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Stefan Krömer[1] · Martin Kružík[1] · Marco Morandotti[2] ·
Elvira Zappale[3]

Abstract

Measure-valued structured deformations are introduced to present a unified theory of deformations of continua. The energy associated with a measure-valued structured deformation is defined via relaxation departing either from energies associated with classical deformations or from energies associated with structured deformations. A concise integral representation of the energy functional is provided both in the unconstrained case and under Dirichlet conditions on a part of the boundary.

Contents

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✉ Martin Kružík
  kruzik@utia.cas.cz

  Stefan Krömer
  skroemer@utia.cas.cz

  Marco Morandotti
  marco.morandotti@polito.it

  Elvira Zappale
  elvira.zappale@uniroma1.it

[1] Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 00 Prague 8, Czech Republic

[2] Dipartimento di Scienze Matematiche "G. L. Lagrange", Politecnico di Torino, Corso Duca degli Abruzzi, 24, 101 29 Turin, Italy

[3] Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Via Antonio Scarpa, 16, 00161 Rome, Italy

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1 Introduction

The primary objective of continuum mechanics in solids is to articulate how a solid body will alter its shape when subjected to specified external forces or boundary conditions. A crucial initial step toward achieving this objective involves selecting a category of deformations for the continuum. In describing numerous continua, certain widely accepted criteria for the chosen category of deformations have been established: these deformations should be invertible, with differentiable mappings and inverses, and the combination of two deformations within this category should result in another deformation within the same category. However, classical deformations may not always suffice for describing all continua, requiring alternative selections in many cases. One approach involves introducing additional kinematic variables, such as the director fields in a polar continuum. An alternative approach entails incorporating supplementary fields that, while connected to the deformation, function as internal variables. For instance, in theories concerning plasticity, the plastic deformation tensor follows an evolutionary law outlined in the constitutive equations of the continuum.

Del Piero and Owen (1993) proposed an alternative approach that identifies classes of deformations called *structured deformations*, suited for continua featuring supplementary kinematical variables, as well as for continua featuring internal variables (we refer the reader to Matias et al. (2023) for a comprehensive survey on this topic). In the theory of structured deformations, if $\Omega \subset \mathbb{R}^N$ is the continuum body, the role usually played by the deformation field $u \colon \Omega \to \mathbb{R}^d$ and by its gradient $\nabla u \colon \Omega \to \mathbb{R}^{d \times N}$ is now played by a triple (κ, g, G), where the piecewise differentiable field $g \colon \Omega \setminus \kappa \to \mathbb{R}^d$ is the macroscopic deformation and the piecewise continuous matrix-valued field $G \colon \Omega \setminus \kappa \to \mathbb{R}^{d \times N}$ captures the contribution at the macroscopic level of smooth submacroscopic changes. The (possibly empty) discontinuity set $\kappa \subset \Omega$ of g and G can be regarded as the crack set of the material. The main result obtained by Del Piero and Owen is the Approximation Theorem (Del Piero and Owen 1993, Theorem 5.8) stating that any structured deformation (κ, g, G) can be approximated (in the L^∞ convergence) by a sequence of *simple deformations* $\{(\kappa_n, u_n)\}$. The matrix-valued field $\nabla g - G$ captures the effects of submacroscopic *disarrangements*, which are slips and separations that occur at the submacroscopic level. The spirit with which structured deformations were introduced was that of enriching the existing class of energies suitable for the variational treatment of physical phenomena without having to commit at the outset to a specific mechanical theory such as elasticity, plasticity, or fracture. Ideally, the regime of the deformation is energetically

chosen by the body depending on the applied external loads: if these are small, then the deformation will most likely be elastic, whereas if these are large, a plastic regime or even fracture may occur.

The natural mathematical context to study problems similar to those mentioned above is that of calculus of variations, in which equilibrium configurations of a deforming body subject to external forces are obtained as minimizers of a suitable energy functional. In the classical theories where the mechanics is described by the gradient of the deformation field u, a typical expression of the energy is

$$E(\kappa, u; \Omega) := \int_{\Omega} W(\nabla u) \, \mathrm{d}x + \int_{\Omega \cap \kappa} \psi([u], \nu_u) \, \mathrm{d}\mathcal{H}^{N-1}, \qquad (1.1)$$

where $W \colon \mathbb{R}^{d \times N} \to [0, +\infty)$ and $\psi \colon \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ are continuous functions satisfying suitable structural assumptions and model the bulk and interfacial energy densities, respectively. In the context of Del Piero and Owen, it is not clear how to assign energy to a structured deformation (κ, g, G); the issue was solved by Choksi and Fonseca who, providing a suitable version of the approximation theorem (Choksi and Fonseca 1997, Theorem 2.12), use the technique of relaxation to assign the energy $I(g, G; \Omega)$ as the minimal energy along sequences $\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)$ converging to $(g, G) \in SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N}) =: SD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$ in the following sense:

$$u_n \overset{*}{\rightharpoonup} g \quad \text{in } BV(\Omega; \mathbb{R}^d) \qquad \text{and} \qquad \nabla u_n \overset{*}{\rightharpoonup} G \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}), \qquad (1.2)$$

where ∇u_n denotes the absolutely continuous part of the distributional gradient Du. More precisely, the relaxation process reads

$$I(g, G; \Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} E(S_{u_n}, u_n; \Omega) : u_n \to (g, G) \text{ according to}(1.2) \right\} \qquad (1.3)$$

and is accompanied by integral representation theorems in $SD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$ for the relaxed energy $I(g, G; \Omega)$ (see (Choksi and Fonseca 1997, Theorems 2.16 and 2.17) and (Owen and Paroni 2015, Theorem 3)). The reader might have noticed that the crack set κ has been identified with the jump set S_{u_n} of the field $u_n \in SBV(\Omega; \mathbb{R}^d)$. The variational setting introduced in Choksi and Fonseca (1997) gave rise to numerous applications of structured deformations in various contexts, see Amar et al. (2022), Barroso et al. (2017a), Barroso et al. (2022), Barroso et al. (2024), Carita et al. (2018), Matias et al. (2021), and Matias et al. (2017), in which an explicit form of the energy $I(g, G; \Omega)$ could be provided.

We stress that, although we look at targets (g, G) belonging to $SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$, in general, the convergence (1.2) might lead to limits that are in $BV(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ and that, in assigning the energy (1.3), Choksi and Fonseca make the explicit choice to represent the relaxed energy only in $SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$. Moreover, from the mechanical point of view, one cannot, in principle, exclude that $\{\nabla u_n\}$ develop singularities in the limit, which would reflect on a weaker

regularity of the field G, possibly not even of the same type as those of the singular part $D^s g$ of the distributional derivative Dg, as is the case in Baía et al. (2012), Barroso et al. (2017b), and Matias and Santos (2014). Both these mathematical and mechanical reasons suggest that the definition of structured deformations should be extended from $SD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$ to the larger set

$$mSD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N}) := BV(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d \times N}), \qquad (1.4)$$

which we call *measure-valued structured deformations*, and which we abbreviate here with mSD.

In this paper, we generalize the results of Choksi and Fonseca (1997) to mSD. In particular, denoting with $u_n \overset{*}{\rightharpoonup} (g, G)$ in mSD the convergence in (1.2), we prove the Approximation Theorem 2.3: given any measure-valued structured deformation $(g, G) \in mSD$, there exists a sequence $\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)$ such that $u_n \overset{*}{\rightharpoonup} (g, G)$ in mSD. This serves to define the energy $I : mSD \to [0, +\infty)$ via the relaxation (1.3) in the larger space mSD, see (2.2), for which we prove the integral representation result, Theorem 2.4. This is one of the main results of the paper, in which we recover the same structure of Choksi and Fonseca (1997, Theorems 2.16 and 2.17) and Owen and Paroni (2015, Theorem 3), with the presence of an additional diffuse part. One of the novelties of our setting is that we manage to obtain a concise form of the relaxed energy functional involving only a bulk contribution H and its recession function at infinity H^∞

$$I(g, G; \Omega) = \int_\Omega H\Big(\nabla g, \frac{\mathrm{d}G}{\mathrm{d}\mathcal{L}^N}\Big)\,\mathrm{d}x + \int_\Omega H^\infty\Big(\frac{\mathrm{d}(D^s g, G^s)}{\mathrm{d}|(D^s g, G^s)|}\Big)\,\mathrm{d}|(D^s g, G^s)|(x),$$

where $D^s g$ and G^s are the singular parts of the measures Dg and G, respectively, see (2.13), in the typical form of Goffman and Serrin (1964) for functionals defined on measures for a density, which is a particular case of those treated in Arroyo-Rabasa et al. (2020). The relaxed bulk energy density H turns out to be quasiconvex-convex; see Proposition 4.3. It is interesting to notice that not every quasiconvex-convex function can be obtained as the bulk energy density associated with a structured deformation: ours retains the memory of the specific relaxation process (2.2) (see also the counterexample in Proposition 6.3). In Theorem 6.1 we prove that the energy $I(g, G; \Omega)$ can be obtained by relaxing from $SD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$ to $mSD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$ the energy (1.1) with the addition of a term penalizing the structuredness $\nabla g - G$

$$\hat{E}_R(g, G; \Omega) := E(S_g, g; \Omega) + R \int_\Omega |\nabla g - G|\,\mathrm{d}x,$$

see (6.1). Another relevant result is the possibility of performing the relaxation under trace constraints, see Theorem 5.1, which has the far-reaching potential of studying minimization problems in $mSD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$ with the addition of boundary data.

From the point of view of continuum mechanics, measure-valued structured deformations have the potential of extending the multiscale theory of structured defor-

mations to include the contributions to macroscopic deformations of submacroscopic deformations that concentrate on subsets of $\mathbb{R}^N$ of dimension lower than $N - 1$. We envisage that allowing G to be a measure-valued field would allow the theory to include concentration of strain in the crease of a folded sheet of paper (where the resistance to bending, for instance, is different in the flat part of the folded paper and in the crease), as well as to include the presence of dislocation lines in three-dimensional solids. Considering the deformations of a material domain in the shape of a three-quarter cylinder would allow one to consider concentrations located at the axis of the cylinder and possibly lead one to a description of disclinations (see, for reference, Cesana et al. 2024; Olbermann 2018; Seung and Nelson 1988; Volterra 1907) in terms of measure-valued structured deformations. Similarly, the inclusion of a third object $\Gamma \in \mathcal{M}(\Omega; \mathbb{R}^{d \times N \times N})$ that plays the role of a second-order gradient, could lead to a definition of measure-valued second-order structured deformations (g, G, Γ), in the spirit of Owen and Paroni (2000) and Barroso et al. (2017b), and also in the spirit of Fonseca et al. (2019) in case the structuredness is only at the level of the second-order gradient. This investigation will be the object of future work.

2 Setting and the Definition of the Energy in *mSD*

We assume that the main results about functions of bounded variations are known, otherwise we refer the reader to the monograph (Ambrosio et al. 2000) for a thorough introduction; likewise, we refer the reader to Dal Maso (1993) for an introduction to relaxation (see also Braides 2002).

We consider an initial energy as in (1.1), which, since we take $\kappa = S_u$, now can be written as $E \colon SBV(\Omega; \mathbb{R}^d) \to [0, +\infty)$

$$E(u; \Omega) := \int_\Omega W(\nabla u) \, \mathrm{d}x + \int_{\Omega \cap S_u} \psi([u], \nu_u) \, \mathrm{d}\mathcal{H}^{N-1}(x), \qquad (2.1)$$

where $W \colon \mathbb{R}^{d \times N} \to [0, +\infty)$ and $\psi \colon \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ are continuous functions satisfying the following assumptions for $A \in \mathbb{R}^{d \times N}$, $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{N-1}$:

$$c_W |A| \leqslant W(A) \leqslant C_W(1 + |A|); \qquad (W{:}1)$$

$$W \text{ is globally Lipschitz continuous}; \qquad (W{:}2)$$

there exist $c > 0$ and $0 < \alpha < 1$ such that $\qquad (W{:}3)$

$$\left| W^\infty(A) - \frac{W(tA)}{t} \right| \leqslant \frac{c \, |A|^{1-\alpha}}{t^\alpha} \quad \text{whenever } t > 0 \text{ and } t \, |A| \geqslant 1,$$

where $W^\infty(A) := \displaystyle\limsup_{t \to +\infty} \frac{W(tA)}{t}$;

$$c_\psi |\lambda| \leqslant \psi(\lambda, \nu) \leqslant C_\psi |\lambda|; \qquad (\psi{:}1)$$

$$\psi(t\lambda, \nu) = t\psi(\lambda, \nu) \text{ and} \psi(-\lambda, -\mu) = \psi(\lambda, \mu); \qquad (\psi{:}2)$$

$$\psi(\lambda_1 + \lambda_2, \nu) \leqslant \psi(\lambda_1, \nu) + \psi(\lambda_2, \nu). \qquad (\psi{:}3)$$

We consider *measure-valued structured deformations*, that is, pairs $(g, G) \in mSD$, see (1.4); we endow the space mSD with the norm

$$\|(g, G)\|_{mSD} := \|g\|_{BV(\Omega;\mathbb{R}^d)} + |G|(\Omega),$$

the latter term denoting the total variation of the measure G. We are interested in assigning an energy $I: mSD \to [0, +\infty)$ by means of the relaxation

$$I(g, G; \Omega) := \inf \left\{ \liminf_{n \to \infty} E(u_n; \Omega) : \{u_n\} \in \mathscr{R}(g, G; \Omega) \right\}, \qquad (2.2)$$

where, for every open set $U \subset \Omega$,

$$\mathscr{R}(g, G; U) := \left\{ \{u_n\} \subset SBV(U; \mathbb{R}^d) : u_n \overset{*}{\rightharpoonup} (g|_U, G|_U) \text{ as in } (1.2) \right\} \qquad (2.3)$$

is the set of admissible sequences. Our main result is a representation theorem for this energy, namely that $I = J$ with the explicit representation of the limit functional given by

$$
\begin{aligned}
J(g, G; \Omega) := &\int_\Omega H(\nabla g, G^a) \, dx + \int_{\Omega \cap S_g} h^j \left([g], \frac{dG_g^j}{d(\mathcal{H}^{N-1} \llcorner S_g)}, \nu_g \right) d\mathcal{H}^{N-1}(x) \\
&+ \int_\Omega h^c \left(\frac{dD^c g}{d|D^c g|}, \frac{dG_g^c}{d|D^c g|} \right) d|D^c g|(x) + \int_\Omega h^c \left(0, \frac{dG_g^s}{d|G_g^s|} \right) d|G_g^s|(x),
\end{aligned}
$$
$$(2.4)$$

where $H: \mathbb{R}^{d \times N} \times \mathbb{R}^{d \times N} \to [0, +\infty)$, $h^j: \mathbb{R}^d \times \mathbb{R}^{d \times N} \times \mathbb{S}^{N-1} \to [0, +\infty)$, and $h^c: \mathbb{R}^{d \times N} \times \mathbb{R}^{d \times N} \to [0, +\infty)$ are suitable bulk, surface, and Cantor-type relaxed energy densities. In (2.4), we have the following objects: since $g \in BV(\Omega; \mathbb{R}^d)$, we know that, by De Giorgi's structure theorem,

$$Dg = D^a g + D^s g = D^a g + D^j g + D^c g = \nabla g \mathcal{L}^N + [g] \otimes \nu_g \mathcal{H}^{N-1} \llcorner S_g + D^c g,$$

and we can decompose

$$G = G^a + G^s = G^a + G_g^j + G_g^c + G_g^s,$$

where

$$G^a \ll \mathcal{L}^N, \quad dG_g^j = \frac{dG}{d|D^j g|} d|D^j g|, \quad dG_g^c = \frac{dG}{d|D^c g|} d|D^c g|,$$

$$G_g^s := G - G^a - G_g^j - G_g^c.$$

Here, in case of G^a and other measures absolutely continuous with respect to the Lebesgue measure, our notation does not distinguish between the measure and its density with respect to $\mathcal{L}^N$. Also, notice that G_g^s is singular with respect to $\mathcal{L}^N + |Dg|$.

To carry out our program, we will use the following results.

Theorem 2.1 (Alberti 1991, Theorem 3; Choksi and Fonseca 1997, Theorem 2.8) *Let* $G \in L^1(\Omega; \mathbb{R}^{d \times N})$. *Then there exist a function* $f \in SBV(\Omega; \mathbb{R}^d)$, *a Borel function* $\beta \colon \Omega \to \mathbb{R}^{d \times N}$, *and a constant* $C_N > 0$ *depending only on N such that*

$$Df = G\,\mathcal{L}^N + \beta\mathcal{H}^{N-1} \llcorner S_f, \quad \int_{\Omega \cap S_f} |\beta|\,\mathrm{d}\mathcal{H}^{N-1}(x) \leqslant C_N \|G\|_{L^1(\Omega; \mathbb{R}^{d \times N})}. \tag{2.5}$$

Lemma 2.2 (Choksi and Fonseca 1997, Lemma 2.9) *Let* $u \in BV(\Omega; \mathbb{R}^d)$. *Then there exist piecewise constant functions* $\bar{u}_n \in SBV(\Omega; \mathbb{R}^d)$ *such that* $\bar{u}_n \to u$ *in* $L^1(\Omega; \mathbb{R}^d)$ *and*

$$|Du|(\Omega) = \lim_{n \to \infty} |D\bar{u}_n|(\Omega) = \lim_{n \to \infty} \int_{\Omega \cap S_{\bar{u}_n}} |[\bar{u}_n]|\,\mathrm{d}\mathcal{H}^{N-1}(x). \tag{2.6}$$

The following approximation theorem generalizes the one obtained in Silhavý (2015).

Theorem 2.3 (approximation theorem) *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded, open set with Lipschitz boundary. For each* $(g, G) \in mSD$ *there exists a sequence* $\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)$ *such that* $u_n \overset{*}{\rightharpoonup} (g, G)$ *in* mSD *according to* (1.2). *In addition, we have that*

$$\|Du_n\|_{\mathcal{M}(\Omega; \mathbb{R}^{d \times N})} \leqslant C_1 \|(g, G)\|_{mSD}, \tag{2.7a}$$

and

$$\|u_n\|_{BV(\Omega; \mathbb{R}^d)} \leqslant C_2(\Omega) \|(g, G)\|_{mSD}, \tag{2.7b}$$

for constants $C_1 = C_1(N) > 0$ *and* $C_2(\Omega) = C_2(N, \Omega) > 0$ *independent of* $\{u_n\}$ *and* (g, G).

Proof Let $\{G^k\} \subset L^1(\Omega; \mathbb{R}^{d \times N})$ be a sequence of functions such that $G^k \overset{*}{\rightharpoonup} G$ as $k \to \infty$ and $\sup_{k \in \mathbb{N}} \|G^k\|_{L^1(\Omega; \mathbb{R}^{d \times N})} \leqslant |G|(\Omega)$ (see Krömer et al. 2023), and consider the corresponding pairs $(g, G^k) \in BV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$. By Theorem 2.1, for each $k \in \mathbb{N}$, there exists $f^k \in SBV(\Omega; \mathbb{R}^d)$ such that $\nabla f^k = G^k$ and, by the estimate in (2.5),

$$\left|Df^k\right|(\Omega) \leqslant C_N \left\|G^k\right\|_{L^1(\Omega; \mathbb{R}^{d \times N})} \leqslant C_N |G|(\Omega). \tag{2.8a}$$

Since Ω is a Lipschitz set, we can use the Poincaré inequality and obtain that

$$\sup_{k \in \mathbb{N}} \left\| f^k \right\|_{BV(\Omega; \mathbb{R}^d)} \leqslant C_P(\Omega) C_N \left| G \right| (\Omega). \tag{2.8b}$$

By Lemma 2.2, for each $k \in \mathbb{N}$ there exists a sequence $\{\bar{v}_n^k\} \subset SBV(\Omega; \mathbb{R}^d)$ of piecewise constant functions such that $\bar{v}_n^k \to g - f^k$ in $L^1(\Omega; \mathbb{R}^d)$ and, by (2.6), $\left| D\bar{v}_n^k \right| (\Omega) \to \left| D(g - f^k) \right| (\Omega)$ as $n \to \infty$. Now, the sequence of functions $v_n^k := \bar{v}_n^k + f^k$ is such that $v_n^k \to g$ in L^1 and $\nabla v_n^k = G^k$, as $n \to \infty$, for every $k \in \mathbb{N}$. The convergences in (2.7) and the estimates in (2.7) now follow from estimates (2.8) by a diagonal argument, by defining $u_n := v_n^{k(n)}$, with $k(n) \to \infty$ slowly enough. □

Before stating our integral representation result, we define the following classes of competitors for the characterization of the relaxed energy densities below. We let $Q \subset \mathbb{R}^N$ be the unit cube centered at the origin with faces perpendicular to the coordinate axes, and for $\nu \in \mathbb{S}^{N-1}$, we let $Q_\nu \subset \mathbb{R}^N$ be the rotated unit cube so that two faces are perpendicular to ν. For $A, B, \Lambda \in \mathbb{R}^{d \times N}$ and $\lambda \in \mathbb{R}^d$, we define

$$\mathcal{C}^{\text{bulk}}(A, B; Q) := \left\{ u \in SBV(Q; \mathbb{R}^d) : u|_{\partial Q}(x) = (Ax)|_{\partial Q}, \ \fint_Q \nabla u \, dx = B \right\}, \tag{2.9a}$$

$$\mathcal{C}^{\text{surface}}(\lambda, \Lambda; Q_\nu) := \left\{ u \in SBV(Q_\nu; \mathbb{R}^d) : u|_{\partial Q_\nu}(x) = s_{\lambda,\nu}|_{\partial Q_\nu}(x), \ \fint_{Q_\nu} \nabla u \, dx = \Lambda \right\}, \tag{2.9b}$$

where $s_{\lambda,\nu}(x) := \frac{1}{2}\lambda(\text{sgn}(x \cdot \nu) + 1)$. Moreover, for any open set $U \subset \mathbb{R}^N$ and $v \in SBV(U; \mathbb{R}^d)$, we let

$$E^\infty(v; U) := \int_U W^\infty(\nabla v) \, dx + \int_{U \cap S_v} \psi([v], \nu_v) \, d\mathcal{H}^{N-1}(x). \tag{2.10}$$

Theorem 2.4 (integral representation) *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and assume that (W:1)–(W:3) and (ψ:1)–(ψ:3) hold true. Then*

$$I(g, G; \Omega) = J(g, G; \Omega) \ \text{ for all } (g, G) \in mSD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N}),$$

where I and J are defined in (2.2) and (2.4), respectively, and the densities in J are given by

$$H(A, B) := \inf \left\{ E(u; Q) : u \in \mathcal{C}^{\text{bulk}}(A, B; Q) \right\}; \tag{2.11a}$$

$$h^j(\lambda, \Lambda, \nu) := \inf \left\{ E^\infty(u; Q_\nu) : u \in \mathcal{C}^{\text{surface}}(\lambda, \Lambda; Q_\nu) \right\}; \tag{2.11b}$$

$$h^c(A, B) := \inf \left\{ E^\infty(u; Q) : u \in \mathcal{C}^{\text{bulk}}(A, B; Q) \right\}. \tag{2.11c}$$

The proof is given in Sect. 4.1 (upper bound: $I \leqslant J$) and Sect. 4.2 (lower bound: $I \geqslant J$).

Remark 2.5 For the special case $(g, G) \in SD(\Omega; \mathbb{R}^d \times \mathbb{R}^{d \times N})$, Theorem 2.4 reduces to Choksi and Fonseca (1997, Theorem 2.16) (for the functional I_1 in the notation of Choksi and Fonseca (1997)). Unlike Choksi and Fonseca (1997), we assumed coercivity of W in (W:1), but only to avoid additional technicalities.

Remark 2.6 As shown in Proposition 3.1 below, h^c coincides with the recession function H^∞ of H, and h^j can be replaced by $h^c = H^\infty$, more precisely,

$$h^j(\lambda, \Lambda, \nu) = h^c(\lambda \otimes \nu, \Lambda). \tag{2.12}$$

This allows for another, much more elegant representation of J:

$$\begin{aligned} J(g, G; \Omega) &= \int_\Omega \mathrm{d}H(Dg, G) = \int_\Omega H\left(\nabla g, \frac{\mathrm{d}G}{\mathrm{d}\mathcal{L}^N}\right) \mathrm{d}x \\ &+ \int_\Omega H^\infty\left(\frac{\mathrm{d}(D^s g, G^s)}{\mathrm{d}|(D^s g, G^s)|}\right) \mathrm{d}|(D^s g, G^s)|(x), \end{aligned} \tag{2.13}$$

see Definition 3.3 and Proposition 4.1 below.

Remark 2.7 (i) As a consequence of (ψ:1) and (ψ:3), ψ is also globally Lipschitz in λ:

$$|\psi(\lambda_1, \nu) - \psi(\lambda_2, \nu)| \leqslant C_\psi |\lambda_1 - \lambda_2|. \tag{2.14}$$

(ii) We will never use the symmetry condition in (ψ:2) directly, but it is necessary to make E well-defined in SBV, as jump direction and jump normal are only uniquely defined up to a simultaneous change of sign.

Remark 2.8 (*Instability of the contribution of G_g^s in $I = J$*) As in the case of typical integral functionals in BV with $G = 0$, the individual contributions in J handling each of the four components of the measure decomposition

$$(Dg, G) = \frac{\mathrm{d}(Dg, G)}{\mathrm{d}\mathcal{L}^N} \mathcal{L}^N + \frac{\mathrm{d}(Dg, G)}{\mathrm{d}|D^a g|}|D^a g| + \frac{\mathrm{d}(Dg, G)}{\mathrm{d}|D^j g|}|D^j g| + \frac{\mathrm{d}(Dg, G)}{\mathrm{d}|G_g^s|}|G_g^s|$$

are not continuous with respect to strict or area-strict convergence; for instance, Lebesgue-absolutely continuous contributions can generate Cantor or jump contributions in the limit. The last contribution in J of the singular rest G_g^s is even worse than the others, though, because it is not even continuous in the norm topology of $BV(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d \times M})$.

Take, for instance, $N = 1$,

$$\Omega := (-1, 1), \quad W := |\cdot|, \quad \psi(\cdot, \nu) := |\cdot|, \quad g_k := \frac{1}{k}\chi_{(0,1)}, \quad G := \delta_0.$$

In particular, $h^c(0, B) = |B|$ for all $B \in \mathbb{R}$. Then $(g_k, G) \to (g, G) = (0, \delta_0)$ strongly in $BV \times \mathcal{M}$, but $G_{g_k}^s = 0$ for all k while $G_g^s = \delta_0$ (since $D^j g_k = \frac{1}{k}\delta_0$,

the whole singular contribution of G with respect to $\mathcal{L}^N + |Dg_k|$ is captured by $\frac{dG}{d|D^j g_k|}|D^j g_k| = k\frac{1}{k}\delta_0 = \delta_0$, while $g = 0$ so that $G_g^s = \delta_0 = G$). As a consequence, the contribution of $G_{g_k}^s$ in J jumps in the limit as $k \to \infty$:

$$\lim_{k\to\infty} \int_\Omega h^c\left(0, \frac{dG_{g_k}^s}{d|G_{g_k}^s|}\right) d|G_{g_k}^s|(x) = 0 \neq 1 = h^c(0, 1) = \int_\Omega h^c\left(0, \frac{dG_g^s}{d|G_g^s|}\right) d|G_g^s|(x).$$

3 Auxiliary Results

In this section, we present some auxiliary results that are pivotal for the proof of Theorem 2.4. In particular, we show that all three densities H, h^j, and h^c are linked (Proposition 3.1) and we present a sequential characterization for them (Proposition 3.2). In Sect. 3.2, functionals depending on measures are introduced, as well as the notion of area-strict convergence.

3.1 Equivalent Characterizations of the Relaxed Energy Densities

Proposition 3.1 *Assume that (W:3) and (ψ:2) hold true and H, h^j, and h^c are defined as in Theorem 2.4. Then the strong recession function of H,*

$$H^\infty(A, B) := \lim_{t\to+\infty} \frac{H(tA, tB)}{t},$$

exists. Moreover, we have that

$$h^c = H^\infty \tag{3.1}$$

and for all $B \in \mathbb{R}^{d\times N}$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$h^c(\lambda \otimes \nu, B) = h_\nu^c(\lambda \otimes \nu, B) = h^j(\lambda, B, \nu), \tag{3.2}$$

where h_ν^c is obtained from h^c by replacing the standard unit cube Q by the unit cube Q_ν oriented according to the normal ν, i.e.,

$$h_\nu^c(A, B) := \inf\left\{E^\infty(u; Q_\nu) : u \in \mathcal{C}^{\text{bulk}}(A, B; Q_\nu)\right\}. \tag{3.3}$$

Proof We define $H^\infty(A, B) := \limsup_{t\to\infty} \frac{1}{t} H(tA, tB)$. With this definition, we obtain (3.1) as a consequence of (W:3) and (ψ:2). Moreover, this even holds if t is replaced by an arbitrary subsequence. The lim sup above is thus independent of subsequences and, therefore, a limit. It remains to show (3.2).

First equality in (3.2): We claim that $h^c(A, B) = h_\nu^c(A, B)$ for arbitrary $A, B \in \mathbb{R}^{d\times N}$. We will first show that $h_\nu^c(A, B) \leqslant h^c(A, B)$. Let $\varepsilon > 0$ and choose an ε-almost

minimizer $u \in SBV(Q; \mathbb{R}^d)$ for the infimum in the definition of $h^c(A, B)$:

$$E(u; Q) \leqslant h_\nu^c(A, B) + \varepsilon, \quad u = Ax \text{ on } \partial Q, \quad \fint_Q \nabla u \, dx = B. \tag{3.4}$$

Up to a set of measure zero, Q_ν can be covered with countably many shifted and rescaled, pairwise disjoint copies of Q:

$$\bigcup_{i \in \mathbb{N}} x_i + \delta_i Q \subset Q_\nu \subset \bigcup_{i \in \mathbb{N}} (x_i + \delta_i \overline{Q}), \tag{3.5}$$

with suitable $x_i \in Q, 0 < \delta_i \leqslant 1$. Defining

$$\tilde{u}(x) := \sum_i \chi_{x_i + \delta_i Q}(x) \left(Ax_i + \delta u \left(\frac{x - x_i}{\delta} \right) \right), \tag{3.6}$$

we obtain $\tilde{u} \in SBV(Q_\nu; \mathbb{R}^d)$ with

$$|D\tilde{u}|(x_i + \delta_i \partial Q) = 0 \quad \text{for all } i \in \mathbb{N} \tag{3.7}$$

and $\tilde{u} = Ax$ on ∂Q_ν (as well as on $x_i + \delta_i \partial Q$). Observe that by the definition of E^∞ in (2.10), the positive one-homogeneity of W^∞ and ψ and a change of variables,

$$E^\infty(\tilde{u}; x_i + \delta_i Q) = \delta_i^N E^\infty(u; x_i + \delta_i Q) \leqslant \delta_i^N \left(h_\nu^c(A, B) + \varepsilon \right), \tag{3.8}$$

the latter due to (3.4). In addition, (3.5) gives that $\sum_{i \in \mathbb{N}} \delta_i^N = \sum_{i \in \mathbb{N}} \mathcal{L}^N(x_i + \delta_i Q) = \mathcal{L}^N(Q_\nu) = 1$. Using the additivity of the integrals in E, (3.5) and (3.7), we can sum (3.8) over i to conclude that

$$E^\infty(\tilde{u}; Q_\nu) \leqslant h_\nu^c(A, B) + \varepsilon. \tag{3.9}$$

Similarly, we can also check that $\fint_{Q_\nu} \nabla \tilde{u} \, dx = B$. Since $\varepsilon > 0$ was arbitrary and $\tilde{u}$ is admissible for the infimum in the definition of h_ν^c, this implies that $h_\nu^c(A, B) \leqslant h^c(A, B)$. The opposite inequality follows in exactly the same way, with exchanged roles of Q and Q_ν.

Second equality in (3.2): We have to show that $h_\nu^c(\lambda \otimes \nu, B) = h^j(\lambda, B, \nu)$. For $k \in \mathbb{N}$, define the laterally extended cuboid

$$R_\nu(k) := \left\{ x \in \mathbb{R}^N \,\middle|\, |x \cdot \nu| < \frac{1}{2}, \ \left| x \cdot \nu_j^\perp \right| < \frac{2k + 1}{2} \text{ for } j = 1, \ldots, N - 1 \right\},$$

where $\nu_j^\perp, j = 1, \ldots, N - 1$, are the pairwise orthogonal unit vectors perpendicular to ν corresponding to the lateral faces of Q_ν. Notice that up to a set of measure zero

formed by overlapping boundaries, $R_\nu(k)$ can be written as a pairwise disjoint union of $(2k + 1)^{N-1}$ shifted copies of Q_ν:

$$\overline{R}_\nu(k) = \bigcup_{\xi \in Z(k)} (\xi + \overline{Q}_\nu), \quad Z(k) := \left\{ \xi = \sum_{i=1}^{N-1} j(i) v_{j(i)}^\perp \,\middle|\, j(i) \in \{-k, \dots, k\} \right\}.$$

Now let $\varepsilon > 0$ and choose an ε-almost minimizer $u \in SBV(Q_\nu; \mathbb{R}^d)$ for the infimum in the definition of $h_\nu^c(\lambda \otimes \nu, B)$:

$$h_\nu^c(\lambda \otimes \nu, B) + \varepsilon \geqslant E^\infty(u; Q_\nu), \tag{3.10}$$

with E^∞ defined in (2.10). Since $v_j^\perp \cdot \nu = 0$, the affine function $x \mapsto (\lambda \otimes \nu)x$ determining the boundary values of u is constant direction $v_j^\perp$ for each $j = 1, \dots, N-1$. We can therefore extend u periodically in the $(N-1)$ directions $v_j^\perp$ to a function $u_k \in SBV(R_\nu(k); \mathbb{R}^d)$, without creating jumps at the interfaces between elementary cells of periodicity: $u_k|_{Q_\nu} = u$, $u_k(x + v_j^\perp) = u_k(x)$ whenever $x, x + v_j^\perp \in R_\nu(k)$, $u_k = (\lambda \otimes \nu)x$ on $\xi + \partial Q_\nu$ for each $\xi \in Z(k)$ (in the sense of traces), and $|Du_k|(\xi + \partial Q_\nu) = 0$ for each $\xi \in Z(k)$. As a consequence, (3.10) is equivalent to

$$h_\nu^c(\lambda \otimes \nu, B) + \varepsilon \geqslant \frac{1}{\#Z(k)} E^\infty(u_k; R_\nu(k)) \tag{3.11}$$

for all $k \in \mathbb{N}$. Analogously, we can also extend the elementary jump function $s_{\lambda,\nu}$ used in the definition of h^j periodically to $s_{\lambda,\nu,k} \in SBV(R_\nu(k); \mathbb{R}^d)$, again without creating jumps at the interfaces since $s_{\lambda,\nu}$ is constant in directions perpendicular to ν.

Now choose functions $\varphi_k \in C_c^\infty(R_\nu(k); [0, 1])$ such that

$$\varphi_k = 1 \text{ on } R_\nu(k-1) \quad \text{and} \quad |\nabla \varphi_k| \leqslant 2 \text{ on } R_\nu(k) \setminus R_\nu(k-1)$$

Defining

$$\tilde{u}_k := \varphi_k u_k + (1 - \varphi_k) s_{\lambda,\nu,k},$$

we obtain that $\tilde{u}_k = s_{\lambda,\nu,k}$ on $\partial R_\nu(k)$, $\tilde{u}_k = u_k$ on $R_\nu(k-1)$ and

$$
\begin{aligned}
|D\tilde{u}_k|(R_\nu(k) \setminus R_\nu(k-1))| &\leqslant 2 \left\| u_k - s_{\lambda,\nu,k} \right\|_{L^1(R_\nu(k) \setminus R_\nu(k-1); \mathbb{R}^d)} \\
&\quad + |D\tilde{u}_k - Ds_{\lambda,\nu,k}|(R_\nu(k) \setminus R_\nu(k-1)) \\
&\leqslant 2(N-1)(2k+1)^{N-2} \left\| u - s_{\lambda,\nu} \right\|_{BV(Q_\nu; \mathbb{R}^d)}.
\end{aligned}
$$

Since $\#Z(k) = (2k + 1)^{N-1}$, we conclude that $\frac{1}{\#Z(k)} |D\tilde{u}_k|(R_\nu(k) \setminus R_\nu(k-1))| = O(1/k) \to 0$ as $k \to \infty$. Using the Lipschitz properties of W (W:2) and ψ (2.14), we

can thus replace u_k with $\tilde{u}_k$ in (3.11), with an error that converges to zero as $k \to \infty$:

$$h_\nu^c(\lambda \otimes \nu, B) + \varepsilon + O(1/k) \geqslant \frac{1}{\#Z(k)} E^\infty(\tilde{u}_k; R_\nu(k)) \tag{3.12}$$

Since $\tilde{u}_k = s_{\lambda,\nu}$ on $\partial R_\nu(k)$, we can define

$$\hat{u}_k(\tilde{x}) := \begin{cases} \tilde{u}_k\big((2k+1)\tilde{x}\big) & \text{if } \tilde{x} \in \frac{1}{2k+1} R_\nu(k), \\ s_{\lambda,\nu}(\tilde{x}) & \text{if } (2k+1)\tilde{x} \in Q_\nu \setminus \frac{1}{2k+1} R_\nu(k), \end{cases}$$

without creating a jump at the interface between $\frac{1}{2k+1} R_\nu(k)$ and the rest. As defined, $\hat{u}_k$ is now admissible for the infimum defining h^j, and by a change of variables on the right-hand side of (3.12), we see that

$$h_\nu^c(\lambda \otimes \nu, B) + \varepsilon + O(1/k) \geqslant E^\infty\Big(\hat{u}_k; \frac{1}{2k+1} R_\nu(k)\Big) = E^\infty\big(\hat{u}_k; Q_\nu\big)$$
$$\geqslant h^j(\lambda, B, \nu). \tag{3.13}$$

As $\varepsilon > 0$ and $k \in \mathbb{N}$ were arbitrary, (3.13) implies that $h_\nu^c(\lambda \otimes \nu, B) \geqslant h^j(\lambda, B, \nu)$. The reverse inequality can be shown analogously. $\qquad\square$

In the following proposition, we prove a sequential characterization of the relaxed energy densities defined in (2.11). To do so, we define the classes of sequences of competitors (see (2.9))

$$\mathcal{C}_{\text{seq}}^{\text{bulk}}(A, B; Q) := \{\{u_n\} \subset SBV(Q; \mathbb{R}^d) : u_n \overset{*}{\rightharpoonup} Ax \text{ in } BV, \nabla u_n \overset{*}{\rightharpoonup} B\mathcal{L}^N \text{ in } \mathcal{M}\}, \tag{3.14a}$$

$$\mathcal{C}_{\text{seq}}^{\text{surface}}(\lambda, \Lambda; Q_\nu) := \{\{u_n\} \subset SBV(Q_\nu; \mathbb{R}^d) : u_n \overset{*}{\rightharpoonup} s_{\lambda,\nu} \text{ in } BV, \nabla u_n \overset{*}{\rightharpoonup} \Lambda\mathcal{L}^N \text{ in } \mathcal{M}\}. \tag{3.14b}$$

Proposition 3.2 *Suppose that* (W:1)–(W:3) *and* (ψ:1)–(ψ:3) *hold true. Then*

$$H(A, B) := \inf \Big\{ \liminf_{n\to\infty} E(u_n; Q) : \{u_n\} \in \mathcal{C}_{\text{seq}}^{\text{bulk}}(A, B; Q) \Big\}; \tag{3.15a}$$

$$h^j(\lambda, \Lambda, \nu) := \inf \Big\{ \liminf_{n\to\infty} E^\infty(u_n; Q_\nu) : \{u_n\} \in \mathcal{C}_{\text{seq}}^{\text{surface}}(\lambda, \Lambda; Q_\nu) \Big\}; \tag{3.15b}$$

$$h^c(A, B) := \inf \Big\{ \liminf_{n\to\infty} E^\infty(u_n; Q) : \{u_n\} \in \mathcal{C}_{\text{seq}}^{\text{bulk}}(A, B; Q) \Big\}. \tag{3.15c}$$

Proof The formulae (3.15a) and (3.15c) are obtained in the same way as in Choksi and Fonseca (1997, Proposition 3.1) (for the latter, notice that $W = W^\infty$ is an admissible choice in (3.15a)); formula (3.15b) is obtained in the same way as in Choksi and Fonseca (1997, Proposition 4.1), i.e., by applying the technique of Choksi and Fonseca (1997, Proposition 3.1) in a vanishing strip around the jumps of $s_{\lambda,\nu}$.. $\qquad\square$

3.2 Nonlinear Transformation of Measures and Area-Strict Convergence

The following shorthand notation will prove useful below.

Definition 3.3 *(nonlinear transformation of measures)* For any Borel set $U \subset \mathbb{R}^N$, any Borel function $h\colon \mathbb{R}^m \to \mathbb{R}$ with strong recession function h^∞ and any Radon measure $\mu \in \mathcal{M}(U; \mathbb{R}^m)$, we define

$$\int_U \mathrm{d}h(\mu) := \int_U h\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^N}\Big)\,\mathrm{d}|\mu| + \int_U h^\infty\Big(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s|,$$

where μ^s denotes the singular part of the Radon-Nikodym decomposition of μ with respect to the Lebesgue measure $\mathcal{L}^N$: $\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^N}\mathcal{L}^N + \mu^s$.

Definition 3.4 *(area-strict convergence, cf.* Kristensen and Rindler 2010*)* For a Borel set V, a sequence $(G_k) \subset \mathcal{M}(V; \mathbb{R}^{d \times N})$ and $G \in \mathcal{M}(V; \mathbb{R}^{d \times N})$, we say that $G_k \to G$ area-strictly if $G_k \overset{*}{\rightharpoonup} G$ in $\mathcal{M}$ and

$$\int_V \mathrm{d}a(G_k) \to \int_V \mathrm{d}a(G), \quad \text{where} \quad a(\xi) := \sqrt{1 + |\xi|^2},\ \xi \in \mathbb{R}^{d \times N}.$$

Analogously, if V is open, for a sequence $(g_k) \subset BV(V; \mathbb{R}^d)$ and $g \in BV(V; \mathbb{R}^d)$, we say that $g_k \to g$ area-strictly if $g_k \overset{*}{\rightharpoonup} g$ in BV and $\int_V \mathrm{d}a(Dg_k) \to \int_V \mathrm{d}a(Dg)$.

The following lemma is a generalized Reshetnyak continuity theorem; see Kristensen and Rindler (2010, Theorem 4) or Rindler (2018, Theorem 10.3).

Lemma 3.5 *If $H\colon \mathbb{R}^{d \times N} \times \mathbb{R}^{d \times N} \to \mathbb{R}$ is continuous and has a recession function in the strong uniform sense (see Proposition 4.2), then the functional defined on $\mathcal{M}(\Omega; \mathbb{R}^{d \times N}) \times \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ by*

$$(F, G) \mapsto \int_\Omega \mathrm{d}H(F, G)$$

is sequentially continuous with respect to the area-strict convergence of measures.

We also need the following well-known lemma combining area-strict approximations in BV and $\mathcal{M}$.

Lemma 3.6 *Let $(g, G) \in mSD$. Then there exists a sequence $\{(g_k, G_k)\} \subset W^{1,1}(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$ such that $g_k \to g$ area-strictly in BV and $G_k \to G$ area-strictly in $\mathcal{M}$.*

Proof The sequences $\{g_k\}$ and $\{G_k\}$ can be defined separately, essentially by mollification. As to $\{g_k\}$, see, for instance, Rindler (2018, Lemma 11.1), while the case of $\{G_k\}$ is simpler. □

4 Proof of Theorem 2.4

The proof of Theorem 2.4 is divided into two parts, each of which is carried out in the following section.

4.1 Upper Bound

Proof We have to show that $I(g, G; \Omega) \leqslant J(g, G; \Omega)$, which is equivalent to the existence of a "recovery" sequence $\{u_n\}$ admissible in the definition of I, i.e., such that $u_n \overset{*}{\rightharpoonup} (g, G)$ in mSD and $E(u_n; \Omega) \to J(g, G; \Omega)$. The proof here is presented using a series of auxiliary results collected below.

First observe that based on Proposition 3.1, our candidate J for the limit functional, introduced in (2.4) using H, h^j, and h^c, can be expressed as a standard integral functional of the measure variable (Dg, G) using only H and its recession function H^∞ as integrands (Proposition 4.1). As H is continuous and its recession function exists in a strong enough sense (cf. Proposition 4.2), J is sequentially continuous with respect to the area-strict convergence of measures (Lemma 3.5). Since any $(g, G) \in mSD$ can be approximated area-strictly by sequences in $W^{1,1} \times L^1$ (Lemma 3.6), a diagonalization argument allows us to reduce the construction of the recovery sequence to the case $(g, G) \in W^{1,1} \times L^1 \subset SBV \times L^1$. This special case was already obtained in Choksi and Fonseca (1997), see Remark 2.5. $\qquad\square$

Proposition 4.1 *Suppose that $(W{:}1)$–$(W{:}3)$ and $(\psi{:}1)$–$(\psi{:}3)$ hold. Then with the notation of Definition 3.3,*

$$J(g, G; \Omega) = \int_\Omega dH(Dg, G)$$

where J is defined in (2.4) and H is given by (2.11a).

Proof Clearly, $H\left(\frac{d(Dg, G)}{d\mathcal{L}^N}\right) = H\left(\nabla g, \frac{dG}{d\mathcal{L}^N}\right)$. In addition,

$$
\begin{aligned}
|(Dg, G)^s| &= \left(1 + \frac{dG}{d|D^s g|}\right)|D^s g| + \theta \\
&= \left(1 + \frac{dG}{d|D^c g|}\right)|D^c g| + \left(1 + \frac{dG}{d|D^j g|}\right)|D^j g| + |G_g^s|,
\end{aligned}
$$

since $|G_g^s|$ and $|Dg^s| + \mathcal{L}^N$ are mutually orthogonal by definition of G_g^s, and the Cantor and jump parts of Dg are mutually orthogonal as well. Since H^∞ is positively 1-homogeneous, the definition of J implies the asserted representation once we use Proposition 3.1 to replace H^∞ by h^j and h^c, respectively. $\qquad\square$

We need the following regularity properties of H, in particular at infinity.

Proposition 4.2 *Suppose that $(W{:}1)$, $(W{:}3)$ and $(\psi{:}1)$–$(\psi{:}3)$ hold. Then H is globally Lipschitz and the recession function H^∞ exists in the strong uniform sense, i.e., the*

limit

$$\lim_{\substack{(A',B')\to(A,B)\\ t\to+\infty}} \frac{H(tA',tB')}{t} \tag{4.1}$$

exists for all $(A, B) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}\backslash\{(0,0)\}$.

Proof The Lipschitz property of H was proved in Barroso et al. (2022, Theorem 2.10) (the case $p = 1$). Concerning (4.1), first observe that since H is Lipschitz with some constant $L > 0$,

$$\left|\frac{H(tA',tB')}{t} - H^\infty(A,B)\right| \leqslant L\left|(A',B') - (A,B)\right| + \left|\frac{H(tA,tB)}{t} - H^\infty(A,B)\right| \tag{4.2}$$

Here, $H^\infty(A,B) = \limsup_{t\to\infty} \frac{1}{t}H(tA,tB)$ as before. It, therefore suffices to show that

$$\lim_{t\to+\infty} \frac{H(tA,tB)}{t} = H^\infty(A,B).$$

We claim that in fact, we even have that

$$\left|\frac{H(tA,tB)}{t} - H^\infty(A,B)\right| \leqslant C(A,B)\left(\frac{1}{t^\alpha} + \frac{1}{t}\right) \quad \text{for all } t > 0, \ A, B \in \mathbb{R}^{d\times N}, \tag{4.3}$$

where $C(A,B) > 0$ is a constant independent of t and $H^\infty(A,B) = \limsup_{t\to+\infty} \frac{1}{t} H(tA,tB)$.

For a proof of 4.3, first fix $\varepsilon > 0$ and choose ε-almost optimal sequence $\{u_{t,n}\}_n$ for the sequential characterization of $H(tA,tB)$ in Proposition 3.2, dependent on $t > 0$ (and A,B). This choice yields that

$$H(tA,tB) + \varepsilon \geqslant \int_Q W(\nabla u_{t,n})\,dx + \int_{Q\cap S_{u_{t,n}}} \psi([u_{t,n}], \nu_{u_{t,n}})\,d\mathcal{H}^{N-1}(x). \tag{4.4}$$

The sequence $v_{t,n} := t^{-1}u_{t,n}$ then is also in the class of admissible sequences for the sequential characterization of $h^c(A,B)$ in Proposition 3.2, and since $h^c(A,B) = H^\infty(A,B)$ by Proposition 3.1, this entails that

$$H^\infty(A,B) \leqslant \int_Q W^\infty(\nabla v_{t,n})\,dx + \int_{Q\cap S_{v_{t,n}}} \frac{1}{t}\psi([v_{t,n}], \nu_{v_{t,n}})\,d\mathcal{H}^{N-1}(x), \tag{4.5}$$

where we exploited that W^∞ and $\psi(\cdot, \nu)$ are positively 1-homogeneous. Multiplying (4.4) by t^{-1} and combining it with (4.5) yields

$$H^\infty(A, B) - \frac{1}{t}H(tA, tB) \leqslant \frac{1}{t}\varepsilon + \int_Q \left(W^\infty(\nabla v_{t,n}) - \frac{1}{t}W(t\nabla v_{t,n}))\right) dx. \quad (4.6)$$

Analogously, we can also choose and ε-almost optimal sequence $\tilde{v}_h$ for the sequential characterization of $h^c(A, B) = H^\infty(A, B)$, which makes $\tilde{u}_{t,h} := t\tilde{v}_h$ admissible for the sequential characterization of $H(tA, Bt)$. With this, get that

$$\frac{1}{t}H(tA, tB) - H^\infty(A, B) \leqslant \varepsilon + \int_Q \left(\frac{1}{t}W(t\nabla\tilde{v}_{t,n})) - W^\infty(\nabla\tilde{v}_{t,n})\right) dx. \quad (4.7)$$

The right hands sides of (4.6) and (4.7) can now be estimated in the same fashion: by (W:1) and the homogeneity of W^∞ we have that

$$\left|\frac{W(tA)}{t} - W^\infty(A)\right| \leqslant \left|\frac{W(tA)}{t}\right| + |W^\infty(A)| \leqslant C_W\left(|A| + \frac{1}{t}\right) + C_W|A|$$
$$\leqslant 3C_W\frac{1}{t} \quad \text{if } t|A| < 1.$$

This is exactly the case excluded in (W:3), so that together with (W:3), we obtain that

$$\left|\frac{W(tA)}{t} - W^\infty(A)\right| \leqslant \frac{c|A|^{1-\alpha}}{t^\alpha} + 3C_W\frac{1}{t}$$
$$\leqslant \frac{c(1 + |A|)}{t^\alpha} + 3C_W\frac{1}{t} \quad \text{for all } t > 0 \text{ and } A \in \mathbb{R}^{d \times N}, \quad (4.8)$$

since $0 < \alpha < 1$. Moreover, (4.4) implies that $\left\|\nabla v_{t,n}\right\|_{L^1} = t^{-1}\left\|\nabla u_{t,n}\right\|_{L^1}$ is equi-bounded for $t \geqslant 1$ since H is globally Lipschitz, $\psi \geqslant 0$ and W is coercive by (W:1). Similarly, $\left\|\nabla\tilde{v}_{t,n}\right\|_{L^1}$ is equi-bounded. Thus,

$$M(A, B) := \sup_{t \geqslant 1}\sup_{n \in \mathbb{N}}\left(\left\|\nabla v_{t,n}\right\|_{L^1} + \left\|\nabla\tilde{v}_{t,n}\right\|_{L^1}\right) < \infty, \quad (4.9)$$

Now we can use (4.8) to obtain upper bounds for the right-hand sides of (4.6) and (4.7) and combine them. By (4.9), this yields that

$$\left|\frac{1}{t}H(tA, tB) - H^\infty(A, B)\right| \leqslant \varepsilon\left(\frac{1}{t} + 1\right) + \frac{c(1 + M(A, B))}{t^\alpha} + 3\mathcal{L}^N(\Omega)C_W\frac{1}{t}$$
$$(4.10)$$

for all $t \geqslant 1$. Since $\varepsilon > 0$ was arbitrary, (4.10) implies (4.3). □

4.2 Lower Bound

Our proof of the lower bound relies on the following lower semicontinuity property.

Proposition 4.3 *Assume that $(W{:}1)$–$(W{:}3)$ and $(\psi{:}1)$–$(\psi{:}3)$ hold. Then the integrand H defined in (2.11a) is quasiconvex-convex in the sense that for all $A, B \in \mathbb{R}^{d \times N}$,*

$$\fint_Q H(A + \nabla v, B + w)\, \mathrm{d}x \geqslant H(A, B)$$

for all $(v, w) \in W_0^\infty(Q; \mathbb{R}^d) \times L^\infty(Q; \mathbb{R}^{d \times N})$ with $\fint_Q w\, \mathrm{d}x = 0$.

Moreover, the functional $(g, G) \mapsto \int_\Omega \mathrm{d}H(Dg, G)$ is sequentially lower semicontinuous with respect to the convergence in (1.2).

Proof We will first show that $(g, G) \mapsto \int_\Omega H(\nabla g, G)\, \mathrm{d}x$ is sequentially lower semi-continuous with respect to weak convergence in $W^{1,1} \times L^1$. Take $(g, G) \in W^{1,1}(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$ and $\{(g_n, G_n)\} \subset W^{1,1}(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$ with $(g_n, G_n) \rightharpoonup (g, G)$ weakly in $W^{1,1} \times L^1$. By Choksi and Fonseca (1997, Theorem 2.16) (recovery sequence for the case of I_1 therein), for each n there exists a sequence $\{u_{n,k}\}_k \subset SBV(\Omega; \mathbb{R}^d)$ such that as $k \to \infty$,

$$u_{n,k} \to g_n \text{ in } L^1(\Omega; \mathbb{R}^d) \quad \text{and} \quad \nabla u_{n,k} \xrightarrow{*} G_n \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}),$$

and

$$E(u_{n,k}; \Omega) \to \int_\Omega H(\nabla g_n, G_n)\, \mathrm{d}x.$$

In addition, we may assume that up to a (not relabeled) subsequence

$$\liminf_{n \to \infty} \int_\Omega H(\nabla g_n, G_n)\, \mathrm{d}x = \lim_{n \to \infty} \int_\Omega H(\nabla g_n, G_n)\, \mathrm{d}x < +\infty.$$

Since E is coercive as a consequence of $(W{:}1)$ and $(\psi{:}1)$, the latter implies that $\{Du_{n,k}\}$ is equi-bounded in $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$. We can therefore find a diagonal subsequence $u_{n,k(n)}$ with $k(n) \to \infty$ fast enough, such that

$$(u_{n,k(n)}, \nabla u_{n,k(n)}) \xrightarrow{*} (g, G) \text{ in } mSD \text{ and } \lim_{n \to \infty} E(u_{n,k(n)}; \Omega)$$

$$= \lim_{n \to \infty} \int_\Omega H(\nabla g_n, G_n)\, \mathrm{d}x.$$

Since the sequence $\{u_{n,k(n)}\}_n$ is admissible for the lower bound in Choksi and Fonseca (1997, Theorem 2.16) (for the functional I_1), we conclude that

$$\lim_{n \to \infty} \int_\Omega H(\nabla g_n, G_n)\, \mathrm{d}x = \lim_{n \to \infty} E(u_{n,k(n)}; \Omega) \geqslant \int_\Omega H(\nabla g, G)\, \mathrm{d}x,$$

i.e., the functional with integrand H is weakly lower semi-continuous in $W^{1,1} \times L^1$. Since H also has at most linear growth by Proposition 4.2 and is non-negative as a consequence of (W:1), Carita et al. (2010, Theorem 1.1) (see also Carita et al. 2011) implies that H is quasiconvex-convex.

It remains to show that $(g, G) \mapsto \int_\Omega \mathrm{d}H(Dg, G)$ is sequentially lower semi-continuous with respect to the convergence (1.2). This follows from Arroyo-Rabasa et al. (2020, Theorem 1.7). Here, notice that with $\mathcal{A} := \mathrm{diag}(\mathrm{Curl}, 0)$, i.e., with $\mathcal{A}(\nabla g^\top, G)^\top := ((\mathrm{Curl}\,\nabla g)^\top, 0)^\top$, $\mathcal{A}(\nabla g_k^\top, G_k)^\top = 0$ in the sense of distributions, and the $\mathcal{A}$-quasiconvexity of H for this special case is equivalent to quasiconvexity-convexity of H. The latter can equivalently be tested with periodic functions on the simply connected U where all curl-free fields are gradients.

$\square$

Proof of Theorem 2.4, lower bound Let H be the integrand in (2.11a). Moreover, let $(g, G) \in mSD$ and $\{v_n\} \subset SBV(\Omega; \mathbb{R}^d)$ be such that $v_n \overset{*}{\rightharpoonup} (g, G)$ in the sense of (1.2). Observing that for each n, v_n can be interpreted as a constant sequence converging to itself in mSD, by Proposition 3.1 and Choksi and Fonseca (1997, Theorem 2.16) (its lower bound for the case of I_1 therein), we have that

$$\int_\Omega \mathrm{d}H(Dv_n, \nabla v_n \mathcal{L}^N) \leqslant \int_\Omega W(\nabla v_n)\mathrm{d}x + \int_{\Omega \cap S_{v_n}} \psi([v_n], v_{v_n})\mathrm{d}\mathcal{H}^{N-1}(x).$$

(4.11)

In addition, $(u, G) \mapsto \int_\Omega \mathrm{d}H(Du, G)$ is weak*-sequentially lower semi-continuous in mSD by Proposition 4.3. In particular,

$$\int_\Omega \mathrm{d}H(Dg, G) \leqslant \liminf_{n\to\infty} \int_\Omega \mathrm{d}H(Dv_n, \nabla v_n \mathcal{L}^N).$$

(4.12)

Taking Proposition 4.1 into account, the lower bound inequality now follows from (4.12) and (4.11):

$$J(g, G; \Omega) = \int_\Omega \mathrm{d}H(Dg, G) \leqslant \liminf_{n\to\infty} \int_\Omega \mathrm{d}H(Dv_n, \nabla v_n \mathcal{L}^N)$$
$$\leqslant \liminf_{n\to\infty} \int_\Omega W(\nabla v_n)\mathrm{d}x + \int_\Omega \psi([v_n], v_{v_n})\mathrm{d}\mathcal{H}^{N-1}(x) = \liminf_{n\to\infty} E(v_n).$$

$\square$

5 Relaxation Under Trace Constraints

Let Ω' be a bounded Lipschitz domain such that $\Omega \subset \Omega'$, and let

$$\Gamma := \Omega' \cap \partial\Omega.$$

Let $u_0 \in W^{1,1}(\Omega'; \mathbb{R}^d)$ and let $(g, G) \in mSD$. The relaxed functional subject to the Dirichlet condition $u = u_0$ on Γ is defined as

$$I_\Gamma(g, G; \Omega) := \inf \left\{ \liminf_{n \to \infty} E(u_n; \Omega) \;\middle|\; \begin{array}{l} u_n \in SBV(\Omega; \mathbb{R}^d),\, u_n = u_0 \text{ on } \Gamma, \\ u_n \overset{*}{\rightharpoonup} g \text{ in } BV(\Omega; \mathbb{R}^d), \\ \nabla u_n \overset{*}{\rightharpoonup} G \text{ in } \mathcal{M}(\Omega \cup \Gamma; \mathbb{R}^{d \times N}) \end{array} \right\}, \quad (5.1)$$

where, for every open subset A of Ω', $E(\cdot; A)$ is the functional given by (2.1), with W and ψ satisfying $(W{:}1)$–$(W{:}1)$ and $(\psi{:}1)$–$(\psi{:}3)$,

We have the following integral representation for I_Γ.

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and assume that $(W{:}1)$–$(W{:}1)$ and $(\psi{:}1)$–$(\psi{:}3)$ hold. Moreover, let $\Omega' \supset \Omega$ be a bounded domain and $u_0 \in W^{1,1}(\Omega'; \mathbb{R}^d)$. In addition, for $\Gamma := \Omega' \cap \partial\Omega$ assume that $\mathcal{H}^{N-1}(\overline{\Gamma} \setminus \Gamma) = 0$. Then,*

$$I_\Gamma(g, G; \Omega) = J_\Gamma(g, G; \Omega) \quad \text{for every } (g, G) \in mSD,$$

where

$$\begin{aligned} J_\Gamma(g, G; \Omega) := &\int_\Omega dH(g, G) \\ &+ \int_\Gamma H^\infty \left(\frac{d([g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1} \llcorner \Gamma, G)}{d|([g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1} \llcorner \Gamma, G)|} \right) \\ &d|([g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1} \llcorner \Gamma, G)|, \end{aligned}$$

and H is the function defined in (2.11a).

The proof will be given in two parts. We immediately start with the lower bound, and the proof of the upper bound will follow after an auxiliary result needed there.

Proof of Theorem 5.1, the lower bound We have to show that $I_\Gamma(g, G; \Omega) \geqslant J_\Gamma(g, G; \Omega)$. For every $k \in \mathbb{N}$, let $\Omega_k := \{x \in \mathbb{R}^N : \text{dist}(x, \overline{\Omega}) \leqslant \frac{1}{k}\}$, and consider

$$\Omega'_k := \Omega_k \cap \Omega'.$$

Thus $\Gamma = \Omega'_k \cap \partial\Omega$, for every k and Ω'_k shrinks to $\Omega \cup \Gamma$ as $k \to \infty$. As for $I_\Gamma(g, G; \Omega)$, define for every $k \in \mathbb{N}$

$$\begin{aligned} \hat{I}_\Gamma(g, G; \Omega'_k) := &\int_{\Omega'_k \setminus \Omega} W(\nabla u_0) \, dx \\ &+ \inf \Big\{ \liminf_{n \to \infty} E(u_n; \Omega) : u_n \in SBV(\Omega'_k; \mathbb{R}^d),\, u_n = u_0 \text{ on } \partial\Omega, \\ &\qquad u_n \overset{*}{\rightharpoonup} g \text{ in } BV(\Omega; \mathbb{R}^d),\, \nabla u_n \overset{*}{\rightharpoonup} \\ &\qquad G \text{ in } \mathcal{M}(\Omega \cup \Gamma; \mathbb{R}^{d \times N}) \Big\}. \end{aligned}$$

Thus

$$I_\Gamma(g, G; \Omega) = \hat{I}_\Gamma(g, G; \Omega'_k) - \int_{\Omega'_k \setminus \Omega} W(\nabla u_0)\mathrm{d}x. \qquad (5.2)$$

On the other hand,

$$\hat{I}_\Gamma(g, G; \Omega'_k) = \inf \Big\{ \liminf_{n\to\infty} E(v_n; \Omega'_k) : v_n \in SBV(\Omega'_k; \mathbb{R}^d), v_n = u_0 \text{ in } \Omega'_k \setminus \overline{\Omega},$$

$$v_n \overset{*}{\rightharpoonup} \hat{g} \text{ in } BV(\Omega'_k; \mathbb{R}^d), \nabla v_n \overset{*}{\rightharpoonup} \hat{G} \text{ in } \mathcal{M}(\Omega'_k; \mathbb{R}^{d\times N}) \Big\},$$

where

$$\hat{g} := \begin{cases} g & \text{in } \Omega, \\ u_0 & \text{in } \Omega'_k \setminus \overline{\Omega} \end{cases} \quad \text{and} \quad \hat{G} := \begin{cases} G & \text{in } \Omega \cup \Gamma, \\ \nabla u_0 & \text{in } \Omega'_k \setminus \overline{\Omega}. \end{cases}$$

In particular,

$$D\hat{g}\lfloor_\Gamma = [g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1}\lfloor\Gamma.$$

Clearly, for every Ω'_k,

$$\begin{aligned}
&\hat{I}_\Gamma(g, G; \Omega'_k) \\
&\geqslant I(\hat{g}, \hat{G}; \Omega'_k) = \int_{\Omega'_k} \mathrm{d}H(\hat{g}, \hat{G}) \\
&\geqslant \int_\Omega \mathrm{d}H(g, G) + \int_\Gamma H^\infty\left(\frac{\mathrm{d}([g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1}\lfloor\Gamma, G)}{\mathrm{d}|([g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1}\lfloor\Gamma, G)|} \right) \\
&\quad \mathrm{d}|([g - u_0] \otimes \nu_\Gamma \, \mathcal{H}^{N-1}\lfloor\Gamma, G)|,
\end{aligned} \qquad (5.3)$$

where $I(\hat{g}, \hat{G}; \Omega'_k)$ is the functional introduced in (2.2), and in the equality we have exploited Theorem 2.4 and Remark 2.6. The proof is concluded by letting $k \to \infty$, in the above inequality, taking into account (5.2) and the fact that $\lim_{k\to\infty} \int_{\Omega'_k \setminus \Omega} W(\nabla u_0)\,\mathrm{d}x = 0$. $\qquad \square$

Below, we will reduce the construction of the recovery sequence needed for the upper bound to that of Theorem 2.4. This relies on the following lemma.

Lemma 5.2 (domain shrinking (Krömer and Valdman 2023, Lemma 3.1)) *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists an open neighborhood $U \supset \overline{\Omega}$ and a sequence of maps $\{\Psi_j\} \subset C^\infty(\overline{U}; \mathbb{R}^N)$ such that for every $j \in \mathbb{N}$,*

$$\Psi_j \colon \overline{U} \to \Psi_j(\overline{U}) \text{ is invertible and } \Psi_j(\Omega) \subset\subset \Omega. \qquad (5.4)$$

In addition, $\Psi_j \to \mathrm{id}$ in $C^m(\overline{U}; \mathbb{R}^N)$ as $j \to \infty$, for all $m \in \mathbb{N} \cup \{0\}$.

Proof This is the case $\Gamma = \emptyset$ in Krömer and Valdman (2023). The statement there has Ψ_j only defined on $\overline{\Omega}$, but the proof also provides the extension to $\overline{U}$ (as long as $\overline{U}$ is still fully covered by the union of Ω and the open cuboids covering $\partial\Omega$ in which $\partial\Omega$ can be seen as a Lipschitz graph). $\qquad\square$

Remark 5.3 If Ω is strictly star-shaped with respect to some $x_0 \in \Omega$, Lemma 5.2 is easy to show with $\Psi_j(x) := x_0 + \frac{j}{j+1}(x - x_0)$. The proof of Krömer and Valdman (2023, Lemma 3.1) for the general case glues local constructions near the boundary using a decomposition of unity, exploiting that everything happens uniformly C^1-close to the identity to preserve invertibility.

Proof of Theorem 5.1, the upper bound We have to show that $I_\Gamma(g, G; \Omega) \leqslant J_\Gamma(g, G; \Omega)$, for each $(g, G) \in BV(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega \cup \Gamma; \mathbb{R}^{d \times N})$. For this, it suffices to find a recovery sequence, i.e., a sequence (u_n) admissible in the definition of $I_\Gamma(g, G; \Omega)$ such that $E(u_n; \Omega) \to J_\Gamma(g, G; \Omega)$. In particular, we must have $u_n = u_0$ on Γ in the sense of traces in BV. The proof is divided into three steps. In the first two steps, we define a suitable approximating sequence of limit states $(\hat{g}_j, \hat{G}_j)$ such that $(\hat{g}_j, \hat{G}_j) \overset{*}{\rightharpoonup} (g, G)$ in mSD, $J_\Gamma(\hat{g}_j, \hat{G}_j; \Omega) \to J_\Gamma(g, G; \Omega)$ and $\hat{g}_j = u_0$ on Γ. In the final step, we will then use the upper bound in Theorem 2.4, which for each j gives a "free" recovery sequence $\{u_{j,n}\}_n \subset BV$ for $I(\hat{g}_j, \hat{G}_j; \Omega)$ that again can be modified to match the trace of its weak* limit $\hat{g}_j$ on Γ. The assertion then follows by a diagonal subsequence argument.

   Step 1: Approximating limit states (g_j, G_j) **with values "close" to** u_0 **near** Γ. Choose a bounded neighborhood U of Ω according to Lemma 5.2 and an extension

$$\tilde{g} \in BV(U; \mathbb{R}^d) \quad \text{with} \quad \tilde{g}|_\Omega = g, \quad |D\tilde{g}|(\partial\Omega) = 0, \quad \tilde{g} \in W^{1,1}(U \setminus \overline{\Omega}; \mathbb{R}^d).$$

With this, we define

$$BV(U; \mathbb{R}^d) \ni g_0 := \chi_\Omega g + \chi_{U \cap (\Omega' \setminus \overline{\Omega})} u_0 + \chi_{U \setminus \overline{\Omega'}} \tilde{g},$$
$$\mathcal{M}(U; \mathbb{R}^{d \times N}) \ni G_0 := \chi_{\Omega \cup \Gamma} G.$$

In particular, with the outer normal ν_Γ to $\partial\Omega$ on Γ,

$$G_0 \llcorner (\Omega \cup \Gamma) = G, \qquad Dg_0 \llcorner (\Omega \cup \Gamma) = Dg \llcorner \Omega + (u_0 - g) \otimes \nu_\Gamma \, \mathcal{H}^{N-1} \llcorner \Gamma,$$

$G_0 \llcorner (U \setminus (\Omega \cup \Gamma)) = 0$, $g_0|_\Omega = g$, and g_0 jumps at Γ from (the trace of) g to u_0 and at $U \cap (\partial\Omega' \setminus \Omega)$ from $\tilde{g}$ to u_0.

   With the maps Ψ_j from Lemma 5.2, we define

$$\Phi_j := \Psi_j^{-1} \quad \text{and} \quad (g_j, G_j) := \big(g_0 \circ \Phi_j, (G_0 \circ \Phi_j)\nabla\Phi_j\big) \in BV(\Psi_j(U); \mathbb{R}^d)$$
$$\times \mathcal{M}(\Psi_j(U); \mathbb{R}^{d \times N}).$$

Here, in the definition of G_j, $G_0 \circ \Phi_j$ is the measure defined as $(G_0 \circ \Phi_j)(A) := G_0(\Phi_j(A))$ for all Borel sets $A \subset \Psi_j(U)$, and $\nabla\Phi_j \in C^0(\overline{U}; \mathbb{R}^{N \times N})$ is interpreted

as a continuous density function attached to it by matrix multiplication from the right. Altogether, G_j is the measure satisfying $\mathrm{d}G_j(z) = \mathrm{d}(G_0 \circ \Phi_j)(z) \nabla \Phi_j(z)$, similar to Dg_j which satisfies $\mathrm{d}Dg_j(z) = (Dg \circ \Phi_j)(z) \nabla \Phi_j(z)$ by the chain rule. Also notice that as a consequence of Lemma 5.2 (where we only need the case $m = 1$), for all j big enough, $\nabla \Psi_j(x)$ is an invertible matrix for all $x \in U$, $\overline{\Omega} \subset \Psi_j(U)$ and $\Phi_j(\partial \Omega) \cap \overline{\Omega} = \emptyset$. Passing to a subsequence (not relabeled), we thus may assume that

$$\Psi_j : U \to \Psi_j(U) \text{ is a diffeomorphism}, \quad \overline{\Omega} \subset \Psi_j(U) \quad \text{and} \quad \Phi_j(\partial \Omega) \cap \overline{\Omega} = \emptyset$$
$$\text{for all } j \in \mathbb{N}. \tag{5.5}$$

We claim that the sequence $\{(g_j, G_j)\}_j$ has the following properties:

$$\left\| T_\Omega(g_j - u_0) \right\|_{L^1(\Gamma; \mathbb{R}^d)} \xrightarrow[j \to \infty]{} 0, \quad |G_j|(\Gamma) = |G_j|(\partial \Omega) = 0, \tag{5.6}$$

where $T_\Omega : BV(\Omega; \mathbb{R}^d) \to L^1(\partial \Omega; \mathbb{R}^d)$ denotes the trace operator,

$$g_j|_\Omega \xrightarrow{*} g \text{ in } BV(\Omega; \mathbb{R}^d), \quad (Dg_j \llcorner \Omega, G_j \llcorner \Omega) \xrightarrow{*} (Dg_0 \llcorner (\Omega \cup \Gamma), G_0)$$
$$\text{in } \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d \times N})^2 \tag{5.7}$$

and

$$\int_\Omega \mathrm{d}H(Dg_j, G_j) \xrightarrow[j \to \infty]{} \int_{\Omega \cup \Gamma} \mathrm{d}H(Dg_0 \llcorner (\Omega \cup \Gamma), G). \tag{5.8}$$

The second part of (5.6) follows from the definition of G_j because $\Phi_j(\partial \Omega) \subset U \backslash \overline{\Omega}$ and $|G_0|(U \backslash \overline{\Omega}) = 0$. As to the first part of (5.6), first notice that since $u_0 \in W^{1,1}(U; \mathbb{R}^d)$, we do not have to distinguish between the inner and outer traces $T_\Omega u_0$ and $T_{U \backslash \Omega} u_0$ of u_0 on $\partial \Omega$. Moreover,

$$g_j - u_0 = \left(g_0 \circ \Phi_j - u_0 \circ \Phi_j \right) + \left(u_0 \circ \Phi_j - u_0 \right)$$

and $u_0 \circ \Phi_j \to u_0$ in $W^{1,1}(\Omega; \mathbb{R}^d)$, so that the asserted convergence of traces follows from the continuity of the trace operator in $W^{1,1}$ once we see that $(g_0 - u_0) \circ \Phi_j = 0$ in some neighborhood of Γ (which may depend on j). The latter is trivial by definition of g_0 if

$$\Phi_j(\overline{\Gamma}) \subset \Omega' \backslash \overline{\Omega} \quad \text{for all } j \in \mathbb{N}; \tag{5.9}$$

here, we already have that $\Phi_j(\overline{\Gamma}) \cap \overline{\Omega} = \emptyset$. We can therefore assume (5.9) without loss of generality: otherwise, if $\Phi_j(\overline{\Gamma}) \not\subset \Omega'$, we can define $r(j) := \frac{1}{2} \operatorname{Dist}(\Phi_j(\overline{\Gamma}), \Omega) > 0$ and take

$$\tilde{\Omega}' := \Omega' \cup \left\{ x \in \mathbb{R}^N \mid \operatorname{dist}(x, \Phi_j(\overline{\Gamma})) < r(j) \text{ for a } j \in \mathbb{N} \right\}$$

instead of Ω'. Here, recall that Ω' is just an auxiliary object to define Γ (and g_0 above, outside of $\overline{\Omega}$), and by construction, $\tilde{\Omega}'$ still has all the properties we required for Ω': $\tilde{\Omega}' \supset \Omega$ is a bounded domain and $\tilde{\Omega}' \cap \partial\Omega = \Gamma = \Omega' \cap \partial\Omega$.

For the proof of (5.7) and (5.8), fix $\varphi \in C(\overline{\Omega})$, continuously extended to $\varphi \in C(\mathbb{R}^N)$. By the definition of (g_j, G_j) and the change of variables $x = \Phi_j(z)$, we get that for every Borel set $V \subset U$,

$$
\begin{aligned}
&\int_\Omega \varphi(z)\mathrm{d}H(Dg_j, G_j)(z) \\
&= \int_\Omega \varphi(z)\mathrm{d}H\big((Dg_0 \circ \Phi_j)\nabla\Phi_j, (G_0 \circ \Phi_j)\nabla\Phi_j\big)(z) \\
&= \int_{\Phi_j(\Omega)} \varphi(\Psi_j(x)) H\Big(\nabla g_0(\nabla\Psi_j)^{-1}, \frac{\mathrm{d}G_0}{\mathrm{d}\mathcal{L}^N}(\nabla\Psi_j)^{-1}\Big) \det(\nabla\Psi_j(x))\, \mathrm{d}x \\
&\quad + \int_{\Phi_j(\Omega)} \varphi(\Psi_j(x))\mathrm{d}H^\infty\big(D^s g_0(\nabla\Psi_j)^{-1}, G_0^s(\nabla\Psi_j)^{-1}\big)(x) \\
&= \Big(\int_{\Phi_j(\Omega)\cap V} (\varphi \circ \Psi_j) H\Big(\nabla g_0(\nabla\Psi_j)^{-1}, \frac{\mathrm{d}G_0}{\mathrm{d}\mathcal{L}^N}(\nabla\Psi_j)^{-1}\Big) \det(\nabla\Psi_j(x))\, \mathrm{d}x \\
&\quad + \int_{\Phi_j(\Omega)\cap V} (\varphi \circ \Psi_j)\, \mathrm{d}H^\infty\big(D^s g_0(\nabla\Psi_j)^{-1}, G_0^s(\nabla\Psi_j)^{-1}\big)(x) \Big) \\
&\quad + \Big(\int_{\Phi_j(\Omega)\setminus V} (\varphi \circ \Psi_j) H\Big(\nabla g_0(\nabla\Psi_j)^{-1}, \frac{\mathrm{d}G_0}{\mathrm{d}\mathcal{L}^N}(\nabla\Psi_j)^{-1}\Big) \det(\nabla\Psi_j(x))\, \mathrm{d}x \\
&\quad + \int_{\Phi_j(\Omega)\setminus V} (\varphi \circ \Psi_j)\, \mathrm{d}H^\infty\big(D^s g_0(\nabla\Psi_j)^{-1}, G_0^s(\nabla\Psi_j)^{-1}\big)(x) \Big) \\
&=: S_j(\varphi; V) + T_j(\varphi; V)
\end{aligned}
\tag{5.10}
$$

As to the second term $T_j(\varphi; V)$ (integrals on $\Phi_j(\Omega) \setminus V$), we exploit that $(\nabla\Psi_j)^{-1}$ is uniformly bounded and H has at most linear growth. Hence, there is a constant $C > 0$ such that with $C_\varphi := C \|\varphi\|_{L^\infty(U)}$,

$$
\begin{aligned}
\limsup_{j\to\infty} \big|T_j(\varphi; V)\big| &\leqslant C_\varphi \limsup_{j\to\infty}(\mathcal{L}^N + |Dg_0| + |G_0|)(\Phi_j(\Omega) \setminus V) \\
&\leqslant C_\varphi (\mathcal{L}^N + |Dg_0| + |G_0|)(\overline{\Omega} \setminus V) \\
&= C_\varphi (\mathcal{L}^N + |Dg_0| + |G_0|)((\Omega \cup \Gamma) \setminus V),
\end{aligned}
\tag{5.11}
$$

by dominated convergence and the fact that $\Phi_j \to \mathrm{id}$ in C^1. Here, we also used that

$$
|Dg_0|(\partial\Omega \setminus \Gamma) = 0 = |G_0|(\partial\Omega \setminus \Gamma),
$$

by definition of g_0, G_0 and our assumption that $\mathcal{H}^{N-1}(\overline{\Gamma} \setminus \Gamma) = 0$.

For the term $S_j(\varphi; V)$ (integrals on $\Phi_j(\Omega) \cap V$) on the right-hand side of (5.10), we again use that $\Phi_j \to \mathrm{id}$ in C^1; in particular, $(\nabla\Psi_j)^{-1} \to I$ (identity matrix) uniformly.

In addition, $H \geqslant 0$ is Lipschitz and φ is uniformly continuous. Consequently, for all $\varphi \geqslant 0$,

$$\int_{\overline{\Omega} \cap V} \varphi \, dH(Dg_0, G_0)(x) \leqslant \liminf_{j \to \infty} S_j(\varphi; V) \leqslant \limsup_{j \to \infty} S_j(\varphi; V)$$
$$\leqslant \int_{\overline{\Omega} \cap V} \varphi \, dH(Dg_0, G_0)(x). \tag{5.12}$$

Here, to handle the limit in the domain of integration $\Phi_j(\Omega) \cap V$, for the lower bound we used monotonicity and the fact that $\overline{\Omega} \subset \Phi_j(\Omega)$ for all j by (5.5), while for the upper bound we used that $\Phi_j(\Omega) \searrow \overline{\Omega}$ and dominated convergence.

By splitting a general φ into positive and negative parts, (5.12) immediately implies that

$$\lim_{j \to \infty} S_j(\varphi; V) = \int_{\overline{\Omega} \cap V} \varphi \, dH(Dg_0, G_0)(x) = \int_{(\Omega \cup \Gamma) \cap V} \varphi \, dH(Dg_0, G)(x) \tag{5.13}$$

for all $\varphi \in C(\overline{U})$. Combining (5.10), (5.11) and (5.13) for the case $V = \Omega \cup \Gamma$, we infer that

$$\int_{\Omega} \varphi \, dH(Dg_j, G_j)(z) \to \int_{\Omega \cup \Gamma} \varphi \, dH(Dg_0, G)(x) \quad \text{as } j \to \infty. \tag{5.14}$$

In particular, (5.14) yields (5.8) when we choose $\varphi \equiv 1$.

In addition, we can analogously obtain (5.14) for other functions instead H (globally Lipschitz with a uniform strong recession function in the sense of (4.1); if needed, H can be temporarily split into a positive and a negative part for the proof of (5.13), just like φ). With the choices

$$H(A, B) := A_{ij} \quad \text{and} \quad H(A, B) := B_{ij}, \quad \text{where } A = (A_{ij}) \text{ and } B = (B_{ij}),$$

for $i = 1, \ldots, d$ and $j = 1, \ldots, N$, (5.14) implies the second part of (5.7), in particular that $Dg_j \llcorner \Omega \overset{*}{\rightharpoonup} Dg_0 \llcorner (\Omega \cup \Gamma)$ in $\mathcal{M}(\overline{\Omega}; \mathbb{R}^{d \times N})$.

Finally, it is not hard to see that $g_j \to g$ in $L^1(\Omega; \mathbb{R}^d)$. We conclude that $g_j \overset{*}{\rightharpoonup} g$ in $BV(\Omega; \mathbb{R}^d)$, which completes the proof of (5.7).

Step 2: Approximating limit states $(\hat{g}_j, \hat{G}_j)$ **with** $\hat{g}_j = u_0$ **on** Γ.

The functions g_j defined in the previous step do not yet satisfy $g_j = u_0$ on Γ, although their traces converge to u_0 by (5.6). We can correct this using the trace extension theorem: Choose $\{v_j\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ such that

$$T_\Omega v_j|_\Gamma = T_\Omega(g_j - u_0)|_\Gamma \quad \text{and} \quad \|v_j\|_{W^{1,1}(\Omega; \mathbb{R}^d)} \leqslant C_{\partial \Omega} \|T_\Omega(g_j - u_0)\|_{L^1(\Gamma; \mathbb{R}^d)}. \tag{5.15}$$

By (5.6), we infer that $\left\|v_j\right\|_{W^{1,1}(\Omega;\mathbb{R}^d)} \to 0$. Consequently, for

$$\hat{g}_j := g_j - v_j \quad \text{and} \quad \hat{G}_j := G_j,$$

instead of (g_j, G_j) we still have (5.6), (5.7) and (5.8), and in addition, $\hat{g}_j = u_0$ on Γ. Namely, defining

$$\Theta := (T_\Omega g - u_0) \otimes \nu_\Gamma \mathcal{H}^{N-1} \llcorner \Gamma$$

so that $Dg_0 = Dg + \Theta$ on $\Omega \cup \Gamma$, we have that

$$\hat{g}_j = u_0 \text{ on } \Gamma, \quad |\hat{G}_j|(\Gamma) = |\hat{G}_j|(\partial\Omega) = 0, \tag{5.16}$$

$$\hat{g}_j|_\Omega \overset{*}{\rightharpoonup} g \text{ in } BV(\Omega;\mathbb{R}^d), \quad (D\hat{g}_j \llcorner \Omega, \hat{G}_j \llcorner \Omega)$$

$$\overset{*}{\rightharpoonup} (Dg + \Theta, G) \text{ in } \mathcal{M}(\Omega \cup \Gamma; \mathbb{R}^{d \times N})^2, \tag{5.17}$$

and

$$\lim_{j \to \infty} \int_\Omega \mathrm{d}H(D\hat{g}_j, \hat{G}_j) = \int_\Omega \mathrm{d}H(g, G) + \int_\Gamma \mathrm{d}H^\infty(\Theta, G). \tag{5.18}$$

Step 3: Recovery by diagonalizing free recovery sequences for $(\hat{g}_j, \hat{G}_j)$

We first observe that I in (2.2) admits the following equivalent representation

$$I_g(g, G; \Omega) := \inf \left\{ \liminf_{n \to \infty} E(u_n; \Omega) : \{u_n\} \subset SBV(\Omega;\mathbb{R}^d), u_n \overset{*}{\rightharpoonup} (g, G), u_n \equiv g \text{ on } \partial\Omega \right\}, \tag{5.19}$$

for every $(g, G) \in mSD$.

Clearly $I(g, G; \Omega) \leqslant I_g(g, G; \Omega)$. The opposite one can be obtained following an argument of Bouchitté et al. (2002). The details are provided below for the reader's convenience.

For any $SBV(\Omega;\mathbb{R}^d) \ni u_n \overset{*}{\rightharpoonup} (g, G)$ in the sense of (2.7), almost optimal for $I(g, G; \Omega)$, i.e., for every $\varepsilon > 0$,

$$\liminf_{n \to \infty} E(u_n; \Omega) \leqslant I(g, G; \Omega) + \varepsilon.$$

Without loss of generality, assume that the above lower limit is indeed a limit and consider the sequence of measures $\nu_n := \mathcal{L}^N + |Du_n| + |Dg|$, which converges weakly* to some Radon measure ν.

Denoting, for every $t > 0$, $\Omega_t := \{x \in \Omega | \mathrm{dist}(x, \partial\Omega) > t\}$, we fix some $\eta > 0$ and for every $0 < \delta < \eta$ we define the subsets $L_\delta := \Omega_{\eta-2\delta} \backslash \overline{\Omega_{\eta+\delta}}$. Consider a smooth cut-off function $\varphi_\delta \in C_0^\infty(\Omega_{\eta-\delta}; [0, 1])$ such that $\varphi_\delta = 1$ on Ω_η. As the thickness of the strip L_δ is of order δ, we have an upper bound of the form $\|\nabla\varphi_\delta\|_{L^\infty(\Omega_{\eta-\delta})} \leq C/\delta$.

Define

$$w_n^\delta := u_n \varphi_\delta + g(1 - \varphi_\delta).$$

Clearly this sequence converges to g in $L^1(\Omega; \mathbb{R}^d)$ and satisfies $T_\Omega w_n^\delta = T_\Omega g$ on $\partial\Omega$. Moreover

$$\nabla w_n^\delta \overset{*}{\rightharpoonup} G \quad \text{and} \quad D w_n^\delta \overset{*}{\rightharpoonup} Dg \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$$

as $n \to \infty$ and then $\delta \to 0$. Indeed

$$\nabla w_n^\delta = \nabla\varphi_\delta \otimes (u_n - g) + \varphi_\delta(\nabla u_n - \nabla g) + \nabla g,$$
$$D w_n^\delta = \nabla\varphi_\delta \otimes (u_n - g) + \varphi_\delta(Du_n - Dg) + Dg.$$

Concerning the energies, we have

$$E(w_n^\delta; \Omega) \le E(w_n^\delta; \Omega_\eta) + E(w_n^\delta; \Omega \setminus \overline{\Omega_{\eta-\delta}}) + E(u_n^\delta; \Omega_{\eta-2\delta} \setminus \overline{\Omega_{\eta+\delta}})$$
$$\le E(u_n; \Omega_\eta) + E(g; \Omega \setminus \overline{\Omega_{\eta-\delta}})$$
$$+ C_{W,\psi}\left((\mathcal{L}^N + |Du_n| + |Dg|)(L_\delta) + \frac{1}{\delta}\int_{L_\delta}|u_n - g|\,dx \right),$$

where $C_{W,\psi}$ is any bigger constant which bounds from above the constants appearing in (W:1), (ψ:1) and in L^∞ bound of $\nabla\psi_\delta$ on L_δ. Taking the limit as $n \to \infty$ we have

$$\liminf_{n\to\infty} E(w_n^\delta; \Omega) \le \lim_{n\to\infty} E(u_n; \Omega) + C_{W,\psi}\nu(\Omega \setminus \overline{\Omega_{\eta-\delta}}) + C_{W,\psi}\nu(\overline{L_\delta})$$
$$\le I(g, G; \Omega) + \varepsilon + C_{W,\psi}\nu(u; \Omega \setminus \overline{\Omega_{\eta-\delta}}) + C_{W,\psi}\nu(\overline{L_\delta}).$$

Letting $\delta \to 0$ we obtain

$$I_g(g, G; \Omega) \le I(g, G; \Omega) + \varepsilon + C_{W,\psi}\nu(\Omega \setminus \Omega_\eta) + C_{W,\psi}\nu(\partial\Omega_\eta).$$

Choose a subsequence $\{\eta_n\}$ such that $\eta_n \to 0^+$ and $\nu(\partial A_{\eta_n}) = 0$. By letting first $n \to \infty$ and then $\varepsilon \to 0^+$ we conclude that $I_g(g, G; \Omega) \le I(g, G; \Omega)$.

Then, for any $(\hat{g}_j, \hat{G}_j) \overset{*}{\rightharpoonup} (g, G)$ as in Step 2, satisfying (5.16), (5.17), and (5.18), we can apply Theorem 2.4 and find a recovery sequence for $I_{\hat{g}_j}(\hat{g}_j, \hat{G}_j; \Omega) = I(\hat{g}_j, \hat{G}_j; \Omega)$ for each j, i.e., $\{u_n^j\}_n \subset SBV(\Omega; \mathbb{R}^d)$ such that $u_n^j \overset{*}{\rightharpoonup} (\hat{g}_j, \hat{G}_j)$ in the sense of (2.7), $T_\Omega u_n^j = T_\Omega \hat{g}_j$ on $\partial\Omega$, in particular $T_\Omega u_n^j = u_0$ on Γ, and

$$\lim_{n\to\infty} E(u_n; \Omega) = J(g_j, G_j; \Omega) = \int_\Omega dH(\hat{g}_j, \hat{G}_j).$$

Since $|\hat{G}_j|(\partial\Omega) = 0$, we also have that $\nabla u_n^j \overset{*}{\rightharpoonup} \hat{G}_j$ in $\mathcal{M}(\Omega \cup \Gamma; \mathbb{R}^{d \times N})$.

A standard diagonalization argument, exploiting the coercivity of E given by (W:1) and (ψ:1) to obtain bounds uniform in n and j, now concludes the proof. $\square$

6 Further Properties and Examples

As shown below, we also have an alternative way of interpreting I, as a more classic relaxation problem of a functional on $SBV \times L^1$ in $BV \times \mathcal{M}$.

Theorem 6.1 *Assume (W:1)–(W:3) and (ψ:1)–(ψ:3). For $(g, G) \in SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$ and $R > 0$, we define*

$$\hat{E}_R(g, G; \Omega) := \int_\Omega (W(\nabla g) + R|\nabla g - G|)\,\mathrm{d}x + \int_{S_g \cap \Omega} \psi([g], \nu_g)\,\mathrm{d}\mathcal{H}^{N-1}(x)$$

(6.1)

and its relaxation

$$\hat{I}_R(g, G; \Omega)$$
$$:= \inf\left\{ \liminf_{n \to \infty} \hat{E}_R(g_n, G_n; \Omega) \,\middle|\, SBV \times L^1 \ni (g_n, G_n) \overset{*}{\rightharpoonup} (g, G) \text{ in } BV \times \mathcal{M} \right\}$$

for $(g, G) \in mSD$. Then there exists $R_0 = R_0(N, W, \psi) > 0$ such that

$$\hat{I}_R(\cdot, \cdot; \Omega) = I(\cdot, \cdot; \Omega) \quad \text{for all } R \geqslant R_0,$$

where $I(\cdot, \cdot; \Omega)$ is the relaxation of $E(\cdot; \Omega)$ defined in (2.2).

Remark 6.2 Theorem 6.1 in principle opens another route to proving Theorem 2.4, our representation theorem for I, via a relaxation theorem characterizing $\hat{I}_R$. However, the closest available results in this direction seem to be Arroyo-Rabasa et al. (2020) and Baía et al. (2013) (for the case $\mathcal{A} = \mathrm{diag}(\mathrm{Curl}, 0)$, cf. the proof of Proposition 4.3) and Barroso et al. (1996). However, the former does not allow us to choose ψ freely, and the latter does not allow us to include G.

Proof of Theorem 6.1 We first observe that, for every $R > 0$, $\hat{E}_R(u, \nabla u, \Omega) = E(u; \Omega)$ for every $u \in SBV(\Omega; \mathbb{R}^d)$. Let $(g, G) \in mSD$ and let $SBV(\Omega; \mathbb{R}^d) \ni g_n \overset{*}{\rightharpoonup} (g, G)$ according to (1.2). Since $\{(g_n, \nabla g_n)\}$ is an admissible sequence for $\hat{E}_R(g, G; \Omega)$,

$$\hat{I}_R(g, G; \Omega) \leqslant \liminf_{n \to \infty} E(g_n; \Omega).$$

Hence, passing to the infimum over all the admissible sequences $\{g_n\}$, we have

$$\hat{I}_R(g, G, \Omega) \leqslant I(g, G, \Omega).$$

To prove the opposite inequality for $R \geqslant R_0$ with a suitable R_0 to be chosen later, take $\{(g_n, G_n)\}$ admissible for $\hat{I}_R(g, G; \Omega)$, so that $g_n \overset{*}{\rightharpoonup} g$ in BV, $G_n \overset{*}{\rightharpoonup} G$ in $\mathcal{M}$. We choose a sequence $\{v_n\}$ given by Šilhavý (2015, Theorem 1.1) such that

$$v_n \overset{*}{\rightharpoonup} 0 \quad \text{in } BV, \qquad \nabla v_n = -\nabla g_n + G_n,$$

and

$$|Dv_n|(\Omega) \leqslant C(N) \int_\Omega |G_n - \nabla g_n| \, dx = C(N) \int_\Omega |\nabla v_n| \, dx. \tag{6.2}$$

In particular, the sequence $u_n := g_n + v_n$ is admissible for $I(g, G; \Omega)$.

Taking into account that

$$S_{g_n} = (S_{g_n} \setminus S_{g_n+v_n}) \cup (S_{g_n} \cap S_{g_n+v_n}) \quad \text{and} \quad S_{g_n+v_n} = (S_{g_n} \cap S_{g_n+v_n}) \cup (S_{g_n+v_n} \setminus S_{g_n}),$$

also using $(W{:}1)$, $(\psi{:}1)$ and (2.14) we obtain that

$$\hat{E}_R(g_n, G_n; \Omega) - \hat{E}_R(g_n + v_n, \nabla(g_n + v_n); \Omega)$$
$$= \int_\Omega \left(W(\nabla g_n) - W(\nabla g_n + \nabla v_n) \right) dx + \int_\Omega R|\nabla g_n - G_n| \, dx$$
$$+ \int_{\Omega \cap S_{g_n}} \psi([g_n], \nu_{g_n}) \, d\mathcal{H}^{N-1}(x) - \int_{\Omega \cap S_{g_n+v_n}} \psi([g_n + v_n], \nu_{g_n+v_n}) \, d\mathcal{H}^{N-1}(x)$$
$$\geqslant -\int_\Omega L|\nabla v_n| \, dx + \int_\Omega R|\nabla v_n| dx + \int_{\Omega \cap (S_{g_n} \setminus S_{g_n+v_n})} \psi([g_n], \nu_{g_n}) d\mathcal{H}^{N-1}(x)$$
$$-\int_{\Omega \cap (S_{g_n} \cap S_{g_n+v_n})} C_\psi |[v_n]| d\mathcal{H}^{N-1}(x) - \int_{\Omega \cap (S_{g_n+v_n} \setminus S_{g_n})} C_\psi |[v_n]| d\mathcal{H}^{N-1}(x)$$
$$\geqslant \int_\Omega (R - L)|\nabla v_n| dx - \int_{\Omega \cap S_{v_n}} C_\psi |[v_n]| \, d\mathcal{H}^{N-1}(x)$$
$$\geqslant (R - (L + C_\psi C(N))) \int_\Omega |\nabla v_n| \, dx \geqslant 0,$$

as long as $R \geqslant R_0 := L + C_\psi C(N)$. Here, L, C_ψ, and $C(N)$ denote the Lipschitz constant of W, the Lipschitz and growth constant of ψ in (2.14) and $(\psi{:}1)$, and the constant appearing in (6.2), respectively. Passing to the limit as $n \to \infty$, we conclude that

$$\liminf_{n\to\infty} \hat{E}_R(g_n, G_n; \Omega) \geqslant \liminf_{n\to\infty} \hat{E}_R(g_n + v_n, \nabla(g_n + v_n); \Omega)$$
$$= \liminf_{n\to\infty} E(g_n + v_n; \Omega) \geqslant I(g, G; \Omega)$$

for all $R \geqslant R_0$. As this holds for all sequences $\{(g_n, G_n)\}$ that are admissible for $\hat{I}_R(g, G; \Omega)$, the thesis follows. $\qquad \square$

In view of Theorem 6.1, it is a natural question to what degree our relaxed functional I is influenced by its origin from E, defined on structured deformations. The following example shows that this special background is still present in the relaxed I at least in the sense that not all quasiconvex-convex densities H (that could be obtained by general relaxation in $BV \times \mathcal{M}$) can be obtained in I.

Proposition 6.3 *For all W and ψ satisfying the assumptions of Theorem 2.4, there exists $B_0 \in \mathbb{R}^{d \times N}$ and $\xi \in \mathbb{R}^d$, $\nu \in \mathbb{R}^N$ with $|\xi| = |\nu| = 1$ so that for the function H defined in (2.11a),*

$$H(B_0 + t\xi \otimes \nu, B_0) = W(B_0) + \psi(t\xi, \nu) \quad \text{for all } t > 0. \tag{6.3}$$

In particular, for any possible choice of W and ψ,

$$H \neq H_0 \quad \text{with} \quad H_0(A, B) := \sqrt{|A|^2 + 1} + |B|$$

because the function $(0, +\infty) \ni t \mapsto H_0(B_0 + t\xi \otimes \nu, B_0)$ is not affine.

Proof To see "$\leqslant$" in (6.3), it suffices to choose a suitable admissible sequence in (3.15a), the sequential characterization of H: On Q, we have $\nabla u_n \overset{*}{\rightharpoonup} B_0$ in $\mathcal{M}$ and $u_n \overset{*}{\rightharpoonup} A_0 x$ for $A_0 := B_0 + t\xi \otimes \nu$ for

$$u_n(x) := B_0 x + \frac{t\xi}{n}[nx \cdot \nu],$$

where $[s] := \min\{z \in \mathbb{Z} : |z - s| = \text{dist}(s, \mathbb{Z})\}$ denotes rounding of s to the closest integer. An upper bound for $H(A_0, B_0)$ is therefore given by $\lim_n E(u_n; Q) = W(B_0) + \psi(t\xi, \nu)$ (by 1-homogeneity of ψ), for any possible choice of B_0, ξ and ν.

To obtain "$\geqslant$" in (6.3), we use a particular choice: Since both W and ψ are continuous and W is coercive, there always exists global minima B_0 of W on $\mathbb{R}^{d \times N}$ and (ξ, ν) of ψ on the compact set $\mathbb{S}^{d-1} \times \mathbb{S}^{N-1} := \{(\xi, \nu) \in \mathbb{R}^d \times \mathbb{R}^N \mid |\xi| = |\nu| = 1\}$. As a consequence,

$$W(B_0) = W^{**}(B_0) \quad \text{and} \quad \psi(\xi, \nu) \leqslant \psi(\tilde{\xi}, \tilde{\nu}) \text{ for all } (\tilde{\xi}, \tilde{\nu}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{N-1}. \tag{6.4}$$

Here, W^{**} denotes the convex hull of W. For any u admissible in the definition (2.11a) of $H(A_0, B_0)$ with $A_0 := B_0 + t\xi \otimes \nu$, we now have that

$$\int_Q W(\nabla u) \, dx \geqslant \int_Q W^{**}(\nabla u) \, dx \geqslant W^{**}(B_0) = W(B_0) \tag{6.5}$$

by Jensen's inequality, since $\fint_Q \nabla u \, dx = B_0$ for all admissible u in (2.11a). Moreover, since $u \in SBV$ and $u = A_0 x$ on ∂Q, we have that

$$\int_{S_u \cap Q} [u] \otimes \nu_u \, d\mathcal{H}^{N-1}(x) = \int_Q dDu - \int_Q \nabla u \, dx = A_0 - B_0 = t\xi \otimes \nu.$$

Multiplied with the fixed unit vector ν from the right, this reduces to

$$\int_{S_u \cap Q} [u](\nu_u \cdot \nu) \, d\mathcal{H}^{N-1}(x) = t\xi. \tag{6.6}$$

By the positive 1-homogeneity of ψ, the minimality property of (ξ, ν) in (6.4) and another application of Jensen's inequality with (6.6) to the convex function $|\cdot|$, we infer that

$$
\begin{aligned}
\int_{S_u \cap Q} \psi([u], \nu_u) \, d\mathcal{H}^{N-1}(x) &= \int_{S_u \cap Q} |[u]| \psi\left(\frac{[u]}{|[u]|}, \nu_u\right) d\mathcal{H}^{N-1}(x) \\
&\geqslant \int_{S_u \cap Q} |[u]|(\nu_u \cdot \nu)|\psi(\xi, \nu) \, d\mathcal{H}^{N-1}(x) \\
&\geqslant |t\xi|\psi(\xi, \nu) = \psi(t\xi, \nu)
\end{aligned} \tag{6.7}
$$

for all $t > 0$. Combining (6.5) and (6.7), we conclude that $E(u; Q) \geqslant W(B_0) + \psi(t\xi, \nu)$ for all u admissible in (2.11a) with $(A, B) = (A_0, B_0)$. This implies the asserted lower bound for $H(A_0, B_0)$. $\qquad\square$

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