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Schauder Regularity Theory for Degenerate and Singular PDEs

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In memory of Vincenza Gitto.

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CONTENTS

INTRODUCTION

This thesis is devoted to the study of certain regularity properties for a class of weighted partial differential equations, where the coefficients fail to satisfy the standard uniform ellipticity condition.

Starting from the very beginning, partial differential equations (PDEs) have proven to be an effective mathematical tool for modelling a wide range of natural phenomena. Some of the earliest and most well-known examples of PDEs are the Laplace equation,

$$-\Delta u = 0,$$

introduced by Pierre-Simon Laplace in his *Traité de Mécanique Céleste* [92], and the Poisson equation,

$$-\Delta u = f,$$

formulated by Siméon Denis Poisson in his work on the theory of magnetism [124]. These equations describe stationary phenomena such as gravitational and electrostatic potentials, and form the basis of the classical theory of elliptic PDEs. Another fundamental example is the heat equation,

$$\partial_t u - \Delta u = 0,$$

introduced by Joseph Fourier in his seminal monograph *Théorie analytique de la chaleur* [71], which models the evolution of temperature in a medium over time, given initial conditions.

Since then, over the past two centuries, the theory of PDEs has grown immensely, enriched by a vast body of results and theoretical developments. When studying a partial differential equation, two fundamental questions naturally arise. First: does a solution exist? Second: if so, how *good* is the solution? The first question concerns *existence*, while the second concerns *regularity* - that is, the smoothness and qualitative behaviour of the solution. In the classical cases of the Laplace and heat equations, regularity is well understood. For instance, thanks to explicit representation formulas - such as integral expressions involving fundamental solutions - it is often straightforward to show that solutions inherit the smoothness of the data.

However, many physical processes exhibit anisotropic behaviour, meaning that the diffusion depends on the direction. Such anisotropies can be modelled by replacing the Laplacian operator with a more general divergence form operator:

$$-\operatorname{div}(A(x)\nabla u) = f,$$

where $A(x)$ is a symmetric, positive-definite matrix whose entries may vary with the spatial variable x, reflecting how the directional dependence influences the behaviour of solutions.

To obtain meaningful regularity results in this more general setting, one typically requires that the matrix $A(x)$ satisfies a *uniform ellipticity* condition: there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2,$$

for all vectors $\xi \in \mathbb{R}^d$ and for almost every point x, which ensures that the operator avoids degeneracies or singularities.

Similarly, in the study of time-dependent processes such as heat diffusion, anisotropic behaviour can be described via parabolic operators in divergence form

$$\partial_t u - \operatorname{div}(A(x,t)\nabla u) = f,$$

where the matrix $A(x,t)$ again satisfies a uniform ellipticity condition.

The regularity theory for elliptic and parabolic operators in divergence form (although analogous results also hold for operators in non-divergence form) was developed after the initial studies on classical equations such as Laplace's and Poisson's equations. There is a rich body of results in this area, among which the Schauder estimates stand out as a fundamental breakthrough. Schauder estimates, developed by Juliusz Schauder in the 1930s [131, 132], originally dealt with elliptic operators in non-divergence form, providing a priori bounds on the Hölder continuity of the second derivatives of solutions in terms of the Hölder regularity of the coefficients and the forcing term. Specifically, if the coefficients of the operator and the right-hand side are Hölder continuous, then the solution itself enjoys an improved Hölder regularity. Beyond characterizing the regularity of solutions, Schauder estimates have proven to be an analytic tool for establishing existence results through the Schauder fixed point theorem [130]. The regularity theory was further advanced in the 1960s by Campanato [30, 31], who introduced an approach based on controlling mean oscillations, giving rise to what are now known as Campanato spaces. This approach allowed for extending regularity results to elliptic operators in divergence form. Today, Schauder-type estimates form a fundamental part of regularity theory for elliptic and parabolic operators, both in divergence and non-divergence form, and have been extensively studied and generalized. For a modern and comprehensive presentation of Schauder estimates and related topics we refer to [91, 75, 96, 69].

The regularity results mentioned above rely crucially on the assumption that the coefficients of the operator are at least continuous. When this assumption is dropped, the problem of obtaining regularity results for PDEs with bounded measurable and uniformly elliptic coefficients was resolved by the breakthrough works of De Giorgi [76] and Nash [116], where they established that solutions to such elliptic and parabolic equations are Hölder continuous. These results were later complemented by Moser's contributions [112, 113], where he established the Harnack inequality for such equations. We emphasize that the optimal regularity for solutions to elliptic equations in divergence form with bounded measurable coefficients is currently known only in dimension $d = 2$, as proved by Piccinini and Spagnolo in [123]. In higher dimensions $d \geq 3$, the problem remains open. One of the main motivations for studying these problems comes from Hilbert's XIX problem, which asked whether the minimizers of regular variational problems are necessarily analytic. As previously mentioned, this question was independently solved by De Giorgi and Nash.

While the regularity for uniformly elliptic operators is by now well understood, new challenges arise when the assumption of uniform ellipticity is no longer satisfied. In such case, the operator may become degenerate or singular in certain regions of the domain, meaning that the ellipticity constants λ could be 0 and/or Λ could be ∞. A prototypical example is the weighted divergence form equation

$$-\operatorname{div}(\omega A(x)\nabla u) = 0,$$

where A satisfies the uniformly ellipticity condition as before, while the weight ω is a nonnegative function that may vanish or diverge. One of the cornerstone result in the study of such weighted equations is due to Fabes, Kenig, and Serapioni in their seminal work [65], where they extended the De Giorgi-Nash-Moser theory to the previous degenerate weighted elliptic equations, assuming that the weight ω either arises from quasiconformal mappings or belongs to the A_2 Muckenhoupt class (we remark, however, that these are not the only admissible classes of weights for which similar regularity results can be established).

More generally, for $1 < p < \infty$, a nonnegative, locally integrable function ω is said to belong to the A_p Muckenhoupt class, introduced in [114], if there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^d$,

$$\Big(\frac{1}{|B|}\int_B \omega\, dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}}\, dx\Big)^{p-1} \leq C.$$

Under the A_2 assumption, Fabes, Kenig and Serapioni established weighted versions of key analytic tools such as Sobolev and Poincaré-Wirtinger inequalities, which allow them to derive Harnack inequality and Hölder continuity of solutions. Along these lines, we refer to the works [63, 64, 86] for the elliptic case, to [36] for the parabolic case, and to the monograph [82] for a general overview.

A classical and widely studied example of a degenerate weight is

$$\omega = y^a, \quad \text{for } y > 0, \quad \text{where } a \in (-1, 1) \text{ is a fixed parameter.}$$

The weight ω vanishes on $\{y = 0\}$ if $a > 0$, and diverges if $a < 0$. Nonetheless, the range of the parameter a guarantees that ω belongs to the A_2 Muckenhoupt class. As we explain in a moment, this kind of weight naturally arises in the study of the fractional Laplacian through the Caffarelli-Silvestre extension.

The fractional Laplacian is the prototypical example of a *nonlocal* operator. Up to this point, all the PDEs under consideration involve *local* operators, in the sense that the behaviour of the solution at a point depends only on its values in a neighbourhood of that point. In contrast, *nonlocal* operators take into account the interaction of the function with values far away from the point under consideration.

For $s \in (0, 1)$, we denote by $(-\Delta)^s$ the fractional Laplacian, which can be defined, for sufficiently regular functions $u : \mathbb{R}^d \to \mathbb{R}$, by the singular integral

$$(-\Delta)^s u(x) = C_{d,s}\, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(z)}{|x - z|^{d+2s}}\, dz,$$

where P.V. denotes the Cauchy principal value and $C_{d,s}$ is a normalization constant.

Alternatively, it can be defined via the Fourier transform as

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s}\, \mathcal{F}[u](\xi),$$

which shows that the fractional Laplacian is a pseudo-differential operator with symbol $|\xi|^{2s}$.

For a general background on fractional calculus and nonlocal operators, we refer to [129, 52].

A celebrated approach to study the fractional Laplacian $(-\Delta)^s$, introduced by Caffarelli and Silvestre in [28], consists in realizing it as a Dirichlet-to-Neumann map for an extension problem in one higher dimension. More precisely, for $s \in (0,1)$, one considers the solution $U(x,y)$ to the degenerate elliptic problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ U(x,0) = u(x) & \text{on } \mathbb{R}^d, \end{cases}$$

and the fractional Laplacian of u is then recovered via the formula

$$(-\Delta)^s u(x) = -c \lim_{y \to 0^+} y^{1-2s} \frac{\partial U}{\partial y}(x,y),$$

where $c > 0$ is a normalization constant. This extension problem transforms the nonlocal operator into a local degenerate elliptic operator in one more dimension, allowing the use of the PDE techniques for local operators to analyse the fractional Laplacian.

Analogously, in the context of evolution equations, one of the most studied nonlocal operators is the fractional power of the heat operator, denoted by $(\partial_t - \Delta)^s$ with $s \in (0,1)$. There are several ways to define such operators. One natural approach is via the Fourier transform, through the formula

$$\mathcal{F}[(\partial_t - \Delta)^s u](\theta,\xi) := (i\theta + |\xi|^2)^s \, \mathcal{F}[u](\theta,\xi),$$

where $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$.

As remarked by Caffarelli and Silvestre in [28] for the fractional Laplacian, a similar extension principle holds for the fractional heat operator, which can be realized as a Dirichlet-to-Neumann map for a suitable degenerate parabolic equation in one higher dimension. This result was rigorously established by Stinga and Torrea in [141], and independently by Nyström and Sande in [120]. In particular, the extended function $U = U(x,y,t)$ satisfies the following equation

$$y^{1-2s}\partial_t U - \operatorname{div}(y^{1-2s}\nabla U) = 0, \quad \text{in } \mathbb{R}^d \times (0,\infty) \times \mathbb{R}$$

and

$$(\partial_t - \Delta)^s u(x,t) = -c \lim_{y \to 0^+} y^{1-2s} \frac{\partial U}{\partial y}(x,y,t).$$

Regarding these fractional operators, their extension techniques, and their wide range of applications, there is an extensive literature on the subject, and it is therefore impossible to cite all the relevant contributions. Nevertheless, we would like to mention some representative results: [19, 29, 133] for the fractional Laplacian without relying on extension methods, [21, 26, 28, 27, 34, 48, 66, 68, 73, 74, 137, 145] for the elliptic case through the extension approach, and [4, 6, 9, 11, 13, 14, 17, 25, 44, 120, 141] for the parabolic case.

Introducing the exponent $a := 1 - 2s$, defining the elliptic operator $L_a u := -\operatorname{div}(y^a \nabla u)$ and the parabolic operator $H_a u := y^a \partial_t u - \operatorname{div}(y^a \nabla u)$, we observe that $a \in (-1,1)$, which is equivalent to stating that the weight y^a belongs to the A_2 Muckenhoupt class, as remarked earlier. Consequently, the regularity results discussed above apply to the extension operators in both elliptic and parabolic settings.

One may wonder what happens when $a \geq 1$ or $a \leq -1$, that is, when the weight y^a no longer belongs to the A_2 Muckenhoupt class. In this direction there are some new results showing that the Schauder regularity theory still holds true. In particular we refer to [138, 139, 143, 55] for an elliptic theory of weak solutions to weighted equations with y^a, where the authors shows an exhaustive answer to the Schauder regularity theory of such elliptic equations and to the work [56–60], where complementary results are proved for both elliptic and parabolic weighted equations under weak regularity assumptions on the data of the problem.

Motivated by these developments, the following section outlines the results of this thesis. In Chapter 1, we study degenerate and singular parabolic equations with weight y^a in the range $a > -1$, and we analyse some consequences of the corresponding regularity theory, including the parabolic Higher Order Boundary Harnack Principle. In Chapters 2 and 3, we address degenerate and singular elliptic equations on lower dimensional manifolds, where the weight appears as $|y|^a$ with $y \in \mathbb{R}^n$, $n \geq 2$. These results extend the previous theory to this new setting, introduce new challenges, and provide a basis for further applications and connections, which are discussed in detail later.

Summary of the results

In Chapter 1, based on the joint works [7, 8] with A. Audrito and S. Vita, we develop a complete Schauder regularity theory for weak solutions to a class of degenerate or singular parabolic equations satisfying a conormal-type boundary condition on the flat boundary. This condition plays a crucial role in ensuring the smoothness of solutions whenever the data are smooth, as will be clarified later.

More precisely, let us introduce coordinates $(x, y, t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$, the upper unit half ball $B_1^+ := \{|(x, y)| < 1, \, y > 0\}$, the upper unit half cylinder $Q_1^+ := B_1^+ \times (-1, 1)$ and the flat part of boundary $\partial^0 Q_1^+ := Q_1 \cap \{y = 0\}$. For $a > -1$, we consider the degenerate parabolic equation

$$\begin{cases} y^a \partial_t u - \operatorname{div}(y^a A \nabla u) = y^a f + \operatorname{div}(y^a F) & \text{in } Q_1^+ \\ \lim\limits_{y \to 0^+} y^a (A \nabla u + F) \cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+, \end{cases} \tag{1}$$

where $A = A(x, y, t)$ is a uniformly elliptic matrix, f is a scalar function, and F is a vector field. The main result consists in establishing a complete Schauder theory for weak solutions to this problem up to the boundary $\{y = 0\}$, and under minimal assumptions on the data and the coefficients. A simplified formulation of the main theorem, under the assumptions $f = 0$ and $F = 0$, is given below.

Theorem. Let u be a weak solution to (1) with $f = 0$, $F = 0$. Let $\alpha \in (0, 1)$, $k \in \mathbb{N}$ and $A \in C_p^{k,\alpha}(Q_1^+)$ (for the definition of parabolic Hölder spaces see Section 1.2.1).

Then, $u \in C_p^{k+1,\alpha}(Q_{1/2}^+)$ and

$$\|u\|_{C_p^{k+1,\alpha}(Q_{1/2}^+)} \leq C \|u\|_{L^2(Q_1^+, y^a)},$$

for some constant $C > 0$ depending on the data of the problem.

The previous theorem is proved in two steps. The first one is establishing the $C_p^{1,\alpha}$ regularity of weak solutions to (1). To achieve this, we adopt a perturbative method following the elliptic counterpart developed in [138]. Specifically, for $\varepsilon > 0$ we introduce the family of regularized weights $\rho_\varepsilon^a := (\varepsilon^2 + y^2)^{a/2}$, which satisfies $\rho_\varepsilon^a \to y^a$ a.e. as $\varepsilon \to 0$, and consider the regularized equations (with $f = 0$ and $F = 0$)

$$\begin{cases} \rho_\varepsilon^a \partial_t u_\varepsilon - \operatorname{div}(y^a A \nabla u_\varepsilon) = 0 & \text{in } Q_1^+ \\ \lim\limits_{y \to 0^+} \rho_\varepsilon^a A \nabla u_\varepsilon \cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+. \end{cases}$$

The advantage of introducing these regularized equations is that they are uniformly parabolic, in the sense described in the first part of the introduction, so that the classical Schauder regularity theory applies. However, the constants involved may depend on the parameter ε.

The crucial steps in our analysis are therefore the following:

i) Prove that the Schauder estimates for the regularized problem are uniform in ε (a stability property). This is achieved via a contradiction argument in the spirit of [136], relying on new Liouville-type theorems for entire solutions to the corresponding equation in the whole space.

ii) Establish suitable approximation results ensuring the existence of a family $\{u_\varepsilon\}_\varepsilon$ of solutions to the regularized problems that converges, in an appropriate sense, to the weak solution of the degenerate equation (1) as $\varepsilon \to 0$.

Once the $C^{1,\alpha}_p$ regularity is established, the second part of the proof addresses higher order regularity for solutions to (1), providing $C^{k,\alpha}_p$ estimates for $k \geq 2$, and emphasizing that these estimates fail to be stable with respect to the regularization of the weight. The strategy of the proof presented here is substantially different from the classical literature on uniformly parabolic equations and weighted elliptic equations, in which such estimates are generally obtained by iteration methods relying on difference quotients and ODE techniques. In our setting, a new regularization scheme is required, together with *a priori* estimates and a fine analysis involving second-order weighted derivatives of the solution. Further details on this method are provided later.

In addition to the significant applications of the extension of the fractional heat equation already discussed, one of the main motivations for studying such equations, particularly in the *superdegenerate* regime $a \geq 1$, comes from their applications to the Boundary Harnack Principle, as shown in [143]. The classical Boundary Harnack Principle states that, given two positive harmonic functions that vanish on the same portion of the boundary of a domain, their ratio remains bounded near the boundary. This principle holds under very general assumptions on the regularity of the domain: in fact, it is valid in Lipschitz domains (see [87, 42, 3]), NTA (non-tangentially accessible) domains (see [86]), and Hölder domains (see [16, 15]). In the first two cases, the ratio of the harmonic functions is also Hölder continuous up to the boundary. We quote also the works [50, 51, 103, 127]. In the parabolic setting, the Boundary Harnack Principle has been extensively studied for the heat equation, as well as for operators in divergence and non-divergence form; we refer to [88, 128, 5, 83, 122, 146].

All the previous results regarding the Boundary Harnack Principle are proved under mild regularity assumptions on the boundary of the domain (Lipschitz, NTA, or Hölder). If the boundary is smoother, one expects that the ratio of solutions to elliptic and parabolic equations enjoys improved regularity, in accordance with the regularity of the boundary itself. This type of result is known as the Higher Order Boundary Harnack Principle and has various applications, for instance in the study of the regularity of free boundary problems. For such results and applications, we refer to the works [49, 143, 154, 85, 107, 108, 20] in the elliptic case and [12, 90, 54] in the parabolic one. We also cite the works [100, 101, 98, 142, 144] where regularity results are obtained for the ratio of solutions to elliptic PDEs sharing the same zero set, *without* any assumption on the sign of these solutions.

In the final part of Chapter 1, we establish results on the parabolic Higher Order Boundary Harnack Principle, obtained as a corollary of the Schauder regularity estimates for the degenerate parabolic equations introduced earlier. This result can be summarized as follows.

Theorem. Let $k \in \{1,2,3,\dots\}$, $\alpha \in (0,1)$, $A \in C_p^{k,\alpha}$, Ω a domain of class $C_p^{k+1,\alpha}$ and u,v such that

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = 0, & \text{in } \Omega \cap Q_1, \\ \partial_t v - \operatorname{div}(A\nabla v) = 0, & \text{in } \Omega \cap Q_1, \\ u > 0, & \text{in } \Omega \cap Q_1, \\ u = v = 0, & \text{in } \partial\Omega \cap Q_1. \end{cases}$$

Then, $v/u \in C_p^{k+1,\alpha}(\Omega \cap Q_{1/2})$ and

$$\left\| \frac{v}{u} \right\|_{C_p^{k+1,\alpha}(\Omega \cap Q_{1/2})} \leq C \|v\|_{L^2(\Omega \cap Q_1)},$$

for some constant $C > 0$ depending on the data of the problem.

After addressing in Chapter 1 the parabolic setting with weight y^a and its consequences for the Boundary Harnack Principle, we now turn to the second part of this thesis. Chapters 2 and 3 are based respectively on the paper [70], written by the author, and on the joint works [38, 37] with G. Cora and S. Vita.

Let $2 \leq n \leq d$ and consider coordinates $z = (x,y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$. We define the lower dimensional manifold

$$\Sigma_0 := \{z = (x,y) \in \mathbb{R}^d : y = 0\},$$

and we study the elliptic equation

$$-\operatorname{div}(|y|^a A \nabla u) = |y|^a f + \operatorname{div}(|y|^a F), \tag{2}$$

where $A = A(x,y)$ is a uniformly elliptic matrix, f is a scalar function, and F is a vector field. In this setting, the coefficients of the operator fail to satisfy the uniform ellipticity condition along the lower dimensional set Σ_0, so the equation is said to be *degenerate* or *singular*, depending on the value of the parameter $a \in \mathbb{R}$, on lower dimensional manifolds.

In most of the results, we work under the assumption $a + n > 0$, which is equivalent to requiring that the weight $|y|^a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We emphasize that the weight $|y|^a$ belongs to the Muckenhoupt class A_2 if and only if $-n < a < n$. Nevertheless, even beyond this range, we show that the weight is still 2-admissible as long as $a + n > 0$; see for instance [82] for a precise definition. This admissibility condition ensures that the theory developed by Fabes, Kenig and Serapioni in [65] applies to such operators, and in particular, Hölder regularity of solutions - under suitable assumptions on the data - is guaranteed.

The new results presented in this work allow us to establish higher regularity for such solutions. In particular, depending on the value of the parameter a and on the codimension n of Σ_0, we prove $C^{0,\alpha}$ and even $C^{1,\alpha}$ regularity results which, under suitable assumptions on the elliptic matrix A, turn out to be optimal.

More specifically, Chapter 2 is devoted to the analysis of the regularity of solutions to (2) that satisfy a Dirichlet boundary condition on the lower dimensional set Σ_0, namely

$$u = \psi, \qquad \text{on } \Sigma_0 \cap B_1.$$

We assume that the parameter a satisfies $a + n \in (0,2)$. Under this condition, the weighted capacity of Σ_0 is locally finite and positive, which guarantees that the trace of weak solutions to (2) is well defined (see [118, Theorem 2.3]). Consequently, the Dirichlet boundary condition is well posed. The principal result of Chapter 2, in the case $f = 0$, $F = 0$ and the matrix $A = \mathbb{I}$, is presented below.

Theorem. Let u be a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a \nabla u) = 0, & \text{in } B_1 \setminus \Sigma_0, \\ u = \psi, & \text{on } \Sigma_0 \cap B_1. \end{cases}$$

Then the following holds true.

- Let $a + n \in (0, 2)$ and $\alpha \in (0, 1) \cap (0, 2 - a - n)$. If $\psi \in C^{0,1}(\Sigma_0 \cap B_1)$ then $u \in C^{0,\alpha}(B_{1/2})$ and
$$\|u\|_{C^{0,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^2(B_1,|y|^a)} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big),$$
for some constant $C > 0$ depending on the data of the problem.
- Let $a + n \in (0, 1)$ and $\alpha \in (0, 1 - a - n)$. If $\psi \in C^{1,\alpha}(\Sigma_0 \cap B_1)$ then $u \in C^{1,\alpha}(B_{1/2})$ and
$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^2(B_1,|y|^a)} + \|\psi\|_{C^{1,\alpha}(\Sigma_0\cap B_1)}\big),$$
for some constant $C > 0$ depending on the data of the problem.

In Chapter 3, we instead focus on solutions to (2) *across* the thin set Σ_0, which is equivalent to requiring that a formal *homogeneous conormal condition* is satisfied on Σ_0. In this setting, we work under the assumption $a + n > 0$, and the main result can be summarized as follows in the case $f = 0$, $F = 0$ and the matrix $A = \mathbb{I}$.

Theorem. Let $a + n > 0$ and set

$$\alpha_* := \frac{2 - a - n + \sqrt{(2 - a - n)^2 + 4(n - 1)}}{2}.$$

Let u be a weak solution to
$$-\operatorname{div}(|y|^a \nabla u) = 0, \qquad \text{in } B_1.$$

Then the following holds true.

- If $\alpha \in (0, 1) \cap (0, \alpha_*)$ then $u \in C^{0,\alpha}(B_{1/2})$ and
$$\|u\|_{C^{0,\alpha}(B_{1/2})} \le C\|u\|_{L^2(B_1,|y|^a)},$$
for some constant $C > 0$ depending on the data of the problem.
- If $\alpha_* > 1$ and $\alpha \in (0, 1) \cap (0, \alpha_* - 1)$ then $u \in C^{1,\alpha}(B_{1/2})$ and
$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C\|u\|_{L^2(B_1,|y|^a)},$$
for some constant $C > 0$ depending on the data of the problem.

The proof of the previous theorems follows a perturbative approach, as done in the parabolic case. However, the regularization we adopt here is different. At first glance, one might expect that regularizing the weight by introducing $(\varepsilon^2 + |y|^2)^{a/2}$ could suffice. However, in some ranges of the parameter a, this approach leads to a loss of information. For instance, in the Dirichlet problem where $u = \psi$ on the lower dimensional set Σ_0, this trace condition is well defined since weak solutions belong to a suitable energy space, as previously remarked. After regularizing the weight, it is no longer possible to impose boundary data on Σ_0, since its classical H^1-capacity is zero.

To overcome this issue, we adopt a regularization scheme at the level of the domain. For small $\varepsilon > 0$, we introduce $\Sigma_\varepsilon := \{|y| \le \varepsilon\}$ and we approximate the lower dimensional set Σ_0, which has dimension $d-n$, with a *classical* boundary $\partial\Sigma_\varepsilon = \{|y| = \varepsilon\}$ of dimension $d-1$. Therefore, we consider the following equation (in the case $f = 0$, $F = 0$ and $A = \mathbb{I}$)

$$-\operatorname{div}(|y|^a \nabla u_\varepsilon) = 0, \qquad \text{in } B_1 \setminus \Sigma_\varepsilon$$

and we associate to it the homogeneous Dirichlet boundary condition $u_\varepsilon = 0$ on $\partial\Sigma_\varepsilon \cap B_1$ for the first theorem, or the conormal boundary condition $\nabla u_\varepsilon \cdot \nu = 0$ on $\partial\Sigma_\varepsilon \cap B_1$ for the second one.

In order to carry out this analysis, we introduce a general framework for the study of these equations and establish regularity estimates that remain stable as $\varepsilon \to 0$ via a contradiction argument involving Liouville-type theorems. By combining these estimates with suitable approximation results, we are able to prove the main theorems.

The study of such operators can be viewed as a natural continuation of the codimension-one theory. However, there are several additional motivations for developing a corresponding regularity theory in higher codimension. In this thesis, we highlight several areas closely related to the degenerate equations introduced above, which may be enriched by the methods and results developed here. Some of these connections are well established in the literature - for instance, harmonic maps with prescribed singularities in general relativity [148–152, 93–95, 119], or the Dirichlet problem and harmonic measure on lower dimensional boundaries [45–47, 43]. Other connections are more recent or less explored, such as the study of the regularity of critical points for Caffarelli–Kohn–Nirenberg inequalities [24], higher codimensional extensions of the fractional Laplacian, and very thin free boundary problems. We shall return to these topics and discuss them further in the later sections of this thesis.

We conclude this introduction by mentioning that the results presented in this thesis are based on the following works.

[7] A. Audrito, G. Fioravanti, and S. Vita, *Schauder estimates for parabolic equations with degenerate or singular weights*, Calc. Var. Partial Differential Equations, **63** (2024).

[8] A. Audrito, G. Fioravanti, and S. Vita, *Higher order Schauder estimates for degenerate or singular parabolic equations*, Rev. Mat. Iberoam., **41** (2025), pp. 1513–1554.

[70] G. Fioravanti, *The Dirichlet problem on lower dimensional boundaries: Schauder estimates via perforated domains*, Nonlinear Analysis, **263** (2026).

[38] G. Cora, G. Fioravanti, and S. Vita, *Schauder estimates for elliptic equations degenerating on lower dimensional manifolds*, Preprint arXiv:2501.19033, (2025).

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CHAPTER

ONE

REGULARITY FOR DEGENERATE OR SINGULAR PARABOLIC EQUATIONS

1.1 Introduction

The first chapter of this thesis is based on the two papers [7, 8], written in collaboration with A. Audrito and S. Vita. We study a class of weighted parabolic equations in which the weight degenerates or blows up on a characteristic hyperplane Σ, behaving like $\mathrm{dist}(\cdot,\Sigma)^a$, where $a>-1$ is a fixed parameter. More precisely, we provide Schauder type regularity for solutions to the following equation

$$\begin{cases} y^a\partial_t u - \mathrm{div}(y^a A\nabla u) = y^a f + \mathrm{div}(y^a F) & \text{in } Q_1^+ \\ \lim\limits_{y\to 0^+} y^a(A\nabla u + F)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+. \end{cases} \tag{1.1}$$

Here $d\geq 1$, $(z,t)=(x,y,t)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$, $\Sigma=\{y=0\}$ and $\mathrm{dist}(P,\Sigma)^a=y^a$ is locally integrable whenever $a>-1$. $B_1\subset\mathbb{R}^{d+1}$ denotes the unit ball with center at 0 and $B_1^+:=B_1\cap\{y>0\}$ the unit upper-half ball. Similar, if $I_1:=(-1,1)$, then $Q_1:=B_1\times I_1$ is the unit parabolic cylinder, $Q_1^+:=B_1^+\times I_1$ is the unit upper-half cylinder, while $\partial^0 Q_1^+=Q_1\cap\{y=0\}$. The operators ∇ and div denote the gradient and the divergence with respect to the spatial variable z, respectively. Furthermore, $A: Q_1^+\to\mathbb{R}^{d+1,d+1}$ is a symmetric $(d+1)$-dimensional matrix satisfying the following ellipticity condition: there exist $0<\lambda\leq\Lambda<+\infty$ such that

$$\lambda|\xi|^2\leq A(z,t)\xi\cdot\xi\leq\Lambda|\xi|^2, \tag{1.2}$$

for all $\xi\in\mathbb{R}^{d+1}$ and a.e. $(z,t)\in Q_1^+$, while the forcing terms in the r.h.s. $f: Q_1^+\to\mathbb{R}$ and $F: Q_1^+\to\mathbb{R}^{d+1}$ are given functions belonging to some suitable functional spaces.

In the simplest case where $A=\mathbb{I}$ and $f=|F|=0$, problem (1.1) (posed in the whole space) is nothing more than the gradient flow of the energy

$$\int_{\mathbb{R}^{d+1}_+} y^a|\nabla v|^2 dz, \quad v\in H^1(\mathbb{R}^{d+1}_+, y^a).$$

So, as one may imagine, the natural functional setting involves the weighted Sobolev spaces involving time. Precisely, we say that u is a weak solution to (1.1) if $u\in L^2(I_1;H^1(B_1^+,y^a))\cap$

$L^\infty(I_1;L^2(B_1^+,y^a))$ and satisfies

$$\int_{Q_1^+} y^a\big(-u\partial_t\phi + A\nabla u\cdot\nabla\phi\big)dzdt = \int_{Q_1^+} y^a\big(f\phi - F\cdot\nabla\phi\big)dzdt,$$

for every test function $\phi\in C_c^\infty(Q_1)$ (cf. Definition 1.2.18). Notice that weak solutions formally satisfy the *conormal boundary condition*

$$\lim_{y\to0^+} y^a(A\nabla u+F)\cdot e_{d+1}=0 \quad \text{on } \Sigma, \tag{1.3}$$

appearing in (1.1): as standard in Neumann-type problems, one can easily check this integrating by parts and use the fact that the test functions ϕ need not to vanish on Σ. Actually, as a consequence of our main theorem (see (1.7)), we will obtain that, under suitable regularity assumptions on the data, weak solutions satisfy

$$(A\nabla u+F)\cdot e_{d+1}=0 \quad \text{on } \Sigma,$$

which is stronger than (1.3) (at least when $a>0$) and, when $A=\mathbb{I}$ and $F=0$, reduces to the *classical* Neumann boundary condition.

As pointed out above, the regularity theory for *uniformly parabolic* equations - corresponding to the case $a=0$ - is by now well established (see, for instance, [91, 96]). For weighted equations where the uniform ellipticity condition (1.2) fails, we refer to the pioneering works [65, 36, 78], which extend the De Giorgi–Nash–Moser theory to the setting of A_2-Muckenhoupt weights - both in the elliptic and parabolic case - ensuring Hölder continuity of weak solutions with some implicit exponent α. In our case, the weight y^a belongs to the A_2 class if and only if $a\in(-1,1)$. However, we point out that such regularity results can still be extended to the whole range $a>-1$.

The peculiar geometry of the degeneracy/singularity set of our weight - the characteristic hyperplane Σ - allows us to get more information compared to the general theory quoted above. In fact, as already done in [138, 143], in the elliptic setting, the approach we follow here allows us to cover the full range $a>-1$ and eventually will allow us to show Schauder $C_p^{k,\alpha}$ estimates for any $k\in\mathbb{N}$. It is important to remark here that the regularity we obtain strongly relies on the *natural conormal boundary condition* (1.3) we impose on the characteristic hyperplane Σ. As one may imagine, different boundary conditions lead to different regularity estimates: for instance, $v=y^{1-a}$ weakly solves $-\mathrm{div}(y^a\nabla v)=0$ in B_1^+ with homogeneous Dirichlet boundary condition at Σ whenever $a\in(-1,1)$, but it is no more than $C^{\lfloor 1-a\rfloor,1-a-\lfloor 1-a\rfloor}$ regular.

Within the broad literature on the regularity theory for degenerate parabolic equations, we highlight the contributions [58, 60], where the authors established Sobolev-type regularity estimates for a wide class of parabolic equations, including those of the form (1.1). Further recent developments can be found in [109–111, 117].

Moreover, the analysis of weighted problems such as (1.1) is closely related to the theory of edge operators [104, 106], as well as to the nonlocal operators $(\partial_t-\Delta)^{\frac{1-a}{2}}$ and their extension theory, as discussed in the introduction (see [13, 120, 141]). In this context, Schauder estimates for solutions to fractional parabolic equations involving operators of the form $(\partial_t-\mathrm{div}_x(A(x)\nabla_x))^{\frac{1-a}{2}}$ have been obtained in [17]. In terms of our notation, this corresponds to regularity estimates in the (x,t)-variables on Σ, with $a\in(-1,1)$ (see also [19, 29, 57, 134]). Let us also mention that space analyticity (in the full z variable) and smoothness in (z,t) of solutions to equation (1.1) were already available by [14] when $a \in (-1,1)$ and coefficients are analytic and satisfy suitable extra assumptions.

It is worth noting that the study of such operators has played a central role in a number of recent works, including [6, 25] (reaction-diffusion equations), [44, 4, 11] (obstacle problems), [137, 9] (nodal set analysis), and [84] (nonlocal harmonic map flows), among others.

Furthermore, as observed in [143], such *superdegenerate* equations find notable applications in the context of the Higher Order Boundary Harnack Principle; we refer to [12, 90] for the corresponding developments in the parabolic setting. We will return to this topic later in the introduction.

Notably, the weighted elliptic Schauder theory developed in [138, 143] has been recently applied in [2, 126] to establish higher regularity of free boundaries in certain semilinear free boundary problems of Alt–Phillips type. It is then natural to ask whether the parabolic Schauder theory developed in the present work could lead to similar results in the parabolic setting as well.

1.1.1 Main results

The main result we aim to prove is a complete Schauder theory for weak solutions to (1.1). Following the approach of [7, 8], we split the main theorem into two parts. In the first part, we establish regularity in the spaces $C^{0,\alpha}_p$ and $C^{1,\alpha}_p$ (Schauder estimates). Then, we prove regularity in the spaces $C^{k+2,\alpha}_p$ for any $k \in \mathbb{N}$ (higher order Schauder estimates). Afterward, we discuss some applications of our results, including equations with degenerate or singular weights on cylindrical curved characteristic manifolds and the parabolic Higher Order Boundary Harnack Principle.

Schauder estimates

In the first part of the chapter we prove $C^{0,\alpha}_p$ and $C^{1,\alpha}_p$ regularity estimates - up to the characteristic hyperplane Σ - for weak solutions to (1.1). The main idea is to extend to the parabolic framework the regularization argument used in [138, 139] in the elliptic one: for $\varepsilon \in (0,1)$, we introduce the family of smooth weights

$$\rho_\varepsilon^a(y) := (\varepsilon^2 + y^2)^{a/2},$$

and we consider weak solutions to

$$\begin{cases} \rho_\varepsilon^a \partial_t u - \operatorname{div}(\rho_\varepsilon^a A \nabla u) = \rho_\varepsilon^a f + \operatorname{div}(\rho_\varepsilon^a F) & \text{in } Q_1^+ \\ \lim\limits_{y\to 0^+} \rho_\varepsilon^a (A\nabla u + F)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+, \end{cases} \tag{1.4}$$

which corresponds to the problem associated to the regularized weight (notice that by construction $\rho_\varepsilon^a(y) \to y^a$ almost everywhere as $\varepsilon \to 0^+$). Since ρ_ε^a is (locally) bounded and bounded away from zero, problem (1.4) is uniformly parabolic. Consequently, the classical Schauder regularity theory applies (see for instance [96, 91]) and one obtains $C^{0,\alpha}_p$ and $C^{1,\alpha}_p$ regularity estimates with constants *possibly depending* on the parameter ε. The idea is to show that that such estimates are *uniform* in $\varepsilon \in (0,1)$ and pass to the limit as $\varepsilon \to 0^+$. We refer to this property as ε-stability of the estimates. The latter, together with a fine approximation procedure (see Section 1.4) yields the following result.

Theorem 1.1.1. *Let $a > -1$, $r \in (0,1)$, A satisfying (1.2) and u be a weak solution to (1.1), in the sense of Definition 1.2.18. Then,* *i) If A is a continuous matrix, ω is a modulus of continuity such that*

$$\|A\|_{L^\infty(Q_1^+)} + \sup_{(z,t),(z',t')\in Q_1^+} \frac{|A(z,t)-A(z',t')|}{\omega(|z-z'|+|t-t'|^{1/2})} \le L,$$

$f \in L^p(Q_1^+, y^a)$ with $p > \frac{d+3+a^+}{2}$, $F \in L^q(Q_1^+, y^a)^{d+1}$ with $q > d+3+a^+$, $\alpha \in (0,1) \cap (0, 2-\frac{d+3+a^+}{p}] \cap (0, 1-\frac{d+3+a^+}{q}]$, then there exists a constant $C>0$, depending on d, a, λ, Λ, p, q, r, L and α such that

$$\|u\|_{C_p^{0,\alpha}(Q_r^+)} \le C\Big(\|u\|_{L^2(Q_1^+,y^a)} + \|f\|_{L^p(Q_1^+,y^a)} + \|F\|_{L^q(Q_1^+,y^a)}\Big). \tag{1.5}$$

ii) If $A, F \in C_p^{0,\alpha}(Q_1^+)$, $f \in L^p(Q_1^+, y^a)$ with $p > d+3+a^+$, $\alpha \in (0,1) \cap (0, 1-\frac{d+3+a^+}{p}]$, then there exists a constant $C>0$, depending on d, a, λ, Λ, p, α, r and $\|A\|_{C_p^{0,\alpha}(Q_1^+)}$ such that

$$\|u\|_{C_p^{1,\alpha}(Q_r^+)} \le C\Big(\|u\|_{L^2(Q_1^+,y^a)} + \|f\|_{L^p(Q_1^+,y^a)} + \|F\|_{C_p^{0,\alpha}(Q_1^+)}\Big). \tag{1.6}$$

In addition, u satisfies the conormal boundary condition

$$(A\nabla u + F)\cdot e_{d+1} = 0 \quad \text{on } \partial^0 Q_1^+. \tag{1.7}$$

Moreover, the estimates (1.5) and (1.6) are ε-stable in the sense of Theorems 1.6.1 and 1.7.1.

As a first comment, we would like to remark that the ε-stability of the $C_p^{k+2,\alpha}$ estimates with respect to the regularization described above cannot be valid, see [138, Remark 5.4]. Secondly, the integrability and regularity conditions required on the data A, f, F are the standard ones (see [138, 139], in the elliptic setting), in terms of the natural scaling of the problem (one can also recover the regularity results for uniformly parabolic equations, that is without weight terms, by taking $a=0$). Moreover, let us stress again the fact that the $C_p^{0,\alpha}$ regularity above is comparable to the regularity theory in [60] but, respect to this, our approach for the $C_p^{1,\alpha}$ regularity requires the ε-stability of the $C_p^{0,\alpha}$ estimate: to the best of our knowledge, this is completely new in the parabolic setting. Regarding the $C_p^{1,\alpha}$ regularity, other approaches, such as the Campanato method, may also work, see for instance [17] (range $a \in (-1,1)$, trace regularity) and [85] (elliptic setting). In contrast, we opt for a different strategy: the proof of our main theorem is based on a contradiction argument combined with a blow-up procedure (in the spirit of the classic paper by Simon [136]), which crucially exploit the Liouville-type Theorem 1.5.1, which ensures a rigidity result for entire solutions with a certain growth-control at infinity.

Higher order Schauder estimates

After the Schauder estimates are established, the goal is to iterate them to prove higher order Schauder estimates for weak solutions to (1.1). Below is the statement of this result.

Theorem 1.1.2. *Let $a > -1$, $r \in (0,1)$, $\alpha \in (0,1)$ and $k \in \mathbb{N}$. Let $A \in C_p^{k+1,\alpha}(Q_1^+)$ satisfying (1.2), $f \in C_p^{k,\alpha}(Q_1^+)$ and $F \in C_p^{k+1,\alpha}(Q_1^+)$ and let u be a weak solution to (1.1). Then, there exists $C>0$ depending only on d, a, λ, Λ, r, α and $\|A\|_{C_p^{k+1,\alpha}(Q_1^+)}$ such that*

$$\|u\|_{C_p^{k+2,\alpha}(Q_r^+)} \le C\Big(\|u\|_{L^2(Q_1^+,y^a)} + \|f\|_{C_p^{k,\alpha}(Q_1)} + \|F\|_{C_p^{k+1,\alpha}(Q_1)}\Big). \tag{1.8}$$

As mentioned earlier, the $C^{2,\alpha}_p$ (or higher order) estimates cannot rely on the ε-regularization scheme of the weight y^a used to prove Theorem 1.1.1, since the ε-stability of the $C^{2,\alpha}_p$ estimate is generally false, even in the elliptic framework. For more details, see [138, Remark 5.4].

Before sketching the main steps of the proof of Theorem 1.1.2, it is important to highlight the following facts, which substantially differ our strategy from the existing literature:

- In the *weighted elliptic framework* (see [138, 143]), as soon as the $C^{1,\alpha}$ regularity is available, one can iterate it on derivatives. This is obtained in two steps: first, one notice that, since the weighted elliptic operator commutes with all but one derivatives, $\partial_{x_i} u$ is also a solution for any $i = 1, ..., d$ (and so $\partial_{x_i} u$ gains regularity); then, the operator itself gives the regularity of the last derivative $\partial_y u$. Formally, this is because, in the special case $A = \mathbb{I}$, one can re-write the equation as
$$-\partial_{yy} u - \partial_y F \cdot e_{d+1} - \frac{a}{y}(\partial_y u + F \cdot e_{d+1}) = f + \mathrm{div}_x F + \Delta_x u,$$
and thus, if $\Delta_x u$ is smooth, then $\partial_y u$ is smooth by ODE methods (of course, provided that the data are smooth as well).
- In the *non-weighted parabolic framework* (see [96]), the idea is roughly the same: if $\Delta_x u$ is smooth, then the equation
$$\partial_t u = f + \mathrm{div}\, F + \Delta_x u$$
yields smoothness of $\partial_t u$.
- In the present *degenerate parabolic setting*, the degenerate variables are two, y and t, and the above strategies do not apply. In particular, the induction argument requires, as starting point, the $C^{2,\alpha}_p$ regularity of weak solutions (see Proposition 1.10.2).

Given the above remarks, our approach relies on a priori estimates and a regularization procedure by convolution with standard mollifiers. More precisely:

For the $C^{2,\alpha}_p$ regularity:

i) We establish some *a priori* $C^{2,\alpha}_p$ estimates in Proposition 1.10.2 using a blow-up argument combined with a Liouville theorem (see Theorem 1.9.1 below), in the spirit of [136] (see also [138] in the weighted elliptic setting).

ii) We prove $C^{2,\alpha}_p$ regularity of weak solutions when the data are C^∞ (see Lemma 1.10.3). In this step, the $C^{1,\alpha}_p$ regularity of weak solutions (see Theorem 1.1.1) is crucial.

iii) We use an approximation scheme to regularize (1.1), by convolution of the data with a family of standard mollifiers. Along the approximating sequence, the $C^{2,\alpha}_p$ regularity estimate extends to weak solutions with $f \in C^{0,\alpha}_p$ and $A, F \in C^{1,\alpha}_p$. In other words, we prove the *a posteriori* regularity estimate in Theorem 1.1.2 when $k = 0$.

For the $C^{k+2,\alpha}$ regularity for every $k \geq 1$:

iv) When the forcing term is zero, i.e. $f = 0$, we iterate the regularity estimates previously obtained - i.e. the $C^{1,\alpha}_p$ and $C^{2,\alpha}_p$ regularity - on partial derivatives of solutions, by using the same scheme as in the proof of Lemma 1.10.3 and Theorem 1.1.2 follows quite easily.

v) In the case of general forcing terms $f \in C^{k,\alpha}_p$ the argument of iv) doesn't apply (at least for $k=1$), and hence we proceed as follows: we use the procedure described at points i), ii), iii) at any order k. To be more precise, the $C^{k+2,\alpha}_p$ *a priori* estimates are obtained inductively on k, starting from the $C^{2,\alpha}_p$ *a priori* estimates proved at point i). This part crucially uses a delicate analysis of a second order weighted-type derivative of solutions in y (see Lemma 1.11.3). The $C^{k+2,\alpha}_p$ regularity when the data are smooth (the analogous of point ii)) is also proved by induction in Lemma 1.11.2. Finally, with the same regularization argument in iii), we finally obtain Theorem 1.1.2.

Cylindrically curved characteristic manifolds

As a consequence of our main theorems, we can treat more general equations with weights behaving as *distance functions* to a regular hypersurface $\Gamma \subset \mathbb{R}^{d+1}$ (curved characteristic manifolds) that we introduce below in Section 1.12 (see (1.139)). Such equations are set in cylindrical domains $\Omega^+ \times (-1,1)$ of $\mathbb{R}^{d+2}$ which live on one side of $\Gamma \times (-1,1)$. The family of weights $\delta = \delta(z)$ we consider behave as a distance function to Γ in the sense of (1.140). We consider weighted equations of the form

$$\begin{cases} \delta^a \partial_t u - \mathrm{div}(\delta^a A \nabla u) = \delta^a f + \mathrm{div}(\delta^a F) & \text{in } (\Omega^+ \cap B_1) \times (-1,1), \\ \delta^a (A\nabla u + F)\cdot \nu = 0 & \text{on } (\Gamma \cap B_1) \times (-1,1), \end{cases} \tag{1.9}$$

where ν is the unit outward normal vector to Ω^+ on Γ. See Definition 1.12.1 for the definition of solutions to (1.9).

Corollary 1.1.3. *Let $a > -1$, A satisfying (1.2), φ be the parametrization defined in (1.139) δ satisfying (1.140) and u be a weak solution to (1.9), in the sense of Definition 1.12.1. Then,*

i) If $\varphi \in C^{1,\alpha}(B_1 \cap \{y=0\})$, $\delta \in C^{1,\alpha}(\Omega^+ \cap B_1)$, $A, F \in C^{0,\alpha}_p((\Omega^+ \cap B_1) \times (-1,1))$, $f \in L^p((\Omega^+ \cap B_1) \times (-1,1), \delta^a)$ with $p > d+3+a^+$, $\alpha \in (0,1) \cap (0, 1 - \frac{d+3+a^+}{p}]$, then, there exists a constant $C > 0$, depending on d, a, λ, Λ, p, α, c_0, $\|A\|_{C^{0,\alpha}_p((\Omega^+\cap B_1)\times(-1,1))}$, $\|\varphi\|_{C^{1,\alpha}(B_1 \cap \{y=0\})}$ and $\|\delta\|_{C^{1,\alpha}(\Omega^+ \cap B_1)}$ such that

$$\begin{aligned} \|u\|_{C^{1,\alpha}_p((\Omega^+\cap B_{1/2})\times(-1/2,1/2))} \le C\Big(&\|u\|_{L^2((\Omega^+\cap B_1)\times(-1,1),\delta^a)} \\ &+ \|f\|_{L^p((\Omega^+\cap B_1)\times(-1,1),\delta^a)} + \|F\|_{C^{0,\alpha}_p((\Omega^+\cap B_1)\times(-1,1))}\Big). \end{aligned}$$

In addition, u satisfies the conormal boundary condition

$$(A\nabla u + F)\cdot \nu = 0 \quad \text{on } (\Gamma \cap B_1) \times (-1,1),$$

where ν is the unit outward normal vector to Ω^+ on Γ.

ii) Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. If $\varphi \in C^{k+2,\alpha}(B_1 \cap \{y=0\})$, $\delta \in C^{k+2,\alpha}(\Omega^+ \cap B_1)$, $A, F \in C^{k+1,\alpha}_p((\Omega^+ \cap B_1) \times (-1,1))$, $f \in C^{k,\alpha}_p((\Omega^+ \cap B_1) \times (-1,1))$, then, there exists a constant $C > 0$, depending on d, a, λ, Λ, α, c_0, $\|A\|_{C^{k+1,\alpha}_p((\Omega^+\cap B_1)\times(-1,1))}$, $\|\varphi\|_{C^{k+2,\alpha}(B_1\cap\{y=0\})}$ and $\|\delta\|_{C^{k+2,\alpha}(\Omega^+\cap B_1)}$ such that

$$\begin{aligned} \|u\|_{C^{k+2,\alpha}_p((\Omega^+\cap B_{1/2})\times(-1/2,1/2))} \le C\Big(&\|u\|_{L^2((\Omega^+\cap B_1)\times(-1,1),\delta^a)} \\ &+ \|f\|_{C^{k,\alpha}_p((\Omega^+\cap B_1)\times(-1,1))} + \|F\|_{C^{k+1,\alpha}_p((\Omega^+\cap B_1)\times(-1,1))}\Big). \end{aligned}$$

Parabolic Higher Order Boundary Harnack Principle

Finally, following the program of the elliptic setting (see [143]), we provide an alternative proof of some *parabolic Higher Order Boundary Harnack Principle* as in [12, 90]. Such kind of regularity comparison principle between two caloric functions u, v (or solutions to more general parabolic equations), vanishing on the same fixed boundary, can be viewed as the Schauder regularity of their quotient $w = v/u$ which, in turns, satisfies a parabolic equation with degenerate weight u^2, see (1.142). After proper diffeomorphic transformations of the domain, the Schauder theory for the ratio w follows as a byproduct of our main Theorem 1.1.2.

The regularity comparison principle is localized at boundary points which lie on the *lateral parabolic boundary* of a space-time domain. In other words, let us consider u, v solutions of

$$\begin{cases} \partial_t u - \mathrm{div}(A\nabla u) = g + Vu + b\cdot\nabla u & \text{in } \Omega\cap Q_1\\ \partial_t v - \mathrm{div}(A\nabla v) = f + Vv + b\cdot\nabla v & \text{in } \Omega\cap Q_1\\ u(z,t) \geq c_0\, d_p((z,t),\partial\Omega\cap Q_1) & \text{in } \Omega\cap Q_1\\ u = v = 0 & \text{on } \partial\Omega\cap Q_1, \end{cases} \tag{1.10}$$

where A, V, b, g and f are suitable data (see Theorem 1.1.4 below). Here, up to rotations, dilations and translations, 0 belongs to the parabolic lateral boundary of Ω; that is, there exists a parametrization φ such that

$$\Omega\cap Q_1 = \{y > \varphi(x,t)\}, \qquad \partial\Omega\cap Q_1 = \{y = \varphi(x,t)\}, \tag{1.11}$$

with $\varphi(0) = 0$ and $\nabla_x\varphi(0) = 0$. Moreover, the parabolic distance to the boundary is defined as

$$d_p((z,t),\partial\Omega\cap Q_1) = \inf_{(\zeta,\tau)\in\partial\Omega\cap Q_1} d_p((z,t),(\zeta,\tau)),$$

and the parabolic distance between points is defined in (1.12).

We will present here the parabolic Higher Order Boundary Harnack Principle for equations in divergence form in $C^{k+2,\alpha}_p$-domains, $k\in\mathbb{N}$. However, let us stress the fact that the regularity assumptions we make on boundaries, coefficients and right hand sides, always allows to pass from non divergence to divergence form equations and viceversa, interchangeably. So, we are considering the same conditions set in [12], which are slightly more general compared to [90], where the assumptions on the drift terms are suboptimal. Actually, our approach allows us to treat equations with nontrivial forcing terms g in the right hand side of the equation of u.

Theorem 1.1.4. *Let $k\in\mathbb{N}$, $\alpha\in(0,1)$ and u, v be solutions to (1.10). Let $\varphi\in C^{k+2,\alpha}_p(Q_1\cap\{y=0\})$ be the parametrization defined in (1.11). Let $A, f, g\in C^{k+1,\alpha}_p(\Omega\cap Q_1)$, with A satisfying (1.2), $V, b\in C^{k,\alpha}_p(\Omega\cap Q_1)$.*

Then, there exists a positive constant $C>0$, depending on d, λ, Λ, c_0, α, $\|A\|_{C^{k+1,\alpha}_p(\Omega\cap Q_1)}$, $\|g\|_{C^{k+1,\alpha}_p(\Omega\cap Q_1)}$, $\|V\|_{C^{k,\alpha}_p(\Omega\cap Q_1)}$, $\|b\|_{C^{k,\alpha}_p(\Omega\cap Q_1)}$, $\|\varphi\|_{C^{k+2,\alpha}_p(Q_1\cap\{y=0\})}$ and $\|u\|_{L^2(\Omega\cap Q_1)}$ such that

$$\left\|\frac{v}{u}\right\|_{C^{k+2,\alpha}_p(\Omega\cap Q_{1/2})} \leq C\Big(\|v\|_{L^2(\Omega\cap Q_1)} + \|f\|_{C^{k+1,\alpha}_p(\Omega\cap Q_1)}\Big).$$

1.1.2 Organization of the chapter

This chapter is divided into two parts as follows: from Section 1.2 to Section 1.7, we establish the Schauder estimates; from Section 1.8 to Section 1.13, we prove the higher order Schauder estimates and derive the corollaries of the main theorems.

In Section 1.2, we set up the problem by introducing the functional framework and the definition of weak solutions to (1.1). In Section 1.3, we prove some uniform estimates - namely, the Caccioppoli inequality and local L^∞ bounds - using De Giorgi's iteration technique. Section 1.4 is devoted to proving the approximation results, that is, the convergence (in suitable energy spaces) of regularized solutions to weak solutions of (1.1). In Section 1.5, we establish a Liouville-type theorem for entire solutions satisfying a sub-quadratic growth condition. Finally, in Sections 1.6 and 1.7, we prove the ε-stability of the $C^{0,\alpha}_p$ and $C^{1,\alpha}_p$ regularity estimates mentioned above, and conclude with the proof of Theorem 1.1.1.

In Section 1.8, we establish additional energy estimates that will be instrumental in the subsequent analysis. In Section 1.9, we extend the Liouville-type theorem to entire solutions satisfying a general polynomial growth condition. Section 1.10 is devoted to proving the $C^{2,\alpha}_p$ regularity of weak solutions. In Section 1.11, we iterate this result to obtain $C^{k,\alpha}_p$ regularity for all integers $k \in \mathbb{N}$, thus completing the proof of Theorem 1.1.2. In Section 1.12, we generalize the previous regularity results to the case of weights degenerating along curved manifolds. Finally, in Section 1.13, we establish the parabolic Higher Order Boundary Harnack Principle (Theorem 1.1.4).

1.1.3 Notation

Let $d \in \mathbb{N}$ an integer and consider coordinates $(z,t) = (x,y,t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$. We define the characteristic manifold $\Sigma := \{y = 0\}$. Given $r > 0$, we denote by $B_r(z_0)$ the ball of radius r centered at z_0. When $z_0 = 0$ we simply write B_r. Moreover, we denote by $B_r^+ := B_r \cap \{y > 0\}$ the upper half ball of radius r centered a 0. We denote by $Q_r(z_0, x_0) := B_r(z_0) \times (t_0 - r^2, t_0 + r^2)$ the parabolic cylinder of radius $r > 0$ and centered at $(z_0, t_0) \in \mathbb{R}^{d+1} \times \mathbb{R}$. When $z_0 = (0,0)$ we write $Q_r = Q_r(0,0) = B_r \times I_r$ where $I_r := (-r^2, r^2)$. We denote by $Q_r^+ := B_r^+ \times I_r$ the upper half cylinder of radius r centered a 0 and by $\partial^0 Q_r^+ = Q_r \cap \Sigma$ the flat lateral boundary of the cylinder.

Given $a \in \mathbb{R}$ and $\varepsilon \in [0,1)$, we define the family of weights

$$\rho_\varepsilon^a(y) := (\varepsilon^2 + y^2)^{a/2},$$

noting that $\rho_0^a(y) = |y|^a$. For $\varepsilon > 0$, the weight is uniformly bounded from above and below (with constants depending on ε); instead, when $\varepsilon = 0$, the weight may either explode or vanish on the characteristic manifold Σ, depending on the value of a.

1.2 Functional setting

The present section is primarily dedicated to establishing the functional framework of the problem, presenting preliminary results on the relevant spaces, and providing the definitions of solutions to our problem.

1.2.1 Parabolic Hölder spaces

In this section, we introduce parabolic Hölder spaces, following [91, 96] (see also [90]). These spaces are the appropriate functional setting to establish Schauder-type regularity results, as they respect the natural scaling of parabolic equations.

Given an open set $\Omega \subset \mathbb{R}^{d+1} \times \mathbb{R}$ we define the parabolic distance $d_p : \Omega \times \Omega \to \mathbb{R}$ as

$$d_p((z,t),(\zeta,\tau)) := (|z-\zeta|^2 + |t-\tau|)^{1/2}, \tag{1.12}$$

for all points $(z,t),(\zeta,\tau) \in \Omega$, where $z,\zeta \in \mathbb{R}^{d+1}$ and $t,\tau \in \mathbb{R}$. We notice that the parabolic distance is parabolically 1-homogeneous, in the sense that

$$d_p((rz,r^2t),(r\zeta,r^2\tau)) = |r| d_p((z,t),(\zeta,\tau)), \quad \text{for every } r \in \mathbb{R}.$$

Given a function $u : \Omega \to \mathbb{R}$ and $\alpha \in (0,1]$ let us define the seminorms

$$[u]_{C_p^{0,\alpha}(\Omega)} := \sup_{\substack{(z,t),(\zeta,\tau)\in\Omega \\ (z,t)\neq(\zeta,\tau)}} \frac{|u(z,t)-u(\zeta,\tau)|}{(|z-\zeta|^2+|t-\tau|)^{\alpha/2}}, \quad [u]_{C_t^{0,\alpha}(\Omega)} := \sup_{\substack{(z,t),(z,\tau)\in\Omega \\ t\neq\tau}} \frac{|u(z,t)-u(z,\tau)|}{|t-\tau|^{\alpha}},$$

and the $C_p^{0,\alpha}$ norm

$$\|u\|_{C_p^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + [u]_{C_p^{0,\alpha}(\Omega)}.$$

For every multi-index $\beta \in \mathbb{N}^{d+1}$ and $k \geq 1$, we define the seminorms

$$[u]_{C_p^{k,\alpha}(\Omega)} := \sum_{|\beta|+2j=k} [\partial_x^\beta \partial_t^j u]_{C_p^{0,\alpha}(\Omega)} + [u]_{C_t^{k-1,\frac{1+\alpha}{2}}(\Omega)}, \quad [u]_{C_t^{k,\frac{1+\alpha}{2}}(\Omega)} := \sum_{|\beta|+2j=k} [\partial_x^\beta \partial_t^j u]_{C_t^{0,\frac{1+\alpha}{2}}(\Omega)},$$

and the norm

$$\|u\|_{C_p^{k,\alpha}(\Omega)} := \sum_{|\beta|+2j\leq k} \sup_{\Omega} |\partial_x^\beta \partial_t^j u| + [u]_{C_p^{k,\alpha}(\Omega)}.$$

Hence, we set

$$C_p^{k,\alpha}(\Omega) := \{u : \Omega \to \mathbb{R} : \|u\|_{C_p^{k,\alpha}(\Omega)} < \infty\}.$$

Finally, we recall some useful interpolation inequalities in parabolic Hölder spaces (for the elliptic version of this results we refer to [75, 69]).

Lemma 1.2.1 ([96, Proposition 4.2]). *Let $d \geq 1$, and $0 < \beta < \alpha \leq 1$. Then, for every $\varepsilon > 0$ there exists $C > 0$ depending only on d and ε such that*

$$\begin{aligned} &\|u\|_{C_p^{0,\beta}(\Omega)} \leq C\|u\|_{L^\infty(\Omega)} + \varepsilon\|u\|_{C_p^{0,\alpha}(\Omega)}, \\ &\|\nabla u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^\infty(\Omega)} + \varepsilon[u]_{C_p^{1,\alpha}(\Omega)}, \\ &\|D^2u\|_{L^\infty(\Omega)} + \|\partial_t u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^\infty(\Omega)} + \varepsilon[u]_{C_p^{2,\alpha}(\Omega)}. \end{aligned} \tag{1.13}$$

We present two auxiliary results on parabolic Hölder functions, which extend to the parabolic setting the elliptic counterparts given in [143, Lemma 2.3, Lemma 2.4, Remark 2.5] and [138, Theorem 7.5].

Lemma 1.2.2. *Let $k \in \mathbb{N}$ and let $v \in C_p^{k+1,\alpha}(Q_1^+)$ such that $v(x,0,t) \equiv 0$. Then $v/y \in C_p^{k,\alpha}(Q_1^+)$ and $[v/y]_{C_p^{k,\alpha}(Q_1^+)} \leq [v]_{C_p^{k+1,\alpha}(Q_1^+)}$.*

Proof. The proof follows its elliptic counterpart and we skip it. $\square$

Lemma 1.2.3. *Let $a>-1$, $k\in\mathbb{N}$, $\alpha\in(0,1)$ and let $g\in C_p^{k,\alpha}(Q_1^+)$. Then the function*

$$\varphi(x,y,t)=\frac{1}{y^{1+a}}\int_0^y s^a g(x,s,t)ds$$

belongs to $C_p^{k,\alpha}(Q_1^+)$ and $[\varphi]_{C_p^{k,\alpha}(Q_1^+)}\le C[g]_{C_p^{k+1,\alpha}(Q_1^+)}$, for some $C>0$ depending only on a. Moreover, the function

$$\psi(x,y,t)=\frac{1}{y^{a}}\int_0^y s^a g(x,s,t)ds$$

satisfies $\partial_y\psi\in C_p^{k,\alpha}(Q_1^+)$ and $[\partial_y\psi]_{C_p^{k,\alpha}(Q_1^+)}\le C[g]_{C_p^{k+1,\alpha}(Q_1^+)}$, for some $C>0$ depending only on a.

Proof. First, notice that the second statement follows immediately from the first since $\partial_y\psi=-a\varphi+g$.

We prove the first statement by induction. Let $k=0$ and $g\in C_p^{0,\alpha}(Q_1^+)$. The parabolic Hölder continuity in x and t is trivially verified. Indeed, let $P_1=(x_1,y,t_1)$ and $P_2=(x_2,y,t_2)$, then

$$|\varphi(P_2)-\varphi(P_1)|\le\frac{1}{y^{1+\alpha}}\int_0^y s^a|g(x_1,s,t_1)-g(x_2,s,t_2)|ds\le\frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{1+a}d_p(P_2,P_1)^\alpha.$$

For $\delta>0$, let us consider

$$S_1:=\{(y_1,y_2):0<y_1<y_2\le 1,\ \text{and } y_2-y_1\ge\delta y_2\},$$

$$S_2:=\{(y_1,y_2):0<y_1<y_2\le 1,\ \text{and } y_2-y_1<\delta y_2\}.$$

Taking $y_1,y_2\in S_1$, one has

$$\begin{aligned}
&|\varphi(x,y_2,t)-\varphi(x,y_1,t)|=\Big|\frac{1}{y_2^{a+1}}\int_0^{y_2}s^a g(x,s,t)ds-\frac{1}{y_1^{a+1}}\int_0^{y_1}s^a g(x,s,t)ds\Big|\\
&=\Big|\frac{1}{y_2^{a+1}}\int_0^{y_2}s^a(g(x,s,t)-g(x,0,t))ds-\frac{1}{y_1^{a+1}}\int_0^{y_1}s^a(g(x,s,t)-g(x,0,t))ds\Big|\\
&\le\frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{y_2^{a+1}}\int_0^{y_2}s^{a+\alpha}ds+\frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{y_1^{a+1}}\int_0^{y_1}s^{a+\alpha}ds=\frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}(y_2^\alpha+y_1^\alpha)\\
&\le\frac{2[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}y_2^\alpha\le\frac{2[g]_{C_p^{0,\alpha}(Q_1^+)}}{\delta^\alpha(a+\alpha+1)}(y_2-y_1)^\alpha.
\end{aligned}$$

Let now $y_1,y_2\in S_2$. Then,

$$\begin{aligned}
&|\varphi(x,y_2,t)-\varphi(x,y_1,t)|\\
&=\Big|\frac{1}{y_2^{a+1}}\int_0^{y_2}s^a(g(x,s,t)-g(x,0,t))ds-\frac{1}{y_1^{a+1}}\int_0^{y_1}s^a(g(x,s,t)-g(x,0,t))ds\Big|\\
&\le\frac{1}{y_2^{a+1}}\int_{y_1}^{y_2}s^a|g(x,s,t)-g(x,0,t)|ds+\Big(\frac{1}{y_1^{a+1}}-\frac{1}{y_2^{a+1}}\Big)\int_0^{y_1}s^a|g(x,s,t)-g(x,0,t)|ds\\
&\le\frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(\frac{y_2^{a+\alpha+1}-y_1^{a+\alpha+1}}{y_2^{a+1}}+\Big(\frac{1}{y_1^{a+1}}-\frac{1}{y_2^{a+1}}\Big)y_1^{a+\alpha+1}\Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(y_2^\alpha - y_1^\alpha\Big(\frac{y_1}{y_2}\Big)^{a+1} + y_1^\alpha\Big(1-\Big(\frac{y_1}{y_2}\Big)^{a+1}\Big)\Big)\\
&= \frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(y_2^\alpha + y_1^\alpha - 2y_1^\alpha\Big(\frac{y_1}{y_2}\Big)^{a+1}\Big)\\
&= \frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(y_2^\alpha - y_1^\alpha + 2y_1^\alpha\Big(1-\Big(\frac{y_1}{y_2}\Big)^{a+1}\Big)\Big)\\
&\le \frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(y_2^\alpha - y_1^\alpha + C_a y_1^\alpha\Big(1-\frac{y_1}{y_2}\Big)\Big),
\end{aligned}$$

where $C_a > 0$ is a constant which depends only on a. Consequently, since by definition $y_1/y_2 > 1-\delta$, we have

$$\begin{aligned}
\frac{|\varphi(x,y_2,t)-\varphi(x,y_1,t)|}{(y_2-y_1)^\alpha} &\le \frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(\frac{y_2^\alpha - y_1^\alpha}{(y_2-y_1)^\alpha} + C_a\frac{y_1^\alpha(y_2-y_1)}{y_2(y_2-y_1)^\alpha}\Big)\\
&\le \frac{[g]_{C_p^{0,\alpha}(Q_1^+)}}{a+\alpha+1}\Big(1 + C_a\delta^{1-\alpha}\Big),
\end{aligned}$$

and hence, the case $k=0$ follows.

Next, let us assume that the our claim is true for some $k\in\mathbb{N}$ and let us prove it for $k+1$: we assume $g\in C_p^{k+1,\alpha}(Q_1^+)$ and show that $\varphi\in C_p^{k+1,\alpha}(Q_1^+)$.

Since

$$\partial_{x_i}\varphi = \frac{1}{y^{1+a}}\int_0^y s^a\partial_{x_i}g(x,s,t)ds, \quad i=1,\dots,d, \quad \partial_t\varphi = \frac{1}{y^{1+a}}\int_0^y s^a\partial_t g(x,s,t)ds,$$

we immediately have that φ is C^{k+1} in x and t. Moreover, the boundedness of the $C_t^{\frac{1+\alpha}{2}}$-seminorm of the mixed-derivates follows as the case $k=0$.

We are left to prove that $\partial_y\varphi\in C_p^{k,\alpha}(Q_1^+)$. To do this, we can rewrite φ as

$$\varphi(x,y,t) = \frac{1}{y^{1+a}}\int_0^y s^a(g(x,s,t)-g(x,0,t))ds + \frac{g(x,0,t)}{a+1},$$

and observe that

$$\partial_y\varphi(x,y,t) = -\frac{a+1}{y^{2+a}}\int_0^y s^{a+1}\frac{g(x,s,t)-g(x,0,t)}{s}ds + \frac{g(x,y,t)-g(x,0,t)}{y}.$$

By Lemma 1.2.2, one has that $\frac{g(x,y,t)-g(x,0,t)}{y}\in C_p^{k,\alpha}(Q_1^+)$ and our claim follows by the inductive assumption. □

1.2.2 Sobolev spaces

In this section, we introduce weighted Sobolev spaces that do not depend on the t-variable, along with some basic properties of these spaces.

We define the following spaces of smooth functions.

$$\begin{aligned}
C^\infty(B_r) &:= \{u_{|B_r} : u \in C^\infty(\mathbb{R}^{d+1})\},\\
C^\infty_c(B_r) &:= \{u_{|B_r} : u \in C^\infty(\mathbb{R}^{d+1}) \text{ and } \operatorname{spt}(u) \subset\subset B_r\},\\
C^\infty_c(\overline{B_r} \setminus \Sigma) &:= \{u_{|B_r} : u \in C^\infty(\mathbb{R}^{d+1}) \text{ and } \operatorname{spt}(u) \subset\subset \overline{B_r} \setminus \Sigma\},\\
C^\infty_c(B_r \setminus \Sigma) &:= \{u_{|B_r} : u \in C^\infty(\mathbb{R}^{d+1}) \text{ and } \operatorname{spt}(u) \subset\subset B_r \setminus \Sigma\},
\end{aligned}$$

Next, we introduce weighted the L^p-space. For $r > 0$ and $p \geq 1$ we set

$$L^p(B_r, \rho^a_\varepsilon) := \{u : B_r \to \mathbb{R} \text{ measurable: } \int_{B_r} \rho^a_\varepsilon |u|^p dz < \infty\},$$

equipped with the norm

$$\|u\|_{L^p(B_r,\rho^a_\varepsilon)} := \Big(\int_{B_r} \rho^a_\varepsilon |u|^p dz\Big)^{1/p}.$$

For fields, we define

$$L^p(B_r, \rho^a_\varepsilon)^{d+1} := \{U : B_r \to \mathbb{R}^{d+1} \text{ measurable: } \int_{B_r} \rho^a_\varepsilon |U|^p dz < \infty\},$$

normed by $\|U\|_{L^p(B_r,\rho^a_\varepsilon)^{d+1}} := \||U|\|_{L^p(B_r,\rho^a_\varepsilon)}$.

Following the definition in [118] (see also [138]), we define the Sobolev space $H^1(B_r, \rho^a_\varepsilon)$ as the completion of $C^\infty(\overline{B}_r)$ with respect to the norm

$$\|v\|_{H^1(B_r,\rho^a_\varepsilon)} = \Big(\int_{B_r} \rho^a_\varepsilon v^2 dz + \int_{B_r} \rho^a_\varepsilon |\nabla v|^2 dz\Big)^{1/2}, \tag{1.14}$$

while the space $H^1_0(B_r, \rho^a_\varepsilon)$ is the completion of $C^\infty_c(B_r)$ with respect to the seminorm

$$\|v\|_{H^1_0(B_r,\rho^a_\varepsilon)} = \Big(\int_{B_r} \rho^a_\varepsilon |\nabla v|^2 dz\Big)^{1/2}. \tag{1.15}$$

We denote by $H^{-1}(B_r, \rho^a_\varepsilon)$ the topological dual space of $H^1_0(B_r, \rho^a_\varepsilon)$. When $\varepsilon = 0$, so that $\rho^a_0 = |y|^a$, we write $H^1(B_r, |y|^a)$ and $H^1_0(B_r, |y|^a)$, instead of $H^1(B_r, \rho^a_0)$ and $H^1_0(B_r, \rho^a_0)$, respectively.

Let us make some remarks on how the weight affects the nature of the spaces defined above. For a fixed $\varepsilon \in (0, 1)$, the weight ρ^a_ε is bounded both from below and above (with constants depending on ε). Therefore, the space $H^1(B_r, \rho^a_\varepsilon)$ is equivalent to the standard Sobolev space $H^1(B_r)$. On the other hand, as observed in [138], when $\varepsilon = 0$, the nature of these spaces is intrinsically related to the degeneracy or singularity of the weight $|y|^a$, which may vanish or explode on Σ, depending on whether $a > 0$ or $a < 0$, respectively. Heuristically, when $a \leq -1$ the weight $|y|^a$ is not locally integrable and thus the functions in $H^1(B_r, |y|^a)$ are forced to have zero trace on Σ. Conversely, when $a \geq 1$, the weight has a strong degeneracy and the traces on Σ of functions in $H^1(B_r, |y|^a)$ have no sense in general (this is due to the zero $H^1(B_r, |y|^a)$-capacity of Σ). These observations suggest to introduce the space $\tilde{H}^1(B_r, \rho^a_\varepsilon)$, defined as the completion of $C^\infty_c(\overline{B}_r \setminus \Sigma)$ with respect to (1.14) and similarly $\tilde{H}^1_0(B_r, \rho^a_\varepsilon)$ as the completion of $C^\infty_c(B_r \setminus \Sigma)$ with respect to (1.15). As above, when $\varepsilon = 0$, we set $\tilde{H}^1(B_r, |y|^a) := \tilde{H}^1(B_r, \rho^a_0)$ and $\tilde{H}^1_0(B_r, |y|^a) := \tilde{H}^1_0(B_r, \rho^a_0)$.

The Sobolev spaces can be defined using weak derivatives in L^2 spaces, and in the unweighted setting, this definition turns out to be equivalent to the one using the density of smooth functions.

We introduce the space

$$W^{1,2}(B_r,|y|^a) := \{u \in L^2(B_r,|y|^a) : \nabla u \in L^2(B_r,|y|^a)^{d+1}\},$$

where ∇u denotes the weak gradient of u, in the sense that $u \in W^{1,1}_{\mathrm{loc}}(B_r)$.

These considerations are summarized in the following proposition, which characterizes the space $H^1(B_r,|y|^a)$, in the various ranges of the parameter $a \in \mathbb{R}$.

Proposition 1.2.4 ([89, Theorem 2.5], [138, Proposition 2.2])**.** *If $a \in (-1,1)$, then:*

$$H^1(B_r,|y|^a) = W^{1,2}(B_r,|y|^a).$$

If $a \in (-\infty,-1] \cup [1,\infty)$, then:

$$\begin{aligned} H^1(B_r,|y|^a) &= \tilde{H}^1(B_r,|y|^a),\\ H^1_0(B_r,|y|^a) &= \tilde{H}^1_0(B_r,|y|^a); \end{aligned}$$

in particular, $C^\infty_c(\overline{B}_r \setminus \Sigma)$ is dense in $H^1(B_r,|y|^a)$ and $C^\infty_c(B_r \setminus \Sigma)$ is dense in $H^1_0(B_r,|y|^a)$.

The spaces introduced above enjoy interesting Sobolev embedding properties, depending on the value of the parameter a.

Theorem 1.2.5 ([79, Theorem 6], [138, Theorem 2.4])**.** *Let $\varepsilon \in [0,1)$ and $r \in [1/2,1]$. Assume either $a > -1$ and $d \geq 2$, or $a > 0$ and $d = 1$. Then there exists $C > 0$ depending only on d and a such that*

$$\Big(\int_{B_r} \rho_\varepsilon^a |u|^{2^*_a} dz\Big)^{2/2^*_a} \leq C\Big(\int_{B_r} \rho_\varepsilon^a u^2 dz + \int_{B_r} \rho_\varepsilon^a |\nabla u|^2 dz\Big),$$

for every $u \in H^1(B_r,\rho_\varepsilon^a)$, where

$$2^*_a := \frac{2(d+1+a^+)}{d+a^+-1}.$$

Further, if $d = 1$ and $a \in (-1,0]$, the above inequality holds with 2^*_a replaced with any $p \in [1,\infty)$ and a constant $C > 0$ depending only on d, a and p.

Theorem 1.2.6 ([138, Theorem 2.5], and [139, Lemma B.5])**.** *Let $a \leq -1$, $d \geq 2$, $\varepsilon \in [0,1)$ and $r \in [1/2,1]$. Then there exists a constant $C > 0$ depending only on d and a such that*

$$\Big(\int_{B_r} (\rho_\varepsilon^a)^{2^*/2} |u|^{2^*} dz\Big)^{2/2^*} \leq C\Big(\int_{B_r} \rho_\varepsilon^a u^2 dz + \int_{B_r} \rho_\varepsilon^a |\nabla u|^2 dz\Big), \tag{1.16}$$

for every $u \in \tilde{H}^1(B_r,\rho_\varepsilon^a)$, where

$$2^* := \frac{2(d+1)}{d-1}.$$

Moreover, the inequality (1.16) implies that

$$\Big(\int_{B_r} \rho_\varepsilon^a |u|^{2^*} dz\Big)^{2/2^*} \leq C\Big(\int_{B_r} \rho_\varepsilon^a u^2 dz + \int_{B_r} \rho_\varepsilon^a |\nabla u|^2 dz\Big).$$

Further, when $d = 1$, the above inequalities hold with 2^* replaced by any $p \in [1,\infty)$ and a constant $C > 0$ depending only on d, a, p.

Remark 1.2.7. It is worth mentioning that the theorem above (range $a \leq -1$) follows as a consequence of a fine analysis of the isometry

$$T_\varepsilon^a : \tilde{H}^1(B_r, \rho_\varepsilon^a) \to \tilde{H}^1(B_r) \qquad u \to v := \sqrt{\rho_\varepsilon^a} u, \tag{1.17}$$

where $\tilde{H}^1(B_r)$ is the completion of $C_c^\infty(\overline{B}_r \setminus \Sigma)$ with respect to the norm

$$Q_\varepsilon(v) = \int_{B_r} |\nabla v|^2 + \int_{B_r} \Big[\Big(\frac{\partial_y \rho_\varepsilon^a}{2\rho_\varepsilon^a}\Big)^2 + \partial_y\Big(\frac{\partial_y \rho_\varepsilon^a}{2\rho_\varepsilon^a}\Big)\Big] v^2 - \int_{\partial B_r} \frac{\partial_y \rho_\varepsilon^a}{2\rho_\varepsilon^a} y v^2,$$

which turns out to be equivalent to the classical $H^1(B_r)$-norm, uniformly in $\varepsilon \in [0,1)$ (see [139, Lemma B.1]). This fact allows to apply the classical Sobolev inequality to $v = T_\varepsilon^a u$ and recover (1.16) in terms of u.

Remark 1.2.8. Notice that the definitions and theorems above hold true replacing the ball with any open bounded domain Ω, including the case of the half balls B_r^+.

Remark 1.2.9. Let $\Omega \subset \mathbb{R}^{d+1}$ be an open bounded set such that $\Omega \subset\subset \mathbb{R}^{d+1} \setminus \Sigma$ and, for every $\varepsilon \in [0,1)$, let $H_0^1(\Omega, \rho_\varepsilon^a)$ be the completion of $C_c^\infty(\Omega)$ with respect to the seminorm

$$\|u\|_{H_0^1(\Omega,\rho_\varepsilon^a)} := \Big(\int_\Omega \rho_\varepsilon^a |\nabla u|^2 dz\Big)^{1/2}.$$

Then, for every $\varepsilon \in [0,1)$,

$$H_0^1(\Omega, \rho_\varepsilon^a) = H_0^1(\Omega). \tag{1.18}$$

Indeed, $\mathrm{dist}(\Omega, \Sigma) \geq \delta$ for some $\delta > 0$ depending only on Ω, and thus $\delta \leq \rho_\varepsilon^a \leq \delta^{-1}$ in $\mathbb{R}^{d+1}$ uniformly in ε, up to taking δ smaller. This shows that $\|\cdot\|_{H_0^1(\Omega,\rho_\varepsilon^a)} \sim \|\cdot\|_{H_0^1(\Omega)}$ which, in turn, readily implies (1.18) by the definition of $H_0^1(\Omega)$ and $H_0^1(\Omega, \rho_\varepsilon^a)$.

1.2.3 Sobolev spaces involving time

Let $a \in \mathbb{R}$, $\varepsilon \in [0,1)$ and $p, q \in [1, \infty)$. We define

$$L^q(I_r; L^p(B_r, \rho_\varepsilon^a)) := \{u : I_r \to L^p(B_r, \rho_\varepsilon^a) \text{ measurable: } \int_{I_r} \|u(t)\|_{L^p(B_r,\rho_\varepsilon^a)}^q dt < \infty\},$$

equipped with the norm

$$\|u\|_{L^q(I_r;L^p(B_r,\rho_\varepsilon^a))} := \Big(\int_{I_r} \|u(t)\|_{L^p(B_r,\rho_\varepsilon^a)}^q dt\Big)^{1/q}.$$

The special case $p = q$ is the most relevant for our results. In such case, we set $L^p(Q_r, \rho_\varepsilon^a) := L^p(I_r; L^p(B_r, \rho_\varepsilon^a))$ and

$$\|u\|_{L^p(Q_r,\rho_\varepsilon^a)} := \|u\|_{L^p(I_r;L^p(B_r,\rho_\varepsilon^a))} = \Big(\int_{Q_r} \rho_\varepsilon^a |u|^p dz dt\Big)^{1/p}.$$

Similarly, for fields we define

$$L^q(I_r; L^p(B_r, \rho_\varepsilon^a)^{d+1}) := \{U : I_r \to L^p(B_r, \rho_\varepsilon^a)^{d+1} \text{ measurable: } \int_{I_r} \|U(t)\|_{L^p(B_r,\rho_\varepsilon^a)^{d+1}}^q dt < \infty\},$$

normed by

$$\|U\|_{L^q(I_r;L^p(B_r,\rho_\varepsilon^a)^{d+1})} := \Big(\int_{I_r} \|U(t)\|^q_{L^p(B_r,\rho_\varepsilon^a)^{d+1}}dt\Big)^{1/q}.$$

As above, when $p=q$, we set $L^p(Q_r,\rho_\varepsilon^a)^{d+1} := L^p(I_r;L^p(B_r,\rho_\varepsilon^a)^{d+1})$ and

$$\|U\|_{L^p(Q_r,\rho_\varepsilon^a)^{d+1}} := \|U\|_{L^p(I_r;L^p(B_r,\rho_\varepsilon^a)^{d+1})} = \Big(\int_{Q_r} \rho_\varepsilon^a|U|^p dzdt\Big)^{1/p}.$$

We set

$$L^\infty(I_r;L^p(B_r,\rho_\varepsilon^a)) := \{u : I_r \to L^p(B_r,\rho_\varepsilon^a) \text{ measurable: } \operatorname*{ess\,sup}_{t\in I_r} \|u(t)\|_{L^p(B_r,\rho_\varepsilon^a)} < \infty\},$$

equipped with the norm

$$\|u\|_{L^\infty(I_r;L^p(B_r,\rho_\varepsilon^a))} := \operatorname*{ess\,sup}_{t\in I_r} \|u(t)\|_{L^p(B_r,\rho_\varepsilon^a)},$$

and

$$C(\overline{I}_r;L^p(B_r,\rho_\varepsilon^a)) := \{u : \overline{I}_r \to L^p(B_r,\rho_\varepsilon^a) \text{ continuous: } \max_{t\in \overline{I}_r} \|u(t)\|_{L^p(B_r,\rho_\varepsilon^a)} < \infty\},$$

normed by

$$\|u\|_{C(\overline{I}_r;L^p(B_r,\rho_\varepsilon^a))} := \max_{t\in \overline{I}_r} \|u(t)\|_{L^p(B_r,\rho_\varepsilon^a)}.$$

Analogously to the previous section, we define the spaces of smooth functions $C^\infty(Q_r)$, $C^\infty_c(Q_r)$, $C^\infty_c(\overline{Q_r}\setminus\Sigma)$, and $C^\infty_c(Q_r\setminus\Sigma)$. The space $L^2(I_r;H^1(B_r,\rho_\varepsilon^a))$ is defined as the completion of $C^\infty(Q_r)$ with respect to the norm

$$\|u\|_{L^2(I_r;H^1(B_r,\rho_\varepsilon^a))} := \Big(\int_{I_r} \|u(t)\|^2_{H^1(B_r,\rho_\varepsilon^a)}dt\Big)^{1/2} = \Big(\int_{Q_r} \rho_\varepsilon^a u^2 dzdt + \int_{Q_r} \rho_\varepsilon^a|\nabla u|^2 dzdt\Big)^{1/2}, \tag{1.19}$$

while $L^2(I_r;H^1_0(B_r,\rho_\varepsilon^a))$ is the completion of $C^\infty_c(Q_r)$ with respect to the seminorm

$$\|u\|_{L^2(I_r;H^1_0(B_r,\rho_\varepsilon^a))} := \Big(\int_{I_r} \|u(t)\|^2_{H^1_0(B_r,\rho_\varepsilon^a)}dt\Big)^{1/2} = \Big(\int_{Q_r} \rho_\varepsilon^a|\nabla u|^2 dzdt\Big)^{1/2}.$$

Notice that by the Riesz's representation theorem, the topological dual space of $L^2(I_r;H^1_0(B_r,\rho_\varepsilon^a))$ satisfies

$$L^2(I_r;H^1_0(B_r,\rho_\varepsilon^a))^\star = L^2(I_r;H^{-1}(B_r,\rho_\varepsilon^a)).$$

Remark 1.2.10. Later on we will use the following classical fact, see [99, Proposition 2.1, Theorem 3.1]: there exists $C>0$ depending only on r, such that

$$\|u\|_{C(\bar{I}_r;L^2(B_r,\rho_\varepsilon^a))} \le C\big(\|u\|_{L^2(\overline{I}_r;H^1_0(B_r,\rho_\varepsilon^a))} + \|\partial_t u\|_{L^2(I_r;H^{-1}(B_r,\rho_\varepsilon^a))}\big),$$

where $\partial_t u$ denotes the weak time derivative of u. That is, if $u\in L^2(I_r;H^1_0(B_r,\rho_\varepsilon^a))$ and $\partial_t u \in L^2(I_r;H^{-1}(B_r,\rho_\varepsilon^a))$, then $u\in C(\overline{I}_r;L^2(B_r,\rho_\varepsilon^a))$.

Exploiting the Sobolev inequalities above, one can prove their parabolic versions (see for instance [36], or the more recent [6]).

Theorem 1.2.11. *Let $\varepsilon\in[0,1)$ and $r\in[1/2,1]$. Assume either $a>-1$ and $d\geq 2$, or $a>0$ and $d=1$. Then there exists $C>0$ depending only on d and a such that*

$$\int_{Q_r}\rho_\varepsilon^a|u|^{2\gamma}dzdt\leq C\Big(\int_{Q_r}\rho_\varepsilon^a\Big(u^2+|\nabla u|^2\Big)dzdt\Big)\operatorname*{ess\,sup}_{t\in I_r}\Big(\int_{Q_r}\rho_\varepsilon^a u^2dzdt\Big)^{\gamma-1},\tag{1.20}$$

for every $u\in L^2(I_r;H^1(B_r,\rho_\varepsilon^a))$, where

$$\gamma:=2\cdot\frac{2_a^*-1}{2_a^*}.$$

Further, if $d=1$ and $a\in(-1,0]$, the above inequality holds with γ replaced with any $p\in[1,2)$ and a constant $C>0$ depending only on d, a and p.

The same isometry in (1.17) can be easily extended to the parabolic setting as a map

$$\overline{T}_\varepsilon^a:L^2(I_r;\tilde{H}^1(B_r,\rho_\varepsilon))\to L^2(I_r;\tilde{H}^1(B_r))\qquad u\to v:=\sqrt{\rho_\varepsilon^a}u.\tag{1.21}$$

Notice that $\overline{T}_\varepsilon^a$ is still an isometry if $L^2(I_r;\tilde{H}^1(B_r))$ is normed by

$$\|v\|_{L^2(I_r;\tilde{H}^1(B_r))}=\Big(\int_{I_r}Q_\varepsilon(v(t))dt\Big)^{1/2},$$

and $L^2(I_r;\tilde{H}^1(B_r,\rho_\varepsilon))$ stands for the completion of $C_c^\infty(\overline{Q}_r\setminus\Sigma)$ with respect to (1.19). Working as in the stationary (time-independent) framework, one can see that such a norm is equivalent to the standard $L^2(I_r;H^1(B_r))$-norm, uniformly in $\varepsilon\in[0,1)$. As a consequence, we obtain the following Sobolev embeddings when $a\leq-1$.

Theorem 1.2.12. *Let $a\leq-1$, $d\geq2$, $\varepsilon\in[0,1)$, $r\in[1/2,1]$. Then there exists $C>0$ depending only on d and a such that*

$$\int_{Q_r}\rho_\varepsilon^a|u|^{2\gamma}dzdt\leq C\Big(\int_{Q_r}\rho_\varepsilon^a\Big(u^2+|\nabla u|^2\Big)dzdt\Big)\operatorname*{ess\,sup}_{t\in I_r}\Big(\int_{Q_r}\rho_\varepsilon^a u^2dzdt\Big)^{\gamma-1},$$

for every $u\in L^2(I_r;\tilde{H}^1(B_r,\rho_\varepsilon^a))$, where

$$\gamma:=2\cdot\frac{2^*-1}{2^*}.$$

Further, if $d=1$, the above inequality holds with γ replaced with any $p\in[1,2)$ and a constant $C>0$ depending only on d, a and p.

Remark 1.2.13. As explained in Remark 1.2.8, one can define the spaces $L^q(I_r;L^p(B_r^+,\rho_\varepsilon^a))$, $L^q(I_r;L^p(B_r^+,\rho_\varepsilon^a)^{d+1})$ and $C(\overline{I}_r;L^p(B_r^+,\rho_\varepsilon^a))$, and the Sobolev spaces $L^2(I_r;H^1(B_r^+,\rho_\varepsilon^a))$ and $L^2(I_r;\tilde{H}^1(B_r^+,\rho_\varepsilon^a))$, by considering the upper half cylinder Q_r^+ instead of Q_r.

1.2.4 Weak solutions

The energy spaces introduced above allow us to give the notion of weak solutions for our class of problems. Before that, we introduce the space of test functions we will use in the definitions below: such space takes into account the integrability/non-integrability of the weight $|y|^a$ when $\varepsilon=0$.

Definition 1.2.14. Let $a \in \mathbb{R}$, $r > 0$ and $\varepsilon \in [0,1)$. We define

$$\mathcal{D}_c^\infty(Q_r) := \begin{cases} C_c^\infty(Q_r) & \text{if either } \varepsilon \in (0,1), \text{ or } \varepsilon = 0 \text{ and } a \in (-1,1) \\ C_c^\infty(Q_r \setminus \Sigma) & \text{if } \varepsilon = 0 \text{ and } a \in (-\infty,1] \cup [1,\infty). \end{cases}$$

Notice that, in light of Proposition 1.2.4, the set $\mathcal{D}_c^\infty(Q_r)$ is dense in $L^2(I_r; H_0^1(B_r, \rho_\varepsilon^a))$ for every $\varepsilon \in [0,1)$.

Definition 1.2.15. Let $a \in \mathbb{R}$, $r > 0$, $\varepsilon \in [0,1)$ and $f \in L^2(Q_r, \rho_\varepsilon^a)$, $F \in L^2(Q_r, \rho_\varepsilon^a)^{d+1}$. We say that u is a weak solution to

$$\rho_\varepsilon^a \partial_t u - \operatorname{div}(\rho_\varepsilon^a A \nabla u) = \rho_\varepsilon^a f + \operatorname{div}(\rho_\varepsilon^a F) \quad \text{in } Q_r, \tag{1.22}$$

if $u \in L^2(I_r; H^1(B_r, \rho_\varepsilon^a)) \cap L^\infty(I_r; L^2(B_r, \rho_\varepsilon^a))$ and satisfies

$$-\int_{Q_r} \rho_\varepsilon^a u \partial_t \phi dz dt + \int_{Q_r} \rho_\varepsilon^a A \nabla u \cdot \nabla \phi dz dt = \int_{Q_r} \rho_\varepsilon^a (f\phi - F \cdot \nabla \phi) dz dt, \tag{1.23}$$

for every $\phi \in \mathcal{D}_c^\infty(Q_r)$. We say that u is an entire solution to

$$\rho_\varepsilon^a \partial_t u - \operatorname{div}(\rho_\varepsilon^a A \nabla u) = \rho_\varepsilon^a f + \operatorname{div}(\rho_\varepsilon^a F) \quad \text{in } \mathbb{R}^{d+1} \times \mathbb{R},$$

if, for every $r > 0$, u is a weak solution to (1.22).

Next, we provide the definition of solutions that satisfy a zero boundary condition on the lateral boundary $\partial B_r \times I_r$ and assume a prescribed initial value at $t = -r^2$.

Definition 1.2.16. Let $a \in \mathbb{R}$, $r > 0$, $\varepsilon \in [0,1)$ and $f \in L^2(Q_r, \rho_\varepsilon^a)$, $F \in L^2(Q_r, \rho_\varepsilon^a)^{d+1}$, $u_0 \in L^2(B_r, \rho_\varepsilon^a)$. We say that u is a weak solution to

$$\begin{cases} \rho_\varepsilon^a \partial_t u - \operatorname{div}(\rho_\varepsilon^a A \nabla u) = \rho_\varepsilon^a f + \operatorname{div}(\rho_\varepsilon^a F) & \text{in } Q_r \\ u = 0 & \text{in } \partial B_r \times I_r \\ u = u_0 & \text{in } B_r \times \{-r^2\}, \end{cases} \tag{1.24}$$

if $u \in L^2(I_r; H_0^1(B_r, \rho_\varepsilon^a)) \cap L^\infty(I_r; L^2(B_r, \rho_\varepsilon^a))$, satisfies (1.23) for every $\phi \in \mathcal{D}_c^\infty(Q_r)$ and $u(-r^2) = u_0$ in $L^2(B_r, \rho_\varepsilon^a)$.

Remark 1.2.17. Let $\varepsilon \in [0,1)$ and let u be a weak solution to (1.24). Then, by the Hölder inequality, (1.2) and the Poincaré inequality (for the degenerate/singular case we refer to [85, Lemma 3.2], [139, Lemma B.5] and [65, Theorem 1.3]), we have

$$-\int_{Q_r} \rho_\varepsilon^a u \partial_t \phi dz dt \le C\big(\Lambda \|u\|_{L^2(I_r; H_0^1(B_r, \rho_\varepsilon^a))} + \|f\|_{L^2(Q_r, \rho_\varepsilon^a)} + \|F\|_{L^2(Q_r, \rho_\varepsilon^a)^{d+1}}\big) \|\phi\|_{L^2(I_r; H_0^1(B_r, \rho_\varepsilon^a))}$$

for every $\phi \in \mathcal{D}_c^\infty(Q_r)$, for some $C > 0$ depending on d, a and ε. Consequently, a standard density argument, shows that the distribution

$$\langle \partial_t u, \phi \rangle := -\int_{Q_r} \rho_\varepsilon^a u \partial_t \phi dz dt, \quad \phi \in L^2(I_r; H_0^1(B_r, \rho_\varepsilon^a))$$

is well-defined and $\partial_t u \in L^2(I_r; H^{-1}(B_r, \rho_\varepsilon^a))$. In particular, $u \in C(\overline{I}_r; L^2(B_r, \rho_\varepsilon^a))$ by Remark 1.2.10 and thus the equation $u(-r^2) = u_0$ in $L^2(B_r, \rho_\varepsilon^a)$ makes sense.

Finally, we provide the definition of solutions on the upper half-cylinder, which satisfy a conormal boundary condition on $\partial^0 Q_r^+ = Q_r \cap \{y=0\}$.

Definition 1.2.18. Let $a>-1$, $r>0$, $\varepsilon\in[0,1)$ and $f\in L^2(Q_r^+,\rho_\varepsilon^a)$, $F\in L^2(Q_r^+,\rho_\varepsilon^a)^{d+1}$. We say that u is a weak solution to

$$\begin{cases}\rho_\varepsilon^a\partial_t u-\mathrm{div}(\rho_\varepsilon^a A\nabla u)=\rho_\varepsilon^a f+\mathrm{div}(\rho_\varepsilon^a F) & \text{in } Q_r^+\\ \rho_\varepsilon^a\Big(A\nabla u+F\Big)\cdot e_{d+1}=0 & \text{in } \partial^0 Q_r^+,\end{cases}\tag{1.25}$$

if $u\in L^2(I_r;H^1(B_r^+,\rho_\varepsilon^a))\cap L^\infty(I_r;L^2(B_r^+,\rho_\varepsilon^a))$ and satisfies

$$-\int_{Q_r^+}\rho_\varepsilon^a u\partial_t\phi dzdt+\int_{Q_r^+}\rho_\varepsilon^a A\nabla u\cdot\nabla\phi dzdt=\int_{Q_r^+}\rho_\varepsilon^a(f\phi-F\cdot\nabla\phi)dzdt,$$

for every $\phi\in C_c^\infty(Q_r)$. We say that u is an entire solution to

$$\begin{cases}\rho_\varepsilon^a\partial_t u-\mathrm{div}(\rho_\varepsilon^a A\nabla u)=\rho_\varepsilon^a f+\mathrm{div}(\rho_\varepsilon^a F) & \text{in } \mathbb{R}^{d+1}_+\times\mathbb{R}\\ \rho_\varepsilon^a\Big(A\nabla u+F\Big)\cdot e_{d+1}=0 & \text{in } \partial\mathbb{R}^{d+1}_+\times\mathbb{R},\end{cases}$$

if, for every $r>0$, u is a weak solution to (1.25).

Remark 1.2.19. A key tool in the study of weak solutions are the Steklov averages, defined as

$$u_h(z,t):=\frac{1}{h}\int_t^{t+h}u(z,s)ds,\quad u_{-h}(z,t):=\frac{1}{h}\int_{t-h}^t u(z,s)ds,$$

where $h>0$ and u is a given function. It is well-known that if $\varepsilon\in(0,1)$, $u\in L^2(I_r;H^1(B_r,\rho_\varepsilon^a))$ and $\delta>0$, then

$$u_h\to u,\quad \nabla u_h\to\nabla u\quad\text{in } L^2(B_r\times(-r^2,r^2-\delta),\rho_\varepsilon^a),$$

as $h\to0$ (see for instance [96, Lemma 3.2, Lemma 3.3]). Furthermore, if u is a weak solution to (1.22), then u_h satisfies

$$\int_{Q_r}\rho_\varepsilon^a(\partial_t u_h\phi+(A\nabla u)_h\cdot\nabla\phi)dzdt=\int_{Q_r}\rho_\varepsilon^a(f_h\phi-F_h\cdot\nabla\phi)dzdt,\tag{1.26}$$

for every $\phi\in C_c^\infty(B_r\times(-r^2,r^2-h))$: the proof is a standard adaptation of the classical framework (see for instance [96, Theorem 6.1]). Similar for the case $\varepsilon=0$ and for weak solutions to (1.24) or (1.25). We quote [91], [96] and the more recent [33] for further properties of Steklov averages.

1.3 Energy estimates I and local boundedness of solutions

In this section we provide some energy type estimates, which allows us to obtain $L^2\to L^\infty$ estimates of weak solutions to (1.22), by employing a standard De Giorgi-Nash-Moser iteration technique (we refer to [113, 36, 147, 13, 6] in a related context). We begin with the following Caccioppoli-type inequality.

Lemma 1.3.1. *Let $a\in\mathbb{R}$, $\varepsilon\in[0,1)$, $p,q\geq2$ and A satisfying (1.2). Let $f\in L^p(Q_1,\rho_\varepsilon^a)$ and $F\in L^q(Q_1,\rho_\varepsilon^a)^{d+1}$ and let u be a weak solution to (1.22). Then there exists $C>0$ depending* *only on d, a, λ and Λ such that for every $1/2 \le r' < r \le 1$ there holds*

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in(-r'^2,r'^2)}\int_{B_{r'}}\rho_\varepsilon^a u^2+\int_{Q_{r'}}\rho_\varepsilon^a|\nabla u|^2\\ &\quad\le C\Big(\frac{1}{(r-r')^2}\int_{Q_r}\rho_\varepsilon^a u^2+\|f\|_{L^p(Q_r,\rho_\varepsilon^a)}\|u\|_{L^{p'}(Q_r,\rho_\varepsilon^a)}+\int_{Q_r}\rho_\varepsilon^a|F|^2\chi_{\{|u|>0\}}\Big).\end{aligned}\tag{1.27}$$

Moreover, for every $\ell\in\mathbb{R}$ and for functions of the form $v:=(u-\ell)_+=\max\{u-\ell,0\}$ and $v:=(u-\ell)_-=\max\{-u+\ell,0\}$, the following inequality holds

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in(-r'^2,r'^2)}\int_{B_{r'}}\rho_\varepsilon^a v^2+\int_{Q_{r'}}\rho_\varepsilon^a|\nabla v|^2\\ &\quad\le C\Big(\frac{1}{(r-r')^2}\int_{Q_r}\rho_\varepsilon^a v^2+\|f\|_{L^p(Q_r,\rho_\varepsilon^a)}\|v\|_{L^{p'}(Q_r,\rho_\varepsilon^a)}+\int_{Q_r}\rho_\varepsilon^a|F|^2\chi_{\{v>0\}}\Big).\end{aligned}\tag{1.28}$$

Proof. To simplify the notation, let $\rho=\rho_\varepsilon^a$. As in [13], we may work with the Steklov average u_h of u and later take the limit as $h\to0$: equivalently, we may assume that $\partial_t u\in L^2(Q_1,\rho)$ and directly work with u which is what we do next.

Fix $1/2\le r'<r<1$. We test the equation of u with η^2u, where η is a smooth cut-off function we will define later. Then:

$$\begin{aligned}&\int_{Q_1}\rho\Big(\frac{\partial_t(u^2\eta^2)}{2}+\eta^2A\nabla u\cdot\nabla u\Big)\\ &\qquad=\int_{Q_1}\rho\Big(\frac{u^2\partial_t(\eta^2)}{2}-2\eta uA\nabla u\cdot\nabla\eta+f\eta^2u-\eta^2F\cdot\nabla u-2\eta uF\cdot\nabla\eta\Big).\end{aligned}$$

By (1.2), the Hölder's and the Young's inequalities, we get

$$\begin{aligned}&\frac12\int_{Q_1}\rho\partial_t(u^2\eta^2)+\lambda\int_{Q_1}\rho\eta^2|\nabla u|^2\\ &\le\frac12\int_{Q_1}\rho u^2\partial_t(\eta^2)+2\Lambda\Big(\int_{Q_1}\rho\eta^2|\nabla u|^2\Big)^{1/2}\Big(\int_{Q_1}\rho u^2|\nabla\eta|^2\Big)^{1/2}+\|\eta f\|_{L^p(Q_1,\rho)}\|\eta u\|_{L^{p'}(Q_1,\rho)}\\ &+\Big(\int_{Q_1}\rho\eta^2|F|^2\chi_{\{|u|>0\}}\Big)^{1/2}\Big(\int_{Q_1}\rho\eta^2|\nabla u|^2\Big)^{1/2}+2\Big(\int_{Q_1}\rho\eta^2|F|^2\chi_{\{|u|>0\}}\Big)^{1/2}\Big(\int_{Q_1}\rho u^2|\nabla\eta|^2\Big)^{1/2}\\ &\le\frac12\int_{Q_1}\rho u^2\partial_t(\eta^2)+\frac{\lambda}{3}\int_{Q_1}\rho\eta^2|\nabla u|^2+\frac{3\Lambda^2}{\lambda}\int_{Q_1}\rho u^2|\nabla\eta|^2+\|\eta f\|_{L^p(Q_1,\rho)}\|\eta u\|_{L^{p'}(Q_1,\rho)}\\ &+\frac{1}{2\lambda}\int_{Q_1}\rho\eta^2|F|^2\chi_{\{|u|>0\}}+\frac{\lambda}{2}\int_{Q_1}\rho\eta^2|\nabla u|^2+\int_{Q_1}\rho\eta^2|F|^2\chi_{\{|u|>0\}}+\int_{Q_1}\rho u^2|\nabla\eta|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac12\int_{Q_1}\rho\partial_t(u^2\eta^2)+\frac16\lambda\int_{Q_1}\rho\eta^2|\nabla u|^2\\ &\qquad\le\frac12\int_{Q_1}\rho u^2\partial_t(\eta^2)+\Big(\frac{3\Lambda^2}{\lambda}+1\Big)\int_{Q_1}\rho u^2|\nabla\eta|^2\\ &\qquad\qquad+\|\eta f\|_{L^p(Q_1,\rho)}\|\eta u\|_{L^{p'}(Q_1,\rho)}+\Big(1+\frac{1}{2\lambda}\Big)\int_{Q_1}\rho\eta^2|F|^2\chi_{\{|u|>0\}}.\end{aligned}\tag{1.29}$$

A standard approximation technique (see [113], [13, Theorem 5.1]) allows us to integrate over a cylinder of the form $B_1 \times (-1, \hat{t})$, for any $\hat{t} \in (-1, 1)$. Now, let $t^* \in (-r', r')$ such that

$$\frac{1}{2} \operatorname*{ess\,sup}_{t \in (-r'^2, r'^2)} \int_{B_{r'}} \rho u^2 \leq \int_{B'_r} \rho u^2(t^*), \tag{1.30}$$

and take the test function $\eta = \psi_1(|z|)\psi_2(t)$, where

$$\psi_1 \equiv 1 \text{ in } B_{r'}, \quad 0 \leq \psi_1 \leq 1 \text{ in } B_1, \quad \operatorname{spt}(\psi_1) = B_r, \quad |\nabla \psi_1| \leq \frac{C}{r - r'}, \tag{1.31}$$

$$\begin{aligned} &\psi_2 \equiv 1 \text{ in } (-r'^2, t^*), \quad 0 \leq \psi_2 \leq 1 \text{ in } (-1, t^*), \\ &\operatorname{spt}(\psi_2) = (-r^2, t^*), \quad |\partial_t \psi_2| \leq \frac{C}{(r - r')^2}. \end{aligned} \tag{1.32}$$

By (1.31) and (1.32) we have

$$\partial_t(\eta^2) + |\nabla \eta|^2 \leq \frac{C}{(r - r')^2},$$

and thus by (1.29), (1.30) and the last inequality we obtain

$$\operatorname*{ess\,sup}_{t \in (-r'^2, r'^2)} \int_{B_{r'}} \rho u^2 \leq C\Big(\frac{1}{(r - r')^2} \int_{Q_r} \rho_\varepsilon^a u^2 + \|f\|_{L^p(Q_r, \rho_\varepsilon^a)} \|u\|_{L^{p'}(Q_r, \rho_\varepsilon^a)} + \int_{Q_r} \rho |F|^2 \chi_{\{|u| > 0\}} \Big).$$

Combining this inequality with (1.29), then (1.20) follows.

To prove (1.28), let $v = (u - k)_+$ and test the equation of u with $\eta^2 v$. Since $\partial_t u = \partial_t v$ and $\nabla u = \nabla v$ on $\{v > 0\}$, we obtain

$$\begin{aligned} &\int_{Q_1} \rho\Big(\partial_t u(\eta^2 v) + \nabla u \cdot \nabla(\eta^2 v) - f(\eta^2 v) + F \cdot \nabla(\eta^2 v) \Big) \\ &= \int_{Q_1} \rho\Big(\partial_t v(\eta^2 v) + \nabla v \cdot \nabla(\eta^2 v) - f(\eta^2 v) + F \cdot \nabla(\eta^2 v) \Big), \end{aligned}$$

and thus, (1.28) follows from the same argument above. The case $v = (u - k)_-$ is analogue, noticing that $\partial_t u = -\partial_t v$ and $\nabla u = -\nabla v$. □

The second step for proving the local boundedness of solutions is the following no-spikes estimate type.

Lemma 1.3.2. *Let $\varepsilon \in [0, 1)$, $a \in \mathbb{R}$, $p > \frac{d + a^+ + 3}{2}$, $q > d + a^+ + 3$ and A satisfying (1.2). Then there exists a constant $\delta \in (0, 1)$, depending on d, a, λ, Λ, p and q, such that if*

$$\|f\|_{L^p(Q_1, \rho_\varepsilon^a)} + \|F\|_{L^q(Q_1, \rho_\varepsilon^a)} \leq 1,$$

and u is a weak solution to (1.22) with

$$\int_{Q_1} \rho_\varepsilon^a (u_+)^2 dz dt \leq \delta,$$

then

$$u \leq 1 \quad \text{in } Q_{1/2}.$$

Proof. Fix $\varepsilon \in [0,1)$, set $\rho = \rho_\varepsilon^a$, and assume either $d \geq 2$ and $a > -1$, or $d = 1$ and $a > 0$ (the other cases are analogous). For every integer $j \geq 0$ define

$$C_j := 1 - 2^{-j}, \quad r_j := \frac{1}{2} + 2^{-j-1}, \quad B_j := B_{r_j}, \quad Q_j := Q_{r_j}.$$

Notice that $C_j \uparrow 1$, $r_j \downarrow 1/2$ as $j \to \infty$, and $r_j - r_{j+1} = 2^{-j-2}$. Define

$$V_j := (u - C_j)_+, \quad E_j := \int_{Q_j} \rho V_j^2 dzdt,$$

and observe that, for every $j \geq 0$, $E_j \leq E_0 \leq \delta$ by assumption. Applying the Caccioppoli inequality (1.28) to V_{j+1}, with $r' = r_{j+1}$ and $r = r_j$ we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{t \in (-r_{j+1}^2, r_{j+1}^2)} \int_{B_{j+1}} \rho V_{j+1}^2 + \int_{Q_{j+1}} \rho |\nabla V_{j+1}|^2 \\ &\qquad \leq C \Big(2^{2j} \int_{Q_j} \rho V_{j+1}^2 + \|V_{j+1}\|_{L^{p'}(Q_j,\rho)} + \int_{Q_r} \rho |F|^2 \chi_{\{V_{j+1}>0\}} \Big). \end{aligned}$$

Consequently, using the Sobolev embedding (1.20) (with $\gamma = 1 + \frac{2}{d+1+a^+}$),

$$\begin{aligned} &\Big(\int_{Q_{j+1}} \rho |V_{j+1}|^{2\gamma} \Big)^{1/\gamma} \\ &\quad \leq C \Big(\int_{Q_{j+1}} \rho V_{j+1}^2 + \int_{Q_{j+1}} \rho |\nabla V_{j+1}|^2 \Big) \Big)^{1/\gamma} \operatorname*{ess\,sup}_{t \in (-r_{j+1}^2, r_{j+1}^2)} \Big(\int_{B_{j+1}} \rho V_{j+1}^2 \Big)^{(\gamma-1)/\gamma} \\ &\quad \leq C \Big(2^{2j} \int_{Q_j} \rho V_{j+1}^2 + \|V_{j+1}\|_{L^{p'}(Q_j,\rho)} + \int_{Q_r} \rho |F|^2 \chi_{\{V_{j+1}>0\}} \Big). \end{aligned} \tag{1.33}$$

Now, by the Hölder inequality

$$E_{j+1} = \int_{Q_{j+1}} \rho V_{j+1}^2 \leq \Big(\int_{Q_{j+1}} \rho |V_{j+1}|^{2\gamma} \Big)^{1/\gamma} \Big(\int_{Q_{j+1}} \rho \chi_{\{V_{j+1}>0\}} \Big)^{1/\gamma'}, \tag{1.34}$$

where $\gamma' = \frac{d+3+a^+}{2}$ is the conjugate exponent of γ and, using the Hölder inequality again, we obtain

$$\begin{aligned} \|V_{j+1}\|_{L^{p'}}(Q_j, \rho) &\leq \Big(\int_{Q_{j+1}} \rho V_{j+1}^2 \Big)^{1/2} \Big(\int_{Q_{j+1}} \rho \chi_{\{V_{j+1}>0\}} \Big)^{(p-2)/2p} \\ &\leq E_j^{1/2} \Big(\int_{Q_{j+1}} \rho \chi_{\{V_{j+1}>0\}} \Big)^{(p-2)/2p} \end{aligned} \tag{1.35}$$

and

$$\begin{aligned} \int_{Q_{j+1}} \rho |F|^2 \chi_{\{V_{j+1}>0\}} &\leq \Big(\int_{Q_{j+1}} \rho |F|^q \Big)^{2/q} \Big(\int_{Q_{j+1}} \rho \chi_{\{V_{j+1}>0\}} \Big)^{(q-2)/q} \\ &\leq \Big(\int_{Q_{j+1}} \rho \chi_{\{V_{j+1}>0\}} \Big)^{(q-2)/q}. \end{aligned} \tag{1.36}$$

Further, using the definition of V_j, it is easy to see that $V_{j+1} > 0$ if and only if $V_j > 2^{-j-1}$, for every j and thus

$$\int_{Q_{j+1}} \rho \chi_{\{V_{j+1}>0\}} = \int_{Q_{j+1}} \rho \chi_{\{V_j^2>2^{-2j-2}\}} \leq 2^{2j+2} \int_{Q_j} \rho V_j^2 = E_j. \tag{1.37}$$

Combining (1.33), (1.34), (1.35), (1.36) and (1.37), we obtain

$$E_{j+1} \leq C^{1+j}\Big(E_j^{1+\frac{1}{\gamma'}} + E_j^{1-\frac{1}{p}+\frac{1}{\gamma'}} + E_j^{1-\frac{2}{q}+\frac{1}{\gamma'}}\Big),$$

where C depends only on d and a. By the assumptions on p and q it follows that $1/\gamma' - 1/p > 0$ and $1/\gamma' - 2/q > 0$. Let us denote by $\bar{\gamma}$ the minimum of such two positive numbers. Taking into account that $E_j \leq \delta$ for every j, we have

$$\begin{cases} E_{j+1} \leq C^{1+j} E_j^{1+\bar{\gamma}}, \\ E_0 \leq \delta, \end{cases}$$

which implies

$$E_j \leq C^{\sum_{i=0}^j i(1+\bar{\gamma})^{j-i}} E_0^{(1+\bar{\gamma})^j} \leq C^{(1+\bar{\gamma})^j \sum_{i=0}^j \frac{i}{(1+\bar{\gamma})^i}} \delta^{(1+\bar{\gamma})^j} \leq (C\delta)^{(1+\bar{\gamma})^j},$$

since $\sum_{i=0}^j \frac{i}{(1+\bar{\gamma})^i} < \infty$. Now, take δ such that $C\delta < 1$. Then $E_j \to 0$, as $j \to \infty$ and thus, by definition of V_j, $E_j \to \int_{Q_{1/2}} \rho(u-1)_+^2 = 0$, which yields $u \leq 1$ in $Q_{1/2}$, as claimed in the statement. ☐

Finally, by combining Lemma 1.3.1 and Lemma 1.3.2 we get the $L^2 \to L^\infty$ estimates, which are uniform in the parameter ε.

Proposition 1.3.3. *Let $a \in \mathbb{R}$, $\varepsilon \in [0,1)$, $p > \frac{d+a^++3}{2}$, $d > d + a^+ + 3$ and A satisfying (1.2). Let $f \in L^p(Q_1, \rho_\varepsilon^a)$ and $F \in L^q(Q_1, \rho_\varepsilon^a)^{d+1}$ and let u be a weak solution to (1.22). Then there exists $C > 0$ depending only on d, a, λ, Λ, p and q such that*

$$\|u\|_{L^\infty(Q_{1/2})} \leq C\Big(\|u\|_{L^2(Q_1,\rho_\varepsilon^a)} + \|f\|_{L^p(Q_1,\rho_\varepsilon^a)} + \|F\|_{L^q(Q_1,\rho_\varepsilon^a)}\Big).$$

Proof. Define

$$V_+ := \theta_+ u_+, \quad \theta_+ := \frac{\sqrt{\delta}}{\|u_+\|_{L^2(Q_1,\rho_\varepsilon^a)} + \|f\|_{L^p(Q_1,\rho_\varepsilon^a)} + \|F\|_{L^q(Q_1,\rho_\varepsilon^a)}},$$

where $\delta > 0$ is as Lemma 1.3.2. The hypothesis of the Lemma 1.3.2 are satisfied, so

$$\|u_+\|_{L^\infty(Q_{1/2})} \leq \frac{1}{\sqrt{\delta}}\Big(\|u_+\|_{L^2(Q_1,\rho_\varepsilon^a)} + \|f\|_{L^p(Q_1,\rho_\varepsilon^a)} + \|F\|_{L^q(Q_1,\rho_\varepsilon^a)}\Big).$$

Repeating the same reasoning with V_- and taking into account that both the estimate (1.28) and Lemma 1.3.2 hold also for the negative part of solutions, it follows

$$\|u_-\|_{L^\infty(Q_{1/2})} \leq \frac{1}{\sqrt{\delta}}\Big(\|u_-\|_{L^2(Q_1,\rho_\varepsilon^a)} + \|f\|_{L^p(Q_1,\rho_\varepsilon^a)} + \|F\|_{L^q(Q_1,\rho_\varepsilon^a)}\Big).$$

So, putting together these two inequalities, the thesis follows choosing $C = \frac{4}{\sqrt{\delta}}$. □

The last result of this section is an energy estimate for weak solutions to the Cauchy-Dirichlet problem (1.24), which depends only on the data of the problem.

Lemma 1.3.4. *Let* $a \in \mathbb{R}$, $\varepsilon \in [0,1)$ *and* $f \in L^2(Q_1, \rho_\varepsilon)$, $F \in L^2(Q_1, \rho_\varepsilon^a)^{d+1}$, $u_0 \in L^2(B_1, \rho_\varepsilon^a)$. *Let* u_ε *be a weak solution to* (1.24) *in* Q_1. *Then there exists* $C > 0$ *depending only on* d, a *and* λ *such that*

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(-1,1;L^2(B_1,\rho_\varepsilon^a))} + \|u_\varepsilon\|_{L^2(-1,1;H_0^1(B_1,\rho_\varepsilon^a))} \\ &\qquad\qquad \le C(\|f\|_{L^2(Q_1,\rho_\varepsilon^a)} + \|F\|_{L^2(Q_1,\rho_\varepsilon^a)} + \|u_0\|_{L^2(B_1,\rho_\varepsilon^a)}). \end{aligned} \tag{1.38}$$

Proof. Let us set $u := u_\varepsilon$ and notice that $u \in C([-1,1], L^2(B_1, \rho_\varepsilon^a))$ by Remark 1.2.17. In what follows, we prove the existence of $C > 0$ depending only on a, d and λ such that

$$\int_{B_1} \rho_\varepsilon^a u^2(\tau) dz + \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a |\nabla u|^2 dz dt \le C\Big(\int_{Q_1} \rho_\varepsilon^a (f^2 + |F|^2) dz dt + \int_{B_1} \rho_\varepsilon^a u_0^2 dz \Big), \tag{1.39}$$

for every $\tau \in (-1,1)$. The bound in (1.38) easily follows by the arbitrariness of τ.

So, let us fix $\tau \in (-1,1)$, $h \in (0, 1-\tau)$ and consider the Steklov average u_h (see Remark 1.2.19). Using a standard approximation procedure (see [96, Theorem 6.1]) and recalling that $\mathcal{D}_c^\infty(Q_r)$ is dense in $L^2(I_r; H_0^1(B_r, \rho_\varepsilon^a))$, we may test (1.26) with $\phi := u_h \chi_{[-1,\tau]}$ to deduce

$$\int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a (\partial_t u_h u_h + (A\nabla u)_h \cdot \nabla u_h) dz dt = \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a (f_h u_h - F_h \cdot \nabla u_h) dz dt.$$

Now, using Fubini-Tonelli theorem and integrating with respect to t, we obtain

$$\int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a \partial_t u_h u_h dz dt = \frac{1}{2} \int_{B_1} \rho_\varepsilon^a \int_{-1}^{\tau} \partial_t (u_h^2) dt dz = \frac{1}{2} \int_{B_1} \rho_\varepsilon^a u_h^2(\tau) dz - \frac{1}{2} \int_{B_1} \rho_\varepsilon^a u_h^2(-1) dz,$$

and thus, passing to the limit as $h \to 0$ and recalling that $u \in C([-1,1], L^2(B_1, \rho_\varepsilon^a))$, it follows

$$\frac{1}{2} \int_{B_1} \rho_\varepsilon^a u^2(\tau) dz + \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a A\nabla u \cdot \nabla u dz dt = \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a (fu - F \cdot \nabla u) dz dt + \frac{1}{2} \int_{B_1} \rho_\varepsilon^a u_0^2 dz.$$

Recalling that A satysfies (1.2) and applying both Hölder's inequality and Young's inequality, it turns out

$$\begin{aligned} &\frac{1}{2} \int_{B_1} \rho_\varepsilon^a u^2(\tau) dz + \lambda \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a |\nabla u|^2 dz dt \\ &\le \|f\|_{L^2(Q_1,\rho_\varepsilon^a)} \Big(\int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a u^2 \Big)^{1/2} + \|F\|_{L^2(Q_1,\rho_\varepsilon^a)} \Big(\int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a |\nabla u|^2 \Big)^{1/2} + \frac{1}{2} \|u_0\|_{L^2(B_1,\rho_\varepsilon^a)}^2 \\ &\le \frac{1}{2} \|f\|_{L^2(Q_1,\rho_\varepsilon^a)}^2 + \frac{1}{2} \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a u^2 + \frac{1}{2\lambda} \|F\|_{L^2(Q_1,\rho_\varepsilon^a)}^2 + \frac{\lambda}{2} \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a |\nabla u|^2 + \frac{1}{2} \|u_0\|_{L^2(B_1,\rho_\varepsilon^a)}^2, \end{aligned}$$

that is

$$H(\tau) + \lambda \int_{-1}^{\tau} \int_{B_1} \rho_\varepsilon^a |\nabla u|^2 dz dt \le \int_{-1}^{\tau} H(t) dt + K,$$

where

$$H(\tau) := \int_{B_1} \rho_\varepsilon^a u^2(\tau), \quad \text{and} \quad K := \|f\|^2_{L^2(Q_1,\rho_\varepsilon^a)} + \frac{1}{\lambda}\|F\|^2_{L^2(Q_1,\rho_\varepsilon^a)} + \|u_0\|^2_{L^2(B_1,\rho_\varepsilon^a)}.$$

Finally, since the second term in the right hand side is nonnegative, the Gronwall's inequality yields $\int_{-1}^{\tau} H(t)dt \le K(1+e^\tau) \le K(1+e)$ which, in turn, proves (1.39). ☐

1.4 Approximation results

The purpose of this section is to establish some approximation results, in the spirit of [138] (elliptic framework). The main fact is that any weak solution u to (1.22) with $\varepsilon = 0$ can be locally approximated with a family of *classical* solutions $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ to (1.22) (with $\varepsilon > 0$), that is $u_\varepsilon \to u$ as $\varepsilon \to 0$, in a suitable sense (see Lemma 1.4.1 and Lemma 1.4.2). This is a key step of our work that will play an important role in the proofs of the Hölder and Schauder estimates.

In what follows, we will repeatedly use the following elementar fact:

$$\rho_\varepsilon^a \to |y|^a \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}\setminus\Sigma),$$

as $\varepsilon \to 0$.

Lemma 1.4.1. *Let $a \in \mathbb{R}$, $p, q \ge 2$, A satisfying (1.2), $R > 0$ and $I_R := (-R^2, R^2)$. Let $\{f_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^p(Q_R, \rho_\varepsilon^a)$, $\{F_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^q(Q_R, \rho_\varepsilon^a)^{d+1}$ and let $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ be a family of weak solutions to*

$$\rho_\varepsilon^a \partial_t u_\varepsilon - \mathrm{div}(\rho_\varepsilon^a A \nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon + \mathrm{div}(\rho_\varepsilon^a F_\varepsilon) \quad \text{in } Q_R. \tag{1.40}$$

Assume that there exist $C > 0$ independent of ε, $f \in L^p_{\mathrm{loc}}(Q_R\setminus\Sigma)$ and $F \in L^q_{\mathrm{loc}}(Q_R\setminus\Sigma)^{d+1}$ such that

$$\|u_\varepsilon\|_{L^2(I_R;H^1(B_R,\rho_\varepsilon^a))} + \|u_\varepsilon\|_{L^\infty(I_R;L^2(B_R,\rho_\varepsilon^a))} \le C, \tag{1.41}$$

$$\|f_\varepsilon\|_{L^p(Q_R,\rho_\varepsilon^a)} + \|F_\varepsilon\|_{L^q(B_R,\rho_\varepsilon^a)} \le C, \tag{1.42}$$

$$f_\varepsilon \to f \quad \text{in } L^p_{\mathrm{loc}}(Q_R\setminus\Sigma) \quad \text{and} \quad F_\varepsilon \to F \quad \text{in } L^q_{\mathrm{loc}}(Q_R\setminus\Sigma)^{d+1} \tag{1.43}$$

as $\varepsilon \to 0$. Then, $f \in L^p(Q_R, |y|^a)$, $F \in L^q(Q_R, |y|^a)^{d+1}$, and there exist a weak solution u to (1.22) in Q_R (with $\varepsilon = 0$) and a sequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \to u$ in $L^2_{\mathrm{loc}}(I_R; H^1_{\mathrm{loc}}(B_R\setminus\Sigma))$ as $k \to \infty$. Moreover, if we assume that $\{u_\varepsilon\} \subset L^2(I_R; H^1_0(B_R, \rho_\varepsilon^a))$, then $u \in L^2(I_R; H^1_0(B_R, |y|^a))$.

Proof. By scaling, we may assume $R = 1$ and set $I := (-1, 1)$.

Step 1. We have $f \in L^p(Q_1, |y|^a)$ and $F \in L^q(Q_1, |y|^a)^{d+1}$. This easily follows by Fatou's lemma, (1.43) and $\rho_\varepsilon^a \to |y|^a$ a.e. in Q_1.

Step 2. In this step we show the existence of $u \in L^2(I; H^1_{\mathrm{loc}}(B_1\setminus\Sigma))$ and a sequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to u \quad \text{in } L^2(I; L^2_{\mathrm{loc}}(B_1\setminus\Sigma)), \tag{1.44}$$

as $k \to \infty$. Further, for every open set $\omega \subset\subset B_1\setminus\Sigma$, u is a weak solution to (1.22) in $\omega \times I$.

Let $\Omega, \tilde\Omega \subset \mathbb{R}^{d+1}$ be open sets such that $\tilde\Omega \subset\subset \Omega \subset\subset \overline{B}_1\setminus\Sigma$ and let $\xi \in C^\infty_c(\Omega)$ with $0 \le \xi \le 1$, $\xi = 1$ in $\tilde\Omega$ and $|\nabla\xi| \le C_0$, where $C_0 > 0$ depends on d, Ω and $\tilde\Omega$.

Define $v_\varepsilon := \xi u_\varepsilon$. By (1.41), we have $v_\varepsilon \in L^2(I; H^1_0(\Omega;\rho_\varepsilon^a)) \cap L^\infty(I; L^2(\Omega;\rho_\varepsilon^a))$ with

$$\|v_\varepsilon\|_{L^2(I;H^1_0(\Omega,\rho_\varepsilon^a))} + \|v_\varepsilon\|_{L^\infty(I;L^2(\Omega,\rho_\varepsilon^a))} \le C, \tag{1.45}$$

for a new $C > 0$ independent of ε. Setting $Q := \Omega \times I$ and fixing $\phi \in C_c^\infty(Q)$, we compute

$$\begin{aligned}
&-\int_Q \rho_\varepsilon^a v_\varepsilon \partial_t\phi dzdt + \int_Q \rho_\varepsilon^a A\nabla v_\varepsilon \cdot \nabla\phi dzdt \\
&= -\int_Q \rho_\varepsilon^a u_\varepsilon \partial_t(\xi\phi) dzdt + \int_Q \rho_\varepsilon^a \xi A\nabla u_\varepsilon \cdot \nabla\phi dzdt + \int_Q \rho_\varepsilon^a u_\varepsilon A\nabla\xi \cdot \nabla\phi dzdt \\
&= -\int_Q \rho_\varepsilon^a u_\varepsilon \partial_t(\xi\phi) dzdt + \int_Q \rho_\varepsilon^a A\nabla u_\varepsilon \cdot \nabla(\xi\phi) dzdt \\
&\quad -\int_Q \rho_\varepsilon^a \phi A\nabla u_\varepsilon \cdot \nabla\xi dzdt + \int_Q \rho_\varepsilon^a u_\varepsilon A\nabla\xi \cdot \nabla\phi dzdt \\
&= \int_Q \rho_\varepsilon^a \Big(f_\varepsilon \xi\phi - F_\varepsilon \cdot \nabla(\xi\phi) - \phi A\nabla u_\varepsilon \cdot \nabla\xi + u_\varepsilon A\nabla\xi \cdot \nabla\phi \Big) dzdt \\
&= \int_Q \rho_\varepsilon^a \Big(f_\varepsilon \xi\phi - \xi F_\varepsilon \cdot \nabla\phi - \phi F_\varepsilon \cdot \nabla\xi - \phi A\nabla u_\varepsilon \cdot \nabla\xi + u_\varepsilon A\nabla\xi \cdot \nabla\phi \Big) dzdt,
\end{aligned}$$

that is,

$$\rho_\varepsilon^a \partial_t v_\varepsilon - \mathrm{div}(\rho_\varepsilon^a A\nabla v_\varepsilon) = \rho_\varepsilon^a \tilde f_\varepsilon + \mathrm{div}(\rho_\varepsilon^a \tilde F_\varepsilon) \quad \text{in } Q,$$

in the weak sense, where we have set

$$\tilde f_\varepsilon := f_\varepsilon \xi - F_\varepsilon \cdot \nabla\xi - A\nabla u_\varepsilon \cdot \nabla\xi, \quad \tilde F_\varepsilon := F_\varepsilon \xi - u_\varepsilon A\nabla\xi.$$

Proceeding as in Remark 1.2.17, one combines the uniform estimates (1.41), (1.42) and (1.45) with the Hölder's and Young's inequalities, to deduce

$$-\int_Q \rho_\varepsilon^a v_\varepsilon \partial_t\phi dzdt \le C\|\phi\|_{L^2(I;H^1_0(\Omega,\rho_\varepsilon^a))},$$

for some new $C > 0$ depending only on d, Ω, $\tilde\Omega$, a and Λ. Notice that, respect to Remark 1.2.17, C is independent of ε: this is because $H^1_0(\Omega,\rho_\varepsilon^a) = H^1_0(\Omega)$ and we can make use of the Poincaré inequality with constant independent of ε, see Remark 1.2.9. As a consequence of the above inequality, it follows $\partial_t v_\varepsilon \in L^2(I; H^{-1}(\Omega,\rho_\varepsilon^a))$ with $\|\partial_t v_\varepsilon\|_{L^2(I;H^{-1}(\Omega,\rho_\varepsilon^a))} \le C$ and so, since $H^{-1}(\Omega,\rho_\varepsilon^a) = H^{-1}(\Omega)$ by Remark 1.2.9, we obtain

$$\|\partial_t v_\varepsilon\|_{L^2(I;H^{-1}(\Omega))} \le C. \tag{1.46}$$

At this point, combining (1.45), (1.46) and Remark 1.2.9 again, it follows

$$\|v_\varepsilon\|_{L^2(I;H^1_0(\Omega))} + \|\partial_t v_\varepsilon\|_{L^2(I;H^{-1}(\Omega))} \le 2C,$$

and thus the Aubin-Lion lemma (see for instance [135, Corollary 8]) yields the existence of $v \in L^2(I; H^1_0(\Omega))$ such that $v_\varepsilon \to v$ in $L^2(Q)$, along a suitable sequence. Further, since by (1.41) there is $u \in L^2(I; H^1(\Omega))$ such that $u_\varepsilon \rightharpoonup u$ in $L^2(I; H^1(\Omega))$ (along a suitable sequence) and $\xi = 1$ in $\tilde\Omega$, we deduce $v = u$ in $L^2(\tilde\Omega \times I)$. A standard diagonal argument yields both $u \in L^2(I; H^1_{\mathrm{loc}}(B_1 \setminus \Sigma))$

and (1.44) (take for instance $\Omega=\Omega_j:=B_1\setminus\{|y|<\frac{1}{j+3}\}$ and $\tilde\Omega=\tilde\Omega_j:=B_{\frac{j+1}{j+2}}\setminus\{|y|<\frac{1}{j+2}\}$, $j\in\mathbb{N}$).

Now, fix $\omega\subset\subset B_1\setminus\Sigma$. Combining (1.44) and $u_{\varepsilon_k}\rightharpoonup u$ in $L^2(I;H^1(\omega))$ and recalling that $\rho^a_{\varepsilon_k}\to|y|^a$ in $L^2(\omega)$, and testing (1.40) with $\phi\in C^\infty_c(\omega\times I)$, we may pass to the limit as $k\to\infty$ into (the weak formulation of) (1.40) and deduce that u is a weak solution to (1.22) in $\omega\times I$.

Step 3. Now we prove that

$$\nabla u_{\varepsilon_k}\to\nabla u\quad\text{in } L^2_{\rm loc}((B_1\setminus\Sigma)\times I),\tag{1.47}$$

as $k\to\infty$, up to passing to a suitable subsequence.

Let $\Omega\subset\subset B_1\setminus\Sigma$, $\eta\in C^\infty_c(\Omega)$, $-1<t_1<t_2<1$ and $h\in(0,1-t_2)$ and let $(u_{\varepsilon_k})_h$ and u_h be the Steklov averages of u_{ε_k} and u, respectively (see Remark 1.2.19). Similar to the proof of Lemma 1.3.4, we test the equation of $(u_{\varepsilon_k})_h$ with $\eta^2\chi_{[t_1,t_2]}(u_{\varepsilon_k})_h$ to obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\Big(\partial_t(u_{\varepsilon_k})_h\eta^2(u_{\varepsilon_k})_h+(A\nabla u_{\varepsilon_k})_h\cdot\nabla(\eta^2(u_{\varepsilon_k})_h)\Big)\\
&\quad=\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\Big(\tfrac12\partial_t(\eta^2(u_{\varepsilon_k})_h^2)+\eta^2(A\nabla u_{\varepsilon_k})_h\cdot\nabla(u_{\varepsilon_k})_h+2\eta(u_{\varepsilon_k})_h(A\nabla u_{\varepsilon_k})_h\cdot\nabla\eta\Big)\\
&\quad=\frac12\int_\Omega\rho^a_{\varepsilon_k}\eta^2(u_{\varepsilon_k})_h^2\Big|_{t=t_1}^{t=t_2}+\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\Big(\eta^2(A\nabla u_{\varepsilon_k})_h\cdot\nabla(u_{\varepsilon_k})_h+2\eta(u_{\varepsilon_k})_h(A\nabla u_{\varepsilon_k})_h\cdot\nabla\eta\Big)\\
&\quad=\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\Big((f_{\varepsilon_k})_h\eta^2(u_{\varepsilon_k})_h+(F_{\varepsilon_k})_h\cdot\nabla(\eta^2(u_{\varepsilon_k})_h)\Big),
\end{aligned}$$

which, rearranging terms, becomes

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\eta^2(A\nabla u_{\varepsilon_k})_h\cdot\nabla(u_{\varepsilon_k})_h=&-\frac12\int_\Omega\rho^a_{\varepsilon_k}\eta^2(u_{\varepsilon_k})_h^2\Big|_{t=t_1}^{t=t_2}-2\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\eta(u_{\varepsilon_k})_h(A\nabla u_{\varepsilon_k})_h\cdot\nabla\eta\\
&+\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\Big((f_{\varepsilon_k})_h\eta^2(u_{\varepsilon_k})_h+(F_{\varepsilon_k})_h\cdot\nabla(\eta^2(u_{\varepsilon_k})_h)\Big)
\end{aligned}\tag{1.48}$$

Using the properties of the Steklov averages (see Remark 1.2.19), we can take the limit as $h\to0$ in (1.48) to obtain

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\eta^2A\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}=&-\frac12\int_\Omega\rho^a_{\varepsilon_k}\eta^2u_{\varepsilon_k}^2\Big|_{t=t_1}^{t_2}-2\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\eta u_{\varepsilon_k}A\nabla u_{\varepsilon_k}\cdot\nabla\eta\\
&+\int_{t_1}^{t_2}\int_\Omega\rho^a_{\varepsilon_k}\Big(f_{\varepsilon_k}\eta^2u_{\varepsilon_k}+F_{\varepsilon_k}\cdot\nabla(\eta^2u_{\varepsilon_k})\Big),
\end{aligned}$$

for every $k\in\mathbb{N}$. Now, by testing the equation of u_h with $\eta^2\chi_{[t_1,t_2]}u_h$ and repeating the very same argument, one shows that

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega|y|^a\eta^2A\nabla u\cdot\nabla u=&-\frac12\int_\Omega|y|^a\eta^2u^2\Big|_{t=t_1}^{t=t_2}-2\int_{t_1}^{t_2}\int_\Omega|y|^a\eta uA\nabla u\cdot\nabla\eta\\
&+\int_{t_1}^{t_2}\int_\Omega|y|^a\Big(f\eta^2u+F\cdot\nabla(\eta^2u)\Big),
\end{aligned}$$

for a.e. t_1 and t_2 as above. Recalling that $\nabla u_{\varepsilon_k} \rightharpoonup \nabla u$ in $L^2(\Omega \times I)$ and using both (1.44) and $\rho^a_{\varepsilon_k} \to |y|^a$ in $L^2(\Omega)$, we find

$$-\frac{1}{2}\int_\Omega \rho^a_{\varepsilon_k}\eta^2 u^2_{\varepsilon_k}\Big|_{t=t_1}^{t_2} - 2\int_{t_1}^{t_2}\int_\Omega \rho^a_{\varepsilon_k}\eta u_{\varepsilon_k} A\nabla u_{\varepsilon_k}\cdot\nabla\eta$$
$$\longrightarrow -\frac{1}{2}\int_\Omega |y|^a\eta^2 u^2\Big|_{t=t_1}^{t=t_2} - 2\int_{t_1}^{t_2}\int_\Omega |y|^a \eta u A\nabla u\cdot\nabla\eta,$$

as $k \to \infty$, for a.e. t_1 and t_2 as above. On the other hand, since in addition $f_{\varepsilon_k} \to f$ in $L^2(\Omega \times I)$ and $F_{\varepsilon_k} \to F$ in $L^2(\Omega \times I)^{d+1}$, it follows

$$\int_{t_1}^{t_2}\int_\Omega \rho^a_{\varepsilon_k}\Big(f_{\varepsilon_k}\eta^2 u_{\varepsilon_k} + F_{\varepsilon_k}\cdot\nabla(\eta^2 u_{\varepsilon_k})\Big) \to \int_{t_1}^{t_2}\int_\Omega |y|^a\Big(f\eta^2 u + F\cdot\nabla(\eta^2 u)\Big),$$

as $k \to \infty$, for a.e. t_1 and t_2 as above. Consequently,

$$\lim_{k\to\infty}\int_{t_1}^{t_2}\int_\Omega \rho^a_{\varepsilon_k}\eta^2 A\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k} = \int_{t_1}^{t_2}\int_\Omega |y|^a\eta^2 A\nabla u\cdot\nabla u.$$

Since ρ^a_ε is bounded and bounded away from 0 in Ω uniformly in ε and $u_{\varepsilon_k} \rightharpoonup u$ in $L^2(I;H^1(\Omega))$, and A satisfies (1.2), we may let $\eta \to \chi_\Omega$ and use the triangular inequality to deduce $\nabla u_{\varepsilon_k} \to \nabla u$ in $L^2(\Omega \times (t_1,t_2))$ as $k \to \infty$. A diagonal argument as above then shows (1.47).

Step 4. Next, we prove that $u \in L^2(I;H^1(B_1,|y|^a)) \cap L^\infty(I;L^2(B_1,|y|^a))$.

By (1.41), (1.44) and Fatou's lemma, we have that $u \in L^\infty(I;L^2(B_1,|y|^a))$. Indeed, for a.e. $t \in I$, one has

$$\int_{B_1}|y|^a u^2(z,t)dz \le \liminf_k \int_{B_1}\rho_{\varepsilon_k}u^2_{\varepsilon_k}(z,t)dz \le \|u_{\varepsilon_k}\|_{L^\infty(I;L^2(B_1,\rho^a_{\varepsilon_k}))} \le C.$$

To show that $u \in L^2(I;H^1(B_1,|y|^a))$ we distinguish three cases, depending on the value of a.

Assume first $a \ge 0$. Since $|y|^a \le \rho^a_\varepsilon(y)$ for every $\varepsilon \in (0,1)$, one has

$$\|u_\varepsilon\|_{L^2(I;H^1(B_1,|y|^a))} \le \|u_\varepsilon\|_{L^2(I;H^1(B_1,\rho^a_\varepsilon))} \le C,$$

by (1.41). Then, the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2(I;H^1(B_1,|y|^a))$ and thus $u \in L^2(I;H^1(B_1,|y|^a))$ by weak convergence. Moreover, if $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(I;H^1_0(B_1,\rho^a_\varepsilon)) \subset L^2(I;H^1_0(B_1,|y|^a))$, then $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(I;H^1_0(B_1,|y|^a))$ and $u \in L^2(I;H^1_0(B_1,|y|^a))$ by weak convergence.

Second, fix $-1 < a < 0$. In this case $|y|^a$ belongs to the Muckenhoupt class A_2 and, since $a < 0$, one has $|y|^a \ge 1$. Therefore,

$$\|u_\varepsilon\|_{L^2(I;H^1(B_1))} \le \|u_\varepsilon\|_{L^2(I;H^1(B_1,\rho^a_\varepsilon))} \le C,$$

and so $u_{\varepsilon_k} \rightharpoonup u$ $L^2(I;H^1(B_1))$ and that u possesses weak gradient. Now, since $u_{\varepsilon_k} \to u$ and $\nabla u_{\varepsilon_k} \to \nabla u$ a.e. in Q_1 by (1.44) and (1.47), we may invoke Fatou's lemma again to conclude u and $|\nabla u|$ belong to $L^2(Q_1,|y|^a)$. This shows our claim thanks to Proposition 1.2.4.

Furthermore, if $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(I;H^1_0(B_1,\rho^a_\varepsilon))$, then $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(I;H^1_0(B_1))$ and thus there exists a sequence satisfying $u_{\varepsilon_k} \to u$ weakly in $L^2(I;H^1_0(B_1))$. So, $u \in L^2(I;H^1(B_1,|y|^a)) \cap L^2(I;H^1_0(B_1))$.

Now, fix $\delta > 0$ and consider $\psi \in C_c^\infty(Q_1)$ such that $\|u-\psi\|_{L^2(I;H_0^1(B_1))} \le \bar\delta$, where $\bar\delta \in (0,1)$ will be chosen in a moment. Let $\hat y \in (0,1)$ small. Then

$$\begin{aligned}\int_{Q_1} |y|^a|\nabla u-\nabla\psi|^2 &= \int_{\{|y|\ge\hat y\}} |y|^a|\nabla u-\nabla\psi|^2 + \int_{\{|y|<\hat y\}} |y|^a|\nabla u-\nabla\psi|^2\\ &\le \hat y^a\int_{Q_1}|\nabla u-\nabla\psi|^2 + \delta'(\hat y) \le \hat y^a\bar\delta + \delta'(\hat y),\end{aligned}$$

where $\delta'(\hat y) \to 0$ as $\hat y \to 0$, since the function $|y|^a|\nabla u-\nabla\psi|^2 \in L^1(Q_1)$. Choosing $\bar\delta < \delta'(\hat y)/\hat y^a$ and $\hat y$ such that $\delta'(\hat y) < \delta/2$, we finally obtain $\|u-\psi\|_{L^2(I;H_0^1(B_1,|y|^a))} \le \delta$, that is, $u \in L^2(I;H_0^1(B_1,|y|^a))$ thanks to the arbitrariness of δ.

Finally, let $a \le -1$. In this case, we consider the isometry $\bar T_\varepsilon^a$ defined in (1.21) and we set $v_\varepsilon := \sqrt{\rho_\varepsilon^a}u_\varepsilon$. By Remark 1.2.7 and (1.41), the family $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2(I;H^1(B_1))$ and so $v_{\varepsilon_k} \rightharpoonup v$ weakly in $L^2(I;H^1(B_1))$. Further, by (1.44), we have

$$\int_{Q_1} v\phi \leftarrow \int_{Q_1} v_{\varepsilon_k}\phi = \int_{Q_1}\sqrt{\rho_{\varepsilon_k}^a}u_{\varepsilon_k}\phi \to \int_{Q_1}|y|^{a/2}u\phi,$$

for every $\phi \in C_c^\infty(Q_1)$, which implies $v = |y|^{a/2}u$ a.e. in Q_1. So, noticing that $u = (\bar T_0^a)^{-1}v$ and applying Remark 1.2.7 again, we conclude that $u \in L^2(I;H^1(B_1,|y|^a))$. Moreover, if $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(I;H_0^1(B_1,\rho_\varepsilon^a))$ then $\{v_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(I;H_0^1(B_1))$. So, $v_{\varepsilon_k} \to v$ weakly in such space, which implies that $u \in L^2(I;H_0^1(B_1,|y|^a))$.

Step 5. In this last step we show that u satisfies (1.22) (with $\varepsilon = 0$) in the weak sense. Let us fix a test function $\phi \in \mathcal{D}_c^\infty(Q_1)$, see Definition 1.2.14 with $\varepsilon = 0$. By (1.44) and (1.47), we have both

$$\rho_{\varepsilon_k}^a(-u_{\varepsilon_k}\partial_t\phi + A\nabla u_{\varepsilon_k}\cdot\nabla\phi) \to |y|^a(-u\partial_t\phi + A\nabla u\cdot\nabla\phi) \quad \text{a.e. in } Q_1$$

and

$$\rho_{\varepsilon_k}^a(f_{\varepsilon_k}\phi - F_{\varepsilon_k}\cdot\nabla\phi) \to |y|^a(f\phi - F\cdot\nabla\phi) \quad \text{a.e. in } Q_1,$$

as $k \to \infty$. Now, let $E \subset Q_1$ be measurable. By (1.2), (1.41) and the Hölder inequality, we get

$$\int_E \rho_{\varepsilon_k}^a|-u_{\varepsilon_k}\partial_t\phi + A\nabla u_{\varepsilon_k}\cdot\nabla\phi| \le C\|u_\varepsilon\|_{L^2(I;H^1(B_1,\rho_{\varepsilon_k}^a))}\|\nabla_{x,t}\phi\|_{L^\infty(Q_1)}\Big(\int_{E\cap\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a\Big)^{1/2} \le \delta(E),$$

where $\delta(E) \ge 0$ satisfies $\delta(E) \to 0$ as $|E| \to 0$. Indeed, when $a \le -1$, we have $\rho_{\varepsilon_k}^a \le |y|^a \in L^\infty(E\cap\mathrm{spt}(\phi))$, by the definition of $\mathcal{D}_c^\infty(Q_1)$. Instead, when $a > -1$, one has $\rho_{\varepsilon_k}^a \le C|y|^{\min(0,a)} \in L^1(B_1)$. In particular, it follows that the family $-\rho_{\varepsilon_k}^a u_{\varepsilon_k}\partial_t\phi + \rho_{\varepsilon_k}^a A\nabla u_{\varepsilon_k}\cdot\nabla\phi$ is uniformly integrable and the Vitali's theorem (see [18, Theorem 4.5.4]) yields

$$\int_{Q_1}\rho_{\varepsilon_k}^a\Big(-u_{\varepsilon_k}\partial_t\phi + A\nabla u_{\varepsilon_k}\cdot\nabla\phi\Big) \to \int_{Q_1}|y|^a\Big(-u\partial_t\phi + A\nabla u\cdot\nabla\phi\Big),$$

as $k \to \infty$. With a very similar argument, we obtain

$$\int_{Q_1}\rho_{\varepsilon_k}^a\Big(f_{\varepsilon_k}\phi - F_{\varepsilon_k}\cdot\nabla\phi\Big) \to \int_{Q_1}|y|^a\Big(f\phi - F\cdot\nabla\phi\Big),$$

as $k \to \infty$, and our statement follows. □

Lemma 1.4.2. *Let* $a \in \mathbb{R}$, $p,q \geq 2$, A *satisfying* (1.2), $R > 0$ *and* $I_R := (-R^2, R^2)$. *Let* $f \in L^p(Q_R, |y|^a)$, $F \in L^q(Q_R, |y|^a)^{d+1}$ *and let* u *be a weak solution to*

$$|y|^a \partial_t u - \operatorname{div}(|y|^a A \nabla u) = |y|^a f + \operatorname{div}(|y|^a F) \quad \text{in } Q_R.$$

Then, for every $r \in (0,R)$, *there exist* $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{f_\varepsilon\}_{\varepsilon\in(0,1)}$ *and* $\{F_\varepsilon\}_{\varepsilon\in(0,1)}$ *satisfying the assumptions of Lemma 1.4.1 in* Q_r. *Moreover, there exists* $\varepsilon_k \to 0$ *such that* $u_{\varepsilon_k} \to u$ *in* $L^2_{\mathrm{loc}}(I_r; H^1_{\mathrm{loc}}(B_r \setminus \Sigma))$ *as* $k \to \infty$.

Proof. By scaling, we may assume $R = 1$ and set $I := (-1,1)$.

Step 1. Let us fix $r \in (0,1)$ and set $\tilde B := B_r$, $\tilde Q := \tilde B \times I$, $B := B_{\frac{1+r}{2}}$ and $Q := B \times I$. Consider a cut-off function $\xi \in C^\infty_c(B_1)$ such that

$$\operatorname{spt}(\xi) \subset \overline{B}, \quad \xi \equiv 1 \text{ in } \tilde B, \quad 0 \leq \xi \leq 1 \text{ in } B_1, \quad |\nabla \xi| \leq C_0,$$

for some $C_0 > 0$ depending on d and r, and define $\tilde u := \xi u$. Now, given $\phi \in \mathcal{D}^\infty_c(Q)$, the same computations of Lemma 1.4.1 show that

$$\int_Q |y|^a \Big(-\tilde u \partial_t \phi + A \nabla \tilde u \cdot \nabla \phi \Big) \\ = \int_Q |y|^a \Big(f\xi\phi - \xi F \cdot \nabla\phi - \phi F \cdot \nabla \xi - \phi A \nabla u \cdot \nabla \xi + u A \nabla \xi \cdot \nabla \phi \Big),$$

and thus, setting

$$\tilde f := f\xi, \quad \tilde F := F\xi, \quad \tilde g := -F \cdot \nabla\xi - A\nabla u \cdot \nabla \xi, \quad \tilde G := -uA\nabla\xi,$$

we obtain that $\tilde u$ is a weak solution to

$$|y|^a \partial_t \tilde u - \operatorname{div}(|y|^a A \nabla \tilde u) = |y|^a (\tilde f + \tilde g) + \operatorname{div}(|y|^a (\tilde F + \tilde G)) \quad \text{in } Q,$$

where we have used that $\tilde u \in L^2(I; H^1_0(B, |y|^a)) \cap L^\infty(I; L^2(B, |y|^a))$ by construction. Moreover, since $u \in L^2(I; H^1(B_1, |y|^a))$ by definition and $p,q \geq 2$, then $\tilde f \in L^p(Q, |y|^a)$, $\tilde F \in L^q(Q, |y|^a)^{d+1}$, $\tilde g \in L^2(Q, |y|^a)$ and $\tilde G \in L^2(Q, |y|^a)^{d+1}$. Therefore, by Remark 1.2.17, it follows that $\tilde u \in C(\bar I; L^2(B, |y|^a))$. In particular, $\tilde u_0 := \tilde u_{|t=-1} = \xi u_{|t=-1} \in L^2(B, |y|^a)$ is well-defined and $\tilde u$ is a weak solution to

$$\begin{cases} |y|^a \partial_t \tilde u - \operatorname{div}(|y|^a A \nabla \tilde u) = |y|^a (\tilde f + \tilde g) + \operatorname{div}(|y|^a (\tilde F + \tilde G)) & \text{in } Q, \\ \tilde u = 0 & \text{on } \partial B \times I, \\ \tilde u_{|t=-1} = \tilde u_0 & \text{on } B. \end{cases} \tag{1.49}$$

Step 2. In this step, we construct a family of smooth approximations u_ε of $\tilde u$, as in the statement. We distinguish between two cases, depending on the value of a.

First, let $a > 0$. We define

$$f_\varepsilon := \left(\frac{|y|^a}{\rho_\varepsilon^a} \right)^{1/p} \tilde f, \quad F_\varepsilon := \left(\frac{|y|^a}{\rho_\varepsilon^a} \right)^{1/q} \tilde F, \quad g_\varepsilon := \left(\frac{|y|^a}{\rho_\varepsilon^a} \right)^{1/2} \tilde g,$$
$$G_\varepsilon := \left(\frac{|y|^a}{\rho_\varepsilon^a} \right)^{1/2} \tilde G, \quad u_{0,\varepsilon} := \left(\frac{|y|^a}{\rho_\varepsilon^a} \right)^{1/2} \tilde u_0,$$

and consider the family of weak solutions $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ to

$$\begin{cases}\rho_\varepsilon^a\partial_t u_\varepsilon-\mathrm{div}(\rho_\varepsilon^a A\nabla u_\varepsilon)=\rho_\varepsilon^a(f_\varepsilon+g_\varepsilon)+\mathrm{div}(\rho_\varepsilon^a(F_\varepsilon+G_\varepsilon)) & \text{in } Q\\ u_\varepsilon=0 & \text{on } \partial B\times I,\\ u_{\varepsilon|_{t=-1}}=u_{0,\varepsilon} & \text{on } B.\end{cases}$$

By construction, we have

$$\|f_\varepsilon\|_{L^p(Q,\rho_\varepsilon^a)}+\|g_\varepsilon\|_{L^2(Q,\rho_\varepsilon^a)}+\|F_\varepsilon\|_{L^q(Q,\rho_\varepsilon^a)}+\|G_\varepsilon\|_{L^2(Q,\rho_\varepsilon^a)}+\|u_{0,\varepsilon}\|_{L^2(B,\rho_\varepsilon^a)}\leq C, \tag{1.50}$$

for some $C>0$ independent of ε and $f_\varepsilon\to\tilde f$, $F_\varepsilon\to\tilde F$, $g_\varepsilon\to\tilde g$, $G_\varepsilon\to\tilde G$ a.e. in Q and $u_{0,\varepsilon}\to\tilde u_0$ a.e. in B. Furthermore, since $a>0$, we may apply the Lebesgue's dominated convergence theorem to deduce that

$$\begin{aligned} f_\varepsilon\to\tilde f &\quad \text{in } L^p_{\mathrm{loc}}((B\setminus\Sigma)\times I), & F_\varepsilon\to\tilde F &\quad \text{in } L^q_{\mathrm{loc}}((B\setminus\Sigma)\times I)^{d+1},\\ g_\varepsilon\to\tilde g &\quad \text{in } L^2_{\mathrm{loc}}((B\setminus\Sigma)\times I), & G_\varepsilon\to\tilde G &\quad \text{in } L^2_{\mathrm{loc}}((B\setminus\Sigma)\times I)^{d+1},\end{aligned} \tag{1.51}$$

and $u_{0,\varepsilon}\to\tilde u_0$ in $L^2_{\mathrm{loc}}(B\setminus\Sigma)$ as $\varepsilon\to0$.

The case $a\leq0$ is easier: we set

$$f_\varepsilon:=\tilde f,\quad F_\varepsilon:=\tilde F,\quad g_\varepsilon:=\tilde g,\quad G_\varepsilon:=\tilde G,\quad u_{0,\varepsilon}:=\tilde u_0.$$

Since $\rho_\varepsilon^a\leq|y|^a$, we immediately deduce (1.50), while (1.51) is obvious by definition.

Step 3. Combining Lemma 1.3.4 and (1.50), we deduce that the family $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2(I;H^1_0(B,\rho_\varepsilon^a))\cap L^\infty(I;L^2(B,\rho_\varepsilon^a))$. Consequently, by (1.50) again and (1.51), $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{f_\varepsilon+g_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{F_\varepsilon+G_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfy the assumptions of Lemma 1.4.1 in Q and so there exist $\varepsilon_k\to0$ and $\bar u\in L^2(I;H^1_0(B,|y|^a))\cap C(\bar I;L^2(B,|y|^a))$ (see Remark 1.2.17) such that $u_{\varepsilon_k}\to\bar u$ in $L^2_{\mathrm{loc}}(I;H^1_{\mathrm{loc}}(B\setminus\Sigma))$. Since $u_{0,\varepsilon}\to\tilde u_0$ in $L^2_{\mathrm{loc}}(B\setminus\Sigma)$, $\bar u|_{t=-1}=\tilde u_0$ in $L^2(B,|y|^a)$ and therefore $\bar u$ is a weak solution to (1.49).

As consequence, we obtain $\bar u=\tilde u$ a.e. in Q by uniqueness of $\tilde u$ (uniqueness of weak solutions to (1.49) follows by the classical theory of the Cauchy-Dirichlet problem in abstract Hilbert spaces, see [99]) and our statement follows since $\tilde u=u$ a.e. in $\tilde Q$ by definition. □

Remark 1.4.3. Let $a>-1$ and $R>r>0$. Then, Lemma 1.4.1 and Lemma 1.4.2 hold for weak solutions to (1.25) in Q_R^+. That is, if $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is a family of weak solutions to (1.25), such that u_ε, f_ε, F_ε and A satisfy the same assumptions of Lemma 1.4.1 in Q_R^+, then $u_\varepsilon\to u$ in the sense of Lemma 1.4.1 and u is a weak solution to (1.25) in Q_R^+ with $\varepsilon=0$. Further, if u is a weak solution (1.25) in Q_R^+ with $\varepsilon=0$, we can construct families $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{f_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{F_\varepsilon\}_{\varepsilon\in(0,1)}$ such that the assumptions of Lemma 1.4.1 in Q_r^+ and $u_\varepsilon\to u$ in the sense of Lemma 1.4.2.

Indeed, fixed $\varepsilon\in[0,1)$, let us consider a solution u_ε to (1.25) in Q_R^+ and let $\phi\in C_c^\infty(Q_R)$ be a test function. Let us define

$$J:=\begin{pmatrix} I_n & 0\\ 0 & -1\end{pmatrix},\quad \tilde A(x,y,t):=JA(x,-y,t)J,\quad \tilde u_\varepsilon(x,y,t):=u_\varepsilon(x,-y,t),$$

$$\tilde f_\varepsilon(x,y,t):=f_\varepsilon(x,-y,t),\quad \tilde F_\varepsilon(x,y,t):=-F_\varepsilon(x,-y,t),\quad \tilde\phi(x,y,t):=\phi(x,-y,t),$$

for $(x,y,t)\in Q_R^-$. By changing variables,

$$\int_{Q_R^+}\rho_\varepsilon^a\big(-u_\varepsilon\phi_t+A\nabla u_\varepsilon\cdot\nabla\phi-f_\varepsilon\phi+F_\varepsilon\cdot\nabla\phi\big)=\int_{Q_R^-}\rho_\varepsilon^a\big(-\tilde u_\varepsilon\tilde\phi_t+\tilde A\nabla\tilde u_\varepsilon\cdot\nabla\tilde\phi-\tilde f_\varepsilon\tilde\phi+\tilde F_\varepsilon\cdot\nabla\tilde\phi\big),$$

where $Q_R^-:=Q_R\cap\{y<0\}$. Hence, if we define

$$\bar u_\varepsilon:=\begin{cases}u_\varepsilon, & \text{in } Q_R^+\\ \tilde u_\varepsilon, & \text{in } Q_R^-\end{cases},\quad \bar A:=\begin{cases}A & \text{in } Q_R^+\\ \tilde A & \text{in } Q_R^-\end{cases},\quad \bar f_\varepsilon:=\begin{cases}f_\varepsilon & \text{in } Q_R^+\\ \tilde f_\varepsilon & \text{in } Q_R^-\end{cases},\quad \bar F_\varepsilon:=\begin{cases}F_\varepsilon & \text{in } Q_R^+\\ \tilde F_\varepsilon & \text{in } Q_R^-\end{cases},$$

we have that $\bar A$ is a symmetric matrix satisfying (1.2) and, by the conormal boundary condition in (1.25), $\bar u_\varepsilon$ is a weak solution to

$$\rho_\varepsilon^a\partial_t\bar u_\varepsilon-\mathrm{div}(\rho_\varepsilon^a\bar A\nabla\bar u_\varepsilon)=\rho_\varepsilon^a\bar f_\varepsilon+\mathrm{div}(\rho_\varepsilon^a\bar F_\varepsilon),\qquad \text{in } Q_R.$$

Then, Lemmas 1.4.1 and 1.4.2 apply to $\bar u_\varepsilon$ in Q_R and, by definition of $\bar u_\varepsilon$, are valid for weak solutions to (1.25) in Q_R^+.

1.5 Liouville theorems I

In this section, we establish a Liouville-type theorem for entire solutions exhibiting sub-quadratic growth. The result applies to both the singular/degenerate weight ($\varepsilon=0$) and the regularized one ($\varepsilon>0$). This result extends the Liouville theorems established in [9], and can be compared with the elliptic counterparts in [138, 139, 143]. Later on, in Section 1.9, we refine this result for the case $\varepsilon=0$, generalizing it to every polynomial growth condition and providing a complete characterization of the polynomial solutions to (1.52).

Theorem 1.5.1. *Let $a>-1$, $\varepsilon\in[0,1)$, $\gamma\in[0,2)$ and let u be an entire solution to*

$$\begin{cases}\rho_\varepsilon^a\partial_t u-\mathrm{div}(\rho_\varepsilon^a\nabla u)=0 & \text{in } \mathbb{R}^{d+1}_+\times\mathbb{R}\\ \rho_\varepsilon^a\partial_y u=0 & \text{on } \partial\mathbb{R}^{d+1}_+\times\mathbb{R}.\end{cases}\tag{1.52}$$

Assume that

$$|u(z,t)|\le C(1+(|z|^2+|t|)^\gamma)^{1/2}\qquad \text{for a.e. } (z,t)\in\mathbb{R}^{d+1}_+\times\mathbb{R}.\tag{1.53}$$

Then u is a linear function depending only on x. Moreover, if $\gamma\in[0,1)$, then u is constant.

The proof of the theorem above is obtained by iterating a Caccioppoli-type inequality, following the approach in [143], which involves difference quotients in the x-variable, and by applying a duality principle between u and its weighted derivative $\rho_\varepsilon^a\partial_y u$, which solve equations with weight ρ_ε^{-a}, respectively, as in [27].

We begin with the following standard lemma.

Lemma 1.5.2. *Let $a\in\mathbb{R}$, $\varepsilon\in[0,1)$ and let u be an entire solution to*

$$\rho_\varepsilon^a\partial_t u-\mathrm{div}(\rho_\varepsilon^a\nabla u)=0\quad \text{in } \mathbb{R}^{d+1}\times\mathbb{R}.$$

Then, for every $i=1,\dots,d$, the function $\partial_{x_i}u$ is an entire solution to the same problem.

The proof combines difference quotients in x and energy estimates, similar to the elliptic setting, see [143, Corollary 2.2]. The next lemma was established in [141, 13] for $a \in (-1,1)$ and $\varepsilon = 0$. We extend it for all values of $a \in \mathbb{R}$ and $\varepsilon \in (0,1)$, with an independent proof.

Lemma 1.5.3. *Let $a \in \mathbb{R}$, $\varepsilon \in [0,1)$ and let u be an entire solution to*

$$\rho_\varepsilon^a \partial_t u - \mathrm{div}(\rho_\varepsilon^a \nabla u) = 0 \qquad \text{in } \mathbb{R}^{d+1} \times \mathbb{R}. \tag{1.54}$$

Then the function $v = \rho_\varepsilon^a \partial_y u$ is an entire solution to

$$\rho_\varepsilon^{-a} \partial_t v - \mathrm{div}(\rho_\varepsilon^{-a} \nabla v) = 0 \qquad \text{in } \mathbb{R}^{d+1} \times \mathbb{R}. \tag{1.55}$$

Proof. The case $\varepsilon \in (0,1)$ follows by explicit computations, since weak solutions are smooth.

When $\varepsilon = 0$, we proceed by approximation as follows. Fix $R > 0$ and let $I_R := (-R^2, R^2)$. By Lemma 1.4.2, there exist a family of solutions $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ to (1.40) in Q_{3R} (with $f_\varepsilon = 0$ and $F_\varepsilon = 0$), uniformly bounded in $L^2(I_{2R}; H^1(B_{2R}, \rho_\varepsilon^a))$, and a sequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to u \quad \text{ in } L^2_{loc}(I_{2R}; H^1_{loc}(B_{2R} \setminus \Sigma)), \tag{1.56}$$

as $k \to \infty$. Now, since $\varepsilon_k > 0$, the function $v_k := \rho_{\varepsilon_k}^a \partial_y u_{\varepsilon_k}$ is a solution to (1.40) in Q_{2R} (with $f_\varepsilon = 0$ and $F_\varepsilon = 0$), with weight $\rho_{\varepsilon_k}^{-a}$. Further, since $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ is uniformly bounded in $L^2(I_{2R}; H^1(B_{2R}, \rho_\varepsilon^a))$, we have

$$\int_{Q_{2R}} \rho_{\varepsilon_k}^{-a} v_k^2 = \int_{Q_{2R}} \rho_{\varepsilon_k}^a (\partial_y u_{\varepsilon_k})^2 \le \int_{Q_{2R}} \rho_{\varepsilon_k}^a |\nabla u_{\varepsilon_k}|^2 \le C,$$

for some $C > 0$ independent of ε and thus, using the Caccioppoli inequality (1.27), it follows

$$\|v_k\|_{L^\infty(I_R; L^2(B_R, \rho_{\varepsilon_k}^{-a}))} + \|\nabla v_k\|_{L^2(Q_R, \rho_{\varepsilon_k}^{-a})} \le C,$$

for some new $C > 0$ independent of ε. As a consequence, the family $\{v_k\}_{k \in \mathbb{N}}$ satisfies the assumptions of Lemma 1.4.1 which, in turn, allows us to conclude that, up to a subsequence, $v_k \to v$ in $L^2_{loc}(I_R; H^1_{loc}(B_R \setminus \Sigma))$, for some weak solution v to (1.22) in Q_R (with $\varepsilon = 0$, $f = 0$ and $F = 0$). By (1.56), we deduce $v = |y|^a \partial_y u$ and, since $R > 0$ is arbitrary, our statement follows. □

Proof of Theorem 1.5.1. First, we point out that it is enough to prove that u is linear and depends only on x. Then the second part of the statement automatically follows combining (1.53) with the extra-assumption $\gamma \in [0,1)$.

Step 1. By Remark 1.4.3, we notice that the even extension with respect to y of u is an entire solution to (1.54). Therefore, it is enough to establish our statement for an entire solution u to (1.54) which is even in y and satisfy (1.53) a.e. in $\mathbb{R}^{d+1} \times \mathbb{R}$. Choosing $r' = R$ and $r = 2R$ in the Caccioppoli inequality (1.27), we get

$$\int_{Q_R} \rho_\varepsilon^a |\nabla u|^2 dz dt \le \frac{C}{R^2} \int_{Q_{2R}} \rho_\varepsilon^a u^2 dz dt, \tag{1.57}$$

for some $C > 0$ independent of ε and R. We will repeatedly use the above inequality in the next steps.

Step 2. In this step, we show that u is linear in x. By Lemma 1.5.2, for every multi-index $\beta \in \mathbb{N}^d$, the function $\partial_x^\beta u$ solves (1.52). Fixed $R > 1$, by (1.57) and (1.53), it follows

$$\int_{Q_R} \rho_\varepsilon^a (\partial_{x_i} u)^2 \leq \int_{Q_R} \rho_\varepsilon^a |\nabla u|^2 \leq \frac{C}{R^2} \int_{Q_{2R}} \rho_\varepsilon^a u^2 \leq \frac{C}{R^2} R^{a^+ + 2\gamma + d + 3},$$

for every $i = 1, \dots, d$. So, setting

$$\tilde{\gamma} := a^+ + 2\gamma + d + 3,$$

and iterating, it follows

$$\int_{Q_R} \rho_\varepsilon^a (\partial_x^\beta u)^2 \leq C R^{\tilde{\gamma} - 2|\beta|},$$

for every multi-index $\beta \in \mathbb{N}^d$. Consequently, taking β such that $2|\beta| > \tilde{\gamma}$ and passing to the limit as $R \to \infty$, we get $\partial_x^\beta u = 0$, and therefore we easily obtain that u is polynomial in the variable x. By (1.53), it follows that u must be linear in x.

Step 3. In this step we show that u is independent of y. By Lemma 1.5.3, $v := \rho_\varepsilon^a \partial_y u$ is an entire solution to (1.55) while, by Lemma 1.5.3 again,

$$w_1 = \rho_\varepsilon^{-a} \partial_y v = \rho_\varepsilon^{-a} \partial_y (\rho_\varepsilon^a \partial_y u) = \partial_{yy} u + \frac{(\rho_\varepsilon^a)'}{\rho_\varepsilon^a} \partial_y u \tag{1.58}$$

is an entire solution to (1.54). So, using (1.57) twice, we deduce that

$$\int_{Q_R} \rho_\varepsilon^a w_1^2 \leq \int_{Q_R} \rho_\varepsilon^{-a} |\nabla v|^2 \leq \frac{C}{R^2} \int_{Q_{2R}} \rho_\varepsilon^{-a} v^2 \leq \frac{C}{R^2} \int_{Q_{2R}} \rho_\varepsilon^a |\nabla u|^2 \leq \frac{C}{R^4} \int_{Q_{4R}} \rho_\varepsilon^a u^2 \leq C R^{\tilde{\gamma} - 4}.$$

Setting

$$w_{j+1} := \partial_{yy} w_j + \frac{(\rho_\varepsilon^a)'}{\rho_\varepsilon^a} \partial_y w_j, \tag{1.59}$$

and noticing that w_{j+1} is an entire solution to (1.54) for $j \in \mathbb{N}_{\geq 1}$, we may iterate the argument above to show the existence of $k \in \mathbb{N}$ such that $\tilde{\gamma} - 4k < 0$ and

$$\int_{Q_R} \rho_\varepsilon^a w_k \leq C R^{\tilde{\gamma} - 4k}.$$

Hence, taking the limit as $R \to \infty$, we obtain $w_k = 0$, that is

$$\partial_{yy} w_{k-1} + \frac{(\rho_\varepsilon^a)'}{\rho_\varepsilon^a} \partial_y w_{k-1} = 0.$$

This ODE can be explicitly solved:

$$w_{k-1} = c_{2k-1}(x,t) \int_0^y \rho_\varepsilon^{-a}(s) ds + c_{2k-2}(x,t), \tag{1.60}$$

where $c_{2k-1}(x,t)$ and $c_{2k-2}(x,t)$ are unknown functions, linear in x. Now, let us define

$$\begin{cases} g_1(y) = \int_0^y \rho_\varepsilon^{-a}(s)ds, \\ g_2(y) = \int_0^y \rho_\varepsilon^{-a}(s) \int_0^s \rho_\varepsilon^a(\tau)d\tau \\ g_i(y) = \int_0^y \rho_\varepsilon^{-a}(s) \int_0^s \rho_\varepsilon^a(\tau) g_{i-2}(\tau)d\tau, \quad \text{for } i \in \mathbb{N}_{\geq 3}, \end{cases} \tag{1.61}$$

which are linked by the relationship

$$\rho_\varepsilon^{-a}\partial_y(\rho_\varepsilon^a \partial_y g_i) = g_{i-2}, \quad \text{for } i \in \mathbb{N}_{\geq 3}.$$

An iterative argument combined with (1.60) and (1.59) shows that

$$w_j = c_{2j}(x,t) + \sum_{i=1}^{2(k-j)-1} g_i(y) c_{2j+i}(x,t),$$

for every $j = 1, \dots, k-1$, and thus, by (1.58),

$$u = c_0(x,t) + \sum_{i=1}^{2k-1} g_i(y) c_i(x,t),$$

where $c_i(x,t)$ are unknown functions, linear in x.

We claim that $c_i \equiv 0$ for any $i = 1, \dots, 2k-1$, which implies that u doesn't depends on y.

First, since $g_i(y)$ are odd functions for odd i, one has that $c_i(x,t) \equiv 0$ for odd i, being u an even function in y. Moreover, for every $i \geq 1$ the functions g_{2i} are asymptotically equivalent to $b_i y^{2i}$ for $y \to +\infty$, where $b_i \in \mathbb{R}$. Indeed, by using twice de l'Hôpital rule and by observing that

$$\lim_{y \to +\infty} \frac{\rho_\varepsilon(y)}{y} = 1,$$

we have that

$$\lim_{y\to+\infty} \frac{g_2(y)}{y^2} = \lim_{y\to+\infty} \frac{\rho_\varepsilon^{-a}(y) g_1(y)}{2y} = \lim_{y\to+\infty} \frac{\rho_\varepsilon^{-a}(y)}{y^{-a}} \frac{\int_0^y \rho_\varepsilon^a(s)ds}{2y^{1+a}} = \lim_{y\to+\infty} \frac{\rho_\varepsilon^a(y)}{2(1+a)y^a} = \frac{1}{2(1+a)}.$$

By using an inductive argument and (1.61), we can prove that

$$\lim_{|y|\to\infty} \frac{g_{2i}(y)}{y^{2i}} = b_i, \quad \text{where } b_i = \prod_{m=1}^{i} \frac{1}{2m(2m-1+a)}.$$

Hence, g_{2i} is asymptotically equivalent to $b_i y^{2i}$ for $y \to +\infty$. This immediately implies that $c_{2i} \equiv 0$ for every $i \geq 1$, by the parabolic sub-quadratic growth condition (1.53). Then, u does not depend on y and it is linear in x. Using the equation satisfied by u, we have that $\partial_t u = 0$, hence the thesis is proved, that is, $u = u(x)$ is a linear function. □

Remark 1.5.4. Let us highlight that when $a = 0$ (and therefore $\rho_\varepsilon^a = 1$), Theorem 1.5.1 remains valid for entire solutions to the heat equation $\partial_t u - \Delta u = 0$ in $\mathbb{R}^{d+2}$ and the proof above works in this setting as well, with minor changes. Furthermore, Theorem 1.5.1 still holds for entire solutions u to

$$\begin{cases}\rho_\varepsilon^a \partial_t u - \mathrm{div}(\rho_\varepsilon^a A\nabla u) = 0 & \text{in } \mathbb{R}^{d+1}_+ \times \mathbb{R},\\ \rho_\varepsilon^a (A\nabla u)\cdot e_{d+1} = 0 & \text{on } \partial\mathbb{R}^{d+1}_+ \times \mathbb{R},\end{cases}$$

where A is a constant symmetric positive definite matrix (and u satisfies (1.53)). Under such assumptions, u must to be a linear function depending only on z. This is a standard result, which immediately follows by a change of coordinates: since A is a symmetric positive definite matrix, we can consider the change of variables $z' = A^{1/2}z$, which allows us to reduce to the case $A = \mathbb{I}$.

1.6 $C_p^{0,\alpha}$ regularity estimates

In this section we prove the following uniform-in-ε Hölder estimates.

Theorem 1.6.1. *Let $a > -1$, $r \in (0,1)$, $p > \frac{d+3+a^+}{2}$, $q > d+3+a^+$, $\alpha \in (0,1) \cap (0, 2 - \frac{d+3+a^+}{p}] \cap (0, 1 - \frac{d+3+a^+}{q}]$. Let A be a continuous matrix satisfying (1.2) and ω be a modulus of continuity such that*

$$\|A\|_{L^\infty(Q_1^+)} + \sup_{(z,t),(z',t')\in Q_1^+} \frac{|A(z,t) - A(z',t')|}{\omega(|z-z'| + |t-t'|^{1/2})} \le L,$$

As $\varepsilon \to 0^+$ let $\{u_\varepsilon\}$ be a family of solutions to

$$\begin{cases}\rho_\varepsilon^a \partial_t u_\varepsilon - \mathrm{div}(\rho_\varepsilon^a A\nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon + \mathrm{div}(\rho_\varepsilon^a F_\varepsilon) & \text{in } Q_1^+,\\ \rho_\varepsilon^a (A\nabla u_\varepsilon + F_\varepsilon)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+.\end{cases} \tag{1.62}$$

Then, there exists a constant $C > 0$, depending on d, a, λ, Λ, p, q, r, L and α such that

$$\|u_\varepsilon\|_{C_p^{0,\alpha}(Q_r^+)} \le C(\|u_\varepsilon\|_{L^2(Q_1^+,\rho_\varepsilon^a)} + \|f_\varepsilon\|_{L^p(Q_1^+,\rho_\varepsilon^a)} + \|F_\varepsilon\|_{L^q(Q_1^+,\rho_\varepsilon^a)}). \tag{1.63}$$

Proof. Without loss of generality we can assume that $r = 1/2$ and there exists a constant $C > 0$, which is uniform in $\varepsilon \to 0^+$, such that

$$\|u_\varepsilon\|_{L^2(Q_1^+,\rho_\varepsilon^a)} + \|f_\varepsilon\|_{L^p(Q_1^+,\rho_\varepsilon^a)} + \|F_\varepsilon\|_{L^q(Q_1^+,\rho_\varepsilon^a)} \le C.$$

Otherwise (1.63) is trivially verified. From now on, we split the proof into several steps.

Step 1: Contradiction argument and blow-up sequences.

Consider a cut-off function $\eta \in C_c^\infty(Q_1^+)$ such that

$$\eta \equiv 1 \quad \text{in } Q_{1/2}^+, \quad 0 \le \eta \le 1, \quad \mathrm{spt}(\eta) = Q_{3/4}^+.$$

By smoothness of η, it immediately follows that $\eta \in C_p^{0,1}(Q_1^+)$; that is, there exists a constant $M > 0$, which depends only on d, such that

$$|\eta(P) - \eta(Q)| \le M d_p(P,Q), \qquad \text{for every } P = (z,t), Q = (\xi,\tau) \in Q_1^+,$$

where $d_p(\cdot,\cdot)$ is the parabolic distance, which is defined in (1.12).

We argue by contradiction. Let us suppose that there exist $p > \frac{d+3+a^+}{2}$, $q > d+3+a^+$, $\alpha \in (0,1) \cap (0, 2 - \frac{d+3+a^+}{p}] \cap (0, 1 - \frac{d+3+a^+}{q}]$ and a sequence of solutions $\{u_k\}_k := \{u_{\varepsilon_k}\}_k$ as

$\varepsilon_k \to 0^+$ to (1.62), such that

$$L_k := [\eta u_k]_{C_p^{0,\alpha}(Q_1^+)} = \sup_{\substack{P,Q\in Q_1^+\\ P\neq Q}} \frac{|(\eta u_k)(P)-(\eta u_k)(Q)|}{d_p(P,Q)^\alpha} \to \infty.$$

Now, by the definition of the parabolic Hölder seminorm of u_k, we can take two sequences of points $P_k = (z_k, t_k), \bar{P}_k = (\xi_k, \tau_k) \in Q_{3/4}^+$ such that

$$\frac{|(\eta u_k)(P_k)-(\eta u_k)(\bar{P}_k)|}{d_p(P_k,\bar{P}_k)^\alpha} \geq \frac{L_k}{2} \to \infty.$$

Defining $r_k := d_p(P_k, \bar{P}_k)$, one has that $r_k \to 0$ as $k \to \infty$. Indeed, by the local uniform boundedness of solutions, see Proposition 1.3.3, one has

$$\infty \leftarrow L_k \leq \frac{4\|\eta u_k\|_{L^\infty(Q_1^+)}}{r_k^\alpha} \leq C r_k^{-\alpha}.$$

Let $\bar{r} := 4/5$. For k large let us define the blow-up domains

$$Q(k) := \frac{B_{\bar{r}}^+ - z_k}{r_k} \times \frac{(-\bar{r}^2 - t_k, \bar{r}^2 - t_k)}{r_k^2.},$$

and set $Q^\infty := \lim_{k\to\infty} Q(k)$ along an appropriate subsequence. We define two blow-up sequences as

$$\begin{aligned} v_k(z,t) &:= \frac{\eta(r_k z + z_k, r_k^2 t + t_k)}{L_k r_k^\alpha}(u_k(r_k z + z_k, r_k^2 t + t_k) - u_k(z_k,t_k)),\\ w_k(z,t) &:= \frac{\eta(z_k,t_k)}{L_k r_k^\alpha}(u_k(r_k z + z_k, r_k^2 t + t_k) - u_k(z_k,t_k)), \end{aligned}$$

for $(z,t) \in Q(k)$. Then, we distinguish two cases:

Case 1:

$$\frac{y_k}{r_k} = \frac{d_p(P_k,\Sigma)}{r_k} \to \infty,$$

as $k \to \infty$. In this case we have $Q^\infty = \mathbb{R}^{d+2}$.

Case 2:

$$\frac{y_k}{r_k} = \frac{d_p(P_k,\Sigma)}{r_k} \leq C,$$

uniformly in k. In this case, one has $y_k/r_k \to l$ and so $Q^\infty = \mathbb{R}^d \times \{y \geq l\} \times \mathbb{R}$.

Step 2: Estimate of the parabolic Hölder seminorm of v_k.

Let us fix a compact set $K \subset Q^\infty$. Then, $K \subset Q(k)$ for any k large. For every $P = (z,t)$, $Q = (\xi,\tau) \in K$, $P \neq Q$, we have

$$\begin{aligned} |v_k(z,t) - v_k(\xi,\tau)| \leq& \frac{|(\eta u_k)(r_k z + z_k, r_k^2 t + t_k) - (\eta u_k)(r_k \xi + z_k, r_k^2 \tau + t_k)|}{L_k r_k^\alpha}\\ &+ \frac{|u_k(z_k,t_k)||\eta(r_k z + z_k, r_k^2 t + t_k) - \eta(r_k \xi + z_k, r_k^2 \tau + t_k)|}{L_k r_k^\alpha} \end{aligned}$$

$$\leq d_p(P,Q)^\alpha + \frac{\|u_k\|_{L^\infty(Q^+_{3/4})} M d_p((r_k z, r_k^2 t),(r_k\xi, r_k^2\tau))}{L_k r_k^\alpha}$$
$$\leq d_p(P,Q)^\alpha + \frac{CMr_k^{1-\alpha} d_p(P,Q)}{L_k}.$$

Then, as $k\to\infty$

$$\frac{|v_k(P)-v_k(Q)|}{d_p(P,Q)^\alpha} \leq 1 + o(1). \tag{1.64}$$

Step 3: The sequences v_k and w_k converge to the same limit w.

Notice that $v_k(0)=0$ for every k. Then, by (1.64), we have that $\|v_k\|_{C_p^{0,\alpha}(K)}$ is uniformly bounded for every compact subset $K \subset Q^\infty$. By the Arzelà-Ascoli theorem, we can pass to a subsequence v_k satisfying $v_k \to w$ uniformly in K and, taking the limit in (1.64), one has $w \in C_p^{0,\alpha}(K)$ with $[w]_{C_p^{0,\alpha}(K)} \leq 1$. Moreover, by a countable compact exhaustion of Q^∞, we have that w is globally $C_p^{0,\alpha}$-continuous in Q^∞, that is,

$$[w]_{C_p^{0,\alpha}(Q^\infty)} \leq 1. \tag{1.65}$$

Furthermore, fixed $K\subset Q^\infty$ compact, for every $P=(z,t)\in K$ one has

$$|v_k(P)-w_k(P)| = \frac{|(u_k(r_k z+z_k, r_k^2 t+t_k)-u_k(z_k,t_k))(\eta_k(r_k z+z_k, r_k^2 t+t_k)-\eta_k(z_k,t_k))|}{L_k r_k^\alpha}$$
$$\leq \frac{2\|u_k\|_{L^\infty(Q^+_{4/5})} r_k M d_p(P,0)}{L_k r_k^\alpha} \to 0.$$

In other words, the sequences v_k and w_k have the same asymptotic behavior as $k\to\infty$ on $K\subset Q^\infty$ and this implies that $w_k \to w$ uniformly in K.

Step 4: w is not constant.

First, $w(0)=0$, since $v_k(0)=0$ for every k. Let us consider the sequence of points

$$S_k := \Big(\frac{\xi_k - z_k}{r_k}, \frac{\tau_k - t_k}{r_k^2}\Big) \in Q(k).$$

Since $d_p(S_k,0)=1$ for any k, we have $S_k\to\bar S$, up to consider a subsequence. Then, as $k\to\infty$

$$|v_k(S_k)| = \Big|\frac{\eta(\bar P_k)(u_k(\bar P_k)-u_k(P_k))}{L_k r_k^\alpha}\Big|$$
$$= \Big|\frac{(\eta u_k)(\bar P_k)-(\eta u_k)(P_k)+(\eta u_k)(P_k)-\eta(\bar P_k)u_k(P_k)}{L_k r_k^\alpha}\Big|$$
$$\geq \Big|\frac{(\eta u_k)(\bar P_k)-(\eta u_k)(P_k)}{L_k r_k^\alpha}\Big| - \Big|\frac{u_k(P_k)(\eta(\bar P_k)-\eta(P_k))}{L_k r_k^\alpha}\Big|$$
$$\geq \frac{1}{2} - \frac{\|u_k\|_{L^\infty(Q^+_{3/4})} M r_k}{L_k r_k^\alpha} = \frac{1}{2} + o(1).$$

Then, as $k\to\infty$, we obtain that $w(\bar S)\geq \frac{1}{2}$; that is, w is not constant.

Step 5: w is an entire solution to a homogeneous equation with constant coefficients.

First, we observe that, defining $A_k(z,t) := A(r_kz+z_k, r_k^2t+t_k)$ and $(\bar{z},\bar{t}) := \lim_{k\to\infty}(z_k,t_k)$, by continuity we can define $\bar{A} := \lim_{k\to\infty} A_k(z,t) = A(\bar{z},\bar{t})$, which is a constant coefficients symmetric matrix satisfying (1.2).

Let us consider $\phi \in C_c^\infty(Q^\infty)$, such that $\mathrm{spt}(\phi) \subset Q(k)$ for any k large, and define $\tilde{\phi}(z,t) := \phi(\frac{z-z_k}{r_k}, \frac{t-t_k}{r_k^2}) \in C_c^\infty(Q_1^+)$. Since u_k is a solution to (1.62), by explicit computations, we have

$$\begin{aligned}
&-\int_{Q(k)} \rho_{\varepsilon_k}^a(r_ky+y_k)w_k\partial_t\phi + \int_{Q(k)} \rho_{\varepsilon_k}^a(r_ky+y_k)A_k\nabla w_k\cdot\nabla\phi\\
&= \frac{(\eta u_k)(z_k,t_k)}{L_kr_k^\alpha}\int_{Q(k)} \rho_{\varepsilon_k}^a(r_ky+y_k)\partial_t\phi\\
&+ \frac{\eta(z_k,t_k)}{L_kr_k^\alpha}\Big(-\int_{Q_1^+}\rho_{\varepsilon_k}^a(y)u_k\partial_t\tilde{\phi} + \int_{Q_1^+}\rho_{\varepsilon_k}^a(y)A\nabla u_k\cdot\nabla\tilde{\phi}\Big)r_k^{-d-1}\\
&= \frac{\eta(z_k,t_k)r_k^{-d-1-\alpha}}{L_k}\int_{Q_1^+}\rho_{\varepsilon_k}^a(y)(f_{\varepsilon_k}\tilde{\phi} - F_{\varepsilon_k}\cdot\nabla\tilde{\phi})\\
&= \frac{\eta(z_k,t_k)r_k^{2-\alpha}}{L_k}\int_{Q(k)}\rho_{\varepsilon_k}^a(r_ky+y_k)f_{\varepsilon_k}(r_kz+z_k, r_k^2t+t_k)\phi\\
&+ \frac{\eta(z_k,t_k)r_k^{1-\alpha}}{L_k}\int_{Q(k)}\rho_{\varepsilon_k}^a(r_ky+y_k)F_{\varepsilon_k}(r_kz+z_k, r_k^2t+t_k)\cdot\nabla\phi.
\end{aligned}$$

So, w_k is a solution in $Q(k)$ to

$$\begin{aligned}
&\rho_{\varepsilon_k}^a(r_k\cdot+y_k)\partial_tw_k - \mathrm{div}(\rho_{\varepsilon_k}^a(r_k\cdot+y_k)A_k\nabla w_k)\\
&\quad= \rho_{\varepsilon_k}^a(r_k\cdot+y_k)\frac{\eta(z_k,t_k)r_k^{2-\alpha}}{L_k}f_{\varepsilon_k}(r_k\cdot+z_k, r_k^2\cdot+t_k)\\
&\qquad+ \frac{\eta(z_k,t_k)r_k^{1-\alpha}}{L_k}\mathrm{div}(\rho_{\varepsilon_k}^a(r_k\cdot+y_k)F_{\varepsilon_k}(r_k\cdot+z_k, r_k^2\cdot+t_k)).
\end{aligned} \tag{1.66}$$

Notice that in **Case 2** the function w_k satisfies a conormal boundary condition on the hyperplane $\{y = y_k/r_k\}$ too.

Next, we normalize the equation (1.66) in the following way: let us define $\Gamma_k := (\varepsilon_k, y_k, r_k)$ and $\nu_k := |\Gamma_k|$, which is bounded from above, since $r_k \to 0$, $\varepsilon_k \to 0$ and $y_k \to \bar{y} \in [0,1)$. Let

$$\tilde{\Gamma}_k := \frac{\Gamma_k}{\nu_k} = \Big(\frac{\varepsilon_k}{\nu_k}, \frac{y_k}{\nu_k}, \frac{r_k}{\nu_k}\Big) = (\tilde{\varepsilon_k}, \tilde{y}_k, \tilde{r}_k).$$

Since $|\tilde{\Gamma_k}| = 1$ for every k, up to consider a subsequence, $\tilde{\Gamma}_k \to \tilde{\Gamma} = (\tilde{\varepsilon}, \tilde{y}, \tilde{r})$. Denoting

$$\tilde{\rho}_k^a(y) := \frac{\rho_{\varepsilon_k}^a(r_ky+y_k)}{\nu_k^a} = (\tilde{\varepsilon}_k^2 + (\tilde{r}_ky+\tilde{y}_k)^2)^{a/2},$$

and

$$\tilde{\rho}^a(y) := (\tilde{\varepsilon}^2 + (\tilde{r}y+\tilde{y})^2)^{a/2},$$

we have that $\tilde\rho_k^a \to \tilde\rho^a$ a.e. in Q^∞. By multiplying the equation (1.66) by ν_k^{-a} we get that w_k solves

$$\begin{aligned}\tilde\rho_k^a\partial_t w_k - \mathrm{div}(\tilde\rho_k^a A_k\nabla w_k) = \tilde\rho_k^a\frac{\eta(z_k,t_k)r_k^{2-\alpha}}{L_k}f_{\varepsilon_k}(r_k\cdot+z_k,r_k^2\cdot+t_k)\\ +\frac{\eta(z_k,t_k)r_k^{1-\alpha}}{L_k}\mathrm{div}(\tilde\rho_k^a F_{\varepsilon_k}(r_k\cdot+z_k,r_k^2\cdot+t_k)).\end{aligned}\tag{1.67}$$

We claim that the right hand side of (1.67) vanishes in a distributional sense as $k\to\infty$. Indeed, fixed $\phi\in C_c^\infty(Q^\infty)$ with $\mathrm{spt}(\phi)\subset Q(k)$ for any k large, we have the following estimate

$$\begin{aligned}&\Big|\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_ky+y_k)f_{\varepsilon_k}(r_kz+z_k,r_k^2t+t_k)\phi(z,t)dzdt\Big|\\ &\le\|\phi\|_{L^\infty(\mathbb{R}^{d+2})}\Big(\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_ky+y_k)|f_{\varepsilon_k}(r_kz+z_k,r_k^2t+t_k)|^p\Big)^{1/p}\cdot\Big(\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_ky+y_k)\Big)^{1/p'}\\ &\le C\|\phi\|_{L^\infty(\mathbb{R}^{d+2})}\Big(\int_{Q_1^+}(\varepsilon_k^2+\xi_{d+1}^2)^{a/2}|f_{\varepsilon_k}(\xi,\tau)|^p r_k^{-(d+3)}d\xi d\tau\Big)^{1/p}\nu_k^{a/p'}\\ &\le C\|f_{\varepsilon_k}\|_{L^p(Q_1^+,\rho_\varepsilon^a)}r_k^{-\frac{d+3}{p}}\nu_k^{\frac{a}{p'}}\le Cr_k^{-\frac{d+3}{p}}\nu_k^{\frac{a}{p'}}.\end{aligned}$$

So, we can estimate the first member of the right hand side of (1.67) as follows

$$\begin{aligned}&\frac{\eta(z_k,t_k)r_k^{2-\alpha}\nu_k^{-a}}{L_k}\Big|\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_ky+y_k)f_{\varepsilon_k}(r_kz+z_k,r_k^2t+t_k)\phi(z,t)dzdt\Big|\\ &\le C\nu_k^{-a}\frac{\eta(z_k,t_k)r_k^{2-\alpha}}{L_k}r_k^{-\frac{d+3}{p}}\nu_k^{\frac{a}{p'}}\le Cr_k^{2-\alpha-\frac{d+3+a^+}{p}}\Big(\frac{r_k^{a^+}}{\nu_k^a}\Big)^{1/p}\to 0,\end{aligned}$$

since $r_k\le\nu_k$ and $\alpha<2-\frac{d+3+a^+}{p}$. Similarly, the second term of the right hand side of (1.67) vanishes as well.

Finally, we prove that the left hand side of (1.67) converges in the following sense

$$\int_{Q^\infty}\tilde\rho_k^a(-w_k\partial_t\phi+A_k\nabla w_k\cdot\nabla\phi)\to\int_{Q^\infty}\tilde\rho^a(-w\partial_t\phi+\bar A\nabla w\cdot\nabla\phi).\tag{1.68}$$

Let us fix $R>0$ such that $\mathrm{spt}(\phi)\subset Q_R\cap Q^\infty$ and observe that $Q^\infty=B^\infty\times\mathbb{R}$. Since $\{w_k\}$ is uniformly bounded in $L^\infty(Q_{2R}\cap Q^\infty)$ one has that $\{w_k\}$ is uniformly bounded in $L^2(Q_{2R}\cap Q^\infty,\tilde\rho_k^a)$. Then, by using the Caccioppoli inequality (1.27), we get that $\{w_k\}$ is uniformly bounded in $L^2(-R^2,R^2;H^1(B_R\cap B^\infty,\tilde\rho_k^a))\cap L^\infty(-R^2,R^2;L^2(B_R\cap B^\infty,\tilde\rho_k^a))$. Using the a.e. convergences $A_k(z,t)\to\bar A$ and $\tilde\rho_k^a\to\tilde\rho^a$, we are able to apply Lemma 1.4.1, with minor changes, and the convergence (1.68) follows. Hence, we have proved that w is an entire solution to

$$\tilde\rho^a\partial_t w-\mathrm{div}(\tilde\rho^a\bar A\nabla w)=0,\quad\text{in }\mathbb{R}^{d+2},\tag{1.69}$$

in **Case 1** while, in **Case 2**, w is an entire solution to

$$\begin{cases}\tilde\rho^a\partial_t w-\mathrm{div}(\tilde\rho^a\bar A\nabla w)=0, & \text{in }\mathbb{R}^{d+1}_+\times\mathbb{R},\\ \tilde\rho^a\bar A\nabla w\cdot e_{d+1}=0 & \text{on }\mathbb{R}^d\times\{y=l\}\times\mathbb{R}.\end{cases}\tag{1.70}$$

Step 6: Liouville theorems.

Summarizing, we have that w solves (1.69) or (1.70), is globally $C_p^{0,\alpha}$-continuous in Q^∞ and is not constant. By the global $C_p^{0,\alpha}$-continuity (1.65), it follows that

$$|w(z,t)| \le |w(z,t)-w(0,0)| + |w(0,0)| \le (|z|^2+|t|)^{\alpha/2}.$$

In **Case 1**, since $y_k/r_k \to \infty$, we have

$$\tilde\rho_k^a(y) = \left(\frac{1}{\nu_k^2}\left(\varepsilon_k^2 + y_k^2\left(\frac{r_k}{y_k}y+1\right)^2\right)\right)^{a/2} = \left(\tilde\varepsilon_k^2 + \tilde{y_k}^2\left(\frac{r_k}{y_k}y+1\right)^2\right)^{a/2} \to (\tilde\varepsilon^2+\tilde y^2)^{a/2},$$

which is a positive constant. Then, by the classical Liouville theorem for the heat equation, see Remark 1.5.4, and the above growth condition, the solution w must be constant and this is a contradiction.

In **Case 2**, $y_k/r_k \le C$, uniformly in k and $\tilde{y_k}/\tilde{r_k} = y_k/r_k \to \tilde y/\tilde r = l$. Up to consider a translation of $\tilde y/\tilde r = l$, we can assume $\tilde y = 0$ and then $\tilde\rho^a(y) = (\tilde\varepsilon^2 + \tilde r^2 y^2)^{a/2}$. There are three possibilities:

i) $\tilde\varepsilon = 0, \tilde r \neq 0$, $\tilde\rho^a(y) = |y|^a$.

ii) $\tilde\varepsilon \neq 0, \tilde r = 0$, $\tilde\rho^a(y) = 1$.

iii) $\tilde\varepsilon \neq 0, \tilde r \neq 0$, $\tilde\rho^a(y) = (1+y^2)^{a/2}$, up to a dilation of $\tilde\varepsilon/\tilde r$.

In any case, we can invoke Liouville Theorem 1.5.1 in $\mathbb{R}^{d+1}_+ \times \mathbb{R}$ and by Remark 1.5.4 we obtain again a contradiction. The proof is complete. ☐

1.7 $C_p^{1,\alpha}$ regularity estimates

In this section, we first prove the following uniform-in-ε Hölder estimates for the gradient. Then, we establish the main theorem in the spaces $C_p^{0,\alpha}$ and $C_p^{1,\alpha}$.

Theorem 1.7.1. *Let $a > -1$, $r \in (0,1)$ $p > d+3+a^+$, $\alpha \in (0, 1-\frac{d+3+a^+}{p})$. Let $A \in C_p^{0,\alpha}(Q_1^+)$ be a matrix satisfying (1.2). As $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$\begin{cases} \rho_\varepsilon^a \partial_t u_\varepsilon - \mathrm{div}(\rho_\varepsilon^a A\nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon + \mathrm{div}(\rho_\varepsilon^a F_\varepsilon) & \text{in } Q_1^+,\\ \rho_\varepsilon^a(A\nabla u_\varepsilon + F)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+. \end{cases} \tag{1.71}$$

Then, there exists a constant $C>0$ depending on d, a, λ, Λ, p, r, α and $\|A\|_{C_p^{0,\alpha}(Q_1^+)}$ such that

$$\|u_\varepsilon\|_{C_p^{1,\alpha}(Q_{1/2}^+)} \le C(\|u_\varepsilon\|_{L^2(Q_1^+,\rho_\varepsilon^a)} + \|f_\varepsilon\|_{L^p(Q_1^+,\rho_\varepsilon^a)} + \|F_\varepsilon\|_{C_p^{0,\alpha}(Q_1^+)}).$$

Proof. To simplify the notation, let $\partial_i := \partial_{x_i}$ for $i = 1,\dots,d$ and $\partial_{d+1} := \partial_y$. As in Theorem 1.6.1, without loss of generality, we can assume that $r = 1/2$ and there exists $C > 0$, which is uniform in $\varepsilon \to 0^+$, such that

$$\|u_\varepsilon\|_{L^2(Q_1^+,\rho_\varepsilon^a)} + \|f_\varepsilon\|_{L^p(Q_1^+,\rho_\varepsilon^a)} + \|F_\varepsilon\|_{C_p^{0,\alpha}(Q_1^+)} \le C.$$

Step 1: Contradiction argument and blow-up sequences.

Consider a cut-off function $\eta \in C_c^\infty(Q_1^+)$ such that

$$\eta \equiv 1 \quad \text{in } Q_{1/2}^+, \quad 0 \leq \eta \leq 1, \quad \operatorname{spt}(\eta) = Q_{3/4}^+.$$

By smoothness of η, it immediately follows that $\eta \in C_p^{1,1}(Q_1^+)$; that is, there exists a constant $M > 0$, which depends only on d, such that $\|\eta\|_{C_p^{1,1}(Q_1^+)} \leq M$.

By contradiction, let us suppose that there exist $p > d + 3 + a^+$, $\alpha \in (0, 1 - \frac{d+3+a^+}{p})$ and a sequence of solutions $\{u_k\} := \{u_{\varepsilon_k}\}$ as $\varepsilon_k \to 0^+$ to (1.71), such that

$$\|\eta u_k\|_{C_p^{1,\alpha}(Q_1^+)} \to \infty.$$

Define

$$L_k := \max\left\{\{[\partial_i(\eta u_k)]_{C_p^{0,\alpha}(Q_1^+)} : i = 1, \ldots, d+1\}, [\eta u_k]_{C_t^{0,\frac{1+\alpha}{2}}(Q_1^+)}\right\},$$

and distinguish two cases: first, let us suppose that there exists $i \in \{1, \ldots, d+1\}$ such that $L_k = [\partial_i(\eta u_k)]_{C_p^{0,\alpha}(Q_1^+)}$ (later we will deal with the second case, when $L_k = [\eta u_k]_{C_t^{0,\frac{1+\alpha}{2}}(Q_1^+)}$). Notice that it cannot be $\|\nabla(\eta u_k)\|_{L^\infty(Q_1^+)} \to \infty$ and $[\eta u_k]_{C_p^{1,\alpha}(Q_1^+)}$ remains bounded, since the functions ηu_k are identically zero outside $Q_{3/4}^+$, for every k.

Next, we take two sequences of points $P_k = (z_k, t_k), \bar{P}_k = (\xi_k, \tau_k) \in Q_{3/4}^+$ such that

$$\frac{|\partial_i(\eta u_k)(P_k) - \partial_i(\eta u_k)(\bar{P}_k)|}{d_p(P_k, \bar{P}_k)^\alpha} \geq \frac{1}{2} L_k \to \infty.$$

Let $r_k := d_p(P_k, \bar{P}_k)$, $\hat{z}_k := (\hat{x}_k, \hat{y}_k) \in B_{3/4}^+$ be a sequence of points which will specify below. Let $\bar{r} := 4/5$. For k large let us define

$$Q(k) := \frac{B_{\bar{r}}^+ - \hat{z}_k}{r_k} \times \frac{(-\bar{r}^2 - t_k, \bar{r}^2 - t_k)}{r_k^2.},$$

and set $Q^\infty := \lim_{k\to\infty} Q(k)$. We define two blow-up sequences as follows

$$\begin{aligned} v_k(z,t) &:= \frac{\eta(r_k z + \hat{z}_k, r_k^2 t + t_k)}{L_k r_k^{1+\alpha}} (u_k(r_k z + \hat{z}_k, r_k^2 t + t_k) - u_k(\hat{z}_k, t_k)), \\ w_k(z,t) &:= \frac{\eta(\hat{z}_k, t_k)}{L_k r_k^{1+\alpha}} (u_k(r_k z + \hat{z}_k, r_k^2 t + t_k) - u_k(\hat{z}_k, t_k)), \end{aligned} \tag{1.72}$$

for $(z,t) \in Q(k)$. Then, we distinguish two cases:

Case 1:

$$\frac{y_k}{r_k} = \frac{d_p(P_k, \Sigma)}{r_k} \to \infty,$$

as $k \to \infty$. Since y_k is uniformly bounded, we have that $r_k \to 0$ and $Q^\infty = \mathbb{R}^{d+2}$. In this case we set $\hat{z}_k = z_k$.

Case 2:

$$\frac{y_k}{r_k} = \frac{d_p(P_k, \Sigma)}{r_k} \leq C,$$

uniformly in k. We set $\hat{z}_k = (x_k, 0)$ and we will show later that also in this case $r_k \to 0$, which implies that $Q^\infty = \mathbb{R}^{d+1}_+ \times \mathbb{R}$.

Step 2: Parabolic Hölder estimates.

Let us fix a compact set $K \subset Q^\infty$. Then, $K \subset Q(k)$ for any k large. For every $P = (z,t), Q = (\xi, \tau) \in K$, $P \neq Q$ and for every $j = 1, \dots, d+1$, we have

$$\begin{aligned}
|\partial_j v_k(P) - \partial_j v_k(Q)| &\le \frac{|\partial_j(\eta u_k)(r_k z + \hat{z}_k, r_k^2 t + t_k) - \partial_j(\eta u_k)(r_k \xi + \hat{z}_k, r_k^2 \tau + t_k)|}{L_k r_k^\alpha} \\
&+ \frac{|u_k(\hat{z}_k, t_k)||\partial_j \eta(r_k z + \hat{z}_k, r_k^2 t + t_k) - \partial_j \eta(r_k \xi + \hat{z}_k, r_k^2 \tau + t_k)|}{L_k r_k^\alpha} \\
&\le \frac{[\partial_j(\eta u_k)]_{C^{0,\alpha}_p(Q_1^+)} d_p(P,Q)^\alpha}{L_k} + \frac{\|u_k\|_{L^\infty(Q^+_{3/4})} r_k^{1-\alpha} M d_p(P,Q)}{L_k} \\
&\le d_p(P,Q)^\alpha + \frac{CM}{L_k},
\end{aligned}$$

since $[\partial_j(\eta u_k)]_{C^{0,\alpha}_p(Q_1^+)} \le L_k$, $r_k \le C$, $\|u_k\|_{L^\infty(Q^+_{3/4})} \le C$ and $d_p(P,Q)^{1-\alpha} \le C$ in K. By dividing the previous inequality by $d_p(P,Q)^\alpha$ and using $L_k \to \infty$, we get

$$\sup_{\substack{P,Q \in K \\ P \neq Q}} \frac{|\partial_j v_k(P) - \partial_j v_k(Q)|}{d_p(P,Q)^\alpha} \le 1 + o(1). \tag{1.73}$$

as $k \to \infty$. On the other hand, for every $P = (z,t), Q = (z,\tau) \in K$, $t \neq \tau$, we have that

$$\begin{aligned}
|v_k(P) - v_k(Q)| &\le \frac{|(\eta u_k)(r_k z + \hat{z}_k, r_k^2 t + t_k) - (\eta u_k)(r_k z + \hat{z}_k, r_k^2 \tau + t_k)|}{L_k r_k^{1+\alpha}} \\
&+ \frac{|u_k(\hat{z}_k, t_k)||\eta(r_k z + \hat{z}_k, r_k^2 t + t_k) - \eta(r_k z + x_k, r_k^2 \tau + t_k)|}{L_k r_k^{1+\alpha}} \\
&\le |t - \tau|^{\frac{1+\alpha}{2}} + \frac{\|u_k\|_{L^\infty(Q^+_{3/4})} r_k^{1-\alpha} M |t-\tau|}{L_k},
\end{aligned}$$

so

$$\sup_{\substack{(z,t),(z,\tau) \in K \\ t \neq \tau}} \frac{|v_k(z,t) - v_k(z,\tau)|}{|t-\tau|^{\frac{1+\alpha}{2}}} \le 1 + o(1). \tag{1.74}$$

Putting together this inequality with (1.73), we obtain the uniform boundedness of $[v_k]_{C^{1,\alpha}_p(K)}$, noticing that these considerations are valid in both **Case 1** and **Case 2**.

Step 3: Convergence of blow-ups.

For $P = (z,t) \in Q(k)$, let us define

$$\bar{v}_k(P) := v_k(P) - \nabla v_k(0) \cdot z, \quad \bar{w}_k(P) := w_k(P) - \nabla w_k(0) \cdot z. \tag{1.75}$$

Notice that $\bar{v}_k(0) = 0 = \bar{w}_k(0)$ and $|\nabla \bar{v}_k(0)| = 0 = |\nabla \bar{w}_k(0)|$. For every $K \subset Q^\infty$ compact, since $[\bar{v}_k]_{C^{1,\alpha}_p(K)} = [v_k]_{C^{1,\alpha}_p(K)}$, we have that $\|\bar{v}_k\|_{C^{1,\alpha}_p(K)}$ is uniformly bounded. Then, we can apply the Arzelá-Ascoli theorem and infer that $\bar{v}_k \to \bar{v}$ in $C^{1,\gamma}_p(K)$, for any $\gamma < \alpha$. Now, passing to the limit in (1.73) and in (1.74) and by a countable compact exhaustion of Q^∞, we obtain that the limit function $\bar{v}$ satisfies

$$[\bar{v}]_{C_p^{1,\alpha}(Q^\infty)} \leq C,$$

that is, $\bar{v}$ is globally $C_p^{1,\alpha}$-continuous in Q^∞.

Next, we want to show that the sequence $\{\bar{w}_k\}$ converges uniformly to $\bar{v}$ on compact sets. Let us fix $K \subset Q^\infty$, such that $K \subset Q(k)$ for any k large. Since $\nabla \bar{v}_k(0) = \nabla \bar{w}_k(0)$, for every $P = (z,t) \in K$, we have

$$\begin{aligned}
|\bar{v}_k(P) - \bar{w}_k(P)| &= |v_k(P) - w_k(P)| \\
&\leq \frac{|\eta(r_k z + \hat{z}_k, r_k^2 t + t_k) - \eta(\hat{z}_k, t_k)| \cdot |u_k(r_k z + \hat{z}_k, r_k^2 t + t_k) - u_k(\hat{z}_k, t_k)|}{L_k r_k^{1+\alpha}} \\
&\leq \frac{C r_k d_p(P,0) \cdot M r_k^\alpha d_p(P,0)^\alpha}{L_k r_k^{1+\alpha}} = \frac{CM d_p(P,0)^{1+\alpha}}{L_k} \to 0,
\end{aligned}$$

as $k \to \infty$, by the properties of η and the Theorem 1.6.1, which ensures local uniform bound of u_k in $C_p^{0,\alpha}$-space. This implies that $\bar{w}_k \to \bar{v}$ uniformly in K.

Step 4: $\nabla \bar{v}$ is not constant.

Let us define two sequences of points as

$$S_k := \Big(\frac{\xi_k - \hat{z}_k}{r_k}, \frac{\tau_k - t_k}{r_k^2}\Big), \quad \bar{S}_k := \Big(\frac{z_k - \hat{z}_k}{r_k}, 0\Big) \in Q(k).$$

In **Case 1**, one has $\hat{z}_k = z_k$, then $S_k \to S \in Q^\infty$, up to consider a subsequence, and $\bar{S}_k = 0$. Let $i \in \{1, \dots, d+1\}$ be the one that realizes the maximum of L_k. We can compute, as $k \to \infty$

$$\begin{aligned}
|\partial_i \bar{v}_k(S_k) - \partial_i \bar{v}_k(\bar{S}_k)| &= |\partial_i v_k(S_k) - \partial_i v_k(0)| \\
&= \frac{|\partial_i(\eta u_k)(\bar{P}_k) - \partial_i(\eta u_k)(P_k) - u_k(P_k)(\partial_i \eta(\bar{P}_k) - \partial_i \eta(P_k))|}{L_k r_k^\alpha} \\
&\geq \frac{1}{2} - \frac{\|u_k\|_{L^\infty(Q^+_{3/4})} M r_k^{1-\alpha}}{L_k} = \frac{1}{2} + o(1).
\end{aligned}$$

Then, as $k \to \infty$, we obtain that $|\partial_i \bar{v}(S) - \partial_i \bar{v}(0)| \geq \frac{1}{2}$, which implies that $\nabla \bar{v}$ is not constant.

Instead, in **Case 2**, we have $\bar{S}_k = y_k/r_k e_{d+1}$, which converge to a point $\bar{S}$, up to consider a subsequence, by the fact that $y_k/r_k \leq C$ uniformly in k. The sequence S_k can be written as

$$S_k = \Big(\frac{\xi_k - z_k}{r_k}, \frac{\tau_k - t_k}{r_k^2}\Big) + \frac{y_k}{r_k} e_{d+1},$$

and still converges, up to a subsequence, to a point $S \in Q^\infty$. So, also in this case, we have

$$|\partial_i \bar{v}_k(S_k) - \partial_i \bar{v}_k(\bar{S}_k)| = |\partial_i v_k(S_k) - \partial_i v_k(0)| \geq \frac{1}{2} + o(1),$$

which allows us to conclude that $\bar{v}$ has non constant gradient exactly as in **Case 1**.

Step 5: $r_k \to 0$ in **Case 2**.

By contradiction, let us suppose that, up to consider a subsequence, $r_k \to \tilde{r} > 0$ in **Case 2**. Then, if $K \subset Q^\infty$ is a fixed compact set, we have

$$\sup_{P\in K} |v_k(P)| \leq 2\frac{\|\eta\|_{L^\infty(Q_1^+)}\|u_k\|_{L^\infty(Q_{3/4}^+)}}{L_k r_k^{1+\alpha}} \to 0,$$

as $k \to \infty$, which means that $v_k \to 0$ uniformly on compact sets of Q^∞. For every $P = (z,t) \in K$, by using the convergence $\bar{v}_k \to \bar{v}$ obtained in *Step 3*, one has

$$\bar{v}(P) = \lim_{k\to\infty} \nabla v_k(0) \cdot z.$$

We claim that the sequence $\{\nabla v_k(0)\}_k$ is bounded. Indeed, assume by contradiction that there exists $j \in \{1, \dots, d+1\}$ such that $\{\partial_j v_k(0)\}$ is unbounded. Fix $R > 0$ sufficiently small such that Q_R^+ is contained in Q^∞. Then

$$|\bar{v}(Re_j)| = R \lim_{k\to\infty} |\nabla v_k(0) \cdot e_j| = R|\partial_j v_k(0)| \to \infty,$$

which is in contradiction to the fact $\bar{v} \in C_p^{1,\alpha}(Q_R^+)$ and hence bounded in Q_R^+. Hence, up to consider a subsequence, we have that $\nabla v_k(0) \to \nu \in \mathbb{R}^{d+1}$ and $\bar{v}(z,t) = \nu \cdot z$, which is in contradiction to the fact that $\bar{v}$ has non constant gradient. So, we have shown that $r_k \to 0$ also in **Case 2**, which implies that $Q^\infty = \mathbb{R}^{d+1}_+ \times \mathbb{R}$.

Step 6: $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients.

First, we look at the equation satisfied by $\bar{w}_k$ in $Q(k)$. As in Theorem 1.6.1, let us define $\nu_k = |(\varepsilon_k, \hat{y}_k, r_k)|$ and $(\tilde{\varepsilon}_k, \tilde{\hat{y}}_k, \tilde{r}_k) = (\frac{\varepsilon_k}{\nu_k}, \frac{\hat{y}_k}{\nu_k}, \frac{r_k}{\nu_k})$, which converges, up to consider a subsequence, to $(\tilde{\varepsilon}, \tilde{y}, \tilde{r})$. Defining

$$\tilde{\rho}_k^a(y) = \frac{\rho_{\varepsilon_k}^a(r_k y + \tilde{y}_k)}{\nu_k^a} = (\tilde{\varepsilon}_k^2 + (\tilde{r}_k y + \tilde{\hat{y}}_k)^2)^{a/2},$$

and

$$\tilde{\rho}^a(y) := (\tilde{\varepsilon}^2 + (\tilde{r}y + \tilde{y})^2)^{a/2},$$

we have that $\tilde{\rho}_k^a \to \tilde{\rho}^a$ a.e. in Q^∞.

Let us fix $\phi \in C_c^\infty(Q^\infty)$, with $\mathrm{spt}(\phi) \subset Q(k)$ for any k large. Then,

$$\begin{aligned}
&\int_{\mathrm{spt}(\phi)} \tilde{\rho}_k^a(y)\Big(- \bar{w}_k \partial_t \phi + A(r_k z + \hat{z}_k, r_k^2 t + t_k)\nabla \bar{w}_k \cdot \nabla\phi\Big) \\
&= \frac{\eta(\hat{z}_k, t_k) r_k^{1-\alpha}\nu_k^{-a}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_{\varepsilon_k}^a(r_k y + \hat{y}_k) f_{\varepsilon_k}(r_k z + \hat{z}_k, r_k^2 t + t_k)\phi \\
&- \frac{\eta(\hat{z}_k, t_k)\nu_k^{-a}}{L_k r_k^\alpha} \int_{\mathrm{spt}(\phi)} \rho_{\varepsilon_k}^a(r_k y + \hat{y}_k)\Big(F_{\varepsilon_k}(r_k z + \hat{z}_k, r_k^2 t + t_k) - F_{\varepsilon_k}(\hat{z}_k, t_k)\Big)\cdot\nabla\phi \qquad (1.76) \\
&- \frac{\eta(\hat{z}_k, t_k)\nu_k^{-a}}{L_k r_k^\alpha} \int_{\mathrm{spt}(\phi)} \rho_{\varepsilon_k}^a(r_k y + \hat{y}_k)\Big(A(r_k z + \hat{z}_k, r_k^2 t + t_k) - A(\hat{z}_k, t_k)\Big)\nabla u_k(\hat{z}_k, t_k)\cdot\nabla\phi \\
&+ \frac{\eta(\hat{z}_k, t_k)\nu_k^{-a}}{L_k r_k^\alpha} \int_{\mathrm{spt}(\phi)} \rho_{\varepsilon_k}^a(r_k y + \hat{y}_k)\Big(A(\hat{z}_k, t_k)\nabla u_k(\hat{z}_k, t_k) + F_{\varepsilon_k}(\hat{z}_k, t_k)\Big)\cdot\nabla\phi.
\end{aligned}$$

Next, we show that the right hand side of (1.76) vanishes in a distributional sense as $k \to \infty$. The first member can be estimate exactly as in Theorem 1.6.1, and by the hypothesis on p and α, we obtain the desired convergence to zero. The second can be bounded as follows

$$\begin{aligned}&\Big|\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_k r_k^{\alpha}}\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)\Big(F_{\varepsilon_k}(r_k z+\hat{z}_k, r_k^2 t+t_k)-F_{\varepsilon_k}(\hat{z}_k,t_k)\Big)\cdot\nabla\phi\Big|\\ &\leq \frac{\nu_k^{-a}\|\nabla\phi\|_{L^\infty(Q^\infty)}}{L_k r_k^{a}}\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)Cr_k^{\alpha}(|z|+|t|^{1/2})^{\alpha}\leq\frac{C}{L_k}\to 0,\end{aligned}$$

as $k \to \infty$. In the previous inequalities, we have used the uniform boundedness of F_{ε_k} in $C_p^{0,a}$-space and the estimate $\int_{\mathrm{spt}(\phi)}\tilde{\rho}_k^a \leq C$.

Next, we show that the fourth member vanishes. First, we can rewrite it as

$$\begin{aligned}&\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_k r_k^{\alpha}}\int_{\mathrm{spt}(\phi)}\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)\Big(A(\hat{z}_k,t_k)\nabla u_k(\hat{z}_k,t_k)+F_{\varepsilon_k}(\hat{z}_k,t_k)\Big)\cdot\nabla\phi\\ &=\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_k r_k^{\alpha}}\int_{\mathrm{spt}(\phi)}\mathrm{div}\Big(\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)(A(\hat{z}_k,t_k)\nabla u_k(\hat{z}_k,t_k)+F_{\varepsilon_k}(\hat{z}_k,t_k))\phi\Big)\\ &+\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_k r_k^{\alpha}}\int_{\mathrm{spt}(\phi)}\partial_y\big(\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)\big)(A(\hat{z}_k,t_k)\nabla u_k(\hat{z}_k,t_k)+F_{\varepsilon_k}(\hat{z}_k,t_k))\cdot e_{d+1}\phi.\end{aligned}\tag{1.77}$$

By using the divergence theorem, we can rewrite the first member in (1.77) as

$$\begin{aligned}&\int_{\mathrm{spt}(\phi)}\mathrm{div}\Big(\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)(A(\hat{z}_k,t_k)\nabla u_k(\hat{z}_k,t_k)+F_{\varepsilon_k}(\hat{z}_k,t_k))\phi\Big)dzdt\\ &=\int_{\partial\{\mathrm{spt}(\phi)\}}\Big(\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)(A(\hat{z}_k,t_k)\nabla u_k(\hat{z}_k,t_k)+F_{\varepsilon_k}(\hat{z}_k,t_k))\phi\Big)d\sigma,\end{aligned}$$

and observe that this is equal to zero. In fact, in **Case 1** we have $Q^\infty = \mathbb{R}^{d+2}$ and ϕ has compact support. Instead, in **Case 2**, since $\hat{z}_k$ lies on the flat boundary, the term vanishes by the conormal boundary condition satisfied by u_k.

The second term in the right hand side of (1.77) vanishes too. In **Case 2** it is identically zero since $(A\nabla u_k + F_{\varepsilon_k})(\hat{z}_k, t_k) = 0$ by the conormal boundary condition. Let us consider **Case 1** and recall that $\hat{y}_k = y_k$ and $r_k/y_k \to 0$ as $k \to \infty$. Then, on compact subsets of $\mathbb{R}^{d+2}$, one has the following estimate

$$\begin{aligned}\big|\nu_k^{-a}\partial_y[\rho_{\varepsilon_k}^a(r_k y+y_k)]\big|&=\Big|\nu_k^{-a}ar_k\rho_{\varepsilon_k}^a(r_k y+y_k)\frac{r_k y+y_k}{\varepsilon_k^2+(r_k y+y_k)^2}\Big|\\ &\leq|a|\tilde{\rho}_k^a(y)\frac{r_k}{y_k}\frac{\frac{r_k}{y_k}y+1}{\frac{\varepsilon_k^2}{y_k^2}+\Big(\frac{r_k}{y_k}y+1\Big)^2}\leq C\frac{r_k}{y_k}.\end{aligned}$$

Next, let $(\zeta_k, t_k) = (x_k, 0, t_k)$ be the projection of $(\hat{z}_k, t_k) = (z_k, t_k)$ on the hyperplane $\{y = 0\}$. By the conormal boundary condition, we have that $[\eta(A\nabla u_k + F_{\varepsilon_k})](\zeta_k, t_k) \cdot e_{d+1} = 0$, so

$$\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_k r_k^{\alpha}}\int_{\mathrm{spt}(\phi)}\partial_y\big(\rho_{\varepsilon_k}^a(r_k y+\hat{y}_k)\big)(A\nabla u_k+F_{\varepsilon_k})(\hat{z}_k,t_k)\cdot e_{d+1}\phi(z,t)dzdt$$

$$
\begin{aligned}
&= \frac{\nu_k^{-a}}{L_k r_k^\alpha}\int_{\mathrm{spt}(\phi)} \partial_y\big(\rho^a_{\varepsilon_k}(r_k y+\hat{y}_k)\big)\big[\eta(A\nabla u_k+F_{\varepsilon_k})\big](\hat{z}_k,t_k)\cdot e_{d+1}\phi(z,t)dzdt\\
&- \frac{\nu_k^{-a}}{L_k r_k^\alpha}\int_{\mathrm{spt}(\phi)} \partial_y\big(\rho^a_{\varepsilon_k}(r_k y+\hat{y}_k)\big)\big[\eta(A\nabla u_k+F_{\varepsilon_k})\big](\zeta_k,t_k)\cdot e_{d+1}\phi(z,t)dzdt.
\end{aligned}
$$

We can estimate

$$
\begin{aligned}
&\big|\big[\eta(A\nabla u_k+F_{\varepsilon_k})\big](\hat{z}_k,t_k)-\big[\eta(A\nabla u_k+F_{\varepsilon_k})\big](\zeta_k,t_k)\big| \le \big|A\nabla(\eta u_k)(\hat{z}_k,t_k)-A\nabla(\eta u_k)(\zeta_k,t_k)\big|\\
&+\big|u_kA\nabla\eta(\hat{z}_k,t_k)-u_kA\nabla\eta(\zeta_k,t_k)\big|+\big|\eta F_{\varepsilon_k}(\hat{z}_k,t_k)-\eta F_{\varepsilon_k}(\zeta_k,t_k)\big|\le CL_ky_k^\alpha.
\end{aligned}
$$

We remark here that in order to estimate the second term above we have used the uniform $C^{0,\gamma}_p$ regularity of u_k for some chosen $\gamma\in(\alpha,1)$. Finally, we obtain that

$$
\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_kr_k^\alpha}\int_{\mathrm{spt}(\phi)}\partial_y\big(\rho^a_{\varepsilon_k}(r_ky+\hat{y}_k)\big)(A\nabla u_k+F_{\varepsilon_k})(\hat{z}_k,t_k)\cdot e_{d+1}\phi(z,t)dzdt\le C\Big(\frac{r_k}{y_k}\Big)^{1-\alpha}\to 0.
$$

So, also in **Case 1** we obtain that the fourth member of the right hand side of (1.76) vanishes.

To conclude, we prove that the third member of (1.76) goes to zero as $k\to\infty$. Notice that

$$
\begin{aligned}
&\Big|\eta(\hat{z}_k,t_k)\Big(A(r_kz+\hat{z}_k,r_k^2t+t_k)-A(\hat{z}_k,t_k)\Big)\nabla u_k(\hat{z}_k,t_k)\Big|\\
&\le\Big|\Big(A(r_kz+\hat{z}_k,r_k^2t+t_k)-A(\hat{z}_k,t_k)\Big)\nabla(\eta u_k)(\hat{z}_k,t_k)\Big|\\
&+\Big|\big(A(r_kz+\hat{z}_k,r_k^2t+t_k)-A(\hat{z}_k,t_k)\big)\nabla\eta(\hat{z}_k,t_k)u_k(\hat{z}_k,t_k)\Big|\\
&\le r_k^\alpha\|\nabla(\eta u_k)\|_{L^\infty(Q^+_{3/4})}+r_k^\alpha\|\nabla\eta\|_{L^\infty(Q^+_{3/4})}\|u_k\|_{L^\infty(Q^+_{3/4})}\le Cr_k^\alpha L_k,
\end{aligned}
$$

where we have used the parabolic Hölder interpolation inequality (1.13), that is,

$$
\|\nabla(\eta u_k)\|_{L^\infty(Q^+_{3/4})}\le C\Big(\|\eta u_k\|_{L^\infty(Q^+_{3/4})}+[\eta u_k]_{C^{1,\alpha}_p(Q^+_{3/4})}\Big)\le C(1+L_k).
$$

Then, in order to make vanish the full term we need to reason in two steps: first, one proves a uniform $C^{1,\beta}$ estimate with a given suboptimal $\beta\in(0,\alpha)$. In fact, in this case the third term vanishes as follows

$$
\begin{aligned}
\Big|\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_kr_k^\beta}&\int_{\mathrm{spt}(\phi)}\rho^a_{\varepsilon_k}(r_ky+\hat{y}_k)\Big(A(r_kz+\hat{z}_k,r_k^2t+t_k)-A(\hat{z}_k,t_k)\Big)\nabla u_k(\hat{z}_k,t_k)\cdot\nabla\phi(z,t)dzdt\Big|\\
&\le r_k^{\alpha-\beta}\int_{\mathrm{spt}(\phi)}\tilde{\rho}^a_k(y)\|\nabla\phi\|_{L^\infty(Q^\infty)}dzdt\le Cr_k^{\alpha-\beta}\to 0,
\end{aligned}
$$

as $k\to\infty$. Then we can proceed with the suboptimal exponent β up to the end of the present proof. This provides uniform boundedness of the sequence ∇u_k. Then, restarting the proof with the optimal α and the additional information above, in the previous computation we get

$$
\begin{aligned}
\Big|\frac{\eta(\hat{z}_k,t_k)\nu_k^{-a}}{L_kr_k^\alpha}&\int_{\mathrm{spt}(\phi)}\rho^a_{\varepsilon_k}(r_ky+\hat{y}_k)\Big(A(r_kz+\hat{z}_k,r_k^2t+t_k)-A(\hat{z}_k,t_k)\Big)\nabla u_k(\hat{z}_k,t_k)\cdot\nabla\phi(z,t)dzdt\Big|\\
&\le\frac{C}{L_k}\|\nabla u_k\|_{L^\infty(Q^+_{3/4})},
\end{aligned}
$$

which converges to zero. Putting together all previous information, we have proved that the right hand side in (1.76) vanishes as $k \to \infty$.

Let $(\bar{z}, \bar{t}) = \lim_{k\to\infty}(\hat{z}_k, t_k)$ and $\bar{A} := \lim_{k\to\infty} A(r_k z + \hat{z}_k, r_k^2 t + t_k)$. Arguing as in Theorem 1.6.1, we can prove the convergence of the left hand side of (1.76) in the following sense

$$\int_{\mathrm{spt}(\phi)} \tilde{\rho}_k^a\big(- w_k \partial_t \phi + A(r_k z + \hat{z}_k, r_k^2 t + t_k) \nabla w_k \cdot \nabla \phi\big) \to \int_{\mathrm{spt}(\phi)} \tilde{\rho}^a\big(- \bar{v} \partial_t \phi + \bar{A} \nabla \bar{v} \cdot \nabla \phi\big),$$

and obtain that $\bar{v}$ is an entire solution to

$$\partial_t \bar{v} - \mathrm{div}(\bar{A}\nabla \bar{v}) = 0 \quad \text{in } \mathbb{R}^{d+2}, \tag{1.78}$$

in **Case 1** and $\bar{v}$ is an entire solution to

$$\begin{cases} \tilde{\rho}^a \partial_t \bar{v} - \mathrm{div}(\tilde{\rho}^a \bar{A} \nabla \bar{v}) = 0 & \text{in } \mathbb{R}^{d+1}_+ \times \mathbb{R}, \\ \tilde{\rho}^a \bar{A} \nabla \bar{v} \cdot e_{d+1} = 0 & \text{on } \mathbb{R}^d \times \{0\} \times \mathbb{R}, \end{cases} \tag{1.79}$$

in **Case 2**.

Step 7: Liouville theorems.

Since $\bar{v}$ is globally $C_p^{1,\alpha}$-continuous in Q^∞, it follows that

$$\begin{aligned} |2\bar{v}(z,t)| &\le |2\bar{v}(z,t) - \bar{v}(0,t) - \nabla \bar{v}(0,t)\cdot z - \bar{v}(z,0)| + |\bar{v}(0,t)| + |\nabla \bar{v}(0,t) \cdot z| + |\bar{v}(z,0)| \\ &\le |\bar{v}(z,t) - \bar{v}(0,t) - \nabla \bar{v}(0,t)\cdot z| + |\bar{v}(z,t) - \bar{v}(z,0)| + C + C|z| + C \\ &\le C|z|^{1+\alpha} + C|t|^{\frac{1+\alpha}{2}} + C(1+|z|) \\ &\le C(1 + (|z|^2 + |t|))^{\frac{1+\alpha}{2}}, \end{aligned}$$

The estimate above exploit a first-order expansion in the spacial variable z for t fixed. However, the constant $C > 0$ can be chosen independently from the point (z,t).

Hence, as in Theorem 1.6.1, by the growth condition above, we can apply the Liouville Theorem 1.5.1 in both **Case 1** and **Case 2**, keeping in mind Remark 1.5.4, and obtain that $\bar{v}$ is a linear function, independent of t, in contradiction with the fact that $\nabla \bar{v}$ is not constant.

Step 8: The case $L_k = [\eta u_k]_{C_t^{0,\frac{1+\alpha}{2}}(Q_1^+)}$.

In this case, the argument is similar with minor differences. As above, we take two sequences of points $P_k = (z_k, t_k), \bar{P}_k = (z_k, \tau_k) \in Q^+_{3/4}$, such that

$$\frac{|(\eta u_k)(z_k,t_k) - (\eta u_k)(z_k, \tau_k)|}{|t_k - s_k|^{\frac{1+\alpha}{2}}} \ge \frac{L_k}{2} \to \infty. \tag{1.80}$$

Defining $r_k := d_p(P_k, \bar{P}_k) = |t_k - \tau_k|^{1/2}$, by (1.80) and the local uniform boundedness of solutions, see Proposition 1.3.3, we get $r_k \to 0$.

We define two blow-up sequences v_k and w_k as in (1.72), centered in the new blow-up sequence P_k, defined on the domains $Q(k)$, which are the same as above and set $Q^\infty := \lim_{k\to\infty} Q(k)$.

Since $[\partial_j(\eta u_k)]_{C_p^{0,\alpha}(Q_1^+)} \le L_k$, for every $j = 1, \dots, d+1$, we obtain that the estimates (1.73) and (1.74) holds; that is, $[v_k]_{C_p^{1,\alpha}(K)} \le C$, uniformly in k, for every compact set $K \subset Q^\infty$. Defining

$\bar v_k$ and $\bar w_k$ as in (1.75), we can use the Arzelá-Ascoli theorem to obtain $\bar v_k \to \bar v$ in $C_p^{1,\gamma}(K)$, for any $\gamma \in (0,\alpha)$, $\bar w_k \to \bar v$ uniformly on K and that $\bar v$ is globally $C_p^{1,\alpha}$ continuous in Q^∞.

The crucial difference between this case and the previous one is in *Step 4*: in this case we claim that $\bar v$ is non constant in the variable t. Indeed, we have that

$$\begin{aligned}\left|\bar v_k\left(0,\frac{t_k-\tau_k}{r_k^2}\right)-\bar v_k(0,0)\right| &= \left|v_k\left(0,\frac{t_k-\tau_k}{r_k^2}\right)\right| = \frac{|\eta(z_k,\tau_k)(u_k(z_k,\tau_k)-u_k(z_k,t_k))|}{L_k r_k^{1+\alpha}}\\ &\geq \frac{|(\eta u_k)(z_k,\tau_k)-(\eta u_k)(z_k,t_k)|}{L_k r_k^{1+\alpha}} - \frac{|(\eta(z_k,\tau_k)-\eta(z_k,t_k))u_k(z_k,t_k)|}{L_k r_k^{1+\alpha}}\\ &\geq \frac12 - \frac{M r_k^{1-\alpha}\|u_k\|_{L^\infty(Q^+_{3/4})}}{L_k} = \frac12 + o(1),\end{aligned}$$

as $k\to\infty$. Observing that $\frac{t_k-\tau_k}{r_k^2} \to \bar t \neq 0$, up to consider a subsequence, we can take the limit as $k\to\infty$ in the previous computation to obtain $|\bar v(0,\bar t)-\bar v(0,0)| \geq \frac12$; that is, $\bar v$ is non constant in the t-variable.

With the same argument of *Step 6* we can prove that $\bar v$ is an entire solution to (1.78) or (1.79). Moreover, since $\bar v$ is globally $C_p^{1,\alpha}$-continuous in Q^∞, then it satisfies a parabolic sub-quadratic growth condition. Hence, by the Liouville theorem 1.5.1, we find that $\bar v$ should be a linear function not depending on t, and this is a contradiction. □

1.7.1 Main result in $C_p^{0,\alpha}$ and $C_p^{1,\alpha}$ spaces

By combining the approximation results from Section 1.4 with the uniform estimates in ε in the parabolic Hölder spaces, we are able to prove the main theorem in the spaces $C_p^{0,\alpha}$ and $C_p^{1,\alpha}$.

Proof of Theorem 1.1.1. Let u be a weak solution to (1.1) in Q_1^+ in the sense of Definition 1.2.18. We prove that (1.5) and (1.6) hold in $Q^+_{1/2}$ (the case for a general $r\in(0,1)$ is analogous).

By Lemma 1.4.2 and Remark 1.4.3, we can find sequences $\{u_{\varepsilon_k}\}_k$, $\{f_{\varepsilon_k}\}_k$, $\{F_{\varepsilon_k}\}_k$ as $\varepsilon_k\to0^+$, such that every u_{ε_k} is a solution to

$$\begin{cases}\rho^a_{\varepsilon_k}\partial_t u_{\varepsilon_k} - \mathrm{div}(\rho^a_{\varepsilon_k}A\nabla u_{\varepsilon_k}) = \rho^a_{\varepsilon_k} f_{\varepsilon_k} + \mathrm{div}(\rho^a_{\varepsilon_k}F_{\varepsilon_k}) & \text{in } Q^+_{3/4}\\ \rho^a_{\varepsilon_k}\left(A\nabla u_{\varepsilon_k}+F_{\varepsilon_k}\right)\cdot e_{d+1} = 0 & \text{in } \partial^0 Q^+_{3/4},\end{cases}$$

and $u_{\varepsilon_k}\to u$ in $L^2_{loc}(I_{3/4};H^1_{loc}(B_{3/4}\setminus\Sigma))$ as $\varepsilon_k\to0^+$. Furthermore, f_{ε_k} and F_{ε_k} satisfy the assumptions of Theorem 1.6.1 (respectively of Theorem 1.7.1): this implies uniform boundedness of the $C_p^{0,\alpha}(Q^+_{1/2})$-norm of u_{ε_k} (respectively of the $C_p^{1,\alpha}(Q^+_{1/2})$-norm). Then, by the Arzelà-Ascoli Theorem and by the a.e. convergences $u_{\varepsilon_k}\to u$ and $\nabla u_{\varepsilon_k}\to\nabla u$, we obtain that the estimates (1.5) and (1.6) hold true.

Finally, in the $C_p^{1,\alpha}$ case, the boundary condition (1.7) follows by the $C^1(Q^+_{1/2})$-convergence $u_{\varepsilon_k}\to u$. □

1.8 Energy estimates II

At this point, we have proved the Theorem 1.1.1, which corresponds to the main result in the paper [7]. From now on, we will establish the higher order Schauder estimates (Theorem 1.1.2), which are contained in [8], and we will no longer work with the regularized weights ρ_ε^a, but only with y^a.

We begin with a local L^2 bound for difference quotients of weak solutions with respect to the time variable.

Lemma 1.8.1. *Let $a > -1$ and let A satisfying (1.2) such that $\partial_t A \in L^\infty(Q_1^+)$. Let $f \in L^2(Q_1^+, y^a)$ and $F \in L^2(Q_1^+, y^a)^{d+1}$ such that $\partial_t f \in L^2(Q_1^+, y^a)$ and $\partial_t F \in L^2(Q_1^+, y^a)^{d+1}$, and let u be a weak solution to (1.1). Consider the difference quotient of u with respect to to t:*

$$u^h(z,t) := \frac{u(z,t+h) - u(z,t)}{h}, \quad h > 0. \tag{1.81}$$

Then, there exists $C > 0$ depending only on d, a, λ and Λ such that, for every $r', r \in \mathbb{R}$ satisfying $1/2 \le r' < r < 1$ and $h > 0$, there holds

$$\begin{aligned}\int_{Q_{r'}^+} y^a (u^h)^2 \le C\Big(&\frac{1}{(r-r')^2}\int_{Q_r^+} y^a|\nabla u|^2 + \|f\|^2_{L^2(Q_1^+,y^a)} + \|F\|^2_{L^2(Q_1^+,y^a)} \\ &+ \|\partial_t A\|^2_{L^\infty(Q_1^+)}\int_{Q_r^+} y^a|\nabla u|^2 + \|\partial_t f\|^2_{L^2(Q_1^+,y^a)} + \|\partial_t F\|^2_{L^2(Q_1^+,y^a)}\Big).\end{aligned} \tag{1.82}$$

Proof. Fix r, r' such that $\frac{1}{2} \le r' < r < 1$. For $h > 0$, such that $r < 1 - h$, let us consider the Steklov average of u

$$u_h(z,t) = \frac{1}{h}\int_t^{t+h} u(z,s)dz,$$

which, by definition, satisfies $\partial_t u_h = u^h$ a.e. in Q_1 and the equation

$$\int_{Q_r^+} y^a(\partial_t u_h \phi + (A\nabla u)_h \cdot \nabla\phi) = \int_{Q_r^+} y^a(f_h\phi - F_h \cdot \nabla\phi)dzdt, \quad \forall \phi \in C_c^\infty(Q_1^+). \tag{1.83}$$

Now, for simplicity of the exposition, we assume $f = 0$, $F = 0$, and we discuss how treat the general case in a second step.

Let us take $\phi = \eta^2 u^h$ as test function in (1.83), where η is a smooth cut-off function which will define later. Using the Hölder and Young inequalities, the properties of Steklov averages and (1.2), we obtain

$$\begin{aligned}\int_{Q_1^+} y^a\eta^2(u^h)^2 &= \int_{Q_1^+} y^a\Big(\eta^2(A\nabla u)_h \cdot \nabla u^h + 2\eta u^h (A\nabla u)_h \cdot \nabla\eta\Big) \\ &\le \Big(\int_{Q_1^+} y^a\eta^2|(A\nabla u)_h|^2\Big)^{1/2}\Big(\int_{Q_1^+} y^a\eta^2|\nabla u^h|^2\Big)^{1/2} \\ &\quad + 2\Big(\int_{Q_1^+} y^a\eta^2(u^h)^2\Big)^{1/2}\Big(\int_{Q_1^+} y^a|(A\nabla u)_h|^2|\nabla\eta|^2\Big)^{1/2} \\ &\le \frac{C}{\delta}\int_{Q_1^+} y^a|\nabla u|^2 + \delta\int_{Q_1^+} y^a\eta^2|\nabla u^h|^2 + \frac{1}{2}\int_{Q_1^+} y^a\eta^2(u^h)^2 + C\int_{Q_1^+} y^a|\nabla\eta|^2|\nabla u|^2,\end{aligned} \tag{1.84}$$

for any fixed $\delta > 0$ and $C > 0$ depending only on d, a, λ and Λ.

In the spirit of [56, Lemma 3.3], we set

$$r_0 = r', \quad r_n = r' + \sum_{k=1}^n \frac{r-r'}{2^k}, \quad s_n = \frac{r_n + r_{n+1}}{2}, \quad n \in \mathbb{N},$$

and notice that r_n and s_n are increasing sequences satisfying $r_n < s_n < r_{n+1}$, $r_n \to r$ and $s_n \to r$.

For a given $n \in \mathbb{N}$, taking a cut-off function $\eta_n \in C_c^\infty(Q_1^+)$ in (1.84) such that

$$\operatorname{spt} \eta_n \subset Q_{s_n}^+, \quad \eta_n \equiv 1 \quad \text{in } Q_{r_n}^+, \quad 0 \le \eta_n \le 1, \quad |\nabla \eta_n| \le C\frac{2^n}{r-r'},$$

we deduce

$$\frac{1}{2}\int_{Q_{r_n}^+} y^a (u^h)^2 \le \delta \int_{Q_{s_n}^+} y^a |\nabla u^h|^2 + C\Big(\frac{2^{2n}}{(r-r')^2} + \frac{1}{\delta}\Big)\int_{Q_r^+} y^a |\nabla u|^2. \tag{1.85}$$

Now, noticing that u^h is a weak solution to

$$y^a \partial_t u^h - \operatorname{div}(y^a A \nabla u^h) = \operatorname{div}(y^a A^h \nabla u) \quad \text{in } Q_r^+,$$

we may apply the Caccioppoli inequality (1.27) to u^h, to obtain

$$\int_{Q_{s_n}^+} y^a |\nabla u^h|^2 \le \frac{C' 2^{2n}}{(r-r')^2}\int_{Q_{r_{n+1}}^+} y^a (u^h)^2 + C' \int_{Q_r^+} y^a |A^h \nabla u|^2, \tag{1.86}$$

for some $C' > 0$ independent of h, r, r'. Then, setting $\delta = \frac{1}{9}\frac{(r-r')^2}{C' 2^{2n}}$ in (1.85) and using (1.86), it follows

$$\int_{Q_{r_n}^+} y^a (u^h)^2 \le \frac{1}{9}\int_{Q_{r_{n+1}}^+} y^a (u^h)^2 + \frac{C 2^{2n}}{(r-r')^2}\int_{Q_r^+} y^a |\nabla u|^2 + \frac{C\|\partial_t A\|^2_{L^\infty(Q_1^+)}}{2^{2n}}\int_{Q_r^+} y^a |\nabla u|^2.$$

Now, multiplying both sides by 3^{-2n} and summing over n, we see that

$$\begin{aligned}\sum_{n=0}^\infty 3^{-2n}\int_{Q_{r_n}^+} y^a (u^h)^2 &\le \sum_{n=0}^\infty 3^{-2n-2}\int_{Q_{r_{n+1}}^+} y^a (u^h)^2 \\ &+ \frac{C}{(r-r')^2}\sum_{n=0}^\infty \Big(\frac{2}{3}\Big)^{2n}\int_{Q_r^+} y^a |\nabla u|^2 + \sum_{n=0}^\infty \frac{C\|\partial_t A\|^2_{L^\infty(Q_1^+)}}{6^{2n}}\int_{Q_r^+} y^a |\nabla u|^2,\end{aligned}$$

which implies that

$$\int_{Q_{r'}^+} y^a (u^h)^2 \le \frac{C}{(r-r')^2}\int_{Q_r^+} y^a |\nabla u|^2 + C\|\partial_t A\|^2_{L^\infty(Q_1^+)}\int_{Q_r^+} y^a |\nabla u|^2,$$

for some new $C > 0$, which is exactly (1.82) in the case $f = 0$ and $F = 0$.

For non-trivial f and F in the right hand side, we have two additional terms: one in (1.84) and one in (1.86). Both of them can be estimated using the arguments above, namely

$$\begin{aligned}&\int_{Q_1^+} y^a (f_h \eta^2 u^h + F_h \cdot \nabla(\eta^2 u^h)) \\ &\le C\|f\|^2_{L^2(Q_1^+, y^a)} + C_\delta \|F\|^2_{L^2(Q_1^+, y^a)} + \frac{1}{4}\int_{Q_1^+} y^a \eta^2 (u^h)^2 + \delta \int_{Q_1^+} y^a \eta^2 |\nabla u^h|^2,\end{aligned}$$

for every $\delta > 0$, where we have implicitly used that

$$\int_{Q_1^+} y^a\big((f^h)^2 + |F^h|^2\big) \leq \int_{Q_1^+} y^a\big((\partial_t f)^2 + |\partial_t F|^2\big),$$

for every $h \in (0,1)$. With such estimate at hand, the argument above can be slightly adapted to obtain (1.82) in the general case. □

An immediate consequence of the above estimates is that, under suitable regularity assumptions on the data, derivatives (with respect to t and x) of weak solutions to (1.1) are still weak solutions (of a suitable problem of the class (1.1).

Lemma 1.8.2. *Let $a > -1$, $r \in (0,1)$ and let A satisfying (1.2) such that $\partial_t A \in L^\infty(Q_1^+)$. Let $f \in L^2(Q_1^+, y^a)$ and $F \in L^2(Q_1^+, y^a)^{d+1}$ such that $\partial_t f, \partial_t F \in L^2(Q_1^+, y^a)^{d+1}$, and let u be a weak solution to (1.1). Then $v := \partial_t u$ is a weak solution to*

$$\begin{cases} y^a\partial_t v - \operatorname{div}(y^a A\nabla v) = y^a\partial_t f + \operatorname{div}(y^a(\partial_t A\nabla u + \partial_t F)) & \text{in } Q_r^+, \\ \lim\limits_{y\to 0^+} y^a(A\nabla v + \partial_t A\nabla u + \partial_t F)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_r^+. \end{cases} \tag{1.87}$$

Proof. Let us fix $0 < r < r' < r'' < 1$ and $h > 0$ such that $r'' < 1 - h$. Let u^h be the difference quotient of u with respect to to t defined in (1.81). By Lemma 1.8.1, $\|u^h\|_{L^2(Q_{r''}^+, y^a)}$ is bounded independently of $h > 0$. Further, since u^h is a weak solution to

$$y^a\partial_t u^h - \operatorname{div}(y^a A\nabla u^h) = y^a f^h + \operatorname{div}(y^a(F^h + A^h\nabla u)) \quad \text{in } Q_{r''}^+, \tag{1.88}$$

we may use Lemma 1.3.1 to deduce that $\|u^h\|_{L^\infty(I_{r'}, L^2(B_{r'}^+, y^a))}$ and $\|u^h\|_{L^2(I_{r'}, H^1(B_{r'}^+, y^a))}$ are bounded independently of $h > 0$ as well.

Now, let $\xi \in C_c^\infty(B_{r'})$ be a cut-off function such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ in B_r and set $v^h := \xi u^h \in L^2(I_{r'}, H_0^1(B_{r'}^+, y^a))$. Arguing as in Lemma 1.4.1 and Remark 1.2.19, we obtain that v^h is a weak solution to

$$y^a\partial_t v^h - \operatorname{div}(y^a A\nabla v^h) = y^a\tilde{f} + \operatorname{div}(y^a\tilde{F}), \quad \text{in } Q_{r'}^+,$$

where

$$\tilde{f} := f^h\xi - (F^h + A^h\nabla u)\cdot\nabla\xi - A\nabla u^h\cdot\nabla\xi, \quad \tilde{F} := (F^h + A^h\nabla u)\xi - u^h A\nabla\xi,$$

satisfying also $\|\partial_t v^h\|_{L^2(I_{r'}, H^{-1}(B_{r'}^+, y^a))} \leq C$, for some $C > 0$ independent of $h > 0$. Consequently,

$$\|v^h\|_{L^2(I_{r'}, H_0^1(B_{r'}^+, y^a))} + \|\partial_t v^h\|_{L^2(I_{r'}, H^{-1}(B_{r'}^+, y^a))} \leq C,$$

for some $C > 0$ independent of $h > 0$. Consequently, the Aubin-Lion lemma (see e.g. [135, Corollary 8]) yields the existence of $v \in L^2(I_{r'}, H_0^1(B_{r'}^+, y^a))$ such that $v^h \to v$ in $L^2(Q_{r'}^+, y^a)$ and $\nabla v^h \rightharpoonup \nabla v$ in $L^2(Q_{r'}^+, y^a)$. Since $\xi \equiv 1$ in Q_r^+, one has that $u^h \to \partial_t u$ in $L^2(Q_r^+, y^a)$ and $\nabla u^h \rightharpoonup \nabla(\partial_t u)$ in $L^2(Q_r^+, y^a)$. Furthermore, by the (H=W) property (see [155, 143]), one has $\partial_t u \in L^2(I_r, H^1(B_r^+, y^a))$ and $\partial_t u \in L^\infty(I_r, L^2(B_r^+, y^a))$ by Fatou's lemma.

Finally, let us fix a test function $\phi \in C_c^\infty(Q_r)$ if $a \in (-1,1)$ or $\phi \in C_c^\infty(Q_r^+)$ if $a \geq 1$. By the same argument of Lemma 1.4.1, we can take the limit as $h \to 0^+$ in the weak formulation of

(1.88), to deduce

$$
\begin{aligned}
0 &= \int_{Q_r^+} y^a\big(- u^h\phi_t + A\nabla u^h\cdot\nabla\phi - f^h\phi + (F^h + A^h\nabla u)\cdot\nabla\phi\big)\\
&\to \int_{Q_r^+} y^a\big(- \partial_t u\phi_t + A\nabla\partial_t u\cdot\nabla\phi - \partial_t f\phi + (\partial_t F + \partial_t A\nabla u)\cdot\nabla\phi\big),
\end{aligned}
$$

as $h\to 0^+$, that is $\partial_t u$ is a weak solution to (1.87). ☐

Analogously, we obtain the equations of the partial derivatives with respect to x.

Lemma 1.8.3. *Let $a>-1$, $r\in(0,1)$, $i\in\{1,\dots,d\}$ and let A satisfying (1.2) such that $\partial_{x_i}A\in L^\infty(Q_1^+)$. Let $f\in L^2(Q_1^+,y^a)$ and $F\in L^2(Q_1^+,y^a)^{d+1}$ such that $\partial_{x_i}f,\partial_{x_i}F\in L^2(Q_1^+,y^a)^{d+1}$, and let u be a weak solution to (1.1). Then $v_i:=\partial_{x_i}u$ is a weak solution to*

$$
\begin{cases}
y^a\partial_t v_i - \mathrm{div}(y^a A\nabla v_i) = y^a\partial_{x_i}f + \mathrm{div}(y^a(\partial_{x_i}A\nabla u + \partial_{x_i}F)) & \text{in } Q_r^+,\\
\lim\limits_{y\to0^+} y^a(A\nabla v_i + \partial_{x_i}A\nabla u + \partial_{x_i}F)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_r^+.
\end{cases}
\tag{1.89}
$$

Proof. The proof closely follows the above one and we skip it. ☐

1.9 Liouville theorems II

In this section, we establish a Liouville-type theorem for entire solutions that satisfy a polynomial growth at infinity.

Theorem 1.9.1. *Let $a>-1$, $m\in\mathbb{N}$, $\gamma\in[0,m+1)$ and let u be an entire solution to*

$$
\begin{cases}
y^a\partial_t u - \mathrm{div}(y^a\nabla u) = 0 & \text{in } \mathbb{R}^{d+1}_+\times\mathbb{R},\\
\lim\limits_{y\to0+} y^a\partial_y u = 0 & \text{on } \partial\mathbb{R}^{d+1}_+\times\mathbb{R}.
\end{cases}
\tag{1.90}
$$

Assume that

$$
|u(z,t)|\le C(1+(|z|^2+|t|)^\gamma)^{1/2} \qquad \text{for a.e. } (z,t)\in\mathbb{R}^{d+1}_+\times\mathbb{R}. \tag{1.91}
$$

Then u is a polynomial with degree at most m in z and at most $\lfloor\frac{m}{2}\rfloor$ in t.

Proof. Let us fix $R>1$ and define

$$
\tilde\gamma := a^+ + 2\gamma + d + 3.
$$

Step 1. Choosing $r'=R$ and $r=2R$ in (1.27) and using (1.91), we get

$$
\int_{Q_R^+} y^a|\nabla u|^2 \le \frac{C}{R^2}\int_{Q_{2R}^+} y^a u^2 \le CR^{\tilde\gamma-2}, \tag{1.92}
$$

for some $C>0$ depending only on d and a. On the other hand, choosing $r'=R$ and $r=2R$ in (1.82) and combining (1.92) and (1.91), we obtain

$$
\int_{Q_R^+} y^a(\partial_t u)^2 \le \frac{C}{R^4}\int_{Q_{4R}^+} y^a u^2 \le CR^{\tilde\gamma-4}, \tag{1.93}
$$

for some new $C > 0$.

Step 2. In this step we prove that u is a polynomial in x. By Lemma 1.8.3, for every multi-index $\beta \in \mathbb{N}^d$, $\partial_x^\beta u$ is a weak solution to (1.90). Then, by iterating (1.92), one has

$$\int_{Q_R} y^a (\partial_x^\beta u)^2 \leq \int_{Q_R} y^a |\nabla u|^2 \leq C R^{\tilde{\gamma} - 2|\beta|}.$$

Consequently, taking β such that $\tilde{\gamma} - 2|\beta| < 0$ and passing to the limit as $R \to \infty$, it follows $\partial_x^\beta u = 0$ and therefore u is a polynomial in the variable x, with degree less or equal than m (the bound on the degree immediately follows by (1.91)).

Step 3. A slight modification of the above argument, which uses (1.93) instead of (1.92), shows that u is a polynomial in the variable t, with degree less or equal than $\lfloor \frac{m}{2} \rfloor$.

Step 4. The last step is to prove that u is polynomial in y. By Remark 1.4.3, we notice that the even extension of u with respect to y is an entire solution to

$$|y|^a \partial_t u - \operatorname{div}(|y|^a \nabla u) = 0 \quad \text{in } \mathbb{R}^{d+1} \times \mathbb{R}. \tag{1.94}$$

Further, by Lemma 1.5.3, $v := |y|^a \partial_y u$ is an entire solution to

$$|y|^{-a} \partial_t u - \operatorname{div}(|y|^{-a} \nabla u) = 0 \quad \text{in } \mathbb{R}^{d+1} \times \mathbb{R},$$

while

$$w_1 := |y|^{-a} \partial_y v = \partial_{yy} u - a \frac{\partial_y u}{y}, \tag{1.95}$$

is an entire solution to (1.94). Now, applying (1.27) twice, we deduce

$$\begin{aligned} \int_{Q_R} |y|^a w_1^2 &\leq \int_{Q_R} |y|^{-a} |\nabla v|^2 \leq \frac{C}{R^2} \int_{Q_{2R}} |y|^{-a} v^2 \\ &\leq \frac{C}{R^2} \int_{Q_{2R}} |y|^a |\nabla u|^2 \leq \frac{C}{R^4} \int_{Q_{4R}} |y|^a u^2 \leq C R^{\tilde{\gamma} - 4}. \end{aligned}$$

Setting

$$w_{j+1} := \partial_{yy} w_j + a \frac{\partial_y w_j}{y}, \tag{1.96}$$

and noticing that w_{j+1} is an entire solution to (1.94) for $j \in \mathbb{N}_+$, we may iterate the argument above to show the existence of $k \in \mathbb{N}$ such that $\tilde{\gamma} - 4k < 0$ and

$$\int_{Q_R} |y|^a w_k \leq C R^{\tilde{\gamma} - 4k}.$$

Hence, taking the limit as $R \to \infty$, we obtain $w_k = 0$, that is

$$\partial_{yy} w_{k-1} + a \frac{\partial_y w_{k-1}}{y} = 0.$$

The above ODE can be explicitly solved:

$$w_{k-1} = c_{2k-1}(x,t) y |y|^{-a} + c_{2k-2}(x,t),$$

where $c_{2k-1}(x,t)$ and $c_{2k-2}(x,t)$ are polynomials. Now, iteratively solving the ODEs in (1.96) and (1.95), we obtain an explicit formula for u:

$$u = c_0(x,t) + \sum_{i\geq 1} y^{2i} c_{2i}(x,t) + \sum_{i\geq 1} y^{2i-1}|y|^{-a} c_{2i-1}(x,t), \tag{1.97}$$

where $c_i(x,t)$ are polynomial. All solutions to (1.94) satisfying a polynomial growth condition (without imposing any symmetry condition) have the form (1.97). Since u is an even solution (which comes from the conormal condition at the hyperplane), $c_{2i-1} \equiv 0$ for every $i \geq 1$. Therefore, our statement follows from the growth assumption (1.91). ☐

In the following proposition, we also provide a classification of the entire solutions to (1.90) satisfying the growth condition (1.91). Such classification was already obtained in [11, Lemma 3.2] in the range $a \in (-1,1)$ (see also [74, Lemma 5.2] in the elliptic setting). We present the proof for completeness.

Proposition 1.9.2. *Let $a > -1$ and let $q_\kappa = q_\kappa(x,t)$ be a polynomial of parabolic degree κ in $\mathbb{R}^d \times \mathbb{R}$. Then, there exist a unique polynomial $\tilde q_\kappa = \tilde q_\kappa(x,y,t)$ of parabolic degree κ in $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}$ such that $\tilde q_\kappa$ satisfies (1.90) and $\tilde q_\kappa(x,0,t) = q_\kappa(x,t)$ for every $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Moreover,*

$$\tilde q_\kappa(x,y,t) = q_\kappa(x,t) + \sum_{i=1}^{\lfloor \kappa/2 \rfloor} \frac{y^{2i}}{2i!} c_{2i} (\partial_t - \Delta_x)^i q_\kappa(x,t), \quad \textit{where } c_{2i} = \prod_{j=1}^{i} \frac{2j-1}{2j-1+a}. \tag{1.98}$$

Proof. We denote with $\Delta_{(x,y)}$ the Laplacian in the variables (x,y), Δ_x the Laplacian in the variable x and $(\partial_t - \Delta_x)^i$ the heat operator applied i times. Let $M := \lfloor \kappa/2 \rfloor$.

If such a polynomial $\tilde q_\kappa$ exists and satisfies the Neumann boundary condition $y^a \partial_y \tilde q_\kappa \to 0$ as $y \to 0^+$, then

$$\tilde q_\kappa(x,y,t) = q_\kappa(x,t) + \sum_{i=1}^{M} y^{2i} q_i(x,t),$$

where $q_i(x,t)$ are polynomials such that $y^{2i} q_i(x,t)$ have parabolic degree at most κ. Indeed, according to Theorem 1.1.1, $\tilde q_k$ satisfies the stronger Neumann boundary condition $\partial_y \tilde q_\kappa = 0$ on $y = 0$. This implies that $\tilde q_\kappa$ cannot contain a non-trivial term $y q_1(x,t)$. As in the proof of Theorem 1.9.1, we may iterate this argument to show that any term of the form $y^{2i+1} q_i(x,t)$ is identically zero.

Notice that

$$\begin{aligned} 0 &= (\partial_t - \Delta_{(x,y)})\tilde q_\kappa - \frac{a}{y}\partial_y \tilde q_\kappa \\ &= (\partial_t - \Delta_x) q_\kappa + \sum_{i=1}^{M} y^{2i} (\partial_t - \Delta_x) q_i - \sum_{i=1}^{M} 2i(2i-1) y^{2i-2} q_i - a \sum_{i=1}^{M} 2i y^{2i-2} q_i \\ &= (\partial_t - \Delta_x) q_\kappa - (2+2a) q_1 + \sum_{i=1}^{M-1} y^{2i}\big((\partial_t - \Delta_x) q_i - (2i+2)(2i+1+a) q_{i+1}\big) \\ &\quad + y^{2M} (\partial_t - \Delta_x) q_M. \end{aligned} \tag{1.99}$$

Now, by iteratively solving the equation in (1.99) we obtain

$$\begin{aligned} q_1 &= \frac{(\partial_t - \Delta_x)q_\kappa}{2(1+a)}, \\ q_2 &= \frac{(\partial_t - \Delta_x)q_1}{4(3+a)} = \frac{(\partial_t - \Delta_x)^2 q_\kappa}{4(3+a)2(1+a)}, \\ q_i &= (\partial_t - \Delta_x)^i q_\kappa \prod_{j=1}^{i} \frac{1}{2j(2j-1+a)} = \frac{(\partial_t - \Delta_x)^i q_\kappa}{2i!} \prod_{j=1}^{i} \frac{2j-1}{2j-1+a}, \end{aligned} \tag{1.100}$$

for $i \in \{1, \dots, M\}$. By construction, the function $\tilde{q}_\kappa$ defined in (1.98) satisfies our statement. The uniqueness of $\tilde{q}_\kappa$ immediately follows by the explicit formula (1.100) and the linearity of the differential operator. □

1.10 $C_p^{2,\alpha}$ regularity estimates

The goal of this section is to prove Theorem 1.1.2 when $k = 0$. Specifically, we establish the following result.

Theorem 1.10.1. *Let $a > -1$, $r \in (0,1)$, $\alpha \in (0,1)$. Let $A \in C_p^{1,\alpha}(Q_1^+)$ satisfying (1.2), $f \in C_p^{0,\alpha}(Q_1^+)$ and $F \in C_p^{1,\alpha}(Q_1^+)$ and let u be a weak solution to (1.1). Then, there exists $C > 0$ depending only on d, a, λ, Λ, r, α and $\|A\|_{C_p^{1,\alpha}(Q_1^+)}$ such that*

$$\|u\|_{C_p^{2,\alpha}(Q_r^+)} \le C\Big(\|u\|_{L^2(Q_1^+, y^a)} + \|f\|_{C_p^{0,\alpha}(Q_1)} + \|F\|_{C_p^{1,\alpha}(Q_1)}\Big). \tag{1.101}$$

The proof is based on some a priori estimates and an approximation argument we present below.

1.10.1 A priori $C_p^{2,\alpha}$ estimates

We begin by showing the a priori $C_p^{2,\alpha}$ estimates, which is stated in the following result.

Proposition 1.10.2. *Let $a > -1$, $\alpha \in (0,1)$ and $r \in (0,1)$. Let $A \in C_p^{1,\alpha}(Q_1^+)$ satisfying (1.2), $f \in C_p^{0,\alpha}(Q_1^+)$, $F \in C_p^{1,\alpha}(Q_1^+)$ and let $u \in C_p^{2,\alpha}(Q_1^+)$ be a weak solution to (1.1). Then, there exists $C > 0$ depending only on d, a, λ, Λ, r, α, $\|A\|_{C_p^{1,\alpha}(Q_1^+)}$ such that (1.101) holds.*

Proof. The proof is divided in several steps as follows.

Step 1. Preliminaries.

Without loss of generality we prove the statement for $r = 1/2$. To simplify the notation, let $x_{d+1} = y$, $\partial_i := \partial_{x_i}$ for $i = 1, \dots, d+1$, and $\partial_{ij} := \partial_i \partial_j$ for $i, j = 1, \dots, d+1$. In the following, we will refer to the variable y as either y or x_{d+1} depending on what seems more convenient. We begin with some preliminary observations.

• By the regularity assumptions and Theorem 1.1.1, one has that u satisfies the equation pointwise in Q_1^+, and so

$$\Big(\partial_t u - \sum_{i,j=1}^{d+1} A_{i,j} \partial_{ij} u - \frac{a}{y} \sum_{j=1}^{d+1} A_{d+1,j} \partial_j u\Big) = \Big(g + \frac{a}{y} F_{d+1}\Big) \quad \text{in } Q_1^+, \tag{1.102}$$

where $g := \sum_{i,j=1}^{d+1} \partial_i A_{i,j} \partial_j u + f + \sum_{i=1}^{d+1} \partial_i F \in C_p^{0,\alpha}(Q_1^+)$ satisfies

$$\begin{aligned} \|g\|_{C_p^{0,\alpha}(Q_1^+)} &\leq 3\|A\|_{C_p^{1,\alpha}(Q_1^+)} \|u\|_{C_p^{1,\alpha}(Q_1^+)} + \|f\|_{C_p^{0,\alpha}(Q_1^+)} + \|F\|_{C_p^{1,\alpha}(Q_1^+)} \\ &\leq C\Big(\|u\|_{L^\infty(Q_1^+)} + \|D^2 u\|_{L^\infty(Q_1^+)} + \|f\|_{C_p^{0,\alpha}(Q_1^+)} + \|F\|_{C_p^{1,\alpha}(Q_1^+)} \Big), \end{aligned} \tag{1.103}$$

for some $C > 0$ depending on $\|A\|_{C_p^{1,\alpha}(Q_1^+)}$, thanks to the interpolation inequality (1.13).

- By the regularity assumptions on the data and u, and using the conormal boundary condition in (1.1) (which is satisfied pointwise by Theorem 1.1.1), we can take the limit as $y \to 0^+$ in (1.102) to get

$$\begin{aligned} \lim_{y \to 0^+} \frac{a\big(\sum_{j=1}^{d+1} A_{d+1,j} \partial_j u + F_{d+1} \big)(x,y,t)}{y} &= a\partial_y \Big(\sum_{j=1}^{d+1} A_{d+1,j} \partial_j u + F_{d+1} \Big)(x,0,t) \\ &= \Big(\partial_t u - \sum_{i,j} A_{i,j} \partial_{ij} u - g \Big)(x,0,t), \end{aligned} \tag{1.104}$$

for every $(x,0,t) \in \partial^0 Q_1^+$.

- It is enough prove that for every $\delta > 0$ sufficiently small,

$$\begin{aligned} [u]_{C_p^{2,\alpha}(Q_{1/2}^+)} &\leq \delta [u]_{C_p^{2,\alpha}(Q_1^+)} \\ &\quad + C_\delta \Big(\|D^2 u\|_{L^\infty(Q_1^+)} + \|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_p^{0,\alpha}(Q_1^+)} + \|F\|_{C_p^{1,\alpha}(Q_1^+)} \Big), \end{aligned} \tag{1.105}$$

for some $C_\delta > 0$ depending only on δ, d, a, λ, Λ, α, $\|A\|_{C_p^{1,\alpha}(Q_1^+)}$. We will show later how (1.101) follows by (1.105).

Step 2. Contradiction argument and blow-up sequences.

By contradiction we assume that there exist $\alpha \in (0,1)$, $A^{(k)}, F^{(k)} \in C_p^{1,\alpha}(Q_1^+)$, $f_k \in C_p^{0,\alpha}(Q_1^+)$ with $\|A^{(k)}\|_{C_p^{1,\alpha}(Q_1^+)} \leq C$ and $u_k \in C_p^{2,\alpha}(Q_1^+)$ such that

$$\begin{cases} y^a \partial_t u_k - \mathrm{div}(y^a A^{(k)} \nabla u_k) = y^a f_k + \mathrm{div}(y^a F^{(k)}) & \text{in } Q_1^+, \\ \lim\limits_{y \to 0^+} y^a \Big(A^{(k)} \nabla u_k + F^{(k)} \Big) \cdot e_{d+1} = 0 & \text{on } \partial^0 Q_1^+, \end{cases}$$

and there exists a small $\delta_0 > 0$ such that

$$\begin{aligned} [u_k]_{C_p^{2,\alpha}(Q_{1/2}^+)} &> \delta_0 [u_k]_{C_p^{2,\alpha}(Q_1^+)} \\ &\quad + k\Big(\|D^2 u_k\|_{L^\infty(Q_1^+)} + \|u_k\|_{L^\infty(Q_1^+)} + \|f_k\|_{C_p^{0,\alpha}(Q_1^+)} + \|F^{(k)}\|_{C_p^{1,\alpha}(Q_1^+)} \Big). \end{aligned} \tag{1.106}$$

Let us define

$$\begin{aligned} L_k := \max \Big\{ &\big\{ [\partial_{ij} u_k]_{C_p^{0,\alpha}(Q_{1/2}^+)} : i,j = 1, \dots, d+1 \big\}, [\partial_t u_k]_{C_p^{0,\alpha}(Q_{1/2}^+)}, \\ &\big\{ [\partial_i u_k]_{C_t^{\frac{1+\alpha}{2}}(Q_{1/2}^+)} \ i = 1, \dots, d+1 \big\} \Big\}, \end{aligned}$$

and distinguish two cases: first, we assume that there exist $i, j \in \{1, \dots, d+1\}$ such that

$$L_k = [\partial_{ij} u_k]_{C_p^{0,\alpha}(Q^+_{1/2})}. \tag{1.107}$$

Later we will deal with the case $L_k = [\partial_i u_k]_{C_t^{\frac{1+\alpha}{2}}(Q^+_{1/2})}$. The case $L_k = [\partial_t u_k]_{C_p^{0,\alpha}(Q^+_{1/2})}$ is very similar to (1.107) and we skip it.

Now, we consider two sequences of points $P_k(z_k, t_k), \bar{P}_k(\xi_k, \tau_k) \in Q^+_{1/2}$ such that

$$\frac{|\partial_{ij} u_k(P_k) - \partial_{ij} u_k(\bar{P}_k)|}{d_p(P_k, \bar{P}_k)^\alpha} \geq \frac{L_k}{2},$$

and define $r_k := d_p(P_k, \bar{P}_k)$. Notice that it must be $r_k \to 0$ as $k \to \infty$, since

$$\frac{L_k}{2} \leq \frac{|\partial_{ij} u_k(P_k) - \partial_{ij} u_k(\bar{P}_k)|}{d_p(P_k, \bar{P}_k)^\alpha} \leq 2 \frac{\|\partial_{ij} u_k\|_{L^\infty(Q^+_{1/2})}}{r_k^\alpha} \leq 2 \frac{[u_k]_{C_p^{2,\alpha}(Q^+_{1/2})}}{r_k^\alpha k} \leq 2 \frac{L_k}{r_k^\alpha k},$$

where we have used (1.106) and the definition of L_k.

Let $\hat{z}_k = (\hat{x}_k, \hat{y}_k) \in B^+_{1/2}$ to be specified below. For k large, let us define

$$Q(k) := \frac{B_1^+ - \hat{z}_k}{r_k} \times \frac{(-1 - t_k, 1 - t_k)}{r_k^2},$$

and set $Q^\infty := \lim_{k \to \infty} Q(k)$, along an appropriate subsequence. For $(z, t) \in Q(k)$, consider the blow-up sequence

$$w_k(z, t) := \frac{u_k(r_k z + \hat{z}_k, r_k^2 t + t_k) - T_k(z, t)}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^{2+\alpha}}, \tag{1.108}$$

where T_k is the quadratic parabolic polynomial

$$T_k(z, t) = u_k(\hat{z}_k, t_k) + r_k \sum_{i=1}^{d+1} \partial_i u_k(\hat{z}_k, t_k) x_i + \frac{r_k^2}{2} \sum_{i,j=1}^{d+1} \partial_{ij} u_k(\hat{z}_k, t_k) x_i x_j + r_k^2 \partial_t u_k(\hat{z}_k, t_k) t.$$

Notice that w_k satisfies

$$w_k(0) = |\nabla w_k(0)| = |D^2 w_k(0)| = \partial_t w_k(0) = 0. \tag{1.109}$$

At this point we distinguish two cases:

Case 1:

$$\frac{y_k}{r_k} = \frac{d_p(P_k, \Sigma)}{r_k} \to \infty, \qquad \text{as } k \to \infty.$$

In this case we set $\hat{z}_k = z_k$ and we have $Q^\infty = \mathbb{R}^{d+2}$.

Case 2:

$$\frac{y_k}{r_k} = \frac{d_p(P_k, \Sigma)}{r_k} \leq C,$$

for some $C > 0$ independent of k. In this case we set $\hat{z}_k = (x_k, 0)$ and we have $Q^\infty = \mathbb{R}^{d+1}_+ \times \mathbb{R}$.

Step 3. Hölder estimates and convergence of the blow-up sequences.

Let us fix a compact set $K \subset Q^\infty$. Then, $K \subset Q(k)$ for any k large enough. By definition of the $C_p^{0,\alpha}$ seminorm and the parabolic scaling, for every $P = (z,t)$, $Q = (\xi,\tau) \in K$ and $i,j \in \{1,\dots,d+1\}$, we have

$$\begin{aligned}|\partial_{ij}w_k(P) - \partial_{ij}w_k(Q)| &\le \frac{|\partial_{ij}u_k(r_kz+\hat{z}_k, r_k^2t+t_k) - \partial_{ij}u_k(r_k\xi+\hat{z}_k, r_k^2\tau+t_k)|}{[u_k]_{C_p^{2,\alpha}(Q_1^+)}r_k^\alpha} \\ &\le d_p(P,Q)^\alpha,\end{aligned}$$

and thus

$$\sup_{\substack{P,Q\in K\\ P\neq Q}} \frac{|\partial_{ij}w_k(P) - \partial_{ij}w_k(Q)|}{d_p(P,Q)^\alpha} \le 1. \tag{1.110}$$

In a similar way, it is not difficult to obtain

$$\sup_{\substack{P,Q\in K\\ P\neq Q}} \frac{|\partial_{t}w_k(P) - \partial_{t}w_k(Q)|}{d_p(P,Q)^\alpha} \le 1. \tag{1.111}$$

Further, for every $(z,t),(z,\tau) \in K$ and $i \in \{1,\dots,d+1\}$, there holds

$$\begin{aligned}|\partial_{i}w_k(z,t) - \partial_{i}w_k(z,\tau)| &\le \frac{|\partial_{i}u_k(r_kz+\hat{z}_k, r_k^2t+t_k) - \partial_{i}u_k(r_kz+\hat{z}_k, r_k^2\tau+t_k)|}{[u_k]_{C_p^{2,\alpha}(Q_1^+)}r_k^{1+\alpha}} \\ &\le |t-\tau|^{\frac{1+\alpha}{2}},\end{aligned}$$

which implies

$$\sup_{\substack{(z,t),(z,\tau)\in K\\ t\neq \tau}} \frac{|\partial_{i}w_k(z,t) - \partial_{i}w_k(z,\tau)|}{|t-\tau|^{\frac{1+\alpha}{2}}} \le 1. \tag{1.112}$$

Combining (1.110), (1.111) and (1.112), we deduce that $[w_k]_{C_p^{2,\alpha}(K)}$ is uniformly bounded in k, for every compact set $K \subset Q^\infty$ (notice that the estimates above are valid in both **Case 1** and **Case 2**, by definition of Q^∞). Consequently, in light of (1.109), $\|w_k\|_{C_p^{2,\alpha}(K)}$ is uniformly bounded as well, and so we may apply the Arzelà-Ascoli theorem to conclude that $w_k \to \bar{w}$ in $C_p^{2,\gamma}(K)$, for every $\gamma \in (0,\alpha)$. Finally, a standard diagonal argument combined with (1.110), (1.111) and (1.112), shows that

$$w_k \to \bar{w} \quad \text{in } C_p^{2,\gamma}(K), \quad \text{for every } K \subset\subset Q^\infty,$$

up to passing to a suitable subsequence, and

$$[\bar{w}]_{C_p^{2,\alpha}(Q^\infty)} \le C. \tag{1.113}$$

for some $C > 0$ which depends only on d, by the definition of the $C_p^{2,\alpha}$ seminorm.

Step 4.

The next step is to prove that $\partial_{ij}\bar{w}$ is not constant, where i,j are the indexes fixed in (1.107). To do this, we consider two sequences of points in $Q(k)$, defined as

$$S_k = \Big(\frac{\xi_k - \hat{z}_k}{r_k}, \frac{\tau_k - t_k}{r_k^2}\Big), \quad \bar{S}_k := \Big(\frac{z_k - \hat{z}_k}{r_k}, 0\Big), \quad k \in \mathbb{N}.$$

In **Case 1**, one has $\hat{z}_k = z_k$, then $S_k \to S \in Q^\infty$, up to passing to a subsequence and $\bar{S}_k = 0$ for every k. Then, using the definition of L_k and (1.106), it follows

$$|\partial_{ij} w_k(S_k) - \partial_{ij} w_k(\bar{S}_k)| = |\partial_{ij} u_k(\bar{P}_k) - \partial_{ij} u_k(P_k)| \geq \frac{L_k}{2[u_k]_{C_p^{2,\alpha}(Q_1^+)}} \geq C\delta_0,$$

for some $C > 0$ independent on k and thus, passing to the limit as $k \to \infty$, we obtain $|\partial_{ij}\bar{w}(S) - \partial_{ij}\bar{w}(0)| \geq C\delta_0$, that is, $\partial_{ij}\bar{w}$ is not constant.

In **Case 2** we can argue in a similar way: we have $\hat{z}_k = (x_k, 0)$ and so $\bar{S}_k = \frac{y_k}{r_k} e_{n+1}$. Recalling that $\frac{y_k}{r_k}$ is uniformly bounded by definition, $S_k \to \bar{S}$, for some $\bar{S}$, up to passing to a subsequence. On the other hand, the sequence S_k can be written as

$$S_k = \left(\frac{\xi_k - z_k}{r_k}, \frac{\tau_k - t_k}{r_k^2} \right) + \frac{y_k}{r_k} e_{d+1}.$$

Therefore, $S_k \to S$ as $k \to \infty$, for some $S \in Q^\infty$, up to passing to a subsequence and so, as above, we have $|\partial_{ij}\bar{w}(S) - \partial_{ij}\bar{w}(\bar{S})| \geq C\delta_0$ which shows our claim.

Step 5. The equation of the limit $\bar{w}$.

In this step, we derive the equation of $\bar{w}$: as in the steps above, we divide the proof in two additional steps (**Case 1** and **Case 2**).

Case 1: In this case, we have $r_k/y_k \to 0$ as $k \to \infty$ and $\hat{z}_k = z_k$. Further, if $\bar{A}^{(k)}(z,t) := A^{(k)}(r_k z + \hat{z}_k, r_k^2 t + t_k)$, $\bar{\rho}_k(y) := r_k y + y_k$ and $(\bar{z}, \bar{t}) := \lim_{k\to\infty}(\hat{z_k}, t_k)$, then, by the regularity assumptions on $A^{(k)}$, one has $\bar{A}^{(k)} \to \bar{A}$ as $k \to \infty$, where $\bar{A} := \lim_{k\to\infty} A^{(k)}(\bar{z}, \bar{t})$ is a symmetric matrix with constant coefficients satisfying (1.2).

We claim that $\bar{w}$ is an entire solution to

$$\partial_t \bar{w} - \operatorname{div}(\bar{A}\nabla\bar{w}) = 0 \quad \text{in } \mathbb{R}^{d+2}. \tag{1.114}$$

Let us fix a compact set $K \subset Q^\infty$. By (1.102) and using estimates in parabolic Hölder spaces, w_k satisfies

$$\begin{aligned}
&\Big|\partial_t w_k - \sum_{i,j=1}^{d+1} (\bar{A}_{i,j}^{(k)} \partial_{ij} w_k)\Big| \\
&= \frac{1}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha} \Big| \partial_t u_k(r_k z + z_k, r_k^2 t + t_k) - \sum_{i,j=1}^{d+1} (A_{i,j}^{(k)} \partial_{ij} u_k)(r_k z + z_k, r_k^2 t + t_k) \\
&\quad - \partial_t u_k(z_k, t_k) + \sum_{i,j=1}^{d+1} A_{i,j}^{(k)}(r_k z + z_k, r_k^2 t + t_k) \partial_{ij} u_k(z_k, t_k) \Big| \\
&= \frac{1}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha} \Big| g_k(r_k z + z_k, r_k^2 t + t_k) \\
&\quad + \frac{a\big(\sum_{j=1}^{d+1} A_{d+1,j}^{(k)} \partial_j u_k + F_{d+1}^{(k)}\big)(r_k z + z_k, r_k^2 t + t_k)}{r_k y + y_k} - \partial_t u_k(z_k, t_k) \\
&\quad + \sum_{i,j=1}^{d+1} A_{i,j}^{(k)} \partial_{ij} u_k(z_k, t_k) + \sum_{i,j=1}^{d+1} \big(A_{i,j}^{(k)}(r_k z + z_k, r_k^2 t + t_k) - A_{i,j}^{(k)}(z_k, t_k)\big) \partial_{ij} u_k(z_k, t_k) \Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha} \Big| g_k(r_k z + z_k, r_k^2 t + t_k) \\
&\quad + \frac{a\big(\sum_{j=1}^{d+1} A_{d+1,j}^{(k)} \partial_j u_k + F_{d+1}^{(k)}\big)(r_k z + z_k, r_k^2 t + t_k)}{r_k y + y_k} - g_k(z_k, t_k) \\
&\quad - \frac{a\big(\sum_{j=1}^{d+1} A_{d+1,j}^{(k)} \partial_j u_k + F_{d+1}^{(k)}\big)(z_k, t_k)}{y_k} \Big| + C \frac{\|D^2 u_k\|_{L^\infty(Q_1^+)}}{[u_k]_{C_p^{2,\alpha}(Q_1^+)}} \\
&= \frac{\big|g_k(r_k z + z_k, r_k^2 t + t_k) - g_k(z_k, t_k)\big|}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha} + \frac{C\|D^2 u_k\|_{L^\infty(Q_1^+)}}{[u_k]_{C_p^{2,\alpha}(Q_1^+)}} \\
&\quad + \frac{a}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha} \Big| \frac{H_k(r_k z + z_k, r_k^2 t + t_k)}{r_k y + y_k} - \frac{H_k(z_k, t_k)}{y_k} \Big| = \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}
$$

where $C > 0$ is a new constant independent of k (here and below the constant $C > 0$ depends on K: we omit this dependence to simplify the exposition) and

$$
H_k(z,t) = \sum_{j=1}^{d+1} (A_{d+1,j}^{(k)} \partial_j u_k)(z,t) + F_{d+1}^{(k)}(z,t),
$$

which satisfies $H_k(x_k, 0, t_k) = 0$, $\nabla H_k(x_k, 0, t_k) = \partial_y H_k(x_k, 0, t_k) e_{d+1}$, $H_k(z,t)/y \in C^{0,\alpha}(Q_1^+)$ by Lemma 1.2.2 and $[H_k/y]_{C^{0,\alpha}(Q_1^+)} \leq C[\nabla H_k]_{C^{1,\alpha}(Q_1^+)} \leq C[u_k]_{C_p^{2,\alpha}(Q_1^+)}$, by the assumption (1.106).

Now, by (1.106) and (1.103), we can estimate I as follows

$$
\mathrm{I} := \frac{\big|g_k(r_k z + z_k, r_k^2 t + t_k) - g_k(z_k, t_k)\big|}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha} \leq \frac{[g_k]_{C_p^{0,\alpha}(Q_1^+)}}{[u_k]_{C_p^{2,\alpha}(Q_1^+)}} \leq \frac{1}{k} \to 0,
$$

as $k \to \infty$. The term II vanishes as well as $k \to \infty$, by similar considerations. Finally, let us prove that III vanishes as $k \to \infty$. First,

$$
\begin{aligned}
&\Big| \frac{H_k(r_k z + z_k, r_k^2 t + t_k)}{r_k y + y_k} - \frac{H_k(z_k, t_k)}{y_k} \Big| \\
&\quad = \Big| \frac{H_k(r_k z + z_k, r_k^2 t + t_k)}{r_k y + y_k} - \frac{H_k(z_k, t_k)}{r_k y + y_k} - \frac{r_k y}{y_k} \frac{H_k(z_k, t_k)}{r_k y + y_k} \Big| \\
&\quad \leq \Big| \frac{H_k(r_k z + z_k, r_k^2 t + t_k)}{r_k y + y_k} - \frac{H_k(z_k, t_k)}{r_k y + y_k} - \frac{\nabla H_k(z_k, t_k) \cdot r_k z}{r_k y + y_k} \Big| \\
&\quad + \Big| \frac{\nabla H_k(z_k, t_k) \cdot r_k z}{r_k y + y_k} - \frac{r_k y}{y_k} \frac{H_k(z_k, t_k)}{r_k y + y_k} \Big| = \mathrm{III}_i + \mathrm{III}_{ii}.
\end{aligned}
$$

By using the parabolic first order expansion of H_k, (1.106) and $r_k y + y_k \geq y_k/2$, one has that

$$
|\mathrm{III}_i| \leq C \frac{[H_k]_{C_p^{1,\alpha}(Q_1^+)} r_k^{1+\alpha}}{r_k y + y_k} \leq C[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^{1+\alpha} y_k^{-1}. \tag{1.115}
$$

Instead, we estimate the term III_{ii} in the following way

$$\begin{aligned}|\mathrm{III}_{ii}| &\leq \Big|\frac{\nabla H_k(z_k,t_k)\cdot r_k z}{r_k y + y_k} - \frac{\nabla H_k(x_k,0,t_k)\cdot r_k z}{r_k y + y_k}\Big| \\ &\quad + \Big|\frac{\nabla H_k(x_k,0,t_k)\cdot r_k z}{r_k y + y_k} - \frac{r_k y}{y_k}\,\frac{H_k(z_k,t_k)}{r_k y + y_k}\Big| \\ &\leq C[H_k]_{C_p^{1,\alpha}(Q_1^+)} r_k y_k^{\alpha-1} \leq C[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k y_k^{\alpha-1},\end{aligned} \tag{1.116}$$

where, in order to estimate the second term in the previous inequality we have used the properties of H_k stated above. Hence, combining (1.115) and (1.116) we have that

$$|\mathrm{III}| \leq C\frac{r_k}{y_k} + C\Big(\frac{r_k}{y_k}\Big)^{1-\alpha} \to 0, \quad \text{as } k \to \infty,$$

since in **Case 1**, $r_k/y_k \to 0$.

$$\partial_t w_k - \sum_{i,j=1}^{d+1} (\bar{A}_{i,j}^{(k)} \partial_{ij} w_k) \to \partial_t \bar{w} - \sum_{i,j=1}^{d+1} \bar{A}_{i,j}\partial_{ij}\bar{w} \quad \text{locally uniformly in } \mathbb{R}^{d+2},$$

as $k \to \infty$ and hence, passing to the limit as $k \to \infty$ into the equation of w_k above (1.114) follows.

Case 2: In this case, we have $\hat{z}_k = (x_k, 0)$ and $r_k/y_k \leq C$ for some $C > 0$ independent of k. We claim that $\bar{w}$ is a entire solution to

$$\begin{cases} y^a \partial_t \bar{w} - \operatorname{div}(y^a \bar{A}\nabla \bar{w}) = 0 & \text{in } \mathbb{R}^{d+1}_+ \times \mathbb{R}, \\ \lim\limits_{y\to 0^+} y^a \bar{A}\nabla\bar{w}\cdot e_{d+1} = 0 & \text{on } \partial\mathbb{R}^{d+1}_+ \times \mathbb{R}. \end{cases} \tag{1.117}$$

Let us fix a compact set $K \subset Q^\infty$. By using (1.102), (1.104) and the fact that $(\hat{z}_k, t_k)$ belongs to $\partial^0 Q_1^+$, w_k satisfies

$$\begin{aligned}\mathcal{L}w_k &:= \partial_t w_k - \sum_{i,j=1}^{d+1}(\bar{A}_{i,j}^{(k)}\partial_{ij}w_k) - \frac{a}{y}\sum_{j=1}^{d+1}(\bar{A}_{d+1,j}^{(k)}\partial_j w_k) \\ &= \frac{1}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha}\Big[\partial_t u_k(r_k z + \hat{z}_k, r_k^2 t + t_k) - \sum_{i,j=1}^{d+1}\big(A_{i,j}^{(k)}\partial_{ij}u_k\big)(r_k z + \hat{z}_k, r_k^2 t + t_k) \\ &\quad - \frac{a}{r_k y}\sum_{j=1}^{d+1}\big(A_{d+1,j}^{(k)}\partial_j u_k\big)(r_k z + \hat{z}_k, r_k^2 t + t_k) - \partial_t u_k(\hat{z}_k, t_k) \\ &\quad - \sum_{i,j=1}^{d+1} A_{i,j}^{(k)}(r_k z + \hat{z}_k, r_k^2 t + t_k)\partial_{ij}u_k(\hat{z}_k, t_k) \\ &\quad + \frac{a}{r_k y}\sum_{j=1}^{d+1} A_{d+1,j}^{(k)}(r_k z + \hat{z}_k, r_k^2 t + t_k)\partial_j u_k(\hat{z}_k, t_k) \\ &\quad + \frac{a}{r_k y}\sum_{i,j=1}^{d+1} A_{d+1,j}^{(k)}(r_k z + \hat{z}_k, r_k^2 t + t_k)\partial_{ij}u_k(\hat{z}_k, t_k) r_k x_i\Big] \\ &= \Bigg\{ \frac{g_k(r_k z + z_k, r_k^2 t + t_k) - g_k(\hat{z}_k, t_k)}{[u_k]_{C_p^{2,\alpha}(Q_1^+)} r_k^\alpha}\end{aligned}$$

$$
\begin{aligned}
&\left. - \frac{\sum_{i,j=1}^{d+1} \Big(A^{(k)}_{i,j}(r_k z+\hat z_k, r_k^2 t+t_k) - A^{(k)}_{i,j}(\hat z_k,t_k)\Big)\partial_{ij}u_k(\hat z_k,t_k)}{[u_k]_{C^{2,\alpha}_p(Q_1^+)} r_k^\alpha} \right\} \\
&+ \frac{a}{[u_k]_{C^{2,\alpha}_p(Q_1^+)} r_k^\alpha}\left\{ \frac{F^{(k)}_{d+1}(r_kz+\hat z_k, r_k^2t+t_k)}{r_k y} \right. \\
&- \partial_y\Big(\sum_{j=1}^{d+1} A_{d+1,j}\partial_j u + F_{d+1}\Big)(\hat z_k,t_k) \\
&+ \frac{1}{r_ky}\sum_{j=1}^{d+1} A^{(k)}_{d+1,j}(r_kz+\hat z_k, r_k^2t+t_k)\partial_j u_k(\hat z_k,t_k) \\
&\left. + \frac{1}{r_ky}\sum_{i,j=1}^{d+1} A^{(k)}_{d+1,j}(r_kz+\hat z_k, r_k^2t+t_k)\partial_{ij}u_k(\hat z_k,t_k) r_k x_i \right\} := \mathrm{J}+\mathrm{JJ}.
\end{aligned}
$$

Similar to **Case 1**, J vanishes as $k\to\infty$ (see the proof for I and II above). We are left to treat JJ. By Lemma 1.8.3 and Theorem 1.1.1, we may differentiate u_k with respect to x_i $(i=1,\dots,d)$ and $\partial_i u$ satisfies the following conormal boundary condition

$$
\lim_{y\to 0^+} \partial_i\Big(\sum_{j=1}^{d+1} A^{(k)}_{d+1,j}\partial_j u_k + F^{(k)}_{d+1}\Big) = 0, \tag{1.118}
$$

and thus, recalling the conormal boundary condition of u_k, we deduce

$$
\begin{aligned}
\mathrm{JJJ} := -\frac{1}{r_ky}\Big[&\sum_{j=1}^{d+1}\big(A^{(k)}_{d+1,j}\partial_j u_k\big)(\hat z_k,t_k) + F^{(k)}_{d+1}(\hat z_k,t_k) \\
&+ \sum_{i=1}^{d}\partial_i\Big(\sum_{j=1}^{d+1} A^{(k)}_{d+1,j}\partial_j u_k + F^{(k)}_{d+1}\Big)(\hat z_k,t_k) r_k x_i\Big] = 0.
\end{aligned}
$$

Adding JJ and JJJ, expanding $F^{(k)}$ and $A^{(k)}$ at order one and using the estimates in parabolic Hölder spaces, we obtain

$$
\begin{aligned}
&\Big|\frac{F^{(k)}_{d+1}(r_kz+\hat z_k, r_k^2t+t_k)}{r_k y} - \partial_y\Big(\sum_{j=1}^{d+1} A_{d+1,j}\partial_j u + F_{d+1}\Big)(\hat z_k,t_k) \\
&\quad + \frac{1}{r_ky}\sum_{j=1}^{d+1} A^{(k)}_{d+1,j}(r_kz+\hat z_k, r_k^2t+t_k)\partial_j u_k(\hat z_k,t_k) \\
&\quad + \frac{1}{r_ky}\sum_{i,j=1}^{d+1} A^{(k)}_{d+1,j}(r_kz+\hat z_k, r_k^2t+t_k)\partial_{ij}u_k(\hat z_k,t_k) r_k x_i\Big| \\
&= \Big|\frac{F^{(k)}_{d+1}(r_kz+\hat z_k, r_k^2t+t_k) - F^{(k)}_{d+1}(\hat z_k,t_k) - \sum_{i=1}^{d+1}\partial_i(F^{(k)}_{d+1})(\hat z_k,t_k) r_k x_i}{r_k y} \\
&\quad + \frac{1}{r_ky}\sum_{j=1}^{d+1}\partial_j u_k(\hat z_k,t_k)\Big(A^{(k)}_{d+1,j}(r_kz+\hat z_k, r_k^2t+t_k) - A^{(k)}_{d+1,j}(\hat z_k,t_k)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\sum_{i=1}^{d+1}\partial_i A^{(k)}_{d+1,j}(\hat z_k,t_k)r_kx_i\Big)\\
&\quad +\frac{1}{r_k y}\sum_{i,j=1}^{d+1}\Big(A^{(k)}_{d+1,j}(r_kz+\hat z_k,r_k^2t+t_k)-A^{(k)}_{d+1,j}(\hat z_k,t_k)\Big)\partial_{i,j}u_k(\hat z_k,t_k)r_kx_i\Big|\\
&\le Cr_k^\alpha\Big([F^{(k)}]_{C^{1,\alpha}_p(Q_1^+)}+[A^{(k)}]_{C^{1,\alpha}_p(Q_1^+)}\|\nabla u_k\|_{L^\infty(Q_1^+)}+[A^{(k)}]_{C^{0,1}_p(Q_1^+)}\|D^2u_k\|_{L^\infty(Q_1^+)}\Big)\\
&\le \frac{Cr_k^\alpha[u_k]_{C^{2,\alpha}_p(Q^+_{1/2})}}{k}.
\end{aligned}
$$

Consequently, $|\mathcal{L}w_k| = o(1)$, as $k\to\infty$. As in **Case 1**, by *Step 3*, one has

$$
\mathcal{L}w_k \to \partial_t\bar w-\sum_{i,j=1}^{d+1}\bar A_{i,j}\partial_{ij}\bar w-\frac{a}{y}\sum_{j=1}^{d+1}\bar A_{d+1,j}\partial_j\bar w \quad \text{locally uniformly in } \mathbb{R}^{d+1}_+,
$$

as $k\to\infty$, and so $\bar w$ satisfies the equation in (1.117) in the classical sense. It remains to prove that $\bar w$ satisfies the conormal boundary condition in (1.117). Since u_k satisfies (1.118) and following the arguments above, we find

$$
\begin{aligned}
\Big|\sum_{j=1}^{d+1}\big(\bar A^{(k)}_{d+1,j}\partial_j w_k\big)\Big| &= \frac{1}{[u_k]_{C^{2,\alpha}_p(Q_1^+)}r_k^{1+\alpha}}\Big|\sum_{j=1}^{d+1}\big(A^{(k)}_{d+1,j}\partial_j u_k\big)(r_kz+\hat z_k,r_k^2t+t_k)\\
&\quad -\sum_{j=1}^{d+1}A^{(k)}_{d+1,j}(r_kz+\hat z_k,r_k^2t+t_k)\partial_j u_k(\hat z_k,t_k)\\
&\quad -\sum_{j=1}^{d+1}A^{(k)}_{d+1,j}(r_kz+\hat z_k,r_k^2t+t_k)\partial_{ij} u_k(\hat z_k,t_k)r_kx_i\Big|\\
&= \frac{1}{[u_k]_{C^{2,\alpha}_p(Q_1^+)}r_k^{1+\alpha}}\Big|-F^{(k)}_{d+1}(r_kz+\hat z_k,r_k^2t+t_k)\\
&\quad -\sum_{j=1}^{d+1}\partial_j u_k(\hat z_k,t_k)\Big(A^{(k)}_{d+1,j}(r_kz+\hat z_k,r_k^2t+t_k)-A^{(k)}_{d+1,j}(\hat z_k,t_k)\\
&\quad -\sum_{i=1}^{d+1}\partial_i\big(A^{(k)}_{d+1,j}\big)(\hat z_k,t_k)r_kx_i\Big)\\
&\quad -\sum_{j=1}^{d+1}\big(A^{(k)}_{d+1,j}\partial_j u_k\big)(\hat z_k,t_k)-\sum_{i,j=1}^{d+1}\big(\partial_i A^{(k)}_{d+1,j}\partial_j u_k\big)(\hat z_k,t_k)r_kx_i\\
&\quad -\sum_{i,j=1}^{d+1}\big(A^{(k)}_{d+1,j}\partial_{ij} u_k\big)(\hat z_k,t_k)r_kx_i\\
&\quad -\sum_{i,j=1}^{d+1}\Big(A^{(k)}_{d+1,j}(r_kz+\hat z_k,r_k^2t+t_k)-A^{(k)}_{d+1,j}(\hat z_k,t_k)\Big)\partial_{ij} u_k(\hat z_k,t_k)r_kx_i\Big|\\
&\le \frac{1}{[u_k]_{C^{2,\alpha}_p(Q_1^+)}r_k^{1+\alpha}}\Big|-F^{(k)}_{d+1}(r_kz+\hat z_k,r_k^2t+t_k)+F^{(k)}_{d+1}(\hat z_k,t_k)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{d+1}\partial_i F^{(k)}_{d+1}(\hat z_k,t_k)r_k x_i-\sum_{i=1}^{d+1}\partial_i F^{(k)}_{d+1}(\hat z_k,t_k)r_k x_i\\
&-\sum_{i,j=1}^{d+1}\partial_i\big(A^{(k)}_{d+1,j}\partial_j u_k\big)(\hat z_k,t_k)r_k x_i\Big|+o(1)\\
\le\ &\frac{\Big|\partial_y\Big(\sum_{j=1}^{d+1}A^{(k)}_{d+1,j}\partial_j u_k+F^{(k)}_{d+1}\Big)(\hat z_k,t_k)r_k y\Big|}{[u_k]_{C^{2,\alpha}_p(Q_1^+)}r_k^{1+\alpha}}+o(1)=o(1),
\end{aligned}
$$

as $k\to\infty$. Thus, passing to the limit as $y\to 0^+$, we obtain

$$
\lim_{y\to 0^+}\Big|\sum_{j=1}^{d+1}\big(\bar A^{(k)}_{d+1,j}\partial_j w_k\big)\Big|\le o(1),
$$

and thus, taking the limit as $k\to\infty$, it follows

$$
\lim_{y\to 0^+}\bar A\nabla\bar w\cdot e_{d+1}=0.
$$

Combining this with the fact that $\bar w\in C^{2,\alpha}_p$ by (1.113) and recalling that $a>-1$, we have

$$
\begin{aligned}
\lim_{y\to 0^+}y^a\bar A\nabla\bar w\cdot e_{d+1}&=\lim_{y\to 0^+}y^{1+a}\lim_{y\to 0^+}\frac{\bar A\nabla\bar w\cdot e_{d+1}}{y}\\
&=\partial_y(\bar A\nabla\bar w\cdot e_{d+1})|_{y=0}\cdot\lim_{y\to 0^+}y^{1+a}=0,
\end{aligned}
$$

and so the proof of (1.117) is completed.

Step 6. Liouville theorems.

Since $\bar w\in C^{2,\alpha}_p(Q^\infty)$, see (1.113), it satisfies the growth condition

$$
|\bar w(z,t)|\le C(1+(|z|^2+|t|)^{2+\alpha})^{1/2}.
$$

Moreover, $\bar w$ has at least one non-constant second derivative and is an entire solution to (1.114) or (1.117). Then, in **Case 1** we can invoke the Liouville Theorem for the heat equation (see Remark 1.5.4) and in **Case 2** we can invoke the Liouville Theorem 1.9.1 to reach the desired contradiction.

Step 7.

We complete the analysis, considering the case when

$$
L_k=[\partial_i u]_{C_t^{0,\frac{1+\alpha}{2}}(Q^+_{1/2})},
$$

for some $i\in\{1,\dots,d+1\}$. We give a short sketch, pointing out the main differences respect to what did above.

We take two sequences of points $P_k=(z_k,t_k),\bar P_k=(z_k,\tau_k)\in Q^+_{1/2}$, such that

$$
\frac{|\partial_i u_k(z_k,t_k)-\partial_i u_k(z_k,\tau_k)|}{|t_k-s_k|^{\frac{1+\alpha}{2}}}\ge\frac{L_k}{2},
$$

and set $r_k := d_p(P_k, \bar{P}_k) = |t_k - \tau_k|^{1/2}$. We define the blow-up sequence w_k as in (1.108), centered in P_k.

The *Steps 3, 5, 6* are the same as above. The only crucial difference is in *Step 4*: in this case, one has that $\partial_i \bar{w}$ is non-constant in t. Indeed,

$$\left|\partial_i w_k\Big(0, \frac{t_k - \tau_k}{r_k^2}\Big) - \partial_i w_k(0,0)\right| \geq \frac{L_k}{2[u_k]_{C_p^{2,\alpha}(Q_1^+)}} \geq C_N \delta_0.$$

Taking the limit as $k \to \infty$, we obtain that $|\partial_i \bar{w}(0,\bar{t}) - \partial_i \bar{w}(0,0)| \geq C_N \delta_0$, where $\bar{t} = \lim_{k\to\infty} \frac{t_k - \tau_k}{r_k^2}$. This allows as to conclude the proof of (1.105) by applying Theorem 1.9.1.

Step 8. Conclusion.

Finally, we briefly explain why (1.105) implies (1.101) (see [69, Theorem 2.20, Lemma 2.27] in the elliptic setting). First, by using a covering argument and the interpolation inequalities in (1.13), we have that (1.105) is satisfied in every $Q_\rho^+(P_0) \subset Q_1^+$, that is,

$$[u]_{C_p^{2,\alpha}(Q_{\rho/2}^+(P_0))} \leq \delta [u]_{C_p^{2,\alpha}(Q_\rho^+(P_0))} + C_\delta \big(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_p^{0,\alpha}(Q_1^+)} + \|F\|_{C_p^{1,\alpha}(Q_1^+)} \big). \tag{1.119}$$

Now, let us define the seminorm

$$[u]^*_{2,\alpha,Q_1^+} := \sup_{Q_\rho^+(P_0) \subset Q_1^+} \rho^{2+\alpha} [u]_{C_p^{2,\alpha}(Q_{\rho/2}^+(P_0))}. \tag{1.120}$$

By using the sub-additivity of the Hölder seminorms respect to unions of convex sets, one can prove that

$$[u]^*_{2,\alpha,Q_1^+} \leq C \sup_{Q_\rho^+(P_0) \subset Q_1^+} \rho^{2+\alpha} [u]_{C_p^{2,\alpha}(Q_{\rho/4}^+(P_0))}, \tag{1.121}$$

for some constant $C > 0$ depending only on d and α. Then, by (1.119) and (1.120), we obtain

$$\rho^{2+\alpha} [u]_{C_p^{2,\alpha}(Q_{\rho/4}^+(P_0))} \leq \delta [u]^*_{2,\alpha,Q_1^+} + C_\delta \big(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_p^{0,\alpha}(Q_1^+)} + \|F\|_{C_p^{1,\alpha}(Q_1^+)} \big).$$

Taking the supremum over $Q_\rho^+(P_0) \subset Q_1^+$ and recalling (1.121), it follows

$$\frac{1}{C} [u]^*_{2,\alpha,Q_1^+} \leq \delta [u]^*_{2,\alpha,Q_1^+} + C_\delta \big(\|u\|_{L^\infty(Q_1^+)} + \|f\|_{C_p^{0,\alpha}(Q_1^+)} + \|F\|_{C_p^{1,\alpha}(Q_1^+)} \big).$$

Hence our statement follows by taking $\delta > 0$ small enough and using the interpolation inequality (1.13). □

1.10.2 A regularization scheme

In this second step, we proceed with a regularization argument: this allows to apply the a priori estimates above and prove Theorem 1.10.1.

Lemma 1.10.3. *Let $a > -1$, $r \in (0,1)$, $\alpha \in (0,1)$. Let $A \in C^\infty(Q_1^+)$ satisfying (1.2) and $f, F \in C^\infty(Q_1^+)$, and let u be a weak solution to (1.1). Then $u \in C_p^{2,\alpha}(Q_r^+)$.*

Proof. We fix $0 < r < r' < 1$. For every $i = 1, \dots, d$, by the regularity assumption on A, f and F and Lemma 1.8.3, we have that $\partial_{x_i} u$ solves (1.89) in $Q_{r'}^+$ and, by Theorem 1.1.1, we deduce that $\partial_{x_i}u \in C_p^{1,\alpha}(Q_r^+)$. Analogously, by Lemma 1.8.2, $\partial_t u$ solves (1.87) in $Q_{r'}^+$ and, by Theorem 1.1.1, we deduce that $\partial_t u \in C_p^{1,\alpha}(Q_r^+)$. To conclude, we need to prove that $\partial_y u \in C_p^{1,\alpha}(Q_r^+)$.

Using the regularity of ∇u and $\partial_t u$ obtained above, we may rewrite the equation of u as

$$\partial_y\big(y^a(A\nabla u+F)\big)\cdot e_{d+1} = y^a\Big[\partial_t u - f - \sum_{i=1}^{d}\partial_{x_i}((A\nabla u+F)\cdot e_i)\Big] := y^a g, \tag{1.122}$$

in the weak sense, where $g \in C_p^{0,\alpha}(Q_r^+)$. Then, integrating in y and using that

$$\lim_{y\to 0^+}(A\nabla u+F)\cdot e_{d+1}=0,$$

see Theorem 1.1.1, one has

$$\psi(x,y,t) := (A\nabla u+F)\cdot e_{d+1}(x,y,t) = \frac{1}{y^a}\int_0^y s^a g(x,s,t)ds.$$

Since $\partial_{x_i}u, \partial_t u \in C_p^{1,\alpha}(Q_r^+)$, we have $\partial_{x_i}\psi \in C_p^{0,\alpha}(Q_r^+)$ by definition, for every $i=1,\dots,d$, and $\partial_t\psi \in C_p^{0,\alpha}(Q_r^+)$. Consequently, $\psi \in C_t^{0,\frac{1+\alpha}{2}}(Q_r^+)$. Now, since $g \in C_p^{0,\alpha}(Q_r^+)$, Lemma 1.2.3 yields $\partial_y\psi \in C_p^{0,\alpha}(Q_r^+)$ and thus $\psi \in C_p^{1,\alpha}(Q_r^+)$. Noticing that, by (1.2), we have

$$\partial_y u = \frac{\psi - \sum_{j=1}^d A_{d+1,j}\partial_j u - F_{d+1}}{A_{d+1,d+1}}, \tag{1.123}$$

it follows $\partial_y u \in C_p^{1,\alpha}(Q_r^+)$ and thus $u \in C_p^{2,\alpha}(Q_r^+)$. □

We are now ready to show Theorem 1.10.1.

Proof of Theorem 1.10.1. Let us fix $0<r<R<1$ and let u be a weak solution to (1.1). Let us consider a smooth cut-off function $\xi \in C_c^\infty(B_R)$, such that $0\le\xi\le 1$ and $\xi=1$ in B_r. Then, $v:=\xi u$ is a weak solution to

$$\begin{cases} y^a\partial_t v - \mathrm{div}(y^a A\nabla v) = y^a g + \mathrm{div}(y^a G) & \text{in } Q_R^+\\ \lim\limits_{y\to 0^+} y^a(A\nabla v+G)\cdot e_{d+1}=0 & \text{on } \partial^0 Q_R^+\\ v=0 & \text{on } \partial B_R^+\times I_R\\ v=\eta u & \text{on } B_R^+\times\{-R^2\}, \end{cases}$$

where

$$g := \xi f - F\cdot\nabla\xi - A\nabla u\cdot\nabla\xi, \quad G := \xi F - uA\nabla\xi.$$

Let us denote with $\bar A$, $\bar f$ and $\bar F$ the even extensions of A, f and F with respect to y, respectively and let $A_\varepsilon := \bar A * \rho_\varepsilon$, $f_\varepsilon := \bar f * \rho_\varepsilon$ and $F_\varepsilon := \bar F * \rho_\varepsilon$, where $\{\rho_\varepsilon\}_{\varepsilon>0}$ is a family of smooth mollifiers. Then, up to choose ε small enough, $A_\varepsilon, f_\varepsilon, F_\varepsilon \in C_c^\infty(Q_R^+)$ and A_ε satisfies (1.2). For every $\varepsilon \in (0,1)$, let v_ε be the weak solution to

$$\begin{cases} y^a\partial_t v_\varepsilon - \mathrm{div}(y^a A_\varepsilon \nabla v_\varepsilon) = y^a g_\varepsilon + \mathrm{div}(y^a G_\varepsilon) & \text{in } Q_R^+ \\ \lim\limits_{y\to 0^+} y^a(A_\varepsilon \nabla v_\varepsilon + G_\varepsilon)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_R^+ \\ v_\varepsilon = 0 & \text{on } \partial B_R^+ \times I_R \\ v_\varepsilon = v & \text{on } B_R^+ \times \{-R^2\}, \end{cases}$$

where

$$g_\varepsilon := \xi f_\varepsilon - F_\varepsilon \cdot \nabla \xi - A_\varepsilon \nabla u \cdot \nabla \xi, \quad G_\varepsilon := \xi F_\varepsilon - u A_\varepsilon \nabla \xi.$$

By the same compactness argument of Lemma 1.8.2 (see also Lemma 1.4.2 and Remark 1.4.3), and by the classical theory of the Cauchy-Dirichlet problem in abstract Hilbert spaces, see [99], we have that $v_\varepsilon \to v$ in $L^2(Q_R^+, y^a)$, which implies that $v_\varepsilon \to u$ in $L^2(Q_r^+, y^a)$ by the definition of v. On the other hand, since $\xi \equiv 1$ in B_r, one has that v_ε is a weak solution to

$$\begin{cases} y^a\partial_t v_\varepsilon - \mathrm{div}(y^a D_\varepsilon \nabla v_\varepsilon) = y^a f_\varepsilon + \mathrm{div}(y^a F_\varepsilon) & \text{in } Q_r^+, \\ \lim\limits_{y\to 0^+} y^a(D_\varepsilon \nabla v_\varepsilon + F_\varepsilon)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_r^+. \end{cases}$$

So, up to rescaling, Lemma 1.10.3 yields that $v_\varepsilon \in C_p^{2,\alpha}(Q_1^+)$. On the other hand, by Proposition 1.10.2, we deduce that v_ε satisfies the desired estimate (1.101) in Q_r^+, uniformly in $\varepsilon > 0$. By the Arzelà-Ascoli theorem, we may thus take the limit as $\varepsilon \to 0^+$ and complete the proof of (1.101). □

1.11 $C_p^{k,\alpha}$ regularity estimates

In this section, we prove Theorem 1.1.2 for any $k \geq 1$ by combining some a priori estimates and an approximation argument. As anticipated in the introduction, we first deal with the case of a zero forcing term in the equation (1.1), i.e. $f = 0$. In this case, the main result follows by a simple iteration of the $C_p^{1,\alpha}$ and $C_p^{2,\alpha}$ estimates on partial derivatives. Secondly, we treat forcing terms $f \in C_p^{k,\alpha}$. In this case, the strategy is more involved and requires some additional and delicate steps (see Lemma 1.11.3).

1.11.1 Higher order Schauder estimates when $f = 0$

We begin by treating the simpler case $f = 0$.

Proof of Theorem 1.1.2 when $f = 0$. We proceed by induction. The initial step $k = 0$ follows by Theorem 1.10.1.

Let us fix $0 < r < r' < 1$ and assume that $A, F \in C_p^{j+2,\alpha}(Q_r^+)$ imply that (1.8) holds for $j = 0, \dots, k$ and prove it for $k+1$. By Lemma 1.8.3 and the induction step we may differentiate the equation of u with respect to x_i to obtain $\partial_{x_i} u \in C_p^{k+2,\alpha}(Q_r^+)$ for every $i = 1, \dots, d$ and

$$\begin{aligned} \|\partial_{x_i} u\|_{C_p^{k+2,\alpha}(Q_r^+)} &\leq C\big(\|\partial_{x_i} u\|_{L^2(Q_{r'}^+, y^a)} + \|\partial_i F\|_{C_p^{k+1,\alpha}(Q_1^+)}\big) \\ &\leq C\big(\|u\|_{L^2(Q_1^+, y^a)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)}\big), \end{aligned} \tag{1.124}$$

for some $C>0$ which depends on d, a, λ, Λ, r, α and $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$. On the other hand, By Lemma 1.8.2 and the induction step (noticing that in the case $k=0$ we use Theorem 1.1.1) we may differentiate the equation of u with respect to t to obtain $\partial_t u\in C_p^{k+1,\alpha}(Q_r^+)$ and

$$\begin{aligned}\|\partial_t u\|_{C_p^{k+1,\alpha}(Q_r^+)}&\le C\big(\|\partial_t u\|_{L^2(Q_{r'}^+,y^a)}+\|\partial_t F\|_{C_p^{k,\alpha}(Q_1^+)}\big)\\&\le C\big(\|u\|_{L^2(Q_1^+,y^a)}+\|F\|_{C_p^{k+2,\alpha}(Q_1^+)}\big),\end{aligned}\tag{1.125}$$

d, a, λ, Λ, r, α and $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$. Repeating exactly the same argument of Lemma 1.10.3 we obtain that the function g defined in (1.122) belongs to $C_p^{k+1,\alpha}(Q_r^+)$ and thus $\partial_y u\in C_p^{k+2,\alpha}(Q_r^+)$ which, in turn, implies $u\in C_p^{k+3,\alpha}(Q_r^+)$. Moreover, by using (1.123), (1.124), (1.125) one has

$$\|\partial_y u\|_{C_p^{k+2,\alpha}(Q_r^+)}\le C\big(\|u\|_{L^2(Q_1^+,y^a)}+\|F\|_{C_p^{k+2,\alpha}(Q_1^+)}\big),\tag{1.126}$$

d, a, λ, Λ, r, α and $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$. Then, combining (1.124), (1.125) and (1.126) our statement follows. □

1.11.2 Higher order Schauder estimates

Now, we consider the case $f\in C_p^{k,\alpha}$. As remarked in the introduction, when $k=1$, we can not use the same argument of the case $f=0$, since the function $\partial_t f$ is not well defined. In order to overcome this problem, we prove a priori $C_p^{3,\alpha}$-estimates and combining these with Lemma 1.11.2 and Lemma 1.11.3, we obtain our statement in the case $k=1$. For the general case $k\ge 2$, one could possibly iterate the estimates obtained to prove the main result, as done in the case $f=0$. However, in order to keep the presentation uniform, we choose to iterate the full procedure (a priori estimates plus approximation) at any step.

Proposition 1.11.1. *Let $a>-1$, $\alpha\in(0,1)$, $r\in(0,1)$ and $k\in\mathbb{N}$. Let $A\in C_p^{k+1,\alpha}(Q_1^+)$ satisfying (1.2), $f\in C_p^{k,\alpha}(Q_1^+)$ and $F\in C_p^{k+1,\alpha}(Q_1^+)$ and let $u\in C_p^{k+2,\alpha}(Q_1^+)$ be a weak solution to (1.1). Then, there exists $C>0$, depending on d, a, λ, Λ, r, α and $\|A\|_{C_p^{k+1,\alpha}(Q_1^+)}$ such that*

$$\|u\|_{C_p^{k+2,\alpha}(Q_r^+)}\le C\big(\|u\|_{L^2(Q_1^+,y^a)}+\|f\|_{C_p^{k,\alpha}(Q_1^+)}+\|F\|_{C_p^{k+1,\alpha}(Q_1^+)}\big).\tag{1.127}$$

The proof of Proposition 1.11.1 crucially uses Lemma 1.11.3 below. In turn, in the proof of Lemma 1.11.3, we exploit an approximation argument which relies on the following auxiliary result.

Lemma 1.11.2. *Let $a>-1$, $r\in(0,1)$, $k\in\mathbb{N}$. Let $A\in C_p^{k+2,\alpha}(Q_1^+)$ satisfying (1.2), $f\in C^\infty(Q_1^+)$, $F\in C_p^{k+2,\alpha}(Q_1^+)$, and let u be a weak solution to (1.1). Then $u\in C_p^{k+3,\alpha}(Q_r^+)$.*

Proof. It is enough to slightly modify the arguments of the proof of Theorem 1.1.2 in the case $f=0$. □

Lemma 1.11.3. *Let $a>-1$, $\alpha\in(0,1)$ and $k\in\mathbb{N}$. Let $D\in C_p^{k+2,\alpha}(Q_1^+)$ be a diagonal matrix satisfying (1.2), $f\in C_p^{k+1,\alpha}(Q_1^+)$ and $F\in C_p^{k+2,\alpha}(Q_1^+)$. Let $\mu:=D_{d+1,d+1}$ and $g:=F_{d+1}$. Let $u\in C_p^{k+3,\alpha}(Q_1^+)$ be a weak solution to*

$$\begin{cases}y^a\partial_t u-\mathrm{div}(y^aD\nabla u)=y^af+\mathrm{div}(y^aF) & \text{in } Q_1^+,\\ \lim\limits_{y\to0^+}y^a(\mu\partial_y u+g)=0 & \text{on } \partial^0Q_1^+.\end{cases}\tag{1.128}$$

Then, the function

$$w := y^{-a}\partial_y\Big(y^a\Big(\partial_y u + \frac{g}{\mu}\Big)\Big) \in C_p^{k+1,\alpha}(Q_1^+),$$

is a weak solution to

$$\begin{cases} y^a\partial_t w - \operatorname{div}(y^a D\nabla w) = \operatorname{div}(y^a\tilde F) & \text{in } Q_r^+,\\ \lim\limits_{y\to0^+} y^a\big(D\nabla w + \tilde F\big)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_r^+, \end{cases} \tag{1.129}$$

where

$$\tilde F := \partial_y D\nabla\Big(\partial_y u + \frac{g}{\mu}\Big) + \Big[\tilde f + \partial_y\mu\frac{a}{y}\Big(\partial_y u + \frac{g}{\mu}\Big)\Big]e_{d+1}, \tag{1.130}$$

and

$$\tilde f := \partial_y f + \partial_y \operatorname{div} g + \operatorname{div}(\partial_y D\nabla u) + \partial_t\Big(\frac{g}{\mu}\Big) - \operatorname{div}\Big(D\nabla\Big(\frac{g}{\mu}\Big)\Big). \tag{1.131}$$

Moreover,

$$\|\tilde F\|_{C_p^{k,\alpha}(Q_r^+)} \le C\big(\|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)} + \|u\|_{C_p^{k+2,\alpha}(Q_r^+)}\big), \tag{1.132}$$

for some $C>0$ depending only on d, a, λ, Λ, r, α and $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$.

Proof. Step 1. First, we prove that

$$w := y^{-a}\partial_y\Big(y^a\Big(\partial_y u + \frac{g}{\mu}\Big)\Big) = \partial_y\Big(\partial_y u + \frac{g}{\mu}\Big) + \frac{a}{y}\Big(\partial_y u + \frac{g}{\mu}\Big) \in C_p^{k+1,\alpha}(Q_1^+).$$

By (1.2), we have $\mu \ge \lambda > 0$ and so $\partial_y u + \frac{g}{\mu} \in C_p^{k+2,\alpha}(Q_1^+)$, thanks to the regularity assumptions on μ, g and μ. By Theorem 1.1.1, u satisfies the conormal boundary condition

$$\lim_{y\to0^+} \mu\partial_y u + g = 0, \tag{1.133}$$

and hence, by Lemma 1.2.2, we deduce that $\frac{a}{y}(\partial_y u + \frac{g}{\mu}) \in C_p^{k+1,\alpha}(Q_1^+)$, which implies that $w \in C_p^{k+1,\alpha}(Q_1^+)$ by definition of w.

By similar considerations, it follows that $\tilde f \in C_p^{k,\alpha}(Q_1^+)$, where $\tilde f$ is defined in (1.131). Consequently, $\tilde F \in C_p^{k,\alpha}(Q_1^+)$ (defined in (1.130)) and (1.132) directly follows by definition.

Step 2. From this point we distinguish two cases as follows. If $k=0$, we assume that $D, f, F \in C^\infty(Q_1^+)$ and thus, by Lemma 1.11.2, $u \in C^\infty(Q_1^+)$ as well. We will recover our statement under the weaker assumptions $D \in C_p^{2,\alpha}(Q_1^+)$, $f \in C_p^{1,\alpha}(Q_1^+)$ and $F \in C_p^{2,\alpha}(Q_1^+)$ throughout an approximation argument (see *Step 3*). If $k \ge 1$ we such approximation argument is not needed (this is because, when $k\ge1$, the equation of w is satisfied in the classical sense).

We may rewrite (1.128) as

$$\partial_t u - \operatorname{div}(D\nabla u) - \frac{a}{y}(\mu\partial_y u + g) = f + \operatorname{div} F \quad \text{in } Q_1^+.$$

Differentiating the above equation with respect to y, we obtain

$$\partial_t(\partial_y u) - \operatorname{div}(D\nabla(\partial_y u)) - \operatorname{div}(\partial_y D\nabla u) - \partial_y\Big(\frac{a}{y}(\mu\partial_y u + g)\Big) = \partial_y f + \partial_y \operatorname{div} F \quad \text{in } Q_1^+. \tag{1.134}$$

Taking in account (1.134) and setting $v := y^a\Big(\partial_y u + \frac{g}{\mu}\Big)$, we obtain the equation of v

$$\begin{aligned}
&y^{-a}\partial_t v - \operatorname{div}(y^{-a}D\nabla v) = \partial_t(\partial_y u) + \partial_t\Big(\frac{g}{\mu}\Big) - \operatorname{div}\Big(D\nabla\Big(\partial_y u + \frac{g}{\mu}\Big)\Big) - \partial_y\Big(\frac{a}{y}(\mu\partial_y u + g)\Big)\\
&= \partial_t(\partial_y u) + \partial_t\Big(\frac{g}{\mu}\Big) - \operatorname{div}(D\nabla(\partial_y u)) - \operatorname{div}\Big(D\nabla\Big(\frac{g}{\mu}\Big)\Big) - \partial_y\Big(\frac{a}{y}(\mu\partial_y u + g)\Big)\\
&= \partial_y f + \partial_y \operatorname{div} \bar F + \operatorname{div}(\partial_y D\nabla u) + \partial_t\Big(\frac{g}{\mu}\Big) - \operatorname{div}\Big(D\nabla\Big(\frac{g}{\mu}\Big)\Big) := \tilde f \quad \text{in } Q_1^+,
\end{aligned}$$

and thus, recalling that $\mu \geq \lambda > 0$ and (1.133), v satisfies

$$\begin{cases} y^{-a}\partial_t v - \operatorname{div}(y^{-a}D\nabla v) = y^{-a}(y^a\tilde f) & \text{in } Q_1^+,\\ v = 0 & \text{on } \partial^0 Q_1^+. \end{cases} \tag{1.135}$$

Differentiating (1.135) with respect to y, we get

$$\partial_t\partial_y v - \operatorname{div}(D\nabla\partial_y v) - \operatorname{div}(\partial_y D\nabla v) - \partial_y\Big(\frac{a}{y}(\mu\partial_y v)\Big) = \partial_y(y^a\tilde f) \quad \text{in } Q_1^+.$$

Consequently, $w = y^{-a}\partial_y v$ and satisfies

$$\begin{aligned}
&y^a\partial_t w - \operatorname{div}(y^a D\nabla w) = \partial_t\partial_y v - \operatorname{div}(D\nabla\partial_y v) + \Big(\frac{a}{y}(\mu\partial_y v)\Big)\\
&= \partial_y(y^a\tilde f) + \operatorname{div}(\partial_y D\nabla v)\\
&= \partial_y(y^a\tilde f) + \operatorname{div}\Big(y^a\partial_y D\nabla\Big(\partial_y u + \frac{g}{\mu}\Big)\Big) + \partial_y\Big(y^a\partial_y\mu\frac{a}{y}\Big(\partial_y u + \frac{g}{\mu}\Big)\Big) \quad \text{in } Q_1^+.
\end{aligned}$$

We need to establish that w satisfies the boundary condition in (1.129). By the regularity assumptions and the fact that $v = 0$ on $\{y = 0\}$, we can take the limit as $y \to 0^+$ in the equation (1.135) to get

$$\lim_{y\to 0^+}\Big[\mu\partial_{yy}v - \frac{a}{y}\mu\partial_y v + \partial_y\mu\partial_y v + y^a\tilde f\Big] = \lim_{y\to 0^+}\Big[\partial_t v - \sum_{i=1}^d \partial_{x_i}(D_{i,i}\partial_{x_i}v)\Big] = 0,$$

which turns out to be the boundary condition

$$\begin{aligned}
0 &= \lim_{y\to 0^+}\Big[y^a(\mu\partial_y w + \tilde f) + \partial_y\mu\partial_y v\Big]\\
&= \lim_{y\to 0^+} y^a\Big[\mu\partial_y w + \tilde f + \partial_y\mu\partial_y\Big(\partial_y u + \frac{g}{\mu}\Big) + \partial_y\mu\frac{a}{y}\Big(\partial_y u + \frac{g}{\mu}\Big)\Big].
\end{aligned}$$

Hence, defining $\tilde F$ as in (1.130), it follows that w is solution to (1.129) as claimed.

Step 3. In this final step, we present the approximation argument which allows to complete the proof when $k = 0$. First, by Theorem 1.10.1, we have that

$$\begin{aligned}
\|\tilde F\|_{C_p^{0,\alpha}(Q_r^+)} &\leq C\big(\|f\|_{C_p^{1,\alpha}(Q_1^+)} + \|F\|_{C_p^{2,\alpha}(Q_1^+)} + \|u\|_{C_p^{2,\alpha}(Q_r^+)}\big)\\
&\leq C\big(\|f\|_{C_p^{1,\alpha}(Q_1^+)} + \|F\|_{C_p^{2,\alpha}(Q_1^+)} + \|u\|_{L^2(Q_1^+,y^a)}\big),
\end{aligned} \tag{1.136}$$

for some $C > 0$ depending only on d, a, λ, Λ, r, α and $\|D\|_{C_p^{2,\alpha}(Q_1^+)}$.

The proof follows the approximation scheme done in the proof of Theorem 1.10.1: it is enough to replace the matrix A with the matrix D. Indeed, after regularizing the data (which we call $f_\varepsilon, F_\varepsilon, A_\varepsilon \in C^\infty(Q_1^+)$), and using Lemma 1.11.2, we can find a family of smooth solutions $v_\varepsilon \in C_c^\infty(Q_r^+)$ to

$$\begin{cases} y^a \partial_t v_\varepsilon - \mathrm{div}(y^a D_\varepsilon \nabla v_\varepsilon) = y^a f_\varepsilon + \mathrm{div}(y^a F_\varepsilon) & \text{in } Q_r^+, \\ \lim\limits_{y\to 0^+} y^a (D_\varepsilon \nabla v_\varepsilon + F_\varepsilon)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_r^+. \end{cases}$$

which converges to the original solution u as $\varepsilon \to 0^+$. Consequently, *Step 2* yields that

$$w_\varepsilon := y^{-a}\partial_y \Big(y^a \Big(\partial_y v_\varepsilon + \frac{\bar f_\varepsilon}{\mu_\varepsilon} \Big) \Big)$$

is a solution to (1.129) (with D and $\tilde F$ replaced by D_ε and $\tilde F_\varepsilon$) and $\tilde F_\varepsilon$, defined accordingly to (1.130), satisfies (1.136). By Proposition 1.10.2 and the Arzelá-Ascoli theorem, one has that $v_\varepsilon \to u$ in $C_p^{2,\alpha}(Q_r^+)$, which implies that $w_\varepsilon \to w$ in $C_p^{0,\alpha}(Q_r^+)$. Then a slight modification of the argument in Lemma 1.4.1 shows that w_ε converges to w in the energy spaces and that w is a weak solution to (1.129), as claimed. □

Proof of Proposition 1.11.1. Let $\partial_i := \partial_{x_i}$ for $i = 1, \dots, d$. We proceed with an induction argument. The step $k = 0$ has been proved in Proposition 1.10.2. Let us assume that (1.127) holds for $j = 1, \dots, k \in \mathbb{N}$ and let us prove that it is valid for $k+1$. So, let $u \in C_p^{k+3,\alpha}(Q_1^+)$, $A, F \in C_p^{k+2,\alpha}(Q_1^+)$ and $f \in C_p^{k+1,\alpha}(Q_1^+)$.

Let us fix $0 < r < r' < 1$. First, for every $i = 1, \dots, d$, by Lemma 1.8.3, one has that $u_i := \partial_i u$ solves (1.89) in $Q_{r'}^+$. Noticing that $u_i \in C_p^{k+2,\alpha}(Q_1^+)$, $A \in C_p^{k+2,\alpha}(Q_1^+)$, $\partial_i f \in C_p^{k,\alpha}(Q_1^+)$ and $\partial_i F, \partial_i A\nabla u \in C_p^{k+1,\alpha}(Q_1^+)$, we can use the inductive step to obtain

$$\begin{aligned} \|u_i\|_{C_p^{k+2,\alpha}(Q_r^+)} &\le C\Big(\|u_i\|_{L^2(Q_{r'}^+, y^a)} + \|\partial_i f\|_{C_p^{k,\alpha}(Q_1^+)} + \|\partial_i F\|_{C_p^{k+1,\alpha}(Q_1^+)} \Big) \\ &\le C\Big(\|u\|_{L^2(Q_1^+, y^a)} + \|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)} \Big), \end{aligned} \tag{1.137}$$

where $C > 0$ depends only on d, a, α, λ, Λ, $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$. It remains to prove that

$$\begin{aligned} &[u_{yyy}]_{C_p^{k,\alpha}(Q_r^+)} + [u_{yy}]_{C_t^{k,\frac{1+\alpha}{2}}(Q_r^+)} + [u_{ty}]_{C_p^{k,\alpha}(Q_r^+)} + [u_t]_{C_t^{k,\frac{1+\alpha}{2}}(Q_r^+)} \\ &\le C\Big(\|u\|_{L^2(Q_1^+, y^a)} + \|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)} \Big). \end{aligned}$$

Let $D := \mathrm{diag}(A)$. It is immediate to check that u solves

$$\begin{cases} y^a \partial_t u - \mathrm{div}(y^a D \nabla u) = y^a \bar f + \mathrm{div}(y^a \bar F) & \text{in } Q_r^+ \\ \lim\limits_{y\to 0^+} y^a (\mu \partial_y u + g) = 0, & \text{on } \partial^0 Q_r^+ \end{cases}$$

where $\bar F := ((A - D)\nabla u \cdot e_{d+1}) e_{d+1} + F$ and $\bar f := f + \sum_{i,j=1}^{d+1} \partial_i((A-D)_{i,j}\partial_j u)$, $g = \bar F_{d+1}$ and $\mu = A_{d+1,d+1}$. Furthermore, by (1.137) and the definition of $\bar F$ and $\bar f$, we have that $\bar F \in C_p^{k+2,\alpha}(Q_{r'}^+)$, $\bar f \in C_p^{k+1,\alpha}(Q_{r'}^+)$ and

$$\|\bar F\|_{C_p^{k+2,\alpha}(Q_{r'}^+)} + \|\bar f\|_{C_p^{k+1,\alpha}(Q_{r'}^+)} \le C\big(\|u\|_{L^2(Q_1^+, y^a)} + \|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)} \big), \tag{1.138}$$

for some $C>0$ which depending only on d, a, α, λ, Λ, $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$.

By Lemma 1.11.3, the function $w := y^{-a}\partial_y\big(y^a(\partial_y u + g/\mu)\big)$ belongs to $C_p^{k+1,\alpha}(Q_{r'}^+)$ and is a weak solution to

$$\begin{cases} y^a\partial_t w - \mathrm{div}(y^a D\nabla w) = \mathrm{div}(y^a\tilde{F}) & \text{in } Q_{r'}^+,\\ \lim\limits_{y\to 0^+} y^a\big(D\nabla w + \tilde{F}\big)\cdot e_{d+1} = 0 & \text{on } \partial^0 Q_{r'}^+,\end{cases}$$

where $\tilde{F}$ is defined in (1.130), with f and F replaced by $\bar{f}$ and $\bar{F}$ respectively. Furthermore, $\tilde{F}\in C_p^{k,\alpha}(Q_{r'}^+)$, so, by the inductive assumption (noticing that in the case $k=0$ we use Theorem 1.1.1), by (1.132), (1.137) and (1.138), we obtain that $w\in C_p^{k+1,\alpha}(Q_r^+)$ and

$$\|w\|_{C_p^{k+1,\alpha}(Q_r^+)} \le C\big(\|u\|_{L^2(Q_1^+,y^a)} + \|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)}\big),$$

for some $C>0$ which depends only on d, a, α, λ, Λ, $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$. Now, by the same arguments of Lemma 1.10.3 and by Lemma 1.2.3, it follows

$$(\partial_y u + g/\mu)(x,y,t) = \frac{1}{y^a}\int_0^y s^a w(x,s,t)ds,$$

satisfies $\partial_y(\partial_y u + \bar{f}/\mu)\in C^{k+1,\alpha}$ and, by the regularity of g and μ, we deduce

$$[u_{yyy}]_{C_p^{k,\alpha}(Q_r^+)} + [u_{yy}]_{C_t^{k,\frac{1+\alpha}{2}}(Q_r^+)} \le C\big(\|u\|_{L^2(Q_1^+,y^a)} + \|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)}\big),$$

for some $C>0$ depending only on d, a, α, λ, Λ, $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$.

To conclude the proof, it is sufficient to observe that

$$\partial_t u = y^{-a}\mathrm{div}(y^a(A\nabla u + F)) + f \in C_p^{k+1,\alpha}(Q_r^+),$$

which immediately implies

$$\|\partial_t u\|_{C_p^{k+1,\alpha}(Q_r^+)} \le C\big(\|u\|_{L^2(Q_1^+,y^a)} + \|f\|_{C_p^{k+1,\alpha}(Q_1^+)} + \|F\|_{C_p^{k+2,\alpha}(Q_1^+)}\big),$$

for some $C>0$ depending only on d, a, α, λ, Λ, $\|A\|_{C_p^{k+2,\alpha}(Q_1^+)}$. □

Proof of Theorem 1.1.2. Once established Proposition 1.11.1 and Lemma 1.11.2, our statement follows by approximation as in in Theorem 1.10.1. □

1.12 Weights degenerating on curved manifolds

In this last section, we show how to extend the $C^{1,\alpha}$ regularity estimates to weak solutions of a class of equations having weights vanishing or exploding on curved characteristic manifolds Γ, as in (1.9). Let us begin with the notion of weak solutions to (1.9).

Specifically, we consider equations set in cylindrical domains of the form $\Omega^+\times(-1,1)\subset\mathbb{R}^{d+2}$, which lie on one side of $\Gamma\times(-1,1)$. Here, $\Gamma\subset\mathbb{R}^{d+1}$ is a C^1 hypersurface parametrized by $\varphi\in C^1(B_1\cap\{y=0\})$, in the sense that, up to perform a dilation, a translation and a rotation, one has

$$\varphi(0)=0, \quad \nabla_x\varphi(0)=0, \quad \Omega^+\cap B_1 = \{y>\varphi(x)\}\cap B_1, \quad \Gamma\cap B_1 = \{y=\varphi(x)\}\cap B_1. \quad (1.139)$$

The family of weights $\delta=\delta(z)$ we consider behave as a distance function to Γ in the sense that

$$\begin{cases}\delta>0 & \text{in } \Omega^+\cap B_1\\ |\nabla\delta|\geq c_0>0 & \text{in } \Omega^+\cap B_1\\ \delta=0 & \text{on } \Gamma\cap B_1.\end{cases} \tag{1.140}$$

Hence, we give the definition of weak solutions to (1.9).

Definition 1.12.1. Let $a>-1$. Let $\varphi\in C^1(B_1\cap\{y=0\})$ be the parametrization defined in (1.139), $\delta\in C^1(\Omega^+\cap B_1)$ satisfying (1.140), $f\in L^2((\Omega^+\cap B_1)\times(-1,1),\delta^a)$ and $F\in L^2((\Omega^+\cap B_1)\times(-1,1),\delta^a)^{d+1}$. We say that u is a weak solution to (1.9) if $u\in L^2(I_1;H^1(\Omega^+\cap B_1,\delta^a))\cap L^\infty(I_1;L^2(\Omega^+\cap B_1,\delta^a))$ and satisfies

$$-\int_{(\Omega^+\cap B_1)\times(-1,1)}\delta^a u\partial_t\phi+\int_{(\Omega^+\cap B_1)\times(-1,1)}\delta^a A\nabla u\cdot\nabla\phi=\int_{(\Omega^+\cap B_1)\times(-1,1)}\delta^a(f\phi-F\cdot\nabla\phi),$$

for every $\phi\in C^\infty_c(Q_1)$.

Proof of Corollary 1.1.3. We divide the proof into several steps. First, we establish the $C^{1,\alpha}_p$ regularity. Then, using an inductive argument, we prove the higher order regularity.

Step 1. Reducing to flat characteristic manifolds by a local diffeomorphism. Let $\varphi\in C^{1,\alpha}(B_1\cap\{y=0\})$ and consider a classical diffeomorphism which straighten the hypersurface Γ,

$$\Phi(x,y)=(x,y+\varphi(x)),$$

which is of class $C^{1,\alpha}$ and then $C^{1,\alpha}_p$ extending constantly in the time variable. Actually, Φ^{-1} locally flattens Γ to Σ. In fact, there exists a small radius $R>0$ such that $\Phi(B_R\cap\{y>0\})\subseteq B_1\cap\{y>\varphi(x)\}$, $\Phi(0)=\Phi^{-1}(0)=0$ and $\Phi(B_R\cap\{y=0\})\subseteq B_1\cap\{y=\varphi(x)\}$. The Jacobian associated to Φ is

$$J_\Phi(x)=\begin{pmatrix}\mathbb{I}_d & 0\\ \nabla\varphi(x)^\top & 1\end{pmatrix},\quad \text{with } |\det J_\Phi|\equiv 1.$$

Up to a dilation, one has that $\tilde u:=u\circ(\Phi(x),t)$ is a weak solution to

$$\begin{cases}\tilde\delta^a\partial_t\tilde u-\operatorname{div}(\tilde\delta^a\tilde A\nabla\tilde u)=\tilde\delta^a\tilde f+\operatorname{div}(\tilde\delta^a\tilde F), & \text{in } Q_1^+,\\ \lim\limits_{y\to0^+}\tilde\delta^a(\tilde A\nabla\tilde u+\tilde F)\cdot e_{d+1}=0 & \text{on } \partial^0Q_1^+.\end{cases}$$

where $\tilde\delta=\delta\circ\Phi$, $\tilde f=f\circ(\Phi(x),t)$ and $\tilde F=J_\Phi^{-1}F\circ(\Phi(x),t)$ and $\tilde A=(J_\Phi^{-1})(A\circ(\Phi(x),t))(J_\Phi^{-1})^\top$.

By [143, Lemma 2.3] (equivalently, by Lemma 1.2.2), $\tilde\delta\in C^{1,\alpha}(B_1^+)$ and satisfies

$$\tilde\delta>0 \text{ in } B_1^+,\quad \tilde\delta=0 \text{ on } \partial^0B_1^+,\quad \partial_y\tilde\delta>0 \text{ on } \partial^0B_1^+,\quad \frac{\tilde\delta}{y}\in C^{0,\alpha}(B_1^+),\quad \frac{\tilde\delta}{y}\geq\mu>0 \text{ in } \overline{B_1^+},$$

where the last nondegeneracy condition is a consequence of the assumption $|\nabla\delta|\geq c_0>0$. Now, noticing that $\tilde u$ is a weak solution to

$$\begin{cases}y^a\left(\frac{\tilde\delta}{y}\right)^a\partial_t\tilde u-\operatorname{div}(y^a\bar A\nabla\tilde u)=y^a\bar f+\operatorname{div}(y^a\bar F), & \text{in } Q_1^+,\\ \lim\limits_{y\to0^+}y^a(\bar A\nabla\tilde u+\bar F)\cdot e_{d+1}=0 & \text{on } \partial^0Q_1^+.\end{cases} \tag{1.141}$$

where $\bar{A}=\tilde{A}(\tilde{\delta}/y)^a\in C_p^{0,\alpha}(Q_1^+)$, $\bar{f}=\tilde{f}(\tilde{\delta}/y)^a\in L^p(Q_1^+,y^a)$ and $\bar{F}=\tilde{F}(\tilde{\delta}/y)^a\in C_p^{0,\alpha}(Q_1^+)$, we are taken back to an equation with the standard degenerate or singular weight y^a as in (1.1), but with a new nondegenerate term $(\tilde{\delta}/y)^a$ in front of the time derivative.

Step 2. Regularity for flat characteristic manifolds with an extra term in front of the time derivative. In what follows we show that our regularity theory applies with minor changes to weak solutions to (1.141); that is, where an extra term b appears in front of the time derivative in the parabolic equation. The term needs to be uniformly continuous in B_1^+ and bounded away from zero $b\geq\mu>0$. In the present case $b(z):=(\tilde{\delta}(z)/y)^a$, which is even Hölder continuous.

First, the energy results obtained in Sections 1.2, 1.3, 1.4 can be easily extended just using the fact that the positive term b is bounded and bounded away from zero. These bounds ensure invariance of the norms involved in the functional setting.

Let us focus on the only difference, respect to the proof of Theorem 1.7.1; that is, the $C_p^{1,\alpha}$ ε-stable regularity of solutions with regularized weights ρ_ε^a (the proof of Theorem 1.6.1, the ε-stability for the $C_p^{0,\alpha}$ estimate, is analogous): in order to prove that the blow-up sequence $\{\bar{w}_k\}$ (see (1.75)) converges to an entire solution with constant coefficients, one considers the limit in the equation (1.76) satisfied by $\bar{w}_k$ with the necessary modifications for the present case. The l.h.s. converges in the following sense: by using the same considerations of Lemma 1.4.1 we have that

$$\int\tilde{\rho}_k^a(y)\big(-b(r_kz+\hat{z}_k)\bar{w}_k\partial_t\phi+A(r_kz+\hat{z}_k,r_k^2t+t_k)\nabla\bar{w}_k\cdot\nabla\phi\big)\to\int\tilde{\rho}^a\big(-\bar{b}\bar{v}\partial_t\phi+\bar{A}\nabla\bar{v}\cdot\nabla\phi\big),$$

where $\bar{b}=\lim_{k\to\infty}b(r_kz+\hat{z}_k)$ is a positive constant and $\bar{A}=\lim_{k\to\infty}A(r_kz+\hat{z}_k,r_k^2t+t_k)$ is a constant coefficient matrix. Therefore, the contradiction argument ends up again with the use of the Liouville Theorem 1.5.1. Finally, by Lemma 1.4.1, with the same considerations done in the proof of Theorem 1.1.1, the statement follows.

Step 3. Higher order regularity. Let $\varphi\in C^{k+2,\alpha}(B_1\cap\{y=0\})$, where $k\in\mathbb{N}$. At this step, the function $b(z):=(\tilde{\delta}(z)/y)^a\in C^{k+1,\alpha}(B_1^+)$, by Lemma 1.2.2, so $b\phi$ is an admissible test function in (1.141). Hence, we compute

$$\begin{aligned}0&=\int_{Q_1^+}y^ab\big(-\tilde{u}\partial_t\phi+\tilde{A}\nabla\tilde{u}\cdot\nabla\phi-\tilde{f}\phi+\tilde{F}\cdot\nabla\phi\big)\\&=\int_{Q_1^+}y^a\big(-\tilde{u}\partial_t(\phi b)+\tilde{A}\nabla\tilde{u}\cdot\nabla(\phi b)-\tilde{A}\nabla\tilde{u}\cdot\nabla b\phi-\tilde{f}(\phi b)+\tilde{F}\cdot\nabla(\phi b)-\tilde{F}\cdot\nabla b\phi\big),\end{aligned}$$

that is, $\tilde{u}$ is a weak solution to

$$\begin{cases}y^a\partial_t\tilde{u}-\mathrm{div}(y^a\tilde{A}\nabla\tilde{u})=y^a\tilde{g}+\mathrm{div}(y^a\tilde{F}), & \text{in } Q_1^+,\\ \lim\limits_{y\to0^+}y^a(\tilde{A}\nabla\tilde{u}+\tilde{F})\cdot e_{d+1}=0 & \text{on } \partial^0Q_1^+,\end{cases}$$

where

$$\tilde{g}:=\tilde{f}+\frac{\tilde{A}\nabla\tilde{u}\cdot\nabla b}{b}+\frac{\tilde{F}\cdot\nabla b}{b}.$$

Finally, we apply a recursive argument to prove the $C_p^{k+2,\alpha}$-regularity of $\tilde{u}$, which in turns extends to the same regularity for the original u by composing back with the diffeomorphism Φ^{-1}.

For $k=0$, we observe that $u \in C^{1,\alpha}_p$ by *Step 1* and *Step 2*. Consequently, after composing with the $C^{2,\alpha}_p$ diffeomorphism, we obtain $\nabla\tilde{u} \in C^{0,\alpha}_p$, which implies that $\tilde{g} \in C^{0,\alpha}_p$. The $C^{2,\alpha}_p$-regularity of $\tilde{u}$ then follows from Theorem 1.1.2.

Finally, this argument can be iterated for any $k \geq 1$, replacing the initial result from *Step 1* and *Step 2* with *Step 3* at a lower step. ☐

1.13 Parabolic Higher Order Boundary Harnack Principle

This last section, is devoted to the proof of the Higher Order Boundary Harnack Principle in Theorem 1.1.4 (see also [12, 90]).

Proof of Theorem 1.1.4. First, the regularity assumptions of boundaries, coefficients and data for the equations in (1.10) do guarantee that $u, v \in C^{k+2,\alpha}_{\mathrm{loc}}(\overline{\Omega} \cap Q_1)$, by classical theory of uniformly parabolic equations (for instance, see [96]). Hence, the equations in (1.10) are satisfied both in the weak sense and pointwisely in $\Omega \cap Q_1$. From this, we deduce a pointwise equation for the quotient $w = v/u$ in $\Omega \cap Q_1$; that is,

$$u^2\partial_t w - \mathrm{div}(u^2 A\nabla w) = uf - vg + u^2 b \cdot \nabla w. \tag{1.142}$$

Now, let us define the standard diffeomorphism

$$\Phi(x,y,t) := (x, y + \varphi(x,t), t),$$

which is of class $C^{k+2,\alpha}_p$. Let us set $\tilde{u} = u \circ \Phi, \tilde{v} = v \circ \Phi, \tilde{f} = f \circ \Phi, \tilde{g} = g \circ \Phi$ and define

$$\tilde{A} = (J^{-1}_{z,\Phi})^\top (A \circ \Phi) J^{-1}_{z,\Phi}, \quad \tilde{b} = J^{-1}_{z,\Phi} b \circ \Phi,$$

where $J_{z,\Phi}$ is the square block $[c_{ij}]_{i,j=1,\dots,d+1}$ of the Jacobian $J_\Phi := [c_{ij}]_{i,j=1,\dots,d+2}$.

Since w solves (1.142), then, up to dilations, $\tilde{w} = w \circ \Phi = \tilde{v}/\tilde{u}$ solves

$$y^2\mu^2\partial_t\tilde{w} - \mathrm{div}(y^2\mu^2\tilde{A}\nabla\tilde{w}) = y\Big(\mu\tilde{f} - \frac{\tilde{v}}{y}\tilde{g}\Big) + y^2\mu^2\tilde{b}\cdot\nabla\tilde{w} + y^2\mu^2 c \cdot \nabla\tilde{w}, \tag{1.143}$$

pointwisely in Q_1^+, where $\mu = \tilde{u}/y$ and $c = \partial_t\varphi e_{d+1}$. Now we need to do some remarks on regularity of the data of the weighted equation above. First, by Lemma 1.2.2 and the non degeneracy condition $u(z,t) \geq c_0\, d_p((z,t), \partial\Omega \cap Q_1)$ in (1.10), we can infer that

$$0 < \overline{c}_0 \leq \mu \in C^{k+1,\alpha}_p(B_1^+).$$

Thanks to the previous information, we can rewrite (1.143) dividing by μ^2 as

$$y^2\partial_t\tilde{w} - \mathrm{div}(y^2\tilde{A}\nabla\tilde{w}) = yh + y^2\overline{b}\cdot\nabla\tilde{w}, \tag{1.144}$$

where

$$\overline{b} = \tilde{b} + c + 2\tilde{A}^\top\frac{\nabla\mu}{\mu} \in C^{k,\alpha}_p, \quad h = \frac{\mu\tilde{f} - \frac{\tilde{v}}{y}\tilde{g}}{\mu^2} \in C^{k+1,\alpha}_p.$$

Moreover, since

$$\tilde{w} = \frac{\tilde{v}/y}{\tilde{u}/y},$$

again by Lemma 1.2.2 and the $C_p^{k+2,\alpha}$-regularity of $\tilde{u},\tilde{v}$, we have $\tilde{w}\in C_p^{k+1,\alpha}(Q_1^+)$ which has two implications: first, the drift term in (1.144) can be considered as a forcing term; that is, $\overline{b}\cdot\nabla\tilde{w}=\overline{f}\in C_p^{k,\alpha}(Q_1^+)$; secondly, $\tilde{w}$ belongs to $L^2(I_1;H^1(B_1^+,y^a))\cap L^\infty(I_1;L^2(B_1^+,y^a))$ and, by multiplying the equation (1.144) by test functions $\phi\in C_c^\infty(Q_1^+)$ and integrating by parts, one gets that $\tilde{w}$ is a weak solution to

$$\begin{cases} y^2\partial_t\tilde{w}-\mathrm{div}(y^2\tilde{A}\nabla\tilde{w})=\mathrm{div}(y^2H)+y^2\overline{f}, & \text{in } Q_1^+,\\ \lim\limits_{y\to0^+}y^2(\tilde{A}\nabla\tilde{w}+H)\cdot e_{d+1}=0, & \text{on } \partial^0Q_1^+, \end{cases}$$

where the field

$$H(x,y,t)=\frac{e_{d+1}}{y^2}\int_0^y sh(x,s,t)\,ds$$

belongs to $C_p^{k+1,\alpha}(Q_1^+)$ by Lemma 1.2.3.

Then, the regularity $C_p^{k+2,\alpha}$-regularity of $\tilde{w}$ follows by Theorem 1.1.2. Finally, the same regularity is inherited by w by composing back with the diffeomorphism. □

CHAPTER

TWO

REGULARITY FOR THE DIRICHLET PROBLEM ON LOWER DIMENSIONAL MANIFOLDS

2.1 Introduction

The second chapter is based on the paper [70] by the author. Let $2 \leq n \leq d$ be two integers and $z = (x, y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$. Let us define the lower dimensional manifold

$$\Sigma_0 := \{(x, y) \in \mathbb{R}^d : |y| = 0\},$$

which has dimension $d - n$, and the weight $|y|^a = \mathrm{dist}^a_{\Sigma_0}(z)$, where the real parameter a satisfies $a + n \in (0, 2)$. We study the following equation

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_0, \\ u = \psi, & \text{on } \Sigma_0 \cap B_1, \end{cases} \tag{2.1}$$

where $B_1 \subset \mathbb{R}^d$ denotes the unit ball with center at 0, $A : B_1 \to \mathbb{R}^{d,d}$ is a symmetric d-dimensional matrix satisfying the following ellipticity condition

$$\lambda|\xi|^2 \leq A(z)\xi \cdot \xi \leq \Lambda|\xi|^2, \tag{2.2}$$

for all $\xi \in \mathbb{R}^d$ and a.e. $z \in B_1$, where $0 < \lambda \leq \Lambda < \infty$ are fixed constants.

The terms $f : B_1 \to \mathbb{R}$, $F : B_1 \to \mathbb{R}^d$ and $\psi : \Sigma_0 \cap B_1 \to \mathbb{R}$ belong to suitable spaces, which will be introduced later. We notice that, when $n = d$, the lower-dimensional boundary $\Sigma_0 = \{0\}$ reduces to a single point. In this case, we assume the boundary condition is simply $u(0) = 0$. The operators ∇ and div denote the gradient and the divergence with respect to the variable z, respectively. Weak solutions to this equation are naturally defined within the framework of weighted Sobolev spaces, which will be introduced in Section 2.2.1. Thus, we say that u is a weak solution to (1.1) if $u \in H^{1,a}(B_1) := H^1(B_1, |y|^a dz)$, satisfies

$$\int_{B_1} |y|^a A\nabla u \cdot \nabla \phi dz = \int_{B_1} |y|^a (f\phi - F \cdot \nabla \phi) dz,$$

for every $\phi \in C^\infty_c(B_1 \setminus \Sigma_0)$ and $u = \psi$ in the sense of the trace (see Definition 2.2.6).

Our primary goal is to establish local regularity estimates up to Σ_0 for weak solutions to (1.1). Specifically, we show that, under suitable assumptions on the data, solutions are $C^{0,\alpha}(B_{1/2})$ and, in some cases, may be $C^{1,\alpha}(B_{1/2})$. As we will see later, our results are *sharp* with respect to the assumptions on the data. Specifically, the function $|y|^{2-a-n}$ is a solution to (1.1), when $A = \mathbb{I}$, $f = 0$, $F = 0$ and $\psi = 0$ and it is C^{-a-n} if $a+n \in [1,2)$ and $C^{1,1-a-n}$ if $a+n \in (0,1)$. The idea behind our theorems is that this solution is the *worst regular* solution to (1.1) when $A = \mathbb{I}$, $f = 0$, $F = 0$ and $\psi = 0$.

The Dirichlet boundary condition $u = \psi$ on Σ_0 requires some clarifications. For general values of $a \in \mathbb{R}$, the trace of functions which belongs to $H^{1,a}(B_1)$ on Σ_0 might not be well defined. In the paper [118], the author shows the existence of a trace operator for a large class of weighted Sobolev spaces on lower dimensional boundaries. Specifically, the following result holds: let Γ be a $(d-n)$-dimensional C^1-manifold, dist_Γ be the distance from Γ and $a+n \in (0,2)$. Then, there exits a unique bounded linear operator

$$T : H^1(B_1, \mathrm{dist}_\Gamma^a) \to L^2(\Gamma \cap B_1)$$

such that

$$Tu = u_{|\Gamma},$$

for every $u \in C^\infty(\overline{B_1})$.

Hence, the restriction we impose on the parameter $a+n \in (0,2)$ ensures that the boundary condition $u = \psi$ on Σ_0 makes sense. In other words, introducing the weight as a power of the distance to the boundary provides a natural framework for studying the Dirichlet problem for linear elliptic operators on lower-dimensional boundaries. Without such a weight, the solutions *do not see* the lower-dimensional sets due to capacitary reasons: for instance, a harmonic function in $B_1 \setminus \Sigma_0$ is the same as a harmonic function in the whole B_1.

As previously observed in the introduction of this thesis, the study of such equations falls within the theory of non uniformly elliptic operators, as the presence of the singular weight causes the operator's coefficients to blow up on the manifold Σ_0. In the seminal paper [65], the authors extended the De Giorgi-Nash-Moser theory to weighted elliptic equations, where the weight arises from quasi-conformal mappings or belongs to the Muckenhoupt A_2 class. In particular, under suitable assumptions, they proved the validity of the Harnack inequality, ensuring Hölder regularity of solutions with a non explicit Hölder exponent. Our weight $|y|^a$ belongs to the Muckenhoupt A_2 class when $a+n \in (0,2n)$, so our assumption $a+n \in (0,2)$ ensures that the known results for this class of weights apply, guaranteeing that the solutions to our problem satisfy some Hölder estimates. However, the peculiar geometry of the singular set of our weight $|y|^a$, combined with its homogeneity property, allows us to obtain more refined results compared to the general theory mentioned above.

In recent years, there have been significant contributions to the study weighted elliptic equations, where the weight behaves like the power of the distance from the boundary of a set. The most notable case is $n = 1$, which is closely related to the extension theory for fractional operators developed by Caffarelli and Silvestre in their seminal paper [28] (we refer also to [34], where the authors study fractional operators in conformal geometry). This setting is now well understood, and there is a rich literature on the regularity properties of such equations. Concerning Schauder type estimates, a notable reference can be found in [29]. Moreover, we highlight the works [138, 139, 143, 55], where the authors establish a complete Schauder regularity theory for degenerate/singular elliptic equations, and its parabolic counterpart [7, 8]. Additionally, we mention [58–60], where elliptic and parabolic weighted equations are studied, yielding alternative regularity results.

In the paper [46], David, Feneuil and Mayboroda developed an elliptic theory for equations which are degenerate/singular on lower dimensional boundaries. In particular, in [43], the authors extensively studied our operator in the case $a+n=1$, under weaker assumptions on the coefficients A. They proved the solvability of the Dirichlet problem in this setting. We also refer to [47] and the references therein for a broader overview of this topic. Recently, in [61], the authors investigated the boundary behaviour of solutions in this high-codimensional setting and established estimates on the singular set near the boundary. Moreover, our class of operators also arises in the context of singular harmonic maps to study equations related to black holes (see [148, 149, 93–95]). In particular, in [119], the author highlights the connection between these singular harmonic maps and differential operators like ours, in the case where $a<0$. Additionally, our work is related to Mazzeo's theory of edge operators [104, 106], which emerges in boundary problems with higher codimensional boundaries and provides essential insights into solution regularity by establishing Fredholmness in degenerate Hölder or Sobolev spaces.

Finally, it seems that our operator could be a good model for free boundary problems of the obstacle type (see, for example, the classical papers [22, 23]), where the obstacle is very thin (with dimension less than $d-2$), see also [68]. As already noted above, classical elliptic operators do not allow for such problems, as they cannot *see* small sets like these due to capacitary reasons.

2.1.1 Main results

The main goal of the chapter is to establish local $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity estimates up to the singular set Σ_0 for weak solutions to (2.1). These results are presented in two main theorems: the first provides Hölder estimates for the solutions, while the second establishes Hölder estimates for the gradient.

Theorem 2.1.1. *Let $2\le n\le d$, $a+n\in(0,2)$, $p>d/2$, $q>d$ and*

$$\alpha\in(0,2-a-n)\cap(0,2-d/p]\cap(0,1-d/q]\cap(0,1). \tag{2.3}$$

Let A be a continuous symmetric matrix satisfying (3.5) and ω be a modulus of continuity such that

$$\|A\|_{C^{0,\omega}(B_1)}:=\|A\|_{L^\infty(B_1)}+\sup_{z,z'\in B_1,z\neq z'}\frac{|A(z)-A(z')|}{\omega(|z-z'|)}\le L.$$

Let $f\in L^{p,a}(B_1)$, $F\in L^{q,a}(B_1)^d$ and $\psi\in C^{0,1}(\Sigma_0\cap B_1)$. Let u be a weak solution to (2.1).

Then, $u\in C^{0,\alpha}(B_{1/2})$ and there exists a constant $c>0$, depending only on d, n, a, λ, Λ, p, q, α and L such that

$$\|u\|_{C^{0,\alpha}(B_{1/2})}\le c\big(\|u\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{L^{q,a}(B_1)}+\|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big). \tag{2.4}$$

Theorem 2.1.2. *Let $2\le n\le d$, $a+n\in(0,1)$, $p>d$ and*

$$\alpha\in(0,1-a-n)\cap(0,1-d/p]. \tag{2.5}$$

Let A be a α-Hölder continuous matrix satisfying (2.2) and $\|A\|_{C^{0,\alpha}(B_1)}\le L$, $f\in L^{p,a}(B_1)$, $F\in C^{0,\alpha}(B_1)$ and $\psi\in C^{1,\alpha}(\Sigma_0\cap B_1)$. Let u be a weak solution to (2.1).

Then, $u \in C^{1,\alpha}(B_{1/2})$ *and there exists a constant* $c > 0$*, depending only on* d*,* n*,* a*,* λ*,* Λ*,* p*,* α *and* L *such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)} + \|\psi\|_{C^{1,\alpha}(\Sigma_0 \cap B_1)}\big). \tag{2.6}$$

In addition, u *satisfies the following boundary condition*

$$\begin{cases} \nabla_x u(x,0) = \nabla_x \psi(x,0), \\ (A\nabla u + F)(x,0)\cdot e_{y_i} = 0, \end{cases} \quad \text{for every } (x,0) \in \Sigma_0 \cap B_{1/2}, \text{ and } i = 1,\dots,n. \tag{2.7}$$

Before presenting the proof idea, we recall a method developed in [140, 138] to establish regularity estimates for weighted elliptic equations that are degenerate or singular on a set of codimension one, that is, when $n = 1$. We notice that this method extends well to parabolic equations, as shown in the previous chapter. This case, where the singular set is an hyperplane of dimension $d-1$, does not fall under the scope of our study, which focuses on $n \ge 2$. The authors develop a regularity theory for such equations by regularizing the weight and using an approximation argument: they first establish ε-stable regularity results for solutions with the regularized weight $(\varepsilon^2 + y^2)^{a/2}$, and then take the limit as $\varepsilon \to 0$. However, this approach seems not to work in our case, where $n \ge 2$. Indeed, by regularizing the weight, that is, considering $\rho_\varepsilon^a(y) := (\varepsilon^2 + |y|^2)^{a/2}$ and solutions to uniformly elliptic problems with this type of weight, the $H^1(\rho_\varepsilon^a)$-capacity of Σ_0 will be zero. As we discussed earlier, in this situation, the *classical* Sobolev spaces will fail to capture information about the boundary condition on the lower dimensional set Σ_0. Consequently, the approximation problem with this type of weight loses critical information about the boundary condition, making it impossible to recover uniform regularity estimates for the weighted problem.

We adopt a different approach based on perforated domains, aiming to establish uniform estimates in this framework. For small $0 < \varepsilon \ll 1$, we define $\Sigma_\varepsilon := \{|y| \le \varepsilon\}$ as the ε-neighbourhood of Σ_0, noting that its boundary $\partial\Sigma_\varepsilon = \{|y| = \varepsilon\}$ has dimension $d-1$ and $\partial\Sigma_\varepsilon \to \Sigma_0$ as $\varepsilon \to 0$. The central idea is to approximate the lower dimensional boundary, which has dimension $d-n$, with a *classical* boundary of dimension $d-1$ and impose the Dirichlet condition on this set. Hence, we consider solutions to the problem

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \partial\Sigma_\varepsilon \cap B_1. \end{cases}$$

In this context, since the singular set Σ_0 is sufficiently far from $B_1 \setminus \Sigma_\varepsilon$, we can apply classical regularity theory to obtain $C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)$ and $C^{1,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)$ regularity estimates, with constants that may depend on ε. The crucial step is proving that these estimates are uniform as $\varepsilon \to 0$, as shown in Theorems 2.6.1 and 2.7.1. Once we have these uniform estimates, we employ an approximation argument (see Section 2.4) to pass to the limit as $\varepsilon \to 0$, thereby recovering the desired regularity results for the original problem.

We emphasize that proving $C^{1,\alpha}$ regularity for solutions to (2.1) requires a refined approach. Specifically, the uniform estimates in perforated domains require an additional assumption on the field F, as highlighted in Remark 2.7.2. Consequently, proving the main theorem necessitates a double approximation strategy: the first via perforated domains and the second through a standard mollification argument. This is combined with some *a priori* estimates for solutions to (2.1) under an additional boundary condition on Σ_0, as detailed in Proposition 2.7.3.

The strategy for establishing the ε-uniform estimates is based on a contradiction argument combined with a blow-up procedure, drawing inspiration from Simon's work [136]. A key component of this approach is the Liouville-type Theorem 2.5.1, which applies to entire solutions satisfying a specific growth-control condition at infinity.

Finally, as a consequence of our main theorems, we extend our results to more general weighted equations, where the weight δ behaves like a distance function from a regular manifold Γ of dimension $d-n$ (see Definition 2.8.1). Specifically, in Corollaries 2.8.3 and 2.8.4, we are able to prove local $C^{0,\alpha}(B_{1/2})$ and $C^{1,\alpha}(B_{1/2})$ regularity estimates for weak solutions to the following equation

$$\begin{cases} -\operatorname{div}(\delta^a A\nabla u) = \delta^a f + \operatorname{div}(\delta^a F), & \text{in } B_1 \setminus \Gamma, \\ u = \psi, & \text{on } \Gamma \cap B_1. \end{cases} \tag{2.8}$$

The precise definition of solutions to (2.8) will be given later in Section 2.8.

2.1.2 Organization of the chapter

The chapter is organized as follows. In Section 2.2, we introduce the problem by defining the appropriate weighted Sobolev spaces, discussing their fundamental properties, and presenting the notion of weak solutions to our equation. Section 2.3 is devoted to establishing uniform boundedness of solutions. In Section 2.4, we prove a key approximation lemma. Section 2.5 contains the proof of a Liouville-type theorem, both in perforated and non-perforated domains. In Sections 2.6 and 2.7, we prove the main results - Theorems 2.1.1 and 2.1.2 - concerning the $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity of solutions, respectively. Finally, in Section 2.8, we extend these results to solutions of the more general equation (2.8).

2.1.3 Notation

Let $d, n \in \mathbb{N}$ be two integers such that $2 \leq n \leq d$. Let us consider coordinates $z = (x, y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$. Given $R > 0$, we denote by $B_R(z_0)$ the ball of radius R centered at z_0. When $z_0 = 0$ we simply write B_R. We define the lower dimensional characteristic manifold

$$\Sigma_0 := \{y = 0\}.$$

We write $\varepsilon \ll 1$, to denote that there exists a small $\varepsilon_0 > 0$ such that $\varepsilon \in (1, \varepsilon_0)$. Hence, for $\varepsilon \ll 1$ we define the ε-neighbourhood of Σ_0 as

$$\Sigma_\varepsilon := \{|y| \leq \varepsilon\},$$

and its topological boundary

$$\partial\Sigma_\varepsilon := \{|y| = \varepsilon\},$$

noticing that $\partial\Sigma_0 = \Sigma_0$.

2.2 Functional setting and preliminary results

2.2.1 Weighted Sobolev Spaces

Let $a+n \in (0,2)$ and $R>0$. For $p \in [1,\infty)$, let us define the weighted Lebesgue spaces

$$L^{p,a}(B_R) := L^p(B_R, |y|^a dz),$$

and for vector field

$$L^{p,a}(B_R)^d := L^p(B_R, |y|^a dz)^d.$$

The Sobolev space $H^{1,a}(B_R)$ is defined as the completion of $C^\infty(\overline{B_R})$ with respect to the norm

$$\|u\|_{H^{1,a}(B_R)} = \Big(\int_{B_R} |y|^a u^2 dz + \int_{B_R} |y|^a |\nabla u|^2 dz\Big)^{1/2},$$

where $C^\infty(\overline{B_R}) = \{u_{|_{B_R}} : u \in C_c^\infty(\mathbb{R}^d)\}$.

Since we are interested in functions which vanish on $\Sigma_0 = \{|y|=0\}$, we define the Sobolev space $\tilde{H}^{1,a}(B_R)$ as the completion of $C_c^\infty(\overline{B_R} \setminus \Sigma_0)$ with respect to the norm $\|\cdot\|_{H^{1,a}(B_R)}$.

Additionally, we define the Sobolev space $H_0^{1,a}(B_R)$ as as the completion of $C_c^\infty(B_R \setminus \Sigma_0)$ with respect to the norm $\|\cdot\|_{H^{1,a}(B_R)}$, which contains functions having zero trace on $\partial(B_R \setminus \Sigma_0)$.

2.2.2 Weighted Sobolev Spaces in perforated domains

Let $a+n \in (0,2)$, $R>0$ and $\varepsilon \ll 1$. For $p \in [1,\infty)$, we set

$$L^{p,a}(B_R \setminus \Sigma_\varepsilon) := L^p(B_R \setminus \Sigma_\varepsilon, |y|^a dz),$$

and

$$L^{p,a}(B_R \setminus \Sigma_\varepsilon)^d := L^p(B_R \setminus \Sigma_\varepsilon, |y|^a dz)^d.$$

Let us define the norm

$$\|u\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)} = \Big(\int_{B_R\setminus\Sigma_\varepsilon} |y|^a u^2 dz + \int_{B_R\setminus\Sigma_\varepsilon} |y|^a |\nabla u|^2 dz\Big)^{1/2}.$$

We define Sobolev spaces in perforated domains as follows.

$H^{1,a}(B_R \setminus \Sigma_\varepsilon)$ as the completion of $C^\infty(\overline{B_R \setminus \Sigma_\varepsilon})$ w.r.t. the norm $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$,

$\tilde{H}^{1,a}(B_R \setminus \Sigma_\varepsilon)$ as the completion of $C_c^\infty(\overline{B_R} \setminus \Sigma_\varepsilon)$ w.r.t. the norm $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$,

$H_0^{1,a}(B_R \setminus \Sigma_\varepsilon)$ as the completion of $C_c^\infty(B_R \setminus \Sigma_\varepsilon)$ w.r.t. the norm $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$.

When $\varepsilon = 0$, we identify the spaces

$$L^{p,a}(B_R \setminus \Sigma_0) = L^{p,a}(B_R), \quad \tilde{H}^{1,a}(B_R \setminus \Sigma_0) = \tilde{H}^{1,a}(B_R), \quad H_0^{1,a}(B_R \setminus \Sigma_0) = H_0^{1,a}(B_R).$$

Moreover, since the Poincaré inequality holds true (see Proposition 2.2.2) in $\tilde{H}^{1,a}(B_R \setminus \Sigma_\varepsilon)$ for every $0 \le \varepsilon \ll 1$, we have that

$$\|u\|_{H_0^{1,a}(B_R\setminus\Sigma_\varepsilon)} = \Big(\int_{B_R\setminus\Sigma_\varepsilon} |y|^a |\nabla u|^2 dz\Big)^{1/2},$$

defines an equivalent norm to $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$ in $\tilde{H}^{1,a}(B_R\setminus\Sigma_\varepsilon)$.

Remark 2.2.1. For $\varepsilon>0$, functions in $\tilde{H}^{1,a}(B_R\setminus\Sigma_\varepsilon)$ can be identified with their trivial extensions in the whole B_R. Consequently, we have the following inclusion of spaces:

$$\tilde{H}^{1,a}(B_R\setminus\Sigma_\varepsilon)\subset\tilde{H}^{1,a}(B_R).$$

The following Proposition establishes several fundamental inequalities in the space $\tilde{H}^{1,a}(B_R\setminus\Sigma_\varepsilon)$, namely the Hardy inequality, the Poincaré inequality, the Poincaré trace inequality, and a Sobolev-type inequality, which are uniform with respect to the parameter ε. All these results are, in fact, based on the Hardy inequality (which is nowadays a standard result). We also refer to [40, 115] for other functional type inequalities in this direction.

Proposition 2.2.2. *Let $2\le n\le d$, $a+n\in(0,2)$, $R>0$ and $0\le\varepsilon\ll1$. Then, there exist a constant $c>0$, depending only on d, n, a and R such that*

$$\int_{B_R}|y|^a\frac{u^2}{|y|^2}dz\le c\int_{B_R}|y|^a|\nabla u|^2dz, \tag{2.9}$$

$$\int_{B_R}|y|^au^2dz\le c\int_{B_R}|y|^a|\nabla u|^2dz, \tag{2.10}$$

$$\int_{\partial B_R}|y|^au^2d\sigma\le c\int_{B_R}|y|^a|\nabla u|^2dz, \tag{2.11}$$

$$\Big(\int_{B_R}|y|^a|u|^{2^*}dz\Big)^{2/2^*}dz\le c\int_{B_R}|y|^a|\nabla u|^2dz, \tag{2.12}$$

for every $u\in C_c^\infty(\overline{B_R}\setminus\Sigma_\varepsilon)$. In the last inequality $2^*:=2d/(d-2)$ if $d>2$ and 2^* can be replaced by any $p\in[1,\infty)$ if $d=2$, and in this case, the constant $c>0$ also depends on p.

Proof. The Hardy inequality (2.9) is well known nowadays, see, for example [104].

The Poincaré inequality (2.10) immediately follows by the validity of the Hardy inequality (2.9), in fact

$$\int_{B_R}|y|^au^2dz\le c\int_{B_R}|y|^a\frac{u^2}{|y|^2}dz,$$

for some $c>0$ depending only on R.

By using the classical embedding $H^1(B_R)\hookrightarrow L^2(\partial B_R)$ to the function $|y|^{a/2}u\in C_c^\infty(\overline{B_R}\setminus\Sigma_\varepsilon)$, the Hardy inequality (2.9), combined with Hölder and Young inequalities, we have that

$$\begin{aligned}\int_{\partial B_R}|y|^au^2d\sigma&\le c\int_{B_R}|\nabla(|y|^{a/2}u)|^2dz\le c\int_{B_R}\Big(|y|^a|\nabla u|^2+\frac{a^2}{4}|y|^a\frac{u^2}{|y|^2}+a|y|^a\frac{|u|}{|y|}|\nabla u|\Big)dz\\&\le c\int_{B_R}|y|^a|\nabla u|^2dz,\end{aligned}$$

and (2.11) holds.

Finally, let's prove the Sobolev embedding (2.12). By using $a<0$, the classical Sobolev embedding $H^1(B_R)\hookrightarrow L^{2^*}(B_R)$ to the function $|y|^{a/2}u\in C_c^\infty(\overline{B_R}\setminus\Sigma_\varepsilon)$, the Hardy inequality (2.9), the Poincaré inequality (2.10), combined with Hölder and Young inequalities, we obtain

$$\Big(\int_{B_R}|y|^a|u|^{2^*}dz\Big)^{2/2^*}\le c\Big(\int_{B_R}(|y|^{a/2}|u|)^{2^*}dz\Big)^{2/2^*}\le c\int_{B_R}\Big(|y|^au^2+|\nabla(|y|^{a/2}u)|^2\Big)dz$$

$$\leq c\int_{B_R}\Big(|y|^a u^2+|y|^a|\nabla u|^2+\frac{a^2}{4}|y|^a\frac{u^2}{|y|^2}+a|y|^a\frac{|u|}{|y|}|\nabla u|\Big)dz\leq c\int_{B_R}|y|^a|\nabla u|^2dz.$$

Hence, the proof is complete. □

2.2.3 Trace inequality on lower dimensional manifolds

In the paper [118], the author shows the existence of a trace operator for a large class of weighted Sobolev spaces on lower dimensional boundaries. Specifically, the following result holds.

Theorem 2.2.3 ([118, Theorem 2.3])**.** *Let* $2\leq n<d$*,* Γ *be a* $(d-n)$*-dimensional* C^1*-manifold,* d_Γ *be the distance from* Γ*,* $a+n\in(0,2)$*. Then, there exits a unique bounded linear operator*

$$T:H^1(B_1,d_\Gamma^a)\to L^2(\Gamma\cap B_1)$$

such that

$$Tu=u_{|\Gamma},$$

for every $u\in C^\infty(\overline{B_1})$*.*

We would like to remark that the condition $a+n\in(0,2)$ cannot be removed in the previous theorem. Moreover, let us stress the fact that in the *straight* case, that is, when $\Gamma=\Sigma_0$ and $d_\Gamma=|y|$, the trace operator

$$T:H^{1,a}(B_1)\to L^2(\Sigma_0\cap B_1)$$

is well defined, linear and continuous.

2.2.4 Weak solutions

In this section we give the definition of weak solutions.

Definition 2.2.4. Let $2\leq n\leq d$, $a+n\in(0,2)$, $R>0$ and $0\leq\varepsilon\ll1$. Let A be matrix satisfying (2.2), $f\in L^{2,a}(B_R\setminus\Sigma_\varepsilon)$ and $F\in L^{2,a}(B_R\setminus\Sigma_\varepsilon)^d$. We say that u is a weak solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } B_R\setminus\Sigma_\varepsilon,\\ u=0, & \text{on } \partial\Sigma_\varepsilon\cap B_R,\end{cases}\tag{2.13}$$

if $u\in\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ and satisfies

$$\int_{B_R}|y|^aA\nabla u\cdot\nabla\phi dz=\int_{B_R}|y|^a(f\phi-F\cdot\nabla\phi)dz,\tag{2.14}$$

for every $\phi\in C_c^\infty(B_R\setminus\Sigma_\varepsilon)$.

We say that u is an entire solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } \mathbb{R}^d\setminus\Sigma_\varepsilon,\\ u=0, & \text{on } \partial\Sigma_\varepsilon,\end{cases}$$

if u is a weak solution to (2.13) for every $R>0$.

Remark 2.2.5. Using the validity of the Poincaré inequality (2.10), we have existence and uniqueness for solutions to (2.13) which also satisfy a boundary condition on $\partial B_R\setminus\Sigma_\varepsilon$. In fact, if u is a weak solution to (2.13) satisfying $u-\bar{u}\in H_0^{1,a}(B_R\setminus\Sigma_\varepsilon)$, for some $\bar{u}\in\tilde{H}^{1,a}(B_R\setminus\Sigma_\varepsilon)$, then u is a minimizer to the functional

$$J(v):=\int_{B_R\setminus\Sigma_\varepsilon}|y|^a\Big(\frac{A\nabla v\cdot\nabla v}{2}-fv+F\cdot\nabla v\Big)dz,$$

over

$$X:=\{v\in H^{1,a}(B_R\setminus\Sigma_\varepsilon):v-\bar{u}\in H_0^{1,a}(B_R\setminus\Sigma_\varepsilon)\},$$

and J is coercive. By a standard application of the Weierstrass Theorem, we have existence and uniqueness of solutions to (2.13) with prescribed trace on $\partial B_R\setminus\Sigma_\varepsilon$.

When $\varepsilon=0$, recalling the trace Theorem 2.2.3, we also give a definition of weak solutions with prescribed trace on the lower dimensional boundary Σ_0.

Definition 2.2.6. Let $2\le n\le d$, $a+n\in(0,2)$, $R>0$ and A satisfies (2.2). Let $f\in L^{2,a}(B_R)$, $F\in L^{2,a}(B_R)^d$ and $\psi\in L^2(\Sigma_0\cap B_R)$. We say that u is a weak solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } B_R\setminus\Sigma_0,\\ u=\psi, & \text{on } \Sigma_0\cap B_R,\end{cases}$$

if $u\in H^{1,a}(B_R)$, satisfies (2.14) for every $\phi\in C_c^\infty(B_R\setminus\Sigma_0)$ and $u=\psi$ on $\Sigma_0\cap B_R$, in the sense of the trace.

2.3 Local boundedness of solutions

The goal of this section is to prove $L^2\to L^\infty$ estimates for weak solutions to (2.13). The proof is fairly standard and employs an iterative technique based on a Caccioppoli-type inequality and the Sobolev embeddings (2.12) (see the proof of Proposition 1.3.3 in the parabolic setting and, for instance, see also [147]). We include the proof for completeness. We start with the following Caccioppoli-type inequality.

Lemma 2.3.1. *Let $2\le n\le d$, $a+n\in(0,2)$, $R>0$, $0\le\varepsilon\ll1$, $p\ge(2^*)'$ and $q\ge2$. Let A be a matrix satisfying (2.2), $f\in L^{p,a}(B_R\setminus\Sigma_\varepsilon)$, $F\in L^{q,a}(B_R\setminus\Sigma_\varepsilon)^d$ and u be a weak solution to (2.13). Then, there exists $c>0$ depending only on d, λ and Λ such that for every $0<R_1<R_2<R$ there holds*

$$\begin{aligned}\int_{B_{R_1}\setminus\Sigma_\varepsilon}|y|^a|\nabla u|^2dz\le c\Big[\frac{1}{(R_2-R_1)^2}\int_{B_{R_2}\setminus\Sigma_\varepsilon}|y|^a|u|^2dz+\|f\|_{L^{p,a}(B_{B_{R_2}}\setminus\Sigma_\varepsilon)}\|u\|_{L^{p',a}(B_{B_{R_2}}\setminus\Sigma_\varepsilon)}\\ +\int_{B_{B_{R_2}}\setminus\Sigma_\varepsilon}|y|^a|F|^2\chi_{\{|u|>0\}}dz\Big].\end{aligned} \tag{2.15}$$

Moreover, for every $b\in\mathbb{R}$ and for $v:=(u-b)_+=\max\{u-b,0\}$ or $v:=(u-b)_-=\max\{-u+b,0\}$ the same inequality (2.15) holds, with u replaced by v.

Proof. Fix $0<R_1<R_2<R$ and consider a smooth cut-off function $\eta\in C_c^\infty(B_R)$ such that

$$\operatorname{spt}(\eta)\subset B_{R_2},\quad \eta=1\text{ on }B_{R_1},\quad 0\le\eta\le1,\quad |\nabla\eta|\le\frac{c}{|R_2-R_1|},$$

for some constant $c>0$ depending only on d. Let us test the equation satisfied by u with $\eta^2 u$ (which is an admissible test function). Then, we obtain

$$\int_{B_R} |y|^a \eta^2 A\nabla u\cdot\nabla u dz = \int_{B_R} |y|^a\big(-2\eta u A\nabla u\cdot\nabla\eta + f\eta^2 u - \eta^2 F\cdot\nabla u - 2\eta u F\cdot\nabla\eta\big)dz.$$

By using (2.2) and applying Hölder and Young inequalities, we obtain

$$\begin{aligned}
&\lambda\int_{B_R}|y|^a\eta^2|\nabla u|^2dz \le 2\Lambda\Big(\int_{B_R}|y|^a\eta^2|\nabla u|^2dz\Big)^{1/2}\Big(\int_{B_R}|y|^a|u|^2|\nabla\eta|^2dz\Big)^{1/2}\\
&+\|\eta f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)}\|\eta u\|_{L^{p',a}(B_R\setminus\Sigma_\varepsilon)}+\Big(\int_{B_R}|y|^a\eta^2|F|^2\chi_{\{|u|>0\}}dz\Big)^{1/2}\Big(\int_{B_R}|y|^a\eta^2|\nabla u|^2dz\Big)^{1/2}\\
&+2\Big(\int_{B_R}|y|^a\eta^2|F|^2\chi_{\{|u|>0\}}dz\Big)^{1/2}\Big(\int_{B_R}|y|^a|u|^2|\nabla\eta|^2dz\Big)^{1/2}\\
&\le\frac{5}{6}\lambda\int_{B_R}|y|^a\eta^2|\nabla u|^2dz+\frac{3\Lambda^2}{\lambda}\int_{B_R}|y|^a|u|^2|\nabla\eta|^2dz+\|\eta f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)}\|\eta u\|_{L^{p',a}(B_R\setminus\Sigma_\varepsilon)}\\
&+\Big(1+\frac{1}{2\lambda}\Big)\int_{B_R}|y|^a\eta^2|F|^2\chi_{\{|u|>0\}}dz+\int_{B_R}|y|^a|u|^2|\nabla\eta|^2dz.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{\lambda}{6}\int_{B_{R_1}}|y|^a|\nabla u|^2dz &\le \Big(1+\frac{3\Lambda^2}{\lambda}\Big)\frac{c}{(R_2-R_1)^2}\int_{B_{R_2}}|y|^a|u|^2dz\\
&\quad+\|f\|_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)}\|u\|_{L^{p',a}(B_{R_2}\setminus\Sigma_\varepsilon)}+\Big(1+\frac{1}{2\lambda}\Big)\int_{B_{R_2}}|y|^a|F|^2\chi_{\{|u|>0\}}dz,
\end{aligned}$$

that is, (2.15) holds true.

Finally, to prove that (2.15) holds true for $v:=(u-b)_+$ it is enough to choose $\eta^2 v$ as test function in (2.13), noting that $\nabla v=\nabla u$ on the set $\{v>0\}$ and performing similar computations as above. The case $w=(u-b)_-$ is analogous, observing that $\nabla w=-\nabla u$ on the set $\{w>0\}$. □

The next lemma is to establish a no-spike estimate type.

Lemma 2.3.2. *Let $2\le n\le d$, $a+n\in(0,2)$, $0<r<R$, $0\le\varepsilon\ll1$, $p>d/2$, $q>d$. Let A be a matrix satisfying (2.2), $f\in L^{p,a}(B_R\setminus\Sigma_\varepsilon)$, $F\in L^{q,a}(B_R\setminus\Sigma_\varepsilon)^d$ satisfying*

$$\|f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)}+\|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)}\le 1.$$

Then, there exists a constant $\delta\in(0,1)$, depending only on d, n, a, λ, Λ, p, q, r and R, such that if u is a weak solution to (2.13) and it satisfies

$$\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u_+|^2dz\le\delta,$$

then

$$u\le 1\qquad a.e.\ in\ B_r\setminus\Sigma_\varepsilon.$$

Conversely, if

$$\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u_-|^2dz\le\delta,$$

then

$$u \geq -1 \qquad a.e. \ \ in \ B_r \setminus \Sigma_\varepsilon.$$

Proof. Let us assume that $d \geq 3$ (the case $d = 2$ is similar and relies only on the Sobolev-type inequality (2.12)) and consider the positive part u_+ (the case u_- is exactly the same).

For every $j \in \mathbb{N}$, set

$$C_j := 1 - 2^{-j}, \quad r_j := (R-r)2^{-j} + r, \quad D_j := B_{r_j} \setminus \Sigma_\varepsilon,$$

noting that $C_0 = 0$, $r_0 = R$, $C_j \uparrow 1$, $r_j \downarrow r$, $D_j \supset D_{j+1}$ and $r_j - r_{j+1} = (R-r)2^{-(j+2)}$. We define

$$V_j := (u - C_j)_+, \quad E_j := \int_{D_j} |y|^a V_j^2 dz,$$

which satisfy, for every $j \in \mathbb{N}$, $E_{j+1} \leq E_j \leq E_0 \leq \delta$ by assumption.

Applying the Sobolev inequality (2.12) and the Caccioppoli-type inequality (2.15) to V_{j+1}, with $R_1 = r_{j+1}$, $R_2 = r_j$, thanks to the assumption $\|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon)} \leq 1$, we get

$$\begin{aligned} \Big(\int_{D_{j+1}} |y|^a |V_{j+1}|^{2^*} dz \Big)^{2/2^*} &\leq c \int_{D_{j+1}} |y|^a |\nabla V_{j+1}|^2 dz \\ &\leq c \Big[2^{2j} \int_{D_j} |y|^a |V_{j+1}|^2 dz + \|V_{j+1}\|_{L^{p',a}(D_j)} + \int_{D_j} |y|^a |F|^2 \chi_{\{V_{j+1} > 0\}} dz \Big]. \end{aligned} \tag{2.16}$$

Next, by using Hölder inequality in the definition of E_{j+1} it follows that

$$E_{j+1} = \int_{D_{j+1}} |y|^a V_{j+1}^2 dz \leq \Big(\int_{D_{j+1}} |y|^a |V_{j+1}|^{2^*} dz \Big)^{2/2^*} \Big(\int_{D_{j+1}} |y|^a \chi_{\{V_{j+1}>0\}} dz \Big)^{(2^*-2)/2^*}. \tag{2.17}$$

Again, we use Hölder inequality, noting that $V_{j+1} \leq V_j$, to estimate

$$\begin{aligned} \|V_{j+1}\|_{L^{p',a}(D_j)} &\leq \Big(\int_{D_j} |y|^a V_{j+1}^2 dz \Big)^{1/2} \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz \Big)^{(p-2)/2p} \\ &\leq E_j^{1/2} \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz \Big)^{(p-2)/2p}, \end{aligned} \tag{2.18}$$

and, by also using $\|F\|_{L^{q,a}(B_R \setminus \Sigma_\varepsilon)} \leq 1$, we obtain

$$\begin{aligned} \int_{D_j} |y|^a |F|^2 \chi_{\{V_{j+1}>0\}} dz &\leq \Big(\int_{D_j} |y|^a |F|^q dz \Big)^{2/q} \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz \Big)^{(q-2)/q} \\ &\leq \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz \Big)^{q-2/q}. \end{aligned} \tag{2.19}$$

Moreover, one has that

$$\{V_{j+1} > 0\} = \{u - C_{j+1} > 0\} = \{u - C_j > 2^{-(j+1)}\} = \{V_j > 2^{-(j+1)}\},$$

hence,

$$\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz = \int_{D_j} |y|^a \chi_{\{V_j^2 > 2^{-2(j+1)}\}} dz \leq 2^{2(j+1)} \int_{D_j} |y|^a V_j^2 dz = 2^{2(j+1)} E_j. \tag{2.20}$$

Putting together (2.16), (2.17), (2.18), (2.19) and (2.20) we obtain

$$E_{j+1} \le C^{j+1}(E_j + E_j^{1-\frac{1}{p}} + E_j^{1-\frac{2}{q}})E_j^{\frac{2^*-2}{2^*}},$$

where $\frac{2^*-2}{2^*} - \frac{1}{p} > 0$ and $\frac{2^*-2}{2^*} - \frac{2}{p} > 0$ by the assumptions $p > d/2$ and $q > d$. By denoting with $\bar{\gamma} > 0$ the minimum of these two numbers we have that

$$\begin{cases} E_{j+1} \le C^{j+1} E_j^{1+\bar{\gamma}}, \\ E_0 \le \delta, \end{cases}$$

which implies

$$E_j \le C^{\sum_{i=0}^{j} i(1+\gamma)^{j-i}} E_0^{(1+\gamma)^j} \le (C\delta)^{(1+\gamma)^j}.$$

Finally, by choosing δ such that $C\delta < 1$, and taking the limit as $j \to \infty$ we obtain that $E_j \to 0$, that is, $\int_{B_r \setminus \Sigma_\varepsilon} |y|^a (u-1)_+^2 = 0$, which yields $u \le 1$ a.e. in $B_r \setminus \Sigma_\varepsilon$. □

Finally, the next lemma states the L^∞_{loc} boundedness of weak solutions to (2.13).

Lemma 2.3.3. *Let* $2 \le n \le d$, $a + n \in (0, 2)$, $R > 0$, $0 \le \varepsilon \ll 1$, $p > d/2$, $q > d$. *Let* A *be a matrix satisfying* (2.2), $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon)$, $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon)^d$ *and let* u *be a weak solution to* (2.13). *Then, for every* $r \in (0, R)$, *there exists* $c > 0$ *depending only on* d, n, a, λ, Λ, p, q *and* r *such that*

$$\|u\|_{L^\infty(B_r \setminus \Sigma_\varepsilon)} \le c\big(\|u\|_{L^{2,a}(B_R \setminus \Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R \setminus \Sigma_\varepsilon)}\big). \tag{2.21}$$

Proof. Let us define

$$v := \theta u, \quad \theta := \frac{\sqrt{\delta}}{\|u\|_{L^{2,a}(B_R \setminus \Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R \setminus \Sigma_\varepsilon)}},$$

where $\delta > 0$ is as in Lemma 2.3.2. One has that v satisfies the hypothesis of Lemma 2.3.2, hence, $v_+ \le 1$ in $B_r \setminus \Sigma_\varepsilon$, which implies

$$\|u_+\|_{L^\infty(B_r \setminus \Sigma_\varepsilon)} \le \frac{1}{\sqrt{\delta}}(\|u\|_{L^{2,a}(B_R \setminus \Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R \setminus \Sigma_\varepsilon)}).$$

Repeating the same argument with v_- one has that

$$\|u_-\|_{L^\infty(B_r \setminus \Sigma_\varepsilon)} \le \frac{1}{\sqrt{\delta}}(\|u\|_{L^{2,a}(B_R \setminus \Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R \setminus \Sigma_\varepsilon)}).$$

Hence, we have that (2.21) holds true, by choosing $C = 2/\sqrt{\delta}$. The proof is complete. □

2.4 Approximation result

In the spirit of Lemma 1.4.1 and [138, Lemma 2.12, Lemma 2.15], the goal of this section is to provide an approximation result, which allows us to construct a family of solutions to (2.13) in perforated domains ($0 < \varepsilon \ll 1$), which converges in a suitable sense to weak solutions of (2.13), when $\varepsilon = 0$.

Lemma 2.4.1. *Let* $2 \leq n \leq d$, $a+n \in (0,2)$, $R>0$. *Let* A *be a matrix satisfying* (2.2), $f \in L^{2,a}(B_R)$, $F \in L^{2,a}(B_R)^d$ *and let* u *be a weak solution to* (2.13) *with* $\varepsilon = 0$.

Then, for every $r \in (0,R)$, *there exists a family* $\{u_\varepsilon\}_{0<\varepsilon\ll 1}$, *such that* u_ε *are weak solutions to*

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u_\varepsilon) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_r \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \partial\Sigma_\varepsilon \cap B_r, \end{cases} \tag{2.22}$$

satisfying

$$\|u_\varepsilon\|_{H^{1,a}(B_r\setminus\Sigma_\varepsilon)} \leq c\big(\|u\|_{H^{1,a}(B_R)} + \|f\|_{L^{2,a}(B_R)} + \|F\|_{L^{2,a}(B_R)}\big), \tag{2.23}$$

for some constant $c>0$ *depending only on* d, n, a, λ, Λ, R, r *and, up to consider the trivial extension of* u_ε *in the whole* B_r *(see Remark* 2.2.1*), there exists a sequence* $\varepsilon_k \to 0$ *such that*

$$u_{\varepsilon_k} \to u \ \text{in } H^{1,a}(B_r).$$

Proof. Let us fix $r \in (0,R)$ and consider a cut-off function $\xi \in C_c^\infty(B_R)$ such that

$$\xi = 1 \text{ in } B_r, \quad \operatorname{spt}(\xi) \subset B_{\frac{R+r}{2}}, \quad 0 \leq \xi \leq 1, \quad |\nabla\xi| \leq c_0,$$

for some $c_0>0$ depending only on d, R and r, and define $\tilde u = \xi u \in H_0^{1,a}(B_R)$.

Fixed $\phi \in C_c^\infty(B_R \setminus \Sigma_0)$, and using the equation (2.14) satisfied by u, we get

$$\begin{aligned} \int_{B_R} |y|^a A\nabla\tilde u \cdot \nabla\phi dz &= \int_{B_R} |y|^a \Big(\xi A\nabla u\cdot\nabla\phi u A\nabla\xi\cdot\nabla\phi\Big)dz \\ &= \int_{B_R} |y|^a \Big(A\nabla u\cdot\nabla(\phi\xi) - \phi A\nabla u\cdot\nabla\xi + uA\nabla\xi\cdot\nabla\phi\Big)dz \\ &= \int_{B_R} |y|^a \Big(f\phi\xi - F\cdot\nabla(\phi\xi) - \phi A\nabla u\cdot\nabla\xi + uA\nabla\xi\cdot\nabla\phi\Big)dz \\ &= \int_{B_R} |y|^a \Big(f\phi\xi - \xi F\cdot\nabla\phi - \phi F\cdot\nabla\xi - \phi A\nabla u\cdot\nabla\xi + uA\nabla\xi\cdot\nabla\phi\Big)dz, \end{aligned}$$

that is, $\tilde u$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla\tilde u) = |y|^a \tilde f + \operatorname{div}(|y|^a \tilde F), & \text{in } B_R \setminus \Sigma_0, \\ u = 0, & \text{on } \partial B_R \setminus \Sigma_0, \\ u = 0, & \text{on } \Sigma_0 \cap B_R, \end{cases} \tag{2.24}$$

where we have set

$$\tilde f = f\xi - F\cdot\nabla\xi - A\nabla u\cdot\nabla\xi, \qquad \tilde F = F\xi - uA\nabla\xi.$$

We estimate the right hand side in (2.24) in the following way

$$\begin{aligned}\int_{B_R}|y|^a\tilde f^2dz &\le 2\int_{B_R}|y|^a\Big((\xi f)^2+(F\cdot\nabla\xi)^2+(A\nabla u\cdot\nabla\xi)^2\Big)dz\\ &\le 2\int_{B_R}|y|^a\Big(f^2+c_0^2|F|^2+|A\nabla\xi|^2|\nabla u|^2\Big)dz\\ &\le c\int_{B_R}|y|^a\Big(f^2+|F|^2+|\nabla u|^2\Big)dz,\end{aligned}\tag{2.25}$$

for some $c>0$ depending only on d, Λ, R and r. By performing similar computations, we get

$$\int_{B_R}|y|^a|\tilde F|^2dz \quad\le c\int_{B_R}|y|^a\Big(|F|^2+u^2\Big)dz,\tag{2.26}$$

for some $c>0$ depending only on d, Λ, R and r

Fixed $0<\varepsilon_0\ll 1$, for every $0<\varepsilon<\varepsilon_0$, recalling Remark 2.2.5, let u_ε be the unique weak solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u_\varepsilon)=|y|^a\tilde f+\operatorname{div}(|y|^a\tilde F), & \text{in } B_R\setminus\Sigma_\varepsilon,\\ u_\varepsilon=0, & \text{on } \partial B_R\setminus\Sigma_\varepsilon,\\ u_\varepsilon=0, & \text{on } \Sigma_\varepsilon\cap B_R.\end{cases}\tag{2.27}$$

By using $u_\varepsilon\in H_0^{1,a}(B_R\setminus\Sigma_\varepsilon)$ as test function in (2.27), up to consider the trivial extension in the whole B_r (see Remark 2.2.1), combined with (2.2), Poincaré inequality (2.10) and Hölder inequality, we get

$$\begin{aligned}&\lambda\int_{B_R}|y|^a|\nabla u_\varepsilon|^2dz\le\int_{B_R}|y|^aA\nabla u_\varepsilon\cdot\nabla u_\varepsilon dz=\int_{B_R}|y|^a\Big(\tilde f u_\varepsilon+\tilde F\cdot\nabla u_\varepsilon\Big)dz\\ &\le\Big(\int_{B_R}|y|^a|\tilde f|^2dz\Big)^{1/2}\Big(\int_{B_R}|y|^a|u_\varepsilon|^2dz\Big)^{1/2}+\Big(\int_{B_R}|y|^a|\tilde F|^2dz\Big)^{1/2}\Big(\int_{B_R}|y|^a|\nabla u_\varepsilon|^2dz\Big)^{1/2}\\ &\le c\Big(\int_{B_R}|y|^a|\nabla u_\varepsilon|^2dz\Big)^{1/2}\Big(\|\tilde f\|_{L^{2,a}(B_R)}+\|\tilde F\|_{L^{2,a}(B_R)}\Big),\end{aligned}$$

and then, by using (2.25) and (2.26), we have that there exists a constant $c>0$ depending only on d, n, a, λ and Λ such that

$$\|u_\varepsilon\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}\le c\big(\|f\|_{L^{2,a}(B_R)}+\|F\|_{L^{2,a}(B_R)}+\|u\|_{H^{1,a}(B_R)}\big).\tag{2.28}$$

So, we get that $\{u_\varepsilon\}\subset H_0^{1,a}(B_R\setminus\Sigma_\varepsilon)\subset H_0^{1,a}(B_R)$ is uniformly bounded. Hence, there exists $\bar u\in H_0^{1,a}(B_R)$ and a sequence $\varepsilon_k\to 0$, such that

$$u_{\varepsilon_k}\rightharpoonup\bar u,\quad \text{weakly in } H^{1,a}(B_R).\tag{2.29}$$

Next, we prove that $\bar u=\tilde u$. Let $\phi\in C_c^\infty(B_R\setminus\Sigma_0)$ be a test function in the equation (2.27) satisfied by u_ε. Then, $\operatorname{spt}(\phi)\subset B_R\setminus\Sigma_\varepsilon$ for every ε small enough. By using (2.29), we have

$$\int_{\operatorname{spt}(\phi)}|y|^a\tilde f\phi-\tilde F\cdot\nabla\phi=\int_{\operatorname{spt}(\phi)}|y|^aA\nabla u_{\varepsilon_k}\cdot\nabla\phi\to\int_{\operatorname{spt}(\phi)}|y|^aA\nabla\bar u\cdot\nabla\phi,\quad \text{as } \varepsilon_k\to 0,$$

so, $\bar{u}$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla\bar{u}) = |y|^a\tilde{f} + \operatorname{div}(|y|^a\tilde{F}), & \text{in } B_R \setminus \Sigma_0, \\ \bar{u} = 0, & \text{on } \partial B_R \setminus \Sigma_0, \\ \bar{u} = 0, & \text{on } \Sigma_0 \cap B_R. \end{cases}$$

By uniqueness of weak solution to (2.24) (see Remark 2.2.5), we get that $\bar{u} = \tilde{u}$ in $H_0^{1,a}(B_R)$.

Finally, we prove that $u_{\varepsilon_k} \to \tilde{u}$ strongly in $H^{1,a}(B_R)$. By testing (2.24) with $\tilde{u}$, we get

$$\int_{B_R} |y|^a A\nabla\tilde{u}\cdot\nabla\tilde{u}dz = \int_{B_R} |y|^a\Big(\tilde{f}\tilde{u} - \tilde{F}\cdot\nabla\tilde{u}\Big)dz, \tag{2.30}$$

and, by testing (2.27) with u_ε combined with (2.29), we have

$$\int_{B_R} |y|^a A\nabla u_\varepsilon\cdot\nabla u_\varepsilon dz = \int_{B_R} |y|^a\Big(\tilde{f}\tilde{u}_\varepsilon - \tilde{F}\cdot\nabla\tilde{u}_\varepsilon\Big)dz \to \int_{B_R} |y|^a\Big(\tilde{f}\tilde{u} - \tilde{F}\cdot\nabla\tilde{u}\Big)dz, \tag{2.31}$$

along a subsequence $\varepsilon_k \to 0$. Putting together (2.30) and (2.31) we obtain

$$\lim_{\varepsilon_k\to 0}\int_{B_R} |y|^a A\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k} dz = \int_{B_R} |y|^a A\nabla\tilde{u}\cdot\nabla\tilde{u}dz.$$

Since A satisfies (2.2), one has that $\|\nabla u_{\varepsilon_k}\|_{L^{2,a}(B_R)} \to \|\nabla\tilde{u}\|_{L^{2,a}(B_R)}$. This, combined with (2.29), allows us to assert that

$$u_{\varepsilon_k} \to \tilde{u}, \quad \text{strongly in } H^{1,a}(B_R). \tag{2.32}$$

Finally, since $\tilde{u} = u$, $\tilde{f} = f$, $\tilde{F} = F$ in B_r, we have that u_ε is a weak solution to (2.22) in B_r and, by using (2.28) and (2.32), our statement follows. □

2.5 Liouville theorems

In this section we prove the following Liouville-type theorem. The proof is based on a spectral trace inequality which is stable with respect to ε, using an argument similar to [138, Theorem 3.4].

Theorem 2.5.1. *Let $2 \le n \le d$, $a + n \in (0, 2)$, $\varepsilon \ge 0$. Let A be a constant symmetric matrix satisfying (2.2) and u be an entire solution to*

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = 0, & \text{in } \mathbb{R}^d \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \partial\Sigma_\varepsilon, \end{cases} \tag{2.33}$$

(see Definition 2.2.4). Assume that there exist constants $c > 0$, $\gamma \in (0, 2 - a - n)$ such that

$$|u(z)| \le c(1 + |z|^\gamma), \quad \text{for a.e. } z \in \mathbb{R}^d \setminus \Sigma_\varepsilon, \tag{2.34}$$

Then, u is identically zero.

We start with a couple of results which are crucial to treat the case $n = d$.

Lemma 2.5.2. *Let $n=d$, $a+n\in(0,2)$, $0\leq\varepsilon\ll 1$ and $r>0$. Let $A\in\mathbb{R}^{n,n}$ be a positive definite symmetric matrix and define $\Omega_r:=\{A^{-1}y\cdot y<r^2\}$. Then,*

$$\int_{\Omega_r}|y|^aA\nabla v\cdot\nabla v dz\geq(2-a-n)\int_{\partial\Omega_r}|y|^av^2d\sigma,$$

for every $v\in C^\infty_c(\overline{\Omega}_r\setminus\Sigma_\varepsilon)$.

Proof. We provide the result for $r=1$. The case for generic $r>0$ follows by a scaling argument.

Since A is a positive definite symmetric matrix, it is well defined the square root $A^{1/2}$, which is a positive definite symmetric matrix too. The homogeneous function

$$\bar{u}(y):=|A^{-1/2}y|^{2-a-n},$$

is solution (in a point-wise sense) to

$$-\operatorname{div}(|y|^aA\nabla\bar{u})=0,\quad \text{in } \mathbb{R}^n\setminus\Sigma_0, \tag{2.35}$$

and satisfies

$$\nabla\bar{u}(y)=(2-a-n)|A^{-1/2}y|^{-a-n}A^{-1}y,\qquad \nabla\bar{u}(y)\cdot y=(2-a-n)\bar{u}(y). \tag{2.36}$$

Indeed, equations (2.35) and (2.36) are verified by a straightforward computation.

Fix $v\in C^\infty_c(\overline{\Omega}_1\setminus\Sigma_\varepsilon)$. Then,

$$\int_{\Omega_1}|y|^aA\nabla\bar{u}\cdot\nabla\Big(\frac{v^2}{\bar{u}}\Big)dy=\int_{\Omega_1}|y|^a\Big(A\nabla v\cdot\nabla v-\Big|A^{1/2}\nabla v-\frac{v}{\bar{u}}A^{1/2}\nabla\bar{u}\Big|^2\Big)dy\leq\int_{\Omega_1}|y|^aA\nabla v\cdot\nabla v dy, \tag{2.37}$$

On the other hand, by using the divergence theorem, (2.35) and (2.36), we have

$$\int_{\Omega_1}|y|^aA\nabla\bar{u}\cdot\nabla\Big(\frac{v^2}{\bar{u}}\Big)dy=\int_{\partial\Omega_1}|y|^a\frac{v^2}{\bar{u}}A\nabla\bar{u}\cdot\nu d\sigma=(2-a-n)\int_{\partial\Omega_1}|y|^av^2d\sigma, \tag{2.38}$$

so, putting together (2.37) and (2.38), our statement follows. □

Lemma 2.5.3. *Let $n=d$, $a+n\in(0,2)$, $R>0$ and $0\leq\varepsilon\ll 1$. Let $A\in\mathbb{R}^{n,n}$ be a positive definite diagonal matrix and define $\Omega_r:=\{A^{-1}y\cdot y<r^2\}$, for every $r>0$ such that $\partial\Omega_r\subset B_R\setminus\Sigma_\varepsilon$. Let u be a weak solution to*

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u)=0, & \textit{in } B_R\setminus\Sigma_\varepsilon,\\ u=0, & \textit{on } \partial\Sigma_\varepsilon\cap B_R,\end{cases} \tag{2.39}$$

Up to consider the trivial extension of u in the whole B_R (see Remark 2.2.1), let us define

$$E(u,r):=\frac{1}{r^{n+a-2}}\int_{\Omega_r}|y|^aA\nabla u\cdot\nabla u dy,$$
$$H(u,r):=\frac{1}{r^{n+a-1}}\int_{\partial\Omega_r}|y|^au^2d\sigma.$$

Then,

$$\partial_rH(v,r)=\frac{2}{r}E(v,r),\qquad \textit{for every } r\in(0,R).$$

Proof. When $\varepsilon > 0$, classical regularity theory ensures that the function u is smooth in $\overline{\Omega_r \setminus \Sigma_\varepsilon}$. Consequently, the result immediately follows trough explicit computations.

When $\varepsilon = 0$, we proceed by an approximation argument. Fixed $0 < \delta \ll 1$, by using the approximation Lemma 2.4.1, we find a family $\{u_\varepsilon\}_{0<\varepsilon\ll 1}$ of solutions to (2.39) in $B_{R-\delta} \setminus \Sigma_\varepsilon$ such that $u_{\varepsilon_k} \to u$ in $H^{1,a}(B_{R-\delta})$ along a sequence $\varepsilon_k \to 0$ and, by applying the trace Poincaré inequality (2.11) (which also holds in Ω_r) we get that $v_{\varepsilon_k} \to v$ in $L^{2,a}(\partial\Omega_r)$. Hence, we have that

$$\begin{aligned} &\int_{\Omega_r} |y|^a A\nabla u_{\varepsilon_k} \cdot \nabla u_{\varepsilon_k} dy \to \int_{\Omega_r} |y|^a A\nabla u \cdot \nabla u dy, \\ &\int_{\partial\Omega_r} |y|^a u_{\varepsilon_k}^2 d\sigma \to \int_{\partial\Omega_r} |y|^a u^2 d\sigma. \end{aligned} \tag{2.40}$$

By utilizing the result obtained in the case $\varepsilon > 0$ one finds that

$$\partial_r H(u_{\varepsilon_k}, r) = \frac{2}{r} E(u_{\varepsilon_k}, r). \tag{2.41}$$

By applying (2.40), we can take the limit as $\varepsilon_k \to 0$ in (2.41) to obtain $\partial_r H(u,r) = \frac{2}{r}E(u,r)$. ☐

The following lemma allows us to handle the unweighted variables x. Its proof relies on the method of difference quotients and an iterative application of the Caccioppoli-type inequality (2.15). For a detailed proof, see [143, Corollary 4.2, Lemma 4.3] in a quite similar context.

Lemma 2.5.4. *Let $2 \le n < d$, $a + n \in (0,2)$, $\varepsilon \ge 0$. Let A be a constant symmetric matrix satisfying (2.2) and u be an entire solution to (2.33). Then, the following holds true.*

i) For every $i = 1, \dots, d-n$, the function $\partial_{x_i} u$ is an entire solution to the same problem.

ii) If u satisfies the growth condition (2.34) for some $\gamma > 0$, then u must be a polynomial in the variable x of degree almost $\lfloor \gamma \rfloor$.

Proof of Theorem 2.5.1. Let u be an entire solution to (2.33) and let us suppose that $n = d$ and so $z = y \in \mathbb{R}^n$.

By contradiction let us suppose that $u \not\equiv 0$. Let $r_0 > 0$ such that $\Sigma_\varepsilon \subset \Omega_r := \{A^{-1}y \cdot y \le r^2\}$ for every $r \ge r_0$ and define

$$E(u,r) = \frac{1}{r^{n+a-2}} \int_{\Omega_r} |y|^a A\nabla u \cdot \nabla u dy,$$

$$H(u,r) = \frac{1}{r^{n+a-1}} \int_{\partial\Omega_r} |y|^a u^2 d\sigma.$$

By applying Lemma 2.5.2 and Lemma 2.5.3, we get

$$\partial_r H(u,r) = \frac{2}{r} E(u,r) \ge \frac{2(2-a-n)}{r} H(u,r),$$

which implies

$$H(u,r) \ge H(u,r_0) r^{2(2-a-n)}, \qquad \text{for every } r > r_0,$$

by Gronwall's inequality. On the other hand, since A satisfies (2.2), the growth condition (2.34) implies

$$H(u,r) \le c(1 + r^{2\gamma}).$$

Combining these two inequalities we get

$$H(u,r_0)\leq cr^{2(\gamma-(2-a-n))}.$$

Taking the limit as $r\to\infty$ and using $\gamma<2-a-n$ we get $H(u,r_0)=0$. Since $r_0>0$ is arbitrary, we deduce that $u\equiv 0$ in $\mathbb{R}^d\setminus\Omega_{r_0}$. Moreover, since u is a solution to (2.33) and satisfies $u=0$ on $\partial(\Omega_{r_0}\setminus\Sigma_\varepsilon)$, we apply the existence and uniqueness result (see Remark 2.2.5) to conclude that $u\equiv 0$ in $\Omega_{r_0}\setminus\Sigma_\varepsilon$. Therefore, $u\equiv 0$ in $\mathbb{R}^d\setminus\Sigma_\varepsilon$, which leads to a contradiction.

Let us consider the case $n<d$. By Lemma 2.5.4, one has that u is polynomial in the variable x. Hence, if $\gamma\in(0,1)$, the function u must be constant in x, so $u(x,y)=u(y)$ and our statement follows by using the result obtained in the case $n=d$. If $\gamma\in[1,2)$, we have that u must be linear in x, that is

$$u(x,y)=u_i(y)+\sum_{i=1}^{d-n}x_iu_i(y),$$

for some unknown functions $u_i(y)$. First,

$$|u_0(y)|=|u(0,y)|\leq c(1+|y|^\gamma).$$

On the other hand,

$$|u(e_{x_i},y)|=|u_i(y)+u_0(y)|\leq c(1+|y|^\gamma),$$

and so

$$|u_i(y)|\leq|u_0(y)|+c(1+|y|^\gamma)\leq c(1+|y|^\gamma).$$

Hence, every u_i satisfies the growth condition (2.34) for every $i=0,\dots,d-n$.

Next, for every $i=1,\dots,d-n$, by applying Lemma 2.5.4 we have that $\partial_{x_i}u(x,y)=u_i(y)$ is an entire solution to (2.33) and satisfies (2.34). Then, the result obtained in the case $d=n$ allows us to conclude that $u_i=0$ for every $i=1,\dots,d-n$ and so $u(x,y)=u_0(y)$. Hence, using again the case $d=n$, we have that u must be zero and our statement follows. ☐

2.6 $C^{0,\alpha}$ regularity estimates

The goal of this section is to prove Theorem 2.1.1, which we obtain as a by-product of ε-uniform Hölder estimates for solutions in perforated domains and the approximation Lemma 2.4.1.

Theorem 2.6.1. *Let $2\leq n\leq d$, $a+n\in(0,2)$, $p>d/2$, $q>d$ and α satisfying (2.3). Let A be a continuous symmetric matrix satisfying (3.5) and ω be a modulus of continuity such that*

$$\|A\|_{L^\infty(B_1)}+\sup_{z,z'\in B_1,z\neq z'}\frac{|A(z)-A(z')|}{\omega(|z-z'|)}\leq L.$$

Let $f\in L^{p,a}(B_1)$ and $F\in L^{q,a}(B_1)^d$. For $0<\varepsilon\ll 1$, let $\{u_\varepsilon\}$ be a family of solutions to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u_\varepsilon)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } B_1\setminus\Sigma_\varepsilon,\\ u_\varepsilon=0, & \text{on } \partial\Sigma_\varepsilon\cap B_1.\end{cases}\tag{2.42}$$

Then, there exists a constant $c>0$, depending only on d, n, a, λ, Λ, p, q, α and L such that

$$\|u_\varepsilon\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)}\leq c\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{L^{q,a}(B_1)^d}\big).\tag{2.43}$$

Proof. By classical regularity theory, we know that solutions to (2.42) are $C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)$ and that (2.43) holds with a constant $c>0$ that may also depend on ε. Our goal is to show that it is possible to provide a constant $c>0$ that is uniform in ε.

Without loss of generality, we can assume that

$$\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{L^{q,a}(B_1)^d}\le c,$$

for some $c>0$, which not depends on ε. Moreover, by using the local uniform bound of weak solutions in (see Lemma 2.3.3), it follows that

$$\|u_\varepsilon\|_{L^\infty(B_{3/4}\setminus\Sigma_\varepsilon)}\le c, \tag{2.44}$$

for some $c>0$, which not depends on ε.

Step 1. Contradiction argument and blow-up sequences.

By contradiction let us suppose that there exist $p>d/2$, $q>d$, α satisfying (2.3), $\{u_k\}_k:=\{u_{\varepsilon_k}\}_k$ as $\varepsilon_k\to0$ such that

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u_k)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } B_1\setminus\Sigma_{\varepsilon_k},\\ u_k=0, & \text{on } \partial\Sigma_{\varepsilon_k}\cap B_1,\end{cases} \tag{2.45}$$

and

$$\|u_k\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})}\to\infty.$$

Let us fix a smooth cut-off function $\eta\in C_c^\infty(B_1)$ such that

$$\operatorname{spt}(\phi)\subset B_{3/4},\quad \eta=1 \text{ in } B_{1/2},\quad 0\le\eta\le1.$$

By (2.44), one has that

$$L_k:=[\eta u_k]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})}\to\infty.$$

By definition of Hölder seminorm, take two sequences of points $z_k=(x_k,y_k),\hat z_k=(\hat x_k,\hat y_k)\in B_1\setminus\Sigma_{\varepsilon_k}$ such that

$$\frac{|(\eta u_k)(z_k)-(\eta u_k)(\hat z_k)|}{|z_k-\hat z_k|^\alpha}\ge\frac{L_k}{2}, \tag{2.46}$$

define $r_k:=|z_k-\hat z_k|$ and observe that at least one of z_k or $\hat z_k$ belongs to $B_{3/4}\setminus\Sigma_{\varepsilon_k}$. First, by using the local uniform bound of weak solutions (2.44), we have that $r_k\to0$, in fact

$$L_k\le\frac{4\|\eta u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{r_k^\alpha}\le\frac{c}{r_k^\alpha},$$

which implies

$$r_k\le\frac{c^{1/\alpha}}{L_k^{1/\alpha}}\to0,\qquad \text{as } k\to\infty.$$

From now on we distinguish three cases.

Case 1: $\dfrac{|y_k|}{r_k}\to\infty,\quad \dfrac{|y_k|-\varepsilon_k}{r_k}\to\infty,$

Case 2: $\dfrac{|y_k|}{r_k}\to\infty,\quad \dfrac{|y_k|-\varepsilon_k}{r_k}\le c,$

Case 3: $\dfrac{|y_k|}{r_k} \le c$,

for some constant $c > 0$ which not depends on k. Let $z_k^0 := (x_k, y_k^0)$ be the projection of z_k on $\partial\Sigma_{\varepsilon_k}$ and define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) := \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k, y_k^0), & \text{in } \textbf{Case 2}, \\ (x_k, 0), & \text{in } \textbf{Case 3}. \end{cases}$$

Define the sequence of domains

$$\Omega_k := \frac{B_1 \setminus \Sigma_{\varepsilon_k} - \tilde{z}_k}{r_k} = \Big\{ z = (x,y) : |\tilde{z}_k + r_k z| < 1, \text{ and } |\tilde{y}_k + r_k y| > \varepsilon_k \Big\},$$

and, for every $z \in \Omega_k$, let us define the sequence of functions

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k)}{r_k^\alpha L_k}, \qquad w_k(z) := \frac{\eta(\tilde{z}_k)\big(u_k(\tilde{z}_k + r_k z) - u_k(\tilde{z}_k)\big)}{r_k^\alpha L_k},$$

in **Case 1** and **Case 2**, and

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z)}{r_k^\alpha L_k}, \qquad w_k(z) := \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z)}{r_k^\alpha L_k},$$

in **Case 3**.

Step 2. Blow-up domains.

Let us define

$$\Omega_\infty := \big\{ z = (x,y) \in \mathbb{R}^d : \text{exists } \hat{k} \text{ such that } z \in \Omega_k \text{ for every } k \ge \hat{k} \big\}. \tag{2.47}$$

In this section we show who is the limit domain $\Omega_\infty := \lim_{k\to\infty} \Omega_k$, along a suitable subsequence. First, in every case, for every $z \in \mathbb{R}^d$, one has that

$$|\tilde{z}_k + r_k z| < |z_k| + r_k |z| \le 3/4 + o(1) < 1.$$

Hence, to prove that $z \in \Omega_\infty$, we only need to show that $\tilde{z}_k + r_k z \notin \Sigma_{\varepsilon_k}$, that is,

$$|\tilde{y}_k + r_k y| > \varepsilon_k. \tag{2.48}$$

Let us start with **Case 1**, recalling that $\tilde{z}_k = z_k$. Fix $z \in \mathbb{R}^d$ and by contradiction let us suppose that (2.48) does not hold. Then, since $|\cdot|$ is a Lipschitz function, one has that

$$\frac{|y_k| - \varepsilon_k}{r_k} \le \frac{|y_k| - |y_k + r_k y|}{r_k} \le c|y|,$$

and taking the limit as $k \to \infty$, it follows

$$|y| \ge \infty,$$

which is a contradiction. Hence, $\Omega_\infty = \mathbb{R}^d$.

Next, let us consider the **Case 2** and recall that $\tilde{z}_k = (x_k, y_k^0)$, $|y_k^0| = \varepsilon_k$, $r_k/\varepsilon_k \to 0$. Defining

$$\bar{e} := \lim_{k\to\infty} \frac{y_k^0}{|y_k^0|},$$

we claim that $\Omega_\infty = \Pi := \{(x,y) : y \cdot \bar{e} > 0\}$, which is an half-space. We observe that, for every $y \in \mathbb{R}^n$,

$$\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - \frac{y_k^0}{|y_k^0|} \cdot y \leq c\frac{r_k}{\varepsilon_k} \to 0, \tag{2.49}$$

as $k \to \infty$. Indeed, by using Lagrange's Theorem to the function $|\cdot|$, there exists y^* (which could depend on k) such that $|y^*| \leq |y|$ and denoting $y_k^* := y_k^0 + r_k y^*$, we have

$$\begin{aligned}\Big|\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - \frac{y_k^0 \cdot y}{|y_k^0|}\Big| &= \Big|\frac{y_k^* \cdot y}{|y_k^*|} - \frac{y_k^0 \cdot y}{|y_k^0|}\Big| \leq \Big|\frac{y_k^* \cdot y}{|y_k^*|} - \frac{y_k^* \cdot y}{|y_k^0|}\Big| + \Big|\frac{y_k^* \cdot y}{|y_k^0|} - \frac{y_k^0 \cdot y}{|y_k^0|}\Big| \\ \leq c\Big|\frac{|y_k^0| - |y_k^*|}{|y_k^0|}\Big| + c\Big|\frac{|y_k^* - y_k^0|}{|y_k^0|}\Big| &\leq c\frac{r_k}{|y_k^0|} = c\frac{r_k}{\varepsilon_k} \to 0, \quad \text{as } k \to \infty,\end{aligned}$$

so (2.49) holds true. Let us fix $z = (x,y)$ such that $\bar{e} \cdot y = \delta > 0$ and suppose by contradiction that (2.48) doesn't hold, that is

$$\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - \frac{y_k^0}{|y_k^0|} \cdot y + \frac{y_k^0}{|y_k^0|} \cdot y \leq 0.$$

So, by taking the limit as $k \to \infty$ and using (2.49), we obtain

$$\bar{e} \cdot y \leq 0,$$

which is a contradiction. In analogous way, we have that every $z = (x,y)$ such $|\bar{e} \cdot y| = -\delta < 0$ satisfies $z \notin \Omega_k$. Hence, since $\delta > 0$ is arbitrary, the claim follows, that is, $\Omega_\infty = \Pi$.

Finally, let us consider the **Case 3**, recall that $\varepsilon_k < |y_k| \leq c r_k$ and $\tilde{z}_k = (x_k, 0)$. First, up to consider a subsequence, the following limit is well defined

$$\bar{\varepsilon} := \lim_{k\to\infty} \frac{\varepsilon_k}{r_k} \in [0, c].$$

Let us fix $z = (x,y)$ such that $|y| = \bar{\varepsilon} + \delta$ for some $\delta > 0$ and suppose by contradiction that (2.48) doesn't hold. Then,

$$\bar{\varepsilon} + \delta = |y| \geq \frac{\varepsilon_k}{r_k} \to \bar{\varepsilon},$$

which is a contradiction, so $z \in \Omega_k$. Instead, fixed $z = (x,y)$ such that $|y| = \bar{\varepsilon} - \delta$ for some $\delta > 0$, one has $z \notin \Omega_k$. Since $\delta > 0$ is arbitrary, we have that

$$\Omega_\infty = \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}} = \{(x,y) : |y| > \bar{\varepsilon}\}.$$

Resuming, we have shown that the limit domain is

$$\Omega_\infty = \begin{cases} \mathbb{R}^d, & \text{in } \textbf{Case 1}, \\ \Pi, & \text{in } \textbf{Case 2}, \\ \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}}, & \text{in } \textbf{Case 3}, \end{cases} \tag{2.50}$$

where $\Pi := \{(x,y) \in \mathbb{R}^d : \bar{e} \cdot y \geq 0\}$ is an half-space.

Step 3. Hölder estimates and convergence of the blow-up sequences.

Let us fix a compact set $K \subset \Omega_\infty$. For every $z, z' \in K$ such that $z \neq z'$, we have

$$|v_k(z) - v_k(z')| = \frac{|(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k + r_k z')|}{r_k^\alpha L_k} \leq |z - z'|^\alpha,$$

that is,

$$[v_k]_{C^{0,\alpha}(K)} \leq 1. \tag{2.51}$$

In **Case 1** and **Case 2**, by using $v_k(0) = 0$, we get the uniform bound $\|v_k\|_{C^{0,\alpha}(K)} \leq c$, for every compact subset $K \subset \Omega_\infty$. Instead, in **Case 3**, since $\varepsilon_k / r_k \leq c$, one has that

$$|v_k(x,y)| = |v_k(x,y) - v_k(0, y_k^0/r_k)| \leq [v_k]_{C^{0,\alpha}(K)}(|x| + |y - y_k^0/r_k|)^\alpha \leq c,$$

for some $c > 0$ which depends only on K, where we have used the boundary condition $\eta u_k = 0$ on $\partial\Sigma_{\varepsilon_k}$. Hence, we have that $\|v_k\|_{L^\infty(K)} \leq c$, which implies that $\|v_k\|_{C^{0,\alpha}(K)} \leq c$ also in this case.

By applying the Arzelà-Ascoli Theorem we conclude that $v_k \to \bar{v}$ uniformly in K. By a standard diagonal argument, we can take the the limit as $k \to \infty$ in (2.51) to obtain

$$[\bar{v}]_{C^{0,\alpha}(\Omega_\infty)} \leq 1,$$

which implies that $\bar{v}$ satisfies the growth condition

$$|\bar{v}| \leq c(1 + |z|^\alpha), \quad \text{a.e. in } \Omega_\infty. \tag{2.52}$$

Moreover, since $u_{\varepsilon_k} = 0$ on $\partial\Sigma_{\varepsilon_k}$, and then $v_k = 0$ on $(\partial\Sigma_{\varepsilon_k} - \tilde{z}_k)/r_k$, by employing the local uniform convergence we have that $\bar{v} = 0$ on $\partial\Pi$ in **Case 2** and $\bar{v} = 0$ on $\partial\Sigma_{\bar{\varepsilon}}$ in **Case 3**.

Furthermore, the sequences v_k and w_k converge to the same limit function. Let us fix a compact set $K \subset \Omega_\infty$. For every $z \in K$, by using (2.44), we have

$$|v_k(z) - w_k(z)| \leq \frac{(\eta(\tilde{z}_k + r_k z) - \eta(\tilde{z}_k))\|u_k\|_{L^\infty(B_{3/4} \setminus \Sigma_{\varepsilon_k})}}{r_k^\alpha L_k} \leq \frac{c r_k^{1-\alpha}}{L_k} \to 0, \quad \text{as } k \to \infty.$$

Hence, the sequences v_k and w_k have the same asymptotic behavior as $k \to \infty$ on every $K \subset \Omega_\infty$, which implies that $w_k \to \bar{v}$ uniformly on K.

Step 4. The limit function $\bar{v}$ is not constant.

Let us consider the sequences of points

$$\xi_k^1 := \frac{z_k - \tilde{z}_k}{r_k}, \quad \xi_k^2 := \frac{\hat{z}_k - \tilde{z}_k}{r_k}.$$

By (2.46), we have

$$|v_k(\xi_k^1) - v_k(\xi_k^2)| = \frac{|(\eta u_k)(\hat{z}_k) - (\eta u_k)(z_k)|}{r_k^\alpha L_k} \geq \frac{1}{2}.$$

In **Case 1**, we have that

$$\xi_k^1 = 0, \quad \xi_k^2 = \frac{\hat{z}_k - z_k}{r_k} \in \partial B_1,$$

then, $\xi_k^1 \to 0$, $\xi_k^2 \to \xi^2 \neq 0$.

In **Case 2**,

$$\xi_k^1 = \frac{(0, y_k - y_k^0)}{r_k}, \quad \xi_k^2 = \frac{(\hat{x}_k - x_k, \hat{y}_k - y_k^0)}{r_k}.$$

Since $(|y_k| - \varepsilon_k)/r_k \leq c$ uniformly in k, $|\xi_k^1 - \xi_k^2| = 1$, we have that $\xi_k^1 \to \xi^1$, $\xi_k^2 \to \xi^2$ and $\xi^1 \neq \xi^2$.

In **Case 3**,

$$\xi_k^1 = \frac{(0, y_k)}{r_k}, \quad \xi_k^2 = \frac{(\hat{x}_k - x_k, \hat{y}_k)}{r_k}.$$

Since $|y_k|/r_k \leq c$ uniformly in k, $|\xi_k^1 - \xi_k^2| = 1$, we have that $\xi_k^1 \to \xi^1$, $\xi_k^2 \to \xi^2$ and $\xi^1 \neq \xi^2$.

Hence, by the local uniform convergence $v_k \to \bar{v}$ we get that $|\bar{v}(\xi^1) - \bar{v}(\xi^2)| \geq 1/2$, so $\bar{v}$ is not constant.

Step 5. $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients.

Let us define $A_k(x, y) := A(\tilde{z}_k + r_k z)$ and $(\bar{x}, \bar{y}) = \lim_{k\to\infty}(\tilde{x}_k, \tilde{y}_k)$. By using the continuity of A, we can define $\bar{A} := \lim_{k\to\infty} A_k(z) = A(\bar{x}, \bar{y})$, which is a constant coefficients symmetric matrix satisfying (2.2). Next, set

$$\rho_k(y) := \begin{cases} \dfrac{|\tilde{y}_k + r_k y|}{|\tilde{y}_k|}, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

observing that in **Case 1** and **Case 2**

$$|\rho_k(y)| = 1 + o(1), \tag{2.53}$$

as $k \to \infty$ on every compact subset $K \subset \mathbb{R}^d$.

Fix $\phi \in C_c^\infty(\Omega_\infty)$. since u_k is a weak solution to (2.45), a straightforward computation shows us that

$$\begin{aligned}\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(z) \nabla w_k(z) \cdot \nabla\phi(z) dz = \frac{r_k^{2-\alpha}\eta(\tilde{z}_k)}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz \\ - \frac{r_k^{1-\alpha}\eta(\tilde{z}_k)}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla\phi(z) dz.\end{aligned} \tag{2.54}$$

Now, we aim to prove that the right-hand side of (2.54) vanishes as $k \to \infty$. First, we focus on the term involving the function f. Let us consider the **Case 1** and **Case 2** together. By using the Hölder inequality, (2.53) and $a < 0$, we get

$$\begin{aligned}&\Big| \int_{\mathrm{spt}(\phi)} \frac{|\tilde{y}_k + r_k y|^a}{|\tilde{y}_k|^a} f(\tilde{z}_k + r_k z)\phi(z) dz \Big| \\ &\leq c\|\phi\|_{L^\infty} \Big(r_k^{-d} |\tilde{y}_k|^{-a} \int_{B_1} |y|^a |f|^p dz \Big)^{1/p} \Big(\int_{\mathrm{spt}(\phi)} \frac{|\tilde{y}_k + r_k y|^a}{|\tilde{y}_k|^a} dz \Big)^{1/p'} \leq c r_k^{d/p}.\end{aligned}$$

Hence,

$$\frac{r_k^{2-\alpha}\eta(\tilde{z}_k)}{L_k} \Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz \Big| \leq c r_k^{2-\alpha-d/p} L_k^{-1} \to 0,$$

as $k \to \infty$, by the hypothesis (2.3), which implies $\alpha \leq 2 - d/p$.

Let us consider now the **Case 3**. By using the Hölder inequality and $a<0$, we get

$$\begin{aligned}&\Big|\int_{\mathrm{spt}(\phi)}|y|^a f(\tilde{z}_k+r_kz)\phi(z)dz\Big|\\ &\leq c\|\phi\|_{L^\infty}\Big(r_k^{-d-a}\int_{B_1}|y|^a|f|^pdz\Big)^{1/p}\Big(\int_{\mathrm{spt}(\phi)}|y|^adz\Big)^{1/p'}\leq cr_k^{-d/p},\end{aligned}$$

and then

$$\frac{r_k^{2-\alpha}\eta(\tilde{z}_k)}{L_k}\Big|\int_{\mathrm{spt}(\phi)}|y|^a f(\tilde{z}_k+r_kz)\phi(z)dz\Big|\leq cr_k^{2-\alpha-d/p}L_k^{-1}\to 0,$$

as before.

The second member of the the right hand side of (2.54) vanishes as $k\to\infty$ by using similar computations. In fact, in every cases, we have

$$\frac{r_k^{1-\alpha}\eta(\tilde{z}_k)}{L_k}\Big|\int_{\mathrm{spt}(\phi)}\rho_k^a(y)F(\tilde{z}_k+r_kz)\cdot\nabla\phi(z)dz\Big|\leq cr_k^{1-\alpha-d/q}L_k^{-1}\to 0,$$

by the assumption (2.3), which implies that $\alpha\leq 1-d/q$.

Finally, we prove that the left hand side of (2.54) converges in the following sense

$$\int_{\mathrm{spt}(\phi)}\rho_k^a(y)A_k(z)\nabla w_k(z)\cdot\nabla\phi(z)dz\to\int_{\mathrm{spt}(\phi)}\bar{\rho}^a(y)\bar{A}\nabla\bar{v}(z)\cdot\nabla\phi(z)dz, \tag{2.55}$$

where

$$\bar{\rho}(y):=\begin{cases}1, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2},\\ |y|, & \text{in } \textbf{Case 3}.\end{cases}$$

Let us fix $R>0$ such that $\mathrm{spt}(\phi)\subset B_{2R}\cap\Omega_k$ for every k large enough. Since w_k is uniformly bounded in $L^\infty(B_{2R}\cap\Omega_\infty)$, one has that w_k is uniformly bounded in $L^2(B_{2R}\cap\Omega_\infty,\rho_k^a(y)dz)$ and, by applying the Caccioppoli-type inequality (2.15), we get that w_k is uniformly bounded in $H^1(B_{2R}\cap\Omega_\infty,\rho_k^a(y)dz)$. Using $\rho_k^a\to\bar{\rho}$ and $A_k\to\bar{A}$ a.e. in Ω_∞ and arguing as in the proof of Lemma 2.4.1, we can conclude that (2.55) holds true and $\bar{v}\in H^1(B_R\cap\Omega_\infty,\bar{\rho}^a)$.

Hence, recalling the definition of the limit domain Ω_∞, see (2.50), and using *Step 3* for the boundary condition, we conclude that

in **Case 1**, $\bar{v}$ is an entire solution to

$$-\operatorname{div}(\bar{A}\nabla\bar{v})=0,\quad\text{in }\mathbb{R}^d,$$

in **Case 2**, $\bar{v}$ is an entire solution to

$$\begin{cases}-\operatorname{div}(\bar{A}\nabla\bar{v})=0, & \text{in }\Pi,\\ \bar{v}=0, & \text{on }\partial\Pi,\end{cases}$$

in **Case 3**, $\bar{v}$ is an entire solution to

$$\begin{cases}-\operatorname{div}(|y|^a\bar{A}\nabla\bar{v})=0, & \text{in }\mathbb{R}^d\setminus\Sigma_{\bar{\varepsilon}},\\ \bar{v}=0, & \text{on }\partial\Sigma_{\bar{\varepsilon}}.\end{cases}$$

Step 6. Liouville Theorems and conclusion.

By (2.52) we have that $\bar{v}$ satisfies the growth condition

$$|\bar{v}(z)| \leq c(1+|z|^\alpha),$$

for every $z \in \Omega_\infty$, where $\alpha < \min\{1, 2-a-n\}$ by hypothesis (2.3). In **Case 1** and **Case 2**, by invoking the classical Liouville Theorem, we can conclude that $\bar{v}$ must be constant and this is a contradiction since $\bar{v}$ is not constant by *Step 4*. In **Case 3**, we have that $\bar{v}$ satisfies the hypothesis of the Liouville Theorem 2.5.1 and so $\bar{v}$ must be identically zero, which is a contradiction. Then, $L_k \leq c$ uniformly in k, which implies that $[u_\varepsilon]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)} \leq c$. The proof is complete. □

Proof of the Theorems 2.1.1. Let u be a weak solution to (2.1) and consider the trivial extension of ψ in B_1, that is, $\psi(x,y) = \psi(x)$, for every $(x,y) \in B_1$. Let us define

$$v := u - \psi.$$

Since ψ is a Lipschitz function, we have that $v \in \tilde{H}^{1,a}(B_1)$ and v is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v) = |y|^a f + \operatorname{div}(|y|^a(F - A\nabla\psi)), & \text{in } B_1 \setminus \Sigma_0, \\ v = 0, & \text{on } \Sigma_0 \cap B_1. \end{cases}$$

By applying Lemma 2.4.1 we find a sequence $\{v_{\varepsilon_k}\}$ as $\varepsilon_k \to 0$, such that every v_{ε_k} is solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v_{\varepsilon_k}) = |y|^a f + \operatorname{div}(|y|^a(F - A\nabla\psi)), & \text{in } B_{3/4} \setminus \Sigma_{\varepsilon_k}, \\ v_{\varepsilon_k} = 0, & \text{on } \partial\Sigma_{\varepsilon_k} \cap B_{3/4}, \end{cases}$$

and $v_{\varepsilon_k} \to v$ in $H^{1,a}(B_{3/4})$ as $\varepsilon_k \to 0$. By applying the Theorem 2.6.1 to the sequences $\{v_{\varepsilon_k}\}$, combined with (2.23), we get that

$$\|v_{\varepsilon_k}\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})} \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)^d} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big),$$

for some $c > 0$ depending only on d, n, a, λ, Λ, p, q, α and L.

By applying Arzelà-Ascoli Theorem, we get that $v_{\varepsilon_k} \to w$ in $C^{0,\gamma}_{\text{loc}}(B_{1/2}\setminus\Sigma_0)$, for every $\gamma \in (0,\alpha)$, and by the a.e. convergence $v_{\varepsilon_k} \to v$ it follows that $v = w$. Furthermore, by taking $z, z' \in B_{1/2}\setminus\Sigma_0$ such that $z \neq z'$, we have that

$$\begin{aligned} \frac{|v(z) - v(z')|}{|z-z'|^\alpha} &= \lim_{\varepsilon_k\to 0} \frac{|v_{\varepsilon_k}(z) - v_{\varepsilon_k}(z')|}{|z-z'|^\alpha} \\ &\leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)^d} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big), \end{aligned}$$

which implies that

$$\begin{aligned} [v]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_0)} &= \sup_{\substack{z,z'\in B_{1/2}\setminus\Sigma_0 \\ z\neq z'}} \frac{|v(z) - v(z')|}{|z-z'|^\alpha} \\ &\leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)^d} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big). \end{aligned}$$

By continuity, we can extend v to the entire $B_{1/2}$ in such a way that $[v]_{C^{0,\alpha}(B_{1/2})} = [v]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_0)}$. Finally, combining the previous estimates with the L^∞_{loc} bounds of solutions (see Lemma 2.3.3), we obtain

$$\begin{aligned}\|u\|_{C^{0,\alpha}(B_{1/2})} &\leq \|v\|_{C^{0,\alpha}(B_{1/2})} + \|\psi\|_{C^{0,\alpha}(\Sigma_0\cap B_{1/2})}\\ &\leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)^d} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big).\end{aligned}$$

that is, $u \in C^{0,\alpha}(B_{1/2})$ and (2.4) holds true. □

2.7 $C^{1,\alpha}$ regularity estimates

This section is devoted to the proof of the Theorem 2.1.2, which establishes the $C^{1,\alpha}_{\mathrm{loc}}$ regularity for weak solutions. To achieve this result, we proceed as follows: first, as in Theorem 2.6.1, we prove ε-uniform estimates for solutions in perforated domains, with an additional assumption on the field F. As we will see in Remark 2.7.2, this condition cannot be removed. Next, we show *a priori* estimates for solutions, which also satisfy an additional boundary condition on Σ_0. Afterwards, we establish the main result through a double approximation process: the first involves perforated domains and using Lemma 2.4.1, while the second one is a standard approximation via convolution with a family of mollifiers.

Theorem 2.7.1. *Let $2 \leq n \leq d$, $a+n \in (0,1)$, $p > d$ and α satisfying (2.5). Let A be a α-Hölder continuous matrix satisfying (2.2) and $\|A\|_{C^{0,\alpha}(B_1)} \leq L$, $f \in L^{p,a}(B_1)$, $F \in C^{0,\alpha}(B_1)$ be a field such that $F(x,0)\cdot e_{y_i} = 0$ for every $(x,0) \in B_1$ and for every $i = 1,\dots,n$. For $0 < \varepsilon \ll 1$, let $\{u_\varepsilon\}$ be a family of solutions to (2.42).*

Then, there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u_\varepsilon\|_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)} \leq c\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big). \tag{2.56}$$

In addition, u_ε satisfies

$$|\nabla u_\varepsilon(z)| \leq c\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big)\varepsilon^\alpha, \quad \textit{for every } z \in \partial\Sigma_\varepsilon \cap B_{1/2}. \tag{2.57}$$

Proof. Under the assumptions of the theorem, classical Schauder theory ensures that solutions to (2.42) are $C^{1,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)$ and that (2.56) holds with a constant $c > 0$ that may also depend on ε. Our goal is to show that it is possible to provide a constant $c > 0$ that not depends on ε.

Without loss of generality, we can assume that

$$\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)^d} \leq c,$$

for some $c > 0$, which not depends on ε. Moreover, for every $\beta \in (0,1)$, the assumptions of Theorem 2.6.1 are satisfied, so

$$\|u_\varepsilon\|_{C^{0,\beta}(B_{3/4}\setminus\Sigma_\varepsilon)} \leq c, \tag{2.58}$$

for some $c > 0$, which not depends on ε.

Step 1. Contradiction argument and blow-up sequences.

By contradiction let us suppose that there exists $p > d$, α satisfying (2.5), $\{u_k\}_k := \{u_{\varepsilon_k}\}_k$, as $\varepsilon_k \to 0$ such that u_k is solution to (2.45) and

$$\|\nabla u_k\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})} \to \infty.$$

Let us fix a smooth cut-off function $\eta \in C_c^\infty(B_1)$ such that

$$\mathrm{spt}(\phi) \subset B_{3/4}, \quad \eta = 1 \text{ in } B_{1/2}, \quad 0 \le \eta \le 1.$$

One has that

$$\|\eta u_k\|_{C^{1,\alpha}(B_1 \setminus \Sigma_{\varepsilon_k})} \to \infty.$$

Let us define

$$L_k := [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1 \setminus \Sigma_{\varepsilon_k})},$$

and notice that it cannot be possible that $[\nabla(\eta u_k)]_{C^{0,\alpha}(B_1)} \le c$ and $\|\nabla(\eta u_k)\|_{L^\infty(B_1)} \to \infty$, since $\nabla(\eta u_k) = 0$ outside of $B_{3/4} \setminus \Sigma_{\varepsilon_k}$. Hence, $L_k \to \infty$.

By definition of Hölder seminorm, take two sequences of points $z_k = (x_k, y_k), \hat{z}_k = (\hat{x}_k, \hat{y}_k) \in B_1 \setminus \Sigma_{\varepsilon_k}$ such that

$$\frac{|\nabla(\eta u_k)(z_k) - \nabla(\eta u_k)(\hat{z}_k)|}{|z_k - \hat{z}_k|^\alpha} \ge \frac{L_k}{2}, \tag{2.59}$$

define $r_k := |z_k - \hat{z}_k|$ and observe that at least one of z_k or $\hat{z}_k$ belongs to $B_{3/4} \setminus \Sigma_{\varepsilon_k}$. We distinguish three cases.

Case 1: $\dfrac{|y_k|}{r_k} \to \infty, \quad \dfrac{|y_k| - \varepsilon_k}{r_k} \to \infty,$

Case 2: $\dfrac{|y_k|}{r_k} \to \infty, \quad \dfrac{|y_k| - \varepsilon_k}{r_k} \le c,$

Case 3: $\dfrac{|y_k|}{r_k} \le c,$

for some constant $c > 0$ which not depends on k. We notice that $r_k \to 0$ in **Case 1** and **Case 2** and we show later that $r_k \to 0$ also in **Case 3**.

Let $z_k^0 := (x_k, y_k^0)$ be the projection of z_k on $\partial\Sigma_{\varepsilon_k}$, define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) := \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k, y_k^0), & \text{in } \textbf{Case 2}, \\ (x_k, 0), & \text{in } \textbf{Case 3}, \end{cases}$$

and the sequence of domains

$$\Omega_k := \frac{B_1 \setminus \Sigma_{\varepsilon_k} - \tilde{z}_k}{r_k}.$$

For every $z \in \Omega_k$, let us define the sequence of functions

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k) - \nabla(\eta u_k)(\tilde{z}_k) \cdot r_k z}{L_k r_k^{1+\alpha}},$$

$$w_k(z) := \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k) - (\eta \nabla u_k)(\tilde{z}_k) \cdot r_k z}{L_k r_k^{1+\alpha}},$$

in **Case 1** and **Case 2**, and

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z)}{L_k r_k^{1+\alpha}}, \quad w_k(z) := \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z)}{L_k r_k^{1+\alpha}},$$

in **Case 3**. Furthermore, let us define the limit domain Ω_∞ as in (2.47).

Step 2. Gradient Hölder estimates and convergence of the blow-up sequences.

Let us fix a compact set $K \subset \Omega_\infty$. For every $z, z' \in K$ such that $z \neq z'$ we have

$$|\nabla v_k(z) - \nabla v_k(z')| = \frac{|\nabla(\eta u_k)(\tilde{z}_k + r_k z) - \nabla(\eta u_k)(\tilde{z}_k + r_k z')|}{r_k^\alpha L_k} \leq |z - z'|^\alpha,$$

that is

$$[\nabla v_k]_{C^{0,\alpha}(K)} \leq 1. \tag{2.60}$$

In **Case 1** and **Case 2**, since $v_k(0) = 0$ and $|\nabla v_k(0)| = 0$, we get the uniform bound of the norm

$$\|v_k\|_{C^{1,\alpha}(K)} \leq c.$$

In **Case 3** we use a different argument to show a uniform bound of $\|v_k\|_{C^{1,\alpha}(K)}$. First, for every point $z_k' = (x_k', y_k') \in \partial\Sigma_{\varepsilon_k} \cap B_{3/4}$, one has that one has that $(\eta u_k)(z_k') = 0$ and, recalling that the normal vector to $\partial\Sigma_{\varepsilon_k}$ at the point z_k' is the vector $(0, y_k')/|y_k'| \in \mathbb{S}^{d-1}$, it follows that

$$\nabla(\eta u_k)(z_k') \cdot \vec{e} = 0, \quad \text{for every vector } \vec{e} \perp (0, y_k').$$

Let us fix $i, h \in \{1, \dots, n\}$ such that $i \neq h$. Then,

$$\begin{aligned}\big|\nabla(\eta u_k)(x_k', y_k') \cdot e_{y_i}\big| &= \big|\big(\nabla(\eta u_k)(x_k', y_k') - \nabla(\eta u_k)(x_k', \varepsilon_k e_{y_h})\big) \cdot e_{y_i}\big| \\ &\leq [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})} |y_k' - \varepsilon_k e_{y_h}|^\alpha \leq 2^\alpha L_k \varepsilon_k^\alpha,\end{aligned} \tag{2.61}$$

and

$$|\nabla(\eta u_k)(x_k', y_k') \cdot e_{x_j}| = 0, \quad \text{for every } j = 1, \dots, d-n. \tag{2.62}$$

Hence, by using (2.61) and (2.62), it follows that

$$|\nabla(\eta u_k)(x_k', y_k')| \leq c L_k \varepsilon_k^\alpha. \tag{2.63}$$

By using interpolation inequality in Hölder spaces [75, Lemma 6.35], one has

$$\|v_k\|_{C^{1,\alpha}(K)} \leq c(\|v_k\|_{L^\infty(K)} + [v_k]_{C^{1,\alpha}(K)}),$$

and, by using the first order expansion of v_k at the point $(0, y_k^0/r_k)$ we obtain

$$\begin{aligned}|v_k(x,y)| &\leq |v_k(x,y) - v_k(0, y_k^0/r_k) - \nabla v_k(0, y_k^0/r_k) \cdot z| + |\nabla v_k(0, y_k^0/r_k)| \\ &\leq c(|x| + |y - y_k^0/r_k|)^{1+\alpha} + |\nabla v_k(0, y_k^0/r_k)| \leq c + |\nabla v_k(0, y_k^0/r_k)|,\end{aligned} \tag{2.64}$$

for some $c > 0$ which depends only on K, noticing that $|y_k^0/r_k| = \varepsilon_k/r_k \leq c$ uniformly in k in **Case 3**. So, if

$$|\nabla v_k(0, y_k^0/r_k)| \leq c, \tag{2.65}$$

uniformly in k, we can take the $\sup_K$ in (2.64) to obtain a uniform of $\|v_k\|_{L^\infty(K)}$ which implies $\|v_k\|_{C^{1,\alpha}(K)} \leq c$. Then, by using (2.63) and recalling that $(x_k, y_k^0) \in \partial\Sigma_{\varepsilon_k} \cap B_{3/4}$, we get that

$$|\nabla v_k(0, y_k^0/r_k)| = \frac{|\nabla(\eta u_k)(x_k, y_k^0)|}{r_k^\alpha L_k} \leq c\frac{\varepsilon_k^\alpha}{r_k^\alpha} \leq c.$$

So, (2.65) holds true and $\|v_k\|_{C^{1,\alpha}(K)} \le c$ uniformly in k.

Then, we may apply the Arzelà-Ascoli Theorem to infer that $v_k \to \bar{v}$ in $C^{1,\gamma}(K)$ for any $\gamma \in (0,\alpha)$. By a standard diagonal argument, we can take the the limit as $k \to \infty$ in (2.60) to obtain

$$[\nabla \bar{v}]_{C^{1,\alpha}(\Omega_\infty)} \le 1,$$

which implies that

$$|\bar{v}(z)| \le c(1 + |z|^{1+\alpha}), \quad \text{a.e. in } \Omega_\infty. \tag{2.66}$$

Moreover, v_k and w_k converge to the same limit function $\bar{w}$. Let us fix a compact set $K \subset \Omega_\infty$. In **Case 1** and **Case 2**, for every $z \in K$, by using (2.58) and exploiting the first order expansion of η, we have

$$\begin{aligned}
|v_k(z) - w_k(z)| &= \frac{|(\eta u_k)(\tilde{z}_k + r_k z) - \eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z) - (u_k \nabla \eta)(\tilde{z}_k) \cdot r_k z|}{r_k^{1+\alpha} L_k} \\
&\le \frac{|u_k(\tilde{z}_k + r_k z)(\eta(\tilde{z}_k + r_k z) - \eta(\tilde{z}_k) - \nabla \eta(\tilde{z}_k) \cdot r_k z)|}{r_k^{1+\alpha} L_k} + c\frac{|\nabla \eta(\tilde{z}_k)||u_k(\tilde{z}_k + r_k z) - u_k(\tilde{z}_k)|}{r_k^{\alpha} L_k} \\
&\le c\frac{\|u_k\|_{L^\infty(B_{3/4}\setminus \Sigma_{\varepsilon_k})} r_k^{1-\alpha}}{L_k} + c\frac{\|u_k\|_{C^{0,\beta}(B_{3/4}\setminus \Sigma_{\varepsilon_k})} r_k^{\beta-\alpha}}{L_k} \le \frac{c r_k^{\beta-\alpha}}{L_k} \to 0,
\end{aligned}$$

as $k \to \infty$, since we can choose $\beta \in (\alpha, 1)$. In **Case 3**, one has that

$$\begin{aligned}
|v_k(z) - w_k(z)| &= \frac{|\eta(\tilde{z}_k + r_k z) - \eta(\tilde{z}_k)| \cdot |u_k(\tilde{z}_k + r_k z)|}{r_k^{1+\alpha} L_k} \\
&\le \frac{c|u_k(\tilde{z}_k + r_k z) - u_k(z_k^0)|}{r_k^{\alpha} L_k} \le \frac{c\|u_k\|_{C^{0,\beta}(B_{3/4}\setminus \Sigma_{\varepsilon_k})} |\tilde{z}_k + r_k z - z_k^0|^\beta}{r_k^{\alpha} L_k} \le \frac{r_k^{\beta-\alpha}}{L_k} \to 0,
\end{aligned}$$

where we have used the facts that $u_k(z_k^0) = 0$ and $|\tilde{z}_k + r_k z - z_k^0| \le r_k|x| + r_k|y| + |y_k^0| \le c r_k + \varepsilon_k \le c r_k$ in **Case 3**. Hence, the sequences v_k and w_k have the same asymptotic behavior as $k \to \infty$ on every $K \subset \Omega_\infty$, which implies that $w_k \to \bar{v}$ uniformly on K.

Step 3. $\nabla \bar{v}$ is not constant.

Let us define the sequence of points

$$\xi_k^1 := \frac{z_k - \tilde{z}_k}{r_k}, \quad \xi_k^2 := \frac{\hat{z}_k - \tilde{z}_k}{r_k}.$$

By using (2.59) we get

$$|\nabla v_k(\xi_k^1) - \nabla v_k(\xi_k^2)| = \frac{|\nabla(\eta u_k)(z_k) - (\nabla \eta u_k)(\hat{z}_k)|}{r_k^{\alpha} L_k} \ge \frac{1}{2} > 0. \tag{2.67}$$

Arguing as Theorem 2.6.1, *Step 4*, we have that $\xi_k^1 \to \xi_1$ and $\xi_k^2 \to \xi_2$ and $\xi^1 \ne \xi^2$. Since $\nabla v_k \to \nabla \bar{v}$ uniformly on compact set by *Step 2*, we can take the limit in (2.67) to obtain $|\nabla \bar{v}(\xi^1) - \nabla \bar{v}(\xi^2)| > \delta_0/2$.

Step 4. $r_k \to 0$ in **Case 3**.

By contradiction let us suppose that $r_k \to \bar{r} > 0$. Fixed $z \in \Omega_\infty$, we have that

$$|\bar{v}(z)| = \Big|\lim_{k\to\infty} v_k(z)\Big| = \Big|\lim_{k\to\infty} \frac{(\eta u_k)(\tilde{z}_k + r_k z)}{L_k r_k^{1+\alpha}}\Big| \leq \frac{2\|u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{r_k^{1+\alpha} L_k} \leq \frac{c}{L_k} \to 0,$$

hence, $\bar{v} = 0$, which is a contradiction with *Step 3*, where we have proved that $\nabla\bar{v}$ is not constant. Then, $r_k \to 0$. By arguing as in Theorem 2.6.1, *Step 3*, we have that the limit domain Ω_∞ is defined by (2.50).

Step 5. $\bar{v}$ satisfies a homogeneous Dirichlet boundary condition in **Case 2** *and* **Case 3**.

First, in **Case 3**, since $v_k = 0$ on $\partial\Sigma_{\varepsilon_k/r_k}$ and $v_k \to \bar{v}$ uniformly on every compact set $K \subset \Omega_\infty$, one can conclude that $\bar{v} = 0$ on $\partial\Sigma_{\bar{\varepsilon}}$.

In **Case 2**, recalling that $\tilde{z}_k = z_k^0 \in \partial\Sigma_{\varepsilon_k}$, let us fix a boundary point $z \in (\partial\Sigma_{\varepsilon_k} - y_k^0)/r_k$ and denote by $z^\perp = (x, y^\perp)$ the projection of z on the hyperplane $\Pi_k = \{y \cdot y_k^0 = 0\}$. Recalling (2.49) and observing that $|y_k^0 + r_k y| = |y_k^0| = \varepsilon_k$, one has that

$$|y - y^\perp| = \Big|y \cdot \frac{y_k^0}{|y_k^0|}\Big| = \Big|\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - y \cdot \frac{y_k^0}{|y_k^0|}\Big| \leq c\frac{r_k}{\varepsilon_k}. \tag{2.68}$$

Then, by using (2.68), (2.63), and noting that $\nabla(\eta u_k)(z_k^0) \cdot y = \nabla(\eta u_k)(z_k^0) \cdot (y - y^\perp)$, we obtain

$$|v_k(z)| = \frac{|\nabla(\eta u_k)(z_k^0) \cdot y|}{r_k^\alpha L_k} \leq \frac{|\nabla(\eta u_k)(z_k^0)|}{r_k^\alpha L_k}|y - y^\perp| \leq c\Big(\frac{\varepsilon_k}{r_k}\Big)^\alpha \frac{r_k}{\varepsilon_k} \to 0,$$

as $k \to \infty$. Therefore, since $v_k \to \bar{v}$ uniformly on every compact set $K \subset \Omega_\infty$, we conclude that $\bar{v} = 0$ on $\partial\Pi$.

Step 6. $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients.

Let $A_k(z) := A(\tilde{z}_k + r_k z)$ and $\bar{z} := \lim_{k\to\infty} \tilde{z}_k$. By using the α-Hölder continuity of A, we can define $\bar{A} := A(\bar{z}) = \lim_{k\to\infty} A_k(z)$, which is a constant coefficients symmetric matrix satisfying (2.2). Let us define

$$\rho_k(y) := \begin{cases} \dfrac{|y_k + r_k y|}{|y_k|}, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

and fix $\phi \in C_c^\infty(\Omega_\infty)$. Since u_k is a solution to (2.45), we have that

$$\begin{aligned} &\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(z)\nabla w_k(z) \cdot \nabla\phi(z) dz = \frac{\eta(\tilde{z}_k) r_k^{1-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz \\ &- \frac{\eta(\tilde{z}_k) r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla\phi(z) dz - \frac{r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(z) P_k \cdot \nabla\phi(z) dz \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned} \tag{2.69}$$

where we have set

$$P_k := \begin{cases} (\eta\nabla u_k)(\tilde{z}_k), & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ 0, & \text{in } \textbf{Case 3}. \end{cases}$$

We want to show that the right hand side vanishes as $k \to \infty$. The term I vanishes exactly as in the Theorem 2.6.1, *Step 5*, by using the integrability assumption $f \in L^{p,a}(B_1)$, with $p > d$ and $\alpha \in (0, 1 - d/p]$ by (2.5).

Next, by using the divergence theorem, we have

$$\begin{aligned}
&\Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla\phi(z) dz \Big| \\
&\le \int_{\mathrm{spt}(\phi)} \rho_k^a(y) |F(\tilde{z}_k + r_k z) - F(\tilde{z}_k)| |\nabla\phi(z)| dz + \Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k) \cdot \nabla\phi(z) dz \Big| \\
&\le c \|F\|_{C^{0,\alpha}(B_1)} r_k^\alpha + \Big| \int_{\mathrm{spt}(\phi)} \nabla\rho_k^a(y) \cdot F(\tilde{z}_k) \phi(z) dz \Big| \\
&\le c r_k^\alpha + \Big| \int_{\mathrm{spt}(\phi)} \nabla\rho_k^a(y) \cdot F(\tilde{z}_k) \phi(z) dz \Big|.
\end{aligned} \tag{2.70}$$

In **Case 3**, since $F(x, 0) \cdot e_{y_i} = 0$, one has that

$$\nabla |y|^a \cdot F(\tilde{z}_k) = a|y|^{a-2} F(x_k, 0) \cdot y = 0,$$

for every $z \in \mathrm{spt}(\phi)$, so the second term in (2.70) is zero and

$$|\mathrm{II}| \le \frac{c}{L_k} \to 0, \quad \text{as } k \to \infty.$$

Instead, in **Case 1** and **Case 2**, since $|\tilde{y}_k + r_k y| \ge |\tilde{y}_k|/2$ for $y \in \mathrm{spt}(\phi)$, we have that

$$\begin{aligned}
\big|\nabla\rho_k^a(y) \cdot F(\tilde{z}_k)\big| &= \Big| \frac{a r_k \rho_k^a(y)(\tilde{y}_k + r_k y)}{|\tilde{y}_k + r_k y|^2} \cdot \Big(F(\tilde{x}_k, \tilde{y}_k) - F(\tilde{x}_k, 0) \Big) \Big| \\
&\le \frac{c r_k [F]_{C^{0,\alpha}(B_1)} |\tilde{y}_k|^\alpha}{|\tilde{y}_k|} \le c r_k |\tilde{y}_k|^{\alpha-1}.
\end{aligned} \tag{2.71}$$

Thus, by using (2.70) and (2.71), one has that

$$|\mathrm{II}| \le \frac{c}{L_k} + \frac{c r_k^{-\alpha}}{L_k} r_k |\tilde{y}_k|^{\alpha-1} \le \frac{c}{L_k} \Big(1 + \Big(\frac{r_k}{|\tilde{y}_k|} \Big)^{1-\alpha} \Big) \le c L_k^{-1} \to 0, \quad \text{as } k \to \infty,$$

in **Case 1** and **Case 2**, since $r_k / |\tilde{y}_k| \le c$.

Finally, we prove that the third member of (2.69) goes to zero as $k \to \infty$. In **Case 3**, $P_k = 0$ so $\mathrm{III} = 0$. Instead, in **Case 1** and **Case 2**, one has

$$\begin{aligned}
|\mathrm{III}| \le &\Big| \frac{r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) A(\tilde{z}_k) (\eta \nabla u_k)(\tilde{z}_k) \cdot \nabla\phi(z) dz \Big| \\
&+ \Big| \frac{r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) (A(\tilde{z}_k + r_k z) - A(\tilde{z}_k)) (\eta \nabla u_k)(\tilde{z}_k) \cdot \nabla\phi(z) dz \Big|.
\end{aligned} \tag{2.72}$$

First, we show that the first member in (2.72) vanishes. Recall that $r_k/|y_k|\to 0$, $\rho_k^a\to 1$ and (x_k,y_k^0) is the projection of z_k on $\partial\Sigma_{\varepsilon_k}$. By using (2.58) and (2.63) we get

$$\begin{aligned}
&|(\eta\nabla u_k)(\tilde z_k))|\le|\nabla(\eta u_k)(\tilde x_k,\tilde y_k)|+|(u_k\nabla\eta)(\tilde x_k,\tilde y_k)|\\
&\le|\nabla(\eta u_k)(\tilde x_k,\tilde y_k)-\nabla(\eta u_k)(x_k,y_k^0)|+|\nabla(\eta u_k)(x_k,y_k^0)|\\
&+|\nabla\eta(\tilde x_k,\tilde y_k)(u_k(\tilde x_k,\tilde y_k)-u_k(x_k,y_k^0))|\\
&\le cL_k|\tilde y_k-y_k^0|^\alpha+cL_k\varepsilon_k^\alpha+c|\tilde y_k-y_k^0|^\alpha\le cL_k|\tilde y_k|^\alpha.
\end{aligned}\tag{2.73}$$

On the other hand, since $|\tilde y_k+r_ky|\ge|\tilde y_k|/2$ for $y\in\mathrm{spt}(\phi)$, it follows

$$|\nabla\rho_k^a(y)|=\Big|\frac{ar_k\rho_k^a(y)(\tilde y_k+r_ky)}{|\tilde y_k+r_ky|^2}\Big|\le c\frac{r_k}{|\tilde y_k|}\rho_k^a(y).\tag{2.74}$$

hence, by combining (2.73) and (2.74), and using the divergence theorem, it follows that

$$\Big|\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)}\nabla\rho_k^a(y)\cdot A(\tilde z_k)(\eta\nabla u_k)(\tilde z_k)\phi(z)dz\Big|\le c\Big(\frac{r_k}{|\tilde y_k|}\Big)^{1-\alpha}\to 0,$$

that is, the first member in (2.72) vanishes. Next, we show that the second member vanishes as $k\to\infty$. In this case, we need to reason in two steps (as done in Theorem 1.7.1 and [138, Remark 5.3]): first, we prove uniform estimates in $C^{1,\alpha'}$ space for some suboptimal $\alpha'\in(0,\alpha)$; then, by using these estimates, we conclude the optimal regularity with exponent α. Let us fix $\alpha'\in(0,\alpha)$. By using interpolation inequality in Hölder spaces [75, Lemma 6.35], we can estimate the second term of (2.72) as follows

$$\begin{aligned}
&\Big|\frac{r_k^{-\alpha'}}{L_k}\int_{\mathrm{spt}(\phi)}\rho_k^a(y)(A(\tilde z_k+r_kz)-A(\tilde z_k))(\eta\nabla u_k)(\tilde z_k)\cdot\nabla\phi(z)dz\Big|\\
&\le\frac{cr_k^{\alpha-\alpha'}\|(\eta\nabla u_k)\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{L_k}\le\frac{cr_k^{\alpha-\alpha'}}{L_k}\big(\|u_k\nabla\eta\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}+\|\nabla(\eta u_k)\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}\big)\\
&\le\frac{cr_k^{\alpha-\alpha'}(\|u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}+[\nabla(\eta u_k)]_{C^{0,\alpha}(B_{3/4}\setminus\Sigma_{\varepsilon_k})})}{L_k}\le cr_k^{\alpha-\alpha'}\to 0,
\end{aligned}$$

since $\alpha'<\alpha$. If we have uniform estimates in $C^{1,\alpha'}$ space, then $\|\eta\nabla u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}\le c$. Hence, restarting the proof with the optimal α and the additional information above, in the previous computation we get

$$\begin{aligned}
&\Big|\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)}\rho_k^a(y)(A(\tilde z_k+r_kz)-A(\tilde z_k))(\eta\nabla u_k)(\tilde z_k)\cdot\nabla\phi(z)dz\Big|\\
&\le\frac{c\|(\eta\nabla u_k)\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{L_k}\le\frac{c}{L_k}\to 0.
\end{aligned}$$

Combining all the previous results, we conclude that the right-hand side of (2.69) vanishes as $k\to\infty$.

Finally, by the same considerations of Theorem 2.6.1, we obtain that the left hand side of (2.69) converges in the following sense

$$\int_{\mathrm{spt}(\phi)}\rho_k^aA_k\nabla w_k\cdot\nabla\phi dz\to\int_{\mathrm{spt}(\phi)}\bar\rho^a\bar A\nabla\bar v\cdot\nabla\phi dz,$$

where

$$\bar\rho(y) := \begin{cases} 1, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2},\\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

and $\bar v \in H^1_{\mathrm{loc}}(\Omega_\infty, \bar\rho^a(y)dz)$.

Then, recalling who is Ω_∞, see (2.50), and using the *Step 5* for the Dirichlet boundary condition, we conclude that

in **Case 1**, $\bar v$ is an entire solution to

$$-\operatorname{div}(\bar A\nabla\bar v) = 0, \quad \text{in } \mathbb{R}^d,$$

in **Case 2**, $\bar v$ is an entire solution to

$$\begin{cases} -\operatorname{div}(\bar A\nabla\bar v) = 0, & \text{in } \Pi,\\ \bar v = 0, & \text{on } \partial\Pi, , \end{cases}$$

in **Case 3**, $\bar v$ is an entire solution to

$$\begin{cases} -\operatorname{div}(|y|^a\bar A\nabla\bar v) = 0, & \text{in } \mathbb{R}^d\setminus\Sigma_{\bar\varepsilon},\\ \bar v = 0, & \text{on } \partial\Sigma_{\bar\varepsilon}, \end{cases}$$

Step 7. Liouville Theorems and conclusion.

Since $\bar v$ satisfies the growth condition (2.66) with $1+\alpha \in (0,2)\cap(0,2-a-n)$, invoking the classical Liouville Theorem in **Case 1** and **Case 2** shows that $\bar v$ must be a linear function. In contrast, applying the Liouville Theorem 2.5.1 in **Case 3** implies that $\bar v$ must be zero.

This contradicts *Step 3*, as $\nabla\bar v$ is not constant. Consequently, $L_k = [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})}$ must be bounded, which implies that $\|\eta u_\varepsilon\|_{C^{1,\alpha}(B_1\setminus\Sigma_\varepsilon)} \le c$. Thus, (2.56) holds true.

Furthermore, recalling (2.63) and using (2.56), we conclude that (2.57) follows. This completes the proof. □

Remark 2.7.2. The homogeneous Dirichlet condition on $\partial\Sigma_\varepsilon$ is too restrictive to handle all possible fields F, thus preventing the validity of ε-uniform $C^{1,\alpha}$ estimates in the general case.

Let us suppose that $n = d = 2$, $-2 < a < -1$ and consider the function $u(y) = u(y_1, y_2) := y_1 + y_2$ which is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a\nabla u) = \operatorname{div}(|y|^a F), & \text{in } B_1\setminus\Sigma_0,\\ u = 0, & \text{on } \Sigma_0\cap B_1. \end{cases}$$

where $F := -\nabla u = (-1,-1)$. By applying Lemma 2.22 we find that there exists a family u_ε which are weak solutions to

$$\begin{cases} -\operatorname{div}(|y|^a\nabla u_\varepsilon) = \operatorname{div}(|y|^a F), & \text{in } B_{3/4}\setminus\Sigma_\varepsilon,\\ u = 0, & \text{on } \Sigma_\varepsilon\cap B_{3/4}, \end{cases}$$

and satisfies $\|u_\varepsilon\|_{H^{1,a}(B_{3/4}\setminus\Sigma_\varepsilon)} \le c$ and $u_\varepsilon \to u$ in $H^{1,a}(B_{3/4})$. If Theorem 2.7.1 holds true for this equation, we obtain that (2.57) works (since it depends only on the Dirichlet boundary condition satisfied by u_ε), that is

$$|\nabla u_\varepsilon| \leq c\varepsilon^\alpha, \qquad \text{on } \partial\Sigma_\varepsilon \cap B_{1/2},$$

and then, taking the limit as $\varepsilon \to 0$ in a suitable sense (see the proof of the Theorem 2.1.2), we deduce that $\nabla u = 0$ on $\Sigma_0 \cap B_{1/2}$, which contradicts $\nabla u = (1,1)$.

The next proposition provides *a priori* estimates for solutions that additionally satisfy an extra boundary condition on the lower dimensional boundary Σ_0.

Proposition 2.7.3. *Let $2 \leq n \leq d$, $a+n \in (0,1)$, $p > d$ and α satisfying (2.5). Let A be a α-Hölder continuous matrix satisfying (2.2) and $\|A\|_{C^{0,\alpha}(B_1)} \leq L$, $f \in L^{p,a}(B_1)$ and $F \in C^{0,\alpha}(B_1)$. Let $u \in C^{1,\alpha}(B_1)$ be a weak solution to*

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_0, \\ u = 0, & \text{on } \Sigma_0 \cap B_1, \end{cases}$$

such that u satisfies the boundary condition

$$\begin{cases} \nabla_x u(x,0) = 0, \\ (A\nabla u + F)(x,0)\cdot e_{y_i} = 0, \end{cases} \qquad \text{for every } (x,0) \in \Sigma_0 \cap B_1, \ \text{ and } i = 1,\dots,n. \tag{2.75}$$

Then, there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big). \tag{2.76}$$

Proof. The proof is quite similar to the one of Theorem 2.7.1, so we avoid some details. Without loss of generality, let us suppose that

$$\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)} \leq c.$$

By contradiction let us suppose that (2.76) doesn't holds, hence, there exist $p > d$, α satisfying (2.5), $\{A_k\}_k$, $\{f_k\}_k$, $\{F_k\}_k$, $\{u_k\}_k$, such that u_k is solution to

$$\begin{cases} -\operatorname{div}(|y|^a A_k\nabla u_k) = |y|^a f_k + \operatorname{div}(|y|^a F_k), & \text{in } B_1 \setminus \Sigma_0, \\ u_k = 0, & \text{on } \Sigma_0 \cap B_1, \end{cases}$$

satisfies the boundary condition (2.75) in $\Sigma_0 \cap B_1$ and

$$\|u_k\|_{C^{1,\alpha}(B_{1/2})} \to \infty.$$

Let us fix a smooth cut-off function $\eta \in C^\infty_c(B_1)$ such that

$$\operatorname{spt}(\phi) \subset B_{3/4}, \quad \eta = 1 \text{ in } B_{1/2}, \quad 0 \leq \eta \leq 1,$$

hence, we have

$$L_k := [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1)} \to \infty.$$

By definition of Hölder seminorm, take two sequences of points $z_k = (x_k, y_k), \hat{z}_k = (\hat{x}_k, \hat{y}_k) \in B_1$ such that

$$\frac{|\nabla(\eta u_k)(z_k) - \nabla(\eta u_k)(\hat{z}_k)|}{|z_k - \hat{z}_k|^\alpha} \geq \frac{L_k}{2},$$

and define $r_k := |z_k - \hat{z}_k|$. Now we distinguish two cases:

$$\textbf{Case 1:} \quad \frac{|y_k|}{r_k} \to \infty, \qquad\qquad \textbf{Case 2:} \quad \frac{|y_k|}{r_k} \leq c,$$

for some $c > 0$ which not depends on k. Let us define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) := \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k, 0), & \text{in } \textbf{Case 2}, \end{cases}$$

the sequence of domains

$$\Omega_k := \frac{B_1 \setminus \Sigma_0 - \tilde{z}_k}{r_k},$$

the sequences of functions

$$\begin{aligned} v_k(z) &:= \frac{(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k) - \nabla(\eta u_k)(\tilde{z}_k) \cdot r_k z}{L_k r_k^{1+\alpha}}, \\ w_k(z) &:= \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k) - (\eta \nabla u_k)(\tilde{z}_k) \cdot r_k z}{L_k r_k^{1+\alpha}}, \end{aligned}$$

for $z \in \Omega_k$ and set Ω_∞ as in (2.47).

By following the argument in Theorem 2.7.1, *Step 2*, we have that $\|v_k\|_{C^{1,\alpha}(K)} \leq 1$ for every compact subset $K \subset \Omega_\infty$. Therefore, we can apply the Arzelà-Ascoli Theorem to conclude that $v_k \to \bar{v}$ in $C^{1,\gamma}(K)$ for every $\gamma \in (0, \alpha)$, and that $|\bar{v}(z)| \leq c(1 + |z|^{1+\alpha})$, for a.e. $z \in \Omega_\infty$. Moreover, it follows that $w_k \to \bar{v}$ as well.

Next, by the same reasoning used in Theorem 2.7.1, *Step 3*, we obtain that $\nabla \bar{v}$ is not constant. Continuing as in *Step 4* of Theorem 2.7.1, we conclude that $r_k \to 0$, which implies that the limit domain Ω_∞ is given by

$$\Omega_\infty := \begin{cases} \mathbb{R}^d, & \text{in } \textbf{Case 1}, \\ \mathbb{R}^d \setminus \Sigma_0, & \text{in } \textbf{Case 2}. \end{cases}$$

Eventually, we prove that $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients and we reach a contradiction by invoking a Liouville type Theorem. Since $\|A_k\|_{C^{0,\alpha}(B_1)} \leq L$, we have that $A_k(\tilde{z}_k + r_k z) \to A(\bar{z}) := \bar{A}$, which is a constant matrix satisfying (2.2). Let us define

$$\rho_k(y) := \begin{cases} \dfrac{|y_k + r_k y|}{|y_k|}, & \text{in } \textbf{Case 1}, \\ |y|, & \text{in } \textbf{Case 2}. \end{cases}$$

and fix $\phi \in C_c^\infty(\Omega_\infty)$. A straightforward computation show us that

$$\begin{aligned} &\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(\tilde{z}_k + r_k z) \nabla w_k(z) \cdot \nabla \phi(z) dz = \frac{\eta(\tilde{z}_k) r_k^{1-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f_k(\tilde{z}_k + r_k z) \phi(z) dz \\ &- \frac{\eta(\tilde{z}_k) r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) \big(F_k(\tilde{z}_k + r_k z) - F_k(\tilde{z}_k)\big) \cdot \nabla \phi(z) dz \\ &- \frac{r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) \big(A_k(\tilde{z}_k + r_k z) - A_k(\tilde{z}_k)\big)(\eta \nabla u_k)(\tilde{z}_k) \cdot \nabla \phi(z) dz \end{aligned}$$

$$- \frac{r_k^{-\alpha}}{L_k}\rho_k^a(y)\big(\eta A_k\nabla u_k + \eta F_k\big)(\tilde{z}_k)\cdot\nabla\phi(z)dz = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

The terms I, II, III vanishes as $k\to\infty$ exactly as in Theorem 2.7.1, *Step 6.*

Next, we show that the fourth member goes to zero. By using the divergence theorem, we get

$$\int_{\mathrm{spt}(\phi)}\rho_k^a(y)\big(\eta A_k\nabla u_k + \eta F_k\big)(\tilde{z}_k)\cdot\nabla\phi(z)dz = \int_{\mathrm{spt}(\phi)}\nabla\rho_k^a(y)\cdot\big(\eta A_k\nabla u_k + \eta F_k\big)(\tilde{z}_k)\phi(z)dz.$$

In **Case 2**, since, $\tilde{z}_k=(x_k,0)\in\Sigma_0$, u_k satisfies the boundary condition (2.75) we have that

$$\eta(\tilde{z}_k)(A_k\nabla u_k + F_k)(\tilde{z}_k)\cdot e_{y_i}=0, \quad \text{for every } i=1,\dots,n,$$

so IV$=0$.

Let us consider the **Case 1**, recalling that $\tilde{z}_k=z_k$, $r_k/|y_k|\to 0$ and $\rho_k^a\to 1$. Arguing as Theorem 2.7.1, *Step 6*, we have that

$$|\nabla\rho_k^a(y)|\le c\frac{r_k}{|y_k|}\rho_k^a(y),$$

and by using the boundary condition (2.75)

$$\begin{aligned}&\left|(\eta A_k\nabla u_k+\eta F_k)(z_k)\cdot\frac{y_k+r_ky}{|y_k+r_ky|}\right|\\ &\le\left|\big((\eta A_k\nabla u_k+\eta F_k)(z_k)-(\eta A_k\nabla u_k+\eta F_k)(x_k,0)\big)\cdot\frac{y_k+r_ky}{|y_k+r_ky|}\right|\\ &\le[\eta A_k\nabla u_k+\eta F_k]_{C^{0,\alpha}(B_{3/4})}|y_k|^\alpha\le cL_k|y_k|^\alpha.\end{aligned}$$

Hence, combining these two inequalities we obtain

$$\left|\nabla\rho_k^a(y)\cdot(\eta A_k\nabla u_k+\eta F_k)(z_k)\right|\le c\frac{r_k}{|y_k|^{1-\alpha}}L_k,$$

so

$$|\mathrm{IV}|=\frac{r_k^{-\alpha}}{L_k}\left|\int_{\mathrm{spt}(\phi)}\nabla\rho_k^a(y)\cdot\big(\eta A_k\nabla u_k+\eta F_k\big)(z_k)\phi(z)dz\right|\le c\Big(\frac{r_k}{|y_k|}\Big)^{1-\alpha}\to 0,$$

as $k\to\infty$.

On the other hand, arguing as in the *Step 5* of the Theorem 2.6.1, we have that

$$\int_{\mathrm{spt}(\phi)}\rho_k^a(y)A_k(\tilde{z}_k+r_kz)\nabla w_k(z)\cdot\nabla\phi(z)dz\to\int_{\mathrm{spt}(\phi)}\bar{\rho}^a\bar{A}\nabla\bar{v}\cdot\nabla\phi dz,$$

where

$$\bar{\rho}(y):=\begin{cases}1, & \text{in } \textbf{Case 1},\\ |y|, & \text{in } \textbf{Case 2}.\end{cases}$$

Then, we have that

in **Case 1**, $\bar{v}$ is an entire solution to

$$-\operatorname{div}(\bar{A}\nabla\bar{v})=0, \quad \text{in } \mathbb{R}^d,$$

in **Case 2**, $\bar{v}$ is an entire solution to

$$\begin{cases} -\operatorname{div}(|y|^a \bar{A}\nabla\bar{v}) = 0, & \text{in } \mathbb{R}^d \setminus \Sigma_0, \\ \bar{v} = 0, & \text{on } \Sigma_0, \end{cases}.$$

where $\bar{A}$ is symmetric constant matrix satisfying (2.2). From this point on, as in Theorem 2.7.1, *Step 7*, by invoking appropriate Liouville type Theorems we get a contradiction and the thesis follows. ☐

Proof of the Theorems 2.1.2. As in the proof of Theorem 2.1.1, without loss of generality, we may consider the function $u - \psi$ and provide the proof of the theorem for solutions to (2.1) with homogeneous Dirichlet boundary conditions. Once this is established, the general case follows directly. We divide the proof in two steps.

Step 1. First, we claim that if $A \in C^{1,\alpha}(B_1)$ and $F \in C^{1,\alpha}(B_1)$ then $u \in C^{1,\alpha}(B_{1/2})$ and u satisfies the boundary condition (2.75) in $\Sigma_0 \cap B_{1/2}$.

Let us split the matrix A in blocks as follows

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^\top & A_3 \end{pmatrix},$$

where $A_1 : B_R \to \mathbb{R}^{d-n,d-n}$, $A_2 : B_R \to \mathbb{R}^{d-n,n}$, $A_3 : B_R \to \mathbb{R}^{n,n}$. Let us consider the decomposition $F = (F_x, F_y)$, where $F_x = (F_{x_1}, \ldots, F_{x_{d-n}})$ and $F_y = (F_{y_1}, \ldots, F_{y_n})$ and define the scalar function

$$g(x,y) = A_3^{-1}(x,0)F_2(x,0) \cdot y,$$

which belongs to $C^{1,\alpha}(B_1)$, since the block A_3 satisfies the uniformly elliptic condition (2.2), and

$$g(x,0) = 0, \qquad \nabla g(x,0) = \big(0, (A_3^{-1}F_2)(x,0)\big).$$

The function $v := u - g \in \tilde{H}^{1,a}(B_1)$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v) = |y|^a f + \operatorname{div}(|y|^a (F - A\nabla g)), & \text{in } B_1 \setminus \Sigma_0, \\ v = 0, & \text{on } \Sigma_0 \cap B_1, \end{cases}$$

where the field $F - A\nabla g \in C^\infty(B_1)$ satisfies

$$(F - A\nabla g)(x,0) \cdot e_{y_i} = 0.$$

Arguing as in the proof of Theorem 2.1.1, by applying Lemma 2.4.1, we can find a sequence $\{v_{\varepsilon_k}\}$ as $\varepsilon_k \to 0$, such that every v_{ε_k} is solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v_{\varepsilon_k}) = |y|^a f + \operatorname{div}(|y|^a (F - A\nabla g)), & \text{in } B_{3/4} \setminus \Sigma_{\varepsilon_k}, \\ v_{\varepsilon_k} = 0, & \text{on } \partial\Sigma_{\varepsilon_k} \cap B_{3/4}, \end{cases}$$

and $v_{\varepsilon_k} \to v$ in $H^{1,a}(B_{3/4} \setminus \Sigma_0)$ as $\varepsilon_k \to 0$. By applying the Theorem 2.7.1 to the sequences $\{v_{\varepsilon_k}\}$, combined with the estimate (2.23), we get that

$$\|v_{\varepsilon_k}\|_{C^{1,\alpha}(B_{1/2} \setminus \Sigma_{\varepsilon_k})} \le c\big(\|v\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F - A\nabla g\|_{C^{0,\alpha}(B_1)}\big),$$

for some $c>0$ which not depends on ε_k.

By applying Arzelà-Ascoli Theorem we get that $v_{\varepsilon_k}\to w$ in $C^{1,\gamma}_{\mathrm{loc}}(B_{1/2}\setminus\Sigma_0)$, for every $\gamma\in(0,\alpha)$, and by the a.e. convergences $v_{\varepsilon_k}\to v$ it follows that $w=v$. Moreover, by taking $z,z'\in B_{1/2}\setminus\Sigma_0$ such that $z\neq z'$, we have that

$$\begin{aligned}\frac{|\nabla v(z)-\nabla v(z')|}{|z-z'|^\alpha}&=\lim_{\varepsilon_k\to0}\frac{|\nabla v_{\varepsilon_k}(z)-\nabla v_{\varepsilon_k}(z')|}{|z-z'|^\alpha}\\&\leq c\big(\|v\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F-A\nabla g\|_{C^{0,\alpha}(B_1)}\big),\end{aligned}$$

which implies that

$$\begin{aligned}[v]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_0)}&=\sup_{\substack{z,z'\in B_{1/2}\setminus\Sigma_0\\ z\neq z'}}\frac{|\nabla v(z)-\nabla v(z')|}{|z-z'|^\alpha}\\&\leq c\big(\|v\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F-A\nabla g\|_{C^{0,\alpha}(B_1)}\big),\end{aligned}$$

By continuity, we can extend v to the whole $B_{1/2}$ in such a way that $[v]_{C^{1,\alpha}(B_{1/2})}=[v]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_0)}$. Combining the previous inequality with the L^∞_{loc} bound of solutions (see Lemma 2.3.3) and interpolation inequality in Hölder spaces [75, Lemma 6.35], we get

$$\|v\|_{C^{1,\alpha}(B_{1/2})}\leq c(\|v\|_{L^\infty(B_{1/2})}+[v]_{C^{1,\alpha}(B_{1/2})})\leq c\big(\|v\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F-A\nabla g\|_{C^{0,\alpha}(B_1)}\big).$$

Hence, $v\in C^{1,\alpha}(B_{1/2})$, which immediately implies $u\in C^{1,\alpha}(B_{1/2})$.

Next, let us prove that u satisfies the boundary condition (2.75). Let us fix $z=(x,y)\in B_{1/2}\setminus\Sigma_0$ and let z_k^0 be the projection of z on $\partial\Sigma_{\varepsilon_k}\cap B_{1/2}$. By using Theorem 2.7.1 and (2.57), we get

$$\begin{aligned}|\nabla v(z)|&\leq|\nabla v(z)-\nabla v_{\varepsilon_k}(z)|+|\nabla v_{\varepsilon_k}(z)-\nabla v_{\varepsilon_k}(z_k^0)|+|\nabla v_{\varepsilon_k}(z_k^0)|\\&\leq|\nabla v(z)-\nabla v_{\varepsilon_k}(z)|+[v_{\varepsilon_k}]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})}|z-z_k^0|^\alpha+c[v_{\varepsilon_k}]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})}\varepsilon_k^\alpha\\&\leq o(1)+c\big(\|v\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F-A\nabla g\|_{C^{0,\alpha}(B_1)}\big)|y|^\alpha,\quad\text{as }\varepsilon_k\to0.\end{aligned}$$

By taking the limit as $|y|\to0$ in the previous inequality, we conclude that $\nabla v=0$ on $\Sigma_0\cap B_{1/2}$ and, by construction, it follows that

$$0=\nabla_x v(x,0)=\nabla_x u(x,0)-\nabla_x g(x,0)=\nabla_x u(x,0),$$

and

$$0=(A\nabla v)(x,0)\cdot e_{y_i}=(A\nabla u-A\nabla g)(x,0)\cdot e_{y_i}=(A\nabla u+F)(x,0)\cdot e_{y_i},$$

that is, (2.75) holds true.

Step 2. Finally, we prove that if $A,F\in C^{0,\alpha}(B_1)$, then $u\in C^{1,\alpha}(B_{1/2})$ and satisfies (2.6), (2.7). Up to consider the function $u-\psi\in\tilde H^{1,a}(B_1)$, we can suppose that u satisfies a Dirichlet homogeneous boundary condition $u=0$ on $\Sigma_0\cap B_1$.

Let $\{\rho_\delta\}_{\delta>0}$ be a family of smooth mollifiers and define $A_\delta:=A*\rho_\delta$ and $F_\delta:=F*\rho_\delta$, which satisfies $\|A_\delta\|_{C^{0,\alpha}(B_{3/4})}\leq\|A\|_{C^{0,\alpha}(B_1)}$ and $\|F_\delta\|_{C^{0,\alpha}(B_{3/4})}\leq\|F\|_{C^{0,\alpha}(B_1)}$. Using an approximation argument similar to the one in Lemma 2.4.1, but with a more standard approach, and following a method analogous to the one in Theorem 1.10.1, we construct a family $\{u_\delta\}_{\delta>0}$ of solutions to

$$\begin{cases} -\operatorname{div}(|y|^a A_\delta \nabla u_\delta) = |y|^a f + \operatorname{div}(|y|^a F_\delta), & \text{in } B_{4/5} \setminus \Sigma_0, \\ u = 0, & \text{on } \Sigma_0 \cap B_{4/5}, \end{cases}$$

satisfies the following properties

$$\|u_\delta\|_{H^{1,a}(B_{4/5})} \leq c(\|u\|_{H^{1,a}(B_1)} + \|f\|_{L^{2,a}(B_1)} + \|F\|_{L^{2,a}(B_1)}),$$

$$u_\delta \to u \text{ strongly in } H^{1,a}(B_{4/5}),$$

for some $c > 0$ depending only on d, n, a, λ and Λ.

Since $A_\delta, F_\delta \in C^\infty(B_{4/5})$, by using *Step 1*, we get that $u \in C^{1,\alpha}(B_{3/4})$ and u satisfies the boundary condition (2.75) on $\Sigma_0 \cap B_{3/4}$. Hence, the assumptions of the Proposition 2.7.3 are satisfies, which implies that u_δ satisfies (2.76), which immediately implies that

$$\|u_\delta\|_{C^{1,\alpha}(B_{1/2})} \leq c(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{2,a}(B_1)} + \|F\|_{L^{2,a}(B_1)}),$$

and the Arzelá-Ascoli Theorem allows us to taking the limit as $\delta \to 0$ to infer that $u \in C^{1,\alpha}(B_{1/2})$, satisfies (2.6) and the boundary condition (2.7). The proof is complete. ☐

2.8 Regularity on curved manifolds

In this section, we prove how to extend Theorems 2.1.1 and 2.1.2, namely the local $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity of weak solutions to (2.1), to a class of equations with weights that are singular on $(d-n)$-dimensional manifolds, that is, for weak solutions to (2.8).

For $2 \leq n < d$, let $\Gamma \subset \mathbb{R}^d$ be a $(d-n)$-dimensional C^1-manifold such that $0 \in \Gamma$ and there exists a parametrization $\varphi \in C^1(\Sigma_0 \cap B_1; \mathbb{R}^n)$ such that, up to perform a dilation, we have

$$B_1 \cap \Gamma = \{(x,y) : y = \varphi(x)\} \cap B_1, \quad \varphi(0) = 0. \tag{2.77}$$

The diffeomorphism

$$\Phi(x,y) := (x, y + \varphi(x)), \tag{2.78}$$

whose inverse straightens the lower dimensional boundary Γ to Σ_0, satisfies $\Phi(\Sigma_0 \cap B_1) \subset \Gamma \cap B_1$ and the Jacobian associated to Φ is

$$J_\Phi(x,y) = \begin{pmatrix} \mathbb{I}_{d-n} & 0 \\ J_\varphi(x) & \mathbb{I}_n \end{pmatrix}, \quad \text{with } |\det J_\Phi| \equiv 1.$$

Given a $(d-n)$-dimensional manifold Γ parametrized by φ, we give the definition of admissible weights with respect to φ.

Definition 2.8.1. Let $2 \leq n < d$ and $\alpha \in [0,1)$. Let Γ be a $(d-n)$-dimensional $C^{1,\alpha}$-manifold and $\varphi \in C^{1,\alpha}(\Sigma_0 \cap B_1; \mathbb{R}^n)$ be a parametrization of Γ in the sense of (2.77) and define the diffeomorphism Φ as in (2.78).

We say that δ is an α-admissible weight with respect to the parametrization φ if $\delta \in C^{0,1}(B_1)$ and the two following condition holds true.

i) There exist constants $0 < c_0 \le c_1$ such that

$$c_0 \le \frac{\delta}{d_\Gamma} \le c_1. \tag{2.79}$$

ii)

$$\tilde{\delta}(x,y) := \frac{\delta(\Phi(x,y))}{|y|} \in C^{0,\alpha}(B_1). \tag{2.80}$$

We point out that the condition i) in the Definition 2.8.1 implies that δ in a equivalent distance from Γ to the standard distance function d_Γ. So, given δ, which satisfies condition *i*), similarly to what was done in Section 2.2.1, we can define the functional spaces $H^1(B_1, \delta^a)$, which is equivalent to $H^1(B_1, d_\Gamma^a)$. Then, by Theorem 2.2.3, it follows that it is well defined the bounded linear trace operator $T : H^1(B_1, \delta^a) \to L^2(\Gamma \cap B_1)$ such that $Tu = u_{|_{\Gamma\cap B_1}}$, for every $u \in C^\infty(\overline{B_1})$. In light of this, we define a notion of weak solutions for the equation (2.8), which satisfy a Dirichlet boundary on Γ.

Definition 2.8.2. Let $2 \le n < d$, $a + n \in (0, 2)$, A be a matrix satisfying (2.2). Let Γ be a $(d-n)$-dimensional C^1-manifold and $\varphi \in C^1(\Sigma_0 \cap B_1; \mathbb{R}^n)$ be a parametrization of Γ in the sense of (2.77), $\delta \in C^{0,1}(B_1)$ satisfying i) of the Definition 2.8.1. Let $f \in L^2(B_1, \delta^a)$, $F \in L^2(B_1, \delta^a)^d$ and $\psi \in L^2(\Gamma \cap B_1)$.

We say that u is a weak solution to (2.8) if $u \in H^1(B_1, \delta^a)$, satisfies

$$\int_{B_1} \delta^a A\nabla u \cdot \nabla\phi dz = \int_{B_1} \delta^a (f\phi - F \cdot \nabla\phi) dz, \tag{2.81}$$

for every $\phi \in C_c^\infty(B_1 \setminus \Gamma)$ and $u = \psi$ in $L^2(\Gamma \cap B_1)$, in the sense of the trace.

Finally, we can state the main results of this section, namely $C^{0,\alpha}_{\rm loc}$ and $C^{1,\alpha}_{\rm loc}$ regularity for weak solutions to (2.8).

Corollary 2.8.3. *Let $2 \le n < d$, $a + n \in (0, 2)$, $p > d/2$, $q > d$ and α satisfying (2.3). Let $\varphi \in C^1(\Sigma_0 \cap B_1; \mathbb{R}^n)$ be the parametrization defined in (2.77). Let δ be a 0-admissible weight with respect to φ in the sense of Definition 2.8.1 and define $\tilde{\delta} \in C^0(B_1)$ as in (2.80). Let A be a continuous symmetric matrix satisfying (3.5), $f \in L^p(B_1, \delta^a)$, $F \in L^q(B_1, \delta^a)^d$ and $\psi \in C^{0,1}(\Gamma \cap B_1)$. Let u be a weak solution to (2.8), in the sense of Definition 2.8.2. Let us suppose that there exists a modulus of continuity ω such that*

$$\|A\|_{C^{0,\omega}(B_1)} + \|\tilde{\delta}\|_{C^{0,\omega}(B_1)} + \|\varphi\|_{C^{1,\omega}(\Sigma_0 \cap B_1; \mathbb{R}^n)} \le L.$$

Then, $u \in C^{0,\alpha}(B_{1/2})$ and there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, q, α and L such that

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \le c\big(\|u\|_{L^2(B_1,\delta^a)} + \|f\|_{L^p(B_1,\delta^a)} + \|F\|_{L^q(B_1,\delta^a)^d} + \|\psi\|_{C^{0,1}(\Gamma\cap B_1)}\big).$$

Corollary 2.8.4. *Let $2 \le n < d$, $a + n \in (0, 1)$, $p > d$ and α satisfying (2.5). Let $\varphi \in C^{1.\alpha}(\Sigma_0 \cap B_1; \mathbb{R}^n)$ be the parametrization defined in (2.77). Let δ be a α-admissible weight with respect to φ in the sense of Definition 2.8.1 and define $\tilde{\delta} \in C^{0,\alpha}(B_1)$, as in (2.80). Let A be a α-Hölder continuous matrix satisfying (2.2), $f \in L^p(B_1, \delta^a)$, $F \in C^{0,\alpha}(B_1)$, $\psi \in C^{1,\alpha}(\Gamma \cap B_1)$*

and u be a weak solution to (2.8), in the sense of Definition 2.8.2. Let us suppose that

$$\|A\|_{C^{0,\alpha}(B_1)} + \|\tilde{\delta}\|_{C^{0,\alpha}(B_1)} + \|\varphi\|_{C^{1,\alpha}(\Sigma_0 \cap B_1;\mathbb{R}^n)} \leq L.$$

Then, $u \in C^{1,\alpha}(B_{1/2})$ and there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq c\big(\|u\|_{L^2(B_1,\delta^a)} + \|f\|_{L^p(B_1,\delta^a)} + \|F\|_{C^{0,\alpha}(B_1)} + \|\psi\|_{C^{1,\alpha}(\Gamma\cap B_1)}\big). \tag{2.82}$$

Moreover, denoting by $T_z\Gamma$ the tangent space to Γ at the point $z \in \Gamma$, we have that u satisfies the following boundary condition for every $z \in \Gamma \cap B_{1/2}$,

$$\begin{cases} \nabla u(z)\cdot\tau(z) = \nabla\psi(z)\cdot\tau(z), & \textit{for every } \tau(z) \in T_z\Gamma,\\ (A\nabla u + F)(z)\cdot\nu(z) = 0, & \textit{for every } \nu(z) \perp T_z\Gamma. \end{cases} \tag{2.83}$$

Since the proofs of Corollaries 2.8.3 and 2.8.4 are quite similar, we will only provide the proof of the second one, as it is more involved.

Proof of Corollary 2.8.4. Let us define the diffeomorphism Φ as in (2.78), which is of class $C^{1,\alpha}(B_1)$, by the assumption $\varphi \in C^{1,\alpha}(\Sigma_0 \cap B_1)$. Defining

$$\tilde{u}(x,y) := u\circ\Phi(x,y), \qquad \tilde{\psi}(x) = \psi\circ\Phi(x,0) \in C^{1,\alpha}(\Sigma_0\cap B_1).$$

we have that $\tilde{u} \in H^{1,a}(B_1)$ and $\tilde{u} = \tilde{\psi}$ on $\Sigma_0 \cap B_1$ in the sense of the traces. Since δ is an α-admissible weight in the sense of Definition 2.8.1, the conditions (2.79) and (2.80) implies that

$$\tilde{\delta}^a(x,y) = \frac{\delta(\Phi(x,y))^a}{|y|^a} \in C^{0,\alpha}(B_1), \quad \tilde{\delta}^a \geq \tilde{c}_0 > 0, \tag{2.84}$$

for some constant $\tilde{c}_0 > 0$, where $\tilde{\delta} = \delta\circ\Phi$.

Let $\phi \in C^\infty_c(B_1\setminus\Gamma)$ be a test function in (2.81). By taking the change of variables $z = \Phi(x,y)$ it follows that

$$0 = \int_{B_1} \delta^a\big(A\nabla u\cdot\nabla\phi - f\phi + F\cdot\nabla\phi\big)dz = \int_{B_1} |y|^a\big(\tilde{A}\nabla\tilde{u}\cdot\nabla\tilde{\phi} - \tilde{f}\tilde{\phi} + \tilde{F}\cdot\nabla\tilde{\phi}\big)dz,$$

where

$$\tilde{A} := \tilde{\delta}^a(J_\Phi^{-1})(A\circ\Phi)(J_\Phi^{-1})^\top, \quad \tilde{f} := \tilde{\delta}^a f\circ\Phi, \quad \tilde{F} := \tilde{\delta}^a(J_\Phi^{-1})F\circ\Phi, \quad \tilde{\phi} := \phi\circ\Phi. \tag{2.85}$$

By using (2.84), we have that

$$\tilde{A} \in C^{0,\alpha}(B_1) \text{ and satisfies (2.2)}, \quad \tilde{f} \in L^{p,a}(B_1), \quad \tilde{F} \in C^{0,\alpha}(B_1).$$

Hence, we have proved that $\tilde{u}$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a\tilde{A}\nabla\tilde{u}) = |y|^a\tilde{f} + \operatorname{div}(|y|^a\tilde{F}), & \text{in } B_1\setminus\Sigma_0,\\ \tilde{u} = \tilde{\psi}, & \text{on } \Sigma_0\cap B_1, \end{cases}$$

and $\tilde{u}$ satisfies the hypothesis of the Theorem 2.1.2. Hence, $\tilde{u}$ satisfies (2.6) and composing back with the diffeomorphism Φ^{-1} we get that u satisfies (2.82).

Eventually, let us prove that u satisfies the boundary condition (2.83). By Theorem 2.1.2, we have that $\tilde{u}$ satisfies the boundary condition

$$\nabla_x \tilde{u} = \nabla_x \tilde{\psi}, \quad (\tilde{A}\nabla\tilde{u} + \tilde{F}) \cdot e_{y_i} = 0, \quad \text{on } \Sigma_0 \cap B_{1/2}, \text{ for every } i = 1, \dots, n. \tag{2.86}$$

Since $\tilde{u}(z) = u(\Phi(z))$, one has that

$$\nabla\tilde{u}(z) = J_\Phi^\top(z)\nabla u(\Phi(z)).$$

and, noting $\Phi(x,0) = (x, \varphi(x)) \in \Gamma$, we have that

$$\nabla\tilde{u}(x,0) \cdot e_{x_j} = J_\Phi^\top(x,0)\nabla u(\Phi(x,0)) \cdot e_{x_j} = \nabla u(x,\varphi(x)) \cdot J_\Phi(x,0)e_{x_j}.$$

Thus, for every $j = 1, \dots, d-n$, and $(x,0) \in \Sigma_0 \cap B_{1/2}$, we have

$$\nabla u(x,\varphi(x)) \cdot J_\Phi(x,0)e_{x_j} = \nabla\psi(x,\varphi(x)) \cdot J_\Phi(x,0)e_{x_j}. \tag{2.87}$$

Next, recalling (2.84), (2.85) and (2.86), we find

$$\begin{aligned} 0 = \tilde{\delta}^{-a}(\tilde{A}\nabla\tilde{u} + \tilde{F})(x,0) \cdot e_{y_i} &= J_\Phi^{-1}(x,0)(A\nabla u + F)(x,\varphi(x)) \cdot e_{y_i} \\ &= (A\nabla u + F)(x,\varphi(x)) \cdot (J_\Phi^{-1})^\top(x,0)e_{y_i}. \end{aligned} \tag{2.88}$$

Finally, observing that the tangent space to Γ at the point $z = (x, \varphi(x))$ is given by

$$T_{(x,\varphi(x))}\Gamma := \{(\xi, J_\varphi(x)\xi) : \xi \in \mathbb{R}^{d-n}\},$$

and noting that

$$J_\Phi(x,0)e_{x_j} \cdot (J_\Phi^{-1})^\top(x,0)e_{y_i} = 0, \quad \text{for every } j = 1, \dots, d-n, \quad i = 1, \dots, n,$$

it follows that (2.87) and (2.88) implies that (2.83) holds true. This complete the proof. □

CHAPTER

THREE

REGULARITY FOR EQUATIONS DEGENERATING ON LOWER DIMENSIONAL MANIFOLDS

3.1 Introduction

The last chapter of this thesis is based on the two works [38, 37], written in collaboration with G. Cora and S. Vita. We study local regularity properties, Hölder $C^{0,\alpha}$ and Schauder $C^{1,\alpha}$ estimates, of weak solutions to

$$-\mathrm{div}(|y|^a A(x,y)\nabla u) = |y|^a f + \mathrm{div}(|y|^a F) \qquad \text{in } B_1 \subset \mathbb{R}^d. \tag{3.1}$$

Here $z = (x,y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$, $2 \le n \le d$ are two integers and $a \in \mathbb{R}$. The second order equation in divergence form above is uniformly elliptic far from a characteristic flat manifold of low dimension $0 \le d-n \le d-2$

$$\Sigma_0 = \{z = (x,y) \in \mathbb{R}^d \mid |y| = 0\},$$

and the weight term is a power of the distance to Σ_0; that is, $|y| = \mathrm{dist}(z, \Sigma_0)$. In other words, there exist $0 < \lambda \le \Lambda$ such that the symmetric d-dimensional matrix $A : B_1 \to \mathbb{R}^{d,d}$ satisfies

$$\lambda|y|^a|\xi|^2 \le |y|^a A(z)\xi \cdot \xi \le \Lambda|y|^a|\xi|^2 \qquad \text{for almost every } z \in B_1, \text{ for every } \xi \in \mathbb{R}^d.$$

The weak solutions to the above problem are elements of the weighed Sobolev space $H^{1,a}(B_1) := H^1(B_1, |y|^a dz)$ which satisfy

$$\int_{B_1} |y|^a A\nabla u \cdot \nabla\phi = \int_{B_1} |y|^a (f\phi - F \cdot \nabla\phi) \qquad \text{for every } \phi \in C_c^\infty(B_1). \tag{3.2}$$

The equation is satisfied *across* the thin manifold Σ_0, and this implies a formal homogeneous conormal condition at Σ_0

$$\lim_{|y|\to 0} |y|^{a+n-1}(A\nabla u + F) \cdot \frac{y}{|y|} = 0, \tag{3.3}$$

see Remark 3.4.2 for the precise meaning of the above expression. Notice also that, when $A = \mathbb{I}$ and $F = 0$, (3.3) corresponds to a vanishing weighted radial derivative with respect to the degenerate variables.

The weighted $H^{1,a}$-capacity of the thin manifold Σ_0 is key to understanding what kind of solutions one may face. Depending on the value of the parameter $a + n \in \mathbb{R}$, the local weighed capacity of the characteristic manifold is infinite when $a+n \leq 0$, positive and finite when $a+n \in (0,2)$ and zero when $a+n \geq 2$. Consequently, weak solutions must vanish at Σ_0 in the *supersingular* case $a+n \leq 0$, and naturally satisfy (3.3) in the *superdegenerate* case $a+n \geq 2$. Finally, in the *mid-range* case $a+n \in (0,2)$, many different boundary conditions can be prescribed at Σ_0, and this corresponds to inhomogeneous Dirichlet and inhomogeneous conormal boundary problems (the first one is addressed in the second chapter of this thesis).

We mostly work under the assumption $a+n > 0$ - which makes the weight term locally integrable - and deal with functions which solve (3.1) across Σ_0 in the sense of (3.2); that is, we study the homogeneous conormal boundary problem. We will see that the regularity for the inhomogeneous conormal problem in the mid-range $a+n \in (0,2)$ follows as a consequence of the theory for the homogeneous one. Concerning Dirichlet type boundary conditions, the inhomogeneous problem in the mid-range $a+n \in (0,2)$ is studied in the previous chapter of this thesis, while the analysis of the homogeneous case, when $a+n < 2$, is partially carried out via a boundary Harnack type principle.

The local regularity theory for weighted degenerate elliptic equations starts with the seminal works by Fabes-Jerison-Kenig-Serapioni [63–65]. Among their motivations for such a regularity theory, the connection with the study of fine qualitative properties of harmonic functions at the boundary of rough domains through quasiconformal mappings from a ball. In [65], the authors extend the De Giorgi-Nash-Moser theory to degenerate elliptic equations where the degeneracy/singularity is carried out by a weight term ω which arises either from quasiconformal mappings or belongs to the A_2-Muckenhoupt class. The latter indicates a combined local integrability property between the weight and its reciprocal. Their regularity theory includes Harnack inequalities and local Hölder continuity of solutions. These results rely primarily in the validity of a general functional framework in weighted Sobolev spaces comprehending Poincaré-Wirtinger and Sobolev inequalities. Later, the conditions which provide this setting have been summarized into four main properties of the weight, characterizing the so-called 2-admissible weights, see for instance [82] and later discussions in Section 3.2.6. The general regularity theory for weighted equations also ends with [65], since Hölder continuity is the optimal regularity if no further geometric assumption on the zero/infinity set $\Sigma_0 = \{\omega(z) = 0 \vee \omega(z) = \infty\}$ is done.

In recent years, there have been significant contributions to the study of degenerate elliptic equations where the weight behaves like a power of the distance from a $(d-1)$-dimensional manifold. The model equation is

$$-\mathrm{div}(y^a A(x,y)\nabla u) = RHS \qquad \text{in } B_1^+ \subset \mathbb{R}^d_+. \tag{3.4}$$

Here $z = (x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ = \mathbb{R}^d_+$, $B_1^+ = B_1 \cap \{y > 0\}$, $a \in \mathbb{R}$ and $\Sigma_0 = \{y = 0\}$ is an hyperplane, i.e. it has codimension $n = 1$. This theory has profound connections with the edge calculus developed in [105, 106], see also [97, 80, 81] and the references therein. However, the recent interest in this problem primarily relies in the connection with fractional Laplacian due to an extension theory via Dirichlet-to-Neumann maps [28].

Concerning the general equation (3.4), it is crucial to comment here its possible connection with Grushin operators. As it is highlighted in [28], a change of coordinates moves the operator into a Grushin type one. However, this is true only for a particular small range of the exponent a, and in any case both the transformation and the power term in the Grushin operator are not smooth. So, hypoellipticity and the Hörmander regularity theory are not valid in general. In fact, the indicial root y^{1-a} always solves the homogeneous ($RHS = 0$) isotropic ($A = \mathbb{I}$) equation (3.4) whenever $a < 1$. This prevents general *harmonic functions for the degenerate operator* from being smooth.

However, the regularity of the solutions may strongly improve by imposing certain boundary conditions at Σ_0. In fact, solutions with a homogeneous conormal boundary condition at Σ_0 enjoy a Schauder theory in $C^{k,\alpha}$ spaces whenever $a > -1$, which in turns leads to smoothness if data are smooth. The literature on the regularity is extensive. We cannot provide a comprehensive list of all contributions on this topic here; however, we refer to the following works [138, 139, 143, 7, 8, 59, 60, 29].

Let us consider now the higher codimensional case presented in the present chapter. The study of (3.1) can be seen at a first glance as a natural continuation of the codimension 1 theory. However, there are many other motivations for developing such a regularity theory. We would like to focus our attention on some topics which strongly relates to the degenerate equation (3.1) and may have further developments due to our results and approach. Some of them are known topics, and have a literature: harmonic maps with prescribed singularities in general relativity, the Dirichlet problem and harmonic measure on lower dimensional boundaries. Some other connections are new: critical points for Caffarelli-Kohn-Nirenberg inequalities, higher codimensional extensions of fractional Laplacian, very thin free boundary problems. We will spend some other words on these topics in the last part of this introduction. Finally, we would like to mention some unique continuation results for Grushin-type operators which are degenerate on lower dimensional manifolds, see for instance [1, 72, 153] and many references therein.

3.1.1 Main results

The main results we are presenting here concern the homogeneous conormal problem for (3.1) in the range $a+n > 0$. The solutions we are referring to are elements of $H^{1,a}(B_1)$ which weakly solve (3.2). Our main objectives are certain Hölder $C^{0,\alpha}$ and Schauder $C^{1,\alpha}$ local regularity estimates up to Σ_0 for this class of weak solutions. In order to state precisely our main results, we need to be more specific regarding the uniform ellipticity properties of the variable coefficient matrix A. The latter is a symmetric d-dimensional matrix satisfying the *global* uniform ellipticity condition

$$\lambda|\xi|^2 \leq A(z)\xi\cdot\xi \leq \Lambda|\xi|^2, \qquad \text{for a.e. } z\in B_1 \text{ and for all } \xi\in\mathbb{R}^d, \tag{3.5}$$

for some ellipticity constants $0 < \lambda \leq \Lambda$. Moreover, A satisfies the *restricted-to-*Σ_0 uniform ellipticity condition

$$\lambda_*|\xi|^2 \leq A_3(x,0)\xi\cdot\xi \leq \Lambda_*|\xi|^2, \qquad \text{for a.e. } (x,0)\in B_1\cap\Sigma_0 \text{ and for all } \xi\in\mathbb{R}^n, \tag{3.6}$$

with ellipticity constants $0 < \lambda \leq \lambda_* \leq \Lambda_* \leq \Lambda$, where A_3 is the $(n\times n)$-dimensional block located in the lower-right corner of the matrix A, see (3.18). Then, let us introduce the exponent

$$\alpha_* = \alpha_*(a,n,\lambda_*/\Lambda_*) = \frac{2-a-n+\sqrt{(2-a-n)^2+4\mu_*}}{2}, \tag{3.7}$$

with

$$\mu_* = \mu_*(a,n,\lambda_*/\Lambda_*) = \begin{cases} \left(\dfrac{\lambda_*}{\Lambda_*}\right)^{\frac{|a|}{2}}(n-1) & \text{if } n\geq 3\,, \\ \left(\dfrac{4}{\pi}\arctan\left(\dfrac{\lambda_*}{\Lambda_*}\right)^{\frac{|a|}{4}}\right)^2 & \text{if } n=2\,. \end{cases}$$

Notice that both α_* and μ_* are monotone increasing with respect to $0<\lambda_*/\Lambda_*\leq 1$. Notice also that in case $\lambda_*/\Lambda_*=1$ (which happens in particular if $A=\mathbb{I}$), $\mu_*=n-1$ which corresponds to the first positive eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Then, our first main result is the following.

Theorem 3.1.1 (Hölder $C^{0,\alpha}$ estimate)**.** *Let $a+n>0$, $p>(d+a_+)/2$, $q>d+a_+$. Let A be a uniformly elliptic matrix satisfying (3.5) and (3.6) with constants $0<\lambda\leq\lambda_*\leq\Lambda_*\leq\Lambda$. Let $\alpha_*=\alpha_*(n,a,\lambda_*/\Lambda_*)$ be defined as in (3.7). Let*

$$\alpha\in(0,1)\cap(0,\alpha_*)\cap(0,2-(d+a_+)/p]\cap(0,1-(d+a_+)/q]. \tag{3.8}$$

Let A be continuous with

$$\|A\|_{C^{0,\omega}(B_1)}:=\|A\|_{L^\infty(B_1)}+\sup_{\substack{z,\zeta\in B_1\\ z\neq\zeta}}\frac{|A(z)-A(\zeta)|}{\omega(|z-\zeta|)}\leq L\qquad\text{for some modulus of continuity } \omega.$$

Let $f\in L^{p,a}(B_1)$, $F\in L^{q,a}(B_1)^d$ and u be a weak solution to (3.1) in B_1.

Then, $u\in C^{0,\alpha}_{\rm loc}(B_1)$. Moreover, there exists a constant $C>0$ depending only on d, n, a, λ, Λ, p, q, L and α such that

$$\|u\|_{C^{0,\alpha}(B_{1/2})}\leq C\big(\|u\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{L^{q,a}(B_1)}\big).$$

Let us notice here that Hölder regularity for some small implicit exponent follows by [65] in the A_2-Muckenhoupt range $a+n\in(0,2n)$ or by the 2-admissibility of the weight (see [82]) even in the full range $a+n>0$ again by [65], once the 2-admissibility condition is established in Section 3.2.6. The latter is true even when the variable coefficient matrix is only assumed to be bounded measurable. However, the above result proves an estimate which comes with an explicit Hölder exponent. As we will see, this additional information is due to the homogeneity property of the weight term together with the peculiar geometry of its nodal set Σ_0.

Then, our second result, is the following Schauder estimate, which holds true any time the exponent α_* exceeds 1.

Theorem 3.1.2 (Schauder $C^{1,\alpha}$ estimate)**.** *Let $a+n>0$, $p>d+a_+$. Let A be a uniformly elliptic matrix satisfying (3.5) and (3.6) with constants $0<\lambda\leq\lambda_*\leq\Lambda_*\leq\Lambda$. Assume that $\alpha_*=\alpha_*(n,a,\lambda_*/\Lambda_*)$ defined as in (3.7) is such that $\alpha_*>1$. Let*

$$\alpha\in(0,1)\cap(0,\alpha_*-1)\cap(0,1-(d+a_+)/p]. \tag{3.9}$$

Let A be α-Hölder continuous with $\|A\|_{C^{0,\alpha}(B_1)}\leq L$, $f\in L^{p,a}(B_1)$, $F\in C^{0,\alpha}(B_1)$ and u be a weak solution to (3.1) in B_1.

Then, $u\in C^{1,\alpha}_{\rm loc}(B_1)$ and satisfies

$$(A\nabla u+F)\cdot e_{y_i}=0,\qquad \text{on } \Sigma_0\cap B_1,\quad \text{for every } i=1,\dots,n. \tag{3.10}$$

Moreover, there exists a constant $C>0$ depending only on d, n, a, λ, Λ, p, L and α such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})}\leq C\big(\|u\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{C^{0,\alpha}(B_1)}\big). \tag{3.11}$$

Let us now comment on the above results and explain the techniques involved. The main idea is to take advantage of a ε-regularization-approximation scheme, which aims to address the loss of uniform ellipticity on the characteristic manifold by approximating with uniformly elliptic problems. In this way, the estimates become available for the ε-approximating problems, and the goal is to prove their uniformity (stability) as the regularization parameter $\varepsilon \to 0$. Returning to the codimension 1 case, in [138] the regularization proposed is at the level of the weight term. Specifically, replacing $|y|^a$ with $\rho_\varepsilon^a(y) = (\varepsilon^2 + |y|^2)^{\frac{a}{2}}$ one immediately obtains a uniformly elliptic problem, and stability can be established as $\varepsilon \to 0$. This approach is not the most effective in this context. To approximate a given solution with a possible boundary condition at Σ_0, the same boundary condition must be prescribed at the level of the regularized equation. However, the local $H^1(\rho_\varepsilon^a)$-capacity of Σ_0 - which coincides with the classical local unweighted H^1-capacity - is always zero. As a result, no boundary condition can be imposed on the thin set. This fact becomes particularly evident in the mid-range $a + n \in (0, 2)$, especially in case of inhomogeneous conditions.

Next, we propose a general strategy suitable for any boundary value problem and valid across different capacitary ranges. The regularization we introduce is performed at the domain level, using ε-perforations around Σ_0. In the isotropic case $A = \mathbb{I}$, the approximating problems are described as follows

$$\begin{cases} -\mathrm{div}(|y|^a \nabla u) = |y|^a f + \mathrm{div}(|y|^a F) & \text{in } B_1 \setminus \Sigma_\varepsilon \\ (\nabla u + F) \cdot \nu = 0 & \text{on } B_1 \cap \partial\Sigma_\varepsilon, \end{cases} \tag{3.12}$$

where

$$\Sigma_\varepsilon = \{|y| \le \varepsilon\} \qquad \text{and} \qquad \partial\Sigma_\varepsilon = \{|y| = \varepsilon\}.$$

As one can see, the conormal boundary condition is now prescribed on a codimension 1 boundary that shrinks toward Σ_0. Given a particular solution of the limiting problem, one can construct an approximating sequence of solutions for the regularized problems as $\varepsilon \to 0$. If the regularity estimates for (3.12) are shown to be uniform in ε, the same regularity is transferred to the limit.

Adding variable coefficients makes things harder, as anisotropy directly affects the approximation. As we will see, the stability of the estimates relies on Liouville theorems on the complementary of a hole, after blow-up. For general variable coefficients, adjusting the shape of this hole to the anisotropy proves to be convenient, and this requires the domain to be perforated accordingly at a macroscopic scale; that is,

$$\Sigma_\varepsilon^A = \{A_3^{-1}(x, y)y \cdot y \le \varepsilon^2\} \qquad \text{and} \qquad \partial\Sigma_\varepsilon^A = \{A_3^{-1}(x, y)y \cdot y = \varepsilon^2\}.$$

However, a preliminary regularization of the coefficients through convolution with mollifiers is also needed at this point. In fact, the approximating problems

$$\begin{cases} -\mathrm{div}(|y|^a A \nabla u) = |y|^a f + \mathrm{div}(|y|^a F) & \text{in } B_1 \setminus \Sigma_\varepsilon^A \\ (A\nabla u + F) \cdot \nu = 0 & \text{on } B_1 \cap \partial\Sigma_\varepsilon^A, \end{cases} \tag{3.13}$$

enjoy the desired Hölder $C^{0,\alpha}$ and Schauder $C^{1,\alpha}$ estimates only requiring additional regularity of coefficients, respectively C^1 and $C^{1,\alpha}$, since the latter ensures the right regularity of the shrinking boundaries (see Remark 3.3.2). This is suboptimal when compared with the requirements stated in Theorems 3.1.1 and 3.1.2. However, this additional regularity is necessary to ensure stability of the estimates with respect to domain perforation. On the other hand, stability with respect to mollification does not require any perforation and holds true under the optimal requirements on coefficients; that is, respectively the C^0 and $C^{0,\alpha}$ regularity imposed by the natural scaling of the equation.

The validity of Theorems 3.1.1 and 3.1.2 is based on the following main result.

Theorem 3.1.3 (Stable regularity estimates in perforated domains)**.** *Let $a+n>0$. Let A be a uniformly elliptic matrix satisfying* (3.5) *and* (3.6) *with constants $0<\lambda\leq\lambda_*\leq\Lambda_*\leq\Lambda$. Let $\alpha_*=\alpha_*(n,a,\lambda_*/\Lambda_*)$ be defined as in* (3.7)*. Then the following points hold true:*

i) Let $p>(d+a_+)/2$, $q>d+a_+$. Let $\alpha\in(0,1)$ satisfying (3.8)*. Let A be C^1 with $\|A\|_{C^{1,\omega}(B_1)}\leq L$ for some modulus of continuity ω, $f\in L^{p,a}(B_1)$ and $F\in L^{q,a}(B_1)^d$. Then there exists $\varepsilon_0<1$ and a constant $C>0$ depending only on d, n, a, λ, Λ, p, q, L, ε_0 and α such that for every $\varepsilon\in(0,\varepsilon_0)$ and for every solution u_ε to* (3.13) *it holds*

$$\|u_\varepsilon\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon^A)}\leq C\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon^A)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{L^{q,a}(B_1)}\big)\,.$$

ii) Let $p>d+a_+$. Let assume that $\alpha_*>1$. Let $\alpha\in(0,1)$ satisfying (3.9)*. Let A be $C^{1,\alpha}$ with $\|A\|_{C^{1,\alpha}(B_1)}\leq L$, $f\in L^{p,a}(B_1)$ and $F\in C^{0,\alpha}(B_1)$. Then there exists $\varepsilon_0<1$ and a constant $C>0$ depending only on d, n, a, λ, Λ, p, L, ε_0 and α such that for every $\varepsilon\in(0,\varepsilon_0)$ and for every solution u_ε to* (3.13) *it holds*

$$\|u_\varepsilon\|_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon^A)}\leq C\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon^A)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{C^{0,\alpha}(B_1)}\big).$$

Moreover, for every points $z\in\partial\Sigma_\varepsilon^A\cap B_{1/2}$ and for every $i=1,\ldots,n$, it holds

$$|(A\nabla u_\varepsilon+F)(z)\cdot e_{y_i}|\leq C\varepsilon^\alpha\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon^A)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{C^{0,\alpha}(B_1)}\big). \tag{3.14}$$

First of all, we would like to point out that the above result remains valid even if the additional C^1 or $C^{1,\alpha}$ regularity is only assumed for the block A_3, see Remark 3.3.1. However, to simplify the notation, we assume the higher regularity for all entries of the matrix A.

Let us also observe that the quantitative condition (3.14) is the key for obtaining the effective conormal boundary condition (3.10) in Theorem 3.1.2, which is a much stronger information compared to the weighted conormal condition (3.3) enjoyed by weak solutions.

The above result, as far as we know, is new even in the case of the Laplacian ($a=0$ and $A=\mathbb{I}$), and provides stability of classical Hölder estimates in perforated domains with Neumann boundary condition on the boundary of the hole. Since our estimates can be extended to second order elliptic equations with general lower order terms, Theorem 3.1.3 *i)* has a remarkable link with the quantitative spectral stability for the Laplacian in domains with holes with prescribed Neumann boundary conditions, see for instance [67, 121, 125] and many references therein. In particular, we can imply stable α-Hölder bounds (for any $\alpha\in(0,1)$) for eigenfunctions in the Neumann-perforated domains, see Remark 3.6.1.

Notice that the stability of the Schauder estimate can not be valid for the Laplacian, since the effective conormal condition that would follow, i.e. $\nabla_y u=0$ on Σ_0, is not a general property of harmonic functions.

In the spirit of the work of Simon [136], the stable estimates are obtained, in both the regularization procedures (Theorem 3.1.3 and Proposition 3.7.1), by a contradiction argument involving a fine blow-up procedure and the classification of entire profiles. The latter is expressed in terms of rigidity results; that is, Liouville type theorems for both uniformly and degenerate elliptic problems on the blow-up domain, which can be the entire space, the half-space, or the space minus an unbounded cylinder around Σ_0. The latter result is of independent interest and can be stated as follows.

Theorem 3.1.4 (Liouville)**.** *Let $a+n>0$, $\varepsilon \geq 0$, and let A be a constant coefficient d-dimensional matrix which satisfies* (3.5)*. Let us denote*

$$\gamma_1^+ = \gamma_1^+(a,n,A) = \frac{2-a-n+\sqrt{(2-a-n)^2+4\mu_1}}{2},$$

where $\mu_1 = \mu_1(a,n,A)$ is a constant, which will be precisely defined in Lemma 3.5.2. Let u be an entire solution (see Definitions 3.4.1 and 3.4.3) to

$$\begin{cases} -\mathrm{div}(|y|^a A\nabla u) = 0 & \text{in } \mathbb{R}^d \setminus \Sigma_\varepsilon^A \\ |y|^a A\nabla u \cdot \nu = 0 & \text{on } \partial\Sigma_\varepsilon^A, \end{cases} \tag{3.15}$$

and assume that there exist two constants $c>0$ and $\gamma \in (0, \gamma_1^+)$ such that

$$|u(z)| \leq c(1+|z|^\gamma)\,. \tag{3.16}$$

If $\gamma < 2$, then u is linear. Moreover, if $\gamma < 1$, then u is constant.

The above result is derived by using spherical coordinates with respect to the degenerate variable y and expressing the solution through a Fourier-type decomposition. This approach requires the anisotropy of the coefficients to align with the geometry of the hole, as radial-in-y (axial) symmetry must emerge after a linear change of coordinates.

Let us elaborate on the regularity results in connection with the Liouville theorem discussed above. The exponent α_* is a threshold for local regularity. It represents the lower bound of the possible homogeneity degrees γ_1^+ that one may encounter at the microscopic scale of the blow-up. Then, we would like to highlight the unexpected similarity between α_* and the optimal Hölder continuity exponent of solutions to equations with bounded measurable (possibly discontinuous) coefficients, see [123]. In the present case, the (restricted-to-Σ_0) ellipticity ratio λ_*/Λ_* also plays a role in the expression for α_*, reducing its value compared to the isotropic case $A = \mathbb{I}$. It remains an open question whether this interference is intrinsic or not, as we currently lack examples of solutions with such reduced regularity.

Let us now comment on the possible higher regularity of solutions. As a becomes more and more negative, depending also on an increasing codimension n, α_* can be made larger than any integer $k \geq 2$. Therefore, $C^{k,\alpha}$ regularity of solutions is expected for any $k \geq 2$, under suitable assumptions on the data and within certain ranges of (a,n).

Unfortunately, we currently lack a general strategy for higher order Schauder estimates, which remains an open problem. On one hand, the regularity estimates obtained in Theorems 3.1.1 and 3.1.2 cannot be directly iterated on derivatives, since the operator is translation invariant only in the tangential directions to Σ_0 (the x variables). This implies that the operator commutes only with the derivatives in $x \in \mathbb{R}^{d-n}$ and not in $y \in \mathbb{R}^n$. However, if one knows additionally that the solution is axially symmetric with respect to Σ_0 (radial-in-y), i.e. $u(x,y) = u(x,|y|)$, the iteration procedure works and one can prove smoothness at least in the isotropic homogeneous case, see Theorem 3.9.3.

On the other hand, one could attempt to construct a regularization-approximation scheme for any order k, by smoothing the data, perforating the domain and finally shrinking at the characteristic manifold. However, this strategy also fails, as the stability of the $C^{2,\alpha}$ estimate with respect to ε-perforation does not hold, see Remark 3.6.2.

The inhomogeneous conormal problem

In the mid-range $a+n\in(0,2)$, the positive weighted capacity of the thin set Σ_0 enables the imposition of inhomogeneous conormal boundary conditions; that is, solutions to

$$\begin{cases} -\operatorname{div}(|y|^a\nabla u)=0, & \text{in } B_1,\\ -\lim\limits_{|y|\to 0}|y|^{a+n-1}\nabla u\cdot\dfrac{y}{|y|}=g(x), & \text{on } \Sigma_0\cap B_1,\end{cases}$$

are well defined (see Definition 3.10.1) as elements of $H^{1,a}(B_1)$ which satisfies

$$\int_{B_1}|y|^a\nabla u\cdot\nabla\phi dz=\omega_n\int_{\Sigma_0\cap B_1}g(x)\phi(x,0)dx,\qquad \text{for every } \phi\in C^\infty_c(B_1).$$

Then, we prove some regularity results for solutions of the above problem, as a corollary of our main theorems, together with the regularity of axially symmetric solutions, see Proposition 3.10.2.

Curved characteristic thin manifolds

As a second corollary of our main theorems, we can deal with the case of curved regular characteristic thin manifolds. Let $2\le n<d$ and consider a $(d-n)$-dimensional $C^{1,\alpha}$ manifold Γ locally embedded in $\mathbb{R}^d$ with $\alpha\in[0,1)$. Then Corollaries 3.8.4 and 3.8.5 provide the extension of the main Theorems 3.1.1 and 3.1.2 to weak solutions (see Definition 3.8.3) to

$$-\operatorname{div}(\delta^a A\nabla u)=\delta^a f+\operatorname{div}(\delta^a F),\quad \text{in } B_1,$$

where the weight δ behaves as a particular distance function to Γ, chosen in accordance to the local parametrization of the thin manifold. See Definition 3.8.1 for detailed assumptions on the *defining function* δ.

The homogeneous Dirichlet problem

As another corollary of our main theorems, whenever $a+n<2$, one can provide regularity results for solutions to the homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div}(|y|^a\nabla u)=0 & \text{in } B_1\setminus\Sigma_0\\ u=0 & \text{on } B_1\cap\Sigma_0.\end{cases}\tag{3.17}$$

As we will explain later in the introduction, the problem above has connections with the work of Nguyen on harmonic maps with prescribed singularities in general relativity [119]. Moreover, it has also profound connections with the works of David-Feneuil-Mayboroda on the Dirichlet problem on lower dimensional boundaries [45, 46], see also [43, 47].

In the same spirit of [139], as $a+n<2$ one can provide regularity results for solutions u to (3.17) by considering the ratio $w=u/u_0$ between u and the characteristic Dirichlet solution

$$u_0(y)=|y|^{2-a-n},$$

which solves again (3.17). The ratio w is a solution of a degenerate problem $-\mathrm{div}(|y|^b\nabla w)=0$ with exponent $b=4-a-2n$ which lies in the superdegenerate range since $b+n=4-a-n>2$. Hence, $C^{0,\alpha}$ and $C^{1,\alpha}$ estimates for w are implied by Theorems 3.1.1 and 3.1.2, see Corollary 3.11.2. This improves some results in [119], and also slightly improves some regularity results of the second chapter of this thesis, ensuring the sharp C^{2-a-n} regularity of solutions under additional requirements on the codimension n. For instance, in the case $a+n=1$ and whenever $n\geq 4$, solutions to (3.17) are provided to be Lipschitz continuous, see Remark 3.11.3.

Higher codimensional extensions of fractional Laplacian

Motivated by [28], one might wonder whether it is possible to extend functions u defined in $\mathbb{R}^{d-n}$ having a well-defined s-fractional Laplacian ($s\in(0,1)$) to the whole of $\mathbb{R}^d$, introducing n additional variables. Under the assumption $d-n>2s$, which allows for the definition of suitable energy spaces, in Section 3.10 we show that such an extension is given by the convolution $u*P$ with the Poisson-type kernel

$$P(x,y)=P(|x|,|y|)=\frac{\Gamma(\frac{d-n+2s}{2})}{\pi^{\frac{d-n}{2}}\Gamma(s)}\frac{|y|^{2s}}{(|x|^2+|y|^2)^{\frac{d-n+2s}{2}}},\qquad (x,y)\in\mathbb{R}^{d-n}\times\mathbb{R}^n.$$

The extension is a radial-in-y solution (axisymmetric with respect to $\Sigma_0=\mathbb{R}^{d-n}$) to

$$\begin{cases}-\mathrm{div}(|y|^a\nabla u)=0 & \text{in } \mathbb{R}^d\setminus\Sigma_0\\ -\lim\limits_{|y|\to 0}|y|^{a+n-1}\nabla u\cdot\dfrac{y}{|y|}=d_{a,n}(-\Delta)^s u(x,0) & \text{on } \Sigma_0,\end{cases}$$

where $a+n=2-2s\in(0,2)$, and $d_{a,n}$ is an explicit positive constant. As it can be seen through the change of variables $|y|=r$, we recover exactly the classical Caffarelli-Silvestre s-harmonic extension.

Some further motivations and applications

In the final part of this introduction, we aim to elaborate further on some noteworthy motivations and significant applications of our theory.

Harmonic maps with prescribed singularities in general relativity

The study of axially symmetric stationary multi-black-hole configurations and the force between co-axially rotating black holes involves an analysis on the boundary regularity of the reduced singular harmonic maps, see [148, 149, 93–95]. In [119], this analysis is carried out by considering those harmonic maps as solutions to some homogeneous divergence systems of partial differential equations with singular coefficients. Then, the model single equation of the system is given by

$$-\mathrm{div}(\omega(z)\nabla u)=0\qquad \text{in } B_1\setminus\Sigma_0,$$

where Σ_0 is a $(d-n)$-dimensional submanifold with $n\geq 2$, ω is a weight that satisfies

$$\frac{1}{C}\mathrm{dist}(z,\Sigma_0)^a\leq\omega(z)\leq C\mathrm{dist}(z,\Sigma_0)^a$$

for some negative power $a < 0$ and some constant $C > 0$. It is worth noting here that the weight's singularity, caused by the negative exponent a, does not confine us to a particular capacitary range. Instead, all three ranges may be crossed since $a + n$ might be any real number.

The Dirichlet problem and harmonic measure on lower dimensional boundaries

In a series of works, see [43, 45–47] and many references therein, David-Feneuil-Mayboroda and collaborators start an elliptic theory on weighted non uniformly elliptic operators which naturally catch the capacity of very thin boundaries. The peculiar operator considered in these works has a weight behaving as $\omega(z) = \mathrm{dist}(z, \Sigma_0)^{1-n}$ where Σ_0 is a $(d - n)$-dimensional manifold, i.e. the power of the weight belongs to the mid-range since $a + n = 1$. Notice that with this choice of the exponent and when $n = 1$ one recovers the classic Laplacian. The main target of the works quoted above is the study of the Dirichlet problem on the lower dimensional boundary, the associated harmonic measure, and its reciprocal absolute continuity with the $(d - n)$-dimensional Hausdorff measure [45], in the spirit of Dahlberg [42]. Although in general the interest in these works relies on rough boundaries, i.e. Lipschitz or less, if the manifold is more regular, C^1 or more, one may infer almost Lipschitz type regularity for the solutions of the inhomogeneous Dirichlet problem (see Chapter 2), or even the sharp Lipschitz regularity for the homogeneous one when $n \geq 4$, see Corollary 3.11.2 and Remark 3.11.3.

Critical points for Caffarelli-Kohn-Nirenberg inequalities

In the particular case $n = d$, semilinear equations involving degenerate weights appear in connection with Caffarelli-Kohn-Nirenberg inequalities [24], i.e. for certain values of α, β, p equations like

$$-\mathrm{div}(|z|^{-2\alpha}\nabla u) = |z|^{-p\beta}|u|^{p-2}u.$$

Here $-2\alpha + n > 2$, and so the weight is superdegenerate and solutions naturally satisfy a homogeneous conormal boundary condition at $\Sigma_0 = \{0\}$. Even if the minimizers in the space $\mathbb{R}^d$ are classified and they are bubbles, the study of the above degenerate equation allows us to imply regularity properties for sign-changing solutions too and for minimizers in domains attaining the singularity at the boundary of a domain [35].

Very thin free boundary problems

Due to the natural property of these degenerate operators of capturing the capacity of very thin sets in the mid-range $a + n \in (0, 2)$, it is possible to formulate obstacle-type free boundary problems with an obstacle that has arbitrary dimension $d - n \in \{0, \dots, d - 2\}$. In other words, it is possible to minimize the energy functional

$$\int_{B_1} |y|^a |\nabla u|^2 \, dz,$$

where u satisfies a trace condition $u = g$ on ∂B_1 and the obstacle condition $u \geq \psi$ on the $(d - n)$-dimensional set Σ_0. We refer to the seminal works of Caffarelli [22, 23] for classical obstacle problems and to the more recent [68], where the authors propose an alternative definition of a *very thin obstacle problem* (obstacles of dimension $d - 2$), related to the Caffarelli-Silvestre extension operator in the regime $a \in (-1, 0)$.

3.1.2 Organization of the chapter

In Section 3.2 we introduce the weighted functional framework for the problem, and we provide many functional inequalities. In Section 3.3, we extend the results of the previous section to a setting for perforated domains. In Section 3.4 we introduce the notion of solutions, and we provide some approximation lemmas for both the regularization procedures. Moreover, we provide stable L^∞ bounds of solutions in perforated domains. Section 3.5 is dedicated to the proof of the main Liouville Theorem 3.1.4. Section 3.6 is the main core of the chapter: it contains the proofs of the main $C^{0,\alpha}$ and $C^{1,\alpha}$ stable estimates in perforated domains, i.e. Theorem 3.1.3. In Section 3.7 we prove the a priori estimates of Proposition 3.7.1, which imply stable estimates with respect to standard smoothing of data. This leads to the proof of the main Theorems 3.1.1 and 3.1.2. Then, in Section 3.8 we generalize our results to the case of curved characteristic thin manifolds. In Section 3.9 we improve the regularity of axially symmetric solutions, achieving smoothness. In Section 3.10 we extend the regularity theory to the case of inhomogeneous conormal boundary problems in the mid-range. In Section 3.11 we prove some regularity results for the homogeneous Dirichlet problem by a boundary Harnack type principle. Finally, in Section 3.12 we establish the connection with fractional Laplacian through Dirichlet-to-Neumann maps, as a higher codimensional analogue of the Caffarelli-Silvestre extension theory. In Appendix 3.A we collect many important technical results concerning the geometry of the perforated domains.

3.1.3 Notation

We establish the notation that will be used throughout the chapter.

Let $d, n \in \mathbb{N}$ be two integers such that $2 \leq n \leq d$. Let us consider coordinates $z = (x, y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$. We denote by $\{e_{z_\ell}\}$, where $\ell = 1, \ldots, d$, the canonical basis of $\mathbb{R}^d$. To distinguish between the variables x and y, we will often denote the basis as $\{e_{x_j}, e_{y_i}\}$, where $j = 1, \ldots, d-n$ and $i = 1, \ldots, n$. For a vector $G = (G_x, G_y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$, we write $G = G_1 + G_2$, where $G_1 = (G_x, 0)$ and $G_2 = (0, G_y)$. We express d-dimensional symmetric uniformly elliptic matrix $A : \Omega \to \mathbb{R}^{d,d}$ in block form as

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^\top & A_3 \end{pmatrix}, \tag{3.18}$$

where $A_1 : \Omega \to \mathbb{R}^{d-n,d-n}$, $A_2 : \Omega \to \mathbb{R}^{d-n,n}$ and $A_3 : \Omega \to \mathbb{R}^{n,n}$. For $m \in \mathbb{N}$ we denote by $B_R^m(\zeta)$ the open m-dimensional ball of radius $R > 0$ centered at $\zeta \in \mathbb{R}^m$. To ease the notation, we simply write $B_R^m = B_R^m(0)$ when $\zeta = 0$, and $B_R(\zeta) = B_R^d(\zeta)$, when the dimension $m = d$. In particular, $B_R = B_R^d(0)$. We write $0 < \varepsilon \ll 1$ to denote that there exists a small $\varepsilon_0 > 0$ such that $\varepsilon \in (0, \varepsilon_0)$.

3.2 Functional setting

3.2.1 Weighted Sobolev spaces

Given a smooth domain $\Omega \subseteq \mathbb{R}^d$, we define the Sobolev space $H^{1,a}(\Omega) = H^1(\Omega, |y|^a dz)$ as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{H^{1,a}(\Omega)} = \left(\int_\Omega |y|^a \left(|u|^2 + |\nabla u|^2\right) dz \right)^{1/2}. \tag{3.19}$$

Here, $C^\infty(\overline{\Omega})$ consists of the restrictions to $\overline{\Omega}$ of smooth functions in $\mathbb{R}^d$. For our purposes, Ω will be mostly a ball B_R with radius $R > 0$. Therefore, from now on we will focus on the space $H^{1,a}(B_R)$. Moreover, we say that $u \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^d)$ if $u \in H^{1,a}(B_R)$ for every $R > 0$.

Next, we define the homogeneous space $H^{1,a}_0(B_R)$ as the completion of $C^\infty_c(B_R)$ with respect to the norm given above. In light of the Poincaré inequality (see Proposition 3.2.3), $H^{1,a}_0(B_R)$ can be equivalently defined as the completion of $C^\infty_c(B_R)$ with respect to the Dirichlet seminorm

$$[u]_{H^{1,a}(B_R)} = \|\nabla u\|_{L^{2,a}(B_R)} = \Big(\int_{B_R} |y|^a |\nabla u|^2\, dz\Big)^{1/2}.$$

The weight term $|y|^a$ is locally integrable at Σ_0 whenever $a+n > 0$. Conversely, when $a+n \le 0$ the weight is *supersingular*; that is, it is not integrable at Σ_0. As we will see in Proposition 3.2.5, this lack of integrability forces elements in the Sobolev space to vanish on the characteristic manifold in order to maintain finite energy.

We recall that a weight $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\omega \ge 0$ is said to belong to the A_2-Muckenhoupt class if and only if

$$M_\omega := \sup_{z_0\in\mathbb{R}^d,\, R>0} \Big(\frac{1}{|B_R(z_0)|}\int_{B_R(z_0)} \omega\, dz\Big)\Big(\frac{1}{|B_R(z_0)|}\int_{B_R(z_0)} \omega^{-1}\, dz\Big) < \infty\,.$$

In our case, the weight $\omega = |y|^a$ belongs to the A_2-Muckenhoupt class if and only if $0 < a+n < 2n$.

Proposition 3.2.1 (A_2-Muckenhoupt weight)**.** *Let $0 < a+n < 2n$. Then the weight $|y|^a$ belongs to the A_2-Muckenhoupt class.*

Proof. Since for every $z_0 = (x_0, y_0) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$ it holds $B_R(z_0) \subset B^{d-n}(x_0) \times B^n(y_0)$, we have

$$M_\omega \le c \sup_{y_0\in\mathbb{R}^n,\, R>0} R^{-2n} \int_{B^n_R(y_0)} |y|^a dy \int_{B^n_R(y_0)} |y|^{-a} dy\,,$$

for a constant $c > 0$ depending only on d, n. To complete the proof, we show that

$$R^{-2n} \int_{B^n_R(y_0)} |y|^a dy \int_{B^n_R(y_0)} |y|^{-a} dy \le c\,, \qquad \text{for every } y_0 \in \mathbb{R}^n, \quad R > 0\,, \tag{3.20}$$

for a constant $c > 0$ not depending either on R or y_0.

We distinguish two cases. If $|y_0| \le 2R$, then $B^n_R(y_0) \subset B^n_{3R}$. Thus

$$\int_{B^n_R(y_0)} |y|^a dy \le \int_{B^n_{3R}} |y|^a dy = c\int_0^{3R} r^{a+n-1} dr = cR^{a+n}\,,$$

where in the last equality we used that $a+n > 0$. Moreover, using that $a+n < 2n$, we obtain

$$\int_{B^n_R(y_0)} |y|^{-a} dy \le cR^{-a+n}\,,$$

thus showing that (3.20) holds in this case.

On the other hand, if $|y_0| > 2R$ then $|y_0|/2 \le |y| \le 3|y_0|/2$ for any $y \in B^n_R(y_0)$. Hence we get

$$\int_{B^n_R(y_0)} |y|^a dy \int_{B^n_R(y_0)} |y|^{-a} dy \le cR^{2n}\,.$$

Therefore, (3.20) holds also in this case. □

3.2.2 Hardy-Poincaré and Poincaré inequalities

The following Hardy-Poincaré-type inequality is well known (see, for instance, [104, §2.1.7, Corollary 2]). The proof of this result is postponed, as it will be included in the proof of the more general Proposition 3.3.4.

Proposition 3.2.2 (Hardy-Poincaré inequality). *Let* $R>0$ *and* $\delta\in\mathbb{R}\setminus\{-n\}$. *Then*

i) if $\delta+n>0$, *then for every* $u\in C^\infty(\overline{B_R})$ *it holds that*

$$\left(\frac{\delta+n}{2}\right)^2\int_{B_R}|y|^\delta|u|^2dz\le\frac{\delta+n}{2}\int_{\partial B_R}|y|^{\delta+1}|u|^2ds+\int_{B_R}|y|^{\delta+2}|\nabla u|^2dz\,;$$

ii) if $\delta+n<0$, *then for every* $u\in C_c^\infty(\overline{B_R}\setminus\Sigma_0)$ *it holds that*

$$\left(\frac{\delta+n}{2}\right)^2\int_{B_R}|y|^\delta|u|^2dz\le\int_{B_R}|y|^{\delta+2}|\nabla u|^2dz\,.$$

As a corollary, we also recover the Poincaré inequality. The proof of this result is similarly postponed, as it will be included in the proof of the more general Proposition 3.3.5.

Proposition 3.2.3 (Poincaré inequality). *Let* $R>0$ *and* $a\in\mathbb{R}$. *Then*

i) if $a+n>0$ *and* $u\in C_c^\infty(B_R)$, *it holds that*

$$\left(\frac{a+n}{2R}\right)^2\int_{B_R}|y|^a|u|^2\,dz\le\int_{B_R}|y|^a|\nabla u|^2dz\,,$$

ii) if $a+n<2$ *and* $u\in C_c^\infty(\overline{B_R}\setminus\Sigma_0)$, *it holds that*

$$\left(\frac{a+n-2}{2R}\right)^2\int_{B_R}|y|^a|u|^2\,dz\le\int_{B_R}|y|^a|\nabla u|^2\,dz\,.$$

3.2.3 Weighted capacity

In this section, we examine how the properties of the weighted Sobolev space $H^{1,a}(B_R)$ depend on the natural weighted capacity of the characteristic manifold.

Given $R>0$ and a bounded domain Ω such that $\Sigma_0\cap B_R\subset\Omega$, we define the local weighted capacity of Σ_0 in the box Ω as

$$\mathrm{Cap}_a(\Sigma_0\cap B_R;\Omega)=\inf\left\{\int_\Omega|y|^a|\nabla u|^2dz\ \middle|\ u\in C_c^\infty(\Omega),\ u=1\text{ on }\Sigma_0\cap B_R\right\}.$$

Thanks to Proposition 3.2.3, one could equivalently consider, in the definition above, the minimization of the full norm in (3.19). As next Lemma shows, we can identify three distinct capacitary ranges: when the weighted capacity of the characteristic manifold is infinite, i.e. $a+n\le 0$, we call the weight $|y|^a$ *supersingular*, as we said before; when the weighted capacity is zero, i.e. $a+n\ge 2$, we call the weight $|y|^a$ *superdegenerate*; finally, when the weighted capacity is locally finite and positive, i.e. $0<a+n<2$, we call the weight $|y|^a$ *mid-range*. Notice that, since $n\ge 2$, the

A_2-Muckenhoupt range $0 < a + n < 2n$ intersects both the mid-range and the superdegenerate intervals. We also note that, within the mid-range interval, the capacity depends significantly on both the radius R and the diameter of the box Ω.

Lemma 3.2.4 (Capacitary ranges)**.** *Let $R > 0$, and let Ω be a open set such that $\Sigma_0 \cap B_R \subset \Omega$.*

i) if $a + n \le 0$, then

$$\mathrm{Cap}_a(\Sigma_0 \cap B_R; \Omega) = \infty\,;$$

ii) if $a + n \ge 2$, then

$$\mathrm{Cap}_a(\Sigma_0 \cap B_R; \Omega) = 0\,;$$

iii) if $0 < a + n < 2$, then there exists a constant $c > 0$ depending only on a, d, n such that

$$0 < c\ \mathrm{diam}(\Omega)^{a+n-2} R^{d-n} \le \mathrm{Cap}_a(\Sigma_0 \cap B_R; \Omega) < \infty\,.$$

Proof. The proof of *i*) is straightforward since, when $a + n \le 0$, it holds that

$$\int_\Omega |y|^a |\nabla u|^2 dz = \infty \qquad \text{for all } u \in C_c^\infty(\Omega),\ u = 1 \text{ on } \Sigma_0 \cap B_R\,.$$

Indeed, assume by contradiction that there exists $u \in C_c^\infty(\Omega)$ such that $u = 1$ on $\Sigma_0 \cap B_R$ and $\int_\Omega |y|^a |\nabla u| dz < \infty$. Since $a + n \le 0$ implies that $|y|^a \notin L^1_{\mathrm{loc}}(\Omega)$, we infer that $\nabla u = 0$ on $\Sigma_0 \cap \Omega$. Therefore, $u = 1$ on $\Sigma_0 \cap \Omega$, which means that u can not be compactly supported on Ω, leading to a contradiction.

Let now prove *ii*). Call $\Omega_0 = \{x \in \mathbb{R}^{d-n} \mid (x, 0) \in \Omega\}$, and take $\varphi \in C_c^\infty(\Omega_0)$ such that $\varphi(x) = 1$ for every $x \in B_R^{d-n} \subset \Omega_0$. Moreover, let $\eta \in C^\infty(\mathbb{R})$ be such that $0 \le \eta \le 1$, $\eta(t) = 1$ for $t \in (2, \infty)$ and $\eta(t) = 0$ for $t \in (-\infty, 1)$. For $h \in \mathbb{N}$, we introduce

$$\Psi_h(z) = \varphi(x)\eta_h(y) \qquad \text{where} \qquad \eta_h(y) = \eta\Big(\frac{-\log|y|}{h}\Big)\,. \tag{3.21}$$

One can readily see that, for h sufficiently large, $\Psi_h \in C_c^\infty(\Omega)$, $\Psi_h(x, y) = 1$ if $|x| \le R$ and $|y| \le e^{-2h}$ (and in particular $\Psi_h = 1$ on $\Sigma_0 \cap B_R$) and $\Psi_h(x, y) = 0$ if $|y| \ge e^{-h}$. Moreover,

$$|\nabla_y \Psi_h(z)| = |\varphi(x)| \Big|\eta'\Big(\frac{-\log|y|}{h}\Big)\frac{y}{h|y|^2}\Big| \le \frac{\|\varphi\|_{L^\infty(\mathbb{R}^{d-n})}\|\eta'\|_{L^\infty(\mathbb{R})}}{h|y|}\chi_{\{e^{-2h} \le |y| \le e^{-h}\}}\,.$$

Using cylindrical coordinates we obtain

$$\begin{aligned}\int_\Omega |y|^a |\nabla \Psi_h|^2 dz &= \int_{\Omega_0} |\nabla_x \varphi|^2 dx \int_{\mathbb{R}^n} |y|^a |\eta_h|^2 dy + \int_{\Omega_0} |\varphi|^2 dx \int_{\mathbb{R}^n} |y|^a |\nabla \eta_h|^2 dy \\ &\le c \int_0^{e^{-h}} r^{a+n-1} dr + \frac{c}{h^2} \int_{e^{-2h}}^{e^{-h}} r^{a+n-3} dr\,.\end{aligned}$$

Thus, since $a + n \ge 2$, it holds that $\|\Psi_h\|_{H^{1,a}(\Omega)} \to 0$ as $h \to \infty$, which proves *ii*).

Finally, we prove *iii*). Let $u \in C_c^\infty(\Omega)$ be such that $u = 1$ on $\Sigma_0 \cap B_R$, that is, $u(x, 0) = 1$ whenever $|x| \le R$. Since $|y|^a \in L^1_{\mathrm{loc}}(\Omega)$ it is clear that $\|u\|_{H^{1,a}} < \infty$, and thus $\mathrm{Cap}(\Sigma_0 \cap B_R) < \infty$.

Next, let $\rho = \mathrm{diam}(\Omega)$, so that $\Omega \subset B_\rho$. Fix $x \in B_R^{d-n}$. By the fundamental theorem of calculus and Hölder's inequality we have that, for every $\sigma \in \mathbb{S}^{n-1}$, it holds

$$\begin{aligned}1 = |u(x,\rho\sigma) - 1|^2 = |u(x,\rho\sigma) - u(x,0)|^2 = \Big|\int_0^\rho \nabla u(x,s\sigma)\cdot\sigma ds\Big|^2 \le \Big(\int_0^\rho |\nabla u|(x,s\sigma)ds\Big)^2 \\ \le \int_0^\rho s^{n+a-1}|\nabla u|^2(x,s\sigma)ds \int_0^\rho s^{1-a-n}ds \le c\rho^{2-a-n}\int_0^\infty s^{a+n-1}|\nabla u|^2(x,s\sigma)ds\,,\end{aligned}$$

where $c > 0$ depends only on a, n. As a consequence we have that

$$n\omega_n = \int_{\mathbb{S}^{n-1}} d\sigma \le c\rho^{2-a-n}\int_{\mathbb{S}^{n-1}}\int_0^\infty s^{n+a-1}|\nabla u|^2(x,s\sigma)dsd\sigma = c\rho^{2-a-n}\int_{\mathbb{R}^n}|y|^a|\nabla u|^2dy\,,$$

and finally, integrating over $x \in B_R^{d-n}$ we find

$$n\omega_n|B_R^{d-n}| \le c\rho^{2-a-n}\int_{B_R^{d-n}\times\mathbb{R}^n}|y|^a|\nabla u|^2dz \le c\rho^{2-a-n}\int_{\mathbb{R}^d}|y|^a|\nabla u|^2dz\,.$$

The conclusion readily follows. ☐

As we have previously remarked, in the supersingular setting $a + n \le 0$, the weight term is not locally integrable. This phenomenon leads to the following result.

Proposition 3.2.5 (Density of smooth functions in the supersingular setting)**.** *Let $a+n \le 0$. The space $H^{1,a}(B_R)$ can be equivalently defined as the completion of $C_c^\infty(\overline{B_R}\setminus\Sigma_0)$ with respect to the norm given in* (3.19).

Proof. It suffices to show that for every $u \in C^\infty(\overline{B_R}) \cap H^{1,a}(B_R)$, there exists a sequence $\{u_\varepsilon\}$ in $C_c^\infty(\overline{B_R}\setminus\Sigma_0)$ such that $\|u_\varepsilon - u\|_{H^{1,a}(B_R)} \to 0$ as $\varepsilon \to 0$.

Let $u \in C^\infty(\overline{B_R})$ be such that $\|u\|_{H^{1,a}(B_R)} < \infty$. Since $|y|^a \notin L^1_{\mathrm{loc}}(B_R)$, it follows that $u = 0$ on $B_R \cap \Sigma_0$.

Now, let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) = 0$ for $|t| < 1$, $\eta(t) = t$ for $|t| > 2$, and $|\eta(t)| \le |t|$ for any $t \in \mathbb{R}$. Define

$$u_\varepsilon(z) = \varepsilon\eta(\varepsilon^{-1}u(z))\,.$$

Clearly, $u_\varepsilon = 0$ whenever $|u| < \varepsilon$, which implies that $u_\varepsilon \in C_c^\infty(\overline{B_R}\setminus\Sigma_0)$. Moreover, $u_\varepsilon = u$ whenever $|u| > 2\varepsilon$, and $\nabla u_\varepsilon = \eta'(\varepsilon^{-1}u)\nabla u$. Thus

$$\int_{B_R}|y|^a|u_\varepsilon - u|^2dz \le 4\int_{B_R\cap\{|u|<2\varepsilon\}\setminus\{u=0\}}|y|^a|u|^2dz \to 0 \quad \text{as } \varepsilon \to 0\,,$$

since $|B_R \cap \{|u| < 2\varepsilon\} \setminus \{u = 0\}| \to 0$. We can estimate the gradient part as

$$\int_{B_R}|y|^a|\nabla(u_\varepsilon - u)|^2dz \le c\int_{B_R\cap\{|u|<2\varepsilon\}\setminus\{u=0,|\nabla u|=0\}}|y|^a|\nabla u|^2dz \to 0 \quad \text{as } \varepsilon \to 0\,,$$

since $|B_R \cap \{|u| < 2\varepsilon\} \setminus \{u = 0, |\nabla u| = 0\}| \to 0$, given that $B_R \cap \{u = 0, |\nabla u| \ne 0\}$ is a set of Hausdorff dimension at most $d - 1$.

To prove this, assume $u \in C^1$ and define $\Omega_k := \overline{B_R} \cap \{u = 0, |\nabla u| \ge \frac{1}{k}\}$, which is a compact set. For each $z \in \Omega_k$, there exists $r_z > 0$ such that $B_{r_z}(z) \cap \Omega_k$ is a $(d-1)$-dimensional regular hypersurface by the implicit function theorem. Then, selecting a finite subcovering of

$\Omega_k \subset \bigcup_{i=1}^{k_0} B_{r_{z_i}}(z_i) \cap \Omega_k$, we have that for any arbitrarily small $\delta > 0$ it holds

$$\mathcal{H}^{d-1+\delta}(\Omega_k) = 0.$$

Thus, since $\overline{B_R} \cap \{u = 0, |\nabla u| \neq 0\} = \bigcup_{k\in\mathbb{N}} \Omega_k$, we have

$$\mathcal{H}^{d-1+\delta}(\overline{B_R} \cap \{u = 0, |\nabla u| \neq 0\}) = 0.$$

The arbitrariness of $\delta > 0$ implies that $\overline{B_R} \cap \{u = 0, |\nabla u| \neq 0\}$ is $(d-1)$-dimensional. Notice that the $(d-1)$-Hausdorff measure could be infinite. □

Similarly to the supersingular setting, i.e. Proposition 3.2.5, whenever the weight is superdegenerate one has the following density result.

Proposition 3.2.6 (Density of smooth functions in the superdegenerate setting)**.** *Let $a + n \geq 2$. Then the space $H^{1,a}(B_R)$ can be equivalently defined as the completion of $C_c^\infty(\overline{B_R} \setminus \Sigma_0)$ with respect to the norm in* (3.19)*.*

Proof. It suffices to prove that for every $u \in C^\infty(\overline{B_R}) \cap H^{1,a}(B_R)$ there exists a sequence $u_h \in C_c^\infty(\overline{B_R} \setminus \Sigma_0)$ such that $\|u_h - u\|_{H^{1,a}(B_R)} \to 0$ as $h \to \infty$.

Let $\Psi_h \in C^\infty(\overline{B_R})$ be the family of functions introduced in (3.21). We recall that $\Psi_h = 1$ in $B_R \cap \{|y| \leq e^{-2h}\}$ and $\Psi_h = 0$ in $B_R \cap \{|y| \geq e^{-h}\}$. Moreover, if $a+n \geq 2$ then $\|\Psi_h\|_{H^{1,a}(B_R)} \to 0$ as $h \to \infty$.

Let us define

$$u_h = u(1 - \Psi_h).$$

Since obviously $u_h \in C_c^\infty(\overline{B_R} \setminus \Sigma_0)$ and

$$\|u_h - u\|_{H^{1,a}(B_R)} \leq c\|\Psi_h\|_{H^{1,a}(B_R)} \,,$$

for a constant $c > 0$ which depends on $\|u\|_{L^\infty(B_R)}$, $\|\nabla u\|_{L^\infty(B_R)}$, the proof is complete. □

3.2.4 H=W property

Sobolev spaces can be defined in terms of weak derivatives in L^p spaces. In the unweighted case, this definition is equivalent to the one based on the density of smooth functions (which we used to introduce $H^{1,a}(B_R)$). However, establishing this equivalence in the context of weighted Sobolev spaces is not as straightforward.

Following the approach of [155], we introduce the set

$$\tilde{W} = \{u \in W^{1,1}_{\mathrm{loc}}(B_R) \mid \|u\|_{H^{1,a}(B_R)} < \infty\}.$$

We then define

$$W^{1,a}(B_R) := \text{the completion of } \tilde{W} \text{ with respect to } \|\cdot\|_{H^{1,a}(B_R)}.$$

We have the following characterization. Note that the two cases intersect.

Lemma 3.2.7. *Let $a \in \mathbb{R}$, $R > 0$. It holds* *i) if* $a+n<2n$*, then*

$$W^{1,a}(B_R)=\{u\in W^{1,1}_{\mathrm{loc}}(B_R)\mid \|u\|_{H^{1,a}(B_R)}<\infty\}\,,$$

ii) if $a+n\geq 2$*, then*

$$W^{1,a}(B_R)=\{u\in W^{1,1}_{\mathrm{loc}}(B_R\setminus\Sigma_0)\mid \|u\|_{H^{1,a}(B_R)}<\infty\}\,.$$

Proof. Let us prove $i)$. The inclusion $\tilde{W}\subset W^{1,a}(B_R)$ is straightforward.

To prove the reverse inclusion, let us take a Cauchy sequence $u_k\subset\tilde{W}$. We note that u_k and ∇u_k are Cauchy sequences in $L^{2,a}(B_R)$ and $L^{2,a}(B_R)^d$, respectively. Due to the completeness of $L^{2,a}(B_R)$, there exist $u\in L^{2,a}(B_R)$ and $V\in L^{2,a}(B_R)^d$ such that

$$u_k\to u \text{ in } L^{2,a}(B_R)\,,\qquad \nabla u_k\to V \text{ in } L^{2,a}(B_R)^d\,. \tag{3.22}$$

In particular, we note that

$$\int_{B_R}|y|^a|u|^2dz+\int_{B_R}|y|^a|V|^2dz<\infty.$$

By Hölder inequality, we have that for every compact set $K\subset B_R$ it holds that

$$\int_K|u|dz\leq\Big(\int_K|y|^a|u|^2dz\Big)^{\frac12}\Big(\int_K|y|^{-a}dz\Big)^{\frac12}<\infty\,.$$

Here we used that $|y|^{-a}\in L^1_{\mathrm{loc}}(B_R)$ due to the assumption $a+n<2n$. Performing the same computation for V, we obtain that $u,V\in L^1_{\mathrm{loc}}(B_R)$.

To conclude, it remains to show that V is the weak gradient of u. Fix $\varphi\in C^\infty_c(B_R)$. Using (3.22) we get

$$\Big|\int_{B_R}(u_k-u)\nabla\varphi dz\Big|\leq\Big(\int_{B_R}|y|^a|u_k-u|^2dz\Big)^{\frac12}\Big(\int_{B_R}|y|^{-a}|\nabla\varphi|^2dz\Big)^{\frac12}\to 0\,,$$
$$\Big|\int_{B_R}(\nabla u_k-V)\varphi dz\Big|\leq\Big(\int_{B_R}|y|^a|\nabla u_k-V|^2dz\Big)^{\frac12}\Big(\int_{B_R}|y|^{-a}|\varphi|^2dz\Big)^{\frac12}\to 0\,.$$

Hence, since $u_k\in W^{1,1}_{\mathrm{loc}}(B_R)$, we have

$$\int_{B_R}(u\nabla\varphi+\varphi V)dz=\int_{B_R}(u_k\nabla\varphi+\varphi\nabla u_k)dz+o(1)=o(1)\,.$$

Therefore, $V=\nabla u$, and $i)$ is proved.

Let us now prove $ii)$. To show that

$$W^{1,a}(B_R)\subset\{u\in W^{1,1}_{\mathrm{loc}}(B_R\setminus\Sigma_0)\mid \|u\|_{H^{1,a}(B_R)}<\infty\}$$

we can proceed in a manner similar to the proof of $i)$, with a few minor modifications. Note that if $a+n\geq 2$, then in general $|y|^{-a}\notin L^1(B_R)$; however, it is always true that $|y|^{-a}\in L^1_{\mathrm{loc}}(B_R\setminus\Sigma_0)$.

Next, let us prove that

$$\{u\in W^{1,1}_{\mathrm{loc}}(B_R\setminus\Sigma_0)\mid \|u\|_{H^{1,a}(B_R)}<\infty\}\subset W^{1,a}(B_R)\,.$$

Let $u \in W^{1,1}_{\mathrm{loc}}(B_R \setminus \Sigma_0)$ with $\|u\|_{H^{1,a}(B_R)} < \infty$ be fixed. Moreover, let $\varepsilon > 0$. For $k \in \mathbb{N}$, consider the sequence u_k defined as

$$u_k = \begin{cases} k & \text{if } u > k \\ u & \text{if } -k \leq u \leq k \\ -k & \text{if } u < -k \end{cases}$$

It is standard to see that $u_k \in L^\infty(B_R) \cap W^{1,1}_{\mathrm{loc}}(B_R \setminus \Sigma_0)$ and $\|u - u_k\|_{H^{1,a}(B_R)} \to 0$ as $k \to \infty$. In particular, there exists $\hat{u} := u_{k_0}$ for some sufficiently large k_0 such that $\|u - \hat{u}\|_{H^{1,a}(B_R)} < \varepsilon/2$.

Let $\Psi_h \in C^\infty(\overline{B_R})$ be the family of functions introduced in (3.21). We recall that $0 \leq \Psi_h \leq 1$, $\Psi_h = 1$ in $B_R \cap \{|y| \leq e^{-2h}\}$ and $\Psi_h = 0$ in $B_R \cap \{|y| \geq e^{-h}\}$. Moreover, since $a + n \geq 2$ then $\|\Psi_h\|_{H^{1,a}(B_R)} \to 0$ as $h \to \infty$.

Define

$$u_h = \hat{u}(1 - \Psi_h).$$

We point out that $\mathrm{spt}(u_h) \subset \{|y| \geq e^{-h}\}$. As a consequence, $u_h, |\nabla u_h| \in L^1_{\mathrm{loc}}(B_R)$, even when $|y|^{-a} \notin L^1_{\mathrm{loc}}(B_R)$. Moreover, for every $\varphi \in C^\infty_c(B_R)$ it holds

$$\int_{B_R} (u_h \nabla\varphi + \varphi \nabla u_h) dz = \int_{B_R} \hat{u} \nabla[(1 - \Psi_h)\varphi] + [(1 - \Psi_h)\varphi] \nabla \hat{u} \, dz = 0\,,$$

where we used that $(1 - \Psi_h)\varphi \in C^\infty_c(B_R \setminus \Sigma_0)$ and $\hat{u} \in W^{1,1}_{\mathrm{loc}}(B_R \setminus \Sigma_0)$. Hence, $u_h \in W^{1,1}_{\mathrm{loc}}(B_R)$.

By Lebesgue dominated convergence theorem (recall that $|\Psi_h| \leq 1$ and $\Psi_h \to 0$ a.e. on B_R as $h \to \infty$) and since $\|\Psi_h\|_{H^{1,a}(B_R)} \to 0$ we obtain, as $h \to \infty$,

$$\begin{aligned} \|\hat{u} - u_h\|^2_{H^{1,a}(B_R)} &\leq 2\int_{B_R} |y|^a(|\nabla \hat{u}|^2 + |\hat{u}|^2)|\Psi_h|^2 dz + 2\int_{B_R} |y|^a |\hat{u}|^2 |\nabla \Psi_h|^2 dz \\ &\leq o(1) + 2\|\hat{u}\|_{L^\infty(B_R)} \|\psi_h\|_{H^{1,a}(B_R)} = o(1)\,. \end{aligned}$$

In particular, fix $h_0 \in \mathbb{N}$ such that $\|\hat{u} - u_{h_0}\|_{H^{1,a}(B_R)} < \frac{\varepsilon}{2}$.

Since $u_h \in W^{1,1}_{\mathrm{loc}}(B_R)$ and

$$\|u - u_{h_0}\|_{H^{1,a}(B_R)} \leq \|u - \hat{u}\|_{H^{1,a}(B_R)} + \|\hat{u} - u_{h_0}\|_{H^{1,a}(B_R)} < \varepsilon\,,$$

thanks to the arbitrariness of ε we conclude that u belongs $W^{1,a}(B_R)$, thus completing the proof. □

It is now straightforward to show that the $H = W$ property holds, at least when $a + n > 0$.

Lemma 3.2.8 (H=W). *If $a + n > 0$, then*

$$H^{1,a}(B_R) = W^{1,a}(B_R).$$

Proof. If $0 < a + n < 2n$ the weight $|y|^a$ belongs to the Muckenhoupt class A_2, see Proposition 3.2.1. Therefore, the result is classical; see for instance [89, Theorem 2.5] or [155, §4].

If $a + n \geq 2$, we first note that every $u \in W^{1,a}(B_R)$ can be approximated by functions $u_h \in W^{1,1}_{\mathrm{loc}}(B_R)$ which are supported away from Σ_0; see the proof of Lemma 3.2.7, *ii*). In turn, the functions u_h can be approximated by functions in $C^\infty_c(\overline{B_R} \setminus \Sigma_0)$ using a standard mollification technique. □

3.2.5 Trace inequalities and compact embeddings

Next we show a trace-type inequality. To this aim, we introduce the projection

$$\Pi : \mathbb{R}^d \to \mathbb{R}^n \qquad \Pi(z) = y\,,$$

and we notice that, using polar coordinates, for any $z = r\sigma$, $r = |z| > 0$, $\sigma = \frac{z}{|z|} \in \mathbb{S}^{d-1}$, it holds

$$y = r\Pi\sigma\,.$$

Lemma 3.2.9 (Trace inequality)**.** *Let $R > 0$ and $a + n > 0$. Then for any $u \in C^\infty(\overline{B_R})$ and $0 < r \leq R$ it holds*

$$c\int_{\partial B_r} |y|^a|u|^2 ds \leq r\int_{B_R} |y|^a|\nabla u|^2 dz + r^{-1}\int_{B_R} |y|^a|u|^2 dz\,,$$

for a constant $c > 0$ depending only on d and a.

Proof. Let $0 < \rho < r \leq R$. By fundamental theorem of calculus and Hölder's inequality we have that for any $\sigma \in \mathbb{S}^{d-1}$ it holds

$$|u(r\sigma) - u(\rho\sigma)|^2 \leq \Big(\int_\rho^r |\nabla u|(\tau\sigma)d\tau\Big)^2 \leq \int_\rho^r \tau^{1-a-d}d\tau \int_\rho^r \tau^{a+d-1}|\nabla u|^2(\tau\sigma)d\tau\,.$$

Therefore

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a|u(r\sigma) - u(\rho\sigma)|^2 d\sigma &\leq \int_\rho^r \tau^{1-a-d}d\tau \int_{B_r\setminus B_\rho} |y|^a|\nabla u|^2 dz \\ &\leq \int_\rho^r \tau^{1-a-d}d\tau \int_{B_R} |y|^a|\nabla u|^2 dz\,,\end{aligned}$$

and thus

$$\begin{aligned}r^{1-a-d}&\int_{\partial B_r} |y|^a|u|^2 ds = \int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a|u(r\sigma)|^2 d\sigma \\ &\leq 2\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a|u(r\sigma) - u(\rho\sigma)|^2 d\sigma + 2\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a|u(\rho\sigma)|^2 d\sigma \\ &\leq 2(r-\rho)\max\{r^{1-a-d}, \rho^{1-a-d}\}\int_{B_R} |y|^a|\nabla u|^2 dz + 2\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a|u(\rho\sigma)|^2 d\sigma\,.\end{aligned}$$

Next, we multiply both sides by ρ^{a+d-1} and integrate over $\rho \in [0, r]$, using that $a + d \geq a + n > 0$. Since

$$\int_0^r \rho^{a+d-1}\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a|u(\rho\sigma)|^2 d\sigma d\rho = \int_{B_r} |y|^a|u|^2 dz\,,$$

the proof is complete. ☐

Lemma 3.2.10 (Compact embedding $H^{1,a} \hookrightarrow L^{2,a}$)**.** *Let $R > 0$ and $a + n > 0$. Let $\{u_k\}, u$ in $H^{1,a}(B_R)$ be such that $u_k \rightharpoonup u$ in $H^{1,a}(B_R)$. Then, up to subsequences, $u_k \to u$ in $L^{2,a}(B_R)$.*

Proof. Whitout loss of generality, we can assume that $u_k \rightharpoonup 0$ in $H^{1,a}(B_R)$. Morever, since u_k is weakly convergent in the Hilbert space $H^{1,a}(B_R)$, it is bounded.

Let $\delta > 0$ be fixed, and consider $\varphi_\delta \in C^\infty(\overline{B_R})$ such that $0 \leq \varphi_\delta \leq 1$ with $\varphi = 1$ when $|y| \leq \delta$ and $\varphi = 0$ when $|y| \geq 2\delta$. In particular, on the support of φ_δ and for every $\sigma > 0$ it holds $1 \leq 2^\sigma \delta^\sigma |y|^{-\sigma}$.

We compute

$$\int_{B_R} |y|^a |u_k|^2 dz \leq 2 \int_{B_R} |y|^a (1-\varphi_\delta)^2 |u_k|^2 + 2 \int_{B_R} |y|^a \varphi_\delta^2 |u_k|^2.$$

Fix $0 < \sigma < 2$ such that $a - \sigma + n > 0$. Using Proposition 3.2.2, $i)$ with $\delta = a - \sigma$, we find

$$\int_{B_R} |y|^a \varphi_\delta^2 |u_k|^2 \leq \delta^\sigma \int_{B_R} |y|^{a-\sigma} |u_k|^2 \leq c\delta^\sigma R^{2-\sigma} \|u_k\|_{H^{1,a}(B_R)} ,$$

where in the last inequality we also used Lemma 3.2.9.

In particular, since $\{u_k\}$ is a bounded sequence in $H^{1,a}(B_R)$, we infer that

$$\int_{B_R} |y|^a |u_k|^2 dz \leq 2 \int_{B_R} |y|^a (1-\varphi_\delta)^2 |u_k|^2 + c\delta^\sigma ,$$

for a constant $c > 0$ not depending on k.

Next, we notice that $(1 - \varphi_\delta)u_k$ is a sequence supported in $\overline{B_R \setminus \Sigma_\delta}$, where the weight $|y|^a$ is bounded and bounded away from zero. Thus, via a standard argument, we can see that $(1 - \varphi_\delta)u_k \rightharpoonup 0$ in the unweighted Sobolev space $H^1(B_R)$, and therefore

$$\int_{B_R} |y|^a (1-\varphi_\delta)^2 |u_k|^2 \leq c \int_{B_R} (1-\varphi_\delta)^2 |u_k|^2 \to 0 \quad \text{as } k \to \infty,$$

by the classical Rellich-Kondrakov theorem. Summing up, we have proved that

$$\lim_{k\to\infty} \int_{B_R} |y|^a |u_k|^2 \, dz \leq c\delta^\sigma,$$

and the conclusion easily follows due to the arbitrariness of δ. □

As a consequence of the previous two results we have the following trace theorem.

Lemma 3.2.11 (Trace operator)**.** *Let $R > 0$ and $a + n > 0$. Then there exists a unique bounded linear operator*

$$T_R : H^{1,a}(B_R) \to L^{2,a}(\partial B_R) ,$$

such that for all $u \in C^\infty(\overline{B_R})$

$$T_R u = u_{|\partial B_R}.$$

Moreover, the following characterization holds

$$H_0^{1,a}(B_R) = \{u \in H^{1,a}(B_R) \mid T_R u = 0\} .$$

Proof. Since $C^\infty(\overline{B_R})$ is dense in $H^{1,a}(B_R)$, the first part of the lemma is a direct consequence of Lemma 3.2.9 and a density argument.

As for the second part, it is obvious by the above definition that every $u \in H_0^{1,a}(B_R)$ satisfies $T_R(u) = 0$. Let then $u \in H^{1,a}(B_R)$ be such that $T_R u = 0$. Since $T_R u = 0$, u can be trivially extended to zero outside of B_R.

Now let us consider, for $\lambda \in (1,2)$ the functions

$$u_\lambda(z) = u(\lambda z)\,,$$

which, by construction, are such that $u_\lambda(z) = 0$ if $\frac{R}{\lambda} \le |z| \le R$. As a consequence, one can easily see that $\|u_\lambda\|_{H^{1,a}(B_R)} \le c\|u\|_{H^{1,a}(B_R)}$ for a constant $c > 0$ not depending on λ. Thus $u_\lambda \in W^{1,a}(B_R)$ by Lemma 3.2.7, and in fact $u_\lambda \in H^{1,a}(B_R)$ by Lemma 3.2.8. Moreover, since $u_\lambda(z) = 0$ if $\frac{R}{\lambda} \le |z| \le R$, each u_λ can be approximated by functions in $C_c^\infty(B_R)$, that is, $u_\lambda \in H_0^{1,a}(B_R)$.

Since $H_0^{1,a}(B_R)$ is complete, to conclude it suffices to show that $\|u_\lambda - u\|_{H^{1,a}(B_R)} \to 0$ as $\lambda \to 1^+$. Since u_λ is bounded in $H_0^{1,a}(B_R)$, it converges weakly to $v \in H_0^{1,a}(B_R)$ and, thanks to Lemma 3.2.10, it also holds $u_\lambda \to v$ a.e. in B_R. In fact, since $u_\lambda \to u$ a.e. in B_R by definition, $v = u$. Finally, we observe that $\|u_\lambda\|_{H^{1,a}(B_R)} \to \|u\|_{H^{1,a}(B_R)}$ as $\lambda \to 1^+$, and this is enough to obtain strong convergence. The proof is complete. □

3.2.6 Poincaré-Wirtinger inequality and 2-admissible weights

In this section we establish the 2-admissibility of the weight term $|y|^a$ whenever $a + n > 0$. The definition of 2-admissible weights can be found in [82, §1.1], and can be resumed into four properties, in order: the doubling condition on the measure $d\mu = |y|^a dz$, a condition on the well-definedness of the weak gradient, a Sobolev inequality, which is proved later in Lemma 3.3.7 in a more general form, and the Poincaré-Wirtinger inequality, which is stated and proved below. Checking the first two conditions is easy and we omit the proofs.

Proposition 3.2.12 (Poincaré-Wirtinger inequality). *Let $a + n > 0$ and $R > 0$. Then there exists a positive constant $c > 0$ depending only on d, a such that for any $u \in H^{1,a}(B_R)$*

$$c\int_{B_R} |y|^a |u - \langle u\rangle_R^a|^2\,dz \le R^2 \int_{B_R} |y|^a |\nabla u|^2\,dz \tag{3.23}$$

where

$$\langle u\rangle_R^a := \frac{1}{\mu_a(B_R)}\int_{B_R} |y|^a u\,dz, \qquad \text{and} \qquad \mu_a(B_R) = \int_{B_R} |y|^a\,dz.$$

The proof is classical and can be found for instance in [62, §5.8.1]. However, we report it in order to show why the inequality holds true whenever the weight is locally integrable and the codimension $n \ge 2$ while it does not hold in the superdegenerate setting when $n = 1$.

Proof. Let us prove the inequality in the unitary ball; that is,

$$c\int_{B_1} |y|^a |u - \langle u\rangle_1^a|^2\,dz \le \int_{B_1} |y|^a |\nabla u|^2\,dz. \tag{3.24}$$

Then, one can recover (3.23) by scaling. Assume by contradiction that (3.24) does not hold. Then, along a sequence $\{u_k\} \subset H^{1,a}(B_1)$

$$\|u_k - \langle u_k\rangle_1^a\|_{L^{2,a}(B_1)} > k\|\nabla u_k\|_{L^{2,a}(B_1)}.$$

Let

$$v_k = \frac{u_k - \langle u_k\rangle_1^a}{\|u_k - \langle u_k\rangle_1^a\|_{L^{2,a}(B_1)}}.$$

Then

$$\|v_k\|_{L^{2,a}(B_1)} = 1, \quad \langle v_k\rangle_1^a = 0, \quad \text{and} \quad \|\nabla v_k\|_{L^{2,a}(B_1)} < 1/k.$$

By Lemma 3.2.10, there exists $v \in H^{1,a}(B_1)$ such that, $v_k \rightharpoonup v$ in $H^{1,a}(B_1)$ and $v_k \to v$ in $L^{2,a}(B_1)$ up to subsequences, with $\|v\|_{L^{2,a}(B_1)} = 1$, $\langle v\rangle_1^a = 0$ and $\nabla v = 0$ almost everywhere in B_1. Since the weak gradient is defined as a $L^1_{\mathrm{loc}}(B_1)$ function whenever $a+n<2n$ and as a $L^1_{\mathrm{loc}}(B_1\setminus\Sigma_0)$ function whenever $a+n\geq 2$, in any case, due to the connectedness respectively of B_1 or $B_1\setminus\Sigma_0$, one can conclude that v must be constant almost everywhere in B_1. Then, the two conditions $\|v\|_{L^{2,a}(B_1)} = 1$ and $\langle v\rangle_1^a = 0$ are in contradiction. □

Notice that, when $n=1$ and in the superdegenerate setting, i.e. $a\geq 1$, $B_1\setminus\Sigma_0$ is not connected. Then, the proof above is not valid. In fact, the Poincaré-Wirtinger inequality is not valid too. Consider the jump function $u=1$ on B_1^+, $u=0$ on B_1^-, see [138, Example 1.4]. However, we would like to stress the fact that in the codimension 1 case, the weight $|y|^a$ is still 2-admissible from one or the other side of the hyperplane Σ_0.

3.3 Functional setting with stability in perforated domains

In this section, we provide the functional setting in perforated domains.

Let us consider a symmetric matrix $A\in C^1(B_R;\mathbb{R}^{d,d})$ which satisfies the uniform ellipticity condition (3.5) and express A in blocks form as in (3.18); that is,

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^\top & A_3 \end{pmatrix},$$

where $A_1\in C^1(B_R;\mathbb{R}^{d-n,d-n})$, $A_2\in C^1(B_R;\mathbb{R}^{d-n,n})$ and $A_3\in C^1(B_R;\mathbb{R}^{n,n})$.

Remark 3.3.1. The regularity assumptions stated above can be relaxed as follows: $A_1\in C^0(B_R;\mathbb{R}^{d-n,d-n})$, $A_2\in C^0(B_R;\mathbb{R}^{d-n,n})$ and $A_3\in C^1(B_R;\mathbb{R}^{n,n})$. In other words, the only block that actually requires one additional degree of regularity is A_3. However, since we will smooth the original coefficients by convolution with standard mollifiers and to simplify the notation, we will assume the stronger condition $A\in C^1(B_R;\mathbb{R}^{d,d})$. In the same way, when we assume $A\in C^{1,\alpha}(B_R;\mathbb{R}^{d,d})$ for some $\alpha\in(0,1)$ we could instead assume $A_1\in C^{0,\alpha}(B_R;\mathbb{R}^{d-n,d-n})$, $A_2\in C^{0,\alpha}(B_R;\mathbb{R}^{d-n,n})$ and $A_3\in C^{1,\alpha}(B_R;\mathbb{R}^{n,n})$.

Let us consider a small $\varepsilon_0>0$ such that Lemma 3.A.1 holds true. The latter depends on R, λ, Λ and L where $\|A\|_{C^1(B_R)}\leq L$. For every $\varepsilon\in(0,\varepsilon_0)$, let us define the (ε, A)-neighborhood of Σ_0 as

$$\Sigma_\varepsilon^A = \{(x,y)\in\mathbb{R}^{d-n}\times\mathbb{R}^n \mid A_3^{-1}(x,y)y\cdot y\leq\varepsilon^2\},$$

and its boundary

$$\partial\Sigma_\varepsilon^A = \{(x,y)\in\mathbb{R}^{d-n}\times\mathbb{R}^n \mid A_3^{-1}(x,y)y\cdot y=\varepsilon^2\}.$$

When $A=\mathbb{I}$ we simply write

$$\Sigma_\varepsilon := \Sigma_\varepsilon^{\mathbb{I}} = \{|y|\leq\varepsilon\} \qquad \partial\Sigma_\varepsilon := \partial\Sigma_\varepsilon^{\mathbb{I}} = \{|y|=\varepsilon\}.$$

Moreover, let us point out that for every matrix A one has that $\{y=0\}=\Sigma_0^A=\partial\Sigma_0^A$.

Remark 3.3.2. The choice of ε_0, given by Lemma 3.A.1, ensures that the normal vector to $\partial\Sigma_\varepsilon^A$ is continuous and well defined in B_R. Hence, the boundary $\partial\Sigma_\varepsilon^A$ is locally of class C^1 whenever $A \in C^1(B_R)$ and of class $C^{1,\alpha}$ whenever $A \in C^{1,\alpha}(B_R)$ for any given $\alpha \in (0,1)$.

We write $0 < \varepsilon \ll 1$ to denote that there exists a possibly small ε_0 such that we consider $\varepsilon \in (0, \varepsilon_0)$. Moreover, in the context of perforated domains, we always assume that the matrix A is at least C^1 to guarantee that the boundary $\partial\Sigma_\varepsilon^A$ is C^1.

3.3.1 Functional spaces on perforated domains

Smooth functions

For $R > 0$, $0 < \varepsilon \ll 1$, we define the following spaces of smooth functions.

$$\begin{aligned}
C^\infty(\overline{B_R \setminus \Sigma_\varepsilon^A}) &= \{u|_{B_R\setminus\Sigma_\varepsilon^A} \ \mid \ u \in C^\infty(\mathbb{R}^d)\},\\
C_c^\infty(\overline{B_R} \setminus \Sigma_\varepsilon^A) &= \{u|_{B_R\setminus\Sigma_\varepsilon^A} \ \mid \ u \in C^\infty(\mathbb{R}^d) \text{ and } \operatorname{spt}(u) \subset\subset \mathbb{R}^d \setminus \Sigma_\varepsilon^A\},\\
C_c^\infty(B_R \setminus \mathring{\Sigma}_\varepsilon^A) &= \{u|_{B_R\setminus\Sigma_\varepsilon^A} \ \mid \ u \in C^\infty(\mathbb{R}^d) \text{ and } \operatorname{spt}(u) \subset\subset B_R\},\\
C_c^\infty(B_R \setminus \Sigma_\varepsilon^A) &= \{u|_{B_R\setminus\Sigma_\varepsilon^A} \ \mid \ u \in C^\infty(\mathbb{R}^d) \text{ and } \operatorname{spt}(u) \subset\subset B_R \setminus \Sigma_\varepsilon^A\}.
\end{aligned}$$

Weighted L^p spaces

For $a \in \mathbb{R}$, and $p \in [1, \infty)$, we define the space

$$L^{p,a}(B_R \setminus \Sigma_\varepsilon^A) := L^p(B_R \setminus \Sigma_\varepsilon^A, |y|^a dz),$$

and for fields

$$L^{p,a}(B_R \setminus \Sigma_\varepsilon^A)^d := L^p(B_R \setminus \Sigma_\varepsilon^A, |y|^a dz)^d.$$

In both cases we denote the norm as $\|\cdot\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon^A)}$, for sake of simplicity.

Weighted Sobolev spaces

Let us define the norm

$$\|u\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon^A)} = \Big(\int_{B_R\setminus\Sigma_\varepsilon^A} |y|^a(u^2 + |\nabla u|^2)dz\Big)^{\frac{1}{2}}. \tag{3.25}$$

We define the following Sobolev spaces:

$$\begin{aligned}
H^{1,a}(B_R \setminus \Sigma_\varepsilon^A) &= \text{the completion of } C^\infty(\overline{B_R \setminus \Sigma_\varepsilon^A}) \text{ with respect to the norm in (3.25)};\\
\tilde{H}^{1,a}(B_R \setminus \Sigma_\varepsilon^A) &= \text{the completion of } C_c^\infty(\overline{B_R} \setminus \Sigma_\varepsilon^A) \text{ with respect to the norm in (3.25)};\\
\hat{H}^{1,a}(B_R \setminus \Sigma_\varepsilon^A) &= \text{the completion of } C_c^\infty(B_R \setminus \mathring{\Sigma}_\varepsilon^A) \text{ with respect to the norm in (3.25)};\\
H_0^{1,a}(B_R \setminus \Sigma_\varepsilon^A) &= \text{the completion of } C_c^\infty(B_R \setminus \Sigma_\varepsilon^A) \text{ with respect to the norm in (3.25)}.
\end{aligned}$$

Note that the following inclusion of spaces hold true

$$H_0^{1,a}(B_R \setminus \Sigma_\varepsilon^A) \subset \hat{H}^{1,a}(B_R \setminus \Sigma_\varepsilon^A) \subset H^{1,a}(B_R \setminus \Sigma_\varepsilon^A),$$

$$H^{1,a}_0(B_R\setminus\Sigma^A_\varepsilon)\subset\tilde H^{1,a}(B_R\setminus\Sigma^A_\varepsilon)\subset H^{1,a}(B_R\setminus\Sigma^A_\varepsilon).$$

Remark 3.3.3. We point out that for every $a\in\mathbb{R}$ it holds

$$H^{1,a}(B_R\setminus\Sigma_0)=H^{1,a}(B_R),\qquad \tilde H^{1,a}(B_R\setminus\Sigma_0)=\tilde H^{1,a}(B_R)$$
$$\hat H^{1,a}(B_R\setminus\Sigma_0)=H^{1,a}_0(B_R\setminus\Sigma_0)=H^{1,a}_0(B_R)\,.$$

If $a+n\in(-\infty,0]\cup[2,\infty)$, by Proposition 3.2.5 and Proposition 3.2.6 we also have

$$H^{1,a}(B_R)=\tilde H^{1,a}(B_R\setminus\Sigma_0),\qquad H^{1,a}_0(B_R)=H^{1,a}_0(B_R\setminus\Sigma_0)\,.$$

3.3.2 Stable Hardy-Poincaré and Poincaré inequalities

Proposition 3.3.4 (ε-stable Hardy-Poincaré inequality)**.** *Let $R>0$, $0\le\varepsilon\ll1$ and $\delta\in\mathbb{R}\setminus\{-n\}$. Then*

i) if $\delta+n>0$, for every $u\in C^\infty(\mathbb{R}^d)$ it holds

$$\Big(\frac{\delta+n}{2}\Big)^2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta|u|^2dz\le\frac{\delta+n}{2}\int_{\partial B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u|^2ds+\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+2}|\nabla u|^2dz$$

ii) if $\delta+n<0$, for every $u\in C^\infty_c(\mathbb{R}^d\setminus\Sigma^A_\varepsilon)$ it holds

$$\Big(\frac{\delta+n}{2}\Big)^2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta|u|^2dz\le\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+2}|\nabla u|^2dz$$

Proof. First we prove i). Let $\delta+n>0$, $\delta\neq-2$. Using that $\Delta|y|^{\delta+2}=(\delta+2)(\delta+n)|y|^\delta$ and integrating by parts, we find that for every $u\in C^\infty(\mathbb{R}^d)$ it holds

$$\begin{aligned}(\delta+n)\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta|u|^2dz&=-2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta u\nabla u\cdot y\,dz\\&\quad+\int_{\partial B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u|^2d\sigma-\int_{\partial\Sigma^A_\varepsilon\cap B_R}|y|^\delta y\cdot\nu(z)|u|^2d\sigma\\&\le2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u||\nabla u|\,dz+\int_{\partial B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u|^2d\sigma\,,\end{aligned}\tag{3.26}$$

where we used that $\nu(z)\cdot y>0$ on $\partial\Sigma^A_\varepsilon$ by ii) in Lemma 3.A.1. In fact, letting $\delta\to-2$ we see that (3.26) holds also for $\delta=-2$.

Next we apply Hölder and Young inequalities to get

$$2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u||\nabla u|\,dz\le\frac{\delta+n}{2}\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta|u|^2dz+\frac{2}{\delta+n}\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+2}|\nabla u|^2dz\,,$$

and i) follows.

Let now $\delta+n<0$ and $u\in C^\infty_c(\mathbb{R}^d\setminus\Sigma^A_\varepsilon)$. Arguing exactly as in the proof of (3.26) we find that

$$\begin{aligned}|\delta+n|\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta|u|^2dz&=-\int_{\partial B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u|^2d\sigma+2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^\delta u\nabla u\cdot y\,dz\\&\le2\int_{B_R\setminus\Sigma^A_\varepsilon}|y|^{\delta+1}|u||\nabla u|dz\,,\end{aligned}$$

and thus we obtain *ii)* via Hölder's inequality. □

Proposition 3.3.5 (ε-stable Poincaré inequality)**.** *Let* $R > 0$, $0 \le \varepsilon \ll 1$ *and* $a \in \mathbb{R}$. *Then*

i) if $a + n > 0$ *and* $u \in C_c^\infty(B_R)$ *it holds*

$$\left(\frac{a+n}{2R}\right)^2 \int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a u^2 \le \int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a |\nabla u|^2. \tag{3.27}$$

ii) if $a + n < 2$ *and* $u \in C_c^\infty(\mathbb{R}^d \setminus \Sigma_\varepsilon^A)$ *it holds*

$$\left(\frac{a+n-2}{2R}\right)^2 \int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a u^2 \le \int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a |\nabla u|^2.$$

Proof. Case *i)* follows from *i)* in Proposition 3.3.4 with $\delta = a$, while case *ii)* follows from *ii)* in Proposition 3.3.4 with $\delta = a - 2$. □

3.3.3 Stable trace inequalities on the boundary of the hole

Lemma 3.3.6 (ε-stable trace inequality on the boundary of the hole)**.** *Let* $a + n > 0$, $R > 0$ *given by Lemma 3.A.3 and* $0 < \varepsilon \ll 1$. *There exists a constant* $c = c(a, n, R, \lambda, \Lambda) > 0$ *such that it holds*

$$c \int_{\partial \Sigma_\varepsilon^A \cap B_R} |y|^a |u|^2 d\sigma \le G_\varepsilon \int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a |\nabla u|^2 dz\,, \qquad u \in \hat{H}^{1,a}(B_R \setminus \Sigma_\varepsilon^A)\,,$$

where

$$G_\varepsilon = \begin{cases} \varepsilon & a+n>2 \\ \varepsilon \log \varepsilon & a+n=2 \\ \varepsilon^{a+n-1} & 0<a+n<2\,. \end{cases}$$

Proof. First, we consider the case $A = \mathbb{I}$. The proof is similar to the proof of Lemma 3.2.9, so we omit some details.

By density, it suffices to prove the result for $u \in C_c^\infty(B_R \setminus \mathring{\Sigma}_\varepsilon)$. Let $x \in B^{d-n}_{\sqrt{R^2-\varepsilon^2}}$ be fixed, and notice that $(x, y) \in B_R \setminus \mathring{\Sigma}_\varepsilon$ if and only if $\varepsilon \le |y| < R_x$, where $R_x = \sqrt{R^2 - |x|^2} \le R$. We point out that $u(x, R_x\sigma) = 0$ for any $\sigma \in \mathbb{S}^{n-1}$. Thus, using the fundamental theorem of calculus and the Hölder inequality we get

$$|u(x, \varepsilon\sigma)|^2 = |u(x, R_x\sigma) - u(x, \varepsilon\sigma)|^2 \le \int_\varepsilon^{R_x} \tau^{a+n-1} |\nabla u|^2(\tau\sigma) d\tau \int_\varepsilon^R \tau^{1-a-n} d\tau\,.$$

Integrating over $\sigma \in \mathbb{S}^{n-1}$ we find

$$\int_{\partial B_\varepsilon^n} |y|^a |u|^2 d\sigma = \varepsilon^{a+n-1} \int_{\mathbb{S}^{n-1}} |u(x, \varepsilon\sigma)|^2 d\sigma \le \left(\varepsilon^{a+n-1} \int_\varepsilon^R \tau^{1-a-n} d\tau\right) \int_{B^n_{R_x} \setminus B^n_\varepsilon} |y|^a |\nabla u|^2 dy\,. \tag{3.28}$$

Notice that

$$\varepsilon^{a+n-1} \int_\varepsilon^R \tau^{1-a-n} d\tau \le c\, G_\varepsilon$$

for a constant $c > 0$ depending only on a, n and R. To conclude, we integrate (3.28) over $x \in B^{d-n}_{\sqrt{R^2-\varepsilon^2}}$.

Let us now consider the general case. Let $u \in C_c^\infty(B_R \setminus \mathring{\Sigma}_\varepsilon^A)$, and define $\tilde{u} = u \circ \Phi^{-1}$, where Φ is the C^1-diffeomorphism from Lemma 3.A.3. For every $z \in B_R$, we have $|\Phi(z)| \leq cR$ with $c > 0$ depending only on λ and Λ. Thus $\tilde{u} \in C_c^1(B_{cR} \setminus \mathring{\Sigma}_\varepsilon)$. Moreover, taking into account *ii)* and *iii)* from Lemma 3.A.3, and using the change of variables $(x, \tau) = \Phi(z)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d \setminus \Sigma_\varepsilon} |\tau|^a |\nabla \tilde{u}(x,\tau)|^2 dx d\tau &= \int_{\mathbb{R}^d \setminus \Sigma_\varepsilon^A} |A_3^{-\frac{1}{2}}(z) y|^a (J_\Phi^\top J_\Phi)^{-1} \nabla u \cdot \nabla u | \det J_\Phi | dz \\ &\leq c \int_{\mathbb{R}^d \setminus \Sigma_\varepsilon^A} |y|^a |\nabla u|^2 dz \,.\end{aligned}$$

On the other hand, by taking the same change of variables in the integration over $\partial\Sigma_\varepsilon$, for instance see [102, §11], and using Lemma 3.A.3, *iv)*, we get

$$\begin{aligned}\int_{\partial \Sigma_\varepsilon} |\tau|^a |\tilde{u}(x,\tau)|^2 d\sigma(x,\tau) &= \int_{\partial \Sigma_\varepsilon^A} |A_3^{-\frac{1}{2}}(z) y|^a |u|^2 | \det(\Pi_\varepsilon \circ J_\Phi^\top J_\Phi \circ \Pi_\varepsilon) | d\sigma(z) \\ &\geq c \int_{\partial \Sigma_\varepsilon^A} |y|^a |u|^2 d\sigma(z) \,.\end{aligned}$$

To conclude, apply to $\tilde{u}$ the previous step, which holds also for functions in $C_c^1(B_{cR} \setminus \mathring{\Sigma}_\varepsilon)$. ☐

3.3.4 Stable Sobolev inequalities

Lemma 3.3.7 (ε-stable Sobolev embeddings)**.** *Let $a + n > 0$, $R > 0$ given by Lemma 3.A.3 and $0 \leq \varepsilon \ll 1$. Set*

$$2_a^* = \frac{2(a_+ + d)}{a_+ + d - 2} \,.$$

i) if $a_+ + d > 2$, then for any $2 \leq q \leq 2_a^*$ it holds

$$c \Big(\int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a |u|^q dz \Big)^{\frac{2}{q}} \leq \int_{B_R \setminus \Sigma_\varepsilon^A} |y|^a |\nabla u|^2 dz \,, \qquad u \in \hat{H}^{1,a}(B_R \setminus \Sigma_\varepsilon),$$

for a constant c not depending on ε nor q.

ii) if $d = n = 2$ and $a \leq 0$, the same result holds true for every $q \in [2, \infty)$.

The proof is a consequence of some lemmas. The first lemma is an ε-stable L^1-version of the Caffarelli-Kohn-Nirenberg inequality.

Lemma 3.3.8 (ε-stable L^1-CKN)**.** *Let $a + n > 0$, $\varepsilon \geq 0$. Then for any $1 \leq q \leq \frac{n}{n-1}$ it holds*

$$c \Big(\int_{\mathbb{R}^n \setminus B_\varepsilon^n} |y|^a |u|^q dy \Big)^{\frac{1}{q}} \leq \int_{\mathbb{R}^n \setminus B_\varepsilon^n} |y|^{\frac{a+n}{q} - (n-1)} |\nabla u| dy \,, \qquad u \in C_c^{0,1}(\mathbb{R}^n),$$

for a constant $c > 0$ that does not depend on ε or q.

Proof. Let $1 \leq q \leq \frac{n}{n-1}$ be fixed. Set, for $\varepsilon \geq 0$,

$$S_{q,\varepsilon} = \inf_{u \in C_c^{0,1}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n \setminus B_\varepsilon^n} |y|^{\frac{a+n}{q} - (n-1)} |\nabla u| dy}{\Big(\int_{\mathbb{R}^n \setminus B_\varepsilon^n} |y|^a |u|^q dy \Big)^{\frac{1}{q}}} \,.$$

Our aim is to show that $S_{q,\varepsilon} \geq c > 0$, for a constant c not depending on ε or q.

First, we notice that by a standard scaling argument it holds that $S_{q,\varepsilon} = S_{q,1}$ for any $\varepsilon > 0$. In fact, given $u \in C_c^{0,1}(\mathbb{R}^n)$, it suffices to consider $v \in C_c^{0,1}(\mathbb{R}^n)$ as $v(y) = u(\varepsilon y)$.

Next we show that $S_{q,0} \leq c S_{q,1}$ for a constant $c > 0$ independent from q. Let $u \in C_c^{0,1}(\mathbb{R}^n)$ be fixed. Arguing as in the proof of Lemma 3.3.4 (take $|u|$ instead of $|u|^2$ in (3.26)), we see that the following Hardy-type inequality holds:

$$\frac{a+n}{q} \int_{\mathbb{R}^n \setminus B_1^n} |y|^{\frac{a+n}{q}-n} |u| dy \leq \int_{\mathbb{R}^n \setminus B_1^n} |y|^{\frac{a+n}{q}-(n-1)} |\nabla u| dy \,. \tag{3.29}$$

Next, we extend u in the unitary ball B_1^n by the Kelvin transform

$$\tilde{u}(y) = \begin{cases} u(y) & |y| \geq 1 \\ |y|^{-\frac{2(a+n)}{q}} u(I(y)) & |y| \leq 1 \,, \end{cases}$$

where $I(y) = |y|^{-2} y$ is the inversion map. Clearly, $\tilde{u} \in C_c^{0,1}(\mathbb{R}^n)$. Since $J_I J_I = |y|^{-4} \mathbb{I}_n$ and thus $|\det J_I| = |y|^{-2n}$, a standard computation yields

$$\int_{\mathbb{R}^n} |y|^a |\tilde{u}|^q dy = 2 \int_{\mathbb{R}^n \setminus B_1^n} |y|^a |u|^q dy$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} |y|^{\frac{a+n}{q}-(n-1)} |\nabla \tilde{u}| dy &\leq c \Big(\int_{\mathbb{R}^n \setminus B_1^n} |y|^{\frac{a+n}{q}-(n-1)} |\nabla u| dy + \int_{\mathbb{R}^n \setminus B_1^n} |y|^{\frac{a+n}{q}-n} |u| dy \Big) \\ &\leq c \int_{\mathbb{R}^n \setminus B_1^n} |y|^{\frac{a+n}{q}-(n-1)} |\nabla u| dy \,. \end{aligned}$$

where we also used (3.29). Notice that the constant $c > 0$ can be taken independent of $1 \leq q \leq \frac{n}{n-1}$.

Thus we infer that

$$S_{q,0} \leq \frac{\int_{\mathbb{R}^n} |y|^{\frac{a+n}{q}-(n-1)} |\nabla \tilde{u}| dy}{\Big(\int_{\mathbb{R}^n} |y|^a |\tilde{u}|^q dy \Big)^{\frac{1}{q}}} \leq c \frac{\int_{\mathbb{R}^n \setminus B_1^n} |y|^{\frac{a+n}{q}-(n-1)} |\nabla u| dy}{\Big(\int_{\mathbb{R}^n \setminus B_1^n} |y|^a |u|^q dy \Big)^{\frac{1}{q}}} \,,$$

for any $u \in C_c^{0,1}(\mathbb{R}^n)$, that is, $S_{q,0} \leq c S_{q,1}$. Since by the classical Caffarelli-Kohn-Nirenberg inequality [24] we have $S_{q,0} > 0$ uniformly in q, the proof is complete. ☐

The second lemma we need is an equivalence between weighted Sobolev-type inequalities and the L^1-Moser weighted inequality. The unweighted version of this equivalence for functions defined on all the space is well known; see, e.g., [10] and the references therein. The presence of the weights and the restriction to subsets does not add any difficulty; nevertheless, we present the proof for the sake of completeness.

Lemma 3.3.9. *Let $d \geq 1$ and let $A \subset \mathbb{R}^d$ be a set with smooth boundary and non empty interior. Let $\mu, \nu \in L^1_{\mathrm{loc}}(A)$, $\mu, \nu \geq 0$. Moreover, let $q \geq 1$. Then*

$$c_0 \Big(\int_A \mu(z) |f|^q dz \Big)^{\frac{1}{q}} \leq \int_A \nu(z) |\nabla f| dz \,, \quad \textit{for every} \quad f \in C_c^{0,1}(\mathbb{R}^d) \tag{3.30}$$

if and only if

$$c_0 \int_A \mu(z)|f|^{1+\frac{q-1}{q}} dz \le \Big(\int_A \mu(z)|f| dz \Big)^{\frac{q-1}{q}} \int_A \nu(z)|\nabla f| dz \quad \textit{for every} \quad f \in C_c^{0,1}(\mathbb{R}^d)\,, \tag{3.31}$$

for the same constant $c_0 > 0$.

Proof. If $q = 1$ the two inequalities coincide. Then, let $q > 1$.

Since (3.30)⇒(3.31) simply follows by the Hölder inequality, we only need to show (3.31)⇒(3.30). Assume then that (3.31) holds, and fix $f \in C_c^{0,1}(\mathbb{R}^n)$, $\delta > 1$, and for $k \in \mathbb{Z}$ define

$$f_k(z) = \begin{cases} \delta^k & |f| \ge \delta^{k+1} \\ |f| - \delta^k & \delta^k \le |f| < \delta^{k+1} \\ 0 & |f| < \delta^k\,. \end{cases}$$

Notice that $f_k \in C_c^{0,1}(\mathbb{R}^n)$ and $f_k \ge 0$. Next we define

$$a_k = \delta^{qk} \int_{A \cap \{\delta^k \le |f|\}} \mu(z) dz\,, \qquad \text{and} \qquad b_k = \int_{A \cap \{\delta^k \le |f| < \delta^{k+1}\}} \nu(z)|\nabla f| dz\,.$$

We have

$$\int_A \mu(z)|f_k|^{1+\frac{q-1}{q}} dz \ge \delta^{k+k\frac{q-1}{q}-k(q-1)} a_{k+1}\,,$$
$$\int_A \mu(z)|f_k| \le \delta^{k+1-qk} a_k\,.$$

Therefore, applying (3.31) to f_k we infer

$$a_{k+1} \le c_0^{-1} \delta^{\frac{q^2+q-1}{q}} b_k a_k^{\frac{q-1}{q}}\,.$$

Summing over $k \in \mathbb{Z}$ and using the Hölder inequality we get

$$\sum_{k\in\mathbb{Z}} a_k = \sum_{k\in\mathbb{Z}} a_{k+1} \le c_0^{-1} \delta^{\frac{q^2+q-1}{q}} \sum_{k\in\mathbb{Z}} b_k a_k^{\frac{q-1}{q}} \le c_0^{-1} \delta^{\frac{q^2+q-1}{q}} \Big(\sum_{k\in\mathbb{Z}} b_k^q \Big)^{\frac{1}{q}} \Big(\sum_{k\in\mathbb{Z}} a_k \Big)^{\frac{q-1}{q}}\,,$$

hence

$$\Big(\sum_{k\in\mathbb{Z}} a_k \Big)^{\frac{1}{q}} \le c_0^{-1} \delta^{\frac{q^2+q-1}{q}} \Big(\sum_{k\in\mathbb{Z}} b_k^q \Big)^{\frac{1}{q}} \le c_0^{-1} \delta^{\frac{q^2+q-1}{q}} \sum_{k\in\mathbb{Z}} b_k = c_0^{-1} \delta^{\frac{q^2+q-1}{q}} \int_\Omega \nu(z)|\nabla f| dz\,.$$

Finally, since

$$\int_\Omega \mu(z)|f|^q dz = \sum_{k\in\mathbb{Z}} \int_{A\cap\{\delta^k \le |f| < \delta^{k+1}\}} \mu(z)|f|^q dz \le \delta^q \sum_{k\in\mathbb{Z}} a_k\,,$$

we infer that

$$c_0 \Big(\int_A \mu(z)|f|^q dz \Big)^{\frac{1}{q}} \le \delta^{\frac{q^2+2q-1}{q}} \int_A \nu(z)|\nabla f| dz\,.$$

Letting $\delta \to 1$ we obtain (3.30) and complete the proof. □

The final preparatory lemma is an L^1-version of Lemma 3.3.7, in the case $A = \mathbb{I}$. In its proof, a technique introduced in [41] is crucial; see also [77]. This allows us to recover L^1-Sobolev inequalities on weighted spaces which are Cartesian products of weighted spaces.

Lemma 3.3.10. *Let $a+n>0$, $R>0$, $\varepsilon\in[0,R)$, and $1\le q\le \frac{a_++d}{a_++d-1}$. It holds*

$$c\Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u|^q dz\Big)^{\frac{1}{q}}\le\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|\nabla u|dz\,,\qquad u\in C_c^\infty(B_R)\,,$$

for a constant c not depending on q or ε.

Proof. The case $d-n=0$ derives directly form Lemma 3.3.8 by noticing that for

$$1\le q\le\min\Big\{\frac{n}{n-1},\frac{a+n}{a+n-1}\Big\}=\frac{a_++n}{a_++n-1}$$

it holds

$$|y|^{\frac{a+n}{q}-(n-1)}\le|y|^a|R|^{\frac{a+n}{q}-(a+n-1)}\le c|y|^a\,,$$

where $c>0$ depends only on a, n, R.

Let then $d-n\ge1$. By classical and well known results (one can use, for instance, the classical Caffarelli-Kohn-Nirenberg inequality when $d-n>1$ and [32, Proposition 1] for the case $d-n=1$) we have that for every $p\ge1$, $(d-n-1)p\le d-n$, it holds

$$c_1\Big(\int_{\mathbb{R}^{d-n}}|f|^p dx\Big)^{\frac{1}{p}}\le\int_{\mathbb{R}^{d-n}}|x|^{\frac{d-n}{p}-(d-n-1)}|\nabla f|dx\,,\qquad f\in C_c^{0,1}(\mathbb{R}^{d-n})\,,$$

for a constant $c_1>0$ not depending on p. Thus, using Lemma 3.3.9 we get that

$$c_1\int_{\mathbb{R}^{d-n}}|f|^{1+\frac{p-1}{p}}dx\le\Big(\int_{\mathbb{R}^{d-n}}|f|dx\Big)^{\frac{p-1}{p}}\int_{\mathbb{R}^{d-n}}|x|^{\frac{d-n}{p}-(d-n-1)}|\nabla_x f|dx\,,\qquad f\in C_c^{0,1}(\mathbb{R}^{d-n})\,,\tag{3.32}$$

whenever $p\ge1$, $(d-n-1)p\le d-n$.

On the other hand, by Lemma 3.3.8 and Lemma 3.3.9 we have that for every $g\in C_c^{0,1}(\mathbb{R}^n)$ it holds

$$c_2\int_{\mathbb{R}^n\setminus B_\varepsilon^n}|y|^a|g|^{1+\frac{q-1}{q}}dy\le\Big(\int_{\mathbb{R}^n\setminus B_\varepsilon^n}|y|^a|g|dy\Big)^{\frac{q-1}{q}}\int_{\mathbb{R}^n\setminus B_\varepsilon^n}|y|^{\frac{a+n}{q}-(n-1)}|\nabla_y g|dy\,,\tag{3.33}$$

whenever $1\le q\le\frac{n}{n-1}$, where the constant $c_2>0$ does not depend neither on q nor ε.

The inequalities (3.32) and (3.33) are equivalent, respectively, to the F-Sobolev and G-Sobolev inequalities

$$\int_{\mathbb{R}^{d-n}}|f|F\left(\frac{|f|}{\|f\|_{L^1(\mathbb{R}^{d-n})}}\right)dx\le\int_{\mathbb{R}^{d-n}}|x|^{\frac{d-n}{p}-(d-n-1)}|\nabla_x f|dx\quad\text{for every}\quad f\in C_c^{0,1}(\mathbb{R}^{d-n})$$
$$\int_{\mathbb{R}^n\setminus B_\varepsilon^n}|y|^a|g|G\left(\frac{|g|}{\|g\|_{L^{1,a}(\mathbb{R}^n\setminus B_\varepsilon^n)}}\right)dy\le\int_{\mathbb{R}^n\setminus B_\varepsilon^n}|y|^{\frac{a+n}{q}-(n-1)}|\nabla_y g|dy\quad\text{for every}\quad g\in C_c^{0,1}(\mathbb{R}^n)\,,\tag{3.34}$$

where

$$F(s)=c_1 s^{\frac{p-1}{p}}\qquad\text{and}\qquad G(t)=c_2t^{\frac{q-1}{q}}\,.$$

Let $u \in C^{0,1}_c(\mathbb{R}^d)$. Arguing as in the proof of [41, Proposition 3.3] with minor modifications, thanks to (3.34) we obtain

$$\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon} |y|^a|u|H\left(\frac{|u|}{\|u\|_{L^{1,a}(\mathbb{R}^d\setminus\Sigma_\varepsilon)}}\right)dxdy \le \int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}(|x|^{\frac{d-n}{p}-(d-n-1)}|y|^a + |y|^{\frac{a+n}{q}-(n-1)})|\nabla u|dxdy\,,$$

where

$$H(r) = \inf_{st=r}[F(s)+G(t)]\,.$$

After some standard computations, we see that for $p\ge 1$, $(d-n-1)p \le d-n$, $1\le q\le \frac{n}{n-1}$, it holds

$$H(r) \ge \min\{c_1,c_2\} r^{\frac{\gamma_{p,q}-1}{\gamma_{p,q}}}\,,$$

where

$$\gamma_{p,q} = \frac{p(q-1)+q(p-1)}{pq-1}\,, \qquad \gamma_{1,1}=1\,.$$

Thus

$$\begin{aligned}\min\{c_1,c_2\}&\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|y|^a|u|^{1+\frac{\gamma_{p,q}-1}{\gamma_{p,q}}}dxdy\\ &\le \Big(\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|y|^a|u|dy\Big)^{\frac{\gamma_{p,q}-1}{\gamma_{p,q}}}\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}(|x|^{\frac{d-n}{p}-(d-n-1)}|y|^a+|y|^{\frac{a+n}{q}-(n-1)})|\nabla u|dxdy\,.\end{aligned}$$

Applying Lemma 3.3.9 once again, we infer that, for every $u\in C^{0,1}_c(\mathbb{R}^d)$ it holds

$$c\Big(\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|y|^a|u|^{\gamma_{p,q}}dz\Big)^{\frac{1}{\gamma_{p,q}}} \le \int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}(|x|^{\frac{d-n}{p}-(d-n-1)}|y|^a+|y|^{\frac{a+n}{q}-(n-1)})|\nabla u|dz\,, \tag{3.35}$$

for some constant $c>0$ not depending on p, q, ε.

Now, let us take $1\le\tau\le\frac{d}{d-1}$, and put $q=\frac{(d-n)-(d-n-1)\tau}{(d-n+1)-(d-n)\tau}$ in (3.35). Further, let $p=\frac{d-n}{d-n-1}$ if $d-n>1$ or take the limit $p\to\infty$ if $d-n=1$. We find that for every $1\le\tau\le\frac{d}{d-1}$ and $u\in C^{0,1}_c(\mathbb{R}^d)$ it holds

$$c\Big(\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|y|^a|u|^{\tau}dz\Big)^{\frac{1}{\tau}} \le \int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|y|^a(1+|y|^{\frac{a+d-\tau(a+d-1)}{d-n-\tau(d-n-1)}})|\nabla u|dz\,.$$

Let now $u\in C^\infty_c(B_R)$. To conclude it suffice to notice that for

$$1\le\tau\le\min\left\{\frac{d}{d-1},\frac{a+d}{a+d-1}\right\} = \frac{a_+ + d}{a_+ + d-1}$$

it holds

$$|y|^{\frac{a+d-\tau(a+d-1)}{d-n-\tau(d-n-1)}} \le R^{\frac{a+d-\tau(a+d-1)}{d-n-\tau(d-n-1)}} \le c\,,$$

for a constant $c>0$ not depending on ε nor q. □

Proof of Lemma 3.3.7. The proof is divided in two steps.

Step 1: The case $A=\mathbb{I}$. Fix q such that $2\le q\le 2^*_a$ if $a_+ + d>2$ or $q\ge 2$ if $a_+ + d = 2$. Let us denote

$$r = \frac{2q}{q+2}\,,$$

and notice that $r<q$ and $1\le r\le \frac{a_++d}{a_++d-1}$.

We fix $u\in C^\infty_c(B_R)$ and define $v=|u|^{\frac{q}{r}}$. Since $q>r$, then v is smooth and we can compute

$$\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u|^q dz=\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|v|^r dz\le c\Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|\nabla v|dz\Big)^r,$$

where the last inequality holds true thanks to Lemma 3.3.10.

Next we use that $|\nabla v|=\frac{q}{r}|u|^{\frac{q-r}{r}}|\nabla u|$ and the Hölder inequality to obtain

$$c\Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u|^q dz\Big)^{\frac{1}{r}}\le\Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|\nabla u|^2 dz\Big)^{\frac{1}{2}}\Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u|^{\frac{2(q-r)}{r}} dz\Big)^{\frac{1}{2}},$$

and since

$$\frac{2(q-r)}{r}=q\quad\text{and}\quad\frac{1}{r}-\frac{1}{2}=\frac{1}{q}\,,$$

we readily conclude.

Step 2: the general case. Let $u\in C^\infty_c(B_R)$, and define $\tilde u=u\circ\Phi^{-1}$, where Φ is the C^1-diffeomorphism from Lemma 3.A.3. For every $z\in B_R$, we have $|\Phi(z)|\le cR$ with $c>0$ depending only on λ, Λ. Thus $\tilde u\in C^1_c(B_{cR})$. Moreover, taking into account *ii)* and *iii)* from Lemma 3.A.3, and using the change of variables $(x,\tau)=\Phi(z)$, we obtain

$$\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|\tau|^a|\tilde u(x,\tau)|^q dx d\tau=\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon^A}|A_3^{-\frac{1}{2}}(z)y|^a|u|^q|\det J_\Phi|dz\ge c\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon^A}|y|^a|u|^q dz\,,$$

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon}|\tau|^a|\nabla\tilde u(x,\tau)|^2 dx d\tau&=\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon^A}|A_3^{-\frac{1}{2}}(z)y|^a(J_\Phi^\top J_\Phi)^{-1}\nabla u\cdot\nabla u|\det J_\Phi|dz\\&\le c\int_{\mathbb{R}^d\setminus\Sigma_\varepsilon^A}|y|^a|\nabla u|^2 dz\,.\end{aligned}$$

The conclusion follows applying to $\tilde u$ the previous step. Notice that the latter holds true for functions in C^1 by a standard density argument. □

3.4 Solutions, stable L^∞ estimates and approximation results

In this section, we provide the notion of weak solutions to (3.1). We define the regularized problems for (3.1), and we prove some approximation lemmas. Moreover, we prove local stable L^∞ estimates for the regularized solutions.

3.4.1 Notions of solutions

Definition 3.4.1. Let $a+n>0$, $R>0$ and A satisfying (3.5). Let $f\in L^{p,a}(B_R)$ where $p=(2^*_a)'$ if $d+a_+>2$ or $p>1$ if $d+a_+=2$ and $F\in L^{q,a}(B_R)^d$ where $q\ge2$. We say that u is a weak solution to

$$-\operatorname{div}(|y|^aA\nabla u)=|y|^af+\operatorname{div}(|y|^aF),\quad\text{in } B_R \tag{3.36}$$

if $u\in H^{1,a}(B_R)$ and satisfies

$$\int_{B_R}|y|^aA\nabla u\cdot\nabla\phi=\int_{B_R}|y|^a(f\phi-F\cdot\nabla\phi), \tag{3.37}$$

for every $\phi \in C_c^\infty(B_R)$.

We say that u is an entire solution to

$$-\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), \quad \text{in } \mathbb{R}^d,$$

if u is a weak solution to (3.36) for every $R > 0$.

Remark 3.4.2. We highlight that our notion of weak solution implies a *weighted conormal boundary condition* on the lower-dimensional set Σ_0. This is a consequence of the fact that the weak formulation (3.37) involves test functions whose support may touch the thin manifold Σ_0. We refer to these solutions as *solutions across* Σ_0. Let us assume for sake of simplicity that $f = 0$, $F = 0$, and fix $\phi \in C_c^\infty(B_R)$. Multiplying (3.36) by ϕ and integrating by parts in $B_R \setminus \Sigma_\varepsilon$, we obtain

$$0 = \int_{B_R\setminus\Sigma_\varepsilon} -\operatorname{div}(|y|^a A\nabla u)\phi \, dz = \int_{B_R\setminus\Sigma_\varepsilon} |y|^a A\nabla u \cdot \nabla\phi \, dz - \int_{\partial\Sigma_\varepsilon\cap B_R} |y|^a \phi A\nabla u \cdot \nu \, d\sigma.$$

Formally, taking the limit as $\varepsilon \downarrow 0$, we find

$$\int_{B_R} |y|^a A\nabla u \cdot \nabla\phi \, dz = \int_{B_R^{d-n}} \mathcal{D}_u(x)\phi(x,0)\, dx,$$

where

$$\mathcal{D}_u(x) := -\lim_{\varepsilon\downarrow 0} \varepsilon^{1-n} \int_{\partial B_\varepsilon^n} |y|^{a+n-2} A\nabla u \cdot y \, d\sigma(y).$$

Hence, in the weak formulation of (3.36), we are assuming that $\mathcal{D}_u = 0$. It is worth noting that if $a + n \geq 2$, due to the zero weighted capacity of the thin manifold, only solutions to (3.36) make sense, since one could not impose any different boundary condition at Σ_0. Conversely, when $a + n \in (0, 2)$, the weighted capacity of Σ_0 is positive and locally finite, allowing the imposition of both inhomogeneous Dirichlet and inhomogeneous conormal boundary conditions, respectively $u = g$ and $\mathcal{D}_u = h$ on Σ_0.

Definition 3.4.3. Let $a + n > 0$, $R > 0$, $A \in C^1(B_R; \mathbb{R}^{d,d})$ satisfying (3.5) and $0 < \varepsilon \ll 1$. Let $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon^A)$, where $p = (2_a^*)'$ if $d > 2$ or $p > 1$ if $d = 2$ and $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon^A)^d$ where $q \geq 2$. We say that u is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_R \setminus \Sigma_\varepsilon^A \\ (A\nabla u + F)\cdot\nu = 0 & \text{on } \partial\Sigma_\varepsilon^A \cap B_R \end{cases} \tag{3.38}$$

if $u \in H^{1,a}(B_R \setminus \Sigma_\varepsilon^A)$ and satisfies

$$\int_{B_R\setminus\Sigma_\varepsilon^A} |y|^a A\nabla u \cdot \nabla\phi dz = \int_{B_R\setminus\Sigma_\varepsilon^A} |y|^a (f\phi - F\cdot\nabla\phi) dz,$$

for every $\phi \in C_c^\infty(B_R)$.

We say that u is an entire solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } \mathbb{R}^d \setminus \Sigma_\varepsilon^A \\ (A\nabla u + F)\cdot\nu = 0 & \text{on } \partial\Sigma_\varepsilon^A \end{cases}$$

if u is a weak solution to (3.38) for every $R > 0$.

Moreover, we adopt the convention that solutions to (3.38) with $\varepsilon = 0$ coincide with those of (3.36).

Remark 3.4.4. Fixed $\bar{u} \in H^{1,a}(B_R)$, a weak solutions to (3.36) satisfying the boundary condition $u - \bar{u} \in H_0^{1,a}(B_R)$ is a minimizer of the functional

$$\mathcal{E}(v) := \int_{B_R} |y|^a \Big(\frac{A\nabla v \cdot \nabla v}{2} - fv + F \cdot \nabla v \Big) dz,$$

over $X := \{ v \in H^{1,a}(B_R) \ | \ v - \bar{u} \in H_0^{1,a}(B_R) \}$.

By the Poincaré inequality and the Sobolev type embedding, see Proposition 3.2.3 and Lemma 3.3.7 respectively, the assumption on the data allows us to obtain that the functional $\mathcal{E}$ is coercive, so by the Weierstrass theorem, we have existence and uniqueness of solutions to (3.36) with prescribed trace on ∂B_R (see also Lemma 3.2.11).

3.4.2 Caccioppoli inequality and local boundedness of solutions

In this section we provide local $L^2 \to L^\infty$ estimates for weak solutions to (3.38) that are uniform with respect to ε. We note that, since the weight $|y|^a$ is 2-admissible, the local L^∞ bounds for weak solutions to (3.36) (when $\varepsilon = 0$ and $A \in L^\infty$) follows from [65].

Furthermore, assuming that the matrix $A \in C^1(B_R)$, where the radius $R > 0$ is taken small enough in order to ensure that the ε-stable Sobolev embedding holds true (see Lemma 3.3.7), we obtain ε-stable L^∞ bounds for weak solutions to (3.38) (when $\varepsilon > 0$). This result is established through a standard argument based on an iteration technique involving the Caccioppoli-type inequality below and Sobolev embeddings. The proof is omitted here, as it has been carried out in a similar context in the previous chapter, see Section 2.3.

Lemma 3.4.5 (Caccioppoli inequality). *Let $a + n > 0$, $R > 0$, $0 \le \varepsilon \ll 1$, $p \ge (2_*^a)'$ if $d + a_+ > 2$ or $p > 1$ if $d + a_+ = 2$, $q \ge 2$ and let $A \in C^1(B_R)$ satisfy (3.5). Let $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon^A)$, $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon^A)^d$, and let u be a weak solution to (3.36) if $\varepsilon = 0$ or to (3.38) if $\varepsilon > 0$. Then, there exists a constant $c > 0$ depending only on d, n, a, λ and Λ such that for every $0 < R_1 < R_2 < R$ it holds*

$$\int_{B_{R_1} \setminus \Sigma_\varepsilon^A} |y|^a |\nabla u|^2 dz \le c \Big((R_2 - R_1)^{-2} \|u\|^2_{L^{2,a}(B_{R_2} \setminus \Sigma_\varepsilon^A)} + \|f\|^2_{L^{p,a}(B_{R_2} \setminus \Sigma_\varepsilon^A)} + \|F\|^2_{L^{q,a}(B_{R_2} \setminus \Sigma_\varepsilon^A)} \Big). \tag{3.39}$$

In the case $d + a_+ = 2$ the constant c depends also on p. Moreover, if $\varepsilon = 0$ it is enough to assume that $A \in L^\infty(B_R)$.

Proposition 3.4.6 (ε-stable L^∞ bounds). *Let $a + n > 0$, $0 \le \varepsilon \ll 1$, and $A \in C^1(B_R)$ satisfy (3.5), where $R > 0$ is given by Lemma 3.A.3. Furthermore, let $p > (a_+ + d)/2$, $q > a_+ + d$ and let $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon^A)$ and $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon^A)^d$ and let u be a weak solution (3.38).*

Then, for every $0 < r < R$, there exists a constant $c > 0$ depending only on d, n, a, λ, Λ, p, q, r, $\|A\|_{C^1(B_R)}$ and R such that

$$\|u\|_{L^\infty(B_r \setminus \Sigma_\varepsilon^A)} \le c \big(\|u\|_{L^{2,a}(B_R \setminus \Sigma_\varepsilon^A)} + \|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon^A)} + \|F\|_{L^{q,a}(B_R \setminus \Sigma_\varepsilon^A)} \big).$$

In addition, if $\varepsilon = 0$ and u is a weak solution to to (3.36), the same estimate holds under the assumption that $A \in L^\infty(B_R)$ and the constant does not depend on $\|A\|_{C^1(B_R)}$.

3.4.3 Approximation results

In this section, we establish two approximation results. The first one, inspired by [138, Lemma 2.15] and Lemma 1.4.2, asserts that any solution u of (3.36) ($\varepsilon = 0$) can be approximated by a family of solutions u_ε of the conormal problem on ε-perforated domains (3.38) ($0 < \varepsilon \ll 1$). Here, we assume that the matrix $A \in C^1$, in order to have a C^1 boundary $\partial\Sigma_\varepsilon^A$, see Remark 3.3.2. The second result shows that any solution u of (3.36) ($\varepsilon = 0$) can be approximated by a family of solutions to the same problem but with smooth coefficients, via a standard mollification technique.

Lemma 3.4.7. *Let $a+n>0$, $R>0$ given by Lemma 3.A.3, $r \in (0,R)$ and $A \in C^1(B_R)$ satisfying (3.5). Let $f \in L^{p,a}(B_R)$ where $p \geq (2_a^*)'$ if $d + a_+ > 2$ or $p > 1$ if $d + a_+ = 2$, $F \in L^{q,a}(B_R)^d$ where $q \geq 2$ and let u be a weak solution to (3.36).*

Then, there exists a family $\{u_\varepsilon\}_{0<\varepsilon\ll 1}$, such that u_ε are weak solutions to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u_\varepsilon) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_r \setminus \Sigma_\varepsilon^A \\ (A\nabla u + F)\cdot \nu = 0, & \text{on } \partial\Sigma_\varepsilon^A \cap B_r, \end{cases}$$

$$\|u_\varepsilon\|_{H^{1,a}(B_r\setminus\Sigma_\varepsilon^A)} \leq c\big(\|u\|_{H^{1,a}(B_R)} + \|f\|_{L^{p,a}(B_R)} + \|F\|_{L^{q,a}(B_R)}\big)\,, \tag{3.40}$$

for some constant $c > 0$ depending only on d, n, a, λ, Λ, r, R, $\|A\|_{C^1(B_R)}$ and there exists a sequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to u \text{ in } H^1_{\mathrm{loc}}(B_r \setminus \Sigma_0).$$

Moreover, in the case $d + a_+ = 2$ the constant c depends also on p.

Proof. We consider the case $d + a_+ > 2$, as the other one can be treated analogously. The proof is divided into several steps.

Step 1. Cut-off solution $\tilde u$.

Let us consider a cut-off function $\xi \in C_c^\infty(B_R)$ such that

$$\xi = 1 \text{ on } B_r\,, \qquad \mathrm{spt}(\xi) \subset B_{\frac{R+r}{2}}\,, \qquad 0 \leq \xi \leq 1\,, \qquad |\nabla\xi| \leq c_0\,,$$

for some $c_0 > 0$ depending only on d, r and R. Define $\tilde u := \xi u$, and notice that $\tilde u \in H_0^{1,a}(B_R)$ by construction.

Fix $\phi \in C_c^\infty(B_R)$. Then,

$$\begin{aligned} \int_{B_R} |y|^a A\nabla\tilde u\cdot\nabla\phi\,dz &= \int_{B_R} |y|^a\Big(\xi A\nabla u\cdot\nabla\phi + uA\nabla\xi\cdot\nabla\phi\Big)\,dz \\ &= \int_{B_R} |y|^a\Big(A\nabla u\cdot\nabla(\phi\xi) - \phi A\nabla u\cdot\nabla\xi + uA\nabla\xi\cdot\nabla\phi\Big)\,dz \\ &= \int_{B_R} |y|^a\Big(f\phi\xi - F\cdot\nabla(\phi\xi) - \phi A\nabla u\cdot\nabla\xi + uA\nabla\xi\cdot\nabla\phi\Big)\,dz \\ &= \int_{B_R} |y|^a\Big(f\phi\xi - \xi F\cdot\nabla\phi - \phi F\cdot\nabla\xi - \phi A\nabla u\cdot\nabla\xi + uA\nabla\xi\cdot\nabla\phi\Big)\,dz\,, \end{aligned}$$

that is, $\tilde u$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla \tilde u) = |y|^a \tilde f + \operatorname{div}(|y|^a \tilde F) & \text{in } B_R,\\ u=0 & \text{on } \partial B_R \end{cases} \tag{3.41}$$

where we have set

$$\tilde f = f\xi - F\cdot\nabla\xi - A\nabla u\cdot\nabla\xi, \qquad \tilde F = F\xi - uA\nabla\xi\,.$$

Step 2. Construction of the approximating solutions u_ε in perforated domains.

Let us consider the family of weak solution $\{u_\varepsilon\}_{0<\varepsilon\ll1}$ to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u_\varepsilon) = |y|^a \tilde f + \operatorname{div}(|y|^a \tilde F), & \text{in } B_R\setminus\Sigma_\varepsilon^A\\ u_\varepsilon=0 & \text{on } \partial B_R\setminus\Sigma_\varepsilon^A\\ (A\nabla u_\varepsilon+\tilde F)\cdot\nu=0 & \text{on } \partial\Sigma_\varepsilon^A\cap B_R. \end{cases} \tag{3.42}$$

By testing (3.42) with u_ε, using (3.5), the Poincaré inequality (3.27), the Sobolev type embedding in Lemma (3.3.7), Hölder and Young inequalities, we get

$$\begin{aligned}
&\lambda\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a|\nabla u_\varepsilon|^2\,dz \le \int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a A\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,dz\\
&=\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a\Big(f\xi u_\varepsilon - F\cdot\nabla\xi u_\varepsilon - A\nabla u\cdot\nabla\xi u_\varepsilon - F\xi\cdot\nabla u_\varepsilon - uA\nabla\xi\cdot\nabla u_\varepsilon\Big)\,dz\\
&\le\Big(\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a|f|^p\,dz\Big)^{\frac1p}\Big(\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a|u_\varepsilon|^{p'}\,dz\Big)^{\frac1{p'}}\\
&+c\Big(\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a|F|^q\,dz\Big)^{\frac1q}\Big(\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a(|u_\varepsilon|^2+|\nabla u_\varepsilon|^2)\,dz\Big)^{\frac12}\\
&+c\Big(\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a(u^2+|\nabla u|^2)\,dz\Big)^{\frac12}\Big(\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a(|u_\varepsilon|^2+|\nabla u_\varepsilon|^2)\,dz\Big)^{\frac12}\\
&\le\frac{\lambda}{2}\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a|\nabla u_\varepsilon|^2\,dz + c\big(\|u\|_{H^{1,a}(B_R)}+\|f\|_{L^{p,a}(B_R)}+\|F\|_{L^{q,a}(B_R)}\big)^2,
\end{aligned}$$

for some $c>0$ depending only on d, n, a, λ, Λ and R. Hence, combining this inequality and the Poincaré inequality (3.27), we have

$$\int_{B_R\setminus\Sigma_\varepsilon^A}|y|^a(|u_\varepsilon|^2+|\nabla u_\varepsilon|^2)\,dz\le c\big(\|u\|_{H^{1,a}(B_R)}+\|f\|_{L^{p,a}(B_R)}+\|F\|_{L^{q,a}(B_R)}\big)^2. \tag{3.43}$$

Step 3. Strong convergence of the sequence u_{ε_k} in $H^1_{\rm loc}(B_R\setminus\Sigma_0)$.

Let us fix two compact sets $K\subset K'\subset B_R\setminus\Sigma_0$. Then, for every ε small enough, we have $K'\subset B_R\setminus\Sigma_\varepsilon^A$. Since $|y|^a$ is uniformly bounded and bounded away from zero on K', by (3.43) we have that $\|u_\varepsilon\|_{H^1(K')}\le c$ for some constant $c>0$ that does not depend on ε. This implies that there exists a sequence $\varepsilon_k\to0^+$ and $\bar u\in H^1(K')$ such that

$$u_{\varepsilon_k}\rightharpoonup\bar u \text{ weakly in } H^1(K'),\qquad u_{\varepsilon_k}\to\bar u\ \text{ strongly in } L^2(K'). \tag{3.44}$$

By testing the equation (3.42) with $\phi^2(u_\varepsilon-\bar u)$, where $\phi\in C^\infty_c(K')$, we obtain

$$\begin{aligned}\int_{K'}|y|^a\phi^2A\nabla u_\varepsilon\cdot\nabla u_\varepsilon\,dz=\int_{K'}|y|^a\Big(&\phi^2A\nabla u_\varepsilon\cdot\nabla\bar u-2(u_\varepsilon-\bar u)\phi A\nabla u_\varepsilon\cdot\nabla\phi+\tilde f\phi^2(u_\varepsilon-\bar u)\\&-\phi^2\tilde F\cdot\nabla(u_\varepsilon-\bar u)-2\phi(u_\varepsilon-\bar u)\tilde F\cdot\nabla\phi\Big)\,dz.\end{aligned}$$

By (3.44) we can take the limit as $\varepsilon_k\to0$ in the right hand side and obtain

$$\lim_{\varepsilon_k\to0}\int_{K'}|y|^a\phi^2A\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}\,dz=\int_{K'}|y|^a\phi^2A\nabla\bar u\cdot\nabla\bar u\,dz\,. \tag{3.45}$$

Fix $\delta>0$. Take $\phi\in C^\infty_c(K')$ such that $0\le\phi\le1$, $\phi=1$ in K and $|\mathrm{spt}(\phi)\setminus K|\ll1$ so that, noticing that $|y|^aA\nabla\bar u\cdot\nabla\bar u\in L^1(B_R)$, we have $\||y|^aA\nabla\bar u\cdot\nabla\bar u\|_{L^1(\mathrm{spt}(\phi)\setminus K)}<\delta$. Then,

$$\begin{aligned}c\int_K|\nabla u_{\varepsilon_k}-\nabla\bar u|^2\,dz&\le\int_K|y|^aA(\nabla u_{\varepsilon_k}-\nabla\bar u)\cdot(\nabla u_{\varepsilon_k}-\nabla\bar u)\,dz\\&=\int_K|y|^aA\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}+|y|^aA\nabla\bar u\cdot\nabla\bar u-2|y|^aA\nabla\bar u\cdot\nabla u_{\varepsilon_k}\,dz\\&\le\int_{K'}|y|^aA\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}\,\phi^2+|y|^aA\nabla\bar u\cdot\nabla\bar u\,\phi^2\,dz-2\int_K|y|^aA\nabla\bar u\cdot\nabla u_{\varepsilon_k}\,dz\\&\le\Big|\int_{K'}|y|^aA\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}\,\phi^2\,dz-\int_{K'}|y|^aA\nabla\bar u\cdot\nabla\bar u\,\phi^2\,dz\Big|+2\int_{K'\setminus K}|y|^aA\nabla\bar u\cdot\nabla\bar u\,\phi^2\,dz\\&\le\Big|\int_{K'}|y|^aA\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}\,\phi^2\,dz-\int_{K'}|y|^aA\nabla\bar u\cdot\nabla\bar u\,\phi^2\,dz\Big|+2\delta\to2\delta\,,\qquad\text{as }\varepsilon_k\to0\,,\end{aligned}$$

thanks to (3.45). By to the arbitrariness of δ, we infer that $\nabla u_{\varepsilon_k}\to\nabla\bar u$ in $L^2(K)^d$, and thus $u_{\varepsilon_k}\to\bar u$ strongly in $H^1(K)$. Finally, a standard diagonal argument yields that

$$u_{\varepsilon_k}\to\bar u\quad\text{strongly in }H^1_{\mathrm{loc}}(B_R\setminus\Sigma_0). \tag{3.46}$$

Step 4. The limit function $\bar u\in H^{1,a}(B_R)$.

Let us now prove that $\bar u\in H^{1,a}(B_R)$. Let us fix $\phi\in C^\infty_c(B_R)$ if $a+n\in(0,2)$ or $\phi\in C^\infty_c(B_R\setminus\Sigma_0)$ if $a+n\ge2$. For every ε_k fixed, one has that $u_{\varepsilon_k}\in W^{1,1}_{\mathrm{loc}}(B_R\setminus\Sigma^A_{\varepsilon_k})$, so it holds

$$\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}\partial_{z_\ell}u_{\varepsilon_k}\phi\,dz=-\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}u_{\varepsilon_k}\partial_{z_\ell}\phi\,dz+\int_{\partial\Sigma^A_{\varepsilon_k}\cap B_R}u_{\varepsilon_k}\phi\,d\sigma\,, \tag{3.47}$$

for every $\ell=1,\dots,d$.

If $a+n\ge2$, $\mathrm{spt}(\phi)\subset B_R\setminus\Sigma^A_{\varepsilon_k}$ for every ε_k small enough, so

$$\int_{\partial\Sigma^A_{\varepsilon_k}\cap B_R}u_{\varepsilon_k}\phi\,d\sigma=0\,.$$

On the other hand, let us fix a measurable set $E\subset B_R$. By using (3.43) and observing that $|y|^{-a}\in L^\infty(\mathrm{spt}(\phi))$, we get

$$\Big|\int_E\partial_{z_\ell}u_{\varepsilon_k}\phi\,dz\Big|\le\Big(\int_{E\cap\mathrm{spt}(\phi)}|y|^a|\partial_{z_\ell}u_{\varepsilon_k}|^2\,dz\Big)^{\frac12}\Big(\int_{E\cap\mathrm{spt}(\phi)}|y|^{-a}|\phi|^2\,dz\Big)^{\frac12}$$

$$\leq C_\phi\big(\|u\|_{H^{1,a}(B_R)} + \|f\|_{L^{p,a}(B_R)} + \|F\|_{L^{q,a}(B_R)}\big)|E|^{\frac{1}{2}}\,,$$

where the constant C_ϕ depends on $\|\phi\|_{L^\infty}$ and $\mathrm{spt}(\phi)$, which implies that $\partial_{z_\ell}u_{\varepsilon_k}\phi$ is uniformly integrable. By the a.e. convergences $\nabla u_{\varepsilon_k}\to\nabla\bar u$ and $\chi_{B_R\setminus\Sigma^A_{\varepsilon_k}}\to\chi_{B_R}$, the Vitali convergence theorem yields

$$\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}\partial_{z_\ell}u_{\varepsilon_k}\phi\,dz\to\int_{B_R}\partial_{z_\ell}\bar u\phi\,dz\,.$$

By employing a similar argument, one has that

$$\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}u_{\varepsilon_k}\partial_{z_\ell}\phi\,dz\to\int_{B_R}\bar u\partial_{z_\ell}\phi\,dz\,,$$

so, by taking the limit as $\varepsilon_k\to0^+$ in (3.47) we have

$$\int_{B_R}\partial_{z_\ell}\bar u\phi\,dz=\int_{B_R}\bar u\partial_{z_\ell}\phi\,dz\,,\tag{3.48}$$

which means that $\bar u\in W^{1,1}_{\mathrm{loc}}(B_R\setminus\Sigma_0)$.

If $a+n\in(0,2)$, by using Lemma 3.3.6 combined with (3.43), we obtain

$$\begin{aligned}\Big|\int_{\partial\Sigma^A_{\varepsilon_k}\cap B_R}u_{\varepsilon_k}\phi\,d\sigma\Big| &\leq\Big(\int_{\partial\Sigma^A_{\varepsilon_k}\cap B_R}|y|^a|u_{\varepsilon_k}|^2\,d\sigma\Big)^{\frac{1}{2}}\Big(\int_{\partial\Sigma^A_{\varepsilon_k}\cap B_R}|y|^{-a}|\phi|^2\,d\sigma\Big)^{\frac{1}{2}}\\ &\leq c\,{\varepsilon_k}^{n-1}\Big(\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}|y|^a|\nabla u_{\varepsilon_k}|^2\,dz\Big)^{\frac{1}{2}}\Big(\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}|y|^{-a}|\nabla\phi|^2\,dz\Big)^{\frac{1}{2}}\\ &\leq c\,{\varepsilon_k}^{n-1}\big(\|u\|_{H^{1,a}(B_R)}+\|f\|_{L^{p,a}(B_R)}+\|F\|_{L^{q,a}(B_R)}\big)\,,\end{aligned}$$

and taking the limit $\varepsilon_k\to0^+$ we get that the term $\int_{\partial\Sigma^A_{\varepsilon_k}\cap B_R}u_{\varepsilon_k}\phi\,d\sigma$ vanishes.

Using the same argument as for the case $a+n\geq2$ and noting that $|y|^{-a}\in L^1(B_R)$, we can take the limit as $\varepsilon_k\to0^+$ in (3.47). This allows us to conclude that (3.48) holds true, which means that $\bar u\in W^{1,1}_{\mathrm{loc}}(B_R)$.

Now, using the Fatou Lemma and (3.43), we have

$$\begin{aligned}\int_{B_R}|y|^a(|\bar u|^2+|\nabla\bar u|^2)\,dz&=\int_{B_R}\lim_{\varepsilon_k\to0^+}|y|^a(|u_{\varepsilon_k}|^2+|\nabla u_{\varepsilon_k}|^2)\chi_{B_R\setminus\Sigma^A_{\varepsilon_k}}\,dz\\ &\leq\liminf_{\varepsilon_k\to0^+}\int_{B_R\setminus\Sigma^A_{\varepsilon_k}}|y|^a(|u_{\varepsilon_k}|^2+|\nabla u_{\varepsilon_k}|^2)\,dz\\ &\leq c\big(\|u\|_{H^{1,a}(B_R)}+\|f\|_{L^{p,a}(B_R)}+\|F\|_{L^{q,a}(B_R)}\big)^2\,,\end{aligned}$$

which means that $\|\bar u\|_{H^{1,a}(B_R)}<\infty$. Thus, by Lemma 3.2.7 and Lemma 3.2.8, we conclude that $\bar u\in H^{1,a}(B_R)$.

Step 5. $\bar u$ has zero trace on ∂B_R.

Finally, we prove that $\bar u$ has zero trace on ∂B_R, meaning that $T_R\bar u=0$. First, notice that for any fixed $\delta\ll1$, it follows from (3.46) that $u_{\varepsilon_k}\to\bar u$ strongly in $H^1(B_R\setminus\Sigma^A_\delta)$. Consequently, we also have $T_Ru_{\varepsilon_k}\to T_R\bar u$ strongly in $L^2(\partial B_R\setminus\Sigma^A_\delta)$. Since $T_Ru_{\varepsilon_k}=0$ by construction, we immediately get that $T_R\bar u=0$ L^2-a.e. in $\partial B_R\setminus\Sigma_0$.

Now, let us fix $\nu > 0$. Since, by Lemma 3.2.11, it holds $|y|^a|T_R\bar{u}|^2 \in L^1(\partial B_R)$, the absolute continuity of the integral ensures that there exists a sufficiently small μ such that

$$\int_{\partial B_R \cap \Sigma_\mu^A} |y|^a|T_R\bar{u}|^2 ds \leq \nu\,.$$

Thus

$$\int_{\partial B_R} |y|^a|T_R\bar{u}|^2 ds \leq \nu + \int_{\partial B_R \setminus \Sigma_\mu^A} |y|^a|T_R\bar{u}|^2 ds \leq \nu + c\int_{\partial B_R \setminus \Sigma_\mu^A} |T_R\bar{u}|^2 ds = \nu\,,$$

where in the last equality holds because $T_R\bar{u} = 0$ L^2-a.e. on $\partial B_R \setminus \Sigma_0$. Since ν is arbitrary, we conclude that $T_R\bar{u} = 0$. Therefore, again by Lemma 3.2.11, we have $u \in H^{1,a}_0(B_R)$.

Step 6. $\bar{u} = \tilde{u}$ and conclusion.

Next, we prove that $\bar{u} = \tilde{u}$. Let us test (3.42) with $\phi \in C_c^\infty(B_R)$. Then, we have

$$\int_{B_R \setminus \Sigma_{\varepsilon_k}^A} |y|^a A\nabla u_{\varepsilon_k} \cdot \nabla\phi\, dz \to \int_{B_R} |y|^a A\nabla\bar{u} \cdot \nabla\phi\, dz\,, \quad \text{as } \varepsilon_k \to 0^+. \tag{3.49}$$

In fact, let us fix $E \subset B_R$ measurable. By using (3.5), (3.43) and $|y|^a \in L^1(B_R)$, we have that

$$\begin{aligned}\int_{E \setminus \Sigma_{\varepsilon_k}^A} |y|^a|A\nabla u_{\varepsilon_k} \cdot \nabla\phi|\, dz &= \int_E |y|^a|A\nabla u_{\varepsilon_k} \cdot \nabla\phi|\chi_{E\setminus\Sigma_{\varepsilon_k}^A}\, dz \\ &\leq c\|u_{\varepsilon_k}\|_{H^{1,a}(B_R\setminus\Sigma_{\varepsilon_k}^A)}\|\nabla\phi\|_{L^\infty(B_R)}\int_E |y|^a\, dz \leq \delta(|E|)\,,\end{aligned}$$

where $\delta(|E|) \to 0$ as $|E| \to 0$. Combining this with the a.e. convergence $\nabla u_{\varepsilon_k} \to \nabla\bar{u}$ we can conclude that (3.49) holds true by applying the Vitali convergence theorem. With similar computations, we get

$$\int_{B_R \setminus \Sigma_{\varepsilon_k}^A} |y|^a(\tilde{f}\phi - \tilde{F} \cdot \nabla\phi)\, dz \to \int_{B_R} |y|^a(\tilde{f}\phi - \tilde{F} \cdot \nabla\phi)\, dz\,, \quad \text{as } \varepsilon_k \to 0^+. \tag{3.50}$$

Combining (3.49) and (3.50) we get

$$\int_{B_R} |y|^a A\nabla\bar{u} \cdot \nabla\phi\, dz = \int_{B_R} |y|^a(\tilde{f}\phi - \tilde{F} \cdot \nabla\phi)\, dz\,,$$

and since $\bar{u} \in H^{1,a}_0(B_R)$, we get that $\bar{u}$ is a weak solution to (3.41). By uniqueness of solutions (see Remark 3.4.4), we get that $\bar{u} = \tilde{u}$ a.e. in B_R. Since $\tilde{u} = u$, $\tilde{f} = f$ and $\tilde{F} = F$ in B_r, and by using (3.43), (3.46) we obtain that our statement holds true. ☐

The next approximation lemma provides the convergence of solutions of equations with regularized coefficients through convolution with standard mollifiers.

Lemma 3.4.8. *Let $a + n > 0$, $R > 0$, $r \in (0, R)$, and $A \in C^0(B_R)$ satisfying (3.5). Let $f \in L^{p,a}(B_R)$ where $p \geq (2^a_*)'$ if $d + a_+ > 2$ or $p > 1$ if $d + a_+ = 2$, $F \in L^{q,a}(B_R)^d$ where $q \geq 2$, let u be a weak solution to (3.36). For $\delta > 0$, let $\{\rho_\delta\}$ be a family of smooth mollifiers and let us define $A_\delta := A * \rho_\delta$. Then, there exists a family $\{u_\delta\}_{0<\delta\ll 1}$, such that u_δ are weak solutions to*

$$-\operatorname{div}(|y|^a A_\delta \nabla u_\delta) = |y|^a f + \operatorname{div}(|y|^a F), \quad \text{in } B_r,$$

$$\|u_\delta\|_{H^{1,a}(B_r)} \le c\big(\|f\|_{L^{p,a}(B_R)} + \|F\|_{L^{q,a}(B_R)} + \|u\|_{H^{1,a}(B_R)}\big)\,,$$

for some constant $c>0$ depending only on d, n, a, λ, Λ, r, R, and there exists a sequence $\delta_k \to 0$ such that

$$u_{\delta_k} \to u \text{ in } H^{1,a}(B_r).$$

Moreover, in the case $d+a_+ = 2$ the constant c depends also on p.

Proof. We consider the case $d+a_+>2$, as the other one can be treated analogously. Let us fix $0<r<R'<R$ and consider a smooth cut-off function $\xi \in C^\infty_c(B_R)$ such that

$$\xi = 1 \text{ in } B_r, \quad 0 \le \xi \le 1, \quad \mathrm{spt}(\phi) \subset B_{R'}.$$

Arguing as in Lemma 3.4.7, the function $\tilde u := \xi u$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla \tilde u) = |y|^a \tilde f + \operatorname{div}(|y|^a \tilde F), & \text{in } B_{R'} \\ \tilde u = 0, & \text{on } \partial B_{R'}, \end{cases} \tag{3.51}$$

where, we set

$$\tilde f = f\xi - F\cdot\nabla\xi - A\nabla u\cdot\nabla\xi, \qquad \tilde F = F\xi - uA\nabla\xi\,.$$

For $\delta>0$, let $\{\rho_\delta\}$ be a family of smooth mollifiers and let us define $A_\delta := A*\rho_\delta$, which is a uniformly elliptic matrix in $B_{R'}$ (choosing δ small enough). For every $0<\delta\ll 1$, recalling Remark 3.4.4, let u_δ be the unique weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A_\delta\nabla u_\delta) = |y|^a \tilde f + \operatorname{div}(|y|^a \tilde F), & \text{in } B_{R'} \\ u_\delta = 0, & \text{on } \partial B_{R'}. \end{cases} \tag{3.52}$$

By testing (3.52) with u_δ and arguing as in Lemma 3.4.7, we obtain

$$\|u_\delta\|_{H^{1,a}(B_{R'})} \le c\big(\|u\|_{H^{1,a}(B_R)} + \|f\|_{L^{p,a}(B_R)} + \|F\|_{L^{q,a}(B_R)}\big),$$

for some $c>0$ depending only on d, λ, Λ and R. Hence, $\{u_\delta\}$ is uniformly bounded in $H^{1,a}_0(B_{R'})$ and by using Lemma 3.2.10, we get

$$u_\delta \to v \text{ strongly in } L^{2,a}(B_{R'})\,, \qquad \nabla u_\delta \rightharpoonup \nabla v \text{ weakly in } L^{2,a}(B_{R'})^d\,, \tag{3.53}$$

for some $v \in H^{1,a}_0(B_R)$. Taking $u_\delta - v$ as test function in (3.52), using (3.53) and recalling that $A_\delta \to A$ uniformly, we get

$$\int_{B_{R'}} |y|^a A_\delta \nabla u_\delta\cdot\nabla u_\delta = \int_{B_{R'}} |y|^a\big(A_\delta\nabla u_\delta\cdot\nabla v + \tilde f(u_\delta - v) - \tilde F\cdot\nabla(u_\delta - v)\big) \to \int_{B_{R'}} |y|^a A\nabla v\cdot\nabla v.$$

Hence, since A_δ satisfies (3.5), we have that $\nabla u_\delta \to \nabla v$ strongly in $L^{2,a}(B_{R'})^d$.

Finally, let us fix $\phi \in C^\infty_c(B_{R'})$. Then,

$$\int_{B_{R'}} |y|^a(\tilde f\phi - \tilde F\cdot\nabla\phi) = \int_{B_{R'}} |y|^a A_\delta\nabla u_\delta\cdot\nabla\phi \to \int_{B_{R'}} |y|^a A\nabla v\cdot\nabla\phi\,,$$

that is, v is a weak solution to (3.51) and by uniqueness of solutions (see Remark 3.4.4), it follows that $v=\tilde u$. Next, since $\tilde f = f$, $\tilde F = F$ and $\tilde u = u$ in B_r, we have proved that $u_\delta \to u$ in $H^{1,a}(B_r)$

and u_δ solves

$$-\operatorname{div}(|y|^a A_\delta \nabla u_\delta) = |y|^a f + \operatorname{div}(|y|^a F) \quad \text{in } B_r.$$

The proof is complete. □

3.5 Liouville theorems

The aim of this section is to prove the Liouville Theorem 3.1.4 for entire solutions to (3.15), see Definition 3.4.3. First, we need some preliminary results. The first result addresses the unweighted tangential variables $x \in \mathbb{R}^{d-n}$, for which the operator is invariant under translation. Its proof uses a standard difference quotients technique and involves an iterative argument based on the Caccioppoli inequality in Lemma 3.4.5 (see for example [143, Corollary 4.2, Lemma 4.3]), and is therefore omitted here.

Proposition 3.5.1. *Let u be an entire solution to (3.15). Then*

i) for every $j = 1, \dots, d-n$, the weak derivative $\partial_{x_j} u$ is also an entire solution to (3.15);

ii) if there exists constants $c, \gamma > 0$ such that

$$|u(z)| \le c(1 + |z|^\gamma),$$

then u is a polynomial of degree at most $\lfloor \gamma \rfloor$ in the variable $x \in \mathbb{R}^{d-n}$, with coefficients depending only on the variable $y \in \mathbb{R}^n$.

In the following crucial lemmas we provide a characterization of the solutions to (3.15) with polynomial growth at infinity in the case $n = d$. The first lemma gives us a basis of $L^2(\mathbb{S}^{n-1})$ which depends on the matrix A.

Lemma 3.5.2. *Let $A \in \mathbb{R}^{n,n}$ be a constant matrix which satisfies (3.5). The following holds:*

i) there exists an increasing, diverging sequence of eigenvalues $\{\mu_k(A)\}_{k\ge 0}$, corresponding to critical levels of the quotient

$$\mathcal{R}(g) = \frac{\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |\nabla_\sigma g|^2 d\sigma}{\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |g|^2 d\sigma}.$$

Each eigenvalue has finite multiplicity, denoted by $m(\mu_k(A))$;

ii) let $g_{k,j}$, for $j = 1, \dots, m(\mu_k(A))$, be the normalized eigenfunctions associated to $\mu_k(A)$. The eigenfunctions $g_{k,j}$ satisfy the following properties:

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a \nabla_\sigma g_{k,j} \cdot \nabla_\sigma \eta \, d\sigma = \mu_k(A) \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a g_{k,j} \eta \, d\sigma, \qquad \eta \in H^1(\mathbb{S}^{n-1}), \\ \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a g_{k,j} g_{k',j'} \, d\sigma = \delta_{kk'}\delta_{jj'}, \end{aligned} \tag{3.54}$$

where δ_{ij} is the Kronecker delta.

Moreover, $g_{k,j}$ provides a basis of $L^2(\mathbb{S}^{n-1})$, which is also orthonormal with respect to the scalar product $\langle u, v\rangle_{L^2(\mathbb{S}^{n-1})} := \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a\, u\, v\, d\sigma$. Thus, for every $u \in L^2(\mathbb{S}^{n-1})$ it holds

$$u(\sigma) = \sum_{k=0}^{\infty} \sum_{j=1}^{m(\mu_k(A))} \Big(\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a u(\eta) g_{k,j}(\eta) d\eta\Big) g_{k,j}(\sigma)\,;$$

iii) the eigenvalue $\mu_0(A) = 0$ is simple and $g_0 = g_{0,1}$ is constant;

iv) the first nontrivial eigenvalue $\mu_1(A)$ is such that

$$\mu_1(A) \geq \begin{cases} \left(\dfrac{\lambda}{\Lambda}\right)^{\frac{|a|}{2}} (n-1) & \textit{if } n \geq 3\,, \\ \left(\dfrac{4}{\pi} \arctan\left(\dfrac{\lambda}{\Lambda}\right)^{\frac{|a|}{4}}\right)^2 & \textit{if } n = 2\,. \end{cases}$$

Proof. By condition (3.5), we have that

$$\lambda \leq |A^{\frac{1}{2}}\sigma|^2 \leq \Lambda\,, \qquad \text{for every } \sigma \in \mathbb{S}^{n-1}\,. \tag{3.55}$$

Thus

$$\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a (|\nabla_\sigma g|^2 + |g|^2) d\sigma$$

provides a norm equivalent to the standard one in $H^1(\mathbb{S}^{n-1})$. Keeping this in mind, we obtain *i*), *ii*) by a standard application of Hilbert-Schmidt theorem.

Since *iii*) is trivial, it remains to prove *iv*). Given $g \in H^1(\mathbb{S}^{n-1})$, let us denote

$$\langle g\rangle_A = \Big(\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a d\sigma\Big)^{-1} \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a g\, d\sigma\,,$$

and recall that

$$\inf_{\xi\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |g - \xi|^2 d\sigma = \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |g - \langle g\rangle_A|^2 d\sigma\,. \tag{3.56}$$

We have that

$$\mu_1(A) = \inf_{g\in H^1(\mathbb{S}^{n-1})\setminus\{0\}, \langle g\rangle_A = 0} \frac{\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |\nabla_\sigma g|^2 d\sigma}{\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |g|^2 d\sigma}\,. \tag{3.57}$$

Notice that the eigenvalues $\mu_k(\mathbb{I})$ of the Laplace-Beltrami operator on the sphere are well known, and in particular $\mu_1(\mathbb{I}) = n - 1$. Now, set

$$m := \min\Big\{\Big(\frac{\lambda}{\Lambda}\Big)^{\frac{a}{2}}, \Big(\frac{\Lambda}{\lambda}\Big)^{\frac{a}{2}}\Big\} = \Big(\frac{\lambda}{\Lambda}\Big)^{\frac{|a|}{2}}$$

and let $\psi \in H^1(\mathbb{S}^{n-1})$ be an eigenfunction associated to $\mu_1(A)$. Notice that $\langle\psi\rangle_A = 0$. Thus, by (3.56) we have that

$$\int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |\psi|^2 d\sigma \leq \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a |\psi - \langle\psi\rangle_{\mathbb{I}}|^2 d\sigma$$

As a consequence, thanks to (3.55) we have

$$\begin{aligned}\mu_1(A)=\frac{\int_{\mathbb{S}^{n-1}}|A^{\frac12}\sigma|^a|\nabla_\sigma\psi|^2d\sigma}{\int_{\mathbb{S}^{n-1}}|A^{\frac12}\sigma|^a|\psi|^2d\sigma}&\geq\frac{\int_{\mathbb{S}^{n-1}}|A^{\frac12}\sigma|^a|\nabla_\sigma(\psi-\langle\psi\rangle_{\mathbb{I}})|^2d\sigma}{\int_{\mathbb{S}^{n-1}}|A^{\frac12}\sigma|^a|\psi-\langle\psi\rangle_{\mathbb{I}}|^2d\sigma}\\&\geq m\frac{\int_{\mathbb{S}^{n-1}}|\nabla_\sigma(\psi-\langle\psi\rangle_{\mathbb{I}})|^2d\sigma}{\int_{\mathbb{S}^{n-1}}|\psi-\langle\psi\rangle_{\mathbb{I}}|^2d\sigma}\geq m(n-1)\,,\end{aligned}$$

where in the last inequality we used that $\langle\psi-\langle\psi\rangle_{\mathbb{I}}\rangle_{\mathbb{I}}=0$ and (3.57) with $A=\mathbb{I}$.

The improved estimate in the case $n=2$ is a consequence of [123, Lemma 1]. □

Lemma 3.5.3. *Let $a+n>0$, $\varepsilon\geq0$, and let $A\in\mathbb{R}^{n,n}$ be a constant matrix which satisfies (3.5). Let $\mu_k(A)$, $g_{k,j}$ be as in Lemma 3.5.2, and define*

$$\gamma_k^{\pm}=\frac{2-a-n\pm\sqrt{(a+n-2)^2+4\mu_k(A)}}{2}\,.$$

Let u be such that $u\in H^{1,a}(B_R\setminus\Sigma_\varepsilon^A)$ for any $R>0$ and

$$\int_{\mathbb{R}^n\setminus\Sigma_\varepsilon^A}|y|^aA\nabla u\cdot\nabla\phi=0\,,\qquad\textit{for any}\qquad\phi\in C_c^\infty(\mathbb{R}^n)\,.$$

Assume there exist constants $c,\gamma>0$ such that

$$|u(y)|\leq c(1+|y|^\gamma)\,.\tag{3.58}$$

Then there exist $C\in\mathbb{R}$, $\hat{k}=\hat{k}(\gamma)\in\mathbb{N}$ such that

$$u(y)=C+\sum_{k=1}^{\hat{k}}\sum_{j=1}^{m(\mu_k^A)}(c_{k,j}^+|A^{-\frac12}y|^{\gamma_k^+}+c_{k,j}^-|A^{-\frac12}y|^{\gamma_k^-})g_{k,j}\Big(\frac{A^{-\frac12}y}{|A^{-\frac12}y|}\Big)\,,\tag{3.59}$$

for some constants $c_{k,j}^+,c_{k,j}^-\in\mathbb{R}$. In particular, if $\varepsilon=0$, then

$$u(y)=C+\sum_{k=1}^{\hat{k}}\sum_{j=1}^{m(\mu_k^A)}c_{k,j}^+|A^{-\frac12}y|^{\gamma_k^+}g_{k,j}\Big(\frac{A^{-\frac12}y}{|A^{-\frac12}y|}\Big)\,.\tag{3.60}$$

Moreover, if $\gamma<\gamma_1^+$, then u is constant.

Proof. Since the proof we assume $d=n$, we will simply write B_R instead of B_R^n to denote general balls in $\mathbb{R}^n$. Let us define $v(\xi)=u(A^{\frac12}\xi)$. Performing some standard computations using the change of variables $\xi=A^{-\frac12}y$ and the uniform ellipticity of A, we see that $v\in H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ for any $R>0$ and satisfies

$$\int_{\mathbb{R}^n\setminus\Sigma_\varepsilon}|A^{\frac12}\xi|^a\nabla v\cdot\nabla\phi=0\,,\qquad\text{for any}\qquad\phi\in C_c^\infty(\mathbb{R}^n)\,.\tag{3.61}$$

The remaining part of the proof is divided in three steps.

Step 1: Spectral decomposition.

In the following we extensively use polar coordinates $\xi = r\sigma$, where $r = |\xi| > 0$ and $\sigma = |\xi|^{-1}\xi \in \mathbb{S}^{n-1}$. Since $v \in H^{1,a}(B_R \setminus \Sigma_\varepsilon)$ for any $R > 0$, then by the Fubini-Tonelli theorem it holds that $v(r\cdot) \in H^1(\mathbb{S}^{n-1})$ for a.e. $r > \varepsilon$. Thus, for a.e. $r > \varepsilon$, we can decompose $v(r\cdot)$ via the basis $\{g_{k,j}\}$ (see Lemma 3.5.2, *ii*)), obtaining

$$v(r\sigma) = \sum_{k=0}^{\infty} \sum_{j=1}^{m(\mu_k^A)} v_{k,j}(r) g_{k,j}(\sigma)\,,$$

where the coefficients $v_{k,j}$ depend on $r \in [\varepsilon, \infty)$, and are such that

$$v_{k,j}(r) = \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a u(r\sigma) g_{k,j} d\sigma\,. \tag{3.62}$$

By previous discussions, the functions $v_{k,j} : [\varepsilon, \infty) \to \mathbb{R}$ are well defined for a.e. $r \geq \varepsilon$. Moreover, using once again that $v \in H^{1,a}(B_R \setminus \Sigma_\varepsilon)$ for any $R > 0$ we can see that

$$v_{k,j} \in H^1((\varepsilon, R), r^{a+n-1}dr) \quad \text{for any } R > 0\,, \qquad v_{k,j}'(r) = \int_{\mathbb{S}^{n-1}} |A^{\frac{1}{2}}\sigma|^a \nabla v(r\sigma) \cdot \sigma g_{k,j}(\sigma) d\sigma\,. \tag{3.63}$$

Step 2: Solutions of an associated ODE.

Let us rewrite (3.61) in polar coordinates as

$$\int_\varepsilon^\infty \int_{\mathbb{S}^{n-1}} r^{a+n-1} |A^{\frac{1}{2}}\sigma|^a \Big(\partial_r v \partial_r \phi + r^{-2} \nabla_\sigma v \cdot \nabla_\sigma \phi\Big) dr\, d\sigma \;= 0\,, \qquad \text{for any } \phi \in C_c^\infty(\mathbb{R}^n)\,. \tag{3.64}$$

Now, let us take any $f \in C_c^\infty([0, \infty))$ such that

$$\mu_k(A) \int_\varepsilon^\infty r^{a+n-3} |f|^2 dr < \infty\,. \tag{3.65}$$

Notice that (3.65) is always satisfied if $\varepsilon > 0$, or if $k = 0$ (by Lemma 3.5.2, *iii*)). Thanks to (3.54) and (3.65) we have that $\phi(r\sigma) = f(r) g_{k,j}(\sigma) \in H^{1,a}(B_R \setminus \Sigma_\varepsilon)$ for any $R > 0$. Moreover, ϕ is compactly supported, and we can use it as a test function in (3.64). Recalling also (3.62) and (3.63), we obtain that $v_{k,j}$ satisfies

$$\int_\varepsilon^\infty r^{a+n-1} (f' v_{k,j}' + r^{-2} \mu_k(A) f v_{k,j}) dr = 0\,, \qquad \text{for every } f \in C_c^\infty([\varepsilon, \infty)) \text{ satisfying (3.65)}\,. \tag{3.66}$$

Now, let us take any $\varphi \in C_c^\infty(\mathbb{R})$, and test (3.66) with $f(r) = \varphi(\log(r))$. Notice that such f is admissible. Performing the change of variables $r = e^\tau$, we find that the function $w_{k,j}(\tau) = v_{k,j}(e^\tau)$ satisfies

$$\int_{\log\varepsilon}^\infty e^{\tau(a+n-2)} (w_{k,j}' \varphi' + \mu_k(A) w_{k,j} \varphi)\, d\tau = 0\,, \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R})\,.$$

Equivalently, we have that $w_{k,j}$ is a solution to the elementary equation

$$w_{k,j}'' + (a + n - 2) w_{k,j}' - \mu_k(A) w_{k,j} = 0 \quad \text{in} \quad [\log\varepsilon, \infty)\,.$$

Recalling that the multiplicity of the first eigenvalue $m(\mu_0(A))=1$, one has that

$$v_{0,1}(r)=\begin{cases} c_1+c_2r^{2-a-n}, & \text{if } 2-a-n\neq 0,\\ c_1+c_2\log r, & \text{if } 2-a-n=0,\end{cases} \tag{3.67}$$

for some constants $c_1,c_2\in\mathbb{R}$. Furthermore, defining

$$\gamma_k^{\pm}=\frac{2-a-n\pm\sqrt{(a+n-2)^2+4\mu_k(A)}}{2},$$

we readily get that $v_{k,j}$ is in the form

$$v_{k,j}(r)=c_{k,j}^{+}r^{\gamma_k^+}+c_{k,j}^{-}r^{\gamma_k^-}, \qquad \text{if } k\geq 1,$$

for some constant $c_{k,j}^+,c_{k,j}^-\in\mathbb{R}$. We point out that

$$\gamma_0^{\pm}=\pm(2-a-n)_{\pm}, \qquad \text{and} \qquad \gamma_k^+>0, \quad \gamma_k^-<0 \quad \text{if } k\geq 1. \tag{3.68}$$

Step 3: Conclusion.

First, let us show that $v_{0,1}$ is constant. Since $\mu_0(A)=0$, condition (3.65) is always satisfied. Thus, we can take $f\in C_c^\infty([\varepsilon,\infty))$ such that $f(\varepsilon)=1$ in (3.66) and recalling (3.67), we find

$$0=\int_\varepsilon^\infty r^{a+n-1}(f'v_{0,1}'+r^{-2}\mu_0(A)fv_{0,1})\,dr=c_2(2-a-n)\int_\varepsilon^\infty f'\,dr=c_2(a+n-2),$$

if $2-a-n\neq 0$, which implies that $c_2=0$. Instead, if $2-a-n=0$, we find

$$0=\int_\varepsilon^\infty r^{a+n-1}(f'v_{0,1}'+r^{-2}\mu_0(A)fv_{0,1})\,dr=c_2\int_\varepsilon^\infty f'\,dr=-c_2,$$

and we conclude as well.

Let us now focus on the case $k\geq 1$. First, let us assume $\varepsilon=0$. By (3.63) we have that $v_{k,j}'\in L^2((0,1),r^{a+n-1}dr)$. As a consequence,

$$(c_{k,j}^-\gamma_k^-)^2\int_0^1 r^{a+n-3+2\gamma_k^-}\,dr<\infty.$$

Thanks to (3.68), this readily implies $c_{k,j}^-=0$. Now, by condition (3.58) and thanks to (3.5) we have that

$$|v_{k,j}|\leq\int_{\mathbb{S}^{n-1}}|A^{\frac12}\sigma|^a|v(r\sigma)||g_{k,j}|d\sigma\leq c(1+r^\gamma).$$

Therefore, since for every γ there exists $\hat{k}$ such that $\gamma_{\hat{k}}^+>\gamma$, we easily infer that $c_{k,j}^+=0$ for every $k\geq\hat{k}$, that is, $v_{k,j}\equiv 0$ for every $k\geq\hat{k}$. Recalling that $u(y)=v(A^{-\frac12}y)$, we readily get (3.60).

Let now consider the case $\varepsilon>0$. Arguing as before, we find that condition (3.58) implies that there exists $\hat{k}$ such that for every $k\geq\hat{k}$ it holds $\gamma_k^+>\gamma$ and

$$v_{k,j}=c_{k,j}^-r^{\gamma_k^-}.$$

Let $k \geq \hat{k}$ be fixed. Since $\varepsilon > 0$, condition (3.65) is always satisfied. Thus we can take $f \in C_c^\infty([\varepsilon, \infty))$ such that $f(\varepsilon) = 1$ in (3.66) and we find, integrating by parts and performing some standard computations

$$c_{k,j}^- \gamma_k^- \varepsilon^{a+n-2+\gamma_k^-} = 0\,.$$

Since $k \geq 1$, then $\gamma_k^- \neq 0$ and we infer that $c_{k,j}^- = 0$, that is, $v_{k,j} \equiv 0$. Also in this case we use $u(y) = v(A^{-\frac{1}{2}}y)$ to complete the proof of (3.59).

Finally, we notice that if $\gamma < \gamma_1^+$ then in both cases it holds $v_{k,j} \equiv 0$ for every $k \geq 1$. Since we have already proved that $v_{0,1}$ is constant, then v is also constant, and u too. □

Proof of Theorem 3.1.4. Let A be as in the assumptions, and recall the notation (3.18). We first assume that $\gamma < 1$. Thus, by Proposition 3.5.1 we have that $u = u(y)$ does not depend on x, which implies that u is an entire solution to (3.15) with $d = n$ and $A = A_3$. Hence, u satisfies the assumptions of Lemma 3.5.3, and using that $\gamma < \gamma_1^+$, we conclude that u must be constant.

Let now assume that $\gamma < 2$. By Proposition 3.5.1 we have that u is a polynomial of degree at most 1 in the x variable; that is, u can be written in the form

$$u(x,y) = u^0(y) + \sum_{j=1}^{d-n} u^j(y) x_j\,.$$

First, we notice that by (3.16) it holds

$$|u^0(y)| = |u(0,y)| \leq c(1 + |y|^\gamma)\,.$$

Thus

$$|u^j(y)| = |u(e_{x_j}, y) - u^0(y)| \leq c(1 + |y|^\gamma)\,; \tag{3.69}$$

that is, each component u^0, u^j still satisfies the growth condition (3.16).

On the other hand, we have that $u^j(y) = \partial_{x_j} u$ for $j = 1, \ldots, d-n$ and thus, again by Proposition 3.5.1, they are entire solutions to (3.15). Keeping (3.69) into account, and using that any u^j does not depend on x, we can argue as in the previous part of the proof and conclude that they are constant.

Summing up, we proved that there exists $\alpha = \alpha(u) \in \mathbb{R}^{d-n}$ constant such that

$$u(x,y) = u^0(y) + \alpha \cdot x\,.$$

We claim that the function u^* given by

$$u^*(x,y) = \alpha \cdot x - (A_3^{-1} A_2^\top \alpha) \cdot y$$

is an entire solution to (3.15). Indeed, we can easily compute

$$A \nabla u^* = (A_1 \alpha - A_2 A_3^{-1} A_2^\top \alpha, 0),$$

and thus for every $\phi \in C_c^\infty(\mathbb{R}^d)$ it holds

$$\int_{\mathbb{R}^d \setminus \Sigma_\varepsilon^A} |y|^a A \nabla u^* \cdot \nabla \phi \, dz = \int_{|A_3^{-\frac{1}{2}} y| \geq \varepsilon} |y|^a (A_1 \alpha - A_2 A_3^{-1} A_2^\top \alpha) \cdot \Big(\int_{\mathbb{R}^{d-n}} \nabla_x \phi dx \Big) dy = 0\,,$$

where in the last equality we used the divergence theorem.

By linearity, the function $\hat{u} = u - u^*$ is an entire solution to (3.15). Moreover, since we are assuming $\gamma \in [1,2)$, it holds

$$|\hat{u}| \le |u| + |u^*| \le c(1+|z|^\gamma) + c|z| \le c(1+|z|^\gamma)\,,$$

that is, $\hat{u}$ satisfies (3.16). Since by construction $\hat{u} = \hat{u}(y)$ does not depend on x, we can argue as in the $\gamma < 1$ case and infer that $\hat{u}$ is constant, by Lemma 3.5.3 and using that $\gamma < \gamma_1^+$. In conclusion, we showed that

$$u = \hat{u} + u^* = c + \alpha \cdot x + \beta \cdot y\,,$$

as needed. $\square$

3.6 Stable regularity estimates in perforated domains

In this section we prove Theorem 3.1.3, namely, the stable regularity estimates in perforated domains for weak solutions to (3.13), that is, for the problem

$$\begin{cases} -\mathrm{div}(|y|^a A\nabla u) = |y|^a f + \mathrm{div}(|y|^a F) & \text{in } B_1 \setminus \Sigma_\varepsilon^A \\ (A\nabla u + F)\cdot \nu = 0 & \text{on } B_1 \cap \partial\Sigma_\varepsilon^A. \end{cases}$$

For simplicity, we set this section in the unit ball B_1; however, the analysis should be carried out in a smaller ball B_R, where $R > 0$ is specified in Proposition 3.4.6. We divide the proof into two parts, obtaining first the Hölder estimate for solutions and then the Hölder estimate for their gradient.

3.6.1 Stable $C^{0,\alpha}$ estimates

Proof of Theorem 3.1.3, i), stable $C^{0,\alpha}$-estimates. Let u_ε be a family of solutions to (3.13). Since the weight $|y|^a$ is uniformly elliptic away from Σ_0, and since the boundary $\partial\Sigma_\varepsilon^a$ is of class C^1 by the assumption $A \in C^1$ with $\|A\|_{C^{1,\omega}(B_1)} \le L$ for some modulus of continuity ω (see Remark 3.3.2), well-known results in elliptic regularity imply that for every $0 < \varepsilon \ll 1$, there exists a constant C_ε (depending only on d, n, a, λ, Λ, p, q, α, L and ε) such that, for any solution u_ε to (3.13), the following estimate holds:

$$\|u_\varepsilon\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon^A)} \le C_\varepsilon\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon^A)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)}\big). \tag{3.70}$$

Our goal is to show that C_ε remains uniformly bounded as $\varepsilon \to 0$. The proof proceeds by contradiction and is divided into several steps.

Step 1. Contradiction argument and blow-up sequences.

Assume, for the sake of contradiction, that there exist two sequences ε_k and u_k such that $\varepsilon_k \to 0$ as $k \to \infty$, the functions u_k are non trivial, satisfy (3.13) with $\varepsilon = \varepsilon_k$ and it holds that

$$\|u_k\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k}^A)} \ge k\big(\|u_k\|_{L^{2,a}(B_1\setminus\Sigma_{\varepsilon_k}^A)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)}\big). \tag{3.71}$$

By Proposition 3.4.6, inequality (3.71) implies the existence of a constant $c > 0$, independent of k, such that

$$[u_k]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k}^A)} \ge c\,k\|u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k}^A)}\,.$$

Now, let $\eta \in C_c^\infty(B_{3/4})$ be a function satisfying $0 \le \eta \le 1$ and $\eta = 1$ on $B_{1/2}$. We define

$$M_k = [\eta u_k]_{C^{0,\alpha}(B_1 \setminus \Sigma^A_{\varepsilon_k})}\,,$$

and observe that

$$M_k \ge [u_k]_{C^{0,\alpha}(B_{1/2} \setminus \Sigma^A_{\varepsilon_k})} \ge c\, k \|u_k\|_{L^\infty(B_{3/4} \setminus \Sigma^A_{\varepsilon_k})}\,. \tag{3.72}$$

Consider two sequences of points $z_k = (x_k, y_k), \hat{z}_k = (\hat{x}_k, \hat{y}_k) \in B_1 \setminus \Sigma^A_{\varepsilon_k}$ such that

$$\frac{|(\eta u_k)(z_k) - (\eta u_k)(\hat{z}_k)|}{|z_k - \hat{z}_k|^\alpha} \ge \frac{1}{2} M_k\,. \tag{3.73}$$

Without loss of generality, we may assume that $z_k \in B_{3/4}$. Define $r_k = |z_k - \hat{z}_k|$. From (3.72) and (3.73) we obtain

$$ck\|u_k\|_{L^\infty(B_{3/4} \setminus \Sigma^A_{\varepsilon_k})} \le \frac{|(\eta u_k)(z_k) - (\eta u_k)(\hat{z}_k)|}{r_k^\alpha} \le \frac{2\|u_k\|_{L^\infty(B_{3/4} \setminus \Sigma^A_{\varepsilon_k})}}{r_k^\alpha}\,.$$

This implies

$$r_k \le ck^{-\frac{1}{\alpha}} \to 0\,, \qquad \text{as } k \to \infty\,.$$

Let us define $d_k = \operatorname{dist}(z_k, \partial\Sigma^A_{\varepsilon_k})$. From now on we distinguish three distinct cases:

Case 1: $\dfrac{d_k}{r_k} \to \infty$ as $k \to \infty$,

Case 2: $\dfrac{d_k}{r_k} \le c$ uniformly in k, and $\dfrac{|y_k|}{r_k} \to \infty$ as $k \to \infty$.

Case 3: $\dfrac{|y_k|}{r_k} \le c$ uniformly in k.

Define

$$Z_k = (x_k, Y_k) = \left(x_k, \tau \frac{y_k}{|y_k|}\right),$$

where τ is chosen such that $Z_k \in \partial\Sigma^A_{\varepsilon_k}$. Such a point exists, is unique, and satisfies $\tau < |y_k|$, thanks to Lemma 3.A.1 point *ii*). Therefore,

$$|y_k| > |y_k| - \tau = |y_k - Y_k| = |z_k - Z_k| \ge d_k\,.$$

As a result, along a suitable subsequence, the three cases are mutually exclusive and collectively exhaustive, covering all possible scenarios. Heuristically, in **Case 1**, the blow-up scale does not capture $\partial\Sigma^A_{\varepsilon_k}$; in **Case 2**, $\partial\Sigma^A_{\varepsilon_k}$ is visible, but Σ_0 is not; and in **Case 3**, both Σ_0 and $\partial\Sigma^A_{\varepsilon_k}$ are visible (although they may or may not coincide in the rescaled limit).

Let $z_k^0 = (x_k^0, y_k^0)$ denote a chosen projection of z_k onto $\partial\Sigma^A_{\varepsilon_k}$, and define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) = \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k^0, y_k^0), & \text{in } \textbf{Case 2}, \\ (x_k, 0), & \text{in } \textbf{Case 3}. \end{cases}$$

We observe that, by construction, $|z_k - \tilde{z}_k| \le cr_k$.

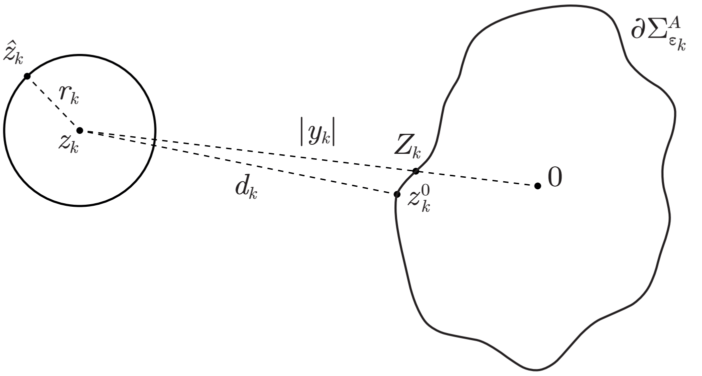

Fig. 3.1 This image describes **Case 1** when $n = d = 2$. In this case we have $d_k/r_k \to \infty$ and $\tilde z_k = z_k$.

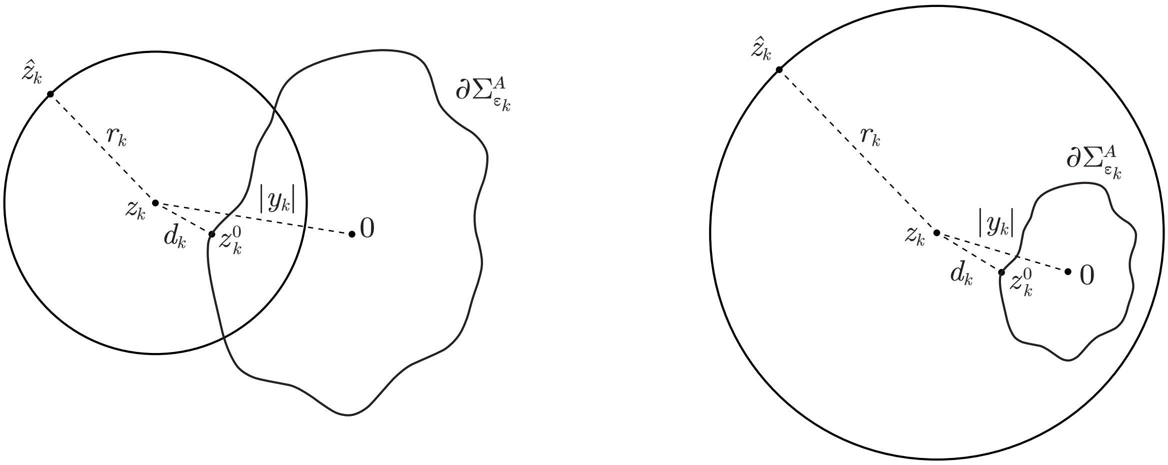

Fig. 3.2 The images on the left and on the right describe respectively **Case 2** and **Case 3** when $n = d = 2$. In **Case 2** we have $d_k/r_k \le c$, $|y_k|/r_k \to \infty$ and $\tilde z_k = z_k^0$. In **Case 3** we have $|y_k|/r_k \le c$ and $\tilde z_k = 0$.

We introduce the sequence of rescaled domains

$$\Omega_k = \frac{B_1 \setminus \Sigma^A_{\varepsilon_k} - \tilde z_k}{r_k} = \Big\{ z = (x,y) \in \mathbb{R}^d \mid |\tilde z_k + r_k z| < 1, A_3^{-1}(\tilde z_k + r_k z)(\tilde y_k + r_k y)\cdot(\tilde y_k + r_k y) \ge \varepsilon_k^2 \Big\},$$

and the limit blow-up domain

$$\Omega_\infty = \{ z = (x,y) \in \mathbb{R}^d \mid \text{ exists } \hat k, r > 0 \text{ s.t. } B_r(z) \subset \Omega_k \text{ for every } k \ge \hat k \}. \tag{3.74}$$

Note that, by this definition, Ω_∞ is an open set.

Then, we introduce the sequences of functions

$$v_k(z) = \frac{(\eta u_k)(\tilde z_k + r_k z) - (\eta u_k)(z_k)}{r_k^\alpha M_k}, \qquad \text{for } z \in \Omega_k\,,$$

and

$$w_k(z) = \frac{\eta(z_k)(u_k(\tilde z_k + r_k z) - u_k(z_k))}{r_k^\alpha M_k}, \qquad \text{for } z \in \Omega_k\,.$$

Step 2. Blow-up limit domains.

Our goal is to characterize Ω_∞ along a suitable subsequence. As previously noted, the three distinct regimes described in **Cases 1, 2, 3** correspond to different scenarios, leading to distinct limit blow-up domains. In **Case 1**, we *do not see* the rescaled boundary of the perforation at any scale of the blow-up. Thus, the hole moves farther and farther away from the blow-up centers and the limit domain Ω_∞ coincides with the entire space $\mathbb{R}^d$. In **Case 2**, we *see* the rescaled boundary of the perforation at all scales, whereas the rescaled center of the perforation remains invisible and increasingly distant. As a result, the rescaled boundary flattens progressively and the limit domain Ω_∞ becomes a half-space. Finally, in **Case 3**, both the rescaled boundary and the center of the perforation are visible at all scales, leading, as limit domain, to a (potentially) perforated space. In the next part, we will rigorously justify this visual intuition, suggested by Figures 1 and 2.

Notice that for every $z \in \mathbb{R}^d$ we have $|\tilde{z}_k + r_k z| < 1$ for sufficiently large k. Indeed, since $|z_k - \tilde{z}_k| \le c r_k$ and $r_k \to 0$, then

$$|\tilde{z}_k + r_k z| \le |z_k| + |z_k - \tilde{z}_k| + r_k |z| \le \frac{3}{4} + c r_k < 1\,.$$

Thus, to characterize Ω_∞, it suffices to determine, for a given $z \in \mathbb{R}^d$, whether a neighborhood of the point $z_k + r_k z$ lies inside or outside $\Sigma^A_{\varepsilon_k}$.

We claim that $\Omega_\infty = \mathbb{R}^d$ in **Case 1**, that is, for every $z \in \mathbb{R}^d$, a neighborhood of $z_k + r_k z$ lies outside of $\Sigma^A_{\varepsilon_k}$ for sufficiently large k. Indeed, let $z \in \mathbb{R}^d$ be fixed, and assume by contradiction that $z_k + r_k z \in \Sigma^A_{\varepsilon_k}$. Since $z_k \notin \Sigma^A_{\varepsilon_k}$, there exists a point $P_k = t_k z_k + (1 - t_k)(z_k + r_k z)$ with $t_k \in [0, 1)$, such that $P_k \in \partial\Sigma^A_{\varepsilon_k}$. As a consequence,

$$\infty \leftarrow \frac{d_k}{r_k} \le \frac{|z_k - P_k|}{r_k} = \frac{|(1 - t_k) r_k z|}{r_k} \le |z|\,,$$

which leads to a contradiction.

Let us now consider **Case 2**. Recall that $\tilde{z}_k = z^0_k$ and $d_k = |z_k - z^0_k| \le c\, r_k$. Furthermore, note that by the uniform ellipticity condition,

$$\infty \leftarrow \frac{|y_k|}{r_k} \le \frac{|y_k - y^0_k|}{r_k} + \frac{|y^0_k|}{r_k} \le \frac{d_k}{r_k} + c\frac{\varepsilon_k}{r_k} \le c\Big(1 + \frac{\varepsilon_k}{r_k}\Big)\,.$$

Therefore, in this case, we have $\varepsilon_k / r_k \to \infty$ as $k \to \infty$.

Since $z^0_k \in \partial\Sigma^A_{\varepsilon_k}$, a point $z \in \mathbb{R}^d$ satisfies $z^0_k + r_k z \notin \Sigma^A_{\varepsilon_k}$ if and only if

$$\frac{\sqrt{A_3^{-1}(z^0_k + r_k z)(y^0_k + r_k y) \cdot (y^0_k + r_k y)} - \sqrt{A_3^{-1}(z^0_k) y^0_k \cdot y^0_k}}{r_k} > 0\,. \tag{3.75}$$

To proceed, we must analyze the behavior of (3.75) as $k \to \infty$.

We define the function $\Psi : \mathbb{R}^d \to \mathbb{R}$ as

$$\Psi(z) = \sqrt{A_3^{-1}(z) y \cdot y}\,.$$

It follows that

$$\Psi(z) = \varepsilon_k, \qquad \text{for every } z \in \partial\Sigma^A_{\varepsilon_k}\,.$$

Moreover, the gradient of Ψ is given by

$$\nabla\Psi(z) = \frac{(0, A_3^{-1}(z)y)}{\Psi(z)} + \frac{G(z)}{\Psi(z)}\,,$$

where the function $G : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$G(z) = \frac{1}{2}\Big(\sum_{i,j=1}^{n} \partial_{z_\ell} b_{i,j}(z) y_i y_j\Big)_{\ell=1,\dots,d}\,, \qquad \text{with} \qquad A_3^{-1}(z) = (b_{i,j}(z))_{i,j=1,\dots,n}.$$

As a result, $\Psi \in C^{0,1}(B_1) \cap C^1(B_1 \setminus \Sigma_0)$, and it satisfies $\|\nabla\Psi\|_{L^\infty(B_1)} \le cL$. Next we show that, for every $z \in \mathbb{R}^d$, it holds

$$\frac{\Psi(z_k^0 + r_k z) - \Psi(z_k^0)}{r_k} - \frac{A_3^{-1}(z_k^0)y_k^0}{\Psi(z_k^0)} \cdot y \to 0, \quad \text{as } k \to \infty. \tag{3.76}$$

By the mean value theorem, there exists a point $z_k^* = (x_k^*, y_k^*) \in \mathbb{R}^d$ of the form $z_k^* = z_k^0 + \theta_k r_k z$ with $\theta_k \in [0,1]$, such that

$$\frac{\Psi(z_k^0 + r_k z) - \Psi(z_k^0)}{r_k} = \nabla\Psi(z_k^*) \cdot z = \frac{A_3^{-1}(z_k^*)y_k^*}{\Psi(z_k^*)} \cdot y + \frac{G(z_k^*)}{\Psi(z_k^*)} \cdot z\,. \tag{3.77}$$

Using (3.5), we estimate

$$\Big|\frac{G(z_k^*)}{\Psi(z_k^*)} \cdot z\Big| \le cL|y_k^*| \le cL(|y_k^0| + cr_k) \le c(\varepsilon_k + r_k)\,, \tag{3.78}$$

and

$$\begin{aligned}
&\Big|\frac{A_3^{-1}(z_k^*)y_k^*}{\Psi(z_k^*)} - \frac{A_3^{-1}(z_k^0)y_k^0}{\Psi(z_k^0)}\Big| \le \Big|\frac{A_3^{-1}(z_k^*)y_k^*}{\Psi(z_k^*)} - \frac{A_3^{-1}(z_k^0)y_k^*}{\Psi(z_k^0)}\Big| + \Big|\frac{A_3^{-1}(z_k^0)y_k^*}{\Psi(z_k^0)} - \frac{A_3^{-1}(z_k^0)y_k^0}{\Psi(z_k^0)}\Big| \\
&\le \Big|\frac{A_3^{-1}(z_k^*)y_k^*}{\Psi(z_k^*)} - \frac{A_3^{-1}(z_k^0)y_k^*}{\Psi(z_k^*)}\Big| + \Big|\frac{A_3^{-1}(z_k^0)y_k^*}{\Psi(z_k^*)} - \frac{A_3^{-1}(z_k^0)y_k^*}{\Psi(z_k^0)}\Big| + \frac{c}{\varepsilon_k}|y_k^* - y_k^0| \\
&\le c|z_k^* - z_k^0|\frac{|y_k^*|}{|\Psi(z_k^*)|} + c|y_k^*|\frac{|\Psi(z_k^0) - \Psi(z_k^*)|}{|\Psi(z_k^0)||\Psi(z_k^*)|} + c\frac{r_k}{\varepsilon_k} \le cr_k + c\frac{r_k}{\varepsilon_k}.
\end{aligned} \tag{3.79}$$

Combining (3.77), (3.78), and (3.79), we obtain

$$\Big|\frac{\Psi(z_k^0 + r_k z) - \Psi(z_k^0)}{r_k} - \frac{A_3^{-1}(z_k^0)y_k^0}{\Psi(z_k^0)} \cdot y\Big| \le c(r_k + \varepsilon_k + \frac{r_k}{\varepsilon_k}) \to 0\,,$$

since $r_k/\varepsilon_k \to 0$. This completes the proof of (3.76).

Next we define, up to subsequences (recall that $z_k^0 \in B_1$),

$$\bar{e} = \lim_{k\to\infty} \frac{A_3^{-1}(z_k^0)y_k^0}{|A_3^{-1}(z_k^0)y_k^0|} \in \mathbb{S}^{n-1}, \qquad \text{and} \qquad \Pi_{\bar{e}} = \{z = (x,y) \mid \bar{e} \cdot y > 0\}.$$

We claim that $\Omega_\infty = \Pi_{\bar{e}}$, which corresponds to a half-space.

Fix $z \in \Pi_{\bar{e}}$. Suppose by contradiction that (3.75) does not hold. Then, by (3.76), we have

$$0 \geq \frac{\Psi(z_k^0)}{|A_3^{-1}(z_k^0)y_k^0|} \frac{\Psi(z_k^0 + r_k z) - \Psi(z_k^0)}{r_k} = o(1) + \frac{A_3^{-1}(z_k^0)y_k^0}{|A_3^{-1}(z_k^0)y_k^0|} \cdot y \to \bar{e} \cdot y > 0\,,$$

where we used that, by the uniform ellipticity condition,

$$\frac{\lambda}{\Lambda^{\frac{1}{2}}} \leq \frac{\Psi(z_k^0)}{|A_3^{-1}(z_k^0)y_k^0|} \leq \frac{\Lambda}{\lambda^{\frac{1}{2}}}\,.$$

Therefore we reach a contradiction, and we infer that $\Pi_{\bar{e}} \subset \Omega_\infty$.

Now, fix $z \in \mathbb{R}^d \setminus \overline{\Pi}_{\bar{e}}$, that is, such that $\bar{e} \cdot y < 0$ and assume that (3.75) holds. Then,

$$0 \leq \frac{\Psi(z_k^0)}{|A_3^{-1}(z_k^0)y_k^0|} \frac{\Psi(z_k^0 + r_k z) - \Psi(z_k^0)}{r_k} \to \bar{e} \cdot y < 0\,,$$

which is a contradiction. Hence $\Omega_\infty \subset \Pi_{\bar{e}}$ and the claim is proved.

Finally, let us consider **Case 3**. Recall that in this case $\tilde{z}_k = (x_k, 0)$, and $|y_k| \leq c r_k$. Moreover, we note that

$$\varepsilon_k < |A_3^{-\frac{1}{2}}(z_k)y_k| \leq \lambda^{-\frac{1}{2}}|y_k| \leq c r_k\,,$$

which implies that $\varepsilon_k/r_k \leq c$ in this case. Thus we define the following limits

$$(\bar{x}, 0) = \lim_{k\to\infty}(x_k, 0), \quad \bar{A} = A(\bar{x}, 0) = \lim_{k\to\infty} A(\tilde{z}_k), \quad \bar{\varepsilon} = \lim_{k\to\infty}\frac{\varepsilon_k}{r_k} \in [0, \infty)\,.$$

We claim that $\Omega_\infty = \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}}^{\bar{A}}$. Let us fix $z \in \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}}^{\bar{A}}$, that is, such that $\bar{A}_3^{-1} y \cdot y > \bar{\varepsilon}^2$. Assume by contradiction that $\tilde{z}_k + r_k z \in \Sigma_{\varepsilon_k}^{A}$, that is,

$$A_3^{-1}(\tilde{z}_k + r_k z) r_k y \cdot r_k y \leq \varepsilon_k^2\,.$$

Dividing the previous inequality by r_k^2 and taking the limit as $k \to \infty$, we obtain

$$\bar{\varepsilon}^2 < \bar{A}_3^{-1} y \cdot y \leq \bar{\varepsilon}^2\,,$$

a contradiction. Thus $\mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}}^{\bar{A}} \subseteq \Omega_\infty$. By performing similar computations, we get that every $z \in \Sigma_{\bar{\varepsilon}}^{\bar{A}}$ does not belong to Ω_k. Therefore, $\Omega_\infty \subseteq \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}}^{\bar{A}}$ and the claim is proved. To summarize, we have shown that

$$\Omega_\infty = \begin{cases} \mathbb{R}^d, & \text{in } \textbf{Case 1}, \\ \Pi_{\bar{e}}, & \text{in } \textbf{Case 2}, \\ \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}}^{\bar{A}}, & \text{in } \textbf{Case 3}. \end{cases} \tag{3.80}$$

Step 3. Hölder estimates and convergence of the blow-up sequences.

Let $z, z' \in \Omega_k$. It holds

$$|v_k(z) - v_k(z')| = \frac{|(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k + r_k z')|}{r_k^\alpha M_k} \leq |z - z'|^\alpha\,.$$

Therefore,

$$[v_k]_{C^{0,\alpha}(\Omega_k)} \leq 1\,. \tag{3.81}$$

Now, fix a compact set $K \subset \Omega_\infty$, and observe that $K \subset \Omega_k$ for sufficiently large k. Let $z \in K$, and consider the sequence of points

$$\xi_k = \frac{z_k - \tilde{z}_k}{r_k}\,.$$

Note that $\xi_k \in \Omega_k$, $|\xi_k| \le c$ uniformly in k, and $v_k(\xi_k) = 0$ for every k. Then, by (3.81), we obtain

$$|v_k(z)| = |v_k(z) - v_k(\xi_k)| \le |z - \xi_k|^\alpha \le c(K)\,,$$

which implies that $\|v_k\|_{C^{0,\alpha}(K)} \le c(K)$. Hence, we can apply the Arzelá-Ascoli theorem to conclude that $v_k \to \bar{v}$ uniformly in K. By an exhaustion of Ω_∞ via compact subsets $K \subset \Omega_\infty$ and a standard diagonal argument, we extend $\bar{v}$ to the entire Ω_∞ and obtain that $\bar{v}$ satisfies

$$[\bar{v}]_{C^{0,\alpha}(\Omega_\infty)} \le 1\,. \tag{3.82}$$

Let us now show that $w_k \to \bar{v}$ uniformly on compact sets. Fix a compact set $K \subset \Omega_\infty$. We claim that

$$\sup_{z\in K} |v_k(z) - w_k(z)| \to 0 \quad \text{as } k \to \infty. \tag{3.83}$$

Let $z \in K \subset \Omega_\infty$ be fixed. Since $z \in \Omega_k$ for large k, we have $\tilde{z}_k + r_k z \in B_\tau \setminus \Sigma^A_{\varepsilon_k}$ for some $\tau < 4/5$ depending only on K. Thus, recalling also that $|z_k - \tilde{z}_k| \le c r_k$, we get

$$\begin{aligned}|v_k(z) - w_k(z)| &= \frac{|u(\tilde{z}_k + r_k z)||\eta(\tilde{z}_k + r_k z) - \eta(z_k)|}{r_k^\alpha M_k}\\ &\le \frac{c\|u\|_{L^\infty(B_\tau\setminus\Sigma^A_{\varepsilon_k})}(|z_k - \tilde{z}_k| + r_k|z|)}{r_k^\alpha M_k} \le c k^{-1} r_k^{1-\alpha}\,,\end{aligned}$$

where in the last inequality we used (3.72). Since $\alpha < 1$, we immediately get (3.83). Therefore, since $v_k \to \bar{v}$ uniformly, we conclude that $w_k \to \bar{v}$ uniformly as well.

Step 4. The limit function is not constant.

Let us consider the sequences of points

$$\xi_k = \frac{z_k - \tilde{z}_k}{r_k}, \quad \hat{\xi}_k = \frac{\hat{z}_k - \tilde{z}_k}{r_k}\,.$$

As already pointed out, $\xi_k \in \Omega_k$, $|\xi_k| \le c$ uniformly in k, and $v_k(\xi_k) = 0$ for every k. In fact, it also holds $\hat{\xi}_k \in \Omega_k$ for every k, and since $|\xi_k - \hat{\xi}_k| = 1$, we also have $|\hat{\xi}_k| \le c$ uniformly in k. Moreover, by (3.73) (recall that $r_k = |z_k - \hat{z}_k|$) we get

$$|v_k(\hat{\xi}_k)| = |v_k(\xi_k) - v_k(\hat{\xi}_k)| = \frac{|\eta u_k(z_k) - \eta u_k(\hat{z}_k)|}{r_k^\alpha M_k} \ge \frac{1}{2}\,.$$

Now, using that $|\xi_k|, |\hat{\xi}_k| \le c$, one can see that there exist $\xi, \hat{\xi} \in \bar{\Omega}_\infty$ such that $\xi_k \to \xi$, $\hat{\xi}_k \to \hat{\xi}$ and $\xi \ne \hat{\xi}$. Thus, by a simple continuity argument, $\bar{v} \le \delta$ in a neighbourhood of ξ, and $\bar{v} \ge 1/2 - \delta$ in a neighbourhood of $\hat{\xi}$, for some small $\delta \in (0, 1/10)$. Therefore $\bar{v}$ is not constant.

Step 5. The limit function is an entire solution to a homogeneous equation.

Let us denote $A_k(z) = A(\tilde{z}_k + r_k z)$. Since $\tilde{z}_k \in B_1$ and $A \in C^1(B_1)$, up to consider a subsequence the Arzelá-Ascoli theorem yields that

$$\bar{z} = \lim_{k\to\infty} \tilde{z}_k \quad \text{and} \quad \bar{A} = A(\bar{z}) = \lim_{k\to\infty} A_k(z)\,,$$

where $\bar{A}$ is a constant coefficients symmetric matrix satisfying (3.5). Next, we define

$$\rho_k(y) = \begin{cases} \dfrac{|\tilde{y}_k + r_k y|}{|\tilde{y}_k|}, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

noticing that, in **Case 1** and **Case 2**, $r_k/|\tilde{y}_k| \to 0$, and thus $\rho_k \to 1$ a.e. in every compact set $K \subset \mathbb{R}^d$.

Let us fix $\phi \in C_c^\infty(\mathbb{R}^d)$, and notice that for k large, $\mathrm{spt}(\phi) \subset \frac{B_1 - \tilde{z}_k}{r_k}$. Since u_k is solution to (3.13) with $\varepsilon = \varepsilon_k$, we obtain that w_k satisfies

$$\begin{aligned}\int_{\Omega_k} \rho_k^a(y) A_k(z) \nabla w_k(z) \cdot \nabla \phi(z)\, dz = \frac{\eta(z_k) r_k^{2-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z)\, dz \\ - \frac{\eta(z_k) r_k^{1-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla\phi(z)\, dz\,.\end{aligned} \tag{3.84}$$

First, we prove that the right-hand side of (3.84) vanishes as $k \to \infty$. We compute

$$\begin{aligned}\Big|\int_{\Omega_k} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz\Big| &\le \Big(\int_{\Omega_k} \rho_k^a(y) |f(\tilde{z}_k + r_k z)|^p dz\Big)^{\frac{1}{p}} \Big(\int_{\mathrm{spt}(\phi)} \rho_k^a(y) |\phi(z)|^{p'} dz\Big)^{\frac{1}{p'}} \\ &\le c\|\phi\|_{L^\infty(\mathbb{R}^d)} \Big(\int_{\Omega_k} \rho_k^a(y) |f(\tilde{z}_k + r_k z)|^p dz\Big)^{\frac{1}{p}}.\end{aligned}$$

By (3.71) and (3.72), and using that $r_k \le c|\tilde{y}_k|$ in **Case 1** and **Case 2**, we get

$$\Big(\int_{\Omega_k} \rho_k^a(y) |f(\tilde{z}_k + r_k z)|^p dz\Big)^{\frac{1}{p}} \le c k^{-1} r_k^{-\frac{d+a_+}{p}} M_k\,.$$

Therefore, thanks to the assumption $\alpha \le 2 - \frac{d+a_+}{p}$, we get

$$\Big|\frac{\eta(z_k) r_k^{2-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz\Big| \le c\|\phi\|_{L^\infty(\mathbb{R}^d)} k^{-1} r_k^{2-\alpha-\frac{d+a_+}{p}} \to 0\,.$$

Performing similar computations, we see that

$$\Big|\frac{\eta(z_k) r_k^{1-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla\phi(z)\, dz\Big| \le c\|\nabla\phi\|_{L^2(\mathbb{R}^d, \rho_k^a dz)} k^{-1} r_k^{1-\alpha-\frac{d+a_+}{q}} \to 0\,,$$

thanks to the assumption $\alpha \le 1 - \frac{d+a_+}{q}$. Thus, the right hand side in (3.84) vanishes as $k \to \infty$.

It is useful to keep explicit track of the dependence of ϕ in the previous computation. Let R be such that $\mathrm{spt}(\phi) \subset R$. Previous part of the proof can be reformulated in the following way: there exists $\delta_k > 0$ such that $\delta_k \to 0$ and

$$\int_{\Omega_k} \rho_k^a(y) A_k(z) \nabla w_k(z) \cdot \nabla \phi(z)\, dz \le \delta_k (\|\phi\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \phi\|_{L^2(\mathbb{R}^d, \rho_k^a dz)}) \,. \tag{3.85}$$

We are now in position to show that the left hand side of (3.84) satisfies

$$\int_{\Omega_k \cap \mathrm{spt}(\phi)} \rho_k^a A_k \nabla w_k \cdot \nabla \phi \, dz \to \int_{\Omega_\infty \cap \mathrm{spt}(\phi)} \bar\rho^a\, \bar A \nabla \bar v \cdot \nabla \phi\, dz \,, \tag{3.86}$$

where

$$\bar\rho(y) = \begin{cases} 1 & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}\,, \\ |y| & \text{in } \textbf{Case 3}\,. \end{cases}$$

First, observe that since $u_k \in H^{1,a}(B_1 \setminus \Sigma^A_{\varepsilon_k})$, it follows that $w_k \in H^1(\Omega_k, \rho_k^a dz)$. Fix $R > 0$. By *Step 3*, the sequence w_k is uniformly bounded in $L^\infty(\Omega_k \cap B_{2R})$. Let $\varphi \in C^\infty_c(B_{2R})$ be such that $0 \le \varphi \le 1$ and $\varphi = 1$ in B_R. Testing (3.85) with $\phi = w_k \varphi^2$, and applying (3.5) along with Hölder and Young inequalities, standard computations yield

$$\int_{\Omega_k} \rho_k^a |\nabla(\varphi^2 w_k)|^2 \, dz \le c(\|w_k\|^2_{L^\infty(\Omega_k \cap B_{2R})} + 1)\,.$$

Using the uniform L^∞–bound of $\{w_k\}$ in $\Omega_k \cap B_{2R}$, and the fact that $\varphi = 1$ in B_R, we conclude that $\{w_k\}$ is uniformly bounded in $H^1(\Omega_k \cap B_R, \rho_k^a dz)$. Finally, arguing as in Lemma 3.4.7 with minor adjustments, and using that $A_k(z) \to \bar A$, $\rho_k^a \to \bar\rho^a$ almost everywhere, we conclude that (3.86) holds, and that $\bar v$ belongs to $H^1(\Omega_\infty \cap B_R, \bar\rho^a dz)$ for every $R > 0$. Summing up, we proved that

$$\int_{\Omega_\infty} \bar\rho^a \bar A \nabla \bar v \cdot \nabla \phi \, dz = 0 \qquad \text{for every } \phi \in C^\infty_c(\mathbb{R}^d)\,.$$

Hence, recalling the definition of the limit blow-up domain Ω_∞, see (3.80), we infer that $\bar v$ is an entire solution to:

Case 1

$$-\operatorname{div}(\bar A \nabla \bar v) = 0, \quad \text{in } \mathbb{R}^d\,;$$

Case 2

$$\begin{cases} -\operatorname{div}(\bar A \nabla \bar v) = 0, & \text{in } \Pi_{\bar\varepsilon} \\ \bar A \nabla \bar v \cdot \nu = 0 & \text{on } \partial \Pi_{\bar\varepsilon}; \end{cases}$$

Case 3, if $\bar\varepsilon = 0$

$$-\operatorname{div}(|y|^a \bar A \nabla \bar v) = 0, \quad \text{in } \mathbb{R}^d\,,$$

Case 3, if $\bar\varepsilon > 0$

$$\begin{cases} -\operatorname{div}(|y|^a \bar A \nabla \bar v) = 0 & \text{in } \mathbb{R}^d \setminus \Sigma^{\bar A}_{\bar\varepsilon}\,, \\ \bar A \nabla \bar v \cdot \nu = 0 & \text{on } \Sigma^{\bar A}_{\bar\varepsilon}\,. \end{cases}$$

Step 6. Liouville theorems and contradiction.

From (3.82) we have that $\bar v$ satisfies

$$|\bar v(z)| \le C(1 + |z|)^\alpha,$$

for every $z \in \Omega_\infty$, where $\alpha < \min\{\alpha_*, 1\}$, by the assumption (3.8). By invoking appropriate Liouville-type Theorems we get that $\bar{v}$ must be constant. In **Case 1**, we use the classical Liouville Theorem in the whole space; in **Case 2**, we use the Liouville Theorem in an half space with an homogeneous conormal boundary condition (for instance, see [143, Theorem 1.6]); in **Case 3**, we use Theorem 3.1.4. Hence, we reach a contradiction, since $\bar{v}$ is not constant by *Step 4*. Thus, (3.70) holds with a constant C uniformly bounded in ε. The proof is complete. □

Remark 3.6.1. We point out that our theory can be extended to equations including lower order terms. In particular, we highlight its connection to the study of eigenvalue problems for the Laplacian in domains with small holes (see [67, 121, 125] and the references therein) - which corresponds to our case $d = n$. More generally, for $2 \leq n \leq d$, let us consider the eigenvalue problem

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon, & \text{in } \Omega \setminus \Sigma_\varepsilon, \\ \nabla u_\varepsilon \cdot \nu = 0, & \text{on } \partial\Sigma_\varepsilon \cap \Omega. \end{cases} \tag{3.87}$$

Theorem 3.1.3, *i)*, which establishes local ε-stable Hölder estimates, extends naturally to equations like (3.87), yielding

$$\|u_\varepsilon\|_{C^{0,\alpha}(K\setminus\Sigma_\varepsilon)} \leq C\lambda_\varepsilon \|u_\varepsilon\|_{L^2(\Omega\setminus\Sigma_\varepsilon)},$$

for every $\alpha \in (0,1)$ and compact set $K \subset \Omega$. The constant $C > 0$ depends only on d, n, α, $\mathrm{dist}(K, \partial\Omega)$.

3.6.2 Stable $C^{1,\alpha}$ estimates

Proof of Theorem 3.1.3, ii), stable $C^{1,\alpha}$-estimates. Let u_ε be a family of solutions to (3.13). Since the weight $|y|^a$ is uniformly elliptic away from Σ_0, well-known results in elliptic regularity imply that for every $0 < \varepsilon \ll 1$ (so that $\partial\Sigma_\varepsilon^A \cap B_1$ is of class $C^{1,\alpha}$ being $A \in C^{1,\alpha}$ with $\|A\|_{C^{1,\alpha}(B_1)} \leq L$, see Remark 3.3.2), there exists a constant C_ε (depending only on d, n, a, λ, Λ, p, α, L and ε) such that for any solution u_ε to (3.13) the following estimate holds

$$\|u_\varepsilon\|_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon^A)} \leq C_\varepsilon\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon^A)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big). \tag{3.88}$$

Our goal is to show that C_ε remains uniformly bounded as $\varepsilon \to 0$. The proof proceeds by contradiction and is divided into several steps.

Step 1. Contradiction argument, preliminary estimates and blow-up sequences.

Assume, for the sake of contradiction, that there exist two sequences ε_k and u_k such that $\varepsilon_k \to 0$ as $k \to \infty$, the functions u_k are non trivial, satisfy (3.13) with $\varepsilon = \varepsilon_k$ and it holds that

$$\|u_k\|_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k}^A)} \geq k\big(\|u_k\|_{L^{2,a}(B_1\setminus\Sigma_{\varepsilon_k}^A)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big). \tag{3.89}$$

Consider $\zeta_k \in B_{1/2} \setminus \Sigma_{\varepsilon_k}^A$ such that

$$\frac{1}{2}\|\nabla u_k\|_{L^\infty(B_{1/2}\setminus\Sigma_{\varepsilon_k}^A)} \leq |\nabla u_k(\zeta_k)|\,,$$

and compute

$$
\begin{aligned}
|\nabla u_k(\zeta_k)|^2 &= \Big(\int_{B_{1/2}\setminus\Sigma^A_{\varepsilon_k}} |y|^a dz\Big)^{-1} \int_{B_{1/2}\setminus\Sigma^A_{\varepsilon_k}} |y|^a |\nabla u_k(\zeta_k)|^2 dz \\
&\leq c\Big(\int_{B_{1/2}\setminus\Sigma^A_{\varepsilon_k}} |y|^a|\nabla u_k(z)-\nabla u_k(\zeta_k)|^2 dz + \int_{B_{1/2}\setminus\Sigma^A_{\varepsilon_k}} |y|^a|\nabla u_k|^2 dz\Big) \\
&\leq c\Big([\nabla u_k]^2_{C^{0,\alpha}(B_{1/2}\setminus\Sigma^A_{\varepsilon_k})} + \int_{B_{1/2}\setminus\Sigma^A_{\varepsilon_k}} |y|^a|\nabla u_k|^2 dz\Big)\,.
\end{aligned}
$$

Combining the last two inequalities with the Caccioppoli inequality (3.39), we conclude that there exists a constant $c>0$ independent on k such that

$$
\|\nabla u_k\|_{L^\infty(B_{1/2}\setminus\Sigma^A_{\varepsilon_k})} \leq c\big([\nabla u_k]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma^A_{\varepsilon_k})} + \|u_k\|_{L^{2,a}(B_1\setminus\Sigma^A_{\varepsilon_k})} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big)\,.
$$

Therefore, by also applying Proposition 3.4.6, we deduce that inequality (3.89) implies the existence of a constant $c>0$, independent of k, such that

$$
[\nabla u_k]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma^A_{\varepsilon_k})} \geq c\,k\Big(\|u_k\|_{L^{2,a}(B_1\setminus\Sigma^A_{\varepsilon_k})} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\Big)\,.
$$

Now, let $\eta\in C^\infty_c(B_{3/4})$ be a function such that $0\leq\eta\leq 1$ and $\eta=1$ on $B_{1/2}$. We define

$$
M_k = [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1\setminus\Sigma^A_{\varepsilon_k})}\,,
$$

and observe that

$$
M_k \geq [\nabla u_k]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma^A_{\varepsilon_k})} \geq c\,k\Big(\|u_k\|_{L^{2,a}(B_1\setminus\Sigma^A_{\varepsilon_k})} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\Big)\,. \tag{3.90}
$$

It is worth noting that, by combining (3.90) with Theorem 3.1.3, *i)*, for every $\tau<1$ and $\beta\in(0,1)$ it holds

$$
\|u_k\|_{C^{0,\beta}(B_\tau\setminus\Sigma^A_{\varepsilon_k})} \leq ck^{-1}M_k\,. \tag{3.91}
$$

Additionally, since $\eta u_k\equiv 0$ in $B_1\setminus B_{3/4}$, we also have

$$
\|\nabla(\eta u_k)\|_{L^\infty(B_1\setminus\Sigma^A_{\varepsilon_k})} \leq cM_k\,. \tag{3.92}
$$

As a consequence of (3.90), (3.91), and (3.92), we immediately infer that

$$
\|\eta A\nabla u_k + \eta F\|_{C^{0,\alpha}(B_1\setminus\Sigma^A_{\varepsilon_k})} \leq cM_k\,. \tag{3.93}
$$

Crucially, the boundary condition satisfied by u_k on $\partial\Sigma^A_{\varepsilon_k}$ gives us an estimate on the y-components of the field $\eta A\nabla u_k+\eta F$. In fact, by Lemma 3.A.2 and (3.93), for every point $z^*_k\in\partial\Sigma^A_{\varepsilon_k}\cap B_1$ and for every $i=1,\ldots,n$, it holds

$$
|(\eta A\nabla u_k+\eta F)(z^*_k)\cdot e_{y_i}| \leq c\,\varepsilon_k^\alpha\|\eta A\nabla u_k+\eta F\|_{C^{0,\alpha}(B_1\setminus\Sigma^A_{\varepsilon_k})} \leq c\,\varepsilon_k^\alpha M_k. \tag{3.94}
$$

Next, consider two sequences of points $z_k=(x_k,y_k)$, $\hat z_k=(\hat x_k,\hat y_k)\in B_1\setminus\Sigma^A_{\varepsilon_k}$ such that

$$
\frac{|\nabla(\eta u_k)(z_k)-\nabla(\eta u_k)(\hat z_k)|}{|z_k-\hat z_k|^\alpha} \geq \frac{1}{2}M_k\,. \tag{3.95}
$$

Without loss of generality, we can always assume that $z_k \in B_{3/4}$. Let $r_k = |z_k - \hat{z}_k|$. Unlike the proof of Theorem 3.1.3, *i)*, establishing that $r_k \to 0$ as $k \to \infty$ is not straightforward here, so we have to proceed more carefully. Let us denote $d_k = \mathrm{dist}(z_k, \partial\Sigma^A_{\varepsilon_k})$. From now on, we distinguish three cases.

$$\textbf{Case 1:} \quad \frac{d_k}{r_k} \to \infty \text{ as } k \to \infty\,,$$

$$\textbf{Case 2:} \quad \frac{d_k}{r_k} \le c \text{ uniformly in } k\,, \quad \text{and} \quad \frac{|y_k|}{r_k} \to \infty \text{ as } k \to \infty\,.$$

$$\textbf{Case 3:} \quad \frac{|y_k|}{r_k} \le c \text{ uniformly in } k\,.$$

We refer to the proof of Theorem 3.1.3, *i)* for remarks on these cases. We recall that $d_k \le |y_k|$, and we note that in **Case 1** and **Case 2**, $r_k \to 0$.

Let $z_k^0 = (x_k^0, y_k^0)$ denote a chosen projection of z_k onto $\partial\Sigma^A_{\varepsilon_k}$, and define

$$\tilde{z}_k = \begin{cases} z_k, & \text{in } \textbf{Case 1}, \\ z_k^0, & \text{in } \textbf{Case 2} \\ (x_k, 0) & \text{in } \textbf{Case 3}. \end{cases}$$

We observe that, by construction, $|z_k - \tilde{z}_k| \le c r_k$. Moreover, in **Case 2** and **Case 3** it holds

$$|z_k^0| \le |z_k| + d_k \le \frac{3}{4} + c r_k\,,$$

so that for sufficiently large k there exists $\tau < 4/5$ such that $z_k^0 \in B_\tau \setminus \Sigma^A_{\varepsilon_k}$.

As done in Theorem 3.1.3, *i)* let us define the sequence of domains

$$\Omega_k = \frac{B_1 \setminus \Sigma^A_{\varepsilon_k} - \tilde{z}_k}{r_k}$$

and the limit domain Ω_∞ as in (3.74). Next we introduce the points

$$\xi_k = \frac{z_k - \tilde{z}_k}{r_k}\,.$$

Notice that $\xi_k \in \Omega_k$ for every k, and $|\xi_k| \le c$. Thus, up to subsequences, $\xi_k \to \xi \in \overline{\Omega}_\infty$.

Finally, we introduce the sequences of functions

$$v_k(z) = \frac{\eta(\tilde{z}_k + r_k z)(u_k(\tilde{z}_k + r_k z) - u_k(z_k)) - \eta(z_k)\nabla u_k(z_k) \cdot r_k(z - \xi_k)}{r_k^{1+\alpha} M_k}, \qquad \text{for } z \in \Omega_k\,,$$

and

$$w_k(z) = \frac{\eta(\tilde{z}_k)(u_k(\tilde{z}_k + r_k z) - u_k(z_k)) - P_k \cdot r_k(z - \xi_k)}{r_k^{1+\alpha} M_k}, \qquad \text{for } z \in \Omega_k\,.$$

where

$$P_k = \begin{cases} \eta(z_k)\nabla u_k(z_k), & \text{in } \textbf{Case 1}, \\ \eta(z_k^0)\nabla u_k(z_k^0), & \text{in } \textbf{Case 2}, \\ A^{-1}(\tilde{z}_k)((\eta A \nabla u_k)_1(z_k^0) - (\eta F)_2(z_k^0)), & \text{in } \textbf{Case 3}. \end{cases}$$

Thanks to (3.90), (3.91), and (3.92), we readily infer

$$|P_k| \le \|\nabla(\eta u_k)\|_{L^\infty(B_1\setminus\Sigma^A_{\varepsilon_k})} + c\|u_k\|_{L^\infty(B_\tau\setminus\Sigma^A_{\varepsilon_k})} + c\|F\|_{L^\infty(B_1)} \le cM_k\,, \tag{3.96}$$

where, in **Case 2** and **Case 3**, we used that $z_k^0 \in B_\tau \setminus \Sigma^A_{\varepsilon_k}$ for some $\tau < 4/5$.

Step 2. Hölder gradient estimates and convergence of v_k.

Let $z, z' \in \Omega_k$. We have

$$\begin{aligned}|\nabla v_k(z) - \nabla v_k(z')| &\le \frac{|\nabla(\eta u_k)(\tilde z_k + r_k z) - \nabla(\eta u_k)(\tilde z_k + r_k z')|}{r_k^\alpha M_k} \\ &\quad + \frac{|u(z_k)||\nabla\eta(\tilde z_k + r_k z) - \nabla\eta(\tilde z_k + r_k z')|}{r_k^\alpha M_k} \\ &\le |z - z'|^\alpha + c\frac{\|u\|_{L^\infty(B_{3/4}\setminus\Sigma^A_{\varepsilon_k})}}{M_k} r_k^{1-\alpha}|z - z'| \\ &\le (1 + ck^{-1}|z - z'|^{1-\alpha})|z - z'|^\alpha\,,\end{aligned}$$

where the last inequality follows from (3.91). Therefore, for every fixed compact set $K \subset \mathbb{R}^d$, we can choose k sufficiently large so that $[\nabla v_k]_{C^{0,\alpha}(\Omega_k\cap K)} \le 2\,$. Moreover, since $\nabla v_k(\xi_k) = 0$ and $|\xi_k| \le c$, it follows that

$$\|\nabla v_k\|_{L^\infty(\Omega_k\cap K)} = \sup_{z\in\Omega_k\cap K} |\nabla v_k(z) - \nabla v_k(\xi_k)| \le 2 \sup_{z\in\Omega_k\cap K} |z - \xi_k|^\alpha \le c(K). \tag{3.97}$$

Next, we prove that for every compact set $K \subset \Omega_\infty$, it holds $\|v_k\|_{L^\infty(K)} \le c(K)$. We claim that there exists a compact set $K' \supset K$ such that, for every $z \in K$ and all sufficiently large k, there exists a curve γ_k, depending on z, satisfying

$$\gamma_k : (0,1) \to \Omega_k, \quad \gamma_k(0) = \xi_k, \quad \gamma_k(1) = z, \quad \|\gamma_k\|_{C^{0,1}(0,1)} \le c(1 + |z|), \quad \mathrm{spt}(\gamma_k) \subset \Omega_k \cap K'\,, \tag{3.98}$$

for some constant $c > 0$ independent of z and k. Assuming the claim, we use (3.97), (3.98), and $v_k(\xi_k) = 0$ to obtain

$$|v_k(z)| := \left|\int_0^1 \nabla v_k(\gamma_k(t))\cdot\gamma_k'(t)\,dt\right| \le \|\nabla v_k\|_{L^\infty(\Omega_k\cap K')}\|\gamma_k\|_{C^{0,1}(0,1)} \le c(1 + |z|^{1+\alpha}) \le c(K), \tag{3.99}$$

which implies $\|v_k\|_{L^\infty(K)} \le c(K)$.

We now prove the claim. In **Case 1** we take γ_k to be the straight-line segment from z to ξ_k. Since $\Omega_\infty = \mathbb{R}^d$ (see Theorem 3.1.3, *i)*, *Step 2*), it follows that $\gamma_k \subset \Omega_k$ for sufficiently large k. The set K' can then be chosen accordingly, noting also that $|\xi_k| \le c$.

In **Case 2**, consider the point $\xi_k^* := \xi_k + y_k/|y_k|$. By Lemma 3.A.1, *ii)*, the segment joining ξ_k and ξ_k^* lies entirely within Ω_k. We next show that $\xi_k^* \in \Omega_\infty = \{y\cdot\bar e > 0\}$. Recalling that $\xi_k \to \xi \in \{y\cdot\bar e \ge 0\}$ and $A_3^{-1}(z_k^0)y_k^0/|A_3^{-1}(z_k^0)y_k^0| \to \bar e$, we compute, for sufficiently large k,

$$\begin{aligned}\bar e\cdot\left(\xi_k + \frac{y_k}{|y_k|}\right) &\ge \bar e\cdot\left(\frac{y_k}{|y_k|}\right) - |\xi_k - \xi| \ge \frac{A_3^{-1}(z_k^0)y_k^0}{|A_3^{-1}(z_k^0)y_k^0|}\cdot\frac{y_k}{|y_k|} - \left|\bar e - \frac{A_3^{-1}(z_k^0)y_k^0}{|A_3^{-1}(z_k^0)y_k^0|}\right| + o(1) \\ &\ge \frac{A_3^{-1}(z_k^0)y_k^0}{|A_3^{-1}(z_k^0)y_k^0|}\cdot\frac{y_k^0}{|y_k^0|} - \left|\frac{y_k}{|y_k|} - \frac{y_k^0}{|y_k^0|}\right| + o(1) \ge \frac{\lambda}{\Lambda} - \left|\frac{y_k|y_k^0| - |y_k|y_k^0}{|y_k^0||y_k|}\right| + o(1)\end{aligned}$$

$$\geq \frac{\lambda}{\Lambda} - 2\frac{|y_k - y_k^0|}{|y_k|} + o(1) = \frac{\lambda}{\Lambda} - 2\frac{d_k}{r_k}\frac{r_k}{|y_k|} + o(1) \geq \frac{\lambda}{\Lambda} + o(1)\,,$$

where we used the uniform ellipticity condition (3.5) and the assumptions of **Case 2**, namely, $d_k/r_k \leq c$ and $|y_k|/r_k \to \infty$. Therefore, for sufficiently large k, we have that $\xi_k^* \cdot \bar{e} \geq c \geq 0$, which implies that $\xi_k^* \in \Omega_\infty$. Since Ω_∞ is convex, the segment connecting z and ξ_k^* lies entirely in Ω_∞, hence in Ω_k for sufficiently large k. We thus define γ_k as the concatenation of the segment from ξ_k to ξ_k^*, and the segment from ξ_k^* to z, and we easily see that γ_k satisfies (3.98).

We now focus on **Case 3**. Given a point $\zeta \in \Omega_k$, let ζ^{**} denote its projection onto the cylinder

$$\mathcal{C}_k := \frac{\partial\Sigma_{2\Lambda^{1/2}\varepsilon_k} - \tilde{z}_k}{r_k} = \{z \in \mathbb{R}^d \mid |y| = 2\Lambda^{1/2}\varepsilon_k/r_k\}\,.$$

Note that, by Lemma 3.A.1, *ii*), the segment joining ζ and ζ^{**} lies entirely within Ω_k, at least for sufficiently large k (otherwise it may intersect the spherical part of $\partial\Omega_k$). Now, fix $z \in K \subset \Omega_\infty$. Consider a path contained in $\mathcal{C}_k \cap \Omega_k$ that connects z^{**} to ξ_k^{**}. This path consist of a translation in the x-variable, followed by a geodesic on the n-dimensional sphere $\partial B_{2\Lambda\varepsilon_k/r_k}$. We then define γ_k as the concatenation of the segment connecting z to z^{**}, the aforementioned path connecting z^{**} and ξ_k^{**} within $\mathcal{C}_k$, and the segment connecting ξ_k^{**} to ξ_k. Since in **Case 3** we have $\varepsilon_k/r_k \leq c$, it is straightforward to verify that γ_k satisfies (3.98). To conclude, it suffices to choose a compact set $K' \supset K$ large enough so that $\mathrm{spt}(\gamma_k) \subset K' \cap \Omega_k$ for every $z \in K$ and all sufficiently large k. The claim is proved.

Summing up, we have shown that for every $K \subset \Omega_\infty$ it holds $\|v_k\|_{C^{1,\alpha}(K)} \leq c(K)$. Applying Arzelá-Ascoli theorem, together with an exhaustion of Ω_∞ by compact subsets and a standard diagonal argument, we conclude that, for every $\gamma \in (0,\alpha)$, it holds $v_k \to \bar{v}$ in $C^{1,\gamma}_{\mathrm{loc}}(\Omega_\infty)$. Moreover, using (3.99) we obtain the growth estimate

$$|\bar{v}(z)| \leq c(1 + |z|^{1+\alpha}). \tag{3.100}$$

Step 3. Convergence of w_k.

First, we show that for sufficiently large k, there exists a constant $c > 0$ such that

$$\frac{|\eta(z_k)\nabla u_k(z_k) - P_k|}{r_k^\alpha M_k} \leq c\,. \tag{3.101}$$

In **Case 1**, this result is straightforward. In **Case 2**, recall that $z_k, z_k^0 \in B_\tau \setminus \Sigma^A_{\varepsilon_k}$ for some $\tau < 4/5$ and $|z_k - z_k^0| \leq c r_k$. By (3.91) and (3.92), we have

$$\begin{aligned}\frac{|(\eta\nabla u_k)(z_k) - (\eta\nabla u_k)(z_k^0)|}{r_k^\alpha M_k} &\leq \frac{|\nabla(\eta u_k)(z_k) - \nabla(\eta u_k)(z_k^0)|}{r_k^\alpha M_k} + \frac{|(u_k\nabla\eta)(z_k) - (u_k\nabla\eta)(z_k^0)|}{r_k^\alpha M_k}\\ &\leq \frac{|z_k - z_k^0|^\alpha}{r_k^\alpha} + c\frac{\|u_k\|_{C^{0,\alpha}(B_\tau\setminus\Sigma^A_{\varepsilon_k})}}{M_k}\frac{|z_k - z_k^0|^\alpha}{r_k^\alpha} \leq c\,.\end{aligned} \tag{3.102}$$

For **Case 3**, note first that $|z_k^0 - \tilde{z}_k| \leq d_k + |y_k| \leq c r_k$, $\varepsilon_k \leq c r_k$, and

$$A^{-1}(\tilde{z}_k)(\eta A\nabla u_k)(z_k^0) - P_k = (\eta A\nabla u_k + \eta F)_2(z_k^0).$$

Therefore, using (3.102), (3.92) and (3.94), we have

$$\begin{aligned}\frac{|(\eta\nabla u_k)(z_k)-P_k|}{r_k^\alpha M_k}&\leq\frac{|(\eta\nabla u_k)(z_k)-(\eta\nabla u_k)(z_k^0)|}{r_k^\alpha M_k}+\frac{|A^{-1}(z_k^0)-A^{-1}(\tilde z_k)||(\eta A\nabla u_k)(z_k^0)|}{r_k^\alpha M_k}\\&+\frac{|A^{-1}(\tilde z_k)(\eta A\nabla u_k)(z_k^0)-P_k|}{r_k^\alpha M_k}\leq c\Big(1+\frac{|(\eta A\nabla u_k+\eta F)_2(z_k^0)|}{r_k^\alpha M_k}\Big)\leq c\Big(1+\Big(\frac{\varepsilon_k}{r_k}\Big)^\alpha\Big)\leq c\,.\end{aligned}$$

Thus, in all three cases, (3.101) is satisfied. As a consequence, up to subsequences, we can define

$$V=\lim_{k\to\infty}\frac{\eta(z_k)\nabla u_k(z_k)-P_k}{r_k^\alpha M_k}\qquad\text{and}\qquad\bar w(z)=\bar v(z)-V\cdot(z-\xi)\,. \tag{3.103}$$

where $\xi\in\bar\Omega_\infty$ is such that $\xi_k\to\xi$. To conclude, we show that $w_k\to\bar w$ uniformly on every compact set $K\subset\Omega_\infty$. Indeed, we have

$$\begin{aligned}\sup_{z\in K}|v_k(z)-V\cdot(z-\xi)-w_k(z)|&\leq\sup_{z\in K}\frac{|\eta(\tilde z_k+r_kz)-\eta(\tilde z_k)||u_k(\tilde z_k+r_kz)-u_k(z_k)|}{r_k^{1+\alpha}M_k}\\&+\Big|\frac{\eta(z_k)\nabla u_k(z_k)-P_k}{r_k^\alpha M_k}-V\Big||z-\xi_k|+|V||\xi-\xi_k|\\&\leq c\Big(\frac{\|u_k\|_{C^{0,\alpha}(B_\tau\setminus\Sigma^A_{\varepsilon_k})}}{M_k}+o(1)\Big)\leq c(k^{-1}+o(1))\to0\,.\end{aligned}$$

Finally, we observe that from (3.100) and (3.103) it follows that

$$|\bar w(z)|\leq c(1+|z|^{1+\alpha})\,. \tag{3.104}$$

Step 4. The limit function is not linear.

Let us consider the sequences of points

$$\xi_k=\frac{z_k-\tilde z_k}{r_k},\qquad\hat\xi_k=\frac{\hat z_k-\tilde z_k}{r_k}.$$

As previously noted, $\xi_k\in\Omega_k$, $|\xi_k|\leq c$ uniformly in k, and $v_k(\xi_k)=\nabla v_k(\xi_k)=0$ for every k. In fact, it also holds that $\hat\xi_k\in\Omega_k$ for every k, and since $|\xi_k-\hat\xi_k|=1$, we have $|\hat\xi_k|\leq c$ uniformly in k. Moreover, by (3.95) (recall that $r_k=|z_k-\hat z_k|$) we get

$$|\nabla v_k(\hat\xi_k)|\geq\frac{|\nabla(\eta u_k)(z_k)-\nabla(\eta u_k)(\hat z_k)|}{r_k^\alpha M_k}-\frac{|u(z_k)||\nabla\eta(z_k)-\nabla\eta(\hat z_k)|}{r_k^\alpha M_k}\geq\frac12-ck^{-1}>\frac14\,.$$

Since $\xi_k\to\xi$, $\hat\xi_k\to\hat\xi$ with $\xi,\hat\xi\in\bar\Omega_\infty$ and $\xi\neq\hat\xi$, a simple continuity argument implies that $\nabla\bar v\leq\delta$ in a neighbourhood of ξ, and $\nabla\bar v\geq1/4-\delta$ in a neighbourhood of $\hat\xi$, for some small $\delta\in(0,1/10)$. Therefore $\nabla\bar v$ is not constant and, by (3.103), we conclude that $\nabla\bar w$ is not constant either.

*Step 5. $r_k\to0$ in **Case 3**.*

Assume by contradiction that, up to a subsequence, $r_k\to\bar r\in(0,2]$. Since $\bar r>0$, it is straightforward to verify that $\Omega_\infty=B_{1/\bar r}(\varsigma)\setminus\Sigma_0$ for some $\varsigma\in\mathbb{R}^d$. Let $K\subset\Omega_\infty$ be a compact set. For every $z \in K$, using (3.91), we estimate

$$\Big|\frac{\eta(\tilde{z}_k + r_k z)(u_k(\tilde{z}_k + r_k z) - u_k(z_k))}{r_k^{1+\alpha} M_k}\Big| \leq 2\frac{\|u\|_{L^\infty(B_{3/4}\setminus\Sigma^A_{\varepsilon_k})}}{r_k^{1+\alpha} M_k} \leq ck^{-1}\,,$$

where in the last inequality we used that $r_k \geq c > 0$ for sufficiently large k. Moreover, from (3.91) and (3.92), we obtain

$$\Big|\frac{\eta(z_k)\nabla u_k(z_k)}{r_k^\alpha M_k}\Big| \leq \frac{c}{\bar{r}^\alpha} \leq c\,.$$

Hence, recalling the definition of v_k and thanks to *Step 2*, we infer that there exists $W \in \mathbb{R}^d$ such that

$$\bar{v}(z) = -W \cdot (z - \xi)\,.$$

This contradicts the fact, established in *Step 4*, that $\nabla\bar{v}$ is not constant. Therefore, we conclude that $r_k \to 0$. At this point, arguing as in Theorem 3.1.3, *i)*, *Step 2*, we find that the limit domain Ω_∞ is given by (3.80).

Step 6. The limit function is an entire solution to a homogeneous equation.

Let us denote $A_k(z) = A(\tilde{z}_k + r_k z)$. Since $\tilde{z}_k \in B_{3/4}$, and $A \in C^{1,\alpha}(B_1)$, we have, up to subsequences

$$\bar{z} = (\bar{x}, \bar{y}) = \lim_{k\to\infty}(\tilde{x}_k, \tilde{y}_k) \qquad \text{and} \qquad \bar{A} = A(\bar{z}) = \lim_{k\to\infty} A_k(z)\,,$$

where $\bar{A}$ is a symmetric matrix with constant coefficients that satisfies (3.5). Next, let us define

$$\rho_k(y) = \begin{cases} \dfrac{|\tilde{y}_k + r_k y|}{|\tilde{y}_k|}, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2},\\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

noting that, in **Case 1** and **Case 2**, $\rho_k \to 1$ uniformly on every compact set $K \subset \mathbb{R}^d$ as $k \to \infty$.

Let us fix $\phi \in C_c^\infty(\mathbb{R}^d)$, and observe that for sufficiently large k, $\mathrm{spt}(\phi) \subset \frac{B_1 - \tilde{z}_k}{r_k}$. Since u_k is solution to (3.13), we find that w_k satisfies

$$\int_{\Omega_k} \rho_k^a(y) A_k(z)\nabla w_k(z)\cdot\nabla\phi(z)\,dz = I_1 - I_2 - I_3 - I_4\,, \tag{3.105}$$

where

$$\begin{aligned}
I_1 &= \frac{r_k^{1-\alpha}}{M_k}\int_{\Omega_k} \rho_k^a(y)\eta(\tilde{z}_k) f(\tilde{z}_k + r_k z)\phi(z)\,dz\,,\\
I_2 &= \frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k} \rho_k^a(y)\eta(\tilde{z}_k)\big(F(\tilde{z}_k + r_k z) - F(\tilde{z}_k)\big)\cdot\nabla\phi(z)\,dz\,,\\
I_3 &= \frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k} \rho_k^a(y)(A(\tilde{z}_k + r_k z) - A(\tilde{z}_k))P_k\cdot\nabla\phi(z)\,dz\,,\\
I_4 &= \frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k} \rho_k^a(y)(A(\tilde{z}_k)P_k + \eta(\tilde{z}_k)F(\tilde{z}_k))\cdot\nabla\phi(z)\,dz\,.
\end{aligned}$$

Following the same argument as in *Step 5* of the proof of Theorem 3.1.3, *i)*, we observe that as $k \to \infty$,

$$|I_1| \leq c\|\phi\|_{L^\infty(\mathbb{R}^d)} k^{-1} r_k^{1-\alpha-\frac{d+a_+}{p}} \to 0\,,$$

due to the assumption $\alpha \le 1 - \frac{d+a_+}{p}$.

Next, using (3.90), we find that as $k \to \infty$,

$$
\begin{aligned}
|I_2| &\le \frac{r_k^{-\alpha}}{M_k} \int_{\Omega_k \cap \mathrm{spt}(\phi)} \rho_k^a(y) \big|F(\tilde z_k + r_k z) - F(\tilde z_k)\big| |\nabla \phi(z)| \, dz \\
&\le \|\nabla \phi\|_{L^1(\mathbb{R}^d, \rho_k^a dz)} \frac{\|F\|_{C^{0,\alpha}(B_1)}}{M_k} \le c k^{-1} \to 0 \, .
\end{aligned}
$$

For I_3, using (3.96), we obtain

$$
\begin{aligned}
|I_3| &\le \frac{r_k^{-\alpha}}{M_k} \int_{\Omega_k \cap \mathrm{spt}(\phi)} \rho_k^a(y) |A(\tilde z_k + r_k z) - A(\tilde z_k)| |P_k| |\nabla \phi(z)| \, dz \\
&\le \|\nabla \phi\|_{L^1(\mathbb{R}^d, \rho_k^a dz)} L r_k^{1-\alpha} \frac{|P_k|}{M_k} \le c r_k^{1-\alpha} \to 0 \, .
\end{aligned}
$$

Finally, it remains to prove that I_4 vanishes as $k \to \infty$.

Let us first examine **Case 1**. Here, recall that $\tilde z_k = z_k$, $r_k/|y_k| \to 0$, $\rho_k^a \to 1$, and $\Omega_\infty = \mathbb{R}^d$. Notice that $|y_k + r_k y| \ge c|y_k|$ for sufficiently large k. Using this, we estimate

$$
|\nabla \rho_k^a(y)| = \left| \frac{a r_k \rho_k^a(y)(y_k + r_k y)}{|y_k + r_k y|^2} \right| \le c \frac{r_k}{|y_k|} \rho_k^a(y) \, .
$$

Moreover, we know that $\mathrm{spt}(\phi) \subset\subset \Omega_k$ for sufficiently large k. Integrating by parts, we obtain

$$
\begin{aligned}
|I_4| &= \frac{r_k^{-\alpha}}{M_k} \Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) (\eta A \nabla u_k + \eta F)(z_k) \cdot \nabla \phi(z) \, dz \Big| \\
&= \frac{r_k^{-\alpha}}{M_k} \Big| \int_{\mathrm{spt}(\phi)} \nabla \rho_k^a(y) \cdot (\eta A \nabla u_k + \eta F)(z_k) \phi(z) \, dz \Big| \\
&\le c \|\phi\|_{L^\infty(\mathbb{R}^d)} \frac{r_k^{1-\alpha}}{|y_k| M_k} |(\eta A \nabla u_k + \eta F)_2(z_k)| \, .
\end{aligned}
$$

As pointed out in Theorem 3.1.3, *i)*, *Step 1*, there exists $Z_k = (x_k, Y_k) \in \partial \Sigma_{\varepsilon_k}^A$ such that $|Y_k| < |y_k|$ and $y_k \cdot Y_k = |y_k||Y_k|$. Moreover, it holds

$$
d_k \le |z_k - Z_k| \le |y_k| \, ,
$$

and, by construction, $Z_k \in B_{3/4}$. By using (3.94) and (3.93), we get

$$
\begin{aligned}
|(\eta A \nabla u_k + \eta F)_2(z_k)| &\le |(\eta A \nabla u_k + \eta F))(z_k) - (\eta A \nabla u_k + \eta F)(Z_k)| + |(\eta A \nabla u_k + \eta F)_2(Z_k)| \\
&\le M_k |z_k - Z_k|^\alpha + c M_k \varepsilon_k^\alpha \le c M_k |y_k|^\alpha \, .
\end{aligned}
$$

As a consequence we have

$$
|I_4| \le c \Big(\frac{r_k}{|y_k|} \Big)^{1-\alpha} \to 0 \quad \text{as } k \to \infty \, ,
$$

as needed.

Let us now turn to **Case 2**. Here, recall that $\tilde{z}_k = z_k^0$, where z_k^0 is such that $d_k = \mathrm{dist}(z_k, \partial\Sigma^A_{\varepsilon_k}) = |z_k - z_k^0| \le cr_k$, $|y_k|/r_k \to \infty$ and $\rho_k^a \to 1$. Additionally, since $|y_k| \le |z_k - z_k^0| + |y_k^0| \le cr_k + |y_k^0|$, it follows that both $|y_k^0|/r_k \to \infty$ and $\varepsilon_k/r_k \to \infty$ in this case.

Integrating by parts we find

$$\begin{aligned} I_4 &= \frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k} \nabla\rho_k^a(y)\cdot(\eta A\nabla u_k + \eta F)_2(z_k^0)\phi(z)\,dz \\ &+ \frac{r_k^{-\alpha}}{M_k}\int_{\partial\Omega_k} \rho_k^a(y)(\eta A\nabla u_k + \eta F)(z_k^0)\cdot\nu(z_k^0 + r_k z)\phi(z)\,dz\,, \end{aligned}$$

where $\nu(z)$ is the normal vector to $\partial\Sigma^A_{\varepsilon_k}$. Using (3.94) and arguing as in **Case 1**, we readily obtain

$$\left|\frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k} \nabla\rho_k^a(y)\cdot(\eta A\nabla u_k + \eta F)_2(z_k^0)\phi(z)\,dz\right| \le c\|\phi\|_{L^\infty(\mathbb{R}^d)}\left(\frac{r_k}{|y_k|}\right)^{1-\alpha} \to 0.$$

Next, using that u_k satisfy a conormal boundary condition on $\partial\Omega_k$, we compute

$$\begin{aligned} &(\eta A\nabla u_k + \eta F)(z_k^0)\cdot\nu(z_k^0 + r_k z) = (\eta A\nabla u_k + \eta F)(z_k^0)\cdot(\nu(z_k^0 + r_k z) - \nu(z_k^0)) \\ &= (\eta A\nabla u_k + \eta F)_2(z_k^0)\cdot(N(z_k^0 + r_k z) - N(z_k^0)) + (\eta A\nabla u_k + \eta F)(z_k^0)\cdot(\tilde{\nu}(z_k^0 + r_k z) - \tilde{\nu}(z_k^0))\,, \end{aligned}$$

where $N(z)$ and $\tilde{\nu}(z)$ are as in (3.134).

As done in Theorem 1.7.1, we split the proof into two parts. Assume to restart the present proof, keeping exactly the same assumptions, but aiming to prove estimates in the form

$$\|u_\varepsilon\|_{C^{1,\alpha'}(B_{1/2}\setminus\Sigma^A_\varepsilon)} \le C\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma^A_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big)\,, \tag{3.106}$$

with a suboptimal exponent $\alpha' < \alpha$. Then, at this point of the proof, using (3.93), (3.94) and Lemma 3.A.1, that is, $\|N\|_{C^{0,1}(\partial\Sigma^A_{\varepsilon_k}\cap B_1)} \le c\varepsilon_k^{-1}$ and $\|\tilde{\nu}\|_{C^{0,\alpha}(\partial\Sigma^A_{\varepsilon_k}\cap B_1)} \le c$, we would obtain

$$|(\eta A\nabla u_k + \eta F)(z_k^0)\cdot\nu(z_k^0 + r_k z)| \le cM_k\frac{r_k}{\varepsilon_k^{1-\alpha}} + cM_k r_k^\alpha\,, \tag{3.107}$$

which implies that

$$\begin{aligned} &\left|\frac{r_k^{-\alpha'}}{M_k}\int_{\partial\Omega_k} \rho_k^a(y)(\eta A\nabla u_k + \eta F)(z_k^0)\cdot\nu(z_k^0 + r_k z)\phi(z)\,dz\right| \\ &\le c\|\phi\|_{L^\infty(\mathbb{R}^d)}\left(\left(\frac{r_k}{\varepsilon_k}\right)^{1-\alpha'} + r_k^{\alpha-\alpha'}\right) \to 0\,. \end{aligned}$$

Therefore, we conclude that $|I_4| \to 0$, as required. From this point on, we can complete the proof of the theorem and obtain stable $C^{1,\alpha'}$ estimates. This, in turn, implies a uniform L^∞ bound for ∇u_k in $B_{3/4}\setminus\Sigma_{\varepsilon_k}$.

We can now return to our previous computation with $\alpha' = \alpha$. Thanks to (3.106) and (3.72), we get

$$\|\eta A\nabla u_k + \eta F\|_{L^\infty(B_1)} \le ck^{-1}M_k\,.$$

Thus we can improve (3.107) and get

$$|(\eta A\nabla u_k + \eta F)(z_k^0)\cdot\nu(z_k^0 + r_k z)| \le cM_k\frac{r_k}{\varepsilon_k^{1-\alpha}} + cM_k k^{-1} r_k^{\alpha}\,.$$

As a consequence,

$$\Big|\frac{r_k^{-\alpha}}{M_k}\int_{\partial\Omega_k}\rho_k^a(y)(\eta A\nabla u_k + \eta F)(z_k^0)\cdot\nu(z_k^0 + r_k z)\phi(z)\,dz\Big| \le c\|\phi\|_{L^\infty(\mathbb{R}^d)}\Big(\Big(\frac{r_k}{\varepsilon_k}\Big)^{1-\alpha} + k^{-1}\Big) \to 0\,,$$

and we can conclude that $|I_4| \to 0$.

Finally, we address **Case 3**. Recall that $\tilde z_k = (x_k, 0)$, $\rho_k = |y|$, and $\varepsilon_k \le c|y_k| \le cr_k$. Let us call

$$I_4' = \frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k}|y|^a(A(\tilde z_k)P_k + (\eta F)(z_k^0))\cdot\nabla\phi(z)\,dz\,.$$

We have

$$\begin{aligned}\Big|I_4 - I_4'\Big| &= \Big|\frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k}|y|^a((\eta F)(\tilde z_k) - (\eta F)(z_k^0))\cdot\nabla\phi(z)\,dz\Big| \\ &\le c\|\nabla\phi\|_{L^1(\mathbb{R}^d,|y|^a dz)}\frac{\|F\|_{C^{0,\alpha}(B_1)}}{M_k}|\tilde z_k - z_k^0|^\alpha r_k^{-\alpha} \le ck^{-1} \to 0\,,\end{aligned}$$

where we used that in **Case 3** it holds

$$|\tilde z_k - z_k^0| \le |\tilde z_k - z_k| + |z_k - z_k^0| = |y_k| + d_k \le cr_k\,.$$

Due to our choice of P_k, an integration by parts yields

$$\begin{aligned}I_4' &= \frac{r_k^{-\alpha}}{M_k}\int_{\Omega_k}|y|^a(\eta A\nabla u_k + \eta F)_1(z_k^0)\cdot\nabla\phi(z)\,dz \\ &= \frac{r_k^{-\alpha}}{M_k}\int_{\partial\Omega_k}|y|^a(\eta A\nabla u_k + \eta F)_1(z_k^0)\cdot\nu(\tilde z_k + r_k z)\phi(z)\,dz\,.\end{aligned}$$

Recall that at a point $z \in \partial\Sigma^A_{\varepsilon_k}$, the normal vector is given by $\nu(z) = (0, N(z)) + \tilde\nu(z)$, see (3.134), where $|\tilde\nu(z)| \le c\varepsilon_k$. Therefore, by (3.93), we get

$$\begin{aligned}|I_4'| &\le \frac{r_k^{-\alpha}}{M_k}\int_{\partial\Omega_k}|y|^a|(\eta A\nabla u_k + \eta F)_1(z_k^0)||\tilde\nu(\tilde z_k + r_k z)||\phi(z)|\,dz \\ &\le c\Big(\frac{\varepsilon_k}{r_k}\Big)r_k^{1-\alpha}\|\phi\|_{L^\infty(\mathbb{R}^d)}\int_{\mathrm{spt}(\phi)\cap\partial\Omega_k}|y|^a\,dz\,.\end{aligned}$$

Notice that there exists $\bar R > 0$ such that

$$\mathrm{spt}(\phi)\cap\partial\Omega_k = \mathrm{spt}(\phi)\cap\Big\{\sqrt{A_3^{-1}(\tilde z_k + r_k z)y\cdot y} = \frac{\varepsilon_k}{r_k}\Big\} \subset (B^{d-n}_{\bar R}\times\mathbb{R}^n)\cap\frac{\partial\Sigma^A_{\varepsilon_k} - \tilde z_k}{r_k}\,.$$

Thus, denoting $E_k = B^{d-n}_{r_k\bar R}(-r_k^{-1}\tilde z_k)\times\mathbb{R}^n$, we have

$$\int_{\mathrm{spt}(\phi)\cap\partial\Omega_k}|y|^a\,dz \le \int_{(B^{d-n}_{\bar R}\times\mathbb{R}^n)\cap\frac{\partial\Sigma^A_{\varepsilon_k} - \tilde z_k}{r_k}}|y|^a\,dz = r_k^{1-d-a}\int_{E_k\cap\partial\Sigma^A_{\varepsilon_k}}|y|^a dz\,.$$

On the other hand, using Lemma 3.A.3, *iv)* and performing the change of variables $(x,\tau)=\Phi(z)$ (for instance see [102, §11]), we obtain

$$\begin{aligned}\int_{E_k\cap\partial\Sigma_{\varepsilon_k}}|\tau|^a d\sigma(x,\tau)&=\int_{E_k\cap\partial\Sigma^A_{\varepsilon_k}}(A_3^{-1}(z)y\cdot y)^{\frac{a}{2}}|\det(\Pi_{\varepsilon_k}\circ J_\Phi^\top J_\Phi\circ\Pi_{\varepsilon_k})|d\sigma(z)\\&\geq c\int_{E_k\cap\partial\Sigma^A_{\varepsilon_k}}|y|^a d\sigma(z)\,.\end{aligned}$$

Summing up, we have that

$$\int_{\mathrm{spt}(\phi)\cap\partial\Omega_k}|y|^a\,dz\leq cr_k^{1-d-a}\int_{E_k\cap\partial\Sigma_{\varepsilon_k}}|\tau|^a d\sigma(x,\tau)=cr_k^{1-n-a}\int_{B^n_{\varepsilon_k}}|\tau|^a d\sigma(\tau)=c\Big(\frac{\varepsilon_k}{r_k}\Big)^{a+n-1}.$$

Therefore

$$|I_4'|\leq c\Big(\frac{\varepsilon_k}{r_k}\Big)^{a+n}r_k^{1-\alpha}\leq cr_k^{1-\alpha}\to 0\,,$$

which implies that $|I_4|\to 0$ also in **Case 3**. Hence, the right hand side of (3.105) vanishes as $k\to\infty$.

From this point on, arguing as in Theorem 3.1.3, *i)*, *Step 5*, we obtain that the left hand side of (3.84) satisfies

$$\int_{\Omega_k\cap\mathrm{spt}(\phi)}\rho_k^a A_k\nabla w_k\cdot\nabla\phi\,dz\to\int_{\Omega_\infty\cap\mathrm{spt}(\phi)}\bar\rho^a\,\bar A\nabla\bar w\cdot\nabla\phi\,dz\,,$$

where

$$\bar\rho(y)=\begin{cases}1 & \text{in }\textbf{Case 1}\text{ and }\textbf{Case 2}\,,\\ |y| & \text{in }\textbf{Case 3}\,,\end{cases}$$

and $\bar w$ belongs to $H^1(\Omega_\infty\cap B_R,\bar\rho^a dz)$ for every $R>0$. Hence, recalling the definition of the limit domain Ω_∞, see (3.80), we infer that $\bar w$ is an entire solution to:

Case 1

$$-\operatorname{div}(\bar A\nabla\bar w)=0,\quad\text{in }\mathbb{R}^d\,;$$

Case 2

$$\begin{cases}-\operatorname{div}(\bar A\nabla\bar w)=0, & \text{in }\Pi_{\bar\varepsilon}\\ \bar A\nabla\bar w\cdot\nu=0 & \text{on }\partial\Pi_{\bar\varepsilon};\end{cases}$$

Case 3, if $\bar\varepsilon=0$

$$-\operatorname{div}(|y|^a\bar A\nabla\bar w)=0,\quad\text{in }\mathbb{R}^d\,,$$

Case 3, if $\bar\varepsilon>0$

$$\begin{cases}-\operatorname{div}(|y|^a\bar A\nabla\bar w)=0 & \text{in }\mathbb{R}^d\setminus\Sigma^{\bar A}_{\bar\varepsilon}\,,\\ \bar A\nabla\bar w\cdot\nu=0 & \text{on }\Sigma^{\bar A}_{\bar\varepsilon}\,.\end{cases}$$

Step 7. Liouville theorems and contradiction.

Since $\bar w$ satisfies the growth condition (3.104), with $1+\alpha<\min\{2,\alpha_*\}$, by the assumption (3.9), by invoking the classical Liouville Theorem in **Case 1**, the Liouville Theorem in an half space with an homogeneous conormal boundary condition (for instance, see [143, Theorem 1.6]) in **Case 2**, and the Liouville Theorem 3.1.4 in **Case 3**, we get that $\bar{w}$ must be a linear function: this is a contradiction with the *Step 3*, since $\nabla\bar{w}$ is not constant. Thus, (3.88) holds with a constant C uniformly bounded in ε. Moreover, recalling (3.94), we get that (3.14) holds true. The proof is complete. ☐

Remark 3.6.2. The exponent α_* given by (3.7) is always strictly positive. In general, given an integer $k \geq 1$, $\alpha_* > k$ if and only if $\mu_* > k(k+a+n-2)$. This implies in particular that $\alpha_* > 1$ if and only if $a < \mu_* + 1 - n \leq 0$, since $0 < \mu_* \leq n-1$. Hence, when α_* is greater than 2, the regularity of solutions to (3.1) is expected to be higher than $C^{1,\alpha}$ ($C^{2,\alpha}$ or even more). In this remark, we give a simple example which shows that our method fails to prove higher order Schauder estimates which are stable in ε with respect to domain perforation.

For $a+n \neq 2$ such that $\alpha_* > 2$, let us consider the family of functions

$$u_\varepsilon(x,y) := (a+n)x_1^2 - |y|^2 + \frac{2}{2-a-n}\varepsilon^{a+n}|y|^{2-a-n},$$

which are solutions to

$$\begin{cases} -\operatorname{div}(|y|^a\nabla u_\varepsilon) = 0, & \text{in } B_1 \setminus \Sigma_\varepsilon, \\ \nabla u_\varepsilon \cdot \nu = 0, & \text{on } \partial\Sigma_\varepsilon \cap B_1, \end{cases}$$

and satisfy $\|u_\varepsilon\|_{L^\infty(B_1\setminus\Sigma_\varepsilon)} \leq c$. Fixed $i,j \in \{1,\dots,n\}$ such that $i \neq j$, we compute

$$\partial^2_{y_iy_j} u_\varepsilon(x,y) = -2(a+n)\varepsilon^{a+n}|y|^{-2-a-n}y_iy_j\,.$$

Let us fix the points

$$z_1 = \frac{\varepsilon}{\sqrt{2}}(e_{y_i} + e_{y_j}), \quad z_2 = \frac{\varepsilon^\beta}{\sqrt{2}}(e_{y_i} + e_{y_j}),$$

where $\beta \in (0,1)$. One has that $z_1, z_2 \in \overline{B_{1/2} \setminus \Sigma_\varepsilon}$ and, for every $\alpha \in (0,1)$, we have

$$\frac{|\partial^2_{y_iy_j}u_\varepsilon(z_1) - \partial^2_{y_iy_j}u_\varepsilon(z_2)|}{|z_1 - z_2|^\alpha} = (a+n)\frac{1-\varepsilon^{(a+n)(1-\beta)}}{\varepsilon^{\alpha\beta}|1-\varepsilon^{1-\beta}|^\alpha} \to \infty\,, \quad \text{as } \varepsilon \to 0\,,$$

since $1-\beta > 0$ and $a+n > 0$. Then, we have proved that ε-uniform $C^{2,\alpha}$ estimates fail.

3.7 A priori estimates and proof of the main result

In this section we prove a priori regularity estimates, which imply stable estimates with respect to standard smoothing of data. This leads to the proof of the main Theorems 3.1.1 and 3.1.2.

3.7.1 A priori estimates

Proposition 3.7.1 (A priori estimates)**.** *Let $a+n > 0$. Let A be a uniformly elliptic matrix satisfying (3.5) and (3.6) with constants $0 < \lambda \leq \lambda_* \leq \Lambda_* \leq \Lambda$. Let $\alpha_* = \alpha_*(n,a,\lambda_*/\Lambda_*)$ be defined as in (3.7). Then the following points hold true:*

i) Let $p > (d+a_+)/2$, $q > d+a_+$. Let $\alpha \in (0,1)$ satisfying (3.8). Let A be C^0 with $\|A\|_{C^{0,\omega}(B_1)} \leq L$ for some given modulus of continuity ω, $f \in L^{p,a}(B_1)$ and $F \in L^{q,a}(B_1)^d$. Let $u \in C^{0,\alpha}(B_1)$ be a weak solution to (3.1) in B_1. Then, there exist a constant $C > 0$

depending only on d, n, a, λ, Λ, p, q α *and* L *such that*

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)}\big).$$

ii) Let $p > d + a_+$. *Let assume that* $\alpha_* > 1$. *Let* $\alpha \in (0,1)$ *satisfying* (3.9). *Let* A *be* $C^{0,\alpha}$ *with* $\|A\|_{C^{0,\alpha}(B_1)} \le L$, $f \in L^{p,a}(B_1)$ *and* $F \in C^{0,\alpha}(B_1)$. *Let* $u \in C^{1,\alpha}(B_1)$ *be a weak solution to* (3.1) *in* B_1 *such that*

$$(A\nabla u + F)\cdot e_{y_i} = 0, \qquad \text{on } \Sigma_0 \cap B_1, \quad \text{for every } i = 1, \dots, n. \tag{3.108}$$

Then, there exist a constant $C > 0$ *depending only on* d, n, a, λ, Λ, p, α *and* L *such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big).$$

We provide the proof only for the estimates in $C^{1,\alpha}$-spaces, as the proof for the other case follows the same argument and is easier to establish.

Proof of Proposition 3.7.1, ii). The proof is similar to that of Theorem 3.1.3, *ii)*, so we omit some details.

By contradiction, let us suppose that there exist sequences $\{u_k\}_k$, $\{f_k\}_k$, $\{F_k\}_k$ and $\{A_k\}_k$, such that $A_k \in C^{0,\alpha}$ with $\|A_k\|_{C^{0,\alpha}(B_1)} \le L$, $f_k \in L^{p,a}(B_1)$, $F_k \in C^{0,\alpha}(B_1)$, and every u_k is a weak solution to

$$-\operatorname{div}(|y|^a A_k \nabla u_k) = |y|^a f_k + \operatorname{div}(|y|^a F_k), \qquad \text{in } B_1,$$

which satisfies the boundary condition (3.108) and

$$\|u_k\|_{C^{1,\alpha}(B_{1/2})} > k\big(\|u_k\|_{L^{2,a}(B_1)} + \|f_k\|_{L^{p,a}(B_1)} + \|F_k\|_{C^{0,\alpha}(B_1)}\big).$$

Let us fix a smooth cut-off function $\eta \in C^\infty_c(B_{3/4})$ such that $\eta = 1$ in $B_{1/2}$ and $0 \le \eta \le 1$. Then, arguing as in *Step 1* in Theorem 3.1.3, *ii)*, we have

$$M_k = [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1)} \ge [\nabla u_k]_{C^{0,\alpha}(B_{1/2})} \ge ck\big(\|u_k\|_{L^{2,a}(B_1)} + \|f_k\|_{L^{p,a}(B_1)} + \|F_k\|_{C^{0,\alpha}(B_1)}\big)\,.$$

Take two sequences of points $z_k = (x_k, y_k)$, $\hat z_k = (\hat x_k, \hat y_k) \in B_1$ such that

$$\frac{|\nabla(\eta u_k)(z_k) - \nabla(\eta u_k)(\hat z_k)|}{|z_k - \hat z_k|^\alpha} \ge \frac{1}{2} M_k,$$

and define $r_k = |z_k - \hat z_k|$. Without loss of generality, we can assume that $z_k \in B_{3/4}$. From now on, we distinguish two cases:

Case 1: $\dfrac{|y_k|}{r_k} \to \infty$ as $k \to \infty$,

Case 2: $\dfrac{|y_k|}{r_k} \le c$ uniformly in k.

Unlike the proofs of the stable estimates in Section 3.6, here we do not introduce any perforation around Σ_0. Consequently, we have only two blow-up regimes: in **Case 1**, the blow-up scale does not capture Σ_0, whereas in **Case 2**, it does.

Let $\zeta_k = (x_k, 0)$ be the projection of z_k onto Σ_0. Let us define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) = \begin{cases} z_k, & \text{in } \textbf{Case 1}, \\ \zeta_k, & \text{in } \textbf{Case 2}, \end{cases} \qquad \xi_k = \frac{z_k - \tilde{z}_k}{r_k},$$

the sequence of rescaled domains

$$\Omega_k = \frac{B_1 - \tilde{z}_k}{r_k},$$

and the sequences of functions

$$v_k(z) = \frac{\eta(\tilde{z}_k + r_k z)\big(u_k(\tilde{z}_k + r_k z) - u_k(\tilde{z}_k)\big) - \eta(z_k)\nabla u_k(z_k) \cdot r_k(z - \xi_k)}{r_k^{1+\alpha} M_k}, \quad \text{for } z \in \Omega_k,$$

$$w_k(z) = \frac{\eta(\tilde{z}_k)(u_k(\tilde{z}_k + r_k z) - u_k(\tilde{z}_k)) - (\eta \nabla u_k)(\tilde{z}_k) \cdot r_k z}{r_k^{1+\alpha} M_k}, \quad \text{for } z \in \Omega_k,$$

and set the limit blow-up domain Ω_∞ as in (3.74).

By arguing as in Theorem 3.1.3, *ii), Step 2* one has that $[v_k]_{C^{1,\alpha}(K \cap \Omega_k)} \le 2$ and $\|v_k\|_{C^{1,\alpha}(K)} \le c(K)$ for every compact set $K \subset \Omega_\infty$. Therefore, by applying the Arzelá-Ascoli theorem, we infer that $v_k \to \bar{v}$ in $C^{1,\gamma}_{\mathrm{loc}}(\Omega_\infty)$ for every $\gamma \in (0, \alpha)$ and

$$|\bar{v}| \le c(1 + |z|^{1+\alpha}).$$

Moreover, employing a similar argument as in Theorem 3.1.3, *ii), Step 3*, it follows that also $w_k \to \bar{v}$ uniformly on compact set of Ω_∞. Next, repeating the argument of Theorem 3.1.3, *ii), Step 4, Step 5*, we conclude that $\nabla \bar{v}$ is not constant and $r_k \to 0$, which immediately implies that $\Omega_\infty = \mathbb{R}^d$ in both **Case 1** and **Case 2**.

Finally, we prove that $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients. Since $\|A_k\|_{C^{0,\alpha}(B_1)} \le L$, we have that $A_k(\tilde{z}_k + r_k z) \to A(\bar{z}) = \bar{A}$, which is a constant matrix satisfying (3.5) and $\bar{z} = \lim_{k \to \infty} \tilde{z}_k$. Let us define

$$\rho_k^a(y) = \begin{cases} \dfrac{|y_k + r_k y|^a}{|y_k|^a}, & \text{in } \textbf{Case 1}, \\ |y|^a, & \text{in } \textbf{Case 2}. \end{cases}$$

Fixed $\phi \in C_c^\infty(\mathbb{R}^d)$, we have that

$$\int_{\Omega_k} \rho_k^a(y) A_k(\tilde{z}_k + r_k z) \nabla w_k(z) \cdot \nabla \phi(z) dz = I_1 - I_2 - I_3 - I_4,$$

where

$$I_1 = \frac{r_k^{1-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) \eta(\tilde{z}_k) f_k(\tilde{z}_k + r_k z) \phi(z) dz$$

$$I_2 = \frac{r_k^{-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) \eta(\tilde{z}_k) \big(F_k(\tilde{z}_k + r_k z) - F_k(\tilde{z}_k)\big) \cdot \nabla \phi(z) dz$$

$$I_3 = \frac{r_k^{-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) \eta(\tilde{z}_k) \big(A_k(\tilde{z}_k + r_k z) - A_k(\tilde{z}_k)\big) \nabla u_k(\tilde{z}_k) \cdot \nabla \phi(z) dz$$

$$I_4 = \frac{r_k^{-\alpha}}{M_k} \int_{\Omega_k} \rho_k^a(y) \eta(\tilde{z}_k) \big(A_k \nabla u_k + F_k\big)(\tilde{z}_k) \cdot \nabla \phi(z) dz.$$

The terms I_1, I_2 vanishes as $k \to \infty$ exactly as in Theorem 3.1.3, *Step 6*.

To show that the term I_3 vanishes, we proceed in two steps, as in the proof of Theorem 3.1.3, *ii), Step 6*. First, we establish local $C^{1,\alpha'}$ estimates for some suboptimal $\alpha' \in (0, \alpha)$. Once this is done, we can then obtain the regularity with the optimal exponent α, using as additional information a L^∞ bound for the gradient of the solutions. Hence, in the first step, we have

$$\begin{aligned}|I_3| &= \Big|\frac{r_k^{-\alpha'}}{M_k}\int_{\Omega_k}\rho_k^a(y)\eta(\tilde{z}_k)(A_k(\tilde{z}_k + r_k z) - A_k(\tilde{z}_k))\nabla u_k(\tilde{z}_k)\cdot\nabla\phi(z)dz\Big|\\ &\le \frac{cLr_k^{\alpha-\alpha'}(\|\eta u_k\|_{L^\infty(B_1)} + \|\nabla(\eta u_k)\|_{L^\infty(B_1)})}{M_k} \le cLr_k^{\alpha-\alpha'} \to 0, \quad \text{as } k \to \infty.\end{aligned}$$

Next, restarting the proof with the optimal exponent α, a priori estimates with a suboptimal exponent imply that $\|\nabla(\eta u_k)\|_{L^\infty(B_1)} \le M_k/k$ and we can conclude in the following way

$$|I_3| \le \frac{cL(\|\eta u_k\|_{L^\infty(B_1)} + \|\nabla(\eta u_k)\|_{L^\infty(B_1)})}{M_k} \le cLk^{-1} \to 0, \quad \text{as } k \to \infty.$$

Finally, we show that the term I_4 goes to zero. Let us consider the **Case 1**, recalling that $\tilde{z}_k = z_k$, $r_k/|y_k| \to 0$, $\rho_k^a \to 1$ and $\zeta_k = (x_k, 0) \in \Sigma_0 \cap B_1$. Arguing as in the proof of Theorem 3.1.3, *ii), Step 6*, and using the boundary condition (3.108), we have that

$$|(\eta A_k\nabla u_k + \eta F_k)_2(z_k)| = |(\eta A_k\nabla u_k + \eta F_k)_2(z_k) - (\eta A_k\nabla u_k + \eta F_k)_2(\zeta_k)| \le cM_k|y_k|^\alpha,$$

and thus

$$|\nabla\rho_k^a(y)\cdot(\eta A_k\nabla u_k + \eta F_k)(z_k)| \le c\frac{r_k}{|y_k|^{1-\alpha}}M_k.$$

Hence, by using the divergence theorem, one has that

$$|I_4| = \frac{r_k^{-\alpha}}{M_k}\Big|\int_{\Omega_k}\nabla\rho_k^a(y)\cdot(\eta A_k\nabla u_k + \eta F_k)(z_k)\phi(z)dz\Big| \le c\Big(\frac{r_k}{|y_k|}\Big)^{1-\alpha} \to 0, \quad \text{as } k \to \infty.$$

In **Case 2**, since u_k satisfies (3.108) and $\tilde{z}_k = (x_k, 0)$, the divergence theorem allows us to conclude directly that $I_4 = 0$. In fact,

$$\int_{\Omega_k}\rho_k^a(y)(A_k\nabla u_k + F_k)(\tilde{z}_k)\cdot\nabla\phi(z)dz = -\int_{\Omega_k}\nabla\rho_k^a(y)\cdot(A_k\nabla u_k + F_k)(\tilde{z}_k)\phi(z)dz = 0.$$

From this point on, arguing as in Theorem 3.1.3 *i), Step 6*, we obtain that

$$\int_{\Omega_k}\rho_k^a(y)A_k(\tilde{z}_k + r_k z)\nabla w_k(z)\cdot\nabla\phi(z)dz \to \int_{\mathbb{R}^d}\bar{\rho}^a(y)\bar{A}\nabla\bar{v}(z)\cdot\nabla\phi(z)dz,$$

where

$$\bar{\rho}(y) = \begin{cases}1, & \text{in } \textbf{Case 1},\\ |y|, & \text{in } \textbf{Case 2}.\end{cases}$$

Then, we have proved that $\bar{v}$ is an entire solution to

$$-\operatorname{div}(\bar{A}\nabla\bar{v}) = 0, \quad \text{in } \mathbb{R}^d, \quad \text{in } \textbf{Case 1}$$

and $\bar v$ is an entire solution to

$$-\operatorname{div}(|y|^a \bar A \nabla \bar v) = 0, \quad \text{in } \mathbb{R}^d, \quad \text{in } \textbf{Case 2}.$$

Finally, as in Theorem 3.1.3, *ii), Step 7*, by invoking appropriate Liouville type theorems we get a contradiction and the thesis follows. □

3.7.2 Proof of Theorems 3.1.1 and 3.1.2

We will prove Theorem 3.1.2 only, as the proof of Theorem 3.1.1 follows by a similar argument.

Proof of Theorem 3.1.2. We divide the proof into two steps.

Step 1. First, we prove that if the matrix $A \in C^{1,\alpha}(B_1)$, then $u \in C^{1,\alpha}(B_{3/4})$ and satisfies (3.10) in $B_{3/4}$.

Let u be a weak solution to (3.1). By applying Lemma 3.4.7 there exists a family of functions $\{u_\varepsilon\}_{0<\varepsilon\ll 1}$, such that every u_ε is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A \nabla u_\varepsilon) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_{4/5} \setminus \Sigma_\varepsilon^A \\ (A\nabla u_\varepsilon + F)\cdot \nu = 0, & \text{on } \partial\Sigma_\varepsilon^A \cap B_{4/5}, \end{cases}$$

and a sequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} \to u$ in $H^1_{\mathrm{loc}}(B_{4/5} \setminus \Sigma_0)$.

By applying Theorem 3.1.3, *ii)* to the sequence $\{u_{\varepsilon_k}\}$, combined with (3.40) and (3.39), we get

$$\|u_{\varepsilon_k}\|_{C^{1,\alpha}(B_{3/4}\setminus\Sigma^A_{\varepsilon_k})} \le c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big),$$

for some $c > 0$ depending only on d, n, a, λ, Λ, p, α and L. By using the Arzelá-Ascoli Theorem and the a.e. convergence $u_{\varepsilon_k} \to u$, we get that $u_{\varepsilon_k} \to u$ in $C^{1,\gamma}_{\mathrm{loc}}(B_{3/4} \setminus \Sigma_0)$, for every $\gamma \in (0,\alpha)$. Moreover, by taking $z, z' \in B_{3/4} \setminus \Sigma_0$ such that $z \neq z'$, we have that

$$\frac{|\nabla u(z) - \nabla u(z')|}{|z-z'|^\alpha} = \lim_{\varepsilon_k \to 0^+} \frac{|\nabla u_{\varepsilon_k}(z) - \nabla u_{\varepsilon_k}(z')|}{|z-z'|^\alpha} \le c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big).$$

Hence, by taking the supremum, we get

$$[\nabla v]_{C^{0,\alpha}(B_{3/4}\setminus\Sigma_0)} = \sup_{z,z' \in B_{3/4}\setminus\Sigma_0} \frac{|\nabla u(z) - \nabla u(z')|}{|z-z'|^\alpha} \le c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big).$$

By a continuity argument, we can extend v in the whole $B_{3/4}$ in such a way that $[\nabla v]_{C^{0,\alpha}(B_{3/4})} = [\nabla v]_{C^{0,\alpha}(B_{3/4}\setminus\Sigma_0)}$, hence, we have that $u \in C^{1,\alpha}(B_{3/4})$.

Further, let us prove that u satisfies the boundary condition (3.10). Fix $z = (x,y) \in B_{3/4} \setminus \Sigma_0$ and let z^ε be a chosen projection of z onto $\partial\Sigma_\varepsilon^A \cap B_{3/4}$. For every $i = 1, \dots, n$, by using (3.14), we get

$$\begin{aligned} \big|(A\nabla u + F)(z)\cdot e_{y_i}\big| &\le \big|(A\nabla u)(z) - (A\nabla u_\varepsilon)(z)\big| + \big|(A\nabla u_\varepsilon + F)(z) - (A\nabla u_\varepsilon + F)(z^\varepsilon)\big| \\ &\quad + \big|(A\nabla u_\varepsilon + F)(z^\varepsilon)\cdot e_{y_i}\big| \\ &\le \Lambda\big|\nabla u(z) - \nabla u_\varepsilon(z)\big| + c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big)\big(|z-z^\varepsilon|^\alpha + \varepsilon^\alpha\big). \end{aligned}$$

By taking the limit as $\varepsilon \to 0^+$ in the previous inequality, we get

$$\big|(A\nabla u + F)(z)\cdot e_{y_i}\big| \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big)|y|^\alpha,$$

therefore, by taking $|y| \to 0$ it follows that $(A\nabla u + F)(x,0)\cdot e_{y_i} = 0$, that is, (3.10) holds true.

Step 2. Finally, we prove that if $A \in C^{0,\alpha}(B_1)$, then $u \in C^{1,\alpha}(B_{1/2})$, satisfies (3.11) and (3.10).

Let u be a weak solution to (3.1). For $\delta > 0$, let $\{\rho_\delta\}_{\delta>0}$ be a family of smooth mollifiers and let us define $A_\delta = A * \rho_\delta$. Choosing δ small enough, we have that $A_\delta \in C^\infty(B_{4/5})$, satisfies (3.5) and $\|A_\delta\|_{C^{0,\alpha}(B_{4/5})} \leq \|A\|_{C^{0,\alpha}(B_1)}$. By using the approximation Lemma 3.4.8, we have that there exists a family $\{u_\delta\}$ such that every u_δ is a weak solution to

$$-\operatorname{div}(|y|^a A_\delta \nabla u_\delta) = |y|^a f + \operatorname{div}(|y|^a F), \text{ in } B_{4/5},$$

satisfies

$$\|u_\delta\|_{H^{1,a}(B_{4/5})} \leq c\big(\|u\|_{H^{1,a}(B_1)} + \|f\|_{L^{2,a}(B_1)} + \|F\|_{L^{2,a}(B_1)}\big),$$

and, up to consider a subsequence, $u_\delta \to u$ in $H^{1,a}(B_{4/5})$.

By applying *Step 1*, we have that $u_\delta \in C^{1,\alpha}(B_{3/4})$ and satisfies (3.10). Hence, by using Proposition 3.7.1, *ii)*, it follows that

$$\|u_\delta\|_{C^{1,\alpha}(B_{1/2})} \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big), \tag{3.109}$$

for some $c > 0$, depending only on d, n, a, λ, Λ, p, α and L. Hence, we can apply the Arzelá-Ascoli theorem to get that $u_\delta \to u$ in $C^{1,\gamma}(B_{1/2})$ (for every $\gamma \in (0,\alpha)$) and passing to the limit as $\delta \to 0$ in (3.109) we obtain that u satisfies (3.11). Moreover, by *Step 1* every u_δ satisfies (3.10), so u also satisfies (3.10), and our statement follows. □

3.8 Equations degenerating on curved manifolds

This section is devoted to the extension of Theorems 3.1.1 and 3.1.2 to a class of equations whose weights are degenerate/singular on lower-dimensional curved manifolds.

Let $2 \leq n < d$ and consider a $(d-n)$-dimensional C^1-manifold Γ which is locally embedded in $\mathbb{R}^d$ and parametrized by $\varphi \in C^1(\Sigma_0 \cap B_1; \mathbb{R}^n)$, in the sense that, up to perform a dilation, a translation and a rotation, one has

$$\Gamma \cap B_1 = \{(x,y) \in \mathbb{R}^d \mid y = \varphi(x)\} \cap B_1, \quad 0 \in \Gamma, \quad \varphi(0) = 0, \quad J_\varphi(0) = 0. \tag{3.110}$$

Let us define the diffeomorphism

$$\Phi(x,y) := (x, y + \varphi(x)), \tag{3.111}$$

which satisfies $\Phi(\Sigma_0 \cap B_R) \subset \Gamma \cap B_1$ for some $R > 0$, $\Phi(0) = \Phi^{-1}(0) = 0$ and the Jacobian associated to Φ is

$$J_\Phi(x,y) = \begin{pmatrix} \mathbb{I}_{d-n} & \mathbf{0} \\ J_\varphi(x) & \mathbb{I}_n \end{pmatrix}, \quad |\det J_\Phi| \equiv 1, \quad J_\Phi(0) = \mathbb{I}_d.$$

Next, we define the class of admissible weights with respect to the parametrization φ.

Definition 3.8.1 (α-defining function)**.** Let $2 \leq n < d$, $\alpha \in [0,1)$ and $\varphi \in C^{1,\alpha}(\Sigma_0 \cap B_1; \mathbb{R}^n)$ be a parametrization in the sense of (3.110). We say that δ is an α-defining function with respect to the parametrization φ if $\delta \in C^{0,1}(B_1)$ and the two following condition holds true:

i) there exist two constants $0 < c_0 \leq c_1$ such that

$$c_0 \leq \frac{\delta}{\mathrm{dist}_\Gamma} \leq c_1;$$

ii)

$$\tilde{\delta}(x,y) := \frac{\delta(\Phi(x,y))}{|y|} \in C^{0,\alpha}(B_1). \tag{3.112}$$

Remark 3.8.2. Let us consider a variable coefficient matrix $A \in C^0(B_1)$ satisfying the global uniform ellipticity condition (3.5) with ellipticity constants $0 < \lambda \leq \Lambda$ and satisfying the restricted-to-Γ uniform ellipticity condition

$$\lambda_* |\xi|^2 \leq A(z)\xi \cdot \xi \leq \Lambda_* |\xi|^2, \qquad \text{for a.e. } z \in B_1 \cap \Gamma \text{ and for all } \xi \in \mathbb{R}^d, \tag{3.113}$$

for some constants $0 < \lambda \leq \lambda_* \leq \Lambda_* \leq \Lambda$. Given a 0-defining function δ, let $\tilde{\delta}$ as in (3.112) and define the matrix

$$\tilde{A} := \tilde{\delta}^a (J_\Phi^{-1})(A \circ \Phi)(J_\Phi^{-1})^\top, \tag{3.114}$$

which is symmetric, continuous and satisfies the global uniform ellipticity condition (3.5). The latter condition follows by *i)* and *ii)* in Definition 3.8.1, which implies $\tilde{\delta} \geq c > 0$. In addition, since $J_\Phi(0) = \mathbb{I}$, one has that $\tilde{A}(0) = \tilde{\delta}^a(0)A(0)$ has ellipticity constants restricted to Σ_0 given by

$$0 < \tilde{\delta}^a(0)\lambda_* \leq \tilde{\delta}^a(0)\Lambda_*.$$

By using the continuity of $\tilde{A}$, for every $\sigma \in (0,1)$ there exists a small radius $\bar{r} \in (0,1)$ such that $\tilde{A}$ has ellipticity constants restricted to $B_{\bar{r}} \cap \Sigma_0$ given by

$$\tilde{\delta}^a(0)\lambda_* - \sigma \leq \tilde{\delta}^a(0)\Lambda_* + \sigma.$$

That is, in a local scale, $\tilde{A}$ and A have the same ellipticity ratio restricted to the thin space, up to an error

$$\frac{\tilde{\delta}^a(0)\lambda_* - \sigma}{\tilde{\delta}^a(0)\Lambda_* + \sigma} = \frac{\lambda_*}{\Lambda_*} + \tilde{\sigma},$$

where $\tilde{\sigma} > 0$ is arbitrarily small.

As done in Section 3.2.1, we can define the spaces $L^p(B_1, \delta^a)$, $L^p(B_1, \delta^a)^d$ and $H^1(B_1, \delta^a)$.

Definition 3.8.3. Let $a + n > 0$, A satisfying (3.5) and δ be a 0-defining function in the sense of Definition 3.8.1. Let $f \in L^{(2_a^*)'}(B_1, \delta^a)$ and $F \in L^2(B_1, \delta^a)^d$. We say that u is a weak solution to

$$-\operatorname{div}(\delta^a A \nabla u) = \delta^a f + \operatorname{div}(\delta^a F), \quad \text{in } B_1, \tag{3.115}$$

if $u \in H^1(B_1, \delta^a)$ and satisfies

$$\int_{B_1} \delta^a A \nabla u \cdot \nabla \phi \, dz = \int_{B_1} \delta^a (f\phi - F \cdot \nabla \phi) dz,$$

for every $\phi \in C_c^\infty(B_1)$.

The main results of this section are the following corollaries, which establish local $C^{0,\alpha}$ and $C^{1,\alpha}$ estimates for weak solutions to (3.115).

Corollary 3.8.4. *Let $a+n>0$, $p>(d+a_+)/2$ and $q>d+a_+$. Let $f\in L^p(B_1,\delta^a)$, $F\in L^q(B_1,\delta^a)^d$, $A\in C^0(B_1)$ be a continuous matrix satisfying (3.5) and (3.113) with constants $0<\lambda\le\lambda_*\le\Lambda_*\le\Lambda$, $\alpha_*=\alpha_*(n,a,\lambda_*/\Lambda_*)$ defined in (3.7) and α satisfying (3.8). Let $\varphi\in C^1(\Sigma_0\cap B_1;\mathbb{R}^n)$ be a parametrization in the sense of (3.110). Let δ be a 0-defining function with respect to φ in the sense of Definition 3.8.1 and set $\tilde\delta\in C^0(B_1)$ as in (3.112).*

Let u be a weak solution to (3.115) in the sense of Definition 3.8.3 and let us suppose that there exists a modulus of continuity ω such that

$$\|A\|_{C^{0,\omega}(B_1)}+\|\tilde\delta\|_{C^{0,\omega}(B_1)}+\|\varphi\|_{C^{1,\omega}(\Sigma_0\cap B_1;\mathbb{R}^n)}\le L.$$

Then, there exists $\bar r\in(0,1/2)$, depending only on α and L, such that $u\in C^{0,\alpha}(B_{\bar r})$ and there exists a constant $c>0$, depending only on d, n, a, λ, Λ, p, q, α and L such that

$$\|u\|_{C^{0,\alpha}(B_{\bar r})}\le c\big(\|u\|_{L^2(B_1,\delta^a)}+\|f\|_{L^p(B_1,\delta^a)}+\|F\|_{L^q(B_1,\delta^a)}\big).$$

Corollary 3.8.5. *Let $a+n>0$ and $p>d+a_+$. Let A be a matrix satisfying (3.5) and (3.113) with constants $0<\lambda\le\lambda_*\le\Lambda_*\le\Lambda$, $\alpha_*=\alpha_*(n,a,\lambda_*/\Lambda_*)$ defined in (3.7) and α satisfying (3.9). Let $\varphi\in C^{1,\alpha}(\Sigma_0\cap B_1;\mathbb{R}^n)$ be a parametrization in the sense of (3.110). Let δ be a α-defining function with respect to φ in the sense of Definition 3.8.1 and set $\tilde\delta\in C^{0,\alpha}(B_1)$ as in (3.112). Let us suppose that A is $C^{0,\alpha}$-continuous, $f\in L^p(B_1,\delta^a)$, $F\in C^{0,\alpha}(B_1)$.*

Let u be a weak solution to (3.115) in the sense of Definition 3.8.3 and let us suppose that

$$\|A\|_{C^{0,\alpha}(B_1)}+\|\tilde\delta\|_{C^{0,\alpha}(B_1)}+\|\varphi\|_{C^{1,\alpha}(\Sigma_0\cap B_1;\mathbb{R}^n)}\le L.$$

Then, there exists $\bar r\in(0,1/2)$, depending only on α and L, such that $u\in C^{1,\alpha}(B_{\bar r})$ and there exists a constant $C>0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u\|_{C^{1,\alpha}(B_{\bar r})}\le C\big(\|u\|_{L^2(B_1,\delta^a)}+\|f\|_{L^p(B_1,\delta^a)}+\|F\|_{C^{0,\alpha}(B_1)}\big). \tag{3.116}$$

Moreover, denoting by $T_z\Gamma$ the tangent space to Γ at the point $z\in\Gamma$, we have that u satisfies the following boundary condition for every $z\in\Gamma\cap B_{\bar r}$,

$$(A\nabla u+F)(z)\cdot\nu(z)=0,\qquad \textit{for every } \nu(z)\perp T_z\Gamma. \tag{3.117}$$

Since the proofs of Corollaries 3.8.4 and 3.8.5 are quite similar, we merely provide the proof of the Corollary 3.8.5. Additionally, we refer to Section 2.8, where the same results are proved in a closely related context.

Proof. The proof relies on making changes of variables to reduce the problem to the *flat* case, where the weight becomes $|y|^a$. The desired result then follows by applying Theorem 3.1.2.

Let Φ be as in (3.111), and define $\tilde u=u\circ\Phi$. Let $\bar r\in(0,1)$ to be chosen later, and let $\phi\in C_c^\infty(B_{2\bar r})$. Using the change of variable $z=\Phi(z)$, we obtain that

$$0=\int_{B_1}\delta^a\big(A\nabla u\cdot\nabla\phi-f\phi+F\cdot\nabla\phi\big)dz=\int_{B_1}|y|^a\big(\tilde A\nabla\tilde u\cdot\nabla\tilde\phi-\tilde f\tilde\phi+\tilde F\cdot\nabla\tilde\phi\big)dz,$$

where $\tilde{A}$ is defined in (3.114) and

$$\tilde{f} := \tilde{\delta}^a f \circ \Phi \in L^{p,a}(B_{2\bar{r}}), \quad \tilde{F} := \tilde{\delta}^a (J_\Phi^{-1}) F \circ \Phi \in C^{0,\alpha}(B_{2\bar{r}}), \quad \tilde{\phi} := \phi \circ \Phi.$$

By using the properties of the defining function (see Definition 3.8.1), one has that $\tilde{A} \in C^{0,\alpha}(B_{2\bar{r}})$. Moreover, by Remark 3.8.2, for every $\sigma > 0$ there exists $\bar{r} > 0$ such that $\tilde{A}$ has ellipticity constant restricted to $B_{2\bar{r}} \cap \Sigma_0$ given by

$$0 < \tilde{\delta}^a(0)\lambda_* - \sigma \leq \tilde{\delta}^a(0)\Lambda_* + \sigma.$$

Let us consider $\bar{r}$ small enough such that

$$\alpha < \alpha_*\Big(n, a, \frac{\tilde{\delta}^a(0)\lambda_* - \sigma}{\tilde{\delta}^a(0)\Lambda_* + \sigma}\Big) - 1,$$

where α_* is defined in (3.7). This choice is always possible since $\tilde{A}(0)$ has ellipticity constant $\tilde{\delta}^a(0)\lambda_* \leq \tilde{\delta}^a(0)\Lambda_*$ on Σ_0 and $\alpha < \alpha_*(n, a, \lambda_*/\Lambda_*) - 1$ (see Remark 3.8.2).

Resuming, we have shown that $\tilde{u}$ is a weak solution to

$$-\operatorname{div}(|y|^a \tilde{A} \nabla \tilde{u}) = |y|^a \tilde{f} + \operatorname{div}(|y|^a \tilde{F}), \qquad \text{in } B_{2\bar{r}}.$$

By the choice of $\bar{r}$, after rescaling the domain, the assumptions of Theorem 3.1.2 are satisfied. Hence, we have that $\tilde{u}$ satisfies (3.11) and (3.10) and composing back with Φ^{-1} we get that u satisfies (3.116) and (3.117) in $B_{\bar{r}}$. □

3.9 Smoothness of axially symmetric solutions

In this section, we prove that axially symmetric - with respect to Σ_0 - solutions (radial-in-y) are locally smooth, by establishing a one-to-one correspondence with solutions to an equation that degenerates on a hyperplane.

Definition 3.9.1. We say that a function $u : B_1 \subset \mathbb{R}^d \to \mathbb{R}$ is axially symmetric in y (i.e. with respect to Σ_0) if there exists a function $\tilde{u} : B_1^+ \subset \mathbb{R}^{d-n+1} \to \mathbb{R}$ such that $u(x,y) = \tilde{u}(x, |y|)$, where $B_1^+ := B_1^{d-n+1} \cap \{(x,r) \in \mathbb{R}^{d-n} \times \mathbb{R} \mid r > 0\}$ denotes the unitary upper half ball in $\mathbb{R}^{d-n+1}$.

Lemma 3.9.2. *Let $a + n > 0$ and let u be an axially symmetric in y function. Then, u is a weak solution to*

$$-\operatorname{div}(|y|^a \nabla u) = 0, \quad \text{in } B_1, \tag{3.118}$$

if and only if the function $\tilde{u}(x,r)$ is a weak solution to

$$\begin{cases} -\operatorname{div}(r^{a+n-1} \nabla \tilde{u}) = 0, & \text{in } B_1^+, \\ \lim\limits_{r \to 0} r^{a+n-1} \partial_r \tilde{u} = 0, & \text{on } B_1 \cap \{r = 0\}, \end{cases} \tag{3.119}$$

in the sense that $\tilde{u} \in H^1(B_1^+, r^{a+n-1} dx dr)$ and

$$\int_{B_1^+} r^{a+n-1} \nabla \tilde{u} \cdot \nabla \phi \, dx dr = 0, \qquad \textit{for every } \phi \in C_c^\infty(B_1)\,.$$

Proof. First, we notice that, by taking the spherical-in-y change of variable, one has

$$\begin{aligned}\int_{B_1^+} r^{a+n-1}(\tilde{u}^2+|\nabla\tilde{u}|^2)dxdr &= c_n\int_{\mathbb{S}^{n-1}}\int_{B_1^+} r^{a+n-1}(\tilde{u}^2+|\nabla\tilde{u}|^2)dxdrd\sigma\\ &= c_n\int_{B_1}|y|^a(u^2+|\nabla u|^2)dxdy,\end{aligned}$$

where $c_n=|\mathbb{S}^{n-1}|^{-1}$. Then, $u\in H^{1,a}(B_1)$ if and only if $\tilde{u}\in H^1(B_1^+, r^{a+n-1}dxdr)$.

Next, let us suppose that u is a weak solution to (3.118). Fix $\tilde{\phi}=\tilde{\phi}(x,r)\in C_c^\infty(B_1^{d-n+1})$ and call $\phi(x,y):=\tilde{\phi}(x,|y|)$. Then,

$$\begin{aligned}\int_{B_1^+} r^{a+n-1}\nabla\tilde{u}\cdot\nabla\tilde{\phi}\,dxdr &= c_n\int_{\mathbb{S}^{n-1}}\int_{B_1^+} r^{a+n-1}\nabla\tilde{u}\cdot\nabla\tilde{\phi}\,dxdrd\sigma\\ &= c_n\int_{B_1}|y|^a\nabla u\cdot\nabla\phi\,dxdy=0\,.\end{aligned}$$

Hence, $\tilde{u}$ is a weak solution to (3.119).

Instead, let us suppose that $\tilde{u}$ is a weak solution to (3.119), fix $\phi=\phi(x,y)\in C_c^\infty(B_1)$ and define $\tilde{\phi}(x,r):=\int_{\mathbb{S}^{n-1}}\phi(x,r\sigma)d\sigma$. Then, one has

$$\begin{aligned}\int_{B_1}|y|^a\nabla u\cdot\nabla\phi\,dxdy &= \int_{B_1^+}\int_{\mathbb{S}^{n-1}} r^{a+n+1}(\nabla_x\tilde{u},\partial_r\tilde{u})\cdot(\nabla_x\phi,\partial_r\phi)\,d\sigma dxdr\\ &= \int_{B_1^+} r^{a+n+1}(\nabla_x\tilde{u},\partial_r\tilde{u})\cdot\Big(\int_{\mathbb{S}^{n-1}}(\nabla_x\phi,\partial_r\phi)d\sigma\Big)dxdr\\ &= \int_{B_1^+} r^{a+n-1}(\nabla_x\tilde{u},\partial_r\tilde{u})\cdot(\nabla_x\tilde{\phi},\partial_r\tilde{\phi})dxdr=0.\end{aligned}$$

Therefore, u is a weak solution to (3.118). The proof is complete. □

Theorem 3.9.3 (Smoothness of axially symmetric solutions)**.** *Let $a+n>0$ and let u be an axially symmetric in y weak solution to* (3.118)*. Then, $u\in C^\infty_{\mathrm{loc}}(B_1)$.*

Proof. By definition $u(x,y)=\tilde{u}(x,|y|)$, for some function $\tilde{u}:B_1^+\subset\mathbb{R}^{d-n+1}\to\mathbb{R}$ and by using Lemma 3.9.2 one has that $\tilde{u}=\tilde{u}(x,r)$ is a weak solution to (3.119). By using the regularity theory of weighted equations degenerating on a hyperplane (see [143]), noting that $a+n-1>-1$, we get that $\tilde{u}\in C^\infty_{\mathrm{loc}}(\overline{B_r^+})$, for every $r\in(0,1)$.

Next, by using [138, Lemma 7.3], the function $\tilde{v}(x,r):=r^{-1}\partial_r\tilde{u}(x,r)$ is a weak solution to

$$\begin{cases}-\mathrm{div}(r^{a+n+1}\nabla\tilde{v})=0, & \text{in } B_{3/4}^+,\\ \lim\limits_{r\to0} r^{a+n+1}\partial_r\tilde{v}=0, & \text{on } B_{3/4}\cap\{r=0\}.\end{cases}$$

Applying again Lemma 3.9.2, the function

$$v(x,y):=\tilde{v}(x,|y|)=\frac{\nabla u(x,y)\cdot y}{|y|^2},$$

is a weak solution to

$$-\mathrm{div}(|y|^{a+2}\nabla v)=0,\quad \text{in } B_{3/4}. \tag{3.120}$$

Next, we prove that $\nabla u \in L^\infty(B_{1/2})$. For every $j = 1, \dots, d-n$, Proposition 3.5.1 ensures that $\partial_{x_j} u$ is a weak solution to (3.118). By using Theorem 3.1.1 we get that $\partial_{x_j} u \in C^{0,\alpha}(B_{1/2})$ and then $\partial_{x_j} u \in L^\infty(B_{1/2})$. On the other hand, since u is an axially symmetric in y function, one has that

$$\nabla_y u(x,y) = \partial_r \tilde{u}(x,|y|)\frac{y}{|y|} \in L^\infty(B_{1/2}),$$

so $\nabla u \in L^\infty(B_{1/2})$. Hence, we have proved that $u \in C^{0,1}(B_{1/2}) \subset H^1(B_{1/2})$ and we can rewrite equation (3.118) as

$$-\Delta u = av, \qquad \text{in } B_{1/2}. \tag{3.121}$$

Since v is a weak solution to (3.120), by using Theorem 3.1.1 one has that $v \in C^{0,\alpha}(B_{1/2})$ for some $\alpha \in (0,1)$. Then, by applying classical Schauder regularity theory to (3.121), we get that $u \in C^{2,\alpha}(B_{1/3})$.

Resuming, we have shown that if u is an axially symmetric in y weak solution to (3.118), then $u \in C^{2,\alpha}(B_{1/3})$. Therefore, since v is an axially symmetric in y weak solution to (3.120), we get that $v \in C^{2,\alpha'}(B_{1/3})$, for some $\alpha' \in (0,1)$ and, by using again classical Schauder regularity theory to (3.121), this implies $u \in C^{4,\alpha'}(B_{1/4})$. By iterating this procedure and using a covering lemma our statement follows. □

3.10 Inhomogeneous conormal problem

In this section, we extend our regularity results by considering equations that satisfy a non-homogeneous conormal condition on the thin set Σ_0 in the case $A = \mathbb{I}$. As discussed in Remark 3.4.2, the weak formulation of equation (3.36), which holds across Σ_0, implies the fact that the solutions formally satisfy a homogeneous conormal condition on Σ_0. In the mid-range $a+n \in (0,2)$, it is possible to impose inhomogeneous conormal boundary conditions on Σ_0.

Definition 3.10.1. Let $2 \le n < d$ and $a+n \in (0,2)$. We say that u is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a \nabla u) = 0, & \text{in } B_1, \\ -\lim\limits_{|y|\to 0} |y|^{a+n-1}\nabla u \cdot \dfrac{y}{|y|} = g(x), & \text{on } \Sigma_0 \cap B_1, \end{cases} \tag{3.122}$$

if $u \in H^{1,a}(B_1)$ and satisfies

$$\int_{B_1} |y|^a \nabla u \cdot \nabla \phi dz = \omega_n \int_{\Sigma_0 \cap B_1} g(x)\phi(x,0)dx, \qquad \text{for every } \phi \in C_c^\infty(B_1),$$

where $\omega_n = |\mathbb{S}^{n-1}|$.

Before stating the main result of this section, we first discuss the expected regularity for the solutions of the equation. Specifically, the function $|y|^{2-a-n}$ solves equation (3.122) with $g(x) = cost$, and this provides an upper bound on the regularity. In particular, when $a+n \in (1,2)$, we expect Hölder continuity of solutions, and when $a+n \in (0,1)$, we expect Hölder continuity of their gradient. Let us remark that $|y|^{2-a-n}$ also solves the homogeneous Dirichlet problem (3.123) whenever $a+n < 2$, see Section 3.11.

Proposition 3.10.2. *Let $2 \le n < d$, $a+n \in (0,2)$ and u be a weak solution to (3.122). Then, the following holds true.*

i) Let $g \in L^p(\Sigma_0 \cap B_1)$*, with* $p > \frac{d-n}{2-a-n}$ *and* $\alpha \in (0, 2-a-n-\frac{d-n}{p}] \cap (0,1)$*. Then,* $u \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$ *and there exists a constant* $C > 0$ *depending only on* d*,* n*,* a*,* p *and* α *such that*

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^{2,a}(B_1)} + \|g\|_{L^p(\Sigma_0 \cap B_1)}\big).$$

ii) Let us suppose that $a + n \in (0,1)$*. Let* $g \in L^p(\Sigma_0 \cap B_1)$*, with* $p > \frac{d-n}{1-a-n}$ *and* $\alpha \in (0, 1-a-n-\frac{d-n}{p}]$*. Then,* $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$ *and there exists a constant* $C > 0$ *depending only on* d*,* n*,* a*,* p *and* α *such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^{2,a}(B_1)} + \|g\|_{L^p(\Sigma_0 \cap B_1)}\big).$$

iii) Let us suppose that $a + n \in (0,1)$ *and let* $\alpha \in (0, 1-a-n)$*. Let* $g \in C^{0,\alpha}(\Sigma_0 \cap B_1)$*. Then,* $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$ *and there exists a constant* $C > 0$ *depending only on* d*,* n*,* a *and* α *such that*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C\big(\|u\|_{L^{2,a}(B_1)} + \|g\|_{C^{0,\alpha}(\Sigma_0 \cap B_1)}\big).$$

Proof. We provide the proof of *iii)* only, as the proofs of *i)* and *ii)* follow from the same arguments.

Let us consider the weak solution $\tilde{u}_1 = \tilde{u}_1(x,r) : B_1^+ \subset \mathbb{R}^{d-n+1} \to \mathbb{R}$ to

$$\begin{cases} -\operatorname{div}(r^{a+n-1}\nabla \tilde{u}_1) = 0, & \text{in } B_1^+, \\ -\lim\limits_{r\to 0} r^{a+n-1}\partial_r \tilde{u}_1 = g(x), & \text{on } B_1 \cap \{r = 0\}, \\ \tilde{u}_1 = 0, & \text{on } \partial B_1 \cap \{r > 0\}. \end{cases}$$

By [138, Theorem 8.4], it follows that $\partial_r \tilde{u}_1 = 0$ on $\Sigma_0 \cap B_1$ and

$$\|\tilde{u}_1\|_{C^{1,\alpha}(B^+_{1/2})} \le C\|g\|_{C^{0,\alpha}(\Sigma_0 \cap B_1)},$$

where $C > 0$ depends only on d, n, a and α. Hence, by applying Lemma 3.9.2 with a minor modification, one has that $u_1(x,y) := \tilde{u}_1(x,|y|)$ is an axially symmetric in y solution to

$$\begin{cases} -\operatorname{div}(|y|^a\nabla u_1) = 0, & \text{in } B_1, \\ -\lim\limits_{|y|\to 0} |y|^{a+n-1}\nabla u_1 \cdot \dfrac{y}{|y|} = g(x), & \text{on } \Sigma_0 \cap B_1, \\ u_1 = 0, & \text{on } \partial B_1. \end{cases}$$

Since $\partial_r \tilde{u}_1 = 0$ in $\Sigma_0 \cap B_1$, following computations analogous to those in Lemma 1.2.3 and in [138, Lemma 7.5], we conclude that u_1 inherits the same regularity as $\tilde{u}_1$; that is,

$$\|u_1\|_{C^{1,\alpha}(B_{1/2})} \le C\|g\|_{C^{0,\alpha}(\Sigma_0 \cap B_1)}.$$

Next, let us define $u_2 := u - u_1$, which is a weak solution in the sense of Definition 3.4.1 to

$$-\operatorname{div}(|y|^a\nabla u_2) = 0, \quad \text{in } B_1.$$

Noticing that $\alpha_* = \alpha_*(n,a,1)$ given by (3.7) satisfies $\alpha_* - 1 > 1 - a - n > \alpha$, Theorem 3.1.2 yields that

$$\|u_2\|_{C^{1,\alpha}(B_{1/2})} \le C\|u\|_{L^{2,a}(B_1)}.$$

Hence, $u = u_1 + u_2$ satisfies the desired estimate and the proof is complete. □

3.11 Homogeneous Dirichlet problem via a boundary Harnack principle

In this section, we prove Hölder $C^{0,\alpha}$ and Schauder $C^{1,\alpha}$ regularity for solutions to the homogeneous Dirichlet problem

$$\begin{cases} -\mathrm{div}(|y|^a\nabla u) = 0 & \text{in } B_1 \setminus \Sigma_0 \\ u = 0 & \text{on } B_1 \cap \Sigma_0\,, \end{cases} \tag{3.123}$$

whenever $a+n<2$. The solutions we are referring to are elements of $\tilde{H}^{1,a}(B_1)$, which is defined as the completion of $C_c^\infty(\overline{B_1}\setminus\Sigma_0)$ with respect to the norm $\|\cdot\|_{H^{1,a}(B_1)}$ and satisfies

$$\int_{B_1} |y|^a \nabla u \cdot \nabla\phi\, dz = 0\,, \qquad \text{for every } \phi \in C_c^\infty(B_1\setminus\Sigma_0).$$

We remark that in the supersingular case $a+n\le 0$, any function in $H^{1,a}(B_1)$ is forced to have trace $u=0$ on Σ_0 (see Proposition 3.2.5). Instead, in the mid-range $a+n\in(0,2)$, the trace condition $u=0$ on Σ_0 is well-defined, since Σ_0 has positive local weighted capacity. See also [118] for some trace type theorems on lower dimensional sets.

The idea is to obtain regularity as a corollary of Theorems 3.1.1 and 3.1.2 via a boundary Harnack principle, in the same spirit as [139]. In fact, the ratio $w := u/u_0$ between a solution u to (3.123) and the *characteristic solution* to (3.123)

$$u_0(y) = |y|^{2-a-n}$$

solves the equation

$$-\mathrm{div}(|y|^b\nabla w) = 0 \qquad \text{in } B_1 \tag{3.124}$$

with exponent $b = 4-a-2n$ lying in the superdegenerate range since $b+n = 4-a-n > 2$.

Lemma 3.11.1. *Let $a+n<2$, u be a solution to (3.123) and $u_0 = |y|^{2-a-n}$. Then, $w = u/u_0$ solves (3.124).*

Proof. The equation is trivially satisfied far from Σ_0 in a classic sense. Moreover, due to the superdegeneracy of the weight $|y|^b$, and combining Lemma 3.2.8 with the density of $C_c^\infty(\overline{B_1}\setminus\Sigma_0)$ in $H^{1,b}(B_1)$, the result is trivially true if one shows that the $H^{1,b}(B_1)$-energy of w is finite. Recalling the Hardy inequality (see Proposition 3.2.2), we easily compute that

$$\begin{aligned} \int_{B_1} |y|^b |\nabla w|^2\, dz &\le c\int_{B_1} |y|^b\Big(\frac{|\nabla u|^2}{|u_0|^2} + \frac{u^2|\nabla u_0|^2}{|u_0|^4}\Big)\, dz \\ &= c\Big(\int_{B_1} |y|^a|\nabla u|^2\, dz + \int_{B_1} |y|^{a-2}u^2\, dz\Big) \le c\int_{B_1} |y|^a |\nabla u|^2\, dz\,. \end{aligned}$$

Furthermore, $\|w\|_{L^{2,b}(B_1)} = \|u\|_{L^{2,a}(B_1)}$, hence $w\in H^{1,b}(B_1)$ and the proof is complete. □

Let us recall that the homogeneity degree appearing (3.7), related to the new exponent b, is given by

$$\alpha_*(b,n) := \alpha_*(b,n,1) = \frac{2-b-n+\sqrt{(2-b-n)^2+4(n-1)}}{2}\,. \tag{3.125}$$

Then, Theorems 3.1.1 and 3.1.2 imply the following result.

Corollary 3.11.2. *Let $a+n<2$ and $b=4-a-2n$. Let $\alpha_*(b,n)>0$ be the homogeneity exponent in (3.125) and*

$$\alpha\in(0,1)\cap(0,\alpha_*(b,n)).$$

Let u be a weak solution to (3.123) in B_1 and $u_0=|y|^{2-a-n}$. Then, $u/u_0\in C^{0,\alpha}_{\mathrm{loc}}(B_1)$ and there exists a constant $c>0$ depending only on d, n, a and α such that

$$\left\|\frac{u}{u_0}\right\|_{C^{0,\alpha}(B_{1/2})}\le c\|u\|_{L^{2,a}(B_1)}.$$

If moreover $\alpha_*(b,n)>1$ and

$$\alpha\in(0,1)\cap(0,\alpha_*(b,n)-1),$$

then $u/u_0\in C^{1,\alpha}_{\mathrm{loc}}(B_1)$ and there exists a constant $c>0$ depending only on d, n, a and α such that

$$\left\|\frac{u}{u_0}\right\|_{C^{1,\alpha}(B_{1/2})}\le c\|u\|_{L^{2,a}(B_1)}.$$

Remark 3.11.3. The previous result improves [119, Theorem 1] in the flat case, which corresponds to a L^∞-bound of u/u_0. Moreover, it slightly improve Theorems 2.1.1 and 2.1.2 in the present homogeneous case. The exponent $\alpha_*(b,n)$, given by (3.125), satisfies $\alpha_*(b,n)>2-a-n$ whenever

$$n-1>2(2-a-n)^2. \tag{3.126}$$

In particular this implies that, under (3.126), one can imply the sharp C^{2-a-n} regularity of solutions to (3.123). In fact, if $w=u/u_0$ has C^β regularity with $\beta\ge2-a-n$, then

$$u=wu_0\in C^{2-a-n}.$$

In particular, when $a+n=1$, which corresponds to the exponent studied by David, Feneuil and Mayboroda (see [45] and its related works), one can prove the sharp Lipschitz continuity of solutions whenever the codimension $n\ge4$.

3.12 Higher codimensional extensions of fractional Laplacian

The aim of this section is to establish a connection between the degenerate equations discussed before and fractional Laplacian on the very thin set Σ_0, in the sense of Dirichlet-to-Neumann maps, in the spirit of [28]. Let $s\in(0,1)$ and $2\le n<d$. The s-fractional Laplacian of sufficiently regular functions can be defined in $\mathbb{R}^{d-n}$ equivalently as a integro-differential operator

$$(-\Delta)^s u(x)=C_{d-n,s}\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^{d-n}\setminus B_\varepsilon(x)}\frac{u(x)-u(\xi)}{|x-\xi|^{d-n+2s}}\,d\xi\,,$$

or via Fourier transform

$$\widehat{(-\Delta)^s u}(\xi)=|\xi|^{2s}\hat{u}(\xi)\,.$$

Assuming $d-n>2s$, and using the fractional Hardy inequality, we can set fractional problems in $\mathcal{D}^s(\mathbb{R}^{d-n})$, defined as the completion of $C^\infty_c(\mathbb{R}^{d-n})$ with respect to the norm

$$\|u\|^2_{\mathcal{D}^s(\mathbb{R}^{d-n})}=\int_{\mathbb{R}^{d-n}}|(-\Delta)^{s/2}u(x)|^2\,.$$

Let $a = 2 - n - 2s$, which satisfies $a + n \in (0, 2)$. This corresponds to the mid-range regime, where the local weighted capacity of Σ_0 is positive and finite, see Lemma 3.2.4. Next, we define $\mathcal{D}^{1,a}(\mathbb{R}^d)$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to

$$\|U\|^2_{\mathcal{D}^{1,a}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |y|^a |\nabla U|^2,$$

which is a norm due to an Hardy-type inequality, see for instance [40, Theorem 1]. In the following result, we establish the existence of a trace operator from $\mathcal{D}^{1,a}(\mathbb{R}^d)$ to $\mathcal{D}^s(\mathbb{R}^{d-n})$. We refer also to [118] for some related local trace type results.

Lemma 3.12.1. *Let $2 \leq n < d$ and $a + n \in (0, 2)$. Then, there exists a constant $C_{a,n} > 0$ such that*

$$C_{a,n} \|\phi_{|\Sigma_0}\|_{\mathcal{D}^s(\mathbb{R}^{d-n})} \leq \|\phi\|_{\mathcal{D}^{1,a}(\mathbb{R}^d)} \tag{3.127}$$

for any $\phi \in C_c^\infty(\mathbb{R}^d)$ with $s = (2 - a - n)/2 \in (0, 1)$. Moreover, if $d + a > 2$ (i.e. $d - n > 2s$), (3.127) defines the continuous trace operator

$$\mathrm{Tr} : \mathcal{D}^{1,a}(\mathbb{R}^d) \to \mathcal{D}^s(\mathbb{R}^{d-n}).$$

Proof. The proof is an adaptation of [39, Theorem 3.8], so we omit most of the details. Let $\phi \in C_c^\infty(\mathbb{R}^d)$, and let $\hat{\phi}$ denote the Fourier transform of ϕ with respect to the x variables. Then we have

$$\int_{\mathbb{R}^d} |y|^a |\nabla\phi|^2 dz = \int_{\mathbb{R}^{d-n}} \int_{\mathbb{R}^n} |y|^a (|\xi|^2 |\hat{\phi}|^2 + |\nabla_y \hat{\phi}|^2) dy d\xi \, .$$

Next, define $v(\xi, t) = \hat{\phi}(\xi, |\xi|^{-1} t)$, and perform the change of variables $y = |\xi|^{-1} t$ in the integral above. This leads to

$$\int_{\mathbb{R}^d} |y|^a |\nabla\phi|^2 dz = \int_{\mathbb{R}^{d-n}} |\xi|^{2-a-n} \int_{\mathbb{R}^n} |t|^a (|v|^2 + |\nabla_t v|^2) dt \, d\xi \, .$$

We now claim that

$$\inf_{v \in C_c^\infty(\mathbb{R}^n), v(0)=1} \int_{\mathbb{R}^n} |t|^a (|v|^2 + |\nabla_t v|^2) dt = C_{a,n} > 0 \, .$$

Assume that this claim is true. Then, recalling that $v(\xi, 0) = \hat{\phi}(\xi, 0)$, we obtain

$$\int_{\mathbb{R}^d} |y|^a |\nabla\phi|^2 dz \geq C_{a,n} \int_{\mathbb{R}^{d-n}} |\xi|^{2-a-n} |\hat{\phi}(\xi, 0)|^2 d\xi = C_{a,n} \int_{\mathbb{R}^{d-n}} |(-\Delta)^{s/2} \phi(x, 0)|^2 dx \, ,$$

with $s = (2 - a - n)/2 \in (0, 1)$. This completes the proof of (3.127). The rest of the proof follows in a standard way.

Let us now prove the claim. To do this, we argue by contradiction. Assume that $C_{a,n} = 0$. Then, for every $\delta > 0$ there exists $v_\delta \in C_c^\infty(\mathbb{R}^n)$ such that $v_\delta(0) = 1$ and

$$\int_{\mathbb{R}^n} |t|^a (|v|^2 + |\nabla_t v|^2) dt < \delta \, . \tag{3.128}$$

Using computations similar to those in the proof of part *iii*) of Lemma 3.2.4, we find that for every $\rho > 0$,

$$n\omega_n = \int_{\mathbb{S}^{n-1}} |v_\delta(0)|^2 d\sigma \leq c\rho^{2-a-n} \int_{\mathbb{R}^n} |t|^a |\nabla_t v_\delta|^2 dt + 2 \int_{\mathbb{S}^{n-1}} |v_\delta(\rho\sigma)|^2 d\sigma \,.$$

Multiplying by ρ^{a+n-1}, integrating over $\rho \in (0,1)$, and using (3.128), we obtain

$$\frac{n\omega_n}{a+n} \leq c \int_{\mathbb{R}^n} |t|^a |\nabla_t v_\delta|^2 dt + 2 \int_{B_1} |t|^a |v_\delta|^2 dt \leq c\delta \,,$$

which is a contradiction for sufficiently small δ. Therefore, the claim holds true and the proof is complete. ☐

As a dual result, we define a unique minimal energy extension in $\mathcal{D}^{1,a}(\mathbb{R}^d)$ for functions in $\mathcal{D}^s(\mathbb{R}^{d-n})$.

Lemma 3.12.2. *Let $2 \leq n < d$ with $d + a > 2$ and $a + n \in (0,2)$. Let $s = (2-a-n)/2$. Then, for any $u \in \mathcal{D}^s(\mathbb{R}^{d-n})$ the minimization problem*

$$\inf_{U \in \mathcal{D}^{1,a}(\mathbb{R}^d)\,,\, \mathrm{Tr}U = u} \|U\|^2_{\mathcal{D}^{1,a}(\mathbb{R}^d)} \tag{3.129}$$

admits a unique minimizer. Moreover, (3.129) defines an extension operator

$$\mathrm{Ext} : \mathcal{D}^s(\mathbb{R}^{d-n}) \to \mathcal{D}^{1,a}(\mathbb{R}^d).$$

Proof. Fix $u \in \mathcal{D}^s(\mathbb{R}^{d-n})$. The existence of a minimizer $\bar{U}$ for (3.129) follows from standard variational arguments. In particular, $\bar{U}$ satisfies

$$\int_{\mathbb{R}^d} |y|^a \nabla \bar{U} \cdot \nabla \phi \, dz = 0 \,, \qquad \text{for every } \phi \in \mathcal{D}^{1,a}(\mathbb{R}^d)\,,\ \mathrm{Tr}\phi = 0 \,. \tag{3.130}$$

Finally, to prove the uniqueness of $\bar{U}$, it is sufficient to use a standard contradiction argument to show that (3.130) admits an unique solution in the set $\{U \in \mathcal{D}^{1,a}(\mathbb{R}^n) \mid \mathrm{Tr}U = u\}$. ☐

Then, the extension result can be summarized as below.

Proposition 3.12.3. *Let $2 \leq n < d$ with $d+a > 2$ and $a+n \in (0,2)$. Let $s = (2-a-n)/2 \in (0,1)$ Then for any $u \in \mathcal{D}^s(\mathbb{R}^{d-n})$, the extension $U = \mathrm{Ext}(u)$ is given by*

$$U(x,y) = u * P(x,y), \qquad P(x,y) = \frac{\Gamma(\frac{d-n+2s}{2})}{\pi^{\frac{d-n}{2}}\Gamma(s)} \frac{|y|^{2s}}{(|x|^2+|y|^2)^{\frac{d-n+2s}{2}}} \,,$$

and is solution to

$$\begin{cases} -\mathrm{div}(|y|^a \nabla U) = 0 & \text{in } \mathbb{R}^d \setminus \Sigma_0 \\ -\lim\limits_{|y|\to 0} |y|^{a+n-1} \nabla U \cdot \dfrac{y}{|y|} = d_{a,n}(-\Delta)^s u & \text{on } \Sigma_0 \,, \end{cases}$$

in the sense that $U \in \mathcal{D}^{1,a}(\mathbb{R}^d)$ and satisfies

$$\int_{\mathbb{R}^d} |y|^a \nabla U \cdot \nabla \phi \, dx\, dy = d_{a,n} \int_{\mathbb{R}^{d-n}} (-\Delta)^s u \, \phi(x,0)\, dx \,, \qquad \textit{for every } \phi \in C_c^\infty(\mathbb{R}^d) \,,$$

where $d_{a,n} = 2^{a+n-1}\Gamma(\frac{a+n}{2})/\Gamma(\frac{2-a-n}{2})$.

Proof. Let $u \in \mathcal{D}^s(\mathbb{R}^n)$. Notice that $P(x,y) = P(|x|,|y|)$, and hence $U(x,y) = \tilde{U}(x,|y|)$, where $\tilde{U}(x,r) = u * P(x,r)$ for $r > 0$. From the codimension 1 extension theory in [28], we have that $\tilde{U} \in \mathcal{D}^{1,1-2s}(\mathbb{R}^{d-n})$, and

$$\begin{cases} -\mathrm{div}(r^{1-2s}\nabla\tilde{U}) = 0 & \text{in } \mathbb{R}^{d-n+1}_+ \\ -\lim\limits_{r\to 0} r^{1-2s}\partial_r\tilde{U} = 2^{1-2s}\dfrac{\Gamma(1-s)}{\Gamma(s)}(-\Delta)^s u & \text{on } \mathbb{R}^{d-n}, \end{cases}$$

or, equivalently, that

$$\int_{\mathbb{R}^{d-n+1}_+} r^{a+n-1}\nabla\tilde{U}\cdot\nabla\tilde{\phi}\,dx\,dr = d_{a,n}\int_{\mathbb{R}^{d-n}} (-\Delta)^s u\,\tilde{\phi}(x,0)\,dx\,, \qquad \text{for every } \tilde{\phi} \in C^\infty_c(\overline{\mathbb{R}^{d-n+1}_+})\,,$$

where we used that $s = (2-a-n)/2$. Thus, adapting the argument of the proof of Lemma 3.9.2 we infer that $U \in \mathcal{D}^{1,a}(\mathbb{R}^d)$, $\mathrm{Tr}U = u$ and

$$\int_{\mathbb{R}^d} |y|^a\nabla U\cdot\nabla\phi\,dx\,dy = d_{a,n}\int_{\mathbb{R}^{d-n}} (-\Delta)^s u\,\phi(x,0)\,dx\,, \qquad \text{for every } \phi \in C^\infty_c(\mathbb{R}^d)\,.$$

In particular, U is the unique solution of (3.130) in the set $\{U \in \mathcal{D}^{1,a}(\mathbb{R}^n) \mid \mathrm{Tr}U = u\}$, and thus, the unique minimizer of (3.129). It immediately follows that $U = \mathrm{Ext}(u)$, and the proof is complete. □

Remark 3.12.4. Let $u \in \mathcal{D}^s(\mathbb{R}^{d-n})$. As shown in Proposition 3.12.3, it holds

$$\int_{\mathbb{R}^d} |y|^a\nabla\mathrm{Ext}(u)\cdot\nabla\phi\,dx\,dy = d_{a,n}\int_{\mathbb{R}^{d-n}} (-\Delta)^s u\,\phi(x,0)\,dx\,, \qquad \text{for every } \phi \in \mathcal{D}^{1,a}(\mathbb{R}^d)\,.$$

By taking $\phi = \mathrm{Ext}(u)$, we immediately infer $\|\mathrm{Ext}(u)\|_{\mathcal{D}^{1,a}(\mathbb{R}^d)} = d_{a,n}\|u\|_{\mathcal{D}^s(\mathbb{R}^{d-n})}$. As a consequence, $C_{a,n} = d_{a,n}$, where $C_{a,n}$ is the constant in Lemma 3.12.1.

Appendix 3.A Geometry of perforated domains

In this section, we work in the unit ball B_1 to simplify the notation, but the same results hold at any scale $R > 0$, in which case the constants involved also depend on R.

Let $A \in C^1(B_1;\mathbb{R}^{d,d})$ (see Remark 3.3.1 for the sharp requirement) be a symmetric matrix satisfying the uniform ellipticity condition (3.5), and recall the notation (3.18). The block A_3 is still symmetric and satisfies

$$\lambda|\zeta|^2 \le A_3(z)\zeta\cdot\zeta \le \Lambda|\zeta|^2\,, \qquad \text{for a.e. } z \in B_1 \quad \text{and every } \zeta \in \mathbb{R}^n\,,$$

where λ and Λ are the uniform ellipticity constants of A. By the ellipticity condition, A_3 is invertible, and $A_3^{-1} \in C^1(B_1;\mathbb{R}^{n,n})$. The matrix A_3^{-1} is also symmetric and satisfies

$$\Lambda^{-1}|\zeta|^2 \le A_3^{-1}(z)\zeta\cdot\zeta \le \lambda^{-1}|\zeta|^2\,, \qquad \text{for a.e. } z \in B_1 \quad \text{and every } \zeta \in \mathbb{R}^n\,. \tag{3.131}$$

The square root of A_3^{-1}, denoted by $A_3^{-\frac{1}{2}}$, is well-defined and belongs to $C^1(B_1;\mathbb{R}^{n,n})$ (see for instance [53, Theorem 2.5]). It is also symmetric and satisfies

$$\Lambda^{-\frac{1}{2}}|\zeta|^2 \leq A_3^{-\frac{1}{2}}(z)\zeta\cdot\zeta \leq \lambda^{-\frac{1}{2}}|\zeta|^2\,, \text{ for a.e. } z\in B_1 \quad \text{and every } \zeta\in\mathbb{R}^n\,. \tag{3.132}$$

We will frequently use the following properties, which follow directly from (3.131) and (3.132).

$$|A_3^{-\frac{1}{2}}(z)y| = \sqrt{A_3^{-1}(z)y\cdot y}\,, \quad \Lambda^{-\frac{1}{2}}|y| \leq |A_3^{-\frac{1}{2}}(z)y| \leq \lambda^{-\frac{1}{2}}|y|\,, \quad \Lambda^{-1}|y| \leq |A_3^{-1}(z)y| \leq \lambda^{-1}|y|\,. \tag{3.133}$$

In the following lemma, we describe some properties of the (ε, A)-neighborhood of Σ_0

$$\Sigma_\varepsilon^A = \{z \mid A_3^{-1}(z)y\cdot y \leq \varepsilon^2\}\,,$$

and its boundary

$$\partial\Sigma_\varepsilon^A = \{z \mid A_3^{-1}(z)y\cdot y = \varepsilon^2\}\,.$$

Lemma 3.A.1. *Let $A\in C^1(B_1;\mathbb{R}^{d,d})$ and $L := \|A_3^{-1}\|_{C^1(B_1)}$. There exists $\varepsilon_0>0$, depending only on d, n, L, λ and Λ, such that for every $\varepsilon\in(0,\varepsilon_0]$ the following properties hold:*

i) the normal vector $\nu(z)$ to $\partial\Sigma_\varepsilon^A$ at $z\in B_1\cap\partial\Sigma_\varepsilon^A$ is well-defined. In particular,

$$\nu(z) = (0, N(z)) + \tilde\nu(z)\,, \tag{3.134}$$

where

$$N(z) = \frac{A_3^{-1}(z)y}{|A_3^{-1}(z)y|} \in \mathbb{S}^{n-1}\,, \tag{3.135}$$

and

$$|\tilde\nu(z)| \leq c\varepsilon\,,$$

for some constant $c = c(d,n,\Lambda,L,\varepsilon_0)>0$;

ii) it holds

$$\nu(z)\cdot\frac{y}{|y|} \geq \frac{\lambda}{2\Lambda}\,; \tag{3.136}$$

iii) there exists a constant $c>0$ depending only on L, d, n, λ and Λ such that

$$[N]_{C^{0,1}(B_1\cap\partial\Sigma_\varepsilon^A)} \leq c\varepsilon^{-1}\,;$$

iv) if in addition $A\in C^{1,\beta}(B_1;\mathbb{R}^{d,d})$ for some $\beta\in(0,1)$, then

$$[\tilde\nu]_{C^{0,\beta}(B_1\cap\partial\Sigma_\varepsilon^A)} \leq c\,,$$

where $c>0$ depends only on $\|A_3^{-1}\|_{C^{1,\beta}(B_1)}$, d, n, λ and Λ.

Proof. Define the function

$$\Psi(z) = \sqrt{A_3^{-1}(z)y\cdot y}\,. \tag{3.137}$$

From (3.133), we have $\Psi\in C^{0,1}(B_1)\cap C^1(B_1\setminus\Sigma_0)$, and $\Psi(x,y)=0$ if and only if $y=0$.

Since $\partial\Sigma_\varepsilon^A=\{\Psi(z)=\varepsilon\}$, the normal vector $\nu(z)$ at a point $z\in\partial\Sigma_\varepsilon^A$ is given by

$$\nu(z)=\frac{\nabla\Psi(z)}{|\nabla\Psi(z)|}=\frac{\nabla(A_3^{-1}(z)y\cdot y)}{|\nabla(A_3^{-1}(z)y\cdot y)|}\,.$$

Let $A_3^{-1}(z)=(b_{ij}(z))_{i,j=1,\dots,n}$. We define the functions $G:B_1\to\mathbb{R}^d$ and $H:B_1\to\mathbb{R}$ as follows

$$G(z)=\frac{1}{2}\Big(\sum_{i,j=1}^{n}\partial_{z_\ell}b_{ij}(z)y_iy_j\Big)_{\ell=1,\dots,d}\,,\qquad H(z)=|A_3^{-1}(z)y|-|(0,A_3^{-1}(z)y)+G(z)|\,.$$

Using this notation, the normal vector can be expressed as

$$\nu(z)=\frac{(0,A_3^{-1}(z)y)+G(z)}{|A_3^{-1}(z)y|-H(z)}\,.$$

There exists a constant $c_1=c_1(d,n)>0$ such that

$$|H(z)|\le|G(z)|\le c_1L|y|^2\,,\qquad\text{for every } z\in B_1\,. \tag{3.138}$$

Fix ε_0 such that $\Lambda^{-1}\lambda^{\frac{1}{2}}-c_1L\Lambda\varepsilon_0>0$. Thus, if $\varepsilon\le\varepsilon_0$ and $z\in B_1\cap\partial\Sigma_\varepsilon^A$, thanks to (3.133) and (3.138) we have

$$|A_3^{-1}(z)y|-H(z)\ge\Lambda^{-1}|y|-cL|y|^2\ge\varepsilon(\Lambda^{-1}\lambda^{\frac{1}{2}}-cL\Lambda\varepsilon)\ge c\varepsilon\,. \tag{3.139}$$

Therefore, ν is well-defined.

Define $\tilde\nu(z):=\nu(z)-(0,N(z))$, where $N(z)$ is as in (3.135). Then

$$\tilde\nu(z)=\frac{H(z)N(z)}{|A_3^{-1}(z)y|-H(z)}+\frac{G(z)}{|A_3^{-1}(z)y|-H(z)}\,.$$

It follows that

$$|\tilde\nu(z)|\le c\varepsilon\,,$$

where $c=c(d,n,\Lambda,\lambda,L,\varepsilon_0)>0$. Thus $i)$ is proved.

To prove $ii)$, we compute

$$\nu\cdot\frac{y}{|y|}=N(z)\cdot\frac{y}{|y|}+\tilde\nu\cdot\frac{y}{|y|}\ge\frac{\lambda}{\Lambda}-|\tilde\nu|\,.$$

Taking if necessary a smaller ε_0, so that $|\tilde\nu|\le\lambda/2\Lambda$, we obtain (3.136).

For $iii)$, note that for every $z=(x,y),z'=(x',y')\in B_1$ it holds

$$|A_3^{-1}(z)y-A_3^{-1}(z')y'|\le|A_3^{-1}(z)-A_3^{-1}(z')||y|+|A_3^{-1}(z')||y-y'|\le 2L|z-z'|\,. \tag{3.140}$$

Therefore, for all $z,z'\in B_1\cap\partial\Sigma_\varepsilon^A$, we have

$$|N(z)-N(z')|\le 2\frac{|A_3^{-1}(z)y-A_3^{-1}(z')y'|}{|A_3^{-1}(z')y'|}\le c\varepsilon^{-1}|z-z'|\,,$$

where $c>0$ depends only on L, Λ, λ.

Finally, we prove *iv*). Assume that $A_3^{-1} \in C^{1,\beta}(B_1;\mathbb{R}^{n,n})$. Using similar computations as before, we obtain

$$|G(z) - G(z')| \le c\varepsilon |z - z'|^\beta \,, \qquad \text{for every} \qquad z, z' \in B_1 \cap \partial\Sigma_\varepsilon^A \,, \tag{3.141}$$

where c depends on d, n, λ, Λ, $\|A_3^{-1}\|_{C^{1,\beta}(B_1)}, \varepsilon_0$.

Define the function

$$h(z) = 2\frac{N(z)\cdot G(z)}{|A_3^{-1}(z)y|} + \frac{|G(z)|^2}{|A_3^{-1}(z)y|^2} \,.$$

By combining (3.138), (3.140), (3.141), and *iii*), we deduce

$$|h(z)| \le c|y| \ \text{ for all } z \in B_1 \,, \quad \text{and} \quad |h(z) - h(z')| \le c|z - z'|^\beta \ \text{ for all } z, z' \in B_1 \,. \tag{3.142}$$

Thus, when ε is sufficiently small, we can apply the Taylor expansion for $(1+t)^{\frac{1}{2}}$ to $|(0, A_3^{-1}(z)y) + G(z)|$, yielding for every $z \in B_1 \cap \partial\Sigma_\varepsilon^A$,

$$H(z) = |A_3^{-1}(z)y| \sum_{j=1}^{\infty} c_j h^j(z) \,,$$

where c_j are the coefficients of the expansion. Using (3.138) and (3.142), we infer that

$$|H(z) - H(z')| \le c\varepsilon |z - z'|^\beta \,. \tag{3.143}$$

Finally, combining (3.138), (3.139), (3.140), (3.141) and (3.143), we obtain

$$|\tilde\nu(z) - \tilde\nu(z)| \le c(|z - z'| + \varepsilon^{-1}|H(z) - H(z')| + \varepsilon^{-1}|G(z) - G(z')|) \le c|z - z'|^\beta \,,$$

and the thesis follows. □

Lemma 3.A.2. *Let* $G : B_1 \setminus \Sigma_\varepsilon^A \to \mathbb{R}^d$ *be such that* $G \in C^{0,\alpha}(B_1 \setminus \Sigma_\varepsilon^A)$ *for some* $\alpha \in (0,1)$, *and* $G(z)\cdot\nu(z) = 0$ *for every* $z \in B_1 \cap \partial\Sigma_\varepsilon^A$. *Then, there exist a constant* $c > 0$ *depending only on* d, n, λ, Λ, α *and* $L := \|A_3^{-1}\|_{C^1(B_1)}$ *such that for every* $0 < \varepsilon \ll 1$ *and* $i = 1, \dots, n$, *it holds*

$$\|G \cdot e_{y_i}\|_{L^\infty(B_{1/2}\cap\partial\Sigma_\varepsilon^A)} \le c\varepsilon^\alpha \|G\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon^A)} \,.$$

Proof. We recall that by Lemma 3.A.1, for $\varepsilon \le \varepsilon_0$, the normal vector to the surface $\partial\Sigma_\varepsilon^A$ at $z = (x,y) \in B_1 \cap \partial\Sigma_\varepsilon^A$ is given by $\nu(z) = (0, N(z)) + \tilde\nu(z)$, where $\tilde\nu(z) \le c\varepsilon$ and

$$N(x,y) = \frac{A_3^{-1}(z)y}{|A_3^{-1}(z)y|} \in \mathbb{R}^n \,.$$

In what follows, we show that there exist $\delta_1 = \delta_1(\varepsilon_0, \Lambda)$, $\delta_2 = \delta_2(L, \Lambda)$, such that for every $x \in B_{1/2}^{d-n}$ and for $\varepsilon \le \varepsilon_0$ there exist $\bar x \in B_{1/2}^{d-n}$ and n vectors $\{y_j\}_{j=1,\dots,n} \subset \mathbb{R}^n$ (depending on $\bar x$) such that, defined $\tau_j = N(\bar x, y_j)$, it holds

$$|x - \bar x| \le \delta_1\varepsilon \,, \qquad (\bar x, y_j) \in B_{1/2} \cap \partial\Sigma_\varepsilon^A \,, \qquad |\tau_i \cdot \tau_j| \le \delta_2\varepsilon \,, \ \text{for every } i \ne j \,.$$

Note that this implies that $\operatorname{span}\{\tau_j\} = \mathbb{R}^n$. Take $\delta_1 > 0$ such that $\Lambda^{\frac{1}{2}} < \delta_1 < (2\varepsilon_0)^{-1}$ and define

$$\bar x = (1 - 2\delta_1\varepsilon)x.$$

By construction $1-2\delta_1\varepsilon>0$ and $|x-\bar{x}|\le\delta_1\varepsilon$. Moreover, let $y\in\mathbb{R}^d$ be such that $(\bar{x},y)\in\partial\Sigma^A_\varepsilon$. It holds

$$|(\bar{x},y)|\le|\bar{x}|+|y|=(1-2\delta_1\varepsilon)|x|+|y|\le\frac{1}{2}-\big(\delta_1-\Lambda^{\frac{1}{2}}\big)\varepsilon<\frac{1}{2}\,.$$

Hence $(\bar{x},y)\in B_{1/2}\cap\partial\Sigma^A_\varepsilon$, as needed.

Now, let us fix any point $(\bar{x},y_1)\in\partial\Sigma^A_\varepsilon$, and define $\tau_1=N(\bar{x},y_1)$. One can choose $\sigma_2\in\mathbb{S}^{n-1}$ such that $\sigma_2\cdot A_3^{-2}(\bar{x},y_1)y_1=0$ (where $A_3^{-2}=(A_3^{-1})^2=(A_3^2)^{-1}$), and define $y_2=r_2\sigma_2$, where r_2 is chosen in such a way that $(\bar{x},y_2)\in\partial\Sigma^A_\varepsilon$.

To show that such r_2 exists, consider the function $f(r)=\Psi(x,r\sigma)$, where Ψ is defined in (3.137). Since f is continuous, $f(0)=0$, and $\lim_{r\to\infty}f(r)=\infty$, there exists r_2 such that $f(r_2)=\varepsilon$, that is, $y_2=r_2\sigma$.

Let us define $\tau_2=N(\bar{x},y_2)$. Note that, by construction,

$$A_3^{-1}(\bar{x},y_1)y_1\cdot A_3^{-1}(\bar{x},y_1)y_2=r_2A_3^{-2}(\bar{x},y_1)y_1\cdot\sigma_2=0\,.$$

Therefore

$$\begin{aligned}|\tau_1\cdot\tau_2|=|N(\bar{x},y_1)\cdot N(\bar{x},y_2)|&=\frac{|A_3^{-1}(\bar{x},y_1)y_1\cdot A_3^{-1}(\bar{x},y_2)y_2|}{|A_3^{-1}(\bar{x},y_1)y_1||A_3^{-1}(\bar{x},y_2)y_2|}\\&\le\frac{|A_3^{-1}(\bar{x},y_2)-A_3^{-1}(\bar{x},y_1)||y_2|}{|A_3^{-1}(\bar{x},y_2)y_2|}\le L\Lambda|y_2-y_1|\le\delta_2\varepsilon\,,\end{aligned}$$

where δ_2 depends only on L, Λ.

Next, fix y_3 such that $(x,y_3)\in\partial\Sigma^A_\varepsilon$ and y_3 is orthogonal to both $A_3^{-2}(\bar{x},y_1)y_1$ and $A_3^{-2}(\bar{x},y_2)y_2$, and define $\tau_3=N(\bar{x},y_3)$. Performing the same computation as before, we find that $\tau_3\cdot\tau_1\le\delta_2\varepsilon$, $\tau_3\cdot\tau_2\le\delta_2\varepsilon$. To conclude, it suffices to iterate this argument a finite number of times.

We are now in position to prove the lemma. Fix $z=(x,y)\in B_{1/2}\cap\partial\Sigma^A_\varepsilon$. Recall that $\tau_j=N(\bar{x},y_j)$, where $(\bar{x},y_j)\in B_{1/2}\cap\partial\Sigma^A_\varepsilon$. Let $T=(t_{ij})$ denote the matrix with entries $t_{ij}=\tau_i\cdot\tau_j$. Let $\|\cdot\|_{\mathbb{R}^{n,n}}$ be a chosen matrix norm. By construction, we have $\|T-\mathbb{I}_n\|_{\mathbb{R}^{n,n}}\le c\varepsilon$, where c depends only on n, δ_2. Thus, for ε sufficiently small, $\det T>0$ and T is invertible.

Fix $i\in\{1,\dots,n\}$. We can decompose $e_{y_i}=\sum_{j=1}^n\alpha_j\tau_j$ with respect to the basis τ_j, where the coefficients $\alpha=(\alpha_j)_{j=1,\dots n}$ satisfy the linear system $T\alpha=\beta$, with $\beta=(e_{y_i}\cdot\tau_j)_{j=1,\dots,n}$. Since T is invertible, it follows that $|\alpha_i|\le c$, where c does not depend on ε. Therefore, we have

$$\begin{aligned}|G(z)\cdot e_{y_i}|=|G(z)\cdot\sum_{j=1}^n\alpha_j\tau_j|&\le c\sum_{j=1}^n|G(z)\cdot(0,N(\bar{x},y_j))|\\&\le c\Big(\sum_{j=1}^n|G(z)\cdot\nu(\bar{x},y_j)|+\sum_{j=1}^n|G(z)\cdot\tilde{\nu}(\bar{x},y_j)|\Big)\,.\end{aligned}$$

Next, using that $G(z)\cdot\nu(z)=0$ for every $z\in B_{1/2}\cap\partial\Sigma^A_\varepsilon$, and that $\tilde{\nu}(z)\le c\varepsilon$, we get

$$\begin{aligned}|G(z)\cdot e_{y_i}|&\le c\Big(\sum_{j=1}^n|(G(z)-G(\bar{x},y_j))\cdot\nu(\bar{x},y_j)|+\varepsilon\|G\|_{L^\infty(B_{1/2}\setminus\Sigma^A_\varepsilon)}\Big)\\&\le c\Big([G]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma^A_\varepsilon)}\sum_{j=1}^n(\delta_1\varepsilon+|y-y_j|)^\alpha+\varepsilon\|G\|_{L^\infty(B_{1/2}\setminus\Sigma^A_\varepsilon)}\Big)\,,\end{aligned}$$

where we used that, by construction, $(\bar{x}, y_i) \in B_{1/2} \cap \partial\Sigma_\varepsilon^A$. Finally, since $|y|, |y_j| \leq c\varepsilon$ by the uniform ellipticity condition, we obtain

$$|G(z) \cdot e_{y_i}| \leq c\varepsilon^\alpha \|G\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon^A)} \, .$$

$\square$

Lemma 3.A.3. *Let* $A \in C^1(B_1; \mathbb{R}^{d,d})$, *and let* $\Phi \in C^1(B_1; \mathbb{R}^d)$ *given by*

$$\Phi(z) = (x, A_3^{-\frac{1}{2}}(z)y) \, .$$

Then, there exists $0 < R \leq 1$ *which depends on* $L := \|A\|_{C^1(B_1)}$, λ, Λ, d, n *such that the following holds:*

i) *the function* $\Phi : B_R \to \mathbb{R}^d$ *is injective. Moreover, there exists* $c > 0$ *such that*

$$c^{-1}|z_1 - z_2| \leq |\Phi(z_1) - \Phi(z_2)| \leq c|z_1 - z_2| \, , \qquad \text{for every } z_1, z_2 \in B_R; \tag{3.144}$$

ii) $\Phi : B_R \to \Phi(B_R)$ *is a* C^1*-diffeomorphism and it holds*

$$\frac{1}{2}\Lambda^{-\frac{n}{2}} \leq \det J_\Phi(z) \leq \frac{3}{2}\lambda^{-\frac{n}{2}} \, ;$$

iii) *the matrix* $J_\Phi^\top J_\Phi$ *is symmetric and uniformly elliptic. In particular*

$$\frac{1}{2}\min\{1, \Lambda^{-1}\}|\zeta|^2 \leq J_\Phi^\top J_\Phi(z)\zeta \cdot \zeta \leq 2\max\{1, \lambda^{-1}\}|\zeta|^2 \qquad \text{for every } \zeta \in \mathbb{R}^d \, ;$$

iv) *Let* $\varepsilon < \varepsilon_0$, *where* ε_0 *is as in Lemma 3.A.1. Given* $z \in B_R \cap \partial\Sigma_\varepsilon^A$, *consider the projection map* $\Pi_\varepsilon(z) : \mathbb{R}^d \to T_z\partial\Sigma_\varepsilon^A \simeq \mathbb{R}^{d-1}$ *given by*

$$\Pi_\varepsilon(z)\xi = \xi - (\xi \cdot \nu(z))\nu(z) \, , \quad \xi \in \mathbb{R}^d \, .$$

Then

$$\frac{(2^{-1}\min\{1, \Lambda^{-1}\})^d}{2\max\{1, \lambda^{-1}\}} \leq \det(\Pi_\varepsilon \circ J_\Phi^\top J_\Phi \circ \Pi_\varepsilon) \leq \frac{(2\max\{1, \lambda^{-1}\})^d}{2^{-1}\min\{1, \Lambda^{-1}\}} \, .$$

Proof. Let us prove *i*). To ease the notations, we call $P = A_3^{-\frac{1}{2}}$. Denote $P = (p_{ij})_{i,j=1,\dots n}$ and $L = \|P\|_{C^1(B_1)}$. Let $z_1, z_2 \in B_{R_1}$, where $0 < R_1 \leq 1$ will be specified later. Obviously

$$\begin{aligned}|\Phi(z_1) - \Phi(z_2)| &\leq |x_1 - x_2| + |(P(z_1) - P(z_2))y_1| + |P(z_2)(y_1 - y_2)| \\ &\leq |x_1 - x_2| + L|z_1 - z_2| + \lambda^{-\frac{1}{2}}|y_1 - y_2| \leq c|z_1 - z_2| \, ,\end{aligned}$$

where $c > 0$ depends only on L, λ.

By similar computations,

$$|P(z_1)y_1 - P(z_2)y_2| \geq |P(z_1)(y_1 - y_2)| - |(P(z_1) - P(z_2))y_2| \geq \Lambda^{-\frac{1}{2}}|y_1 - y_2| - L|z_1 - z_2||y_2|,$$

hence,

$$|\Phi(z_1) - \Phi(z_2)| \geq \min\{1, \Lambda^{-\frac{1}{2}}\}|z_1 - z_2| - LR_1|z_1 - z_2| \geq c^{-1}|z_1 - z_2|,$$

by choosing $R_1 > 0$ small enough, such that $\min\{1, \Lambda^{-\frac{1}{2}}\} - LR_1 > c^{-1}$. Then, (3.144) follows and the map Φ is injective.

Let us prove *ii*). The Jacobian of Φ is given by

$$J_\Phi = \begin{pmatrix} \mathbb{I}_{d-n} & 0 \\ P_x & P + P_y \end{pmatrix},$$

where

$$P_x = (p^x_{ij})_{i=1,\dots n, j=1\dots,d-n} \qquad p^x_{ij} = \sum_{k=1}^{n} \partial_{x_j} p_{i,k}(z) y_k\,,$$

$$P_y = (p^y_{ij})_{i,j=1,\dots,n} \qquad p^y_{ij} = \sum_{k=1}^{n} \partial_{y_j} p_{i,k}(z) y_k\,.$$

We compute (recall that $P = A_3^{-\frac{1}{2}}$ is invertible)

$$\det J_\Phi = \det(P + P_y) = \det(P)\det(\mathbb{I}_n + P^{-1}P_y)\,. \tag{3.145}$$

First we notice that, thanks to (3.132), it holds

$$\Lambda^{-\frac{n}{2}} \le \det P(z) \le \lambda^{-\frac{n}{2}}\,. \tag{3.146}$$

Let now $\|\cdot\|_{\mathbb{R}^{n,n}}$ be a chosen matrix norm. We have

$$\|P^{-1}P_y\|_{\mathbb{R}^{n,n}} \le \|P^{-1}\|_{\mathbb{R}^{n,n}} \|P_y\|_{\mathbb{R}^{n,n}} \le cL^2|y|\,.$$

Since the determinant is continuous with respect to any matrix norm, we infer that there exists $R_2 \le R_1$ such that for every $z \in B_{R_2}$ it holds

$$\frac{1}{2} \le \det(\mathbb{I}_n + P^{-1}P_y) \le \frac{3}{2}\,. \tag{3.147}$$

By (3.145), (3.146) and (3.147) we readily get that

$$\frac{1}{2}\Lambda^{-\frac{n}{2}} \le \det J_\Phi(z) \le \frac{3}{2}\lambda^{-\frac{n}{2}} \tag{3.148}$$

holds for every $z \in B_{R_2}$, thus proving *ii*).

Let now prove *iii*). Call $M = J_\Phi^\top J_\Phi$. Obviously, M is symmetric and thanks to (3.148) is invertible in B_{R_2}. Moreover, via a straightforward computation (recall that P is symmetric and $P^2 = A_3^{-1}$) we find that $M = M_1 + M_2$ where

$$M_1 = \begin{pmatrix} \mathbb{I}_{d-n} & 0 \\ 0 & A_3^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} P_x^\top P_x & (P+P_y)^\top P_x \\ P_x^\top (P+P_y) & P^\top P_y + P_y^\top P + P_y^\top P_y \end{pmatrix}.$$

We immediately see that there exists a constant $c = c(L) > 0$ such that $\|M_2\|_{\mathbb{R}^{d,d}} \le c|y|$. Moreover, thanks to (3.131) we have

$$\min\{1, \Lambda^{-1}\}|\zeta|^2 \le M_1(z)\zeta\cdot\zeta \le \max\{1, \lambda^{-1}\}|\zeta|^2\,.$$

Thus

$$(\min\{1,\Lambda^{-1}\} - c|y|)|\zeta|^2 \leq M(z)\zeta\cdot\zeta \leq (\max\{1,\lambda^{-1}\} + c|y|)|\zeta|^2\,.$$

Hence, there exists $R \leq R_2$ such that for every $z \in B_R$ the matrix M satisfies

$$\frac{1}{2}\min\{1,\Lambda^{-1}\}|\zeta|^2 \leq M(z)\zeta\cdot\zeta \leq 2\max\{1,\lambda^{-1}\}|\zeta|^2 \qquad \text{for every } \zeta\in\mathbb{R}^d\,.$$

As for *iv*), let us call $N = \Pi_\varepsilon \circ M \circ \Pi_\varepsilon$. We can represent Π_ε via a $\mathbb{R}^{d-1,d}$ matrix Q. Thus, $N = QMQ^\top \in \mathbb{R}^{d-1,d-1}$. By the Cauchy interlacing theorem (also known as Poincaré separation theorem), the eigenvalues $\{\mu_j\}_{j=1,\dots,d}$ of M are related to the eignevalues $\{\tilde{\mu}_i\}_{i=1,\dots,d-1}$ of N via the formula

$$\mu_{i+1} \leq \tilde{\mu}_i \leq \mu_i\,.$$

Therefore, using *iii*), we infer

$$\frac{(2^{-1}\min\{1,\Lambda^{-1}\})^d}{2\max\{1,\lambda^{-1}\}} \leq \frac{\det M}{\mu_1} \leq \det N \leq \frac{\det M}{\mu_d} \leq \frac{(2\max\{1,\lambda^{-1}\})^d}{2^{-1}\min\{1,\Lambda^{-1}\}}\,.$$

This completes the proof of *iv*) and of the lemma. □

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