lambda^2(Q^*)$, $S^2\mathcal{L}^*(-1)$, $S^2(\mathcal{O} \otimes \mathcal{C}^*)(-1)$, $Q^*(-1)$ and $\Lambda^2 Q^*(-1)$ are all immaculate; by using the cohomology long exact sequence, we obtain that $Q^* \cdot \mathcal{C}^*(-1)$ and $Q^* \cdot \mathcal{C}^*$ are immaculate as well. Then, by the same argument, the immaculateness $S^2(\mathcal{O} \otimes \mathcal{C}^*)(-1)$ and $Q^* \cdot \mathcal{C}^*|_{\mathcal{E}_\eta}$ also follows.

Therefore, we have shown that only the bundles in the lower right 2×2 square can have nonzero cohomologies and by exactness of the cohomology sequences we have for all i:

$$H^i(X, S^2(\mathcal{L}^*)) = H^i(X, S^2(\mathcal{O} \otimes \mathcal{C}^*))$$

$$= H^i(\mathcal{E}_\eta, S^2(\mathcal{O} \otimes \mathcal{C}^*)) = H^i(\mathcal{E}_\eta, S^2(\mathcal{L}^*)).$$

To finish the proof it is now enough to observe that any of the nontrivial H^0's is the $\mathsf{Sp}(\mathcal{C})$ irreducible representation $W_{(2,0,0)}$: but the claim is obvious for the trivial bundle in the center of diagram (11.5), since the zeroth cohomology of a trivial bundle coincides with the fiber, that is, $S^2(\mathcal{C}^*)$. This concludes the proof. □

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