Implementation of Beam-Type Finite Elements Based on Carrera Unified Formulation

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Preface

What prompts the creation of yet another book on the finite element modeling of beam-like structures? The existing literature is rich with contributions from eminent scientists delving into various beam theories. However, a fresh perspective on a classical subject can ignite the curiosity of both seasoned and budding researchers. While this alone might attract interested readers, the authors' motivations for writing a new book on finite elements are distinct. They can be succinctly summarized as follows:

- Traditional one-dimensional (1D) finite elements typically possess a maximum of six degrees of freedom per node. There are various types of 1D finite elements, such as rods, bars, and beams. These elements have different cross-sections, and their primary distinction lies in the loading conditions they can handle. A rod, for instance, is subjected to axial and torsional loads but not bending, exhibiting axial and torsional stiffness without bending stiffness. On the contrary, bars and beams can accommodate various loads, including tension/compression, torsion, bending, and shear. Consequently, the selection of the appropriate type of 1D structure (rod, bar, or beam) depends on the specific external loads and boundary conditions applied to the real component, making this choice highly problem dependent. This classification may pose challenges for beginners and, at times, even experienced users due to potential confusion. The primary objective of this book is to introduce a unified 1D finite element that can emulate the behavior of a rod, bar, or beam without requiring ad-hoc formulations. This approach aims to simplify the modeling process and alleviate the complexities associated with choosing the right structural type, offering a more versatile and user-friendly solution.
- Over the past three decades, the Carrera Unified Formulation (CUF) has emerged in the literature. This formulation was originally proposed by Carrera, and the *unified* aspect of CUF enables the implementation of a 1D mathematical model capable of functioning as a rod, bar, beam, or any other physical representation. This eliminates the need for specific formulations, and users simply need to choose the degree of refinement for the mathematical model. This approach facilitates the

implementation of 1D analyses ranging from low to higher order, encompassing classical Euler-Bernoulli and Timoshenko beams to high-fidelity 1D models.

- Finite elements stand out as versatile and widely employed numerical methods among both researchers and practitioners, serving as a cornerstone for tackling intricate problems in engineering and science. However, the approximations inherent in finite element formulations can sometimes lead to unexpected behaviors under specific conditions. The assumptions made in displacement functions to represent the actual system behavior and the application of numerical integration techniques may result in unpredictable outcomes. One notable challenge in finite element formulations is shear locking, a phenomenon characterized by a significant underestimation of displacements, rendering the structure excessively stiff. The term "locking" signifies that the structure essentially "locks" itself against deformations. The book delves into the intricacies of this problem, exploring in detail potential solutions found in the literature, such as reduced and selectively reduced integration, as well as mixed formulation techniques. These approaches typically revolve around the adjustment of the number of Gauss points used for integrating volumetric and shear strains. The book provides a comprehensive analysis of this issue.
- This book provides the stiffness matrix of a beam structure subjected to various loading conditions, from axial elongation, to bending around transverse directions, and torsion. The determination of the stiffness matrix is approached in each case using the Principle of Virtual Displacement (PVD), a concept rooted in classical mechanics and variational calculus. Within the framework of Lagrangian mechanics, the PVD is instrumental in deriving the equations of motion for a mechanical system. It asserts that the virtual work carried out by internal and external forces acting on a system is zero for any virtual displacement consistent with the system's constraints. What sets this book apart is its introduction of a recursive notation for the evaluation of the stiffness matrix. This innovative approach allows the construction of a matrix whose form is independent of the adopted structural theory.
- An innovative development stemming from CUF is the Node-Dependent Kinematics (NDK) technique, allowing for different kinematics along the length of the 1D mathematical model. This proves valuable when only a specific portion of the 1D structure experiences significant deformation and, consequently, a complex 3D stress state due to simulation requirements. The NDK technique enables refinement exclusively in that portion of the structure, conserving degrees of freedom and, consequently, reducing computational costs. The book will provide a detailed exploration of the NDK concept.

- For each scenario, a dedicated MATLAB script is included in the Appendix. These scripts serve as valuable tools to assist users in systematically deriving the stiffness matrix, offering step-by-step guidance throughout the process.

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Competing Interests The authors have no competing interests to declare that are relevant to the content of this manuscript.

Ethics Approval The authors declare that this manuscript does not include primary studies involving animals.

Contents

1 Introduction 1
References 7

2 Stiffness Matrix of a Bar Element 11
2.1 Structural Model of a Bar 12
2.2 Finite Element Approximation 14
2.3 Principle of Virtual Displacement 15
2.4 Matrix Notation 15
2.5 Recursive Notation Against Shape Functions 18
2.6 Recursive Notation Against the Theory of Structures 20

3 Stiffness Matrix of a Beam Bending Around z Axis 25
3.1 Structural Model of a Beam 26
3.2 Finite Element Approximation 28
3.3 Principle of Virtual Displacement 28
3.4 Matrix Notation 29
3.5 Recursive Notation Against Shape Functions 34
3.6 Recursive Notation Against the Theory of Structure 38
3.7 Locking Phenomena 41
3.7.1 The Shear Locking 41
3.7.2 Bending and Shear Stiffness Matrix Parts 43
3.7.3 Gauss Quadrature Technique 46
3.7.4 Full Integration 48
3.7.5 Full Integration Applied to the Bending Part of the Stiffness Matrix 49
3.7.6 Full Integration Applied to the Shear Part of the Stiffness Matrix 55
3.7.7 Full Integration: The Complete Stiffness Matrix 67
3.7.8 Full Integration: Considerations 67

4 Numerical Remedies Against Shear Locking 69
4.1 Uniform Reduced Integration 70
4.1.1 Uniform Reduced Integration Applied to the Bending Part of the Stiffness Matrix 70
4.1.2 Uniform Reduced Integration Applied to the Shear Part of the Stiffness Matrix 75
4.1.3 Uniform Reduced Integration: Considerations on Bending Part and Shear Part of the Stiffness Matrix 83
4.1.4 Uniform Reduced Integration: The Complete Stiffness Matrix 84
4.2 Selective Reduced Integration 85
4.2.1 Selective Reduced Integration Applied to the Bending Part of the Stiffness Matrix 85
4.2.2 Selective Reduced Integration Applied to the Shear Part of the Stiffness Matrix 85
4.2.3 Selective Reduced Integration: The Complete Stiffness Matrix 86
4.3 Mixed Interpolations of Tensorial Components 87
4.3.1 Bending Stiffness Matrix Using the Mixed Interpolations of Tensorial Components 88
4.3.2 The Shear Stiffness Matrix Using the Mixed Interpolations of Tensorial Components 90
4.4 Use of Higher-Order Elements 97
4.4.1 Quadratic B3 Element 97
4.4.2 Cubic Element: B4 Element 135
4.5 Numerical Examples of Comparison Between FI and URI Matrices of B2 Element 204
4.5.1 Comparison Between FI and URI Matrices of B2 Element 204
4.5.2 Comparison Between the Full and Selective Reduced Integrated Matrices of B2 Element 215
References 222

5 Stiffness Matrix of a Beam Bending Around *x* Axis 223
5.1 Structural Model of a Beam Bended Around *x* Axis 224
5.2 Finite Element Approximation 226
5.3 Matrix Notation 226
5.4 Recursive Notation Against Shape Functions 231
5.5 Recursive Notation Against the Theory of Structure 235

6 Stiffness Matrix of a Beam Subjected by a Torsion 239
6.1 Structural Model of a Beam Subjected by a Torsion 240
6.2 Finite Element Approximation 242
6.3 Matrix Notation 243
6.4 Recursive Notation Against Shape Functions 245
6.5 Recursive Notation Against the Theory of Structure 247

7 Stiffness Matrix of a Beam Under Combined Load 251
7.1 Structural Model of a Complete Beam 252
7.2 Finite Element Approximation 254
7.3 Matrix Notation 255
7.4 Recursive Notation Against Shape Functions 262
7.5 Recursive Notation Against the Theory of Structure 269

8 Fundamental Nucleus Using Unified Formulation 275
8.1 Structural Model of a Generic Beam 277
8.2 Stiffness Matrix 281
8.3 Fundamental Nucleus for a Bar 283
8.4 Fundamental Nucleus for a Complete Case Study 289
8.4.1 Geometrical Relations 290
8.4.2 Matrices $B_{\tau i}$ and B_{sj} 291
8.4.3 Fundamental Nucleus 293
8.4.4 FN for a Beam Bending Around z Axis 294
8.5 Numerical Examples 298
Reference 304

9 Stiffness Matrix with Node-Dependent Kinematic 305
9.1 Structural Model of a Two-Node Beam with Different Kinematics 306
9.2 Stiffness Matrix via PVD 309
9.3 Fundamental Nucleus for NDK 315
9.3.1 Computation of F_τ^i and F_s^i and F_τ^j and F_s^j 315
9.3.2 FN for i = 1 and j = 1 317
9.3.3 FN for i = 1 and j = 2 318
9.3.4 FN for i = 2 and j = 1 320
9.3.5 FN for i = 2 and j = 2 321
9.3.6 Complete Stiffness Matrix 323
9.4 Different Shape Functions Against Shear Locking 324
9.4.1 Geometric Relation 325
9.4.2 Calculation of Matrices B_i and B_j 327
9.4.3 Calculation of FN for Different Shape Functions to Mitigate Shear Locking Effects 329

10 Application to the Analysis of Real Structures 335
10.1 Industrial Building 337
10.2 Composite Empty Launcher 339
10.3 Boat Structure 340
10.4 Concluding Remarks 342
References 342

Appendix A: Bar Element Under Axial Deformation 343

Appendix B: Beam Element Bending Around the *z* Axis 351

Appendix C: Shear Locking Correction 365

Appendix D: Beam Element Bending Around the *x* Axis 411

Appendix E: Beam Element Subjected by a Torsion 425

Appendix F: Beam Element Under Combined Load 437

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Erasmo Carrera After earning two degrees (Aeronautics, 1986, and Aerospace Engineering, 1988) at the Politecnico di Torino, Erasmo Carrera received his Ph.D. degree in Aerospace Engineering at the Politecnico di Milano—Politecnico di Torino—Università di Pisa in 1991. He began working as a Researcher at the Department of Aerospace for the Politecnico di Torino in 1992 where he held courses on Missiles and Aerospace Structure Design, Plates and Shells, and the Finite Element Method. He became Associate Professor of Aerospace Structures and Aeroelasticity in 2000, and Full Professor at the Politecnico di Torino in 2011. He has visited the Institute fur Statik und Dynamik, Universitat Stuttgart twice, the first time as a Ph.D. student (six months in 1991) and then as visiting Scientist under a GKKS Grant (18 months in 1995–1996). In the Summer of '96, he was Visiting Professor at the ESM Department of Virginia Tech. His main research topics are: composite materials, finite elements, plates and shells, postbuckling and stability, smart structures, thermal stress, aeroelasticity, multibody dynamics and the design and analysis of non classical lifting systems. He is author of more than 200 articles on these topics, many of which have been published in international journals. He serves as a referee for many journals, such as *Applied Mechanics Reviews*, *AIAA Journal*, *Journal of Sound and Vibration*, *International Journal of Solids and Structures*, *International Journal of Numerical Methods in Engineering* and as contributing editor for Mechanics of Advanced Materials and Structures.

Gerlando Augello is an Italian engineer with over 30 years of experience in mechanical and structural analysis, particularly in the aerospace sector. He earned his Master's Degree in Mechanical Engineering in 1986 and has worked extensively with Thales Alenia Space, holding roles such as Head of Mechanical Computer Aided Engineering (MCAE). His expertise includes finite element analysis, structural optimization, and advanced simulation techniques, with significant contributions to several space projects among which EXOMRAS, Space Rider, EUCLID,

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List of Figures

Fig. 1.1 Clamped thin-walled beam subjected to a uniform transverse pressure and related deformation . . . 3
Fig. 1.2 Finite element model application to the study case proposed in Fig. 1.1 . . . 4
Fig. 2.1 Deformation of a bar under traction . . . 12
Fig. 2.2 1D displacement of a bar under traction . . . 12
Fig. 2.3 Representation of the axial approximation (using N_i, N_j), and the cross-section expansion (using F_τ, F_s) of the bar using a two-node finite element . . . 21
Fig. 3.1 Bending around z axis . . . 26
Fig. 3.2 Two-dimensional assumptions of bending around z axis . . . 26
Fig. 3.3 Schematic representation of a beam with 2 nodes . . . 28
Fig. 3.4 Local coordinates along a beam with 2 nodes . . . 42
Fig. 3.5 Natural coordinates along a beam with 2 nodes . . . 42
Fig. 3.6 Position of 2 SPs along the two-node beam . . . 49
Fig. 4.1 Position of 1 sampling points along the two-node beam . . . 70
Fig. 4.2 Tying point position in local coordinates along a beam with 2 nodes . . . 88
Fig. 4.3 Schematic representation of a three-node beam (B3 element) . . . 98
Fig. 4.4 Position of the three sampling points along the B3 beam and related weights . . . 104
Fig. 4.5 Schematic representation of a four-node beam (B4 element) . . . 135
Fig. 4.6 Position of four SPs along the B4 beam and related weights . . . 149
Fig. 5.1 Bending around x axis . . . 224
Fig. 5.2 2D assumption of bending around x axis . . . 224
Fig. 5.3 Schematic representation of a beam with 2 nodes . . . 226
Fig. 6.1 Beam under torsion around y axis . . . 240
Fig. 6.2 Two-dimensional assumptions of torsion around beam axis . . . 240
Fig. 6.3 Schematic representation of a beam with 2 nodes . . . 242
Fig. 7.1 Schematic representation of a complete beam with 2 nodes . . . 255

Fig. 8.1 Displacement fields of a beam defined using classical beam theories 276
Fig. 8.2 Geometric properties of the C-shaped (**a**) and T-shaped (**b**) cross-section beam 299
Fig. 8.3 Values of vertical displacement of the point A of the C-shaped cross-section beam of Fig. 8.2a TE1 theory and B2 (**a**), B3 (**b**) and B4 (**c**) elements 301
Fig. 8.4 Values of vertical displacement of the point A of the C-shaped cross-section beam of Fig. 8.2a using TE15 theory and B2 (**a**), B3 (**b**) and B4 (**c**) elements 302
Fig. 8.5 Values of vertical displacement of the point A of the T-shaped cross-section beam of Fig. 8.2b using TE1 theory and B2 (**a**), B3 (**b**) and B4 (**c**) elements 304
Fig. 9.1 Schematic representation of a two-node beam with NDK application 307
Fig. 10.1 Nine-point elements, L9 and six-point elements, L6 based on Lagrange polynomials 336
Fig. 10.2 Four stringer wing-box (**a**) split into its components (**b**) 336
Fig. 10.3 Component-wise approach to simultaneously model panels, stringers and ribs 337
Fig. 10.4 Industrial building 338
Fig. 10.5 Representative mode shapes of the undamaged and damaged industrial building: **a** f = 11.11 Hz; **b** f = 11.60 Hz 338
Fig. 10.6 Reinforce launcher model 339
Fig. 10.7 Different modal shapes of the composite launcher 340
Fig. 10.8 Boat geometries with tapered walls incorporated in front and rear 341
Fig. 10.9 First deformed mode for each of the three configurations of Fig. 10.8 341

List of Tables

Table 4.1 Percentage difference of the U_2^0 displacement component 209
Table 4.2 Percentage difference of the U_2^0 displacement component using different number of elements 215
Table 4.3 Free end displacement of a three elements cantilever beam (see above Case 1 [4.538]) 218
Table 4.4 Free end displacement of a four elements cantilever beam (see above Case 2 [4.538]) 218
Table 4.5 Free end displacement of a five elements cantilever beam (see above Case 3 [4.538]) 218
Table 4.6 Strain energy of the B2 element 219
Table 8.1 Taylor-like polynomials. N is the order of the Taylor polynomial and m is the number of terms in the expansions . . . 299
Table 8.2 Material properties of the two considered beams, which cross-sections are described in Fig. 8.2a and b 299
Table 8.3 Evaluation of the effect of shear locking on the vertical displacements using B2 FEs at the point A of the beam of C-shaped cross-section beam of Fig. 8.2a 300
Table 8.4 Evaluation of the effect of shear locking on the vertical displacements using B3 FEs at the point A of the beam of C-shaped cross-section beam of Fig. 8.2a 300
Table 8.5 Evaluation of the effect of shear locking on the vertical displacements using B4 FEs at the point A of the C-shaped cross-section beam of Fig. 8.2a 301
Table 8.6 Evaluation of the effect of shear locking on the vertical displacements using B2 FEs at the point A of the beam of T-shaped cross-section beam of Fig. 8.2b 302
Table 8.7 Evaluation of the effect of shear locking on the vertical displacements using B3 FEs at the point A of the beam of T-shaped cross-section beam of Fig. 8.2b 303

Table 8.8 Evaluation of the effect of shear locking on the vertical displacements using B4 FEs at the point A of the beam of T-shaped cross-section beam of Fig. 8.2b 303
Table 10.1 First 10 modes in Hz for empty launcher 339
Table 10.2 First three natural frequencies (in Hz) of the three boat configurations of Fig. 10.8 341